The Hodge Decomposition Theorem

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Contents

Introduction

This text is written as my thesis for the bachelor program of mathematics and physics at Utrecht University. The content of the thesis was supposed to be that of a masters course and the reader is assumed to have followed an introductory course in Differential Geometry. The main source of information for this thesis was the book *Differential analysis on complex manifolds* by Raymond O. Wells and the explanation of Gil Cavalcanti about the subjects covered. The aim of this thesis is to proof the Hodge Decomposition Theorem which says that the de Rham groups can be written as the direct sum of the Dolbeault groups of the same total degree.
The organization of the thesis is as follows. We will start by recalling the basic definitions of manifolds and vector bundles in the first chapter. We assume the reader to be familiar with the most of this. Also we introduce the concepts of almost complex and complex structures and differential forms of type (p, q). We finish the chapter with a discussion of integrability and the Newlander-Nirenberg theorem.
In the second chapter we define sheaves and presheaves. We will skip most of the details regarding this subject and state the most important results mainly for completeness and to define the cohomology in a rather general way, which has some advantages. For the skipped proofs and details we refer the reader to [**4**]. We will also give some explicit representations of these cohomology groups in some given geometric situations.
The third chapter is concerned with the local description of vector bundles and sections of vector bundles. These are used to define Hermitian metrics and inner products, concepts which are well known from the classical $\mathbb{R}^n$ case, in the more general setting of complex manifolds.
In the fourth chapter we will again skip most of the details (here we also refer the reader to [**4**] for the details) since the theory of pseudodifferential operators is much too big to treat in this thesis. The main results from this chapter are that it is possible in the various cases we are interested in to represent the earlier defined cohomology groups in terms of harmonic functions, which are defined to be the kernel of elliptic operators. We will treat in detail the representation of these cohomology groups because they will play an important role in the proof of the Hodge Decomposition Theorem.
In the fifth chapter we will apply the results of the previous two chapters to the study of compact complex manifolds. The most important results in the first part of the chapter are the two duality theorems known as Poincaré and Serre duality. The biggest part of the fifth chapter is concerned with representation theory and a specific representation of $\mathfrak{sl}(2, \mathbb{C})$ which gives us the last results needed for the proof of the decomposition theorem. After this we define what it means for a manifold

to be of Kähler type. These type of manifolds are the setting in which the Hodge decomposition holds. We end the chapter by giving a rather short proof of the Hodge Decomposition Theorem, which is possible by the work done in the previous chapters, and state some of its direct consequences.

CHAPTER 1

Manifold & Vector Bundles

In this first chapter we will be concerned with the basic definitions and concepts used in the rest of the text. Since we assume the reader to be familiar with concepts such as manifolds, differential forms on differentiable manifolds and partitions of unity, the goal of the first part of the chapter is to introduce notation. We will start by recalling the definitions of differentiable and complex manifolds and vector bundles over manifolds. After this we will introduce almost complex structures and differential forms of type (p,q). We finish the chapter with a discussion of integrability and the Newlander-Nirenberg theorem.

1. Manifolds & $\mathcal{S}$-structures

We will begin with some basic definitions. As usual let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers and we write $\mathbb{K}$ if we want to denote either of these fields. Given an open $D \subset \mathbb{K}^n$ we have the following $\mathbb{K}$-vector spaces of functions on D:

- $\mathbb{K} = \mathbb{R}$ We let $\mathcal{E}(D)$ denote the real valued *smooth (differentiable)* functions on D. That is, $f \in \mathcal{E}(D)$ if and only if f is a real valued function and the partial derivatives of all order exist and are continuous at every point in D.
- $\mathbb{K} = \mathbb{C}$ We let $\mathcal{O}(D)$ denote the complex valued *holomorphic* functions on D. That is, $f \in \mathcal{O}(D)$ if and only if f can be represented by a convergent power series around every point of D.

If we want to denote either of these families of functions, we write $\mathcal{S}$. For the elements of $\mathcal{S}$ on an open $D \subset \mathbb{K}^n$ we write $\mathcal{S}(D)$.

Definition 1.1. A *topological n-manifold* is a Hausdorff topological space with a countable basis which is locally homeomorphic to an open subset of $\mathbb{R}^n$. The integer n is called the *topological dimension* of the manifold.

Definition 1.2. An $\mathcal{S}$-structure, $\mathcal{S}_M$ on a $\mathbb{K}$-manifold M is a family of $\mathbb{K}$-valued continuous functions defined on the open sets of M such that the following holds.

(1) For every $p \in M$, there exists an open neighborhood U of p and a homeomorphism $h : U \to U'$, where $U \subset \mathbb{K}^n$ is open, such that for any open set $V \subset U$

$$f : V \to \mathbb{K} \in \mathcal{S}_M \text{ if and only if } f \circ h^{-1} \in \mathcal{S}(h(V)).$$

(2) If $f : U \to \mathbb{K}$, where $U = \bigcup_i U_i$ with U_i open in M, then $f \in \mathcal{S}_M$ if and only if, $f|_{U_i} \in \mathcal{S}_M$ for all i.

Definition 1.3. A manifold M with a $\mathcal{S}$-structure is called an $\mathcal{S}$-*manifold* and is denoted by $(M, \mathcal{S}_M)$. The elements of $\mathcal{S}_M$ are called $\mathcal{S}$-*functions* on M. If N is a subset of M then a $\mathcal{S}$-function on N is defined to be the restriction of an $\mathcal{S}$-function defined on some open containing N.

We denote by $\mathcal{S}_M|_N$ all functions on relatively open subsets of N which are restrictions of $\mathcal{S}$-functions on open subsets of M. An open subset $U \subset M$ and a homeomorphism $h : U \to U' \subset K^n$ as above is called an $\mathcal{S}$-*coordinate system*. A collection $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ is called an *atlas* for M.
For the classes of functions defined above we have the following names for the $\mathcal{S}$-manifolds:

- $\mathcal{S} = \mathcal{E}$: *differentiable* (or smooth) manifold and $\mathcal{E}_M$ is called a *differentiable* structure. The functions in $\mathcal{E}$ are called $\mathbb{C}^\infty$ or *smooth functions* on open subsets of M.
- $\mathcal{S} = \mathcal{O}$: *complex* manifold and $\mathcal{O}_M$ is called a *holomorphic* structure. The functions in $\mathcal{O}$ are called *holomorphic functions* on open subsets of M.

Now we have defined manifolds with some structure on them we can define maps between manifolds preserving this structure.

Definition 1.4. An $\mathcal{S}$-*morphism* $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map from M to N, such that

$$f \in \mathcal{S}_N \text{ implies } f \circ F \in \mathcal{S}_M.$$

An $\mathcal{S}$-*isomorphism* is an $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ such that F is a homeomorphism and $F^{-1} : (N, \mathcal{S}_N) \to (M, \mathcal{S}_M)$ is also a $\mathcal{S}$-morphism.

Given an $\mathcal{S}$-manifold $(M, \mathcal{S}_M)$ and two coordinate systems, $h_1 : U_1 \to K^n$ and $h_2 : U_2 \to K^n$ such that $U_1 \cap U_2 \neq \varnothing$ then it follows that

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2) \tag{1.1}$$

is an $\mathcal{S}$-morphism on open subsets of $(\mathbb{K}^n, \mathcal{S}_{K^n})$. To see this note that if we are given $f \in \mathcal{S}_{\mathbb{K}^n}$, then $f \circ h_2 \in \mathcal{S}_M$ by Definition 1.2 (1) since $f \circ h_2 \circ h_2^{-1} = f \in \mathcal{S}_{\mathbb{K}^n}$. By the same definition, we have $f \circ h_2 \circ h_1^{-1} \in \mathcal{S}_{K^n}$. Conversely, assume we are given an open cover $\{U_\alpha\}_{\alpha \in A}$ of a topological manifold M and a set $\{h_\alpha : U_\alpha \to V \subset \mathbb{K}^n\}$, such that for every $\alpha, \beta \in A$ we have $h_\alpha \circ h_\beta^{-1}$ is an $\mathcal{S}$-morphism. We can pull $\mathcal{S}$ back by the $\{h_\alpha\}_{\alpha \in A}$ to obtain a $\mathcal{S}$-structure on M. That is, we define $\mathcal{S}_M$ to be $\{f : U \to \mathbb{K}\}$ such that $U \subset M$ is open, and $f \circ h_\alpha^{-1} \in \mathcal{S}(h_\alpha(U \cap U_\alpha))$ for all $\alpha \in A$.
Just as in the case of the $\mathcal{S}$-structures we have the following special names for the $\mathcal{S}$-morphisms and the $\mathcal{S}$-isomorphisms respectively:

- $\mathcal{S} = \mathcal{E}$: *differentiable (smooth) mapping* and *diffeomorphism.*
- $\mathcal{S} = \mathcal{O}$: *holomorphic mapping* and *biholomorphism.*

2. Vector bundles & bundle maps

Definition 1.5. A $\mathbb{K}$-*vector bundle of rank* r is a surjective continuous map $\pi : E \to X$ from a Hausdorff space E, the *total space*, onto a Hausdorff space X, the *base space*, which satisfies the following conditions.

(1) For all $p \in X$ the *fiber* over the *basepoint* p, $E_p := \pi^{-1}(p)$, is a $\mathbb{K}$-vector space of dimension r.

(2) For all $p \in X$ there is a neighbourhood U of p and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{K}^r$ that preserves basepoints, i.e. $h(E_p) \subset \{p\} \times \mathbb{K}^r$.

(3) For all $u \in U$ the composition $h^u : E_u \xrightarrow{h} \{u\} \times \mathbb{K}^r \xrightarrow{proj.} \mathbb{K}^r$ is a $\mathbb{K}$-vector space isomorphism.

A pair (U, h) is called a *local trivialization*. If E and X are $\mathcal{S}$-manifolds, π is an $\mathcal{S}$-morphism and the local trivializations are $\mathcal{S}$-isomorphisms we call $\pi : E \to X$ an $\mathcal{S}$-bundle.

Definition 1.6. Let $E \xrightarrow{\pi} X$ be an $\mathcal{S}$-bundle. Then a $\mathcal{S}$-*subbundle*, $F \subset E$ is a $\mathcal{S}$-submanifold that satisfies the following condition:

(1) $F \cap E_x$ is a vector subspace of E_x.
(2) $\pi|_F : F \to X$ has an $\mathcal{S}$-bundle structure induced by the $\mathcal{S}$-bundle structure of E.

The second condition means that there exist local trivialization such that the following diagram commutes.

$$\begin{array}{ccc} E|_U & \xrightarrow{\cong} & U \times \mathbb{K}^r \\ \uparrow i & & \uparrow id \times j \\ F|_U & \xrightarrow{\cong} & U \times \mathbb{K}^s \end{array}$$

Here we have that $s \leq r$, i is the inclusion map of F in E and j is the inclusion map of $\mathbb{K}^s$ as subspace of $\mathbb{K}^r$.
Again we have the following special names for $\mathcal{S}$-bundles

- $\mathcal{S} = \mathcal{E}$: *differentiable vector bundle.*
- $\mathcal{S} = \mathcal{O}$: *holomorphic vector bundle.*

From Definition 1.5 we see that if we are given two local trivializations (U_α, h_α) and (U_β, h_β) we get from the map

$$h_\alpha \circ h_\beta^{-1} : \pi^{-1}(U_\alpha \cap U_\beta) \times \mathbb{K}^r \to (U_\alpha \cap U_\beta) \times \mathbb{K}^r$$

an induced map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{K})$$

called a *transition function*, defined by

$$g_{\alpha\beta}(p) : h_\alpha^p \circ (h_\beta^p)^{-1} : K^r \to K^r.$$

The transition functions satisfy the following conditions which follow immediately from the definition,

$$g_{\alpha\gamma} \circ g_{\gamma\beta} \circ g_{\beta\alpha} = I_r \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \qquad (1.2)$$
$$g_{\alpha\alpha} = I_r \text{ on } U_\alpha.$$

Conversely if we are given a manifold M with an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ where we have given for every nonempty intersection $U_\alpha \cap U_\beta$ an $\mathcal{S}$ function $g_{\alpha\beta}$ which satisfy equation 1.2, we can construct a vector bundle having the $g_{\alpha\beta}$ as transition functions. We define

$$\tilde{E} = \dot{\bigcup_{\alpha \in A}} U_\alpha \times \mathbb{K}^r$$

equiped with the natural topology and $\mathcal{S}$-structure. We define an equivalence relation on $\tilde{E}$ by

$$(x,v) \sim (y,w) \text{ for } (x,v) \in U_\alpha \times \mathbb{K}^r \text{ and } (y,w) \in U_\beta \times \mathbb{K}^r$$

if and only if

$$x = y \qquad \text{and} \qquad w = g_{\beta\alpha} v.$$

By condition 1.2 we see that $v = I_r v = g_{\alpha\alpha} v$. Also, given $w = g_{\beta\alpha} v$ and $u = g_{\gamma\beta} w$ we get by the same condition that $v = I_r v = g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} v = g_{\alpha\gamma} g_{\gamma\beta} w = g_{\alpha\gamma} u$ from which it follows that $u = g_{\gamma\alpha} v$. Therefore we have, using $g_{\beta\alpha}^{-1} = g_{\alpha\beta}$, $v \sim w$ implies $w \sim v$, $v \sim v$, and $v \sim u$ whenever $u \sim w$ and $w \sim v$ so the equivalence relation is well-defined. Now let $E = \tilde{E}/\sim$, the set of all equivalence classes, equipped with the quotient topology and we let $\pi : E \to M$ given by the projection $E \ni [(p,v)] \mapsto p \in M$. To see that E also has a $\mathcal{S}$ structure we note that we can use the $\mathcal{S}$ structures on M (and thus on the U_α's) and $\mathbb{K}^r$ to get coordinate functions that satisfy condition 1.1 and these give then the $\mathcal{S}$ structure on E. From this it follows that the E constructed is actually a $\mathcal{S}$-vector bundle.

Given a differentiable manifold M, we define

$$\mathcal{E}_{M,p} := \varinjlim_{p \in \underset{open}{\subset} M} \mathcal{E}_M(U)$$

to be the *algebra of germs of smooth functions* at the point $p \in M$. An element of $\mathcal{E}_{M,p}$ is called a *germ* of a smooth function at p. We could also have defined two smooth functions, f and g to be equivalent whenever they coincide on some open neighborhood of p and take the set of equivalence classes. It is clear that $\mathcal{E}_{M,p}$ is an algebra over $\mathbb{R}$ under pointwise multiplication and addition of (in the second definition representatives of) smooth functions. Now we define the *tangent space*, T_pM to M at the point p to be the set of all derivations of $\mathcal{E}_{M,p}$. Recall that a derivation $D : \mathcal{E} \to \mathbb{R}$ of the algebra $\mathcal{E}_{M,p}$ is a homomorphism satisfying, $D(fg) = f(p)D(g) + D(f)g(p)$. See for example [**2**, **3**].

Since M is a differentiable manifold we can find a diffeomorphism $h : U \to \mathbb{R}^n$ on an open neighborhood U of p. We let h^* and h_* denote the pullback and push forward by h respectively. Then it is well known that , denoting by x_i the standard coordinate functions in $\mathbb{R}^n$, the set of derivations $\{\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n}\}$ form a basis for $T_{h(p)}\mathbb{R}^n$ and that $h_* : T_pM \xrightarrow{\cong} T_{h(p)}\mathbb{R}^n$ is an isomorphism. So we have that at each point $p \in M$ the tangent space T_pM is a vector space with the same dimension as the manifold.

Definition 1.7. Let $f : M \to N$ be a smooth map between differentiable manifolds. The *differential mapping* of f at p is the linear map

$$df : T_pM \to T_{f(p)}N$$

defined by

$$df(v)(g) = v(g \circ f)$$

for $v \in T_pM$ and $g \in \mathcal{E}_{N,f(p)}$.

Now we construct the *tangent bundle*, let

$$TM = \dot{\bigcup_{p \in M}} T_pM$$

with the natural projection $\pi : TM \to M$ defined by $\pi(v) = p$ if $v \in T_pM$.

Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in A}$ be an atlas for M and denote $TU_\alpha = \pi^{-1}(U_\alpha)$. We know from the definition of the differential that for $v \in TU_\alpha$ we have

$$dh_\alpha(v) = \sum_{i=1}^{n} v_i(p) \left.\frac{\partial}{\partial x_i}\right|_{h_\alpha(p)} \in T_{h_\alpha(p)}\mathbb{R}^n.$$

We define

$$\psi_\alpha : TU_\alpha \to U_\alpha \times \mathbb{R}^n$$

by

$$\psi_\alpha(v) = (p, v_1(p), \dots, v_n(p))$$

for $v \in T_pM \subset TU_\alpha$. It is clear that ψ_α is a fiber preserving bijection. With the same notation as in Definition 1.5 we define $\psi_\alpha^p : T_pM \xrightarrow{\psi_\alpha} U_\alpha \times \mathbb{R}^n \xrightarrow{proj.} \mathbb{R}^n$ which clearly is a vector space isomorphism. Now we can define transition functions

$$g_{\alpha\beta} : U_\beta \cap U_\alpha \to GL(n, \mathbb{R})$$

by

$$g_{\alpha\beta}(p) = \psi_\alpha^p \circ (\psi_\beta^p)^{-1} : \mathbb{R}^n \to \mathbb{R}^n.$$

From the above expressions for ψ_α in local coordinates it is clear that the $g_{\alpha\beta}$ are smooth functions on $U_\alpha \cap U_\beta$. We give TM the topology where $U \subset TM$ is open if and only if $\psi_\alpha(U \cap TU_\alpha) \subset U_\alpha \times \mathbb{R}^n$ is open for all α. This is well defined since the $g_{\alpha\beta}$ are smooth and therefore

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to: (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is a diffeomorphism for any α and β. With this topology TM becomes a differentiable vector bundle where the projection π and the local trivializations $\{(U_\alpha, \psi_\alpha)\}$ are smooth maps.

Analogously if we let $(X, \mathcal{O}_x)$ be a complex manifold of complex dimension n, we let

$$\mathcal{O}_{X,x} := \varinjlim_{x\in U \underset{open}{\subset} X} \mathcal{O}(U)$$

be the $\mathbb{C}$ algebra of germs of holomorphic functions at $x \in X$ and let T_xX be the *holomorphic* (or *complex*) tangent space of X at x, consisting of all the derivations of $\mathcal{O}_{X,x}$. The set of derivations $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$ form a basis for T_xX and in the same way as before we can make TX into a complex vector bundle, with all fibers isomorphic to $\mathbb{C}^n$.

Next we introduce the notion of maps between $\mathcal{S}$ bundles.

Definition 1.8. Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be $\mathcal{S}$-bundles over X. Then,

(1) a *homomorphism of* $\mathcal{S}$*-bundles* $f : E \to F$ is a fiber preserving $\mathcal{S}$-morphism of the total spaces, which is $\mathbb{K}$-linear mapping on each fiber.
(2) a *isomorphism of* $\mathcal{S}$*-bundles* is a fiber preserving $\mathcal{S}$-isomorphism of the total space, which is a $\mathbb{K}$-linear isomorphism on each fiber.

For bundles over different base spaces we have:

Definition 1.9. Let $\pi_E : E \to X$ and $\pi_F : F \to Y$ be two $\mathcal{S}$-bundles over different base spaces. Then an $\mathcal{S}$*-bundle morphism* is an $\mathcal{S}$-morphism of total spaces $f : E \to F$ which is a $\mathbb{K}$-linear map between the fibers. This induces an $\mathcal{S}$-morphism of base spaces $\tilde{f}$, defined by $\tilde{f}(\pi_E e) = \pi_F(f(e))$, which makes the following diagram commutative.

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow{\tilde{f}} & Y \end{array}$$

Note that taking $X = Y$ and $\tilde{f} = Id$ reduces to Definition 1.8.

Proposition 1.10 *Given an* $\mathcal{S}$*-morphism* $f : X \to Y$ *and an* $\mathcal{S}$*-bundle* $\pi : E \to Y$*, then there exists an* $\mathcal{S}$*-bundle* $\pi' : E' \to X$ *and an* $\mathcal{S}$*-bundle morphism* g *such that the following diagram commutes.*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

Moreover, E' *is unique up to equivalence. We call* E' *the pullback of* E *by* f *and denote it by* f^*E*.*

Proof. Define

$$E' = \{(x,e) \in X \times E | f(x) = \pi(e)\}.$$

We have the natural projections $\pi' : E' \to X$ defined by $(x,e) \mapsto x$ and $g : E' \to E$ defined by $(x,e) \mapsto e$. Since each $E_{f(x)}$ is a $\mathbb{K}$- vector space we can make each $E'_x = \{x\} \times E_{f(x)}$ into a $\mathbb{K}$-vector space. Given a local trivialization (U,h) of E which by definition says that

$$E|_U \xrightarrow{h} U \times \mathbb{K}^n$$

then it is clear that

$$E'|_{f^{-1}(X)} \to f^{-1}(U) \times \mathbb{K}^n$$

is a local trivialization of E'. To see the uniqueness suppose that another bundle $\tilde{\pi} : \tilde{E} \to X$ and a bundle morphism $\tilde{g} : \tilde{E} \to E$ are given and that again the following diagram commutes.

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{g}} & E \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

Define $h : \tilde{E} \to E$ by

$$h(\tilde{e}) = (\tilde{\pi}(\tilde{e}), \tilde{g}(\tilde{e})) \in \{\pi(\tilde{e})\} \times E.$$

This is a bundle homomorphism since by commutativity we have $f(\tilde{\pi}(\tilde{e})) = \pi(\tilde{g}(\tilde{e}))$ and therefore $h(\tilde{e}) \in E'$. Since h is also an isomorphism on fibers it is a bundle isomorphism. □

Let $E \xrightarrow{f} F$ be a vector bundle homomorphism of $\mathbb{K}$ vector bundles over a space X then we define

$$\operatorname{Ker} f = \dot{\bigcup_{x \in X}} \operatorname{Ker} f_x$$
$$\operatorname{Im} f = \dot{\bigcup_{x \in X}} \operatorname{Im} f_x$$

where $f_x = f|_{E_x}$.
Suppose now that we have a sequence of vector bundle homomorphisms over the same base space X,

$$\ldots \to E \xrightarrow{f} F \xrightarrow{g} G \to \ldots$$

then we say the sequence is exact at F if $\operatorname{Ker} g = \operatorname{Im} f$. A *short exact sequence* of vector bundles is defined to be a sequence of vector bundles of the form,

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

which is exact at E', E, and E''. It then follows that f is injective, g is surjective and $\operatorname{Im} f = \operatorname{Ker} g$ is a subbundle of E.
Given two vector spaces A and B, we can combine them to form new vector spaces. Some of the important examples include:

- $A \oplus B$, the direct sum.
- $A \otimes B$, the direct product.
- $\wedge^k A$, the antisymmetric tensor product of degree k.
- $S^k(A)$, the symmetric tensor product of degree k.

Since the fibers of vector bundles are vector spaces we can extend the above constructions to vector bundles (over the same base space) by performing them fiber wise. For example, given two vector bundles $\pi_E : E \to X$ and $\pi_F : F \to X$ then we define

$$E \oplus F = \dot{\bigcup_{p \in X}} E_p \oplus F_p$$

with the projection $\pi : E \oplus F \to X$ given by $\pi(v) = p$ if $v \in E_p \oplus F_p$. It is clear that we can also extend the local trivializations and transition functions thus $E \oplus F$ becomes a vector bundle. Moreover if E and F are $\mathcal{S}$-bundles $E \oplus F$ will also be an $\mathcal{S}$-bundle. Using vector bundles we can generalize the notion of a function in the following way.

Definition 1.11. An $\mathcal{S}$-*section* of a $\mathcal{S}$-bundle $E \xrightarrow{\pi} X$ is an $\mathcal{S}$-morphism $s : X \to E$ satisfying

$$\pi \circ s = 1_X,$$

which says that s maps a point $x \in X$ to E_x the fiber over x.

We denote the $\mathcal{S}$-sections of E over X by $\mathcal{S}(X, E)$, the sections of $E|_U$ over $U \subset X$ will be denoted by $\mathcal{S}(U, E)$. Consider the trivial bundle $M \times \mathbb{R}$ over M. Then we can identify $\mathcal{E}(M, M \times \mathbb{R})$ with $\mathcal{E}(M)$, the smooth real valued functions on M. Similarly we can identify $\mathcal{E}(M, M \times \mathbb{R}^n)$ with the smooth vector valued functions on M. As we have seen arbitrary vector bundles are locally of the form $U \times \mathbb{R}^n$ thus we can view sections locally as vector valued functions. Since two different local representations of a vector bundle are related by transition functions we can view sections of vector bundles as “twisted” vector valued functions.

The fact that the fibers of vector bundles are vector spaces also makes it possible to make $\mathcal{S}(X,E)$ into a vector space under pointwise addition and multiplication by scalars, that is:

(1) For $s,t \in \mathcal{S}(X,E)$ we define $(s+t)(x) := s(x)+t(x)$ for all $x \in X$.
(2) For $s \in \mathcal{S}(X,E)$, $\lambda \in \mathbb{K}$ and $x \in X$ we define $(\lambda s)(x) := \lambda s(x)$ for all $x \in X$.

If we consider the tangent bundle $TM \xrightarrow{\pi} M$ we can use the remarks above to construct some important examples of vector bundles:

- The *cotangent bundle*, T^*M, whose fiber at $x \in M$ is given given by the $\mathbb{R}$-linear dual space to T_xM.
- The *exterior algebra bundles*, $\wedge^p TM, \wedge^p T^*M$ whose fiber at $x \in M$ is given by the antisymmetric tensor product (of degreep) of T_xM and T_x^*M respectively.
- The *symmetric algebra bundles*, S^kTM and S^kT^*M whose fiber at $x \in M$ is given by the symmetric tensor product of T_xM and T_x^*M respectively.

We define the *smooth differential forms of degree p* on the open set $U \subset M$ to be the smooth sections of the exterior cotangent bundle of degree p denoted by

$$\mathcal{E}^p(U) = \mathcal{E}(U, \wedge^p T^*M))$$

on which, as usual, we can define the exterior derivative

$$d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U),$$

see for example [**2**,**3**]. Related to sections of vector bundles is the notion of a frame of a vector bundle, which we will later use to obtain local representations for differential forms.

Definition 1.12. Let an $\mathcal{S}$ bundle $E \to X$ of rank r and an open $U \subset X$ be given. Then a *frame for E over U* is a set of r $\mathcal{S}$-sections $\{s_1, \dots, s_r\}$ that form a basis for E_x for any $x \in U$.

We note that any $\mathcal{S}$ bundle, $E \to X$ has a local frame. Moreover the existence of a local frame is equivalent to the existence of a local trivialization and the existence of a global frame is equivalent to the bundle being trivial. To see this let U be a trivializing neighborhood for E, then we have

$$h : E|_U \to U \times \mathbb{K}^r$$

which induces an isomorphism

$$h_* : \mathcal{S}(U, E|_U) \to \mathcal{S}(U, U \times \mathbb{K}^r).$$

It is clear that the set of vector valued functions $e_i : U \to U \times \mathbb{K}^r$ where e_i is the i-th (constant) coordinate functions form a frame for $U \times \mathbb{K}^r$. Since h_* is an isomorphism on fibers it follows that $\{h_*^{-1}(e_i)\}$ form a frame for $E|_U$. Conversely given a local frame $\{w_i\}$ for $E|_U$ it is easy to see that the map

$$\sum_{i=1}^{r} \alpha_i w_i(x) \mapsto (x, \alpha_1, \dots, \alpha_r)$$

sending $\sum_{i=1}^r \alpha_i w_i(x) \in E_x \subset E|_U$ to an element of $U \times \mathbb{K}^r$ defines a local trivialization.

3. Almost Complex Manifolds & $\bar{\partial}$ Operator

Let V be a real vector space then it is possible to construct from V a complex vector space, the *complexification* of V, denoted by V_c, in which we can multiply vectors in V with complex scalars. The complexification of V is given by $V \otimes_{\mathbb{R}} \mathbb{C}$, where multiplication by complex numbers is defined by $\lambda(v \otimes z) = v \otimes (\lambda z)$. Since V_c is obtained by tensoring a real vector space with the complex numbers, we also have a complex conjugation on V_c defined by $\overline{v \otimes z} = v \otimes \overline{z}$, for $v \in V$ and $\lambda, z \in \mathbb{C}$. Note that V sits inside $V \otimes_{\mathbb{R}} \mathbb{C}$ if we identify v with $v \otimes 1$ and that given a basis $\{v_1 \dots v_n\}$ of V the elements $\{v_1 \otimes 1, \dots v_n \otimes 1, v_1 \otimes i, \dots v_n \otimes i\}$ form a basis for $V \otimes_{\mathbb{R}} \mathbb{C}$ over the reals. It follows that $\dim_{\mathbb{C}} V \otimes \mathbb{C} = \dim_{\mathbb{R}} V$ and $\dim_{\mathbb{R}} V \otimes \mathbb{C} = 2 \dim_{\mathbb{R}} V$. Given a $\mathbb{R}$ linear map $f : V \to V$ this extends to a $\mathbb{C}$ linear map $\tilde{f} : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ by defining $\tilde{f}(v \otimes z) = f(v) \otimes z$ for $v \in V$ and $z \in \mathbb{C}$.

Now let V be a real vector space and assume that we are given an $\mathbb{R}$ linear isomorphism $J : V \to V$ with the property that $J^2 = -I$. We call J a *complex structure* on V. In this case there is another way to make V into a complex vector space. Namely define multiplication by complex scalars $z = \alpha + i\beta$ by

$$(\alpha + i\beta)v := \alpha v + \beta Jv, \qquad \alpha, \beta \in \mathbb{R} \qquad i = \sqrt{-1}.$$

It is clear that since $J^2 = -I$, this makes V into a complex vector space which we denote by V_J. Conversely if V is a complex vector space, then we can also view it as a vector space over $\mathbb{R}$ and then multiplication by i is an $\mathbb{R}$ linear endomorphism has all the properties of a complex structure and we denote this endomorphism by J. Note that if $\{v_1, \dots, v_n\}$ is a basis for V over $\mathbb{C}$ then $\{v_1, \dots, v_n, Jv_1, \dots Jv_n\}$ is a basis for V over $\mathbb{R}$.

Example 1.13. Consider $\mathbb{C}^n$ and let $\{z_1, \dots z_n\}$ be a complex basis for this space. Then writing $z_i = x_i + iy_i$ where $x_i = \operatorname{Re} z_i$ and $y_i = \operatorname{Im} z_i$, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \{x_1, y_1, \dots, x_n, y_n\}$. Since we have in $\mathbb{C}^n$ that

$$iz_1 = i(x_1 + iy_1) = (-y_1) + i(x_1)$$

it follows that complex multiplication in $\mathbb{C}^n$ induces a complex structure J on $\mathbb{R}^{2n}$ defined by

$$J(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n).$$

This complex structure is called the *standard complex structure* on $\mathbb{R}^{2n}$

Definition 1.14. Let X be a smooth manifold of dimension $2n$. Let J be a vector bundle isomorphism

$$J : TX \to TX,$$

such that for each $x \in X$, $J_x : T_x X \to T_x X$ is a complex structure for $T_x X$. Then J is an *almost complex structure* for the smooth manifold X. The pair (X, J) is called an *almost complex manifold.*

We see that such a manifold is indeed "almost complex" in the sense that it can be viewed as a real manifold with a complex tangent bundle.

Now let V be a real vector space with a complex structure J. We consider the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of V and extend J to a $\mathbb{C}$ linear map on the complexification as before. Then it still holds that $J^2 = -I$ thus we get that J has two eigenvalues

$\{i,-i\}$ and we denote the corresponding eigenspaces by $V^{1,0}$ corresponding to i and $V^{0,1}$ corresponding to $-i$. From this we get that

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

If $v \otimes \alpha \in V^{1,0}$ then it follows from the way we defined conjugation in $V \otimes_{\mathbb{R}} \mathbb{C}$ that

$$J(v \otimes \overline{\alpha}) = (Jv) \otimes \overline{\alpha} = \overline{(Jv) \otimes \alpha} = \overline{iv \otimes \alpha} = -i\overline{v \otimes \alpha} = -iv \otimes \overline{\alpha},$$

so we have $v \otimes \overline{\alpha} \in V^{0,1}$. We see that conjugation is an $\mathbb{R}$ linear (and $\mathbb{C}$ anti-linear) map, and thus $V^{1,0} \cong_{\mathbb{R}} V^{0,1}$. Furthermore, since we have defined $iv = Jv$ and not $-iv = Jv$ on V_J we get that V_j is $\mathbb{C}$ linearly isomorphic to $V^{1,0}$ and we will identify these two spaces with each other from now on.
Now we consider the exterior algebras of these spaces , $\wedge V_c$, $\wedge V^{1,0}$ and $\wedge V^{0,1}$. There are injections

$$\begin{array}{l} V^{1,0} \searrow \\ \qquad\qquad V_C \\ V^{0,1} \nearrow \end{array}$$

and if we define $\wedge^{p,q}V$ to be the subspace of V_c generated by elements $u \wedge w$ for $u \in \wedge^p V^{1,0}$ and $w \in \wedge^q V^{0,1}$ then we have the direct sum

$$\wedge V_c = \sum_{r=0}^{2n} \sum_{p+q=r} \wedge^{p,q} V$$

where $n = \dim_{\mathbb{C}} V^{1,0}$. We will now apply this to the cotangent bundle. Let smooth manifold X of dimension m be given. Let $TX_c = TX \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*X_c = T^*X \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of the tangent and cotangent bundle respectively. Then after forming the exterior algebra bundle $\wedge T^*X_c$ we define

$$\mathcal{E}^r(X_c) = \mathcal{E}(X, \wedge T^*X_c)$$

the *complex valued differential forms of total degree* r on X. If there is no chance of confusion with the real valued differential forms we will also write $\mathcal{E}^r(T^*X)$.
Now consider (X,J) an almost complex manifold. Then again J extends to an $\mathbb{C}$ linear vector bundle isomorphism on TX_c with eigenvalues $\{i,-i\}$. Note that these eigenvalues are fiberwise since for every $x \in X$ J_x is an isomorphism on T_xX_c. In the same way as discussed before we have $T_xX_c = T_xX^{1,0} \oplus T_xX^{0,1}$ and define $TX^{1,0}$, $TX^{0,1}$ to be the bundles of $\pm i$ eigenspaces of J respectively. We can define conjugation on TX_c to be fiberwise the conjugation defined before and then it follows again $\overline{TX^{1,0}} = TX^{0,1}$. Let $T^*X^{1,0}$ and $T^*X^{0,1}$ be the complex dual bundles of $TX^{1,0}$ and $TX^{0,1}$. Then writing $\wedge T^*X_c$, $\wedge T^*X^{1,0}$ and $\wedge T^*X^{0,1}$ for the exterior algebra bundles, we have as before

$$T^*X_c = T^*X^{1,0} \oplus T^*X^{0,1}$$

also we have injections

$$\begin{array}{l} \wedge T^*X^{1,0} \searrow \\ \qquad\qquad\qquad \wedge T^*X_c \\ \wedge T^*X^{0,1} \nearrow \end{array}$$

and defining the *complex valued differential forms of type (p,q)* to be

$$\mathcal{E}^{p,q} = \mathcal{E}(X, \wedge^{p,q} T^* X),$$

we get again a direct sum composition

$$\wedge \mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p,q}(X).$$

This decomposition gives rise to projections

$$\pi_{p,q} : \mathcal{E}^r \to \mathcal{E}^{p,q}, \qquad p+q=r.$$

Also since $d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1}(X)$ we can use the projections to define

$$\partial : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X)$$
$$\bar{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)$$

by

$$\partial = \pi_{p+1,q} \circ d$$
$$\bar{\partial} = \pi_{p,q+1} \circ d$$

and we can extend them by complex linearity to $\mathcal{E}^* = \sum_{r=0}^{\dim X} \mathcal{E}^r(X)$. Denoting complex conjugation by Q, we have the following proposition.

Proposition 1.15 *For any $f \in \mathcal{E}^*(X)$ we have*

$$Q\bar{\partial}(Qf) = \partial f.$$

Proof. Let $f \in \mathcal{E}^r(X)$, with $r = p+q$, be given. Since $Q : \mathcal{E}^{p,q} \xrightarrow{\cong} \mathcal{E}^{q,p}$ is an isomorphism it follows that $Q\pi_{p,q} f = \pi_{q,p} Qf$. Furthermore since the exterior derivative is complex linear we have that $Q(df) = dQf$. Combining this we get for $f \in \mathcal{E}^r$

$$Q\bar{\partial}(Qf) = \sum_{p+q=r} Q\bar{\partial}(Qf) = \sum_{p+q=r} Q\pi_{p,q+1} Qf = \sum_{p+q=r} QQ\pi_{q+1,p} f = \sum_{q+p=r} \pi_{p+1,q} f = \partial f.$$

Where in the last step we used that $Q^2 = I$ and the fact that p and q are dummy indices since we sum over all p, q such that $p+q=r$. □

From this proposition we see that $\bar{\partial}^2 = 0$ if and only if $\partial^2 = 0$. Consider $d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1}(X)$, using the decomposition of $\mathcal{E}^r$ we see that d can be decomposed as

$$d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \bar{\partial} + \dots$$

There is a special class of complex structures called *integrable* complex structures which have the property that $d = \partial + \bar{\partial}$. If this is the case then we see that

$$d^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2$$

and since we know that $d^2 = 0$ and all the operators in the sum project to a different component of $\mathcal{E}^{p+q+2}$ it follows that for integrable almost complex structures

$$\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0.$$

We now proof a proposition and an important theorem which will be the main result of this section.

Proposition 1.16 *A complex manifold induces an almost complex structure on its underlying differentiable manifold.*

Proof. Let X be a complex manifold and denote by X_0 the underlying differentiable manifold. For every point $x \in X$ we have a local frame for T_xX given $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$. Just as in Example 1.13 complex multiplication on T_xX induces a complex structure J_x on T_xX_0, which has a local frame $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$, defined by

$$J_x(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}) = (-\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_n}).$$

To see that $J : TX_0 \to TX_0$ is a vector bundle isomorphism it remains to show that J_x depends smoothly on x. Let (h, U) be a holomorphic coordinate chart around x then we know that

$$TX_0 \cong h(U) \times \mathbb{R}^{2n}$$

and with respect to this trivialization $J|_U$ takes the form

$$id \times J : h(U) \times \mathbb{R}^{2n} \to h(U) \times \mathbb{R}^{2n}$$

with

$$J(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n)$$

where $\{x_1, y_1, \dots x_n, y_n\}$ is a basis for $\mathbb{R}^{2n}$. Then we see that J is a constant map in local coordinates and thus it depends smoothly on x. Therefore J as defined is indeed an almost complex structure on X_0. □

Theorem 1.17 *The induced almost complex structure on a complex manifold is integrable.*

Proof. Let X be a complex manifold and (X_0, J) the underlying real manifold with the induces almost complex structure as in Proposition 1.16. We have seen that for a vector space (V, J) we have $V \cong V^{1,0}$ and since TX and TX_0 are $\mathbb{C}$-linear isomorphic when we use the complex structure on TX_0 induced by J, we get that

$$TX \cong TX_0^{1,0} \qquad \text{and} \qquad TX^* \cong T^*X_0^{1,0}.$$

If $z_j = x_j + iy_j$ for $j = 1, \dots, n$ are local coordinates for X we set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad j = 1, \dots, n$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \qquad j = 1, \dots, n$$

where $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots \frac{\partial}{\partial y_n}\}$ form a local frame for TX_{0c} and $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$ form a local frame for TX. Since we identified $TX \cong TX_0^{1,0}$ it follows that the $\frac{\partial}{\partial z_j}$ form a local frame for ${TX_0}^{1,0}$ and the $\frac{\partial}{\partial \bar{z}_j}$ form a local frame for ${TX_0}^{0,1}$ (recall that ${TX_0}^{1,0}$ and ${TX_0}^{0,1}$ are related by complex conjugation). It follows that

$$dz_j = dx_j + idy_j, \qquad j = 1, \dots, n$$

$$d\bar{z}_j = dx_j - idy_j, \qquad j = 1, \dots, n$$

which are dual to $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ form a local frame for $T^*X_0^{1,0}$ and $T^*X_0^{0,1}$ respectively. We also get the identities

$$dx_j = \frac{1}{2}\left(dz_j + d\bar{z}_j\right) \qquad dy_j = \frac{1}{2i}\left(dz_j - d\bar{z}_j\right).$$

Let $s \in \mathcal{E}^{p,q}$ be given, then the above remarks imply, using multi-index notation [1]

$$s = \sum_{I,J}{}' a_{IJ} dz^I \wedge d\bar{z}^J.$$

We have

$$\begin{aligned} ds &= \sum_{j=1}^{n} \sum_{I,J}{}' \left(\frac{\partial a_{IJ}}{\partial x_j} dx_j + \frac{\partial a_{IJ}}{\partial y_j} dy_j \right) \wedge dz^I \wedge d\bar{z}^J \\ &= \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial z_j} dz_j \wedge dz^I \wedge d\bar{z}^J + \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^I \wedge d\bar{z}^J. \end{aligned}$$

Since the first term is of type $(p+1,q)$ and the second of type $(p,q+1)$ it follows that

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j \qquad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$$

and hence $d = \partial + \bar{\partial}$ which means by definition that the induces almost complex structure is integrable. □

The converse of this theorem is also true and is known as the Newlander-Nirenberg theorem.

Theorem 1.18 *Let (X, J) be an integrable almost complex manifold. Then there exists a unique complex structure $\mathcal{O}_X$ on X which induces the almost complex structure J.*

Since we do not use this theorem we will omit the proof and instead refer the reader to [**1**].

[1] Recall that letting $N = \{1, \dots, n\}$ we can consider multi-indices $I = \{\mu_1, \dots, \mu_p\}$ where $\mu_1, \dots, \mu_p$ are distinct elements of N and we set $|I| = p$. Using such multi-indices we let

$$z_I = z_{\mu_1} \wedge \cdots \wedge z_{\mu_p}$$

An increasing multi-index $I = \{\mu_1, \dots, \mu_p\}$ has the property that $\mu_1 \le \cdots \le \mu_p$ and we let $\sum_I'$ denote summation over increasing multi-indices only.

CHAPTER 2

Sheaves & Cohomology

In this chapter we will be dealing with sheaves. Sheaves are useful because we can use them to keep track of local information attached to open sets of a topological space and pass to a global solution of a given problem after solving it locally. In the first section we will give the basic definition of presheaves and sheaves. We also introduce resolutions of sheaves and some examples of resolutions. In the second section we define sheaf cohomology and proof some isomorphisms which give us explicit versions of the de Rham and Dolbeault theorems.

1. Presheaves & Sheaves

We begin with some definitions introducing the concepts of presheaves and sheaves.

Definition 2.1. A *presheaf* $\mathcal{F}$ over a topological space X is

(1) An assignment to each nonempty open set $U \subset X$ of a set $\mathcal{F}(U)$.
(2) A collection of mappings (called restriction homomorphisms)

$$r_V^U : \mathcal{F}(U) \to \mathcal{F}(V),$$

for each pair of open sets U and V such that $V \subset U$, satisfying:
  (a) r_U^U = identity on $U (= 1_U)$.
  (b) For $U \supset V \supset W$, we have $r_W^U = r_W^V \circ r_V^U$.

Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ over X, a *morphism of presheaves*

$$h : \mathcal{F} \to \mathcal{G}$$

is a collection of maps

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open set $U \subset X$ such that each h_U commutes with the restriction homomorphisms. The elements of $\mathcal{F}(U)$ will be called *sections* of $\mathcal{F}$ over U. If the $\mathcal{F}(U)$ have some kind of algebraic structure (for example abelian groups) we also require the restriction and sheaf homomorphisms to preserve this structure (for example r_V^U and h_U are commuting group homomorphisms). If the h_U are inclusion mappings then $\mathcal{F}$ is called a *subpresheaf* of $\mathcal{G}$.

Definition 2.2. A presheaf $\mathcal{F}$ is called a *sheaf* if for every collection U_i of open subsets of X with $U = \bigcup U_i$ the following is true

(1) If $s, t \in \mathcal{F}(U)$ and $r_{U_i}^U(s) = r_{U_i}^U(t)$ for all i, then $s = t$.
(2) If $s_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j \neq \varnothing$ we have

$$r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_j)$$

for all i and j, then there exists an $s \in \mathcal{F}(U)$ such that $r_{U_i}^U(s) = s_i$ for all i.

A *morphism of sheaves* is a morphism of presheaves of the underlying sheaf and if the h_U and a *isomorphism of sheaves* is a sheaf morphism where the h_U are isomorphisms.
Some examples of sheaves are:

Example 2.3. Let X and Y be topological spaces. We define a sheaf $\mathcal{C}_{X,Y}$ over X by the presheaf

$$\mathcal{C}_{X,Y}(U) = \{f : U \to Y | f \text{ is continuous}\}.$$

The restriction homomorphisms are given by the natural restriction of functions.

Example 2.4. Let X be a topological space and G an abelian group. For each open, connected $U \subset X$ the assignment $U \to G$ defines a sheaf called the *constant sheaf* (with coefficients in G).

Example 2.5. Let $E \to X$ be a $\mathcal{S}$-bundle. Define a presheaf $\mathcal{S}(E)$ by setting $\mathcal{S}(E)(U) = \mathcal{S}(U, E)$, for $U \subset X$ open with the natural restriction of sections as restriction homomorphisms. We call $\mathcal{S}(E)$ the *sheaf of* $\mathcal{S}$*-sections of the vector bundle* E. Since $\mathcal{S}(E)$ is a subsheaf of $\mathcal{C}_{X,E}$ it is in fact a sheaf. Special cases of this example include $\mathcal{E}_X^*$ the sheaf of differential forms on a differentiable manifold and $\mathcal{E}_X^{p,q}$ the sheaf of differential forms of type (p, q) on a complex manifold X.

A *graded sheaf* is a family of sheaves indexed by integers, $\mathcal{F}^* = \{\mathcal{F}^\alpha\}_{\alpha \in \mathbb{Z}}$. A *sequence of sheaves* is a graded sheaf connected by sheaf mappings

$$\ldots \to \mathcal{F}^0 \xrightarrow{\alpha_0} \mathcal{F}^1 \xrightarrow{\alpha_1} \mathcal{F}^2 \to \ldots$$

A sequence of sheaves is called a *differential sheaf* if the composite of any pair of maps is zero, that is, $\alpha_j \circ \alpha_{j_1} = 0$. A *resolution* of a sheaf $\mathcal{F}$ is an exact sequence [1] of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots \to \mathcal{F}^m \to \ldots$$

which we will denote by

$$0 \to \mathcal{F} \to \mathcal{F}^*.$$

We give some important examples of sheaf resolutions that will be used later on.

Example 2.6. Let X be a differentiable manifold with $\dim_{\mathbb{R}} X = m$ and let $\mathcal{E}_X^p$ be the sheaf of real-valued differential forms of degree p. Then there is a resolution of the constant sheaf $\mathbb{R}$ given by

$$0 \to \mathbb{R} \xrightarrow{i} \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}_X^m \xrightarrow{d} 0,$$

where i is the natural inclusion and d is the exterior differentiation operator. Since $d^2 = 0$ this is a differential sheaf. Using the classical Poincaré Lemma we obtain exactness of the sheaf. We shall denote this sheaf resolution by $0 \to \mathbb{R} \to \mathcal{E}_X^*$ (or $0 \to \mathbb{C} \to \mathcal{E}_X^*$ when using complex coefficients).

Example 2.7. Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$ and let $\mathcal{E}^{p,q}$ be the sheaf of (p, q) forms on X. For $p \geq 0$ fixed we have the sequence of sheaves

$$0 \to \Omega^p \xrightarrow{i} \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \ldots \xrightarrow{\bar\partial} \mathcal{E}^{p,n} \xrightarrow{\bar\partial} 0.$$

[1]By definition a sequence of sheaves is exact if it is exact at *stalk* level. We did not define this and refer the reader to [**4**] for details

Here Ω^p is by definition the kernel of the map $\mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1}$, the sheaf of *holomorphic differential forms of type* $(p,0)$, which we usually call holomorphic forms of degree p. Since $\bar\partial^2 = 0$ we have for each p a differential sheaf

$$0 \to \Omega^p \to \mathcal{E}^{p,*}.$$

Exactness of this differential sheaf can be obtained by the Grothendieck version of the Poincaré lemma for the $\bar\partial$-operator. So we have, in fact, for every p a sheaf resolution of Ω^p.

2. Cohomology theory

We now want to define sheaf cohomology. To do this we will be making use of global sections of sheaves and we would like the following to be true. Given a short exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

we would like that the induced sequence

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0$$

is also exact. It appears however that exactness at $\mathcal{C}(X)$ is not necessarily the case. After some work, which we will skip here, it can be shown that when dealing with sheaves of abelian groups over paracompact Hausdorff spaces (which we will always be doing), the following condition is sufficient for the exactness of the above sequence.

Definition 2.8. A sheaf $\mathcal{F}$ over a space X is *soft* if for any closed subset $S \subset X$ the restriction mapping

$$\mathcal{F}(X) \to \mathcal{F}(S)$$

is surjective. This says that any section of $\mathcal{F}$ over S can be extended to a section of $\mathcal{F}$ over X.

We will now show that if a sheaf has a partition of unity it is soft. In particular this implies that the sheaves $\mathcal{E}^p_X$ and $\mathcal{E}^{p,q}(X)$ from the previous examples are soft.

Definition 2.9. A sheaf of abelian groups $\mathcal{F}$ over a paracompact Hausdorff space X is *fine* if for any locally finite open cover $\{U_i\}$ of X there exists a family of sheaf morphisms

$$\{\eta_i : \mathcal{F} \to \mathcal{F}\}$$

such that

(1) $\sum \eta_i = 1$.
(2) $\eta_i(\mathcal{F}_x)^2 = 0$ for all x in some neighborhood of the complement of U_i.

The family $\{\eta_i\}$ is called a *partition of unity* of $\mathcal{F}$ subordinate to the covering $\{U_i\}$.

The sheaves from the previous examples are fine because multiplication by a differentiable globally defined function defines a sheaf homomorphism in a natural way so we can use the C^∞ partitions of unity to get a sheaf partition of unity. The following proposition gives the desired result that sheaves with a partition of unity are soft.

Proposition 2.10 *Fine sheaves are soft.*

[2]Here $\mathcal{F}_x$ denotes the *stalk* of $\mathcal{F}$ at x. We did not cover this and refer the reader to [**4**].

PROOF. Let $\mathcal{F}$ be a fine sheaf over X and let S be a closed subset of X. Suppose that $s \in \mathcal{F}(S)$. Then there is an opoen cover $\{U_i\}$ of S and there are sections $s_i \in \mathcal{F}(U_i)$ such that

$$S_i|_{S \cap U_i} = S|_{S \cap U_i}.$$

Let $U_0 = X \setminus S$ and $s_0 = 0$, so that $\{U_i\}$ extends to an open covering of all of X. Since X is paracompact, we may assume $\{U_i\}$ to be locally finite and hence there is a sheaf partition of unity $\{\eta_i\}$ subordinate to $\{U_i\}$. Now $\eta_i(s_i)$ is a section on U_i which is zero in a neighborhood of the boundary of U_i, so we can extend it to a section on all of X. Thus we can define

$$\tilde{s} = \sum_i \eta_i(s_i)$$

and obtain the required extension of s. □

Theorem 2.11 *If $\mathcal{A}$ is a soft sheaf and*

$$0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0$$

is a short exact sequence of sheaves. Then the induced sequence

$$0 \to \mathcal{A}(X) \xrightarrow{g_X} \mathcal{B}(X) \xrightarrow{h_X} \mathcal{C}(X) \to 0$$

is exact.

From this theorem we get the following corollary.

Corollary 2.12 *If*

$$0 \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \dots$$

is an exact sequence of soft sheaves, then the induced section sequence of soft sheaves, then the induced section sequence

$$0 \to \mathcal{S}_0(X) \to \mathcal{S}_1(X) \to \mathcal{S}_2(X) \to \dots$$

is also exact.

The last result we need before defining the cohomology groups is that for any sheaf $\mathcal{S}$ over a topological space X it is possible to construct a canonical soft resolution. Let $\mathcal{S}$ be a given sheaf and let $\tilde{\mathcal{S}} \xrightarrow{\pi} X$ be the étalé space associated to $\mathcal{S}$[3]. We let $\mathcal{C}^0(\mathcal{S})$ be the presheaf defined by

$$\mathcal{C}^0(\mathcal{S})(U) = \{f : U \to \tilde{\mathcal{S}} | \pi \circ f = 1_U\}.$$

This presheaf is a sheaf and is called the *sheaf of discontinuous sections of* $\mathcal{S}$ *over* S. There is a natural injection

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}).$$

Now let $\mathcal{F}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{S})/\mathcal{S}$ and define by induction:

$$\mathcal{F}^i(\mathcal{S}) = \mathcal{C}^{i-1}(\mathcal{S})/\mathcal{F}^{i-1}(\mathcal{S})$$

and

$$\mathcal{C}^i(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^i(\mathcal{S})).$$

Using this notation we have the following theorem stating the existence of a canonical soft resolution for any sheaf over a topological space.

[3]For the definition and details about the étalé space associated to a sheaf, see [**4**]

Theorem 2.13 *Let $\mathcal{S}$ be any sheaf over a topological space X. There exists a canonical soft resolution of $\mathcal{S}$ given by*

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to \mathcal{C}^2(\mathcal{S}) \to \dots$$

which we abbreviate by

$$0 \to \mathcal{S} \to \mathcal{C}^*(\mathcal{S}).$$

Again it takes some work to actually proof this and we refer the reader to [**4**]for the details. We remark that examples 2.6 and 2.7 are soft resolutions of the constant sheaf and of the sheaf of holomorphic p-forms respectively.
To define the cohomology groups of a space X with coefficients in a sheaf $\mathcal{S}$ consider the canonical resolution of $\mathcal{S}$ given above. Then we can take global sections and get a sequence of the form

$$0 \to \Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{C}^0(\mathcal{S})) \to \Gamma(X, \mathcal{C}^1(\mathcal{S})) \to \dots \tag{2.1}$$

and if the sheaf $\mathcal{S}$ is soft we see by Corollary 2.12 that this sequence is exact everywhere. We denote

$$C^*(X, \mathcal{S}) := \Gamma(X, \mathcal{C}^*(\mathcal{S}))$$

and write Equation 2.1 in the form

$$0 \to \Gamma(X, \mathcal{S}) \to C^*(X, \mathcal{S}).$$

Definition 2.14. Let $\mathcal{S}$ be a sheaf over a space X and let

$$H^q(X, \mathcal{S}) := H^q(C^*(X, \mathcal{S}))$$

where $H^q(C^*(X, \mathcal{S}))$ is the q-th derived group of the cochain complex $C^*(X, \mathcal{S})$, that is,

$$H^q(C^*) = \frac{\operatorname{Ker}(C^q \to C^{q+1})}{\operatorname{Im}(C^{q-1} \to C^q)}, \qquad \text{where } C^{-1} = 0.$$

The abelian groups $H^q(X, \mathcal{S})$ are defined for $q \geq 0$ and are called the *sheaf cohomology groups of the space X of degree q and with coefficients in $\mathcal{S}$.*

These cohomology groups have the following properties.

Theorem 2.15 *Let X be a paracompact Hausdorff space. Then*

(1) *For any sheaf $\mathcal{S}$ over X*
  (a) $H^0(X, \mathcal{S}) = \Gamma(X, \mathcal{S})$.
  (b) *If $\mathcal{S}$ is soft, then $H^q(X, \mathcal{S}) = 0$ for $q > 0$.*

(2) *For any sheaf morphism*

$$h : \mathcal{A} \to \mathcal{B}$$

there is, for each $q \geq 0$ a group homomorphism

$$h_q : H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$$

such that:
  (a) $h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X)$.
  (b) *h_q is the identity map if h is the idenitity map,for all $q \geq 0$.*
  (c) *$g_q \circ h_q = (g \circ h)_q$ for all $q \geq 0$, if $g : \mathcal{B} \to \mathcal{C}$ is a second sheaf morphism.*

(3) *For each short exact sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

there is a group homomorphism

$$\delta^q : H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A})$$

for all $q \geq 0$ *such that:*

(a) *The induced sequence*

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \xrightarrow{\delta^0} H^1(X, \mathcal{A}) \to \dots$$

is exact.

(b) *A commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}' & \to & \mathcal{B}' & \to & \mathcal{C}' & \longrightarrow & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, \mathcal{A}) & \longrightarrow & H^0(X, \mathcal{B}) & \longrightarrow & H^0(X, \mathcal{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{A}') & \longrightarrow & H^0(X, \mathcal{B}') & \longrightarrow & H^0(X, \mathcal{C}')' & \longrightarrow & 0 \end{array}$$

This definition of the cohomology groups is quite abstract. However we will state a theorem which enables us to represent these groups more explicitly in a given geometric situation. First we make the following definition.

Definition 2.16. A resolution of a sheaf $\mathcal{S}$ over a space X

$$0 \to \mathcal{S} \to \mathcal{A}^*$$

is called *acyclic* if $H^q(X, \mathcal{A}^p) = 0$ for all $q > 0$ and $p \geq 0$.

Note that a soft resolution of a sheaf is acyclic.

Theorem 2.17 *Let* $\mathcal{S}$ *be a sheaf over a space* X *and let*

$$0 \to \mathcal{S} \to \mathcal{A}^*$$

be a resolution of $\mathcal{S}$. *Then there is a natural homomorphism*

$$\gamma^p : H^p(\Gamma(X, \mathcal{A}^*)) \to H^p(X, \mathcal{S})$$

where $H^p(\Gamma(X, \mathcal{A}^*))$ *is the pth derived group of the cochain comples* $\Gamma(X, \mathcal{A}^*)$. *Moreover if*

$$0 \to \mathcal{S} \to \mathcal{A}^*$$

is acyclic then γ^p *is an isomorphism.*

We now obtain the following results.

Theorem 2.18 *Let* X *be a differentiable manifold. Then*

$$H^q(X, \mathbb{C}) \cong \frac{\operatorname{Ker}\left(\mathcal{E}^q(X) \xrightarrow{d} \mathcal{E}^{q+1}(X)\right)}{\operatorname{Im}\left(\mathcal{E}^{q-1}(X) \xrightarrow{d} \mathcal{E}^q(X)\right)}.$$

Proof. The resolution in Example 2.6 is a fine resolution, and we can apply Theorem 2.17. □

Theorem 2.19 *Let X be a complex manifold. Then*

$$H^q(X,\Omega^p) \cong \frac{\operatorname{Ker}\Big(\mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X)\Big)}{\operatorname{Im}\Big(\mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X)\Big)}.$$

Proof. The resolution given in Example 2.7 is a fine resolution, and we can apply Theorem 2.17. □

We can extend this last theorem in the following way. Recall from Example 2.7 that we have the following resolution for Ω^p,

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{E}^{p,n} \to 0.$$

Now if E is a holomorphic vector bundle then we have the sheaf of holomorphic sections of E denoted by $\mathcal{O}(E)$. It can then be shown that we have the following resolution of the sheaf $\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E)$,

$$0 \to \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \to \mathcal{E}^{p,0} \otimes_{\mathcal{O}} \mathcal{O}(E) \xrightarrow{\bar\partial\otimes 1} \dots \xrightarrow{\bar\partial\otimes 1} \mathcal{E}^{p,n} \otimes_{\mathcal{O}} \mathcal{O}(E) \to 0.$$

We note that

$$\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \cong \mathcal{O}(\wedge^p T^*X \otimes_{\mathbb{C}} E)$$

and

$$\begin{aligned}\mathcal{E}^{p,q} \otimes_{\mathcal{O}} \mathcal{O}(E) &\cong \mathcal{E}^{p,q} \otimes_{\mathcal{E}} \mathcal{E}(E)\\ &\cong \mathcal{E}(\wedge^{p,q} T^*X \otimes_{\mathbb{C}} E),\end{aligned}$$

where $\mathcal{E}(E)$ denotes the sheaf of differentiable sections of E. We call $\mathcal{O}(X, \wedge^p T^*X \otimes_{\mathbb{C}} E)$ the (global) *holomorphic p-forms on X with coefficients in E* and we will denote this by $\Omega^p(X,E)$. The sheaf of holomorphic p-forms with coefficients in E by $\Omega^p(E)$. The differentiable (p,q)-forms with coefficiens in E will be denoted by

$$\mathcal{E}^{p,q}(X,E) := \mathcal{E}(X, \wedge^{p,q} T^*X \otimes_{\mathbb{C}} E).$$

Defining $\bar\partial_E = \bar\partial \otimes 1$ we can rewrite the previous resolution in the form

$$0 \to \Omega^p(E) \to \mathcal{E}^{p,0}(E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,1}(E) \xrightarrow{\bar\partial_E} \dots \xrightarrow{\bar\partial_E} \mathcal{E}^{p,n}(E) \xrightarrow{\bar\partial_E} 0$$

since this is a fine resolution we have the following theorem.

Theorem 2.20 *Let X be a complex manifold and let $E \to X$ be a holomorphic vector bundle. Then*

$$H^q(X,\Omega^p(E)) \cong \frac{\operatorname{Ker}(\mathcal{E}^{p,q}(X,E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,q+1}(X,E))}{\operatorname{Im}(\mathcal{E}^{p,q-1}(X,E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,q}(X,E))}.$$

CHAPTER 3

Hermitian Geometry

In this chapter we will use local frames to define Hermitian metrics on vector bundles and prove that every vector bundle admits a Hermitian metric. We will do this in the context of differentiable $\mathbb{C}$-vector bundles over a differentiable manifold X.

1. Hermitian geometry

In this section all vector bundles will be complex vector bundles. Let such a vector bundle $E \to X$ of rank r be given. Suppose that $f = (e_1, \dots, e_r)$ is a frame at $x \in X$. Recall that this means that there is some open neighborhood U of x such that the sections $\{e_1, \dots, e_r\}$, where $e_i \in \mathcal{E}(U, E)$, are linearly independent at each point of U(if we want to denote more explicitly the dependance of f on U we write $f|_U$). Given a map $g : U \to \mathrm{Gl}(r, \mathbb{C})$ there is an action of g on the set of all frames on U defined by

$$f \mapsto fg$$

where

$$(fg)(x) = \left(\sum_{\rho=1}^{r} g_{\rho 1}(x) e_\rho(x), \dots, \sum_{\rho=1}^{r} g_{\rho r}(x) e_\rho(x) \right), \qquad x \in U.$$

That is, we regard f as a row vector and use normal matrix multiplication to get a new row vector fg. Since for every $x \in U$, $g(x)$ is an invertible matrix and the components of f are linearly independent we get that the components of fg are also linearly independent and therefore fg is a new frame. Moreover for every two frames f and $\tilde{f}$ on U it is possible to find a mapping $g : U \to \mathrm{Gl}(r, \mathbb{C})$ such that $\tilde{f} = fg$. The mapping g is called a *change of frame*. We can use frames to give local descriptions of geometric objects. We start by giving a local description of sections of a vector bundle. Let a vector bundle $E \to X$ and a frame $f = (e_1, \dots, e_r)$ over $U \subset X$ be given. We know from the discussion following Definition 1.12 that for every $x \in X$ there exists a frame $f|_U$ around x if we just choose U small enough. Let $\xi \in \mathcal{E}(U, E)$ then we can write

$$\xi = \sum_{\rho=1}^{r} \xi^\rho(f) e_\rho \tag{3.1}$$

where the $\xi^\rho(f) \in \mathcal{E}(U)$ are uniquely defined and dependent on the frame f. We get an induced map

$$\mathcal{E}(U, E) \xrightarrow{\ell_f} \mathcal{E}(U)^r \cong \mathcal{E}(U, U \times \mathbb{C}^r) \tag{3.2}$$

defined by

$$\xi \mapsto \xi(f) = \begin{bmatrix} \xi^1(f) \\ \cdot \\ \cdot \\ \cdot \\ \xi^r(f) \end{bmatrix}.$$

Let $f = (e_1, \dots, e_r)$ and g a change of frame over U such that $fg = (e'_1, \dots, e'_r)$ then we have the identities

$$\xi = \sum_\rho \xi^\rho(fg)e'_\rho = \sum_\sigma \xi^\sigma(f)e'_\sigma \qquad \text{and} \qquad e'_\sigma = \sum_\rho g^{-1}_{\rho\sigma} e_\rho$$

substituting we get

$$\xi = \sum_\sigma \sum_\rho \xi^\sigma(f) g^{-1}_{\rho\sigma} e_\rho$$

but this implies

$$\xi^\rho(fg) = \sum_\sigma g^{-1}_{\rho\sigma} \xi^\sigma(f)$$

or

$$g\xi(fg) = \xi(f) \tag{3.3}$$

where the product is matrix multiplication. Analogously if E is a holomorphic vector bundle, we also have *holomorphic frames*, $f = (e_1, \dots, e_r)$ where $e_i \in \mathcal{O}(U, E)$ are linearly independent, and *holomorphic frames* where $g : U \to \mathrm{Gl}(r, \mathbb{C})$ is holomorphic. Then with respect to a holomorphic frame we have again a map

$$\mathcal{O}(U, E) \xrightarrow{\ell_f} \mathcal{O}(U)^r \cong \mathcal{E}(U, U \times \mathbb{C}^r).$$

We shall now introduce the concept of a metric on a vector bundle.

Definition 3.1. Let $E \to X$ be a vector bundle. A *Hermitian metric*, h, on E is defined to be the assignment of a Hermitian inner product $\langle \cdot, \cdot \rangle_x$, to each fiber E_x of E such that for every open $U \subset X$ and $\xi, \chi \in \mathcal{E}(U, E)$ the map

$$\langle \xi, \chi \rangle : U \to \mathbb{C}$$

defined by

$$x \mapsto \langle \xi(x), \chi(x) \rangle_x$$

is smooth.

The smoothness condition says that the Hermitian inner products on different fibers vary smoothly. Now let a frame $f = (e_1, \dots, e_r)$ for E over some open $U \subset X$ be given. We define $h(f)_{\rho\sigma} = \langle e_\sigma, e_\rho \rangle$ and let $h(f) = \left[h(f)_{\rho\sigma} \right]$ denote the $r \times r$ matrix of smooth functions $h(f)_{\rho\sigma}$. If we write $\eta = \sum_\rho \eta^\rho(f) e_\rho$ and $\xi = \sum_\sigma \xi^\sigma(f) e_\sigma$ for $\eta, \xi \in \mathcal{E}(U, E)$ it follows from the fact that for every $x \in U$ h_x is an Hermitian inner product that

$$\begin{aligned} \langle \xi, \eta \rangle &= \langle \sum_\sigma \xi^\sigma(f) e_\sigma, \sum_\rho \eta^\rho(f) e_\rho \rangle \\ &= \sum_{\rho,\sigma} \overline{\eta^\sigma}(f) h_{\sigma\rho}(f) \xi^\rho(f) \end{aligned}$$

but this is the same as

$$\langle \xi, \eta \rangle = \eta^*(f) h(f) \xi(f) \tag{3.4}$$

where η^* denotes conjugate transpose and the multiplication is matrix multiplication. From the fact that $\langle e_\rho, e_\sigma\rangle = \overline{\langle e_\sigma, e_\rho\rangle}$ it follows that $h(f)^* = h(f)$ and so $h(f)$ is a Hermitian matrix. Similarly from the fact that $\langle \eta, \eta\rangle \geq 0$ it follows that $h(f)$ is a positive definite matrix. The transformation rule for frames, Equation 3.3 implies that for a change of frame g

$$\langle \xi, \eta\rangle = \eta^*(f)h(f)\xi(f) = \eta^*(fg)h(fg)\xi(fg) = \eta^* g^{-1^*} h(fg) g^{-1}\xi(f)$$

which in turn implies that

$$h(fg) = g^* h(f) g. \tag{3.5}$$

This is the transformation rule for local representations of a Hermitian metric.

Theorem 3.2 *Every vector bundle $E \to X$ admits a Hermitian metric.*

PROOF. To prove the theorem we will construct a Hermitian metric. We know there exists a locally finite cover $\{U_\alpha\}$ of X and frames $f_\alpha|_{U_\alpha}$. For every $x \in U_\alpha$ we define a Hermitian inner product on $E_x \subset E|_{U_\alpha}$ by

$$\langle \xi, \eta\rangle_x^\alpha = \eta(f_\alpha)^*(x) \cdot \xi(f_\alpha)(x).$$

It is clear that this definition satisfies all the properties of an Hermitian inner product. Since the U_α can overlap, let $\{\rho_\alpha\}$ be a smooth partition of unit subordinate to $\{U_\alpha\}$ and define

$$\langle \xi, \eta\rangle_x = \sum_\alpha \rho_\alpha(x)\langle \xi, \eta\rangle_x^\alpha.$$

The positivity and symmetry of the Hermitian metric follows from the positivity and symmetry of the Hermitian inner product. The only condition that we have left to check is that this inner product depends smoothly on x but this is clear since

$$x \mapsto \langle \xi(x), \eta(x)\rangle_x = \sum_\alpha \rho(x)_\alpha \eta(f_\alpha)^*(x) \cdot \xi(f_\alpha)(x)$$

is smooth in x. This concludes the construction. □

CHAPTER 4

Elliptic Operator Theory

In this chapter we will develop all the results with regard to differentiable operators needed for the next chapter. We will skip a lot of details since the general theory and background of the stated results is too extensive for this text. In the first section we will discuss the basic structure of differential operators and their symbols. We will also give a few examples of symbol sequences in the cases of the de Rham and Dolbeault complexes. In the second section we will generalize the discussion of the first section to the context of pseudodifferential operators. In the third section we will define what it means for an operator to be elliptic and prove an important decomposition theorem for self-adjoint elliptic operators. In the last section we generalize elliptic operators to elliptic complexes. The main result from this section is that we can represent the de Rham and Dolbeault cohomology groups by harmonic forms.

1. Differential Operators

Suppose we are given two differentiable $\mathbb{C}$-vector bundles E and F both over the same differentiable manifold X. A $\mathbb{C}$-linear map

$$L : \mathcal{E}(X,E) \to \mathcal{E}(X,F)$$

is called a *differential operator* for any choice of local coordinates and trivializations there exists a linear partial differential operator $\tilde{L}$ such that the following diagram commutes.

$$\begin{array}{ccc} [\mathcal{E}(U)]^p & \xrightarrow{\tilde{L}} & [\mathcal{E}(U)]^q \\ \wr\Vert & & \wr\Vert \\ \mathcal{E}(U, U\times\mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U\times\mathbb{C}^q) \\ \cup & & \cup \\ \mathcal{E}(X,E)|_U & \xrightarrow{\mathrm{L}} & \mathcal{E}(X,F)|_U \end{array}$$

Recall that $\tilde{L}$ being a linear partial differential operator means that for $f = (f_1, \dots, f_p) \in [\mathcal{E}(U)]^p$ we have

$$\tilde{L}(f)_i = \sum_{\substack{j=1 \\ |\alpha|\geq k}}^{p} a_\alpha^{ij} D^\alpha f_j, \qquad i = 1, \dots, q.$$

Here we are using the notation $D_j = \frac{\partial}{\partial x_j}$ and $D^\alpha = (-i)^{|\alpha|} D_1^{\alpha_1} \dots D_n^{\alpha_n}$. If there are no derivatives of order $k \geq k+1$ appearing in the local representation then the differential operator is said to be of *order* k. The vector space of all differential operators of order k from $\mathcal{E}(X,E)$ to $\mathcal{E}(X,F)$ is denoted by $\mathrm{Diff}_k(E,F)$. We will also use another class of mappings denoted by $\mathrm{OP}_k(E,F)$ called *operators of order* k. This is defined to be the vector space of $\mathbb{C}$-linear mappings

$$T : \mathcal{E}(X,E) \to \mathcal{E}(X,F)$$

(where X is a differentiable manifold) such that there exists a continuous extension of T

$$\bar{T}_s : W^s(X,E) \to W^{s-k}(X,F)$$

for all s. Here $W^s(X,E)$ is the completion of $\mathcal{E}(X,E)$ with respect to the *sobolev norm* $\|.\|_s$ as defined in [**4**].
We recall the definition of a *formal adjoint operator*.

Definition 4.1. Let

$$L : \mathcal{E}(X,E) \to \mathcal{E}(X,F)$$

be a $\mathbb{C}$-linear map. Then a $\mathbb{C}$-linear map

$$L^* : \mathcal{E}(F,X) \to \mathcal{E}(X,E)$$

is called an *adjoint* of L if

$$(Lf,g) = (f, L^* g)$$

for all $f \in \mathcal{E}(X,E), q \in \mathcal{E}(X,F)$.

We now have the following proposition which we will not prove. (for a proof see [**4**])

Proposition 4.2 *Let* $L \in \mathrm{Diff}_k(E,F)$*. Then* L^* *exists, and* $L^* \in \mathrm{Diff}_k(F,E)$*.*

We now want to introduce the *symbol* of an operator which can be used to classify differential operators. That is, we want to define a mapping which assigns to each differential operator its symbol. In order to do this we first define the set of symbols. Let $L \in \mathrm{Diff}_k(E,F)$ where E and F are given vector bundles over a differentiable manifold X. Given T^*X, the real cotangent bundle to X we denote by $T'X$ the real cotangent bundle with the zero cross section removed, the bundle of nonzero cotangent vectors, with the projection map $\pi : T'X \to X$. As in Proposition 1.10, let π^*E and π^*F denote the pullbacks of E and F over $T'X$. For $k \in \mathbb{Z}$ we define

$$\mathrm{Smbl}_k(E,F) = \{\sigma \in \mathrm{Hom}(\pi^*E, \pi^*F)^1 | \sigma(x,\rho v) = \rho^k \sigma(x,v) \forall (x,v) \in T'X, \rho > 0\}.$$

Now that we have defined the set of symbols we define a linear map

$$\sigma_k : \mathrm{Diff}_k(E,F) \to \mathrm{Smbl}_k(E,F) \tag{4.1}$$

in the following way. First note that by the definition of the pullback of a vector bundle and the fact we are pulling E and F back by the projection map we have that $\sigma(L)(x,v)$ should be a linear mapping from E_x to F_x. Let $(x,v) \in T'X$ and $e \in E_x$ be given. Find $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ such that $dg_x = v$ and $f(x) = e$ then

$$\sigma_k(L)(x,v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) \in F_x$$

[1]Recall that $\mathrm{Hom}(\pi^*E, \pi^*F)$ is the vector bundle whose fiber at x is the space of linear maps from π^*E_x to π^*F_x

we call $\sigma_k(L)$ the *k-symbol* of the differential operator L. Note that this indeed, for every $(x,v) \in T'X$, defines a linear mapping

$$\sigma_k(L)(x,v) : E_x \to F_x$$

which is an element of $\mathrm{Smbl}_k(E,F)$ and independent of the choices made. Indeed, concretely, if we let $L \in \mathrm{Diff}_k(E,F)$ with $\tilde{L} : \mathcal{E}(U)^p \to \mathcal{E}(U)^q$ given by

$$\sum_{|\nu|\le k} A_\nu D^\nu$$

where $\{A_\nu\}$ are $q \times p$ matrices of smooth functions on U. We claim that

$$\sigma_k(L)(x,v) = \sum_{|\nu|=k} A_\mu(x)\xi^\nu$$

where $v = \xi_1 dx_1 + \cdots + \xi_n dx_n$. To see this choose $g \in \mathcal{E}(U)$ such that $v = dg = \sum \xi_j dx_j$. Then for $e \in \mathbb{C}^p$ we have

$$\sigma_k(L)(x,v)e = \sum_{|\nu|\le k} A_\nu D^\nu \left(\frac{i^k}{k!}(g - g(x))^k e \right)(x).$$

Since all the terms with derivative of order $\le k-1$ will contain a factor $(g - g(x))\,|_x = 0$ the only nonzero terms are the ones with $|\nu| = k$. These are given by (recalling $D^\nu = (-i)^{|\nu|} D_1^{\nu_1} \dots D_n^{\nu_n}$)

$$\begin{aligned} &\sum_{|\nu|=k} A_\nu(x) \frac{k!}{k!} (D_1 g(x))^{\nu_1} \dots (D_n(g))^{\nu_n} \\ &= \sum_{|\nu|=k} A_\nu(x) \xi_1^{\nu_1} \dots \xi_n^{\nu_n} = \sum_{|\nu|=k} A_\nu(x)\xi^\nu. \end{aligned} \tag{4.2}$$

We obtain the following proposition.

Proposition 4.3 *The symbol map σ_k gives rise to an exact sequence*

$$0 \to \mathrm{Diff}_{k-1}(E,F) \xrightarrow{j} \mathrm{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E,F)$$

where j is the natural inclusion.

This is a simple consequence from Equation 4.2 since elements of $\mathrm{Diff}_{k-1}(E,F)$ do not have partial derivatives of order k. We also state the following property of symbols. Given $L_1 \in \mathrm{Diff}_k(E,F)$ and $L_2 \in \mathrm{Diff}_m(F,G)$ we have $L_2 L_1 \in \mathrm{Diff}_{k+m}(E,G)$ with

$$\sigma_{k+m}(L_2 L_1) = \sigma_m(L_2)\sigma(L_1).$$

We now look at the symbol maps of some important differential operators.

Example 4.4. Consider the de Rham complex

$$\mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}^n(X)$$

given by exterior differentiation on differential forms. Letting $T^* = T^*X \otimes \mathbb{C}$ we can rewrite this as

$$\mathcal{E}(X, \wedge^0 T^*) \xrightarrow{d} \mathcal{E}(X, \wedge^1 T^*) \xrightarrow{d} \dots$$

We want to compute the associated 1-symbol maps

$$\wedge^0 T_x^* \xrightarrow{\sigma_1(d)(x,v)} \wedge^1 T_x^* \xrightarrow{\sigma_1(d)(x,v)} \wedge^2 T_x^* \to \dots \tag{4.3}$$

Let $e \in \wedge^p T^* x$, $f \in \wedge^p T^*$ with $f(x) = e$ and $g \in \mathcal{E}(X)$ such that $dg = v$ then using the definition of the symbol map we obtain

$$\begin{aligned}\sigma_1(d)(x,v)e &= d\left(i(g-g(x))f\right)(x)\\ &= id\left((g-g(x))f\right)(x)\\ &= id(g-g(x))(x)\wedge f(x) + (g(x)-g(x))df(x)\\ &= idg(x)\wedge f(x)\\ &= iv\wedge e.\end{aligned}$$

We know that $v \wedge v = 0$. It is also true that if for some vector space V we have $a \in \wedge^k V$ and $v \in V$ such that $v \wedge a = 0$, then it follows that $a = v \wedge b$ for some $b \in \wedge^{k-1} V$ [2]. From this we conclude that the symbol sequence is exact.

Example 4.5. Consider the Dolbeault complex on a complex manifold X,

$$\mathcal{E}^{p,0}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}(X) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{E}^{p,n}(X) \to 0.$$

Then we have the associated symbol sequence

$$\to \wedge^{p,q-1} T_x^* X \xrightarrow{\sigma_1(\bar\partial)(x,v)} \wedge^{p,q} T_x^* X \xrightarrow{\sigma_1(\bar\partial)(x,v)} \wedge^{p,q+1} T_x^* X \to .$$

Since $v \in T_x^* X$,where $T_x^* X$ is considered as a real cotangent bundle we have that $v = v^{1,0} + v^{0,1}$, given by the injection

$$\begin{aligned}0 \to T_x^* X \to T_x^* X \otimes_{\mathbb{R}} \mathbb{C} &= T^* X^{1,0} \oplus T^* X^{0,1}\\ &= \wedge^{1,0} T^* X \oplus \wedge^{0,1} T^* X.\end{aligned}$$

The computation for the symbol $\sigma_1(\bar\partial)$ is the same as before only now we have $\bar\partial g = v^{0,1}$ so we get

$$\sigma_1(\bar\partial)(x,v)e = iv^{0,1} \wedge e.$$

Again the symbol sequence is exact.

Example 4.6. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. We then consider the differentiable (p,q)-forms with coefficients in E, $\mathcal{E}^{p,q}(X,E)$ and we have the following complex

$$\to \mathcal{E}^{p,q}(X,E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,q+1}(X,E) \to$$

with the associated symbol sequence

$$\to \wedge^{p,q} T_x^* \otimes E_x \xrightarrow{\sigma_1(\bar\partial_E)(x,v)} \wedge^{p,q+1} T_x^* \otimes E_x \to .$$

Just like in the previous example we let $v = v^{1,0} + v^{0,1}$ and again we have for $f \otimes e \in \wedge^{p,q} T_x^* \otimes E_x$

$$\sigma_1(\bar\partial)(x,v) f \otimes e = (iv^{0,1} \wedge f) \otimes e$$

and the symbol sequence is again exact.

[2] To see this extend v to a basis $\{v = v_1, \dots, v_n\}$ for V. Then a basis for $\wedge^k V$ is given by $v_{i_1} \wedge \dots \wedge v_{i_k}$ with $i_1 < \dots < i_k$. Therefore $a = \sum a_{i_1,\dots,i_k} v_{i_1} \wedge \dots \wedge v_{i_k} = \sum_{i_1 = 1} a_{1,i_2,\dots,i_k} v \wedge \dots \wedge v_{i_k} + \sum_{i_1 \neq 1} a_{i_1,\dots,i_k} v_{i_1} \wedge \dots \wedge v_{i_k}$. Now $v \wedge a = 0$ implies $\sum_{i_1 \neq 1} a_{i_1,\dots,i_k} v^v_{i_1} \wedge \dots v_{i_k} = 0$, since $v \wedge v = 0$, and therefore $a_{i_1,\dots,i_k} = 0$ for $i_1 \neq 1$. It then follows immediatly that $a = \sum a_{1,\dots,i_k} v \wedge \dots v_{i_k} = v \wedge \sum a_{i_2,\dots,i_k} v_{i_2} \wedge \dots v_{i_k} = v \wedge b$ for some $b \in \wedge^{k-1} V$.

2. Pseudodifferential Operators

In the previous section we defined differential operators, in this section we would like to generalize these operators to so called *pseudodifferential operators*. Given $U \subset \mathbb{R}^n$, open and $p(x,\xi)$ a polynomial of degree m in ξ with smooth coefficients depending on $x \in U$ we can define a differential operator $P = p(x,D)$ by replacing $\xi = (\xi_1,\dots,\xi_n)$ by $(-iD_1,\dots,-iD_n)$. For $u \in \mathcal{D}(U)$ (where $\mathcal{D}(U) \subset \mathcal{E}(U)$ denote the compactly supported functions) we can write using the Fourier transform

$$Pu(x) = p(x,D)u(x) = \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi$$

where $\langle x,\xi\rangle$ is the standard Euclidean inproduct and $\hat{u}(\xi) = (2\pi)^{-n}\int u(x)e^{-i\langle x,\xi\rangle}dx$ is the Fourier transform of u. We see from this that we can define differential operators by using polynomials $p(x,\xi)$. To generalize our definition of differential operators we can therefore generalize the functions $p(x,\xi)$.

Definition 4.7. Let U be an open set in $\mathbb{R}^n$ and let m be any integer.

(1) Let $\tilde{S}^m$ be the class of smooth functions $p(x,\xi)$ defined on $U\times\mathbb{R}^n$ such that for any compact set $K \subset U$ and multiindices α,β there exists a constant $C_{\alpha,\beta,K}$, depending on α,β,K and p with
$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\alpha|}, \quad x \in K, \xi \in \mathbb{R}^n.$$
(2) Let $S^m(U)$ denote the subset of $p \in \tilde{S}^m(U)$ such that the limit
$$\sigma_m(p)(x,\xi) = \lim_{\lambda\to\infty}\frac{p(x,\lambda\xi)}{\lambda^m}$$
exists for $\xi \neq 0$ and with
$$p(x,\xi) - \psi(\xi)\sigma_m(p)(x,\xi) \in \tilde{S}^{m-1}(U)$$
where $\psi \in C^\infty(\mathbb{R}^n)$ is a cut off function with $\psi(\xi) = 0$ near $\xi = 0$ and $\psi(\xi) = 1$ outside the unit ball.
(3) Let $\tilde{S}_0^m(U)$ denote the subset of $p \in \tilde{S}^m(U)$ such that $p(x,\xi)$ has uniform compact support in the x variable.
(4) Let $S_0^m(U) = S^m(U) \cap \tilde{S}_0^m(U)$.

Note that $\sigma_m(p)(x,\xi)$ is the m-th order homogeneous part of the polynomial where all the lower order terms have gone to zero in the limit and there are no higher order terms since this would imply that the limit does not exist. Also note that if $p(x,\xi)$ is a polynomial of degree m as before and if the coefficients have compact support in U then $p \in S_0^m(U)$.

Now we have introduced our generalized classes of functions the definition of the pseudodifferential operators is straight forward. For any open $U \subset \mathbb{R}^n$, $p \in \tilde{S}^m(U)$ and $u \in \mathcal{D}(U)$ we define a *canonical pseudodifferential operator of order* m $L(p)$ by (the local representation)

$$L(p)u(x) = \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi.$$

It can be shown that this defines a linear operator of order m mapping $\mathcal{D}(U)$ into $\mathcal{E}(U)$. We will not show this here and direct the reader to [**4**] for a proof. Having defined pseudodifferential operators on $\mathbb{R}^n$ we now define them in the general case.

Definition 4.8. Let L be a linear mapping $L : \mathcal{D}(X,E)$ to $\mathcal{E}(X,F)$. Then L is a *pseudodifferential operator* on X if and only if for any coordinate chart U with trivializations of E and F over U and for any open set $U' \subset U$ there exists a $r \times p$ matrix (p^{ij}), $p^{ij} \in S_0^m(U)$, so that the induced map

$$L_U : \mathcal{D}(U')^p \to \mathcal{E}(U)^r$$

with $u \in \mathcal{D}(U')^p \xrightarrow{L_U} Lu$, extending u by zero to be an element of $\mathcal{D}(X,E)$ (where $p = \operatorname{rank} E$, $r = \operatorname{rank} F$, and we identify $\mathcal{E}(U)^p$ with $\mathcal{E}(U,E)$ and $\mathcal{E}(U)^r$ with $\mathcal{E}(U,F)$), is a matrix of canonical pseudodifferential operators $L(p^{ij})$, $i = 1,\dots,r$, $j = 1,\dots,p$. Moreover, we say that L has *order* m if the canonical pseudodifferential operators $L(p^{ij})$ are of order m. The class of all pseudodifferential operators on X of order m is denoted by $\mathrm{PDiff}_m(X)$.

Again it can be shown that this indeed defines operators of order m. Since we want to make use of the symbol map in the general setting of pseudodifferential operators we have to also generalize the definition of our symbol map. The way to do this is define the so called *local m-symbol* of a pseudodifferential operator on coordinate charts. Then it is possible to proof that this local definition is invariant under local change of coordinates and we obtain a *global m-symbol* . We immediatly state the global m-symbol and again refer the reader to [**4**] for details.

Proposition 4.9 *Let E and F be vector bundles over a differentiable manifold X. There exists a canonical linear map*

$$\sigma_m : \mathrm{PDiff}_m(E,F) \to \mathrm{Smbl}_m(E,F)$$

which is defined locally in a coordinate chart $U \subset X$ by

$$\sigma_m(L_U)(x,\xi) = [\sigma_m(p^{ij})(x,\xi)]$$

where $L_U = [L(p^{ij})]$ is a matrix of canonical pseudodifferential operators, and where $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ is a point in $T'U$ expressed in the local coordinates of U.

3. A parametrix for Elliptic Differenital Operators

Let E and F be vector bundles over a differentiable manifold X.

Definition 4.10. Let $s \in \mathrm{Smbl}_k(E,F)$. Then s is said to be *elliptic* if and only if for any $(x,\xi) \in T'X$, the linear map

$$s(x,\xi) : E_x \to F_x$$

is an isomorphism.

Definition 4.11. Let $L \in \mathrm{PDiff}_k(E,F)$. Then L is said to be *elliptic (of order k)* if and only if $\sigma_k(L)$ is an elliptic symbol.

If $L \in \mathrm{Diff}_m(E,F)$ we set

$$\mathcal{H}_L = \{\xi \in \mathcal{E}(X,E) | L\xi = 0\}$$

and we set

$$\mathcal{H}_L^\perp = \{\eta \in W^0(E) | (\xi,\eta)_E = 0, \xi \in \mathcal{H}_L\}.$$

We define $H_L : \mathcal{E}(X,E) \to \mathcal{E}(X,E)$ to be the orthogonal projection onto the kernel of L. If we restrict L to the orthogonal complement of its kernel, $\mathcal{H}_L^\perp$, then L has no kernel. In fact $L : \mathcal{H}_L^\perp \to \mathcal{H}_L^\perp$ is an isomorphism, hence we can form its inverse. The proof of these claims are stated in the following theorem.

Theorem 4.12 *Let $L \in \mathrm{Diff}_m(E)$ be self-adjoint and elliptic. Then there exist linear mappings H_L and G_L*

$$H_L : \mathcal{E}(X,E) \to \mathcal{E}(X,E)$$
$$G_L : \mathcal{E}(X,E) \to \mathcal{E}(X,E)$$

with the following properties:

(1) $H_L(\mathcal{E}(X,E)) = \mathcal{H}_L(E)$ *and* $\dim_{\mathbb{C}} \mathcal{H}_L(E) < \infty$.
(2) $L \circ G_L + H_L = G_L \circ L + H_L = I_E$, *where I_E is the identity on $\mathcal{E}(X,E)$.*
(3) H_L *and* $G_L \in \mathrm{OP}_0(E)$, *and , in particular extend to bounded operators on* $W^0(E)$.
(4) $\mathcal{E}(X,E) = \mathcal{H}_L(X,E)) \oplus G_L \circ L(\mathcal{E}(X,E)) = \mathcal{H}_L(X,E) \oplus L \circ G_L(\mathcal{E}(X,E))$, *and this decomposition is orthogonal with respect to the inner product in* $W^0(E)$.

The operator G_L is called the *Green's operator* associated to L. The sections in $\mathcal{H}_L$ for a self adjoint elliptic operator are called *L-harmonic sections* or just *harmonic sections*. These harmonic sections will play an important role in the Hodge Decomposition Theorems in Chapter 5 and this is the main reason for stating this theorem.

4. Elliptic Complexes

The basic idea of elliptic complexes is that instead of a pair of vector bundles with a differential operator between them we study sequences of differential operators between sequences of vector bundles all over the same compact differentiable manifold. In more detail, let X be a given compact differentiable manifold and $E_0, \dots, E_N$ be a sequence of vector bundles over X. Suppose we are given a sequence of differential operators, of some fixed order k, $L_0, \dots, L_{N-1}$ mapping as follows

$$\mathcal{E}(E_0) \xrightarrow{L_0} \mathcal{E}(E_1) \xrightarrow{L_1} \mathcal{E}(E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{N-1}} \mathcal{E}(E_N). \tag{4.4}$$

We then have the associated symbol sequence

$$0 \to \pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \pi^* E_2 \to \dots \xrightarrow{\sigma(L_{N-1})} \pi^* E_N \to 0.$$

Definition 4.13. The sequence of operators and vector bundles E in Equation 4.4 is called a *complex* if $L_i \circ L_{i-1} = 0$, $i = 1, \dots N-1$. Such a complex is called an *elliptic complex* if the associated symbol sequence is exact.

If E is a complex as in the above definition then we can define

$$H^q(E) = \frac{\mathrm{Ker}\,(L_q : \mathcal{E}(E_q) \to \mathcal{E}(E_{q+1}))}{\mathrm{Im}\,(L_{q-1} : \mathcal{E}(E_{q-1}) \to \mathcal{E}(E_q))} = \frac{Z^q(E)}{B^q(E)}$$

called the *cohomology groups* of the complex E, $q = 0, \dots, N$. Note that we use the convention $L_{-1} = L_N = E_{-1} = E_{N+1} = 0$ so that the definition makes sense. The de Rham and Dolbeault complexes, which we discussed in Examples 4.4, 4.5 and 4.6 are the examples of elliptic complexes we will mostly be dealing with. For each elliptic complex there is a generalization of the Laplacian. If we have the operators $L_j : \mathcal{E}(E_j) \to \mathcal{E}(_{j+1})$ in the elliptic complex we define the *Laplacian operators* of the elliptic complex E by

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \mathcal{E}(E_j) \to \mathcal{E}(E_j), \quad j = 0, \dots, N.$$

These operators are elliptic and self-adjoint. Using Theorem 4.12 we see that that each Δ_j has a Greens operator G_{Δ_j} and an orthogonal projection operator H_j onto

$$\mathcal{H}(E_j) = \mathcal{H}_{\Delta_j}(E_j) = \ker \Delta_j : \mathcal{E}(E_j) \to \mathcal{E}(E_j)$$

which are the Δ_j-harmonic sections. To simplify the notation we introduce the following. Let

$$\mathcal{E}(E) = \bigoplus_{j=0}^{N} \mathcal{E}(E_j)$$

denote the graded vector space with the natural grading. We can extend the operators L, L^*, Δ, G and H to $\mathcal{E}(E)$ by setting for example

$$L(\xi) = L(\xi_0 + \dots + \xi_N) = L_0\xi_0 + \dots L_N\xi_N.$$

The relations

$$\Delta = LL^* + L^*L$$
$$I = H + G\Delta = H + \Delta G$$

from Theorem 4.12 still hold. Also we note that with respect to this grading L has degree +1, L^* has degree −1 and Δ, G and H have degree 0. We will denote by $\mathcal{H}(E) = \bigoplus \mathcal{H}(E_j)$ the total space of Δ-harmonic sections. Given an inner product on each $\mathcal{E}(E_j)$ we extend the inner product to $\mathcal{E}(E)$ in the obvious way by

$$\langle \xi, \eta \rangle_E = \sum_{j=0}^{N} \langle \xi_j, \eta_j \rangle_{E_j}.$$

The main theorem is now:

Theorem 4.14 *Let $(\mathcal{E}(E), L)$ be an elliptic complex equipped with an inner product. Then we have the following:*

(1) *There is an orthogonal decomposition*

$$\mathcal{E}(E) = \mathcal{H}(E) \oplus LL^*G(\mathcal{E}(E)) \oplus L^*LG(\mathcal{E}(E)).$$

(2) *The following commutation relations are valid:*
   (a) $I = H + \Delta G = H + G\Delta$.
   (b) $HG = GH = H\Delta = \Delta H = 0$.
   (c) $L\Delta = \Delta L, \quad L^*\Delta = \Delta L^*$.
   (d) $LG = GL, \quad L^*G = GL^*$.

(3) $\dim_{\mathbb{C}} \mathcal{H}(E) < \infty$*, and there is a canonical isomorphism*

$$\mathcal{H}(E_j) \cong H^j(E).$$

We will now discuss some examples in which we will see that it is possible to represent various cohomology groups by harmonic forms. This property will play an important role in the proof of the Hodge Decomposition Theorem.

Example 4.15. We have seen in Theorem 2.18 that

$$H^r(X,\mathbb{C}) \cong H^r(\mathcal{E}^*(X))$$

and when X is a differentiable manifold we shall make this isomorphism an identification and just call $H^r(X,\mathbb{C})$ the de Rham group with complex coefficients. If X is also compact then there is a Riemannian metric on X. This induces an inner product on $\wedge^p T^*X$ and therefore $(\mathcal{E}^*(X), d)$ becomes an elliptic complex with inner product. We shall denote the associated Laplacian by $\Delta = \Delta_d = dd^* + d^*d$. The vector space of Δ-harmonic forms is denoted by

$$\mathcal{H}^r(X) = \mathcal{H}_\Delta(\wedge^r T^*X).$$

Now from Theorem 4.14 we get that

$$H^r(X,\mathbb{C}) \cong \mathcal{H}^r(X).$$

This is a very important relation and it says that every cohomology class in the de Rham cohomology, $c \in H^r(X,\mathbb{C})$, can be represented by a unique harmonic form ϕ.

Example 4.16. For a compact complex manifold X we consider again the elliptic complex

$$\dots \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X) \xrightarrow{\bar\partial} \dots$$

We have seen in Theorem 2.19 that

$$H^q(X,\Omega^p) \cong H^q((\wedge^{p,*}T^*X, \bar\partial)).$$

If we equip $\wedge^{p,q}T^*X$ with a Hermitian metric then the complex becomes a elliptic complex with an inner product. We denote the associated Laplacian by

$$\overline{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

and denote by

$$\mathcal{H}^{p,q}(X) = \mathcal{H}_{\overline{\Box}}(\wedge^{p,q}T^*X)$$

the $\overline{\Box}$- harmonic (p,q)-forms. Just as in the previous example we now have by Theorem 4.14 that

$$H^q(X,\Omega^p) \cong \mathcal{H}^{p,q}(X).$$

Just as before, this example can be extended to the case where we have coefficients in a vector bundle.

Example 4.17. Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X and let $(\mathcal{E}^{p,*}(X,E), \bar\partial)$ be the elliptic complex of (p,q)-forms with coefficients in E. By Theorem 2.20 we have

$$H^q(X,\Omega^p(E)) \cong H^q((\mathcal{E}(\wedge^{p,q}T^*X \otimes_{\mathbb{C}} E), \bar\partial_E)).$$

The bundles in the complex are of the form $\wedge^{p,q}T^*X \otimes E$ and equipping them with a Hermitian metric we get again an elliptic complex with an inner product. We define the associated Laplacian by

$$\overline{\Box} = \bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E$$

and let again

$$\mathcal{H}^{p,q}(X) = \mathcal{H}_{\overline{\Box}}(\wedge^{p,q}T^*X \otimes E)$$

denote the $\overline{\square}$-harmonic E-valued (p,q)-forms. Then by Theorem 4.14 we have the isomorphism

$$H^q(X, \Omega^p(E)) \cong \mathcal{H}^{p,q}(X, E).$$

CHAPTER 5

Compact Complex Manifolds & The Hodge Decomposition

In this chapter we will apply the result of the previous chapters to the study of compact complex manifolds. This is the most important chapter, as we will proof the main theorem of the text, the Hodge Decomposition Theorem. This theorem states that the de Rham groups can be expressed as the direct sum of the Dolbeault groups of the same total degree. We will start by introducing the fundamental 2-form and the Hodge $\star$-operator in the first section. Section two will deal with the theory about harmonic forms on compact manifolds. We will also prove two duality theorems known as the Poincaré and Serre duality. In the third section we will be concerned with a finite dimensional representation for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and using this representation we will derive the Lefschetz decomposition theorem for a Hermitian exterior algebra. In the fourth section we introduce the concepts of a Kähler metric and Kähler manifolds. In this section we will also prove various commutation relations for the operators d, ∂, $\bar{\partial}$ and their associated Laplacian operators. These relations are the final ingredients for our proof of the Hodge Decomposition Theorem in the fifth section.

1. Hermitian Exterior Algebra on a Hermitian Vector Space

Let V be a finite dimensional real vector space equiped with an inner product $\langle \cdot, \cdot \rangle$. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis for V where $d = \dim V$ then we know that $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 \leq \cdots \leq i_p \leq d\}$ is a basis for $\wedge V$ (which shall be orthonormal under the induced inner product). Also an *orientation* for V is given by an ordering of the a basis such as $\{e_1, \dots, e_d\}$. This is equivalent to the choice of sign for a particular d form, for example $e_1 \wedge \cdots \wedge e_d$ called a *volume element*, denoted by vol. It is possible to equip the exterior algebra $\wedge V$ of V with an inner product $\langle \cdot | \cdot \rangle$ (notice the difference between the notation $\langle \cdot | \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ for the inner product on $\wedge V$ and V respectively) induced by the inner product on V. Namely define $\langle \cdot | \cdot \rangle$ on the basis elements of $\wedge^k V$ to be

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k} | e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \det \left[\langle e_{i_m}, e_{j_n} \rangle \right]$$

where $\left[\langle e_{i_m}, e_{j_n} \rangle \right]$ denotes the $k \times k$ matrix with elements $\langle e_{i_m}, e_{j_n} \rangle$. We extend this definition to all of $\wedge V$ by linearity. To see that $\langle \cdot | \cdot \rangle$ is symmetric note that $\det A = \det A^\top$. For positive definiteness not that for any basis element $e_{i_1} \wedge \cdots \wedge e_{i_k}$ of $\wedge^k V$ and $\lambda \in \mathbb{R}$ we have

$$\langle \lambda e_{i_1} \wedge \cdots \wedge e_{i_k} | \lambda e_{i_1} \wedge \cdots \wedge e_{i_k} \rangle = \det \lambda^2 \langle e_{i_m}, e_{i,n} \rangle = \lambda^2 \det I = \lambda^2 \geq 0$$

since the $\{e_1, \dots, e_d\}$ form an orthonormal basis of V.

Now let $\{\tilde{e}_1, \dots \tilde{e}_d\}$ be another orthonormal basis for V then we know that

$$\tilde{e}_i = \sum_j A_{ij} e_j$$

where $A = \left[A_i j\right]$ is an orthogonal matrix. Computing the inner product we get for two random basis vector ,$\tilde{e}_{i_1} \wedge \dots \wedge \tilde{e}_{i_k}$ and $\tilde{e}_{j_1} \wedge \dots \wedge \tilde{e}_{j_k}$ that

$$\langle \tilde{e}_{i_1} \wedge \dots \wedge \tilde{e}_{i_k} | \tilde{e}_{j_1} \wedge \dots \wedge \tilde{e}_{j_k} \rangle = \det\left[\sum_{\rho\sigma} A_{i_m\rho} A_{j_n\sigma} \langle e_\rho, e_\sigma \rangle\right]$$

where $1 \le m, n \le k$.
But from $\langle e_i, e_j \rangle = \delta_{ij}$ it follows that

$$\det\left[\sum_{\rho\sigma} A_{i_m\rho} A_{j_n\sigma} \langle e_\rho, e_\sigma \rangle\right] = \det\left[\sum_{\rho} A_{i_m\rho} A_{j_n\rho}\right] = \det\left[A^\top A_{i_m j_n}\right] = \det\left[\delta_{i_m j_n}\right]$$

while on the other hand

$$\langle e_{i_1} \wedge \dots \wedge e_{i_k} | e_{j_1} \wedge \dots \wedge e_{j_k} \rangle = \det\left[\langle e_{i_m}, e_{j,n} \rangle\right] = \det\left[\delta_{i_n j_m}\right]$$

where $\left[\delta_{i_n, j_m}\right]$ denotes the $k \times k$ matrix whose element at the position n, m is given by $\delta_{i_m j_n}$. From linearity it follows that $\langle \cdot | \cdot \rangle$ is independent of the orthonormal basis chosen. Moreover using multi-index notation it is easy to check that given $\alpha = \sum'_{|I|=k} a^I e_I$ $\beta = \sum'_{|I|=k} b^I e_I$ we have

$$\langle \alpha | \beta \rangle = \sum_{|I|=k}' a_I b_I.$$

We now introduce an operator used to extend this inner product. Given an orthonormal basis (of a d dimensional real vector space) with a volume element we define the *Hodge $*$-operator* to be a map

$$* : \wedge^p V \to \wedge^{d-p} V$$

on basis elements by

$$*(e_{i_1} \wedge \dots \wedge e_{i_p}) \mapsto \pm e_{j_1} \wedge \dots \wedge e_{j_{d-p}}$$

where $\{j_1, \dots, j_{d-p}$ is the complement of $\{i_1, \dots, i_p\}$ in $\{1, \dots, d\}$ and the sign is chosen such that

$$e_{i_1} \wedge e_{i_p} \wedge *(e_{i_1} \wedge \dots \wedge e_{i_p}) = \mathrm{vol}\,. \tag{5.1}$$

To define $*$ on all of $\wedge^p V$ we can just extend the definition on the basis elements by using the bilinearity of the wedge product. We claim that given $\alpha = \sum'_{|I|=p} a^I e_I, \beta = \sum'_{|I|=p} b^I e_I \in \wedge^p V$ we have

$$\alpha \wedge *\beta = \langle \alpha | \beta \rangle \,\mathrm{vol} \tag{5.2}$$

where $\langle \cdot | \cdot \rangle$ as before. To see this note that

$$\alpha \wedge *\beta = \sum_{|I|=|J|=p}' a_I b_J e_I \wedge *e_J$$

but the wedge product vanishes unless $I = J$ and in that case we have $e_I \wedge *e_I = \mathrm{vol}$ so we get

$$\alpha \wedge *\beta = \sum_{|I|=p}' a_I b_I \,\mathrm{vol} = \langle \alpha | \beta \rangle \,\mathrm{vol}\,.$$

We also see from this that the Hodge $*$ operator is independent of basis (since $\langle\cdot|\cdot\rangle$ and vol are) and depends only on the inner product structure of V and the choice of orientation.We extend the inner product given by Equation 5.1 to complex valued differential forms in the following way. Let $\alpha, \beta \in \wedge^p V \otimes \mathbb{C}$ then we can write $\alpha = \sum'_{|I|=p} a_I e_I$ and $\beta = \sum'_{|I|=p} b_I e_I$ where $a_I, b_I \in \mathbb{C}$. Define

$$\langle \alpha|\beta\rangle = \sum_{|I|=p}{}' a_I \bar{b}_I.$$

It is clear that in the case that α, β are real we get our old inner product back. Furthermore, also extending the Hodge $*$ operator to $\wedge^p V \otimes \mathbb{C}$ by complex linearity we get

$$\alpha \wedge *\bar{\beta} = \langle \alpha|\beta\rangle \operatorname{vol}. \tag{5.3}$$

Next define a linear map $w : \wedge V \to \wedge V$ by

$$w = \sum (-1)^{dr+r} \Pi_r$$

where $\Pi_r : \wedge V \to \wedge^r V$ denotes projection onto the homogeneous vectors of degree r. Let $\{i_1, \ldots, i_p, j_1, \ldots j_{d-p}\}$ be an ordered set then we see that it takes $p(d-p) = dp - p^2$ transposition to get to the ordered set $\{j_1, \ldots, j_{d-p}, i_1, \ldots, i_p\}$. Using this and noting that $dp - p^2$ has the same parity as $dp - p$ we get that $w = **$. Moreover, in the special case that d is even we have

$$w = \sum_r (-1)^r \prod_r . \tag{5.4}$$

We now introduce the notion of forms on a vector space and use this to define the so called fundamental form which is related to the metric. Let E be a complex vector space with $\dim_{\mathbb{C}} E = n$ (note that we dont assume E to have a metric because a metric is not needed for the notion of general forms). Let E' be the real dual to the underlying real vector space of E, that is E' is the $\mathbb{R}$ vector space of real valued, $\mathbb{R}$ linear maps from E to $\mathbb{R}$. Then we define

$$F = E' \otimes_{\mathbb{R}} \mathbb{C}$$

to be the $\mathbb{C}$ vector space of complex valued $\mathbb{R}$ linear maps from E to $\mathbb{C}$. Note that $\dim_{\mathbb{R}} E' = 2n$ we have $\dim_{\mathbb{C}} F = 2n$. We form the exterior algebra on F denoted by

$$\wedge F = \sum_{p=0}^{2n} \wedge^p F.$$

Elements of $\wedge^p F$ are called *p-forms* or *p-covectors* on E. Conjugation on $\wedge F$ is given by

$$\bar{\omega}(v_1, \ldots, v_p) = \overline{\omega(v_1, \ldots, v_p)} \qquad \text{for } v_i \in E, \omega \in \wedge^p F.$$

Let $\wedge^{1,0} F$ and $\wedge^{0,1} F$ be the subspace of $\wedge^1 F$ consisting of complex linear and anti-linear forms respectively. Then we see that $\overline{\wedge^{1,0} F} = \wedge^{0,1} F$ and

$$\wedge^1 F = \wedge^{1,0} F \oplus \wedge^{0,1} F$$

from which we get, just as in Chapter 1 a bigrading on F

$$\wedge F = \sum_{r=0}^{2n} \sum_{p+q=r} \wedge^{p,q} F.$$

Now suppose our vector space is equipped with a Hermitian metric $\langle\cdot,\cdot\rangle$. Since the inner product is a map from $E\times E$ to $\mathbb{C}$ it is actually Hermitian symmetric positive definite 2-form, which is linear in the first argument and anti-linear in the second (this is called sesquilinear). Letting $\{z_1,\ldots,z_n\}$ be a basis for $\wedge^{1,0}F$ it follows that $\{\bar z_1,\ldots,\bar z_n\}$ is a basis for $\wedge^{0,1}F$ and that we can write

$$\langle u,v\rangle = h(u,v)$$

where

$$h=\sum_{\mu,\nu} h_{\mu\nu} z_m\otimes \bar z_\nu$$

and $h_{\mu\nu}$ is a positive definite Hermitian matrix. From $z_j = x_j + iy_j$ and $\bar z_j = x_j - iy_j$ it follows that

$$\begin{aligned}h &= \sum_{\mu,\nu} h_{\mu\nu}(x_\mu + iy_\mu)\otimes(x_\nu - iy_\nu)\\ &= \sum_{\mu,\nu} h_{\mu\nu}\left(x_\mu\otimes x_\nu\right) + y_\mu\otimes y_\nu) + i\sum_{\mu,\nu} h_{\mu\nu}\left(y_\mu\otimes x_\nu - x_\mu\otimes y_\nu\right).\end{aligned}$$

In other words, we can write

$$h = S + iA$$

where S is a symmetric positive definite bilinear form representing the Euclidean inner product induced on the underlying real vector space of A by the Hermitian metric on E namely $S=\sum_{\mu,\nu} h_{\mu\nu}\left(x_\mu\otimes x_\nu\right) + y_\mu\otimes y_\nu)$. Using

$$x_\mu = \frac{z_\mu+\bar z_\mu}{2},\qquad y_\mu = \frac{z_\mu-\bar z_\mu}{2i}$$

we get that

$$\begin{aligned}A &= \sum_{\mu,\nu} h_{\mu\nu}\left(y_\mu\otimes x_\nu - x_\mu\otimes y_\nu\right)\\ &= \sum_{\mu,\nu}\frac{h_{\mu\nu}}{2}\left((i\bar z_\mu - iz_\mu)\otimes(z_\nu+\bar z_\nu) - (z_\mu+\bar z_\mu)\otimes(i\bar z_\nu - iz_\nu)\right)\\ &= -i\sum_{\mu,\nu} h_{\mu\nu} z_\mu\wedge\bar z_\nu.\end{aligned}$$

We define the *fundamental 2-form* associated to the Hermitian metric to be

$$\Omega = \frac{i}{2}\sum_{\mu,\nu} h_{\mu\nu} z_\mu\wedge\bar z_\nu. \tag{5.5}$$

Using this definition we get

$$h = S - 2i\Omega \tag{5.6}$$

where S is as before. Note that Ω is of type $(1,1)$ and that since

$$\bar\Omega = \overline{\frac{i}{2}\sum_{\mu,\nu} h_{\mu\nu} z_\mu\wedge\bar z_\nu} = -\frac{i}{2}\sum_{\mu,\nu}\bar z_\mu\otimes z_\nu = \frac{i}{2}\sum_{\mu,\nu} z_\nu\otimes\bar z_\mu = \Omega$$

and therefore Ω is a real 2-form of type $(1,1)$. By taking the dual of a orthonormal basis of E we see that it is always possible to find a basis $\{z_\mu\}$ of $\wedge^{1,0}F$ such that $h_{\mu\nu}$ is the identity matrix. In this case we have

$$\begin{aligned} S &= \sum x_\mu \otimes y_\nu + y_\mu \otimes x_\nu, \\ \Omega &= \sum x_\mu \wedge y_\mu = \frac{i}{2}\sum z_\mu \wedge \bar{z}_\mu. \end{aligned} \tag{5.7}$$

We easily see that

$$\Omega^n = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n. \tag{5.8}$$

But this is a volume form on E' and therefore gives an orientation on E'. Moreover, it follows from $S = \sum_\mu x_\mu \otimes x_\mu + y_\mu \otimes y_\mu$ that $\{x_\mu, y_\mu\}$ is an orthonormal basis for E' (because S represents the euclidean inner product on underlying real vector space of E and therefore induces a euclidean inner product on E' and a basis of E' is orthonormal with respect to this induces inner product if it is the dual to a orthonormal basis of the underlying real vector space of E) and thus there is a naturally defined Hodge $*$ operator

$$* : \wedge^p E' \to \wedge^{2n-p} E' \tag{5.9}$$

induced from the Hermitian inner product on E. We define

$$\text{vol} = \frac{1}{n!}\Omega$$

which is independent of the choice of basis (since Ω is). In the special case of the orthonormal basis we used earlier we have that

$$\text{vol} = x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n.$$

We will now use the above discussion to define linear operators mapping $\wedge F \to \wedge F$. Since these operators will be defined in terms of the above structure it follows that their definitions all follow from the Hermitian inner product structure on E. We had already defined w for even dimensional manifold by Equation 5.4. Extending this definition by complex linearity we get

$$w : \wedge F \to \wedge F.$$

Similarly we extend $*$ defined in Equation 5.9 by complex linearity which gives

$$* : \wedge^p F \to \wedge^{2n-p} F.$$

Note that both w and $*$ are real operators which says by definition that they commute with complex conjugation, i.e. for a real operator D we have $D(\bar{x}) = \overline{D(x)}$ Next define

$$J : \wedge F \to \wedge F$$

by

$$J = \sum i^{p-q}\Pi_{p,q}$$

where $\Pi : \wedge F \to \wedge^{p,q} F$ is the usual projection. Note that if $v \in \wedge^{1,0}$ then $Jv = iv$ and if $v \in \wedge^{0,1} F$ then $Jv = iv$. This is exactly the same as the J operator defined in Chapter 1 which represents the complex structure. Therefore we see that the J

here defined is the natural multilinear extension of the complex structure operator J to the exterior algebra $\wedge F$. Lastly we define a linear operator $L : \wedge F \to \wedge F$ by

$$L(v) = \Omega \wedge v.$$

Then we see that $L : \wedge^{p,q} F \to \wedge^{p+1,q+1} F$ since Ω is of type $(1,1)$ and that L is a real operator since Ω is a real 2-form.
As we saw in Equation 5.3, $\wedge^p F$ is equiped with an inner product. Then we have the following proposition.

Proposition 5.1 *With respect to the inner product defined by 5.3 L has a Hermitian adjoint $L^* : \wedge^p F \to \wedge^{p-2} F$ given by*

$$L^* = w * L * .$$

Proof. Let $\alpha \in \wedge^p F$ and $\beta \in \wedge^{p+2} F$ be given then we compute

$$\begin{aligned}
\langle L\alpha, \beta \rangle &= \Omega \wedge \alpha \wedge (*\bar{\beta}) \\
&= \alpha \wedge \Omega \wedge (*\bar{\beta}) \\
&= \alpha \wedge (L * \bar{\beta}) \\
&= \alpha \wedge (*w * L * \bar{\beta}) \\
&= \alpha \wedge (\overline{*w * \beta}) \\
&= \langle \alpha, w * L * \beta \rangle \,\mathrm{vol} \\
&= \langle \alpha, L^* \beta \rangle \,\mathrm{vol}
\end{aligned}$$

where we used that $*w* = * * ** = ww = id$, and that $L, w, *$ are real operators. □

From this proposition we see that L^* is a real operator, homogeneous of degree -2. Using the following proposition we shall see that L^* is bihomogeneous of degree $(-1,-1)$. However we first need to proof a Lemma needed in the proof of the proposition. We shall use the notation, where M is a multi-index,

$$w_M = \prod_{\mu \in M} z_\mu \wedge \bar{z}_\mu = (-2i)^{|M|} \prod_{\mu \in M} x_\mu \wedge y_\mu$$

(here the product denotes the wedge product) where $\{z_1, \dots, z_n\}$ is a basis for $\wedge^{1,0} F$. Also we let $N = \{1, \dots, n\}$ then we have the following lemma.

Lemma 5.2 *Suppose that A, B and M are mutually disjoint increasing multi-indices. Then*

$$*(z_A \wedge \bar{z}_B \wedge w_M) = \gamma(a,b,m) z_A \wedge \bar{z}_B \wedge w_{M'}$$

for a non-vanishing constant $\gamma(a,b,m)$, where $a = |A|$, $b = |B|$, $m = |M|$ and $M' = N - (A \cup B \cup M)$. Moreover

$$\gamma(a,b,m) = i^{a-b} (-1)^{\frac{p(p+1)}{2} + m} (-2i)^{p-n}$$

where $p = a + b + 2m$ is the total degree of $z_A \wedge \bar{z}_B \wedge w_M$.

Proof. Write $\nu = z_A \wedge \bar{z}_B \wedge w_M$ and if for a multi-index A we have $A = A_1 \cup A_2$ define

$$\epsilon_A^{A_1 A_2} = \begin{cases} 0 & \text{if} A_1 \cap A_2 \neq \varnothing. \\ 1 & \text{if } A_1 A_2 \text{ is an even permutation of } A. \\ -1 & \text{if } A_1 A_2 \text{ is an odd permutation of } A. \end{cases}$$

Using this we get

$$z_A = \sum_{A=A_1\cup A_2}' \epsilon_A^{A_1A_2} i^{a_2} x_{A_1} \wedge Y_{A_2}$$

where we sum over increasing multi-indices A_1, A_2 such that $A = A_1 \cup A_2$. To see this note that if $A = \{\mu_1, \dots, \mu_a\}$ then $z_A = (x_{\mu_1} + iy_{\mu_1}) \wedge \cdots \wedge (x_{\mu_a} + iy_{\mu_a})$ and expanding this gives exactly the summation above. Thus we get

$$\nu = (-2i)^m \sum_{\substack{A=A_1\cup A_1 \\ B=B_1\cup B_2}}' \epsilon_A^{A_1A_2} \epsilon_B^{B_1B_2} i^{a_2-b_2} x_{A_1} \wedge y_{A_2} \wedge x_{B_1} \wedge Y_{B_2} \wedge \prod_{\mu\in M} x_\mu \wedge y_\mu.$$

Now we have expressed ν in terms of a real basis we shall use the complex linearity of $*$ to compute $*\nu$ term by term. To simplify the notation we shall assume $B = \varnothing$ note that the computation of the general case is completely analogous to this one. We have

$$*(z_A \wedge w_M) = (-2i)^m \sum_{A=A_1\cup A_2}' \epsilon_A^{A_1A_2} i^{a_2} * \left\{ x_{A_1} \wedge y_{A_2} \wedge \prod_{\mu\in M} x_\mu \wedge y_\mu \right\}.$$

It is easy to see that the result of $*$ acting on the expression in brackets should be of the form

$$\pm x_{A_2} \wedge y_{A_1} \wedge \prod_{\mu\in M'} x_\mu \wedge y_\mu$$

where $M' = N - (A \cup M)$. The only thing we have to do is determine the sign. Since $\prod_{\mu\in M'} x_\mu \wedge y_\mu$ has an even number of terms is irrelevant for the sign since transposing an even number of times doesnt change the sign of the expression and therefore we can always move the products around without changing sign. It follows that it is enough to consider

$$x_{A_1} \wedge y_{A_2} \wedge x_{A_2} \wedge y_{A_1} = (-1)^{a_2^2} x_{A_1} \wedge y_{A_1} \wedge x_{A_2} \wedge y_{A_2}.$$

In general we have that

$$x_C \wedge y_C = (-1)\frac{|C|(|C|-1)}{2} x_{\mu_1} \wedge y_{\mu_1} \wedge \cdots \wedge x_{\mu_{|C|}} \wedge y_{\mu_{|C|}}.$$

To see this note that y_{μ_1} has to move $|C| - 1$ places ,y_{μ_2} has to move $|C| - 2$ places, etc. Therefore the total number of transpositions is $1 + \cdots + |C| - 1 = \frac{|C|(|C|-1)}{2}$. Using this identity we get that the sign in 1 should be

$$(-1)^{a_2^2} + \frac{a_1(a_1-1)}{2} + \frac{a_2(a_2-1)}{2} := (-1)^r.$$

Now substituting this in Equation 1 and using

$$\begin{aligned} \epsilon_A^{A_1A_2} &= (-1)^{a_1a_2} \epsilon_A^{A_2A_1} \\ i^{a_2} &= i^a (-1)^{a_1} i^{a_1} \end{aligned}$$

we get

$$*(z_A \wedge w_M) = i^a (-2i)^m \sum_{A=A_1\cup A_2}' \epsilon_A^{A_2A_1} i^{a_1} \{(-1)^{r+a_1+a_1a_2}\} x_{A_2} \wedge y_{A_1} \wedge \prod_{\mu\in M'} x_\mu \wedge y_\mu.$$

Since we have

$$(-1)^{r+a_1+a_1a_2} = (-1)^{\frac{a(a=1)}{2}} = (-1)^{\frac{p(p+1)}{2}+m}$$

we can pull the bracketed term outside the summation and we obtain the wanted equation for the case $B = \varnothing$. □

Let, as usual, $[M, N] = MN - NM$ denote the commutator of two given endomorphisms on a vector space.

Proposition 5.3 *Let E be a Hermitian vector space with $\dim_{\mathbb{C}} E = n$ with fundamental form Ω and associated operators w, J, L and L^*. Then:*

(1) $*\Pi_{p,q} = \Pi_{n-q,n-p}*$.
(2) $[L, w] = [L, J] = [L^*, w] = [L^*, J] = 0$.
(3) $[L^*, L] = \sum_{p=0}^{2n}(n-p)\Pi_p$.

PROOF. For the first part of the proposition note that any element of $\wedge F$ can be written in the form $\sum'_{A,B,M} c_{A,B,M} z_A \wedge \bar{z}_B \wedge w_M$. The type of a term of the form $z_A \wedge \bar{z}_B w_M$ is $(m+a, m+b)$ where $|A| = a, |B| = b, |M| = m$ and $|N| = n$. Then we see from Lemma 5.2 that since $M' = N-(A \cup B \cup M)$,and therefore $|M'| = m' = n-a-b-m$, that the type of $*(z_A \wedge \bar{z}_B \wedge w_M$ is $(a+m', b+m') = (a+n-a-b-m, b+n-a-b-m) = (n-(b+m), n-(a+m)) = (n-q, n-p)$. This proves the first identity. We note that this is equivalent to

$$*|_{\wedge^{p,q}F} : \wedge^{p,q}F \to \wedge^{n-q,n-p}F$$

is an isomorphism.
The second part of the proposition is an easy computation using the fact that L is a homogeneous operator of type $(1,1)$ and L^* is a homogeneous operator of type $(-1,-1)$. For the third part (we use the same notation as in Lemma 5.2) we use Equation 5.7 which gives

$$\begin{aligned} L(z_A \wedge \bar{z}_B \wedge w_M) &= \Omega \wedge z_A \wedge \bar{z}_B \wedge w_M \\ &= \frac{i}{2}\left(\sum_{\mu=1}^{n} z_\mu \wedge \bar{z}_\mu\right) \wedge z_A \wedge \bar{z}_B \wedge w_M \\ &= \frac{i}{2} z_A \wedge \bar{z}_B \wedge \left(\sum_{\mu \in M'} w_{M \cup \{\mu\}}\right) \end{aligned} \tag{5.10}$$

where, as before $M' = N - (A \cup B \cup M)$. For the second equation we used that $x \wedge x = 0$ and that all terms where $\mu \notin M'$ give such doubled terms. On the other hand using 5.2 and $L^* = w * L*$ we get that

$$L^*(z_A \wedge \bar{z}_B \wedge w_M) = \frac{2}{i} z_A \wedge \bar{z}_B \wedge \left(\sum_{\mu \in M} w_{M-\{\mu\}}\right). \tag{5.11}$$

Using these formulas we get

$$L^* L - L L^* = (n-p) z_A \wedge \bar{z}_B \wedge w_M$$

where p is the degree of $z_A \wedge \bar{z}_B \wedge w_M$. This gives the last part of the proposition. □

2. Harmonic Theory on Compact Manifolds

As we have seen the laplacian on a Riemannian manifold is defined to be $dd^* + d^*d$, where d^* is the adjoint of d with respect to some inner product on the complex $\mathcal{E}^*(X)$ of complex valued differential forms on X. In this section we will use the Hodge star operator to define a particular inner product on the vector space of differential forms of a given degree. The inner product we define will enable us to find explicit formula's for the adjoint of d and other operators.

Assume we have a compact oriented Riemannian manifold X of dimension d. Like in Section 1 the Riemannian structure and the orientation of X define the Hodge $*$ operator

$$* : \wedge^p T_x^* X \xrightarrow{\cong} \wedge^{d-p} T_x^* X$$

for each $x \in X$. But since it is possible to find for all $x \in X$ a neighborhood U of x and a smooth orthonormal frame of $\wedge^p T^* X$ defined on U we see that $*$ actually defines a smooth bundle map. Therefore it follows that, extending $*$ by complex linearity to $\wedge^p T^* X \otimes \mathbb{C} = \mathcal{E}^p(X)$ we get an isomorphism of sections

$$* : \mathcal{E}^p(X) \xrightarrow{\cong} \mathcal{E}^{d-p}(X).$$

We define integration of d-forms in the standard way, see for example [**2**]. Let $\phi \in \mathcal{E}^d(X)$, $\{\phi_\alpha\}$ a partition of unit subordinate to a finite cover of X and $f_\alpha : U_\alpha \subset \mathbb{R}^n \to X$ the associated coordinate mappings, then we set

$$\int_X \phi = \sum_\alpha \int_{U_\alpha} f_\alpha^*(\phi_\alpha \phi) = \sum_\alpha \int_{\mathbb{R}^d} g_\alpha(x) dx_1 \wedge \cdots \wedge dx_d$$

where the smooth function g_α has compact support in U_α and it is a well known fact that this definition is independent of of the covering and partition of unit. Given $\phi, \psi \in \mathcal{E}^p(X)$ we know that $\phi \wedge *\psi$ is a d-form which makes it possible to define the so called *Hodge inner product* on $\mathcal{E}^*(X)$ by

$$(5.12) \qquad \begin{aligned} (\phi, \psi) &= \int_X \phi \wedge *\psi & \phi, \psi \in \mathcal{E}^p(X) \\ (\phi, \psi) &= 0 & \phi \in \mathcal{E}^p(X), \psi \in \mathcal{E}^q(X), p \neq q. \end{aligned}$$

We then have the following properties of the Hodge inner product.

Proposition 5.4 *The form $(\cdot, \cdot)$ defined by Equation 5.12 actually is a positive definite, Hermitian symmetric, sesquilinear form on the complex vector space $\mathcal{E}^*(X) = \bigoplus_{p=0}^d \mathcal{E}^p(X)$.*

Proof. We have seen in Equation 5.3 that the Riemannian mertic on X induces an Hermitian inner product, $\langle \cdot, \cdot \rangle$ on $\wedge^p T_x^* X$ for each $x \in X$ defined by

$$\psi \wedge *\bar{\psi} = \langle \phi, \psi \rangle \operatorname{vol}.$$

Then it is easy to see that

$$(\phi, \psi) = \int_X \phi \wedge *\bar{\psi} = \int_X \langle \phi, \psi \rangle \operatorname{vol}$$

where $\phi, \psi \in \mathcal{E}^P(X)$ is a positive semidefinite, sesquilinear Hermitian form on $\mathcal{E}^p(X)$. We want it to be a positive definite form but this is not clear a priori since it could be the case that $\langle \cdot, \cdot \rangle$ is nonzero only for x in a subset of X that has measure zero and thus contributes nothing to the integral. To see that this cannot happen let

$\phi \in \mathcal{E}^p(X)$ such that ϕ is non zero at $x_0 \in X$. Then it is possible to find an local orthonormal frame for $\{e_1, \ldots, e_d\}$ for $T^*X \otimes \mathbb{C}$ around x_0 such that

$$\phi = \sum_{|I|=p}{}' \phi_I e_I$$

and

$$\phi \wedge *\bar{\phi} = \sum_{|I|=p}{}' |\phi_I|^2 \operatorname{vol}$$

near x_0. Since $\sum'_{|I|=p} |\phi_I|^2 > 0$ it follows that the contribution to the integral

$$(\phi, \phi) = \int_X \phi \wedge *\bar{\phi}$$

will be nonzero and therefore $(\phi, \phi) > 0$ if $\phi \neq 0$ □

Proposition 5.5 *The direct sum decomposition $\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p,q}$ is an orthogonal direct sum decomposition with respect to the Hodge inner product.*

Proof. Let $\phi \in \mathcal{E}^{p,q}(X)$ and $\psi \in \mathcal{E}^{r,s}(X)$ such that $p+q = r+s$ be given. We see that $\bar{\psi}$ is of type (s, r) and with Proposition 5.3 we get that $*\bar{\psi}$ is of type $(n-r, n-s)$ from which it follows that $\phi \wedge *\bar{\psi}$ is of type $(n-r+p, n-s+q)$. It is easy to see that $\phi \wedge *\bar{\psi}$ will be a $2n$ form only if $r = p$ and $s = q$. Otherwise, $\phi \wedge *\bar{\psi}$ is zero. This concludes the proof of the proposition. □

One of the benefits of the Hodge inner product is that it is relatively easy to compute the adjoints of the linear operators we defined on $\mathcal{E}^*(X)$. However to do this we first need to extend our definition of the Hodge $*$ operator. We first define

$$\bar{*} : \mathcal{E}^*(X) \to \mathcal{E}^*(X)$$

by $\bar{*}\phi = *\bar{\phi}$. Then

$$\bar{*} : \wedge^p T^* X_c \to \wedge^{m-p} T^* X_c$$

is a conjugate-linear isomorphism.

Now let X be a Hermitian complex manifold and $E \to X$ a Hermitian vector bundle. We want to expend $\bar{*}$ to differential forms with coefficients in E. To do this we first choose a conjugate bundle isomorphism of E onto its dual bundle E^*

$$\tau : E \to E^*$$

one possibility is to define τ fiberwise by $e \mapsto \langle \cdot, e \rangle$ where $\langle \cdot, \cdot \rangle$ for $e \in E_x$ and where $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product on E_x. We then extend $\bar{*}$,

$$\bar{*}_E : \wedge^p T^* X_c \otimes E \to \wedge^{2n-p} T^* X_c \otimes E^*$$

by

$$\bar{*}_E(\phi \otimes e) = \bar{*}(\phi) \otimes \tau(e)$$

where $\phi \in \wedge^p T^*_x X_c$ and $e \in E_x$. Recalling that $\mathcal{E}^r(X, E)$ are by definition sections of $\wedge^r T^* X_c \otimes E$ we extend the Hodge inner product to $\mathcal{E}^*(X, E)$ in the following way. Let $\phi \in \mathcal{E}^r(X, E)$ and $\psi \in \mathcal{E}^s(X, E)$ be given then

$$(\phi, \psi) = \begin{cases} \int_X \phi \wedge \bar{*}_E \psi & r = s \\ 0 & r \neq s \end{cases}$$

Note that given $\phi \in \wedge^p T_x^* X_c, e \in E_x, \psi \in \wedge^{2n-p} T_x^* X_c, f \in E_x^*$ we have

$$(\phi \otimes e) \wedge (\psi \otimes f) = \phi \wedge \psi \cdot f(e) \in \wedge^{2n} T_x^* X_c$$

so the definition of the Hodge inner product extension indeed makes sense. Moreover it is clear that $\bar{*}_E$ preserves the bigrading on $\mathcal{E}^*(X, E)$ since the extensions only differ from the old inner product in the way they act on the coefficients and this is independent of the bigrading. It follows that

$$\bar{*}_E : \mathcal{E}^{p,q}(X, E) \to \mathcal{E}^{n-p,n-q}(X, E^*)$$

is a conjugate-linear isomorphism. Therefore we see that Proposition 5.5 still holds in this case. Now we have the following propositions regarding the adjoints of various operators with respect to the Hodge inner product.

Proposition 5.6 *Let X be an oriented compact Riemannian manifold of real dimension m and let $\nabla = dd^* + d^*d$ where d^* is the adjoint of d with respect to the Hodge inner product on $\mathcal{E}^*(X)$. Then*

(1) $d^* = (-1)^{m+mp+1} \bar{*} d \bar{*} = (-1)^{m+mp+1} * d *$ *on* $\mathcal{E}^p(X)$.

(2) $*\nabla = \nabla *, \qquad \bar{*}\nabla = \nabla \bar{*}$.

Proof. Let $\phi \in \mathcal{E}^{p-1}(X)$ and $\psi \in \mathcal{E}^p(X)$ be given. Then we have

$$\begin{aligned}(d\phi, \psi) &= \int_X d\phi \wedge \bar{*}\psi \\ &= \int_x d(\phi \wedge \bar{*}\psi) - (-1)^{p-1} \int_X \phi \wedge d\bar{*}\psi\end{aligned}$$

where we used that $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^{p-1}\phi \wedge d\psi$. Using Stokes theorem we see that the first term vanishes. Using $** = \bar{*}\bar{*} = w$ and $*w* = \bar{*}w\bar{*} = 1$ we obtain

$$\begin{aligned}(d\phi, \psi) &= (-1)^p \int_X \phi \wedge \bar{*}(\bar{*}^{-1} d\bar{*})\psi \\ &= (-1)^p \int_X \phi \wedge \ \bar{*}(\bar{*} w d\bar{*})\psi \qquad = (-1)^{m+mp+1}(\phi, \bar{*} d\bar{*}\psi).\end{aligned}$$

For the last equation note that $\psi \in \mathcal{E}^p(X)$ so $\bar{*}\psi \in \mathcal{E}^{m-p}$. From here we get that $d\bar{*} \in \mathcal{E}^{m-p+1}(X)$ and the sign we get from w is given by $(-1)^{m^2-mp+m+m-p+1}$ together with the $(-1)^p$ factor in front of the integral we get a sign $(-1)^{m+mp+1}$. We conclude that $d^* = (-1)^{m+mp+1}\bar{*}d\bar{*}$ and since d is real we also have $d^* = (-1)^{m+mp+1} * d*$. For the second part we write, given $\phi \in \mathcal{E}^p(X)$,

$$\begin{aligned}*\nabla\phi &= (-1)^{m+mp+1} \left(*d * d * + (-1)^m * *d * d\right)\phi \\ \nabla * \phi &= (-1)^{m+m(m-p)+1} \left(d * d * * + (-1)m * d * d*\right)\phi \\ &= (-1)^{m+mp+1} \left(*d * d * + (-1)^m d * d * *\right)\phi\end{aligned}$$

which is a simple computation. So if we show that $wd * d\phi = d * dw\phi$ we are done. To see this recall that $w = \sum(-1)^{p+mp}\Pi_p$ from which it follows that

$$d * dw\phi = (-1)^{p+mp} d * d\phi$$

and since $d * d\phi$ has degree $m - p$, we get

$$wd * d\phi = (-1)^{m-p+m(m-p)} d * d\phi = (-1)^{mp+p} d * d\phi$$

so we are done.

□

In the Hermitian case we have a similar result, given by the following proposition. Note that $\bar{*}_{E^*}$ is defined in the same way as $\bar{*}_E$ by using $\tau^{-1} : E^* \to E$.

Proposition 5.7 *Let X be a Hermitian complex manifold and let $E \to X$ be a Hermitian holomorphic vector bundle. Then*

(1) $\bar{\partial} : \mathcal{E}^{p,q}(X,E) \to \mathcal{E}^{p,q+1}(X,E)$ *has an adjoint $\bar{\partial}^*$ with respect to the Hodge inner product on $\mathcal{E}^{*,*}(X,E)$ given by*

$$\bar{\partial}^* = -\bar{*}_{E^*}\bar{\partial}\bar{*}_E.$$

(2) *If $\overline{\square} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the complex laplacian acting on $\mathcal{E}^{*,*}(X,E)$, then*

$$\overline{\square}_E = \bar{*}_E\overline{\square}.$$

Proof. In this case we also have $\bar{*}_E\bar{*}_{E^*} = w$. Since the real dimension of X is even, this reduces to $\sum(-1)^p\Pi_p$. For the proof of the first statement let $\phi \in \mathcal{E}^{p,q-1}(X,E)$ and $\phi \in \mathcal{E}^{p,q}(X,E)$. Then $\phi \wedge \bar{*}_E\psi$ is of type $(n, n-1)$ it follows that $\bar{\partial}(\phi \wedge \bar{*}_E\psi) = d(\phi \wedge \bar{*}_E\psi)$ since ∂ is zero on type $(n, n-1)$ forms. Moreover,

$$\bar{\partial}(\phi \wedge \bar{*}_E\psi) = \bar{\partial}\phi \wedge \bar{*}_E\psi + (-1)^{p+q-1}\phi \wedge \bar{\partial}\bar{*}_E\psi.$$

Using this identity and Stokes theorem as in the proof of Proposition 5.6 we obtain,

$$\begin{aligned}(\bar{\partial}\phi, \psi) &= (-1)^{p+q}\int_X \phi \wedge \bar{\partial}\bar{*}_E\psi \\ &= (-1)^{p+q}\int \phi \wedge \bar{*}_E(w\bar{*}_{E^*}\bar{\partial}\bar{*}_E\psi) \\ &= -\int \phi \wedge \bar{*}_E(\bar{*}_{E^*}\bar{\partial}\bar{*}_E\psi) \\ &= (\phi, -\bar{*}_{E^*}\bar{\partial}\bar{*}_E\psi).\end{aligned}$$

This proves the first part of the proposition. The proof is the second statement is exxactly the same as in Proposition 5.6 where we replace $\bar{\partial}$ for d, $\overline{\square}$ for Δ and $\bar{*}_E$ for $*$.

□

Using the above results we will now prove two well-known duality theorems. First we state the fact that if E is a finite dimensional complex vector space, then it is conjugate-linearly isomorphic to a complex vector space F if and only if F is complex-linearly isomorphic to E^*. The first duality theorem we will proof is known as the Poincaré duality.

Theorem 5.8 *Let X be a compact m-dimensional orientable differentiable manifold. Then there is a conjugate linear isomorphism*

$$\sigma : H^r(X, \mathbb{C}) \to H^{m-r}(X, \mathbb{C}),$$

and hence $H^{m-r}(X, \mathbb{C})$ is isomorphic to the dual of $H^r(X, C)$.

Proof. We introduce a Riemannian metric on X and choose an orientation. Let $*$ be the associated $*$-operator. Then the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{E}^r(X) & \xrightarrow{\bar{*}} & \mathcal{E}^{m-r}(X) \\
\big\downarrow H_\Delta & & \big\downarrow H_\Delta \\
\mathcal{H}^r(X) & \xrightarrow{\bar{*}} & \mathcal{H}^{m-r}(X) \\
\wr\| & & \wr\| \\
H^r(X,\mathbb{C}) & \xrightarrow{\sigma} & H^{m-r}(X,\mathbb{C})
\end{array}$$

Here H_Δ denotes the projection onto the harmonic forms. From Proposition 5.6 we known that $\bar{*}$ maps harmonic forms to harmonic forms since $\Delta\bar{*} = \bar{*}\Delta$. Moreover, the de Rham groups $H^r(X,\mathbb{C})$ are isomorphic to $\mathcal{H}^r(X)$, and σ denotes the induced conjugate linear isomorphism. □

Now we have the second duality theorem which is known as the Serre duality.

Theorem 5.9 *Let X be a compact complex manifold of complex dimension n and let $E \to X$ be a holomorphic vector bundle over X. Then there is a conjugate linear isomorphism*

$$\sigma : Hr(X,\Omega^p(E)) \to H^{n-r}(X,\Omega^{n-p}(E^*)),$$

and hence these spaces are dual to one another.

Proof. We introduce Hermitian metrics on X and X so we can define the $\bar{*}_E$ operator. Then the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{E}^{p,q}(X,E) & \xrightarrow{\bar{*}_E} & \mathcal{E}^{n-p,n-q}(X,E^*) \\
\big\downarrow H_{\bar{\Box}} & & \big\downarrow H_{\bar{\Box}} \\
\mathcal{H}^{p,q}(X,E) & \xrightarrow{\bar{*}_E} & \mathcal{H}^{n-p,n-q}(X,E^*) \\
\wr\| & & \wr\| \\
H^{p,q}(X,E) & \xrightarrow{\tau} & H^{n-p,n-q}(X,E^*) \\
\wr\| & & \wr\| \\
H^q(X,\Omega^p(E)) & \xrightarrow{\sigma} & H^{n-q}(X,\Omega^{n-p}(E^*))
\end{array}$$

This proves the theorem. Here we used that $\bar{*}_E$ maps harmonic forms to harmonics since we know from Proposition 5.7 that $\bar{\Box}\bar{*}_E = \bar{*}_E\bar{\Box}$. We also used the known fact that the Dolbeault groups, which we denoted by $\{H^{p,q}(X,E)\}$, are isomorphic to $H^q(X,\Omega^p(E))$ (See Theorem 2.20). □

3. Representations of $\mathfrak{sl}(2,\mathbb{C})$ on Hermitian Exterior Algebras

First we recall some basic definitions. A *Lie algebra* is a vector space $\mathfrak{U}$ equipped with a Lie bracket $[\cdot,\cdot]$, which is bilinear anticommutative and satisfies the Jacobi identity. A *represenation* of a lie algebra $\mathfrak{U}$ on a complex vector space V is an algebra homomorphism

$$\pi : \mathfrak{U} \to \mathrm{End}(V)$$

where $\mathrm{End}(V)$ denotes the Lie algebra of enodmorphism of V with the Lie bracket $[A, B] = AB - BA$. The *dimensions* of the representation is the dimension of the vector space V. A representation π is called *irreducible* if there is no proper invariant subspace $V_0 \neq \varnothing$ of V. That is, a subspace $V_0 \subset V$ such that $0 \neq V_0 \neq V$ and

$$\pi(X)V_0 \subset V_0 \qquad \text{for all } X \in \mathfrak{U}.$$

Given two representations π_1 and π_2 on V_1 and V_2 respectively, $\pi = \pi_1 \oplus \pi_2$ is a representation on $V_1 \oplus V_2$. Two representations are called equivalent if there exists an isomorphism $S : V_1 \to V_2$ such that $\pi_1 = S^{-1}\pi_2 S$. A representation is called *reducible* if it is equivalent to a direct sum of irreducible representations. Recall that a *Lie group* is a group which is also a smooth manifold. A representation of a Lie group G of a finite-dimensional complex vector space V is a real analytic homomorphism $\rho : G \to \mathrm{GL}(V)$, where as usual $\mathrm{GL}(V)$ denotes the Lie group of nonsingular endomorphisms of the vector space V. For Lie groups we have the same notions of irreducibility, complete reducibility, equivalence etc. as for Lie algebra's. To every Lie group G we can associate a Lie algebra $\mathfrak{g}$ which is by definition $T_e G$, the tangent space of G at the identity, or equivalently the vector space of left invariant vector fields of G. It is a standard result that Lie brackets preserve left invariance, see for example [**2**]. We will now introduce a specific Lie algebra which will be the only one of interest for our discussion. The lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is by definition the vector space of 2×2 complex matrices with trace zero endowed with the commutator bracket of matrices. It is the associated Lie algebra to the Lie group $\mathrm{SL}(2, \mathbb{C})$ the group of 2×2 matrices with complex coefficients and determinant equal to 1. To see this note that there exists an exponential map

$$\exp : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C}) \tag{5.13}$$

given as usual by,

$$\exp X = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

from which we see that

$$\frac{d}{dt}\, e^{tX}\Big|_{t=0} = X$$

for an arbitrary matrix X. We will also use the well known identity from matrix calculus

$$\det(e^{tX}) = e^{\mathrm{Tr}\, X}.$$

Let $X \in \mathfrak{sl}(2, \mathbb{C})$ then since $\mathrm{Tr}(X) = 0$ it follows that $\det(e^{tX}) = 1$, therefore e^{tX} path in $\mathrm{SL}(2, \mathbb{C})$ going through the identity for $t = 0$ with speed X. On the other hand let Y be any 2×2 matrix such that $\det(e^{tY}) = 1$ then, since that $e^{t\,\mathrm{Tr}\, Y} = 1$, it follows that $t\,\mathrm{Tr}(Y)$ is an integer multiple of $2\pi i$ which is only possible if $\mathrm{Tr}(Y) = 0$. So $\mathfrak{sl}(2, \mathbb{C})$ is indeed the tangent space to $\mathrm{SL}(2, \mathbb{C})$ at the identity. The elements of $\mathfrak{sl}(2, \mathbb{C})$ depend on $4 - 1 = 3$ parameters since they are 2×2 matrices with the condition $\det = 1$, it follows that $\dim \mathfrak{sl}(2, \mathbb{C}) = 3$. A basis is given by

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.14}$$

with the following commutation relations

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y \tag{5.15}$$

We also introduce the special element of $\mathfrak{sl}(2,\mathbb{C})$;

$$w = i(X+Y) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

called the *Weyl element*. Note that we have

$$wHw^{-1} = -H, \qquad wXw^{-1} = Y, \qquad wYw^{-1} = X.$$

We will be using the following result from Lie Theory.

Proposition 5.10 *There is a one-to-one correspondence between representations of* $\mathfrak{sl}(2,\mathbb{C})$ *and* $\mathrm{SL}(2,\mathbb{C})$. *More explicitly, let*

$$\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$$

be given and $g = e^X \in \mathrm{SL}(2,\mathbb{C})$ *where* $X \in \mathfrak{sl}(2,\mathbb{C})$ *then the corresponding representation*

$$\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$$

is given by

$$\pi(e^X) = e^{\rho(X)}.$$

Conversely $\rho = d\pi$ *where* d *denotes the derivative mapping.*

Proposition 5.11 *Let* ρ *be a presentation of* $\mathfrak{sl}(2,\mathbb{C})$ *on a finite-dimensional complex vector space, then* ρ *is completely reducible.*

We will now describe these irreducible representations. Assume a fixed finite dimensional representation ρ of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V to be given.

Definition 5.12. For a finite dimensional representation ρ of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V we define V^λ to be the *eigenvectors* of $\rho(H)$ with eigenvalue λ. A $v \in V^\lambda$ is said to have *weight* λ. We say that a vector $v \in V^\lambda$ is *primitive of weight* λ if v is nonzero and $\rho(X)v = 0$.

Using this terminology we have the following Lemma's.

Lemma 5.13 *The following is true.*

(1) *The sum* $\sum_{\lambda\in\mathbb{C}} V^\lambda$ *is a direct sum.*
(2) *If* v *is of weight* λ, *then* $\rho(X)v$ *is of weight* $\lambda+2$ *and* $\rho(Y)v$ *is of weight* $\lambda-2$.

Proof. The first assertion just says that eigenvectors with different eigenvalues are linearly independent. For the second statement we will use the linearity of ρ and the commutation relations, $[H,X] = 2X$ and $[H,Y] = -2Y$.

$$\begin{aligned} \rho(H)\rho(X)v &= (\rho(H)\rho(X) - \rho(X)\rho(H))\,v + \rho(X)\rho(H)v \\ &= \rho([H,X])v + \lambda\rho(X)v \\ &= \rho(2X)v + \lambda\rho(X)v \\ &= (\lambda+2)\rho(X)v. \end{aligned}$$

Similarly we get $\rho(H)\rho(Y)v = (\lambda-2)\rho(Y)v$. □

Lemma 5.14 *Every representation* ρ *of* $\mathfrak{sl}(2,\mathbb{C})$ *on a finite dimensional complex vector space* V, *has at least one primitive vector.*

Proof. Let v_0 be an eigenvector os $\rho(X)$, then we consider the sequence

$$v_0, \rho(X)v_0, \rho(X)^2 v_0, \dots$$

The nonzero terms in this sequence are all linearly independent (since they have different eigenvalues). Since V is finite dimensional it follows that this sequence must become zero at some point. Therefore, for some k, we have $\rho(X)^k v_0 = 0$ and $\rho(X)^{k-1}v_0 \neq 0$ and we have a primitive vector $v = \rho(X)^{k-1}v_0$. □

We now have the basic description of an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ on a finite dimensional complex vector space.

Theorem 5.15 *Let ρ be a irreducible representation ρ of $\mathfrak{sl}(2,\mathbb{C})$ on a finite dimensional complex vector space V. Let v_0 be a primitive vector of weight λ. Then setting $v_{-1} = 0$ and*

$$v_n = \frac{1}{n!}\rho(Y)^n v_0, \qquad n = 0, 1 \dots, m, \dots$$

we get for $n \geq 0$,

(1) $\rho(H)v_n = (\lambda - 2n)v_n$,
(2) $\rho(Y)v_n = (n+1)v_{n+1}$,
(3) $\rho(X)v_n = (\lambda + n + 1)v_{n-1}$.

Moreover, $\lambda = m$, where $m + 1 = \dim_{\mathbb{C}} V$, and

$$\rho(Y)^n v_0 = 0, \qquad n > m.$$

Proof. (1) follows immediatly from Lemma 5.13. (2) is clear from the definition of v_n. (3) follows from induction on n. For $n = 0$ we have $\rho(X)v_0 = 0$ since v_0 is a primitive vector. Suppose that (3) holds for $n-1$ then we get, using $[X,Y] = H$ and the induction hypothesis for $n-1$,

$$\begin{aligned} n\rho(X)v_n &= \rho(X)\rho(Y)v_{n-1} = \rho(Y)\rho(X)v_{n-1} + \rho([X,Y])v_{n-1} \\ &= (\lambda - n + 2)\rho(Y)v_{n-2} + \rho(H)v_{n-1} \\ &= (\lambda - n + 2)(n-1)v_{n-1} + (\lambda - 2n + 2)v_{n-1} \\ &= n(\lambda - n + 1)v_{n-1} \end{aligned}$$

which yields (3) after dividing by n. Since V is finite dimensional we have $v_0, \dots, v_m$ all nonzero and $v_{m+1}, \dots$ all zero. Using (3) we get

$$0 = \rho(X)v_{m+1} = (\lambda - (m+1) + 1)v_m = (\lambda - m)v_m$$

and since $v_m \neq 0$ it follows that $\lambda = m$. It also follows using (2) that $\rho(Y)^n v_0 = 0$ for $n > m$. We now show that $m + 1 = \dim_{\mathbb{C}} V$ and then we are done. Let $V_m = \mathrm{Span}\{v_0, \dots, v_m\}$. We then claim that V_m is invariant under the action of ρ on V. Since ρ is irreducible this implies that $V_m = V$ and $\dim_{\mathbb{C}} V = m + 1$. To see this let $v = \sum \alpha_n v_n$ be a vector in V_m then we have

$$\begin{aligned} \rho(H)v &= \sum \alpha_n (m - 2n)v_n \\ \rho(Y)v &= \sum \alpha_n (n+1)v_{n+1} \\ \rho(X)v &= \sum \alpha_n (m - n + 1)v_{n-1} \end{aligned}$$

all these vectors are in V_m again and so $\rho(\mathfrak{sl}(2,\mathbb{C}))V_m \subset V_m$ and we conclude that $\dim_{\mathbb{C}} V = m+1$. □

The next theorem states that up to equivalence there is only one irreducible representation of dimension $m+1$.

Theorem 5.16 *Let V be a complex vector space of dimension $m+1$ for $m \geq 0$. Let $\{v_0, \ldots, v_m\}$ be a basis for V. Define a representation ρ of $\mathfrak{sl}(2,\mathbb{C})$ on V by*

(1) $\rho(H)v_n = (m-2n)v_n$,
(2) $\rho(Y)v_n = (n+1)v_{n+1}$,
(3) $\rho(X)v_n = (m-n+1)v_{n-1}$,

for $n = 0, \ldots, m$ and $v_{-1} = v_{m+1} = 0$. This representation is irreducible and any irreducible representation of dimension $m+1$ is equivalent to this one.

Proof. It is easy to see that ρ defined above is indeed a representation. For the second part of the statement let V_0 be a nonzero subspace of V invariant under ρ. Then there is an eigenvector of $\rho(H)$ in V_0. But since the list of eigenvectors $\{v_0, \ldots, v_m\}$ is complete there must be a $0 \leq k \leq m$ such that $v_k \in V_0$. But then we can use the actions of $\rho(Y)$ and $\rho(X)$ to see that V_0 contains v_n for $n = 0, \ldots m$. In other words $V_0 = V$ and ρ is irreducible. Theorem 5.15 gives that any irreducible representation of dimension $m+1$ is equivalent to this one. □

Corollary 5.17 *Given an irreducible representation of dimension $m+1$, $\rho : \mathfrak{sl}(2,\mathbb{C}) \to V$ let $\phi \in V$ be an eigenvector of $\rho(H)$ of weight λ. Then there exists a primitive vector ϕ_0 of weight $\lambda + 2r$, for an integer $r \geq 0$, such that*

$$\phi = \rho(Y)^r \phi_0,$$

and where

$$\phi_0 = \frac{(m-r)!}{m!r!}\rho(X)^r \phi.$$

Proof. We let $\{v_0, \ldots, v_m\}$ be a basis for V as in Theorem 5.16 then for a fixed r, $0 \leq r \leq m$ we have

$$\rho(X)^r v_r = (m-r+1)(m-r+2)\cdots m v_0 = \frac{m!}{(m-r)!}v_0.$$

From this we see that

$$\rho(Y)^r \rho(X)^r v_r = 1 \cdot 2 \cdots r \frac{m!}{(m-r)!}v_r = \frac{m!r!}{(m-r)!}v_r.$$

Now if ϕ is an eigenvector of $\rho(H)$ then ϕ is a multiple of one of the eigenvectors $\{v_0, \ldots, v_m\}$ i.e. $\phi = \alpha v_r$ so

$$\phi = \frac{(m-r)!}{m!r!}\rho(Y)^r \rho(X)^r \phi$$

and defining

$$\phi_0 = \frac{(m-r)!}{m!r!}\rho(X)^r \phi$$

we see that ϕ_0 is primitive. □

We will now introduce specific representations for $SL(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$. To this we consider elements of $\mathbb{C}^2$ as column vectors and let a basis be given by

$$v_{1,0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_{1,1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then there is a natural action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ by matrix multiplication and we can define a representation π_1 of $SL(2,\mathbb{C})$ by

$$\pi_1(g) = (\boldsymbol{v} \mapsto g\boldsymbol{v})$$

for any $v \in \mathbb{C}^2$ and $g \in SL(2,\mathbb{C})$ where the product is matrix multiplication. Let $S^m(\mathbb{C}^2)$ denote the m-fold symmetric tensor product of $\mathbb{C}^2$ with itself. We extend π_1 by multilinearity to a representation of $SL(2,\mathbb{C})$ on $S^m(\mathbb{C}^2)$, denoted by π_m. Note that the dimension of $S^m(\mathbb{C}^2)$ is $m+1$.[1] Moreover we know from Proposition 5.10 that the representation π_m induces a representation $\rho_m = d\pi_m$ of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$ in the following way. Let $t \mapsto A_t$ be a path in $SL(2,\mathbb{C})$ such that $A_0 = I$ and $\frac{d}{dt} A_t\big|_{t=0} = X$. Let $v_1 \otimes \cdots \otimes v_m \in S^m(\mathbb{C}^2)$ then

$$\begin{aligned}
X(v_1 \otimes \cdots \otimes v_m) &= (\frac{d}{dt}A_t)\Big|_{t=0} (v_1 \otimes \cdots \otimes v_m) \\
&= \frac{d}{dt}\left(A_t(v_1 \otimes \cdots \otimes v_m)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(A_t v_1 \otimes \cdots \otimes A_t v_m\right)\Big|_{t=0} \\
&= \sum_{i=1}^{m} \left(A_t v_1 \otimes \cdots \otimes \dot{A}_t v_i \otimes \cdots \otimes A_t v_m\right)\Big|_{t=0} \\
&= \sum_{i=1}^{m} v_1 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_m.
\end{aligned}$$

It follows that ρ_m is just like π_m the multilinear extension of matrix multiplication, but in the case of ρ_m this extension is a *derivation.* Using the basis of $\mathfrak{sl}(2,\mathbb{C})$ given by Equation 5.14 we obtain

$$\rho_1(H)v_{1,0} = v_{1,0}, \qquad \rho_1(X)v_{1,0} = 0, \qquad \rho_1(Y)v_{1,0} = v_{1,1},$$

with similar results for the other basis vector, from which we see that ρ_1 satisfies the relations of Theorem 5.16 and therefore it is irreducible. For the general case define

$$v_{m,k} = v_{1,0}^{m-k} v_{0,1}^{k}, \qquad 0 \le k \le m$$

[1] It is well known that the $m - fold$ symmetric tensor product $S^m(V)$of a n-dimensional vector space V is given by the binomial coefficient $\binom{n+m-1}{m}$ which in this case reduces to $\dim S^m(\mathbb{C}^2) = \binom{m+1}{m} = m+1$

these $m+1$ elements form a basis of $S^m(\mathbb{C}^2)$. For this basis we can compute the following relations:

$$\begin{aligned}
\rho_m(H)v_{m,k} &= (m-2k)v_{m,k} \qquad 0 \le k \le m,\\
\rho_m(X)v_{m,0} &= 0,\\
\rho_m(Y)v_{m,m} &= 0,\\
\rho_m(X)v_{m,k} &= kv_{m,k-1} \qquad 1 \le k \le m,\\
\rho_m(Y)v_{m,k} &= (m-k)v_{m,k+1} \qquad 0 \le k \le m-1.
\end{aligned}$$

Using this relations we can express every basis vector $v_{m,k}$ in terms of the action of ρ_m on $v_{m,0}$(which is primitive), that is $v_{m,0}$ generates $S^m(\mathbb{C}^2)$ and therefore ρ_m is irreducible and equivalent to the representation in Theorem 5.16. Set

$$\phi_k = \rho_m Y^k v_{m,0},$$

then repeated use of $\rho_m(Y)v_{m,k} = (m-k)v_{m,k+1}$ gives

$$\phi_k = \frac{m!}{(m-k)!} v_{1,0}^{m-k} v_{1,1}^k$$

and using this definition it is easy to compute the action of the Weyl element w on ϕ_k,

$$\pi_m(w)\phi_k = \frac{m!}{(m-k)!}\pi_m(w)\left(v_{1,0}^{m-k}v_{1,1}^k\right)$$

but $\pi_m(w)$ was defined to be the multilinear extension of matrix multiplication and since

$$\pi_1(w)v_{1,0} = iv_{1,1}, \qquad \pi_1(w)v_{1,1} = iv_{1,0},$$

it follows that

$$\pi_m(w)\phi_k = \frac{m!}{(m-k)!} i^m v_{0,1}^{m-k} v_{1,0}^k = i^m \frac{k!}{(m-k)!}\phi_{m-k}.$$

From this we obtain

$$\pi_m(w)\rho_m(Y)^k\phi_0 = i^m \frac{k!}{(m-k)!}\rho_m(Y)^{m-k}\phi_0, \tag{5.16}$$

which will be used later in the proof of Lemma 5.22. Now we consider another explicit representation of $\mathfrak{sl}(2,\mathbb{C})$ this time on the exterior algebra of forms on an Hermitian vector space E. Let E be a fixed Hermitian vector space of complex dimension n with its algebra of forms $\wedge F$ and operators L and L^* as in Section 2. We will write

$$\Lambda := L^*,$$

$$B := \sum_{p=0}^{2n}(n-p)\Pi_p,$$

and we can define a representation $\alpha : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\wedge F)$ by

$$\alpha(X) = \Lambda, \qquad \alpha(Y) = L, \qquad \alpha(H) = B.$$

To see that this is indeed a representation we have to show that the commutation relations $[B,L]=-2L, [B,\Lambda]=2\Lambda$ and $[\Lambda,L]=B$ hold. Let v be a k-form then

$$\begin{aligned}[B,L]v=(BL-LB)v &= B(\Omega\wedge v)-L((n-k)v)\\ &=(n-(k+2))\Omega\wedge v-(n-k)\Omega\wedge v\\ &=-2\Omega\wedge v=-2Lv,\end{aligned}$$

which proves the first commutation relation. For the second commutation relation, $[B,\Lambda]=2\Lambda$ note that for a $k-form$ v, the operator B only multiplies v with a number. Furthermore since Λ is the adjoint of L it is a homogeneous operator of degree -2 so we get

$$\begin{aligned}[B,\Lambda]v=(B\Lambda-\Lambda B)v &= (n-k+2)\Lambda v-(n-k)\Lambda v\\ &=2\Lambda v.\end{aligned}$$

The third commutation relation $[\Lambda,L]=B$ follows immediately from Proposition 5.3 so α is a representation.

Definition 5.18. A p-form $\phi\in\wedge^pF$ is said to be *primitive* if $\Lambda\phi=0$.

In the next two theorems we will see that if ϕ is a primitive p-form then the action of α generates a subspace $F_\phi\subset\wedge F$ of dimension $n-p+1$, on which α acts irreducibly. The action of α also leaves the real form $\wedge_RF$ invariant since L,Λ and B are real operators. Thus we get a decomposition of $\wedge_RF$ into irreducible components, which is compatible with the decomposition $F=\oplus\wedge^{p,q}F$ since L,Λ and B are bihomogeneous. This decomposition is called the *Lefschetz decomposition*. Furthermore we know from Proposition 5.10 that α induces a representation of $\mathrm{SL}(2,\mathbb{C})$ on $\wedge F$ wich we will denote by π_α. Also we denote $(x)^+=\max(x,0)$.

Theorem 5.19 *Let E be an Hermitian vector space of complex dimension n.*

(1) *If $\phi\in\wedge^pF$ is a primitive p-form, then $L^q\phi=0$, $q\geq(n-p+1)^+$.*
(2) *There are no primitive forms of degree $p>n$.*

Proof. Let ϕ be a primitive p-form and let F_ϕ be the subspace of $\wedge F$ generated by the action of $\mathfrak{sl}(2,\mathbb{C})$ on ϕ by the representation α. Then clearly F_ϕ is invariant under the action of $\mathfrak{sl}(2,\mathbb{C})$ so α acts irreducibly on F_ϕ. Then using Theorem 5.15 we get that $\alpha(H)\phi=m\phi$ where $m+1=\dim F_\phi$. Since $\alpha(H)=B=\sum(n-p)\Pi_p$ we also have $\alpha(H)\phi=(n-p)\phi$, so $m=n-p$. Again by Theorem 5.15 it follows that $\alpha(Y)^q\phi=L^q\phi=0$ for $q\geq(n-p+1)^+$. The second part of the theorem is a simple consequence from the fact that $\dim F_\phi=n-p+1$. □

The next theorem states the *Lefschetz decomposition* for a Hermitian exterior algebra, as mentioned earlier.

Theorem 5.20 *Let E be an Hermitian vector space of complex dimension n, and let $\phi\in\wedge^pF$ be a p-form, then*

(1) *One can write ϕ uniquely in the form*

$$\phi=\sum_{r\geq(p-n)^+}L^r\phi_r, \tag{5.17}$$

where each ϕ_r is a primitive $(p-2r)$-form. Moreover, each ϕ_r can be expressed in the form

$$\phi_r = \sum_s a_{r,s} L^s \Lambda^{r+s}\phi, \qquad a_{r,s} \in \mathbb{Q}. \tag{5.18}$$

(2) *If $L^m\phi = 0$, then the primitive $(p-2r)$-forms ϕ_r appearing in the decomposition vanish if $r \geq (p-n+m)^+$, that is,*

$$\phi = \sum_{r=(p-n)^+}^{(p-n+m)^+} L^r \phi_r.$$

(3) *If $p \leq n$, and $L^{n-p}\phi = 0$ then $\phi = 0$.*

Proof. By Proposition 5.11 the representation space $V = \wedge F$ decomposes into a direct sum of irreducible subspaces, that is $V = V_1 \oplus \cdots \oplus V_l$. Given a p-form ϕ it follows that

$$\phi = \psi^1 + \cdots + \psi^l$$

where $\psi^j \in V_j$. Using Corollary 5.17 and the fact that each ψ^j is an eigenvector of $\alpha(H)$ of weight $n-p$, we see that

$$\psi^j = L^{r_j}\chi_j$$

where χ_j is a primitive $(p-2r_j)$-form and

$$\chi_j = c_j \Lambda^{r_j}\psi^j, \qquad c_j \in \mathbb{Q}. \tag{5.19}$$

Collecting primitive forms of the same degree we get a decomposition of ϕ of the form

$$\phi = \sum_{r \geq (n-p)^+} L^r \phi_r$$

where each ϕ_r is primitive of degree $(p-2r)$. Suppose there is another such decomposition of ϕ then if we subtract these two we get a decomposition

$$0 = \phi_0 + L\phi_1 + \cdots + L^m\phi_m \tag{5.20}$$

where each ϕ_j is primitive $j = 0, \ldots, m \geq 1$. From Theorem 5.15 we know that

$$\Lambda^k L^k \phi_k = c_k \phi_k, \qquad k = 1, \ldots, m \tag{5.21}$$

for $0 \neq c_k \in \mathbb{Q}$. Now applying Λ^m to Equation 5.20 and using Equation 5.21 we get the following

$$\begin{aligned} 0 &= \Lambda^m\phi_0 + \Lambda^{m-1}(\Lambda L)\phi_1 + \cdots + \Lambda\left(\Lambda^{m-1}L^{m-1}\right)\phi_{m-1} + \Lambda^m L^m \phi_m \\ &= \Lambda^m\phi_0 + \Lambda^{m-1}c_1\phi_1 + \cdots + \Lambda c_{m-1}\phi_{m-1} + c_m\phi_m \\ &= c_m\phi_m \end{aligned}$$

where we used that all ϕ_k are primitive. Since all $c_k \neq 0$ it follows that $\phi_m = 0$ contradicting the assumption that ϕ_m was primitive. Therefore we conclude that the decomposition given by Equation 5.17 is unique. To see Equation 5.18 let a p-form ϕ have the decomposition

$$\phi = \phi_0 + L\phi_1 + \cdots + L^m\phi_m$$

where the ϕ_j are primitive $(p-2j)$-forms. Applying Λ^m to this equation as before we get

$$\Lambda^m \phi = c_m \phi_m$$

from which it follows that

$$\phi_m = \frac{1}{c_m} \Lambda^m \phi.$$

Using induction from above, which means starting with m and then doing the case $m-1$ etc., we get formulas as in Equation 5.18 for each ϕ_j, $j = 0, \ldots, m$. The second and third part of the theorem are consequences of the uniqueness of the decomposition in Equation 5.17. Namely using the first part of the theorem we have

$$0 = L^m \phi = L^m \sum_{r \geq (p-n)^+} L^r \phi_r = \sum_{r \geq (p-n)^+} L^{m+r} \phi_r.$$

From Theorem 5.19 and the primitiveness of ϕ_r it follows that $L^q \phi_r = 0$ if $q \geq (n-(p-2r)+1)^+$ (recall that each ϕ_r is a $(p-2r)$-form). This implies, taking $q = r+m$, that $L^{r+m}\phi_r = 0$ if $r+m \geq (n-p+2r+1)^+$ or equivalently $r < (p-n+m)$. So we have

$$0 = \sum_{r \geq (p-n+m)^+} L^{m+r} \phi_r = \sum_{q \geq (p+2m-n)^+} L^q \phi_q.$$

This is a primitive decomposition of the zero form from which it folllows that $\phi_{q-m} = 0$ for $q \geq (p+2m-n)^+$ that is, $\phi_r = 0$ for $r \geq (p-n+m)^+$, as desired. The third part of the theorem follows as a special case of the second part. □

We will now prove some relations between the operators $*, \Lambda$ and L. Let E be an Hermitian vector space with fundamental form

$$\Omega = \sum_{\mu=1}^{n} x_\mu \wedge y_\mu$$

as before, where $\{x_\mu, y_\mu\}$ is again an orthonormal basis for E'. We define for any p-form $\eta \in \wedge F$ the operator $e(\eta)$ given by

$$e(\eta) := \eta \wedge \phi.$$

If η is a real 1-form we have

$$e^*(\eta) = *e(\eta)* . \tag{5.22}$$

To see this let $\phi \in \wedge^p F$ and $\xi \in \wedge^{p+1} F$ then

$$\begin{aligned}
(e(\eta)\phi, \xi) &= \int \eta \wedge \phi \wedge \bar{*}\xi \\
&= (-1)^p \int \phi \wedge \bar{*}\bar{*}^{-1}\eta \wedge \bar{*}\xi \\
&= (-1)^p \int \phi \wedge \bar{*}\left(\bar{*}w\eta \wedge \bar{*}\xi\right) \\
&= \int \phi \wedge \bar{*}\left(*\eta \wedge *\xi\right) \\
&= (\phi, *e(\eta)*\xi)
\end{aligned}$$

where we used that η is real to replace $\bar{*}$ by $*$ and the sign in front of the integral goes away because $\eta \wedge *\xi$ is a $2n-p$ form. From this computation it is also clear that if η is any 1-form, not necessary real then we have $e^*(\eta) = \bar{\star} e(\eta) \bar{\star}$. Given an oriented orthonormal basis $\{e_1, \dots, e_{2n}\}$ of E' we have

$$e^*(e_{j_1})(e_{j_1} \wedge \cdots \wedge e_{j_k}) = \begin{cases} e_{j_2} \wedge \cdots \wedge e_{j_k}, & \text{if } j_1 \in \{j_2, \dots, j_k\} \\ 0 & \text{otherwise} \end{cases} \tag{5.23}$$

Note that

$$L = e(\Omega) = \sum_{\mu=1}^{n} e(x_\mu) e(y_\mu),$$

$$\Lambda = e^*(\Omega) = \sum_{\mu=1}^{n} e^*(y_\mu) e^*(x_\mu).$$

Since Ω is a 2-form we have

$$[L, e(\eta)] = 0, \qquad \text{for any } \eta \in \wedge F$$

while

$$\begin{aligned} [\Lambda, e(x_\mu)] &= e^*(y_\mu) \\ [\Lambda, e(y_\mu)] &= -e^*(x_\mu) \end{aligned} \tag{5.24}$$

for $\mu = 1, \dots, n$. This can be seen in the following way. To start note that the second equation follows from the first by reversing the role of x_μ and y_μ in the definition of the L operator. To see the first equation we write

$$\begin{aligned} &= \sum_{\mu=1}^{n} e^*(y_\mu) e^*(e_\mu) e(x_j) - e(x_j) \sum_{\mu=1}^{n} e^*(y_\mu) e^*(x_\mu) \\ &= e^*(y_j) e^*(x_j) e(x_j) - e(x_j) - e(x_j) e^*(y_j) e^*(x_j) \end{aligned} \tag{5.25}$$

because, as we know from Equation 5.23, $e(x_j)$ commutes with $e^*(x_\mu)$ and $e^*(y_\mu)$ for $j \neq \mu$. Now let ψ be a given form then

$$\psi = \psi_1 + x_j \wedge \psi_2 + y_j \wedge \psi_3 + x_j \wedge y_j \wedge \psi_4$$

where the ψ_1, ψ_2, ψ_3 and ψ_4 do not contain any x_j or y_j. Then it is clear that

$$[\Lambda, e(x_j)]\psi = \psi_3 - x_j \wedge \psi_4$$

and

$$e^*(y_j)\psi = \psi_3 - x_j \wedge \psi_4$$

and we are done.
Now let η be a $(1,0)$-form. Then

$$\begin{aligned} [\Lambda, e(\eta)] &= -i e^*(\bar{\eta}) \\ [\Lambda, e(\bar{\eta})] &= i e^*(\eta) \end{aligned} \tag{5.26}$$

This follows from a simple computation. It suffices to consider the special case $\eta = x_j + i y_j$ and using Equation 5.24 we have

$$\begin{aligned} [\Lambda, e(x_j + i y_j)] &= [\Lambda, e(x_j)] + i[\Lambda, e(y_j)] \\ &= -i\left(e^*(x_j) + i e^*(y_j)\right) \\ &= -i\left(e^*(x_j - i y_j)\right) \end{aligned}$$

where in the last step we used $e^*(\eta) = \bar{*}e(\eta)\bar{*}$. The second equation follows in a similar way. Moreover, if η is a real 1-form, then

$$[\Lambda, e(\eta)] = -Je^*(\eta)J^{-1}. \tag{5.27}$$

To see this we use the fact that every real 1-form can be written in the form $\eta = \phi + \bar{\phi}$ where ϕ is a $(1,0)$-form. This follows easily since every 1-form is the sum of a $(1,0)$ and a $(0,1)$ form and the requirement that ϕ is real gives that the $(1,0)$ and $(0,1)$ are each others complex conjugate. Then we have using 5.26

$$[\Lambda, e(\eta)] = -ie^*(\bar{\phi}) + ie^*(\phi)$$

but

$$-i^*(\bar{\phi}) = -Je^*(\bar{\phi})J^{-1}$$
$$ie^*(\phi) = -Je^*(\phi)J^{-1}$$

so the relation follows.
We introduce the following operator on $\wedge F$ and proof a Lemma showing the relationship between $*$ and this operator. Define $\# = \pi_\alpha(w) = \exp(\frac{1}{2}(\Lambda + L))$. The following two Lemmas show the relationship between the $*$ and $\#$ operators (recall that $\mathrm{J} = \sum_{p,q} i^{p-q}\Pi_{p,q}$).

Lemma 5.21 *Let η be a real 1-form. Then*

$$\#e(\eta)\#^{-1} = -iJe^*(\eta)J^{-1}.$$

Proof. We define

$$e_t(\eta) = \exp(it(\Lambda + L)) \cdot e(\eta) \cdot \exp(-it(\Lambda + L))$$

and note that $e_{\frac{1}{2}}(\eta) = \#e(\eta)\#^{-1}$ and $e_0(\eta) = e(\eta)$. Now let A and B be operators then using $\exp(A) = \sum_n \frac{A^n}{n!}$ we get

$$\begin{aligned} e^A B e^{-A} &= (1 + A + \frac{A^2}{2} + \dots)B(1 + A + \frac{A^2}{2} + \dots) \\ &= B + AB - BA + \frac{1}{2!}\left(A^2B - 2ABA + BA^2\right) + \dots + \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}A^{k-i}BA^i + \dots \\ &= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \end{aligned}$$

So defining

$$\mathrm{ad}(X)Y = [X,Y]$$

for operators X and Y we obtain

$$e_t(\eta) = \sum_{k=0}^{\infty}\frac{1}{k!}\,\mathrm{ad}^k\left([it(\Lambda + L)]\right)e(\eta). \tag{5.28}$$

We know that $\Lambda L = L\Lambda + B$ and $\mathrm{ad}\,e(\eta) = 0$. Also since η is of degree 1 we have $\mathrm{ad}(-B)e(\eta) = e(\eta)$. Combining this with the fact that $\mathrm{ad}^k(\Lambda + L)$ is a sum of monomials in $\mathrm{ad}(\Lambda)$ and $\mathrm{ad}(L)$ we see that we get

$$e_t(\eta) = \sum_{t=0}^{\infty} a_k(t)\,\mathrm{ad}^k(\Lambda)e(\eta)$$

with $a_k(t)$ real analytic functions in t. Since Λ commutes with J and $e^*(\eta)$ we see, using Equation 5.27, that

$$e_t(\eta) = a_0(t)e(\eta) + a_1(t)\,\mathrm{ad}(\Lambda)e(\eta). \tag{5.29}$$

Differentiating Equation 5.28 with respect to t we obtain the following differential equation for $e_t(\eta)$

$$\begin{aligned} e_t'(\eta) &= i(\mathrm{ad}(\Lambda + L))e_t(\eta) \\ e_0(\eta) &= e(\eta) \end{aligned} \tag{5.30}$$

which we can solve using Equation 5.29. Namely

$$e_t'(\eta) = a_0'(t)e(\eta) + a_1'(t)\,\mathrm{ad}(\Lambda)e(\eta)$$

and according to the differential equation this must be equal to

$$i(\mathrm{ad}(\Lambda{+}L))\left[a_0(t)e(\eta) + a_1(t)\,\mathrm{ad}(\lambda)e(\eta)\right] = ia_0(t)\,\mathrm{ad}(\Lambda)e(\eta){+}ia_1(t)\,\mathrm{ad}(L)\,\mathrm{ad}(\Lambda)e(\eta)$$

where we used the fact that $\mathrm{ad}^2(\Lambda)e(\eta) = 0$ and$\mathrm{ad}(L)e(\eta) = 0$. However

$$\begin{aligned} \mathrm{ad}(L)\,\mathrm{ad}(\Lambda)e(\eta) &= \mathrm{ad}([L,\Lambda]) + \mathrm{ad}(\Lambda)\,\mathrm{ad}(L)e(\eta) \\ &= \mathrm{ad}(-B)e(\eta) = e(\eta) \end{aligned}$$

so we obtain

$$a_0'(t)e(\eta) + a_1'(t)\,\mathrm{ad}(\Lambda)e(\eta) = ia_0(t)\,\mathrm{ad}(\Lambda)e(\eta) + ia_1(t)e(\eta)$$

which implies

$$\begin{aligned} a_0'(t) &= ia_1(t), \\ a_1'(t) &= ia_0(t). \end{aligned}$$

So we let $a_0(t) = \cos(t), a_1(t) = i\sin(t)$ and we conclude that

$$e_t(\eta) = \cos(t)e(\eta) + i\sin(t)\,\mathrm{ad}(\Lambda)e(\eta) \tag{5.31}$$

is the unique solution to the differential equation. Taking $t = \frac{\pi}{2}$ gives

$$e_{\frac{\pi}{2}} = i[\Lambda, e(\eta)]$$

which proves the Lemma using Equation 5.27

□

Lemma 5.22 *Let $\phi \in \wedge^p F$. Then*

$$*\phi = i^{p2-n}J^{-1}\#\phi.$$

Proof. For the $*$-operator we have the following relations when $*$ is acting on p-forms

$$*\,1 = \mathrm{vol} = \frac{1}{n!}L^n(1), \tag{5.32}$$

$$*\,e(\eta) = (-1)^p e^*(\eta)*, \tag{5.33}$$

where η is a real 1-form. Relation 5.32 is clear. To see relation 5.33, let $\phi \in \wedge^p F$ be given then,

$$*e(\eta)\phi = *e(\eta) * w * \phi = (-1)^{2n-p}e^*(\eta) * \phi = (-1)^p e^*(\eta) * \phi.$$

Since any form in $\wedge F$ can be obtained by repeated application of $e(\eta)$ to 1 (for differen η) we get that $*$ is the only operator satisfying 5.32 and 5.33. We now define an operator $\tilde{*}$ on $\wedge^p$ by

$$\tilde{*} = i^{p^2-n} J^{-1} \#,$$

and show that it satisfies 5.32 and 5.33 and therefore equals $*$. Using Equation 5.16 we obtain

$$\#\alpha(Y)^k \phi_0 = i^m \frac{k!}{(m-k)} \alpha(Y)^{m-k} \phi_0 \tag{5.34}$$

where ϕ_0 is primitive of weight m. But $\phi_0 = 1$ is a primitive 0-form of weight n so we get using Equation 5.34 in the case $k = 0$

$$\#1 = \frac{i^n}{n!} L^n(1).$$

Therefore

$$\tilde{*}1 = i^{-n} \frac{i^n}{n!} L^n(1) = \mathrm{vol}\,.$$

Also, for $\eta \in \wedge^1_{\mathbb{R}} F$, and $\phi \in \wedge^p F$, we get

$$\begin{aligned}
\tilde{*} e(\eta)\phi &= i^{(p+1)^2-n} J^{-1} \# e(\eta)\phi \\
&= i^{p^2-n} (-1)^p i J^{-1} \# e(\eta) \#^{-1} \# \phi \\
&= i^{p^2-n} (-1)^p e^*(\eta) J^{-1} \# \phi \\
&= (-1)^p e^*(\eta) \tilde{*} \phi,
\end{aligned}$$

from which we conclude that 5.33 also holds for $\tilde{*}$. Thus $* = \tilde{*}$.

□

4. Differential Operators on a Kähler Manifold

In section 1 we saw that given a Hermitian complex manifold X with an Hermitian metric h we have an associated fundamental form Ω as in Equation 5.5.

Definition 5.23. A Hermitian metric h on X is called a *Kähler metric* if the fundamental form Ω associated with h is closed, that is $d\Omega = 0$.

Definition 5.24. A complex manifold X is said to be of *Kähler type* if it admits at least one Kähler metric. A complex manifold equipped with a Kähler metric is called a *Kähler manifold.*

In our study of Kähler manifolds we will make use of the following Laplacian operators,

$$\begin{aligned}
\Delta &= dd^* + d^*d, \\
\square &= \partial\partial^* + \partial^*\partial, \\
\overline{\square} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.
\end{aligned}$$

We will also make use of the following operators

$$d_c = J^{-1}dJ = wJdJ,$$
$$d_c^* = J^{-1}d^*J = wJd^*J.$$

These are real operators. Let ϕ be a (p,q)-form then we have

$$\begin{aligned} d_c\phi &= wJdJ\phi \\ &= (-1)^{p+q+1}i^{p-q}J(\partial\phi + \bar\partial\phi) \\ &= (-1)^{p+q+1}i^{p-q}(i^{p+1-q}\partial\phi + i^{p-q-1}\bar\partial\phi) \\ &= (-1)^{p+q+1}i^{p-q+p+1-q}(\partial\phi - \bar\partial\phi) \\ &= (-1)^{p+q+p-q} - i(\partial\phi - \bar\partial\phi) \\ &= -i(\partial - \bar\partial)\phi, \end{aligned}$$

from this it follows immediately that

$$dd_c = i\partial\bar\partial - i\bar\partial\partial = 2i\partial\bar\partial, \tag{5.35}$$

which is a real operator of type $(1,1)$ acting on differential forms in $\mathcal{E}^*(X)$.

Theorem 5.25 *Let X be a Kähler manifold. Then*

(1) $[L,d] = 0, [L^*,d^*] = 0.$
(2) $[L,d^*] = d_c, [L^*,d] = -d_c^*.$

Proof. For part (1) let ϕ be a p-form then we compute

$$(Ld - dL)\phi = \Omega\wedge d\phi - d(\Omega\wedge\phi) = \Omega\wedge d\phi - d\Omega\wedge\phi - \Omega\wedge d\phi,$$

and since $d\Omega = 0$ by the Kähler assumption, the sum is zero. The second part of (1) is just the adjoint statement of the first part. Similarly the second statement of part (2) is the adjoint of the first statement, so we will only show that

$$L^*d - dL^* = -J^{-1}d^*J,$$

from Proposition 5.6 we see that when acting on p-forms,

$$d^* = (-1)^{p+1} * d *^{-1}.$$

Let ϕ be a p-form then we get the following two identities from Lemma 5.22

$$\#\phi = i^{-p2+n}J*\phi,$$
$$\#^{-1}\phi = i^{p^2-n} *^{-1} J^{-1}\phi.$$

Therefore we see that

$$\begin{aligned} \#d\#^{-1}\phi &= i^{-(2n-p+1)^2+n}i^{p^2-n}J*d*^{-1}J^{-1}\phi \\ &= i^{2p-1}J*d*^{-1}J^{-1}\phi \\ &= iJ\left((-1)^{p+1}*d*^{-1}\right)J^{-1} \\ &= iJd^*J^{-1}\phi \end{aligned} \tag{5.36}$$

where the last step follows again from Proposition 5.6. Now we repeat the technique used in the proof of Lemma 5.21. Let

$$d_t = \exp\left[it(\Lambda + L)\right] \circ d \circ \exp\left[-it(\Lambda + L)\right],$$

then we have again

$$d_t = \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}^k \left[it(\Lambda + L)\right] d.$$

Also note that $d_{\frac{\pi}{2}} = \# d \#^{-1}$. As before $\operatorname{ad}^k(\Lambda + L)$ is the sum of monomials in $\operatorname{ad}(\Lambda)$ and $\operatorname{ad}(L)$ but from the Kähler condition (according to theorem 5.25) follows that $\operatorname{ad}(L)d = 0$ so we get

$$d_t = \sum_{k=0}^{\infty} a_k(t) \operatorname{ad}^k(\Lambda) d. \tag{5.37}$$

From the relation in Lemma 5.22 between $*$ and $\#$ it follows that $d_{\frac{\pi}{2}} = \# d \#^{-1}$ is an operator of degree -1. Since there is only one term in Equation 5.37 which has degree -1 we get that

$$d_{\frac{\pi}{2}} = a_1(\frac{\pi}{2}) \operatorname{ad}(\Lambda) d. \tag{5.38}$$

Now we can combine 5.36 and 5.38 to conclude that iJd^*J^{-1} is proportional to $\operatorname{ad}(\Lambda)d$, and hence $\operatorname{ad}^k \Lambda d = 0$ for $k \geq 2$ since Λ commutes with d^* and J. So just as in the proof of Lemma 5.21 we get

$$d_t = a_0(t) d + a_1(t) \operatorname{ad}(\Lambda) d,$$

from which we get

$$d_t = \cos(t) d + i \sin(t) \operatorname{ad}(\Lambda) d.$$

Letting $t = \frac{\pi}{2}$ and using that $Jd^*J^{-1} = -J^{-1}d^*J$ concludes the prove. □

Corollary 5.26 *Let X be a Kähler manifold. Then*

$$[L, d_c] = 0, \quad [L^*, d_c^*] = 0, \quad [L, d_c^*] = -d, \quad [L^*, d_c] = d^*$$

Proof. From Proposition 5.3 that J commutes with L and L^*. Also it can be seen directly from the definition that $(d_c)_c = -d$ and $(d_c^*)_c = -d^*$. Then the corollary follows directly from Theorem 5.25. □

Corollary 5.27 *Let X be a Kähler manifold. Then*

$$\begin{aligned} &[L, \partial] = [L, \bar{\partial}] = [L^*, \partial^*] = [L^*, \bar{\partial}^*] = 0, \\ &[L, \partial^*] = i\bar{\partial}, \quad [L, \bar{\partial}^*] = -i\partial, \\ &[L^*, \partial] = i\bar{\partial}^*, \quad [L^*, \bar{\partial}] = -i\partial^*, \end{aligned} \tag{5.39}$$

$$\begin{aligned} d^* d_c &= -d_c d^* = d^* L d^* = -d_c L^* d_c, \\ d d_c^* &= -d_c^* d = d_c^* L d_c^* = -d L^* d, \\ \partial \bar{\partial}^* &= -\bar{\partial}^* \partial = -i \bar{\partial}^* L \bar{\partial}^* = -i \partial L^* \partial, \\ \bar{\partial} \partial^* &= -\partial^* \bar{\partial} = i \partial^* L \partial^* = i \bar{\partial} L^* \bar{\partial}. \end{aligned} \tag{5.40}$$

PROOF. The Equations in 5.39 are a direct result of Theorem 5.25 by using $d_c = i(\bar\partial - \partial)$ and comparing bidegrees. The equations 5.40 can all be seen in a similar way, for example using $d^* = [L^*, d_c]$ we get

$$d^* d_c = L^* d_c d_c - d_c L^* d_c = -d_c L^* d_c,$$
$$-d_c d^* = -d_c L^* d_c + d_c d_c L^* = -d_c L^* d_c.$$

Using the other commutation relations the relations follow. □

With this result we can prove a very important theorem about the Laplacian operators we introduced earlier on Kähler manifolds.

Theorem 5.28 *Let X be a Kähler manifold. If the differential operators $d, d^*, \bar\partial, \bar\partial^*, \partial, \partial^*, \Box, \overline{\Box}$ and Δ are defined with respect to the Kähler metric on X. Then*

$$[\Delta, *] = [\Delta, d] = [\Delta, L] = 0$$

and

$$\Delta = 2\Box = 2\overline{\Box}.$$

In particular,

(1) $\Box$ *and* $\overline{\Box}$ *are real operators.*
(2) $\Delta|_{\mathcal{E}^{p,q}} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q}$.

PROOF. To see that $[\Delta, d] = 0$ note that

$$\Delta d = dd^* d + d^* dd = dd^* d = ddd^* + dd^* d = d\Delta.$$

Using Proposition 5.6 it follows that $[\Delta, *] = 0$. Now we prove that $[\Delta, L] = 0$. We have, using $[L, d] = 0$ and Theorem 5.25,

$$\begin{aligned}\Delta L - L\Delta &= dd^* L + d^* dL - Ldd^* = Ld^* d \\ &= dd^* L + d^* Ld - dLd^* - Ld^* d \\ &= -d[L, d^*] - [L, d^*]d \\ &= -dd_c - d_c d.\end{aligned}$$

But from Equation 5.35 it follows that $dd_c = -d_c d$ and so we conclude $[\Delta, L] = 0$. To see the relations between the Laplacian operators we use Corollary 5.27 to write

$$\begin{aligned}\Delta &= dd^* + d^* d = d[L^*, d_c] + [L^*, d_c]d \\ &= dL^* d_c - dd_c L^* + L^* d_c d - d_c L^* d.\end{aligned}$$

Now multiplying from the left with J and from the right with J^{-1} we obtain

$$\Delta_c = -d_c L^* + d_c dL^* - L^* dd_c + dL^* d_c.$$

But $\Delta_c = \Delta$ since, as we saw earlier, $dd_c = -d_c d$. Note that

$$d + id_c = \partial + \bar\partial - i^2(\partial - \bar\partial) = 2\partial$$

from which it follows that

$$\begin{aligned}4(\partial\partial^* + \partial^*\partial) &= (d + id_c)(d^* - id_c^*) + (d^* - id_c^*)(d + id_c) \\ &= (dd^* + d^* d) + (d_c d_c^* + d_c^* d_c) + i(d_c d^* + d^* d_c) - i(dd_c^* + d_c^* d).\end{aligned}$$

Now using Corollary 5.27 we get that the last two terms vanish. Also we have

$$\Delta = \Delta_c = J^{-1}\Delta J = J^{-1}dd^*J + J^{-1}d^*dJ = d_c d_c^* + d_c^* d_c$$

combining this we conclude

$$4\Box = \Delta + \Delta = 2\Delta$$

so $2\Box = \Delta$. The second relation follows analogously. Since Δ is a real operator so are $\Box$ and $\overline{\Box}$ and since $\Box$ is of bidegree $(0,0)$ so is Δ. This concludes the proof. $\square$

Corollary 5.29 *On a Kähler manifold, the operator Δ commutes with $J, L^*, d, \partial, \bar{\partial}, \partial^*$ and d^*.*

From Theorem 5.28 we see that on a Kähler manifold Δ-harmonic differential forms are the same as $\Box$-harmonic and $\overline{\Box}$-harmonic forms. We will call these *harmonic forms* on X and denote them by $\mathcal{H}^r(X)$ and $\mathcal{H}^{p,q}(X)$. The *primitive harmonic r-forms* shall be denoted by $\mathcal{H}_0^r(X) = \ker L^* : \mathcal{H}^r(X) \to \mathcal{H}^{r-2}(X)$ and the *primitive harmonic (p,q)-forms* shall be denoted by $\mathcal{H}_0^{p,q}(X) = \ker L^* : \mathcal{H}^{p,q}(X) \to \mathcal{H}^{p-1,q-1}(X)$. Note that these maps are well defined since Δ commutes with L^*.

Corollary 5.30 *On a compact Kähler manifold X there are direct sum decompositions:*

$$\mathcal{H}^r(X) = \sum_{s\geq(r-n)^+} L^s\mathcal{H}_0^{r-2s}(X),$$
$$\mathcal{H}^{p,q}(X) = \sum_{s\geq(p+q-n)^+} L^s\mathcal{H}_0^{p-s,q-s}.$$

Proof. Let $\phi \in \mathcal{H}^r(X)$ then we know from Theorem 5.20 that

$$\phi = \sum_{s\geq(r-n)^+} L^s\phi_s$$

where for each $s \geq (r-n)^+$, ϕ_s is a primitive $(r-2s)$-form. Since Δ commutes with L and L^* it follows from Equation 5.18 that

$$\Delta\phi_s = \sum_k a_{k,s}L^k L^* k + s\Delta\phi = 0$$

since $\phi \in \mathcal{H}^r(X)$ but this implies that $\phi_s \in \mathcal{H}^{r-2s}$ and it is in $\mathcal{H}_0^{r-2s}$ since ϕ is primitive. Since ϕ was arbitrary we conclude

$$\mathcal{H}^r(X) = \sum_{s\geq(r-n)^+} L^s\mathcal{H}_0^{r-2s}(X).$$

The second statement is proven similarly. $\square$

Corollary 5.31 *Let X be a compact Kähler manifold, then*

$$L^{n-p} = e(\Omega)^{n-p} : H^p(X,\mathbb{C}) \to H^{2n-p}(X,\mathbb{C})$$

is an iromorphism, where Ω is the Kähler form on X.

Proof. We represent the cohomology groups by harmonic forms. It follows from Corollary 5.30 that L^{n-p} is surjective and from Theorem 5.20 that L^{n-p} is injective. $\square$

5. The Hodge Decomposition Theorem on Compact Kähler Manifolds

We are now in the position to prove the central theorem of this thesis, the Hodge Decomposition Theorem. The proof will be surprisingly short but it is important to realize how much work we have done in order to be able to present such an elegant proof. Let X be a compact complex manifold. We then have the de Rham groups on X, represented by d-closed forms, and which we will denote here by $H^r(X,\mathbb{C})$. We also have the the Dolbeault groups on X, represented by $\bar{\partial}$-closed forms, which we will denote here by $H^{p,q}(X)$. In general it does not have to be the case that d-closed forms are $\bar{\partial}$ closed or vice versa, however on Kähler manifolds we have that this is indeed the case, as is stated in the following theorem.

Theorem 5.32 *Let X be a compact complex manifold of Kähler type. Then there is a direct sum decomposition*

$$H^r(X,\mathbb{C}) = \sum_{p+q=r} H^{p,q}(X) \tag{5.41}$$

and, moreover,

$$\bar{H}^{p,q}(X) = H^{q,p}(X). \tag{5.42}$$

Proof. Since we can represent cohomology by Harmonic forms it is enough to show

$$\mathcal{H}^r(X) = \sum_{p+q=r} \mathcal{H}^{p,q}(X).$$

To see this, let $\phi \in \mathcal{H}^r(X)$. This implies that $\Delta\phi = 0$ but since $\Delta = 2\bar{\square}$ we get $\bar{\square}\phi = 0$. Writing out ϕ in bihomogeneous terms we get

$$\bar{\square}\phi = \bar{\square}\phi^{r,0} + \cdots + \bar{\square}\phi^{0,r}.$$

Since $\bar{\square}$ preserves bidegree it follows that $\bar{\square}\phi^{r,0} = \cdots = \bar{\square}\phi^{0,r} = 0$. Thus we have a map

$$\tau : \mathcal{H}^r(X) \to \sum_{p+q=r} \mathcal{H}^{p,q}(X),$$

defined by

$$\phi \mapsto (\phi^{r,0}, \dots, \phi^{0,r}).$$

It is easy to see that this map is injective. To see it is surjective let $\phi \in \mathcal{H}^{p,q}(X)$ then $\bar{\square}\phi = \frac{1}{2}\Delta\phi = 0$ which implies that $\phi \in \mathcal{H}^{p+q}(X)$ and so τ is surjective. The second statement of the Theorem follows from the fact that $\bar{\square}$ is a real operator and complex conjugation is an isomorphism from $\mathcal{E}^{p,q}$ to $\mathcal{E}^{q,p}$. □

Now we have proved our most important theorem we conclude with some direct consequences. Recall the definitions of the Betti numbers $b_r(X) = \dim_{\mathbb{C}} H^r(X,\mathbb{C})$ and the Hodge numbers $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. We get the following Corollary from the Hodge Decomposition Theorem.

Corollary 5.33 *Let X be a compact Kähler manifold. Then*

(1) $b_r(X) = \sum_{p+q=r} h^{p,q}(X)$.
(2) $h^{p,q}(X) = h^{q,p}(X)$.
(3) $b_q(X)$ *is even when q is odd.*

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