Cohomology theory

bachelor thesis

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July 20, 2012

Abstract

In this thesis we study sheaves over a topological spaces and in particular over differentiable manifolds in order to proof that any sheaf cohomology theory is isomorphic and the existence of such a theory. A couple of classical cohomologies are discussed and explicitly shown to be isomorphic to the sheaf cohomology theory. Furthermore we will use this set up to proof the de Rham theorem and the Hodge theorem. Only some basic knowledge of differential geometry is assumed.

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1 Introduction

In this thesis we will mostly follow Frank Werner's Foundation of Differentiable Manifolds and Lie Groups. This book however uses a definition of sheaves, which is not commonly used anymore. Therefore the first two chapters will be about the definition of a sheaf and the correspondence between Frank Werner's definition and the most commonly used one. Section 4 and 5 will contain some theorems which will be needed later on and in Chapter 6 we will define what a sheaf cohomology theory is and proof the isomorphisms between different ones. There will also be a way to construct such theories, which depend on sheaf resolutions. That these resolutions exist will become clear in section 7, in which we will discuss classical cohomologies like Čech cohomology, singular cohomology and the de Rham cohomology for a differentiable manifold. Finally section 8 and 9 will be about the de Rham theorem and the Hodge theorem. We will conclude this section with the Poincaré duality, which gives another tool on computing cohomology.
The Hodge Theory that we discuss is only for the exterior derivative d on a manifold, and not for general differential operators.

2 Sheaves

Definition 1.1: presheaf
A *presheaf* $\mathcal{F}$ over a topological space X consists of:

I a set $\mathcal{F}(U)$ for each nonempty open set $U \subset X$;

II a *restriction homomorphism* $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for each opens $V \subset U \subset X$ satisfying:

- i $r^U_U = \mathrm{Id}_U$
- ii for $W \subset V \subset U$: $r^U_W = r^V_W \circ r^U_V$.

Definition 1.2: sheaf
A presheaf $\mathcal{F}$ is called a *sheaf* if for every collection U_i of open subsets of X with $U = \cup_i U_i$ the two axioms (S_1) and (S_2) are satisfied:

(S_1) If $s, t \in \mathcal{F}(U)$ and $\forall i : r^U_{U_i}(s) = r^U_{U_i}(t)$, then $s = t$;

(S_2) If $s_i \in \mathcal{F}(U_i) \, \forall i$ such that $U_i \cap U_j \neq \emptyset \Rightarrow r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$, then there exists an $s \in \mathcal{F}(U)$ such that for all i: $r^U_{U_i}(s) = s_i$.

Example 1.1: $\mathcal{C}_{X,Y}$
Let X and Y be topological spaces and for $U \subset X$ define

$$\mathcal{C}_{X,Y}(U) := \{f : U \to Y | f \text{ is continuous}\},$$

with the restriction homomorphism the natural restriction of functions, that is for $f \in \mathcal{C}_{X,Y}(U)$ and $V \subset U$ we get that $r^U_V(f) = f|_V$. Note that it is clearly a presheaf. It is a sheaf as well, for it satisfies (S_1) and (S_2).
(S_1): Let $U = \cup_i U_i$, $f, g \in \mathcal{C}_{X,Y}(U)$ and assume that $\forall i : r^U_{U_i}(f) = r^U_{U_i}(g)$, so $\forall i :\ f|_{U_i} = g|_{U_i}$. Let $x \in U$, then there exists i such that $x \in U_i$ and thus we get that $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$. We conclude that for each $x \in U$ $f(x) = g(x)$, so $f = g$.
(S_2): Let $U = \cup_i U_i$ and let $f_i \in \mathcal{C}_{X,Y}(U_i) \, \forall i$, such that $U_i \cap U_j \neq \emptyset \Rightarrow r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)$. This condition implies that if $x \in U_i \cap U_j$, then $f_i(x) = f_j(x)$. Now define $f : U \to Y$ to be the function such that if $x \in U_i$, then $f(x) = f_i(x)$. This function is clearly well defined. Now we still need that f is continuous, so let V be an open in Y. Then we get that $f^{-1}(V) = \cup_i f_i^{-1}(V)$. Since f_i is continuous for all i we get that $f_i^{-1}(V)$ is open in U for all i and thus is $f^{-1}(V)$ open. We conclude that f is continuous, so that $f \in \mathcal{C}_{X,Y}(U)$.

On a sheaf we can have some more structure, than only that of a set. The case we are interested in is if each $\mathcal{F}(U)$ is a $K-$module where K is a principal ideal domain. In this case the restriction homomorphisms in the definition of a presheaf need to be homomorphisms of $K-$modules. Furthermore we will assume that X is a differentiable manifold, even though this will not always be needed. It is needed however, when we discuss the de Rham cohomology and alike.

Example 1.2: $\mathcal{C}_X$
Letting $Y = \mathbb{R}$ in Example 1 we get the sheaf $\mathcal{C}_X$. This is a sheaf of $\mathbb{R}-$modules when we define $(rf)(x) := r \cdot f(x)$ for each $r \in \mathbb{R}$ and $f \in \mathcal{C}_X(U)$.

Example 1.3: $\mathcal{C}_X^\infty$
For any $U \subset X$ we can define $\mathcal{C}_X^\infty(U)$ to be the smooth functions $f : U \to \mathbb{R}$ with the natural restriction again. Then for each $U \subset X$ we have that $\mathcal{C}^\infty(U)$ is a $\mathbb{R}-$module, when we define $(rf)(x) := r \cdot f(x)$ for each $r \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(U)$. So $\mathcal{C}^\infty$ is a presheaf of $\mathbb{R}-$modules. With the same argument as in example 1.1 we see that it satisfies (S_1).

Example 1.4: the p-th de Rham sheaf Ω^p
Let X be a differentiable manifold. For each open $U \subset X$ we let $\Omega^p(U)$ to be equal to the set of differential $p-$forms on U. For this presheaf, the restriction homomorphisms are the natural restrictions for maps. This presheaf turns out to be a sheaf of $\mathbb{R}-$modules and is called *the p-th de Rham sheaf.*
It is clear that $\Omega^p(U)$ is a vectorspace over $\mathbb{R}$ for each $U \subset X$, by definition of differential forms. Now let $U = \cup U_i$. For (S_1) we suppose that $s, t \in \Omega^p(U)$ such that for each i: $r^U_{U_i}(s) = r^U_{U_i}(t)$. For each $x \in U$ there exists some U_i such that $x \in U_i$ and therefore $s(x) = (r^U_{U_i}(s))(x) = (r^U_{U_i}(t))(x) = t(x)$. This means that $s = t$, so Ω^p satisfies (S_1).
Furthermore if $s_i \in \Omega^p(U_i)$ for each i such that for each i, j and $x \in U_i \cap U_j$ $s_i(x) = s_j(x)$ holds, then we can define $s : U \to \bigwedge^*_k(X); x \mapsto s_i(x)$ when $x \in U_i$. Such an i always exist, since $\{U_i\}$ covers U. It is clear that s is still a smooth section, since each s_i is exactly that. So Ω^p satisfies (S_2) and is therefore a sheaf.

Definition 1.3: (pre)sheaf morphisms
Let $\mathcal{F}, \mathcal{G}$ be (pre)sheaves. Then a *(pre)sheaf morphism* $h : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $h_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{h_U} & \mathcal{G}(U) \\ \downarrow r^U_V & & \downarrow r^U_V \\ \mathcal{F}(V) & \xrightarrow{h_V} & \mathcal{G}(V) \end{array}$$

When $\mathcal{F}$ and $\mathcal{G}$ are (pre)sheaves of $K-$modules, we will call h is an *isomorphism* if h_U is an isomorphism of K-modules for each $U \subset X$.

Definition 1.4: sub(pre)sheaf
A (pre)sheaf $\mathcal{F}$ is called a *sub(pre)sheaf* of $\mathcal{G}$ if the inclusion map $i : \mathcal{F} \to \mathcal{G}$ is a (pre)sheaf morphism.

3 Étalé spaces

We will find that we can associate some kind of topological space to each (pre)sheaf. This space is an étalé space. Furthermore we will see that each étalé space has a canonical associated sheaf, the sheaf of sections, which will inherit the structure of $K-$modules. The main goal of this

section is to show that if the étalé space comes from a sheaf, then the sheaf of sections will be isomorphic to the original sheaf.
Finally we will look at some more similarities and connections between sheaves and their étalé spaces.

Definition 2.1: étalé space
An *étalé space* over a topological space X is a topological space Y together with a continuous surjective mapping $\pi : Y \to X$ such that π is a local homeomorphism. We will denote such a space by $Y \xrightarrow{\pi} X$. Further we will call $\pi^{-1}(x)$ the *stalk of the étalé space at* x. When each stalk is a $K-$module, then Y is called an *étalé space of* $K-$*modules.*

Example 2.1: the constant étalé space
The most simple étalé space is the constant one. Here $Y = X \times K$, with the discrete topology on K and the projection sends $(x, k) \mapsto x$. It is obvious that π is continuous, surjective and a local homeomorphism. Also since $\pi^{-1}(x) = K$ for all $x \in X$ we get that $X \times K$ is actually an étalé space of $K-$modules. In exactly the same way we define $Y_B = X \times B$ for a $K-$module B.

Example 2.2: sheaf of discontinuous sections
Suppose $Y \xrightarrow{\pi} X$ is an étalé space of $K-$modules. Then define the sheaf of discontinuous sections of Y to be $\mathcal{F}_Y(U) = \{s : U \to Y | s(x) \in \pi^{-1}(x) \forall x\}$. This is clearly a presheaf when we use the natural restriction maps of functions. It is a sheaf as well, as we will see that it satisfies (S_1) and (S_2).
(S_1): Suppose $s, t \in \mathcal{F}_Y(U)$, $U = \cup U_i$ such that for all i: $r^U_{U_i}(s) = r_(U_i)^U(t)$. Then we have that for each $x \in U$ there exists some i such that $x \in U_i$ and this immediately implies that $s(x) = r^U_{U_i}(s)(x) = r_(U_i)^U(t)(x) = t(x)$. So we can conclude that $s = t$.
(S_2): Let again $U = \cup U_i$ and suppose that there exist $s_i \in \mathcal{F}(U_i)$ for all i with $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$ whenever $U_i \cap U_j \neq \emptyset$. For $x \in U$ there exists again a i such that $x \in U_i$ and now define $s : U \to Y; x \mapsto s_i(x)$. We note that s is indeed an element of $\mathcal{F}(U)$ and that for all i we have that $r^U_{U_i}(s) = s_i$. So (S_2) is satisfied as well.

Definition 2.2: homomorphism of étalé spaces of $K-$modules
Let $\pi :\to X$ and $\pi' : Y' \to X$ be two étalé spaces of $K-$modules over X. Then a continuous function $\psi : Y \to Y'$ is called a *homomorphism* if $\pi = \pi' \circ \psi$ and $\psi : \pi^{-1}(x) \to \pi'^{-1}(x)$ is a homomorphism of $K-$modules.

Definition 2.3: sections of an étalé space
A *section of an étalé space* $Y \xrightarrow{\pi} X$ over an open set $U \subset X$ is a continuous map $s : U \to Y$ such that $\pi \circ s = \mathrm{Id}_U$. The space of all sections over U is denoted by $\Gamma(U, Y)$.

Lemma 2.1: The sections of an étalé space $Y \xrightarrow{\pi} X$ form a sheaf Γ_Y over X.
Proof: We note that for each $U \subset X$, $\Gamma_Y(U) := \Gamma(U, Y) \subset \mathcal{C}_{X,Y}(U)$ and it is clear that the following diagram commutes for each $V \subset U \subset X$. Here the restriction maps are both the natural restriction of continuous functions.

$$\begin{array}{ccc} \Gamma(U,Y) & \xrightarrow{i} & \mathcal{C}_{X,Y}(U) \\ \downarrow r^U_V & & \downarrow r^U_V \\ \Gamma(V,Y) & \xrightarrow{i} & \mathcal{C}_{X,Y}(V) \end{array}$$

□

Lemma 2.2: Given an étalé space of $K-$modules $Y \xrightarrow{\pi} X$, the induced sheaf is a sheaf of $K-$modules.
Proof: If $s_1, s_2 \in \Gamma_Y(U)$ are sections and $k_1, k_2 \in K$, then we can define that $(k_1 \cdot s_1)(x) := k_1 \cdot s_1(x)$ and $(s_1 + s_2)(x) := s_1(x) + s_2(x)$. Using that $\pi^{-1}(x)$ is a $K-$module we get that

$k_1 \cdot s_1, s_1 + s_2 \in \Gamma_Y(U)$, so the operations are well defined. They make $\Gamma_Y(U)$ into a $K-$module, since:

$$\begin{aligned}(k_1 \cdot (s_1 + s_2))(x) &= k_1 \cdot (s_1 + s_2)(x) = k_1 \cdot (s_1(x) + s_2(x)) = k_1 \cdot s_1(x) + k_1 \cdot s_2(x) \\ &= (k_1 \cdot s_1)(x) + (k_2 \cdot s_2)(x) = (k_1 \cdot s_1 + k_2 \cdot s_2)(x),\end{aligned}$$

$$\begin{aligned}((k_1 + k_2) \cdot (s_1))(x) &= (k_1 + k_2) \cdot s_1(x) = k_1 \cdot s_1(x) + k_2 \cdot s_1(x) = (k_1 \cdot s_1)(x) + (k_2 \cdot s_1)(x) \\ &= (k_1 \cdot s_1 + k_2 \cdot s_1)(x),\end{aligned}$$

$$((k_1 \cdot k_2)(s))(x) = (k_1 \cdot k_2) \cdot s(x) = k_1 \cdot (k_2 \cdot s(x)) = k_1 \cdot (k_2 \cdot s)(x) = (k_1 \cdot (k_2 \cdot s))(x),$$

$$(1_K \cdot s)(x) = 1_K \cdot s(x) = s(x).$$

When we have $V \subset U \subset X$ we get that for $x \in V$ it still holds that:

$$(k_1 \cdot s_1 + k_2 \cdot s_2)(x) = k_1 \cdot s_1(x) + k_2 \cdot s_2(x).$$

And since the restriction map is just the natural restriction to the subset V, we get that it is actually a homomorphism of $K-$modules. We conclude that Γ_Y is a sheaf of $K-$modules. □

The sheaf induced by the constant étalé space will be denoted by $\mathcal{K}$. It turns out that this sheaf will have use in setting up our sheaf cohomology, which is our goal. Similarly, the sheaf induced by the étalé space Y_B will be denoted by $\mathcal{B}$.

Definition 2.4: stalk of a sheaf
Let $\mathcal{F}$ be a (pre)sheaf over X. We can define the *stalk of* $\mathcal{F}$ *at* x, $\mathcal{F}_x$, as following: let $x \in U \cap V$ and $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$ we say that $f \sim g$ if and only if there exists a neighborhood W of x such that $W \subset U \cap V$ and $r^U_W(f) = r^V_W(g) \in \mathcal{F}(W)$. This defines an equivalence relation on $\bigsqcup_{x \in U} \mathcal{F}(U)$. Then we define:

$$\mathcal{F}_x := \bigsqcup_{x \in U} \mathcal{F}(U) \diagup \sim .$$

We will denote the natural projections into this quotient space by $r^U_x : \mathcal{F}(U) \to \mathcal{F}_x$;.

Lemma 2.3: If $\mathcal{F}$ is a (pre)sheaf of $K-$modules, then for each $x \in X$ $\mathcal{F}_x$ is a $K-$module and for each $x \in X$ and $U \subset X$ r^U_x is a homomorphisms of the modules.
Proof: Let $m, n \in \mathcal{F}_x$ and assume that $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are representatives of m and n respectively. Since $x \in U \cap V$ we can find an open neighborhood W of x such that $W \subset U \cap V$. Then we define:

$$m + n := r^W_x(r^U_W(f) + r^V_W(g)),$$

$$k \cdot m := r^U_x(k \cdot f).$$

(i) These operations are well defined. Let $f_1 \in \mathcal{F}(U_1)$, $f_2 \in \mathcal{F}(U_2)$, $g_1 \in \mathcal{F}(V_1)$ and $g_2 \in \mathcal{F}(V_2)$ such that $r^{U_1}_x(f_1) = r^{U_2}_x(f_2)$ and $r^{V_1}_x(g_1) = r^{V_2}_x(g_2)$. Furthermore let $W_1 \subset U_1 \cap V_1$ and $W_2 \subset U_2 \cap V_2$. For the addition to be well defined we need to show that:

$$r^{W_1}_x(r^{U_1}_{W_1}(f_1) + r^{V_1}_{W_1}(g_1)) = r^{W_2}_x(r^{U_2}_{W_2}(f_2)) + r^{V_2}_{W_2}(g_2)) \text{ that is,}$$

$$r^{U_1}_{W_1}(f_1) + r^{V_1}_{W_1}(g_1) \sim r^{U_2}_{W_2}(f_2) + r^{V_2}_{W_2}(g_2).$$

We already know that $f_1 \sim f_2$, so there exists an open neighborhood Y of x such that $Y \subset U_1 \cap U_2$ and $r^{U_1}_Y(f_1) = r^{U_2}_Y(f_2)$ and analogously there exists a $Z \subset V_1 \cap V_2$ such that $r^{V_1}_Z(g_1) = r^{V_2}_Z(g_2)$.

Now $A := Y \cap Z \cap W_1 \cap W_2$ is an open neighborhood of x and on this open we have that $r_A^{U_1}(f_1) = r_A^Y \circ r_Y^{U_1}(f_1) = r_A^Y \circ r_Y^{U_2}(f_2) = r_A^{U_2}(f_2)$ and similarly that $r_A^{V_1}(g_1) = r_A^{V_2}(g_2)$. Then we get that:

$$\begin{aligned} r_A^{W_1}\left(r_{W_1}^{U_1}(f_1) + r_{W_1}^{V_1}(g_1)\right) &= r_A^{W_1} \circ r_{W_1}^{U_1}(f_1) + r_A^{W_1} \circ r_{W_1}^{V_1}(g_1) = r_A^{U_1}(f_1) + r_A^{V_1}(g_1) \\ &= r_A^{U_2}(f_2) + r_A^{V_2}(g_2) = r_A^{W_2} \circ r_{W_2}^{U_2}(f_2) + r_A^{W_2} \circ r_{W_2}^{V_2}(g_2) \\ &= r_A^{W_2}\left(r_{W_2}^{U_2}(f_2) + r_{W_2}^{V_2}(g_2)\right), \end{aligned}$$

from which we conclude that $r_{W_1}^{U_1}(f_1) + r_{W_1}^{V_1}(g_1) \sim r_{W_2}^{U_2}(f_2) + r_{W_2}^{V_2}(g_2)$.
For the multiplication by scalar to be well defined we need to show that for any $k \in K$ $r_x^{U_1}(k \cdot f_1) = r_x^{U_2}(k \cdot f_2)$. This is again an easy computation:

$$r_Y^{U_1}(k \cdot f_1) = k \cdot r_Y^{U_1}(f_1) = k \cdot r_Y^{U_2}(f_2) = r_Y^{U_2}(k \cdot f_2),$$

from which we conclude that $k \cdot f_1 \sim k \cdot f_2$ and thus that $r_x^{U_1}(k \cdot f_1) = r_x^{U_2}(k \cdot f_2)$.
(ii) Under these operations $\mathcal{F}_x$ is a $K-$module, let $m, n, f = f_1, g = g_1, U = U_1, V = V_1$ and $W = W_1$ as before. Then by definition of the operations $r_W^U(f) + r_W^V(g) \in \mathcal{F}(W)$ is a representative of $m + n$, $k \cdot f$ one for $k \cdot m$ and $k \cdot g$ one for $k \cdot n$, so:

$$\begin{aligned} k \cdot (m + n) &= r_x^W(k \cdot (r_W^U(f) + r_W^V(g))) = r_x^W(k \cdot r_W^U(f) + k \cdot r_W^V(g)) \\ &= r_x^W(r_W^U(k \cdot f) + r_W^V(k \cdot g)) = k \cdot m + k \cdot n, \end{aligned}$$

$$(k_1 + k_2) \cdot m = r_x^U((k_1 + k_2) \cdot f) = r_x^U(k_1 \cdot f + k_2 \cdot f) = k_1 \cdot m + k_2 \cdot m,$$

$$(k_1 k_2) \cdot m = r_x^U((k k_1 k_2) \cdot f) = r_x^U(k_1 \cdot (k_2 \cdot f)) = k_2 \cdot (k_2 \cdot m),$$

$$1_K \cdot m = r_x^U(1_K \cdot f) = r_x^U(f) = m.$$

Where we use that $\mathcal{F}(Z)$ is a $K-$module for each $Z \subset X$, and that the restriction maps are homomorphisms of $K-$modules.
Finally we note that r_x^U is a homomorphism for each $U \subset X$. This is easily seen using the definitions of the operations: if $f \in \mathcal{F}(U)$ is the representative for m and $k \in K$, then $k \cdot f \in \mathcal{F}(U)$ is a representative for $k \cdot m$ and furthermore if $f, g \in \mathcal{F}(U)$, then we can omit the restriction maps in the formula for the addition of their classes. □
Remark: The stalk of $\mathcal{F}$ over x is equal to the direct limit:

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

And since $\mathcal{F}(U)$ is a $K-$module for each $U \subset X$ we get that $\mathcal{F}_x$ is a $K-$module as a property of direct limits.

Lemma 2.4: Given a (pre)sheaf $\mathcal{F}$ over a topological space X, $\overline{\mathcal{F}} := \cup_{x \in X} \mathcal{F}_x$ determines an étalé space of $K-$modules over X.
Proof: Let $\pi : \overline{\mathcal{F}} \to X$ be the surjective mapping that maps anything in $\mathcal{F}_x$ to x. Now it is enough to give a topology on $\overline{\mathcal{F}}$, such that π is a local homeomorphism. Therefore we define for $f \in \mathcal{F}(U)$, $U \subset X$: $\mathrm{O}_f := \{r_y^U(f) : y \in U\}$. This topology basis induces a topology, which is the one we will use. To see that it is a topology basis we first note that for any $m \in \overline{\mathcal{F}}$ we have that $m \in \mathcal{F}_x$ for some unique $x \in X$ and therefore we find an representative $f \in \mathcal{F}(U)$ for some open $U \subset X$, such that $x \in U$. It is clear that for this f we have that $m \in \mathrm{O}_f$. Secondly, if $m \in \mathrm{O}_f \cap \mathrm{O}_g$ for some $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, then we have that $r_x^U(f) = m = r_x^V(g)$ and thus

$f \sim g$. By definition we conclude that there exists $W \subset U \cap V$ such that $r^U_W(f) = r^V_W(g)$ and then we have that $m \in \mathrm{O}_{r^U_W(f)} = \mathrm{O}_{r^V_W(g)} \subset \mathrm{O}_f \cap \mathrm{O}_g$.
The last thing we have to show is that π is a local homeomorphism, so let $m \in \overline{\mathcal{F}}$. Like before we get x, U, f such that $m = r^U_x(f)$. This means that $m \in \mathrm{O}_f$ which is open by definition of the topology. Its image under π is equal to U. $\pi|_{\mathrm{O}_f} : \mathrm{O}_f \to U$ is clearly bijective, since $\forall y \in U$ we get that $r^U_y(f) \in \mathrm{O}_f$ and furthermore we have that if $\pi(r^U_{y_1}(f)) = \pi(r^U_{y_2}(f))$, then immediately $y_1 = y_2$. Now let $V \subset U$ be open. Then

$$\pi^{-1}(V) = \{r^U_y(f) : y \in V\} = \{r^V_y(r^U_V(f)) : y \in V\} = \mathrm{O}_{r^U_V(f)} = \text{open},$$

which is to day: $\pi|_{\mathrm{O}_f}$ is continuous. The final part is to show that $\pi|_{\mathrm{O}_f}$ is an open map as well. This uses the same argument as for $\pi|_{\mathrm{O}_f}$ being continuous, since we only have to check that $\pi(\mathrm{O}_g)$ is open, where $g = r^U_V(f)$ for some open $V \subset U$. This is consequence of the fact that our topology is induced by a topology basis. □

Remark: In Warner the definition of a sheaf is this étalé space.

With the projection as defined in lemma 2.4 we see that the stalk of $\mathcal{F}$ over x is the same as the stalk of the étalé space $\overline{\mathcal{F}}$ over x, hence the naming.
We conclude from the lemma's before that when given a sheaf or a presheaf $\mathcal{F}$ of $K-$modules over X, we can construct an étalé space over X, which is associated to $\mathcal{F}$. We also have already seen that the sections of a étalé space form a sheaf. When we denote this sheaf by $\Gamma_{\mathcal{F}}$, so $\Gamma_{\mathcal{F}} := \Gamma_{\overline{\mathcal{F}}}$, then we get the following lemma and theorem:

Lemma 2.5: $\Gamma_{\mathcal{F}}$ is a sheaf of $K-$modules.
Proof: Combine lemma 2.4 with lemma 2.2. □

Theorem 1: If $\mathcal{F}$ is a sheaf, then $\Gamma_{\mathcal{F}}$ is isomorphic to $\mathcal{F}$.
Proof: Let $h : \mathcal{F} \to \Gamma_{\mathcal{F}}$ be the collection of morphisms $h_U : \mathcal{F}(U) \to \Gamma_{\mathcal{F}}(U) = \Gamma(U, \tilde{\mathcal{F}}); f \to s_f$, such that $s_f(x) := r^U_x(f)$. Note that $\pi(s_f(x)) = \pi(r^U_x(f)) = x$. Let $g \in \mathcal{F}(V)$, then we get that

$$s_f^{-1}(\mathrm{O}_g) = \{x \in U | r^U_x(f) \in \{r^V_y(g) | y \in V\}\} = \begin{cases} \emptyset & \text{if } f \nsim g \\ U \cap V & \text{if } f \sim g \end{cases},$$

so s_f is continuous as well. We conclude that s_f is indeed a section and h_U is therefore a well-defined map. When we use lemma 2.2 and lemma 2.4 we get that:

$$\begin{aligned}(h_U(k_1 \cdot f + k_2 \cdot g))(x) &= r^U_x(k_1 \cdot f + k_2 \cdot g) = k_1 \cdot r^U_x(f) + k_2 \cdot r^U_x(g) \\ &= k_1 \cdot (h_U(f))(x) + k_2 \cdot (h_U(g))(x) = (k_1 \cdot h_U(f) + k_2 \cdot h_U(g))(x),\end{aligned}$$

and so h_U is a homomorphism for each $U \subset X$.
If $V \subset U \subset X$ and $x \in V$, then it is clear that $r^V_x \circ r^U_V = r^U_x$, so we conclude that h is a sheaf-morphism. We only need to show that h_U is bijective for each $U \subset X$, and then we have the wanted isomorphism.
(i) For injectivity it is enough to show that the kernel of h_U is trivial. So suppose $0 = (h_U(f)) \in \Gamma_{\mathcal{F}}(U)$, then for all $x \in U$:

$$r^U_x(0) = 0 = (h_U(f))(x) = r^U_x(f),$$

which implies that $\forall x \in U$ there exists $W_x \subset U$ such that $x \in W_x$ and $r^U_{W_x}(0) = r^U_{W_x}(f)$. Furthermore it is clear that $U = \cup_{x \in U} W_x$. Using property (S_1) if the sheaf $\mathcal{F}$ we get that $0 = f$
(ii) Finally we will proof surjectivity. Let $s \in \Gamma(U, \overline{\mathcal{F}})$ be any section. Then for each $x \in U$ we have that there exists some $V_x \subset$ open and a $g_x \in \mathcal{F}(V_x)$ such that $s(x) = r^{V_x}_x(g_x)$. Since s is continuous and $\mathrm{O}_{g_x} = \{r^{V_x}_y(g_x) : y \in V_x\}$ is open we have that $U_x := s^{-1}(\mathrm{O}_{g_x})$ is open in U. Note that it is clear that $x \in U_x$ and therefore we have that $U_x \neq \emptyset$. We have for any $y \in U_x$ that

$s(y) \in \mathrm{O}_{g_x}$, so $s(y) = r_y^{V_x}(f_x) = r_y^{U_x}(r_{U_x}^{V_x}(g_x))$. Denote $f_x := r_{U_x}^{V_x}(g_x)$, then we have that for any $x \in U$ we can find an open U_x and $f_x \in \mathcal{F}(U_x)$ such that for every $y \in U_x$ the following holds:

$$r_y^{U_x}(f_x) = s(y)$$

Now for $x, y \in U$ we will look at $z \in U_x \cap U_y$. For such a z we have that $r_z^{U_x}(f_x) = s(z) = r_z^{U_y}(f_y)$, which means that there exists a $W_z \in U_x \cap U_y$, with $z \in W_z$, such that $r_{W_z}^{U_x}(f_x) = r_{W_z}^{U_y}(f_y)$. From this we conclude that

$$r_{W_z}^{U_x \cap U_y}(r_{U_x \cap U_y}^{U_x}(f_x)) = r_{W_z}^{U_x \cap U_y}(r_{U_x \cap U_y}^{U_y}(f_y)).$$

Since $z \in W_z$ for all $z \in U_x \cap U_y$ we get that $U_x \cap U_y = \cup_{z \in U_x \cap U_y} W_z$ and now property (S_1) implies that

$$r_{U_x \cap U_y}^{U_x}(f_x) = r_{U_x \cap U_y}^{U_y}(f_y).$$

From this, the fact that $x \in U_x$ for all $x \in U$ implies that $\{U_x\}$ covers U and property (S_2) of the sheaf $\mathcal{F}$ we conclude that there exists a $f \in \mathcal{F}(U)$ such that $f_x = r_x^{U_x}(f)$ for all $x \in U$. It is clear that the image of f under h_U and thus is h_U surjective. □

Example 2.3: the quotient sheaf
Suppose $\mathcal{G}$ is a presheaf of the sheaf $\mathcal{F}$ of $K-$modules over X. Then we define the presheaf $(\mathcal{F}/\mathcal{G})(U) := \mathcal{F}(U)/\mathcal{G}(U)$. This does not, in general, give a sheaf. The quotient sheaf is therefore defined as $\Gamma_{\overline{\mathcal{F}/\mathcal{G}}}$, with $(\mathcal{F}/\mathcal{G})_x = \mathcal{F}_x/\mathcal{G}_x$.
Such a quotient sheaf clearly induces a short exact sequence:

$$0 \longrightarrow \mathcal{G} \xrightarrow{\ i\ } \mathcal{F} \xrightarrow{\ \pi\ } \mathcal{F}/\mathcal{G} \longrightarrow 0$$

Not only does the étalé space of a sheaf generate the same sheaf, morphism between étalé spaces are also naturally mapped into morphism between the sheaves and vice versa, as is proven in the two lemma's below. From this we conclude that the étalé spaces of sheaves have the same information in them as the sheaves do.

Lemma 2.6: If $\mathcal{F}$ and $\mathcal{G}$ are (pre)sheaves over X, and $h : \mathcal{F} \to \mathcal{G}$ a (pre)sheaf morphism, then there exists an induced homomorphism of étalé spaces $\psi : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$.
Proof: Let $f \in \mathcal{F}(U)$. Then we define $\psi(r_x^U(f)) := r_x^U(h_U(f))$. Note that by definition we now have that $\pi_f = \pi_g \circ \psi$. ψ is well defined, for let $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ such that $r_x^U(f) = r_x^V(g)$. Then we have $W \subset U \cap V$ such that $r_W^U(f) = r_W^V(g)$. Using the commuting diagrams of homomorphisms h_U we get that:

$$r_W^U(h_U(f)) = h_W(r_W^U(f)) = h_W(r_W^V(g)) = r_W^V(h_V(g)),$$

so $r_x^U(h_U(f)) = r_x^V(h_V(g))$, which means that $\psi(r_x^U(f)) = \psi(r_x^V(g))$ and ψ is well-defined.
Secondly we need that ψ is continuous, but since π_f and π_g are both local homeomorphisms we get that they are both continuous and open maps. Therefore, for any open $U \subset \overline{\mathcal{G}}$ we get that $\psi^{-1}(U) = \pi_f^{-1}(\pi_g(U))$ which is open, so ψ is continuous.
Finally we need to show that ψ is a homomorphism on the stalks. Let again $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ such that f is a representative of m and g of n and let $k_1, k_2 \in K$. Like before we have that $k_1 \cdot m + k_2 \cdot n = r_x^W(k_1 \cdot r_W^U(f) + k_2 \cdot r_W^V(g))$. So:

$$\begin{aligned}
\psi(k_1 \cdot m + k_2 \cdot n) &= r_x^W\left(h_W(k_1 \cdot r_W^U(f) + k_2 \cdot r_W^V(g))\right) \\
&= r_x^W\left(k_1 \cdot h_W(r_W^U(f)) + k_2 \cdot h_W(r_W^V(g))\right) \\
&= r_x^W\left(k_1 \cdot r_W^U(h_U(f)) + k_2 \cdot r_W^V(h_V(g))\right) \\
&= k_1 \cdot r_x^W(r_W^U(h_U(f))) + k_2 \cdot r_x^W(r_W^V(h_V(g))) = k_1 \cdot \psi(m) + k_2 \cdot \psi(n),
\end{aligned}$$

where we needed that the h_U and r_x^U are homomorphisms of $K-$modules for each $U \subset X$. We conclude that ψ is indeed a homomorphism of the étalé spaces. $\square$

Lemma 2.7: Let $Y \xrightarrow{\pi} X$ and $Y' \xrightarrow{\pi'} X$ be two étalé spaces of $K-$modules over X and $\psi : y \to Y'$ a homomorphism between them, then there exists an induced sheaf morphism $h : \Gamma_Y \to \Gamma_{Y'}$.

Proof: Let $U \subset X$ and define $h_U : \Gamma(U, Y) \to \Gamma(U, Y'); s \mapsto \psi \circ s$. Because ψ is continuous and it is a map on the stalks, we get that $\psi \circ s$ is indeed a section, so h_U is well-defined. For h_U to be a sheaf morphism we need that it is homomorphism of the $K-$modules and that the diagram

$$\begin{array}{ccc} \Gamma(U,Y) & \xrightarrow{h_U} & \Gamma(U,Y') \\ \downarrow r_V^U & & \downarrow r_V^U \\ \Gamma(V,Y) & \xrightarrow{h_V} & \Gamma(V,Y') \end{array}$$

commutes, where $V \subset U \subset X$. The first part is an easy calculation, since for each $x \in U$, $k_1, k_2 \in K$ and $s_1, s_2 \in \Gamma(U, Y)$ we have that:

$$\begin{aligned}(h_U(k_1 \cdot s_2 + k_2 \cdot s_2))(x) &= \psi((k_1 \cdot s_2 + k_2 \cdot s_2)(x)) = \psi(k_1 \cdot s_1(x) + k_2 \cdot s_2(x)) \\ &= k_1 \cdot \psi(s_1(x)) + k_2 \cdot \psi(s_2(x)) = k_1 \cdot (h_U(s_1))(x) + k_2 \cdot (h_U(s_2))(x) \\ &= (k_1 \cdot h_U(s_1) + k_2 \cdot h_U(s_2))(x),\end{aligned}$$

where we used that ψ is a homomorphism of $K-$modules. The second part is also easily to be seen to be true, since for any $s \in \Gamma(U, Y)$ and $x \in V$ we have that $(r_V^U(h_U(s)))(x) = (h_U(s))(x) = \psi(s(x))$, while $(h_V(r_V^U(s)))(x) = \psi((r_V^U(s))(x)) = \psi(s(x))$. We conclude that ψ is indeed a sheaf homomorphism. $\square$

Remark: Note that if $h : \mathcal{F} \to \mathcal{F}'$, $h' : \mathcal{F}' \to \mathcal{F}''$ and $h'' : \mathcal{F} \to \mathcal{F}''$ are presheaf homomorphisms, such that $h' \circ h = h''$, that then the induced sheaf homomorphism commute as well. This is true since for an element $r_x^U(f) \in \mathcal{F}_x$ we have that:

$$\psi' \circ \psi(r_x^U(f)) = \psi'(r_x^U(h(f))) = r_x^U(h' \circ h(f)) = r_x^U(h''(f)) = \psi''(r_x^U(f)),$$

and for an element $s \in \Gamma_{\mathcal{F}}(U)$, with some abuse of notation:

$$(h'_U \circ h_U(s))(x) = (h'_U(\psi \circ s))(x) = \psi' \circ \psi \circ s(x) = \psi'' \circ s(x) = h''_U(s).$$

When $\mathcal{F}$ and $\mathcal{F}'$ are sheaves, $h : \mathcal{F} \to \mathcal{F}'$ a sheaf homomorphism and $U \subset X$ we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{h_U} & \mathcal{F}'(U) \\ \downarrow \simeq & & \downarrow \simeq \\ \Gamma_{\mathcal{F}}(U) & \xrightarrow{h_U} & \Gamma_{\mathcal{F}'}(U) \end{array}$$

Since for $f \in \mathcal{F}(U)$ we get that:

$$(h_U(s_f))(x) = \psi \circ s_f(x) = \psi(r_x^U(f)) = r_x^U(h_U(f)) = s_{h_U(f)}(x).$$

Not only do we have this correspondence, some useful definitions are defined with help of this correspondence between sheaves and étalé spaces. For h a homomorphism between sheaves we denoted the induced homomorphism of the étalé spaces by ψ. From now on we will denote ψ

restricted to the stalk of x as h_x. Then we get the next definition.

Definition 2.5: exact sequence of sheaves
If $\mathcal{F}, \mathcal{F}'$ and $\mathcal{F}''$ are sheaves of $K-$modules and $h : \mathcal{F}' \to \mathcal{F}$, $h' : \mathcal{F} \to \mathcal{F}''$ are sheaf morphisms, then this sequence is *exact at* $\mathcal{F}$ if the induced sequence on the stalks:

$$\mathcal{F}'_x \xrightarrow{h_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{F}''_x$$

is exact at $\mathcal{F}_x$ for all $x \in X$. We will call a sequence

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

a *short exact sequence* if it is everywhere exact.

Definition 2.6: morphism of exact sequences of sheaves
A *homomorphism of the short exact sequences of sheaves* $0 \to \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{h'} \mathcal{F}'' \to 0$ and $0 \to \tilde{\mathcal{F}}' \xrightarrow{h} \tilde{\mathcal{F}} \xrightarrow{h'} \tilde{\mathcal{F}}'' \to 0$ consists of sheaf homomorphisms $g : \mathcal{F} \to \tilde{\mathcal{F}}$, $g' : \mathcal{F}' \to \tilde{\mathcal{F}}'$ and $g'' : \mathcal{F}'' \to \tilde{\mathcal{F}}''$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{h} & \mathcal{F} & \xrightarrow{h'} & \mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' & & \\ 0 & \longrightarrow & \tilde{\mathcal{F}}' & \xrightarrow{\tilde{h}} & \tilde{\mathcal{F}} & \xrightarrow{\tilde{h}'} & \tilde{\mathcal{F}}'' & \longrightarrow & 0 \end{array}$$

4 Tensor products and sheaf properties

In this section we will discuss some properties of sheaves and their behavior with respect to tensor products. **Definition 3.1: tensor products of (pre)sheaves**
Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves of $K-$modules over X, then their *tensor product* $\mathcal{F} \otimes \mathcal{G}$ is the presheaf which consists of $K-$modules $(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$ with restriction homomorphism $(r_V^U)_{\mathcal{F}} \otimes (r_V^U)_{\mathcal{G}}$. When $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then their tensor product is the sheaf $\Gamma_{\mathcal{F} \otimes \mathcal{G}}$ associated to this presheaf. We will still call this sheaf $\mathcal{F} \otimes \mathcal{G}$.
Furthermore if $h : \mathcal{F} \to \mathcal{G}$ and $h' : \mathcal{F}' \to \mathcal{G}'$ are two sheaf morphisms, then $h \otimes h' : \mathcal{F} \otimes \mathcal{G} \to \mathcal{F}' \otimes \mathcal{G}'$ will be the sheaf morphism associated to the presheaf morphism, which consists of the module morphisms $h_U \otimes h'_U$.

Definition 3.2: partition of unity for a sheaf
Let $\mathcal{F}$ be a sheaf over X and $\{U_i\}$ an open covering of X. Then a *partition of unity for* $\mathcal{F}$ *subordinated to* $\{U_i\}$ is a family of endomorphism $\{l^i\}$ of $\mathcal{F}$ such that the following holds:

I $\sum_i l^i_U(f) = f$ for all $U \subset X$, open, and $f \in \mathcal{F}(U)$

II $\overline{\{x \in X | l^i_x \not\equiv 0\}} = \overline{X \setminus \{x \in X | l^i_x \equiv 0\}} \subset U_i$

Definition 3.3: fine sheaf
A sheaf $\mathcal{F}$ is called *fine* if for every locally finite cover $\{U_i\}$ of X there exists a partition of unity for $\mathcal{F}$ subordinated to $\{U_i\}$.

Example 3.1: the sheaf of discontinuous sections
The sheaf $\mathcal{F}$ defined in example 2.2 is a fine sheaf. To see this, let $\{U_i\}$ be a locally finite cover of X. Using the shrinking lemma 5.17 of [1]. We can pick an open refinement $\{V_i\}$ such that

$\overline{V_i} \subset U_i$ for all i. Now for $x \in X$ there exists some i such that $x \in V_i$. This i does not have to be unique, but we can choose one. Now we can define ψ_i to be the function which is 1 on x if we have picked i for this x and otherwise we define ψ_i to be zero. Note that $\{x \in X | \psi_i(x) \neq 0\} \subset V_i$ and therefore we get that $\overline{\{x \in X | \psi_i(x) \neq 0\}} \subset U_i$. Furthermore it is clear that $\sum_i \psi_i(x) = 1$ for all x.
Define $l_U^i : \mathcal{F}(U) \to \mathcal{F}(U)$ to be the map such that $(l_U^i(s))(x) = \psi_i(x)s(x)$. This is an endormorphism, since:

$$\begin{aligned}(l_U^i(k_1 \cdot s_1 + k_2 \cdot s_2))(x) &= \psi_i(x)(k_1 \cdot s_1 + k_2 \cdot s_2)(x) = k_1 \cdot \psi_i(x)s_1(x) + k_2 \cdot \psi_i(x)s_2(x) \\ &= k_1 \cdot (l_U^i(s_1))(x) + k_2 \cdot (l_U^i(s_2))(x) = (k_1 \cdot (l_U^i(s_1)) + k_2 \cdot (l_U^i(s_2)))(x).\end{aligned}$$

Furthermore we see that:

$$\left(\sum_i l_U^i(s)\right)(x) = \sum_i \psi_i(x)s(x) = s(x),$$

where we need the locally finiteness of our covers and that $\psi_i(x) = 1$ only once. Finally suppose that $x \notin \overline{V_i}$. Then there exists some open $\tilde{V}$ such that $x \in \tilde{V} \subset X \setminus V_i$. For each $y \in \tilde{V}$ we have that $y \notin V_i$ and therefore we get that $(l_U^i(s))(y) = 0$, where $s \in \mathcal{F}(U)$. For $U \subset X$ such that $x \in U$ we define $W = U \cap \tilde{V}$. Then clearly for all $y \in W$ we have that $(l_U^i(s))(y) = 0$, so $r_W^U(l_U^i(s)) = 0$ and therefore we have that $r_x^U(l_U^i(s)) = 0$. Since l_x^i sends $r_x^U(s)$ into $r_x^U(l_U^i(s))$ we conclude that $l_x^i \equiv 0$. So $\overline{X \setminus \{x \in X | l_x^i \equiv 0\}} \subset \overline{V_i} \subset U_i$. We conclude that l_U^i form a partition of unity for $\mathcal{F}$ subordinated to the locally finite cover $\{U_i\}$. And therefore we have that $\mathcal{F}$ is a fine sheaf.

Remark: To be able to use the shrinking lemma we need that X is a paracompact Hausdorff space.

When $\mathcal{F}$ and $\mathcal{G}$ are sheaves of which $\mathcal{F}$ is fine, then $\mathcal{F} \otimes \mathcal{G}$ is fine as well. This is easily seen, since each partition of unity for $\mathcal{F}$ can be tensored with the identity on $\mathcal{G}$ to form a partition of unity for $\mathcal{F} \otimes \mathcal{G}$. Fine sheaves have some nice properties as the following lemma's will show.

Lemma 3.1: If

$$0 \longrightarrow \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{h'} \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of sheaves and $Z \subset X$ open, then the following sequence is exact:

$$0 \longrightarrow \mathcal{F}'(Z) \xrightarrow{h_Z} \mathcal{F}(Z) \xrightarrow{h'_Z} \mathcal{F}''(Z)$$

Proof: Since the first sequence is exact, we have that for every $x \in Z$ that

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{h_x} \mathcal{F}_x \xrightarrow{h'_x} \mathcal{F}''_x \longrightarrow 0$$

is exact. We will need this repeatedly. The first part of the proof is to show that there is exactness at $\mathcal{F}'(Z)$, which means that $\ker(h_Z) = \{0\}$. So let $f' \in \mathcal{F}'(Z)$ such that $h_Z(f') = 0$. Then we get that for every x:

$$h_x\big(r_x^Z(f')\big) = r_x^Z\big(h_Z(f')\big) = r_x^Z(0).$$

Using the exactness we get that $r_x^Z(f') = 0 = r_x^Z(0)$, which implies that there exists $U_x \subset Z$ such that $x \in U_x$ and $r_{U_x}^Z(f') = r_{U_x}^x(0)$. Using the fact that $Z = \cup_{x \in Z} U_x$ and property $(S1)$ we get

that $f' = 0$, so the kernel is trivial and h_Z is injective for each Z.
The second part is to show exactness at $\mathcal{F}(Z)$, which consists of two parts.
(i) First we will see that $\mathrm{Im}\,(h_Z) \subset \ker(h'_Z)$. Therefore let $f \in \mathcal{F}(Z)$ be such that $f = h_Z(f')$ for some $f' \in \mathcal{F}'(Z)$. We now get that for all $x \in Z$:

$$r_x^Z\big(h'_Z(f)\big) = h'_x\big(r_x^Z(f)\big) = h'_x\big(r_x^Z(h_Z(f'))\big) = h'_x\big(h_x(r_x^Z(f'))\big) = 0 = r_x^Z(0),$$

which implies that for all x there exists some open $U_x \subset Z$ such that $x \in U_x$ and $r_{U_x}^Z(h'_Z(f)) = r_{U_x}^Z(0)$. Again with property $(S1)$ we get that $h'_Z(f) = 0$, which is what we wanted.
(ii) Finally we need that $\ker(h'_Z) \subset \mathrm{Im}\,(h_Z)$. Therefore let $f \in \mathcal{F}(Z)$ such that $h'_Z(f) = 0$. Then we get that for each $x \in Z$ $r_x^Z(f) \in \ker(h'_x)$, because:

$$h'_x\big(r_x^Z(f)\big) = r_x^Z\big(h'_Z(f)\big) = r_x^Z(0) = 0.$$

Using exactness at $\mathcal{F}_x$ we get a $U_x \subset Z$ and $k_x \in \mathcal{F}'(U_x)$ such that $x \in U_x$ and $h_x(r_x^{U_x}(k_x)) = r_x^Z(f)$. And since $h_x(r_x^{U_x}(k_x)) = r_x^{U_x}(h_{U_x}(k_x))$ we have some $W_x \subset U_x$ such that $x \in W_x$ and $r_{W_x}^Z(f) = r_{W_x}^{U_x}(h_{U_x}(k_x)) = h_{W_x}(r_{W_x}^{U_x}(k_x))$, where the last equality is just from commutativity of the homomorphisms. Define $f'_x := r_{W_x}^{U_x}(k_x)$. Now for each $Q := W_x \cap W_y \neq \emptyset$ we get that:

$$h_Q\big(r_Q^{W_x}(f'_x)\big) = r_Q^{W_x}\big(r_{W_x}^Z(f)\big) = r_Q^Z(f) = r_Q^{W_y}\big(r_{W_y}^Z(f)\big) = h_Q\big(r_Q^{W_y}(f'_y)\big).$$

Now, since h_Q is injective for $Q = W_x \cap W_y$, we get that $r_Q^{W_x}(f'_x) = r_Q^{W_y}(f'_y)$. With (S_2) we conclude that there exists a $f' \in \mathcal{F}'(Z)$ such that $r_{W_x}^Z(f') = f'_x$. Furthermore we see that for all $x \in Z$:

$$r_{W_x}^Z\big(h_Z(f')\big) = h_{W_x}\big(r_{W_x}^Z(f')\big) = h_{W_x}(f'_x) = r_{W_x}^Z(f),$$

and using (S_1) this implies that $h_Z(f') = f \in \mathrm{Im}\,(h_Z)$. □

Remark: Note that we have not used the surjectivity of h'_x to conclude this.

Lemma 3.2: If, in the above notation, $\mathcal{F}'$ is a fine sheaf, then h'_X is surjective as well, so we have the short exact sequence:

$$0 \longrightarrow \mathcal{F}'(X) \xrightarrow{h_X} \mathcal{F}(X) \xrightarrow{h'_X} \mathcal{F}''(X) \longrightarrow 0$$

Proof: Let ψ' be as in lemma 2.6, corresponding to h', and let $f \in \mathcal{F}''$ and $s_f \in \Gamma(X, \tilde{\mathcal{F}}'')$ be the (unique) corresponding section. To proof this lemma we will first construct a locally finite cover $\{U_i\}$ of X and a section $t_i : U_i \to \mathcal{F}$ such that $\psi'(t_i) = s_f$ on U_i. Then with help of the partition of unity we can constuct a global section.
For each $x \in X$ we have that $y'' := s_f(x) \in \overline{\mathcal{F}}''$ and since π'' is a local homeomorphism we find an open V'' around y'' such that $\pi'' : V'' \to \pi''(V'')$ is a homeomorphism. Since s_f is continuous we get that $U'' := s_f^{-1}(V'')$ is open in X and note that $U'' \subset \pi''(V'')$ and $x \in U''$. Furthermore we have that $W'' := s_f(U'') \subset V''$ and thus π'' is a homeomorphism on W'' as well. We conclude here that W'' is open and that on U'' we have that s_f is the inverse of π''.
Since h'_x is surjective we can find a $y \in \overline{\mathcal{F}}$ such that $\psi'(y) = y''$. π is a local homeomorphism as well and just like before we find an open V around y such that $\pi : V \to \pi(V)$ is a homeomorphism. Now define $W = V \cap \psi'^{-1}(W'')$. Since ψ' is continuous we get that W is an open neighborhood of y. Furthermore we have that $\pi : W \to \pi(W)$ is a homeomorphism. Finally define $U = U'' \cap \pi(W)$, which is again a non-empty open since is contains x.
Now for $z \in U$ we find a unique element $w \in W$ such that $\pi(w) = z$ and a unique element $w'' \in W''$ such that $s_f(z) = w''$. Here uniqueness comes from the homeomorphisms π and π'' on W and W'' respectively. Since $w \in W$ we get that $h'_z(w) \in W''$, so we can conclude that $h'_z(w) = w''$. Now define the section $t : U \to \tilde{\mathcal{F}}$ which sends z to the corresponding w. Since all

the maps are continuous we can conclude that is is a section and on U it clearly satisfies $\psi' \circ t = s_f$.

Since $x \in U$ we now have an open cover of X and for each open a section. Assuming X is paracompact, we can go over to a locally finite subcover $\{U_i, t_i\}$. When $U_{i,j} := U_i \cap U_j \neq \emptyset$ we can define the section $t_{i,j} := t_i - t_j$. Clearly, for each $x \in U_{i,j}$ we have that $t_{i,j}(x) \in \ker(h'_x)$. Using the given exactness we know that there exists some $y' \in \tilde{\mathcal{F}}'$ such that $h_x(y') = t_{i,j}(x)$. This now gives us a section $t'_{i,j} : U_{i,j} \to \overline{\mathcal{F}}'$. Note that on $U_i \cap U_j \cap U_k$

$$t_{i,j} + t_{j,k} = t_i - t_j + t + j - t_k = t_i - t_k = t_{i,k},$$

which gives that

$$\psi(t'_{i,j} + t'_{j,k} - t'_{i,k}) = t_{i,j} + t_{j,k} - t_{i,k} = 0.$$

Using injectivity of h_x for each x, we get that $t'_{i,j} + t'_{j,k} = t'_{i,k}$. Since $\mathcal{F}'$ is fine and our cover locally finite, we can pick a partition of unity $\{l^i\}$ for $\mathcal{F}'$ subordinated to $\{U_i, t_i\}$. Using this we can extend $t'_{i,j}$ to a section $l^j \circ t'_{i,j}$ on U_i by declaring that $l^j \circ t'_{i,j}(x) = l^j_x(t'_{i,j}(x))$ on $U_i \cap U_j$ and 0 on $U_i \setminus U_j$. This is still continuous, since the support of l^j is within U_j. Finally we can define $s'_i := \sum_k l^k \circ t'_{i,k}$, which is still a section of $\overline{\mathcal{F}}'$ over U_i. We get that on $U_{i,j}$:

$$s'_i - s'_j = \sum_k l^k \circ t'_{i,k} - \sum_k l^k \circ t'_{j,k} = \sum_k l^k \circ t_{i,j} = t'_{i,j}.$$

Using this we and if we set $\psi_h(s'_i) = s_i$, then we get that $s_i - s_j = t_{i,j} = t_i - t_j$. We conclude that $s : X \to \overline{\mathcal{F}}$ which sends $x \in U_i$ to $t_i(x) - s_i(x)$ is a well defined section on whole X. Furthermore we have that $\psi' \circ s = s_f$ by construction. Using the correspondence between $\Gamma_{\mathcal{F}}(X)$ and $\mathcal{F}(X)$ we find that h'_X is surjective. □

Definition 3.4: a torsionless sheaf
A $K-$module A will be called *torsionless* if for each $a \in A$ we have either $a = 0$ or there exists no non-zero element $k \in K$ such that $ka = 0$. Now, let $\mathcal{F}$ be a sheaf of $K-$modules. We will say that $\mathcal{F}$ is *torsionless* if for all $x \in X$ we have that $\mathcal{F}_x$ is torsionless.

Lemma 3.3: Let A, A', B, B' be $K-$modules and $\alpha : A \to A'$ and $\beta : B \to B'$ be surjective homomorphisms. Then $\alpha \otimes \beta : A \otimes B \to A' \otimes B'$ is surjective as well.
Proof: From definition we know that $A' \otimes B'$ is generated by elements $a' \otimes b'$ where $a' \in A'$ and $b' \in B'$. Using surjectivity we can find an $a \in A$ and $b \in B$ for each a' and b' such that $\alpha(a) = a'$ and $\beta b = b'$. We now get that $a' \otimes b' = \alpha(a) \otimes \beta(b) = (\alpha \otimes \beta)(a \otimes b)$, which means that $A' \otimes B'$ is generated by $(\alpha \otimes \beta)(a \otimes b)$ and thus is $\alpha \otimes \beta$ surjective. □

Lemma 3.4: Let $\alpha : A \to A'$ and $\beta : B \to B'$ be as in lemma 3.3. Then $\ker(\alpha \otimes \beta)$ is generated by elements $a \otimes b$ such that $a \in \ker(\alpha)$ or $b \in \ker(\beta)$.
Proof: Let D be the set generated by $\{a \otimes b : a \in \ker(\alpha) \text{ or } b \in \ker(\beta)\}$. Since for all the generating elements $a \otimes b$ of D it holds that $(\alpha \otimes \beta)(a \otimes b) = 0$ we get that $D \subset \ker(\alpha \otimes \beta)$. Furthermore it is obvious that D forms a submodule of $A \otimes B$ and therefore we can look at $C := A \otimes B \big/ D$. Let $p : A \otimes B \to C$ be the projection. Now define f to be the multi-linear map $f : A' \times B' \to C; (a', b') \mapsto p(a \otimes b)$ where $a \in A$ and $b \in B$ are such that $\alpha(a) = a'$ and $\beta(b) = b'$. Note that these a and b always exist, since α, β are surjective. We have to check whether this map is well defined, so let $a_1, a_2 \in \alpha^{-1}(a')$ and $b_1, b_2 \in \beta^{-1}(b')$. Then f is well defined when $p(a_1 \otimes b_1) = p(a_2 \otimes b_2)$, that is when $a_1 \otimes b_1 - a_2 \otimes b_2 \in D$. A calculation shows us however that:

$$a_1 \otimes b_1 - a_2 \otimes b_2 = (a_1 \otimes b_1 - a_1 \otimes b_2) + (a_1 \otimes b_2 - a_2 \otimes b_2) = a_1 \otimes (b_1 - b_2) + (a_1 - a_2) \otimes b_2.$$

And since $\alpha(a_1 - a_2) = a' - a' = 0$ and $\beta(b_1 - b_2) = b' - b' = 0$ we get that both terms are elements of the generating set of D and therefore $a_1 \otimes b_1 - a_2 \otimes b_2 \in D$ holds. By the universal property of $A' \otimes B'$ f induces a homomorphism $\tilde{f} : A' \otimes B' \to C$, such that $\tilde{f}(a' \otimes b') = p(a \otimes b)$. From this we conclude that $p = \tilde{f} \circ (\alpha \otimes \beta)$ and since $\tilde{f}$ is a homomorphism we know that it sends elements of $\ker(\alpha \otimes \beta)$ into $0 \in C$. So these elements lie in D and thus $\ker(\alpha \otimes \beta) \subset D$ holds as well: $\ker(\alpha \otimes \beta) = D$. □

We will use the two lemma's above to look at exactness of tensors of $K-$modules. In the next lemma we will only proof the first part. The second one needs quite some other technical lemma's, which are not really in line with the rest of the text.

Lemma 3.5: Let B be an $K-$module and let

$$0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$$

be an exact sequence of $K-$modules. Then

$$A' \otimes B \xrightarrow{\alpha \otimes \mathrm{Id}} A \otimes B \xrightarrow{\beta \otimes \mathrm{Id}} A'' \otimes B \longrightarrow 0$$

is exact. If moreover either A'' or B is torsionless, then $\alpha \otimes \mathrm{Id} : A' \otimes B \to A \otimes B$ is injective as well.

Proof: From our known exact sequence we get that β is surjective. Clearly $\mathrm{Id} : B \to B$ is surjective as well, so lemma 3.3 gives us that $\beta \otimes \mathrm{Id}$ is surjective. This proves the exactness at $A'' \otimes B$. For exactness at $A \otimes B$ we use lemma 3.4. Since β and Id are surjective we get that

$$\ker(\beta \otimes \mathrm{Id}) = \langle a \otimes b | \beta(a) = 0 \text{ or } \mathrm{Id}(b) = 0 \rangle = \langle a \otimes b | \beta(a) = 0 \rangle.$$

Using that $\ker(\beta) = \mathrm{Im}\,(\alpha)$, we get that $\ker(\beta \otimes \mathrm{Id}) = \langle \alpha(a') \otimes b | a' \in A' \rangle = \mathrm{Im}\,(\alpha \otimes \mathrm{Id})$, which proves the exactness. □

Theorem 2: Let

$$0 \longrightarrow \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of sheaves of $K-$modules over X. Let $\mathcal{G}$ be another sheaf. Then if $\mathcal{G}$ or $\mathcal{F}''$ is torsionless, then

$$0 \longrightarrow \mathcal{F}' \otimes \mathcal{G} \xrightarrow{h \otimes \mathrm{Id}} \mathcal{F} \otimes \mathcal{G} \xrightarrow{g \otimes \mathrm{Id}} \mathcal{F}'' \otimes \mathcal{G} \longrightarrow 0$$

is exact. If $\mathcal{G}$ or $\mathcal{F}'$ is fine as well, then the induced sequence

$$0 \longrightarrow (\mathcal{F}' \otimes \mathcal{G})(X) \xrightarrow{(h \otimes \mathrm{Id})_X} (\mathcal{F} \otimes \mathcal{G})(X) \xrightarrow{(g \otimes \mathrm{Id})_X} (\mathcal{F}'' \otimes \mathcal{G})(X) \longrightarrow 0$$

is exact.

Proof: We will need that for each x $(\mathcal{F} \otimes \mathcal{G})_x \simeq \mathcal{F}_x \otimes \mathcal{G}_x$. Therefore define the map $\psi : (\mathcal{F} \otimes \mathcal{G})_x \to \mathcal{F}_x \otimes \mathcal{G}_x; r_x^U(f \otimes g) \mapsto r_x^U(f) \otimes r_x^U(g)$. It is enough to check that this map is well defined, so let $f' \otimes g' \in \mathcal{F}(V) \otimes \mathcal{G}(V)$ such that $r_x^U(f \otimes g) = r_x^V(f' \otimes g')$. This means that there exists some $W \subset U \cap V$ such that $r_W^U(f \otimes g) = r_W^V(f' \otimes g')$. Now by definition we have that $r_W^U(f \otimes g) = r_W^U(f) \otimes r_W^U(g)$, which means that we get that $r_W^U(f) = r_W^V(f')$ and $r_W^U(g) = r_W^V(g')$. So clearly we now have that ψ is well defined.

Now using this and lemma 3.5 we get the first exact sequence. The second sequence is a result of lemma 3.2 and the fact that $\mathcal{F}' \otimes \mathcal{G}$ is fine when either $\mathcal{F}'$ or $\mathcal{G}$ is, as is discussed before. □

We will discuss one more lemma about torsionless sheaves, which will be handy in section

7.

Lemma 3.6: Let $\mathcal{F}$ be a sheaf of $K-$modules over X such that each $\mathcal{F}(U)$ is torsionless. Then $\mathcal{F}$ is torsionless as well.
Proof: Let $x \in X$. We will show that $\mathcal{F}_x$ is torsionless, so let $r_x^U(f)$ be an element of $\mathcal{F}_x$ and $k \in K \setminus \{0\}$ such that $k \cdot r_x^U(f) = 0$. From this we know that $r_x^U(k \cdot f) = 0$ and thus there exists an open subset W such that $x \in W \subset U$ and $0 = r_W^U(k \cdot f) = k \cdot r_W^U(f)$. $\mathcal{F}(W)$ is torsionless and therefore we are allowed to conclude that $r_W^U(f) = 0$, so that $r_x^U(f) = r_x^W(r_W^U(f)) = r_x^W(0) = 0$. □

5 Cochain complexes

Classical cohomology groups are defined as quotient spaces of cochain complexes. This indicates that cochain complexes will be important for us as well. Therefore we will discuss the notion of such complexes and some results, which we will use later on again.
Definition 4.1: cochain complex
A *cochain complex* C^* consists of a sequence of $K-$modules $\{\mathrm{C}^q\}_{q\in\mathbb{Z}}$ and homomorphisms $d^q : \mathrm{C}^q \to \mathrm{C}^{q+1}$ such that $B^q(\mathrm{C}^*) := \mathrm{Im}\,(d^{q-1}) \subset Z^q(\mathrm{C}^*) := \ker(d^q)$ for all q. We will call d^q the *qth coboundary operator*, elements of Z^q will be called *qth degree cocycles* and elements of B^q will be called *qth degree coboundaries*. Finally we will define the quotient of the module Z^q by B^q as:

$$H^q(\mathrm{C}^*) := Z^q(\mathrm{C}^*) \big/ B^q(\mathrm{C}^*),$$

the *qth cohomology module*.

Definition 4.2: cochain maps
Let C^* and D^* be two cochain complexes. Then a *cochain map* $\alpha : \mathrm{C}^* \to \mathrm{D}^*$ is a collection of $K-$module homomorphisms $\alpha^q : \mathrm{C}^q \to \mathrm{D}^q$ such that the following diagram commutes for all q.

$$\begin{array}{ccc} \mathrm{C}^{q+1} & \xrightarrow{\alpha^{q+1}} & \mathrm{D}^{q+1} \\ \uparrow d^q & & \uparrow d^q \\ \mathrm{C}^q & \xrightarrow{\alpha^q} & \mathrm{D}^q \end{array} \qquad (1)$$

Lemma 4.1: If α is a cohain map between cochain complexes C^* and D^*, then $\alpha^q(B^q(\mathrm{C}^*)) \subset B^q(\mathrm{D}^*)$ and $\alpha^q(Z^q(\mathrm{C}^*)) \subset Z^q(\mathrm{C}^*)$ for all q.
Proof: Suppose that $\sigma_q \in B^q(\mathrm{C}^*)$, then we have $\sigma_{q-1} \in \mathrm{C}^{q-1}$ such that $d^{q-1}(\sigma_{q-1}) = \sigma_q$. Using diagram (1) we find that:

$$\alpha^q(\sigma_q) = \alpha^q(d^{q-1}(\sigma_{q-1})) = d^{q-1}(\alpha^{q-1}(\sigma_{q-1})) \in B^q(\mathrm{D}^*).$$

Next assume that $\sigma_q \in Z^q(\mathrm{C}^*)$, so $d^q(\sigma_q) = 0$. Again using (1) we get that $d^q(\alpha^q(\sigma_q)) = \alpha^{q+1}(d^q(\sigma_q)) = \alpha^{q+1}(0) = 0$, where we use that α^{q+1} is a homomorphism of $K-$modules. And thus we have that $\alpha^q(\sigma_q) \in Z^q(\mathrm{D}^*)$.

Remark: From the above lemma we can conclude that a cochain map α induces a homomorphisms $\overline{\alpha}^q$ on the cohomology modules. For we can define $\overline{\alpha}^q : H^q(\mathrm{C}^*) \to H^q(\mathrm{D}^*); [\sigma] \mapsto [\alpha^q(\sigma)]$, where the $[\sigma]$ with $\sigma \in Z^q(\mathrm{C}^*)$ denotes the class of σ in the quotient. This map is clearly a homomorphism. so we only need to check whether this map is well defined. Let $[\sigma_1] = [\sigma_2]$, then there exists $\zeta \in B^q(\mathrm{C}^*)$ such that $\sigma_2 = \sigma_1 + \zeta$. Now we get that

$$\overline{\alpha}^q([\sigma_2]) = [\alpha^q(\sigma_2)] = [\alpha^q(\sigma_1 + \zeta)] = [\alpha^q(\sigma_1) + \alpha^q(\zeta)] = [\alpha^q(\sigma_1) + 0] = [\alpha^q(\sigma_1)] = \overline{\alpha}^q([\sigma_1]).$$

This induced map of cochain complexes will from now on always be denoted with as $\overline{\alpha}$.

Lemma 4.2: When $\alpha : \mathrm{C}^* \to \mathrm{D}^*$ and $\beta : \mathrm{D}^* \to \mathrm{E}^*$ are two cochain maps, then we get that $\overline{\beta \circ \alpha} = \overline{\beta} \circ \overline{\alpha}$.
Proof: This is an easy calculation. Let $[\sigma] \in H^q(\mathrm{C}^*)$. Then we have that

$$\overline{\beta}^q \circ \overline{\alpha}^q([\sigma]) = \overline{\beta}^q([\alpha^q(\sigma)]) = [\beta^q \circ \alpha^q(\sigma)].$$

On the other hand we have that

$$\overline{\beta \circ \alpha}^q([\sigma]) = [(\beta \circ \alpha)^q(\sigma)] = [\beta^q \circ \alpha^q(\sigma)].$$

Since this is true for each q we conclude that $\overline{\beta \circ \alpha} = \overline{\beta} \circ \overline{\alpha}$ □

Definition 4.3: short exact sequence of cochains
If C^*, D^* and E^* are cochain complexes and $\alpha : \mathrm{C}^* \to \mathrm{D}^*$ and $\beta : \mathrm{D}^* \to \mathrm{E}^*$ are cochain maps, then this sequence forms a short exact sequence if for each q:

$$0 \longrightarrow \mathrm{C}^q \xrightarrow{\alpha^q} \mathrm{D}^q \xrightarrow{\beta^q} \mathrm{E}^q \longrightarrow 0$$

is a short exact sequence of $K-$modules.

Definition 4.4: homomorphism between short exact sequences
If $0 \to \mathrm{C}^* \overset{\alpha}{\to} \mathrm{D}^* \overset{\beta}{\to} \mathrm{E}^* \to 0$ and $0 \to \tilde{\mathrm{C}}^* \overset{\tilde{\alpha}}{\to} \tilde{\mathrm{D}}^* \overset{\tilde{\beta}}{\to} \tilde{\mathrm{E}}^* \to 0$ are short exact sequences of cochain complexes, then a *homomorphism of the exact sequences of cochain complexes* consists of cochain maps $\gamma_1 : \mathrm{C}^* \to \tilde{\mathrm{C}}^*$, $\gamma_2 : \mathrm{D}^* \to \tilde{\mathrm{D}}^*$ and $\gamma_3 : \mathrm{E}^* \to \tilde{\mathrm{E}}^*$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}^* & \xrightarrow{\alpha} & \mathrm{D}^* & \xrightarrow{\beta} & \mathrm{E}^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \\ 0 & \longrightarrow & \tilde{\mathrm{C}}^* & \xrightarrow{\tilde{\alpha}} & \tilde{\mathrm{D}}^* & \xrightarrow{\tilde{\beta}} & \tilde{\mathrm{E}}^* & \longrightarrow & 0 \end{array}$$

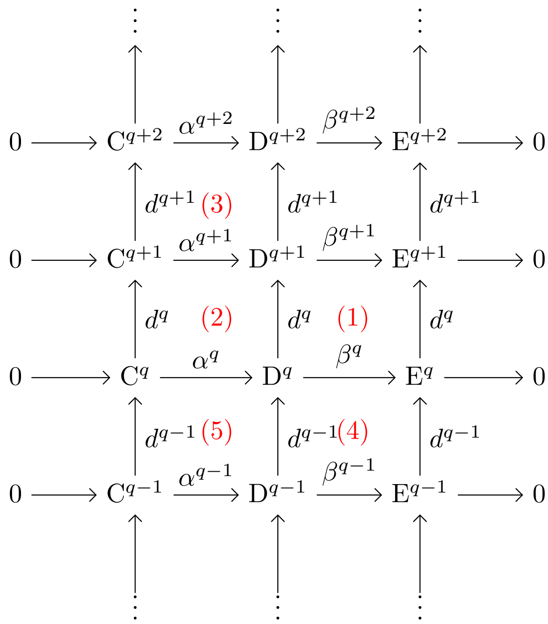
Theorem 3: Given the short exact sequence $0 \to \mathrm{C}^* \to \mathrm{D}^* \to \mathrm{E}^* \to 0$ with cochain maps α and β, then there exists homomorphisms $\partial^q : H^q(\mathrm{E}^*) \to H^{q+1}(\mathrm{C}^*)$ such that

$$... \longrightarrow H^{q-1}(\mathrm{E}^*) \xrightarrow{\partial^{q-1}} H^q(\mathrm{C}^*) \xrightarrow{\overline{\alpha}^q} H^q(\mathrm{D}^*) \xrightarrow{\overline{\beta}^q} H^q(\mathrm{E}^*) \xrightarrow{\partial^q} H^{q+1}(\mathrm{C}^*) \longrightarrow ... \tag{2}$$

is exact. Moreover if there exists a homomorphism γ to another short exact sequence $0 \to \tilde{\mathrm{C}}^* \to \tilde{\mathrm{D}}^* \to \tilde{\mathrm{E}}^* \to 0$, then the following diagram commutes:

$$\begin{array}{ccc} H^q(\mathrm{E}^*) & \xrightarrow{\partial^q} & H^{q+1}(\mathrm{D}^*) \\ \downarrow{\scriptstyle \overline{\gamma}_3^q} & & \downarrow{\scriptstyle \overline{\gamma}_1^{q+1}} \\ H^q(\tilde{\mathrm{E}}^*) & \xrightarrow{\tilde{\partial}^q} & H^{q+1}(\tilde{\mathrm{C}}^*) \end{array} \tag{3}$$

Proof: Consider the commutative diagram:



First we will determine the map ∂. Let $\sigma_3 \in Z^q(\mathrm{E}^*)$, so $d^q(\sigma_3) = 0$. Since each row is a short exact sequence, we know that β^q is surjective and therefore we find a $\sigma_2 \in \mathrm{D}^q$ such that $\beta^q(\sigma_2) = \sigma_3$. Using the commutative square (1) we have that

$$\beta^{q+1}(d^q(\sigma_2)) = d^q(\beta^q\sigma_2) = d^q(\sigma_3) = 0,$$

so $d^q(\sigma_2) \in \ker(\beta^{q+1}) = \mathrm{Im}\,(\alpha^{q+1})$. This means that there exists a $\sigma_1 \in \mathrm{C}^{q+1}$ such that $\alpha^{q+1}(\sigma_1) = d^q(\sigma_2)$ and since α^{q+1} is injective we have that σ_1 is unique for each σ_2. Furthermore we have that $\sigma_1 \in Z^{q+1}(\mathrm{C}^*)$, because we can use the commutative square (3) which gives us that $\alpha^{q+2}(d^{q+1}(\sigma_1)) = d^{q+1}(d^q(\sigma_2)) = 0$. So $d^{q+1}(\sigma_1) \in \ker(\alpha^{q+2}) = \{0\}$, which means that $d^{q+1}(\sigma_1) = 0$ and hence we have that $\sigma_1 \in Z^{q+1}(\mathrm{C}^*)$.
We want to say that $\partial^q([\sigma_3]) := [\sigma_1]$, but since β^q is only surjective we might find a $\sigma_2' \neq \sigma_2$ such that $\beta^q(\sigma_2') = \beta^q(\sigma_2) = \sigma_3$ and then we would have $\sigma_1 \neq \sigma_1' \in \mathrm{C}^{q+1}$ such that $\alpha^{q+1}(\sigma_1) = d^q(\sigma_2)$ and $\alpha^{q+1}(\sigma_1') = d^q(\sigma_2')$ and thus we might have two different images for $\partial([\sigma_3])$. $\sigma_2' \neq \sigma_2 \in (\beta^q)^{-1}(\sigma_3)$ gives us however that $\beta^q(\sigma_2' - \sigma_2) = 0$, so $\sigma_2' - \sigma_2 \in \ker(\beta^q) = \mathrm{Im}\,(\alpha^q)$. Thus we have some $\sigma_4 \in \mathrm{C}^q$ such that $\alpha^q(\sigma_4) = \sigma_2' - \sigma_2$. Using square (2) we find that

$$\alpha^{q+1}(d^q(\sigma_4)) = d^q(\alpha^q(\sigma_4)) = d^q(\sigma_2' - \sigma_2) = \alpha^{q+1}(\sigma_1' - \sigma_1).$$

Now we can use the injectivity of α^{q+1} again and we get that $\sigma_1' - \sigma_1 = d^q(\sigma_4) \in B^{q+1}(\mathrm{C}^*)$. From this it is clear that $[\sigma_1] = [\sigma_1']$.
The map $\overline{\partial}$ we just constructed is actually a map from $Z^q(\mathrm{E}^*) \to H^{q+1}(\mathrm{E}^*)$. It is a homomorphism of $K-$modules, since every map we used in its construction is a homomorphism. To show that this gives a homomorphism $\partial^q : H^q(\mathrm{E}^*) \to H^{q+1}(\mathrm{C}^*)$ we need to show that if $\sigma_3 \in B^q(\mathrm{E}^*)$, then its image under the just constructed map is 0. That σ is in the image of d^{q-1} means that we have a $\sigma_5 \in \mathrm{E}^{q-1}$ such that $\sigma_3 = d^{q-1}(\sigma_5)$. The surjectivity of β^{q-1} gives a $\sigma_6 \in \mathrm{D}^{q-1}$ such that $\beta^{q-1}(\sigma_6) = \sigma_5$. With square (4) we now have that:

$$\beta^q(d^{q-1}(\sigma_6)) = d^{q-1}(\beta^{q-1}(\sigma_6)) = d^{q-1}(\sigma_5) = \sigma_3.$$

In the discussion above we have seen that it does not matter which element σ_2 we take out of the pre-image of σ_3 under β^q, so we can take $\sigma_2 = d^{q-1}(\sigma_6) \in B^q(\mathrm{D}^*)$. Now in $\overline{\partial}$ we picked σ_1 such that $\alpha^{q+1}(\sigma_1) = d^q(\sigma_2)$, which is now 0, since $d^q \circ d^{q-1} = 0$. The injectivity of α^{q+1} implies that $\sigma_1 = 0$ and hence $[\sigma_1] = 0$, which is what we needed.

Next part of the proof is to show that ∂ makes (2) into an exact sequence. For this we need to proof exactness at $H^q(\mathrm{C}^*)$, at $H^q(\mathrm{D}^*)$ and at $H^q(\mathrm{E}^*)$, so in total we need to proof six inclusions.
(i) First we will see that $\mathrm{Im}\,(\partial^{q-1}) \subset \ker(\overline{\alpha}^q)$. Let $[\sigma_1] \in \mathrm{Im}\,(\partial^{q-1})$, then we have a $\sigma_3 \in Z^{q-1}(\mathrm{E}^*)$ such that $[\sigma_1] = \overline{\partial}^{q-1}(\sigma_3)$. By construction we thus have a $\xi \in \mathrm{C}^{q-1}$ and $\sigma_2 \in \mathrm{D}^q$ such that $\beta^{q-1}(\sigma_2) = \sigma_3$ and $\alpha^q(\sigma_1 + d^{q-1}\xi) = d^{q-1}(\sigma_2)$. This last one and the commutative square (5) gives that:

$$\begin{aligned}\alpha^q(\sigma_1) &= \alpha^q(\sigma_1 + d^{q-1}(\xi)) - \alpha^q(d^{q-1}(\xi)) = d^{q-1}(\sigma_2) - d^{q-1}(\alpha^{q-1}(\xi))\\ &= d^{q-1}(\sigma_2 - \alpha^{q-1}(\xi)) \in B^q(\mathrm{D}^*).\end{aligned}$$

Now of this and because $\overline{\alpha}^q([\sigma_1]) = [\alpha^q(\sigma_1)]$ we get that $\overline{\alpha}^q([\sigma_1]) = 0$, and thus $[\sigma_1] \in \ker(\overline{\alpha}^q)$.
(ii) To conclude the exactness at $H^q(\mathrm{C}^*)$ we will proof that $\ker(\overline{\alpha}^q) \subset \mathrm{Im}\,(\partial^{q-1})$, so that they are equal. When $0 = \overline{\alpha}^q([\sigma_1]) = [\alpha^q(\sigma_1)]$ we get that $\alpha^q(\sigma_1) = d^{q-1}(\sigma_2)$ for some $\sigma_2 \in \mathrm{D}^{q-1}$. Furthermore for $\sigma_3 := \beta^{q-1}(\sigma_2) \in \mathrm{E}^{q-1}$ we have with (4) and the exactness of the rows that

$$d^{q-1}(\sigma_3) = d^{q-1}(\beta^{q-1}(\sigma_2)) = \beta^q(d^{q-1}(\sigma_2)) = \beta^q(\alpha^q(\sigma_1)) = 0.$$

From this we conclude that $\sigma_3 \in Z^{q-1}(\mathrm{E}^*)$ and by construction it is clear that $\partial^q([\sigma_3]) = \overline{\partial}^{q-1}(\sigma_3) = [\sigma_1]$.
(iii) For the second exactness we will first proof that $\mathrm{Im}\,(\overline{\alpha}^q) \subset \ker(\overline{\beta}^q)$, so assume $[\sigma_2] \in \mathrm{Im}\,(\overline{\alpha}^q)$. We then have a $\sigma_1 \in Z^q(\mathrm{C}^*)$ such that $\overline{\alpha}^q([\sigma_1]) = [\sigma_2]$. But since $\overline{\alpha}^q([\sigma_1]) = [\alpha^q(\sigma_1)]$ per definition we have that there exists a $\xi \in \mathrm{D}^{q-1}$ such that $\sigma_2 = \alpha^q(\sigma_1) + d^{q-1}\xi$. Again using the exactness of the rows and (4) we get that:

$$\overline{\beta}^q([\sigma_2]) = [\beta^q(\sigma_2)] = [\beta^q(\alpha^q(\sigma_1)) + \beta^q(d^{q-1}(\xi))] = [0 + d^{q-1}(\beta^{q-1}(\xi))] = 0.$$

We conclude that $[\sigma_2] \in \ker(\overline{\beta}^q)$.
(iv) Now let $[\sigma_2] \in \ker(\overline{\beta}^q)$, then we have that $\sigma \in Z^q(\mathrm{D}^*)$ and $0 = \overline{\beta}^q([\sigma_2]) = [\beta^q(\sigma_2)]$. This implies that there exists a $\xi \in \mathrm{E}^{q-1}$ such that $\beta^q(\sigma_2) = d^{q-1}\xi$. Since β^{q-1} is surjective we find a $\sigma_3 \in \mathrm{D}^{q-1}$ such that $\beta^{q-1}(\sigma_3) = \xi$. Using (4) again we get that:

$$\beta^q(d^{q-1}(\sigma_3)) = d^{q-1}(\beta^{q-1}(\sigma_3)) = d^{q-1}(\xi) = \beta^q(\sigma_2),$$

and therefore $\sigma_2 - d^{q-1}(\sigma_3)$ lies in the kernel of β^q. Since the rows are exact we have that it lies in the image of α^q as well and this gives us a $\sigma_1 \in \mathrm{C}^q$ such that $\alpha^q(\sigma_1) = \sigma_2 - d^{q-1}(\sigma_3)$. Now we need to proof that $d^q(\sigma_1) = 0$, so that we can speak of $[\sigma_1]$. Here we need (2):

$$\alpha^{q-1}(d^q(\sigma_1)) = d^q(\alpha^q(\sigma_1)) = d^q(\sigma_2 - d^{q-1}(\sigma_3)) = d^q(\sigma_2) = 0.$$

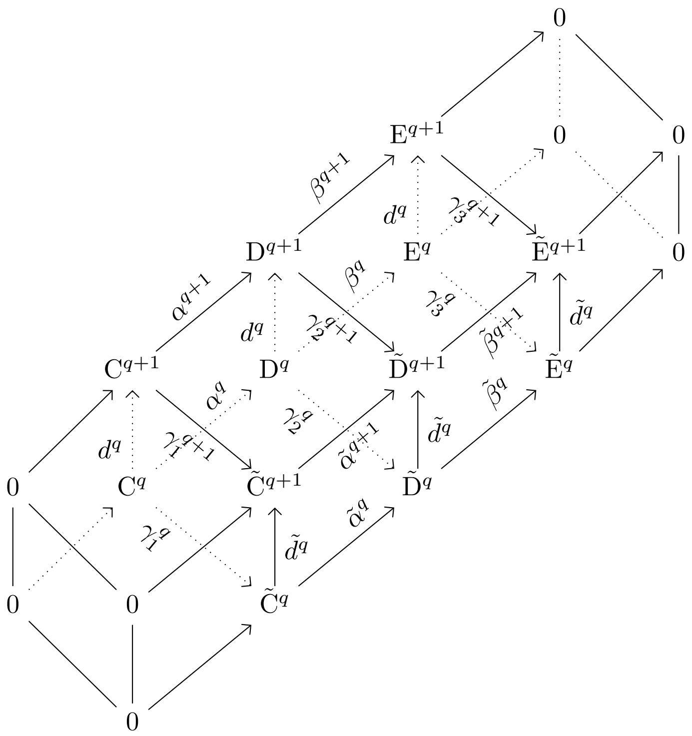
Since α^q is injective we have that $d^q(\sigma_1) = 0$, so $[\sigma_1]$ exists. Now we get that $\overline{\alpha}^q([\sigma_1]) = [\alpha^q(\sigma_1)] = [\sigma_2 - d^{q-1}(\sigma_3)] = [\sigma_2]$, which is exactly what we need to show that $\ker(\overline{\beta}^q) \subset \mathrm{Im}\,(\overline{\alpha}^q)$.
(v) We will show that $\mathrm{Im}\,(\overline{\beta}^q) \subset \ker(\partial^q)$, so let $\overline{\beta}^q([\sigma_2]) = [\sigma_3]$, with $\sigma_2 \in Z^q(\mathrm{D}^*)$. Since $\overline{\beta}^q([\sigma_2]) = [\beta^q(\sigma_2)]$ we get that:

$$\partial^q([\sigma_3]) = \partial^q([\beta^q(\sigma_2)]) = \overline{\partial}^q(\beta^q(\sigma_2)).$$

Now assume that $\sigma_1 \in \overline{\partial}^q(\beta^q(\sigma_2))$, then by construction of ∂ there exists a $\xi \in \mathrm{C}^q$ such that $\alpha^{q+1}(\sigma_1 + d^q(\xi)) = d^q(\sigma_2) = 0$. With the injectivity of α^{q+1} we conclude that $\sigma_1 + d^q(\xi) = 0$ and hence $\partial^q([\sigma_3]) = \overline{\partial}^q(\beta^q(\sigma_2)) = [\sigma_1] = 0$.
(vi) The final part of the exactness is to show that $\ker(\partial^q) \subset \mathrm{Im}\,(\overline{\beta}^q)$. Let us assume therefore that $0 = \partial^q([\sigma_3]) = \overline{\partial}^q(\sigma_3)$. By construction of $\overline{\partial}^q$ we find $\sigma_2 \in \mathrm{D}^q$ and $\xi \in \mathrm{C}^q$ such that $\beta^q(\sigma_2) = \sigma_3$ and $\alpha^{q+1}(d^q(\xi)) = d^q(\sigma_2)$. Using (2) we know that $\alpha^{q+1}(d^q(\xi)) = d^q(\alpha^q(\xi))$, so when we define $\sigma_1 := \sigma_2 - \alpha^q(\xi)$ we immediately get that $d^q(\sigma_1) = 0$ so that $[\sigma_1]$ exists. Furthermore since the rows are exact we also get that $\beta^q(\sigma_1) = \beta^q(\sigma_2) - \beta^q(\alpha^q(\xi)) = \sigma_3$, so that $\overline{\beta}^q([\sigma_1]) = [\sigma_3]$

which is what we wanted.

Finally we will show that diagram (3) commutes. Therefore look at the commuting diagram below, we will use its commutative squares extensively.



Let $\sigma_3 \in Z^q(\mathrm{E}^*)$ and $\tilde{\sigma}_3 := \overline{\gamma}_3^q(\sigma_3) \in Z^q(\tilde{\mathrm{E}}^*)$. By construction of ∂ we find $\sigma_1 \in \mathrm{C}^{q+1}$, $\tilde{\sigma}_1 \in \tilde{\mathrm{C}}^{q+1}$, $\sigma_2 \in \mathrm{D}^q$ and $\tilde{\sigma}_2 \in \tilde{\mathrm{D}}^q$ such that the following holds:

$$\alpha^{q+1}(\sigma_1) = d^q(\sigma_2), \quad \beta^q(\sigma_2) = \sigma_3, \quad \partial([\sigma_3]) = [\sigma_1],$$

$$\tilde{\alpha}^{q+1}(\tilde{\sigma}_1) = \tilde{d}^q(\tilde{\sigma}_2), \quad \tilde{\beta}^q(\tilde{\sigma}_2) = \tilde{\sigma}_3, \quad \tilde{\partial}([\tilde{\sigma}_3]) = [\tilde{\sigma}_1].$$

So to proof that (3) commutes we need to show the second equality in $[\gamma_1^{q+1}(\sigma_1)] = \overline{\gamma}_1^{q+1}([\sigma_1]) = \tilde{\partial}([\tilde{\sigma}_1])$. First note that

$$\tilde{\beta}^q(\gamma_2^q(\sigma_2)) = \gamma_3^q(\beta^q(\sigma_2)) = \gamma_3^q(\sigma_3) = \tilde{\sigma}_3 = \tilde{\beta}^q(\tilde{\sigma}_2),$$

which implies that $\gamma_3^q(\sigma_2) - \tilde{\sigma}_2 \in \ker(\tilde{\beta}^q) = \mathrm{Im}\,(\tilde{\alpha}^q)$. Therefore we find $\xi \in \tilde{\mathrm{C}}^q$ such that $\tilde{\alpha}^q(\xi) = \gamma_3^q(\sigma_2) - \tilde{\sigma}_2$. Now we compute that:

$$\begin{aligned}
\tilde{\alpha}^{q+1}\left(\tilde{d}^q(\xi)\right) &= \tilde{d}^q(\tilde{\alpha}^q(\xi)) = \tilde{d}^q(\gamma_2^q(\sigma_2) - \tilde{\sigma}_2) = \tilde{d}^q(\gamma_2^q(\sigma_2)) - \tilde{\alpha}^{q+1}(\tilde{\sigma}_1) \\
&= \gamma_2^{q+1}(d^q(\sigma_2)) - \tilde{\alpha}^{q+1}(\tilde{\sigma}_1) = \gamma_2^{q+1}\left(\alpha^{q+1}(\sigma_1)\right) - \tilde{\alpha}^{q+1}(\tilde{\sigma}_1) \\
&= \tilde{\alpha}^{q+1}\left(\gamma_1^{q+1}(\sigma_1) - \tilde{\sigma}_1\right)
\end{aligned}$$

Knowing that α^{q+1} is injective, we conclude that $\gamma_1^{q+1}(\sigma_1) - \tilde{\sigma}_1 = \tilde{d}^q(\xi)$, which means that $[\gamma_1^{q+1}(\sigma_1)] = [\tilde{\sigma}_1]$ holds and therefore diagram (3) commutes. □

6 Axiomatic sheaves

Finally, we are able to set op sheaf cohomology. We will start with the definition of a sheaf cohomology theory, which is axiomatic. The rest of the section will be used to show that there is a natural manner in which these theories occur and that any two sheaf cohomology theories are isomorphic.

Definition 5.1: sheaf cohomology theory

A *sheaf cohomology theory* $\mathcal{H}$ for a manifold X with coefficients in sheaves of $K-$modules over X consists of:

I a $K-$module $H^q(X, \mathcal{F})$ for each sheaf $\mathcal{F}$ and integer q,

II a homomorphism $h^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}')$ for each homomorphism $h : \mathcal{F} \to \mathcal{F}'$ and integer q,

III a homomorphism $\partial^q(\mathcal{F}', \mathcal{F}, \mathcal{F}'') : H^q(X, \mathcal{F}'') \to H^{q+1}(X, \mathcal{F}')$ for each short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ and integer q,

such that

(a) For each sheaf $\mathcal{F}$

1 if $q < 0$, then $H^q(X, \mathcal{F}) = 0$

2 there exist an isomorphism $h_\mathcal{F} : H^0(X, \mathcal{F}) \to \mathcal{F}(X)$, such that if $h : \mathcal{F} \to \mathcal{F}'$ is a homomorphism, then the following diagram commutes:

$$\begin{array}{ccc} H^0(X,\mathcal{F}) & \xrightarrow{h_{\mathcal{F}}} & \mathcal{F}(X) \\ \downarrow h^0 & & \downarrow h_X \\ H^0(X,\mathcal{F}') & \xrightarrow{h_{\mathcal{F}'}} & \mathcal{F}'(X) \end{array}$$

(b) If $\mathcal{F}$ is a fine sheaf, then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$.

(c) For each short exact sequence $0 \to \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{h'} \mathcal{F}'' \to 0$, the sequence

$$\ldots \to H^q(X, \mathcal{F}') \to H^q(X, \mathcal{F}) \to H^q(X, S'') \to H^{q+1}(M, \mathcal{F}') \to \ldots$$

is exact.

(d) If $h : \mathcal{F} \to \mathcal{F}$ is equal to the identity, then so is h^q for all q.

(e) If

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{h} & \mathcal{F}' \\ & \searrow h' & \downarrow h'' \\ & & \mathcal{F}'' \end{array}$$

commutes, then so does

$$\begin{array}{ccc} H^q(X,\mathcal{F}) & \xrightarrow{h^q} & H^q(X,\mathcal{F}') \\ & \searrow^{h'^q} & \Big\downarrow h''^q \\ & & H^q(X,\mathcal{F}'') \end{array}$$

(f) When there exists a homomorphism between two exact short sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{h} & \mathcal{F} & \xrightarrow{h'} & \mathcal{F}'' & \longrightarrow & 0 \\ & & \Big\downarrow g' & & \Big\downarrow g & & \Big\downarrow g'' & & \\ 0 & \longrightarrow & \tilde{\mathcal{F}}' & \xrightarrow{\tilde{h}} & \tilde{\mathcal{F}} & \xrightarrow{\tilde{h}'} & \tilde{\mathcal{F}}'' & \longrightarrow & 0 \end{array}$$

then for each q we have the following commuting diagram:

$$\begin{array}{ccc} H^q(X,\mathcal{F}'') & \xrightarrow{\partial^q} & H^{q+1}(X,\mathcal{F}') \\ \Big\downarrow g''^q & & \Big\downarrow g'^q \\ H^q(X,\tilde{\mathcal{F}}'') & \xrightarrow{\tilde{\partial}^q} & H^{q+1}(X,\tilde{\mathcal{F}}') \end{array}$$

Definition 5.2: resolution of a sheaf
A *resolution of a sheaf* $\mathcal{K}$ is an exact sheaf sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{g^0} \mathcal{G}^0 \xrightarrow{g^1} \mathcal{G}^1 \xrightarrow{g^2} \mathcal{G}^2 \longrightarrow ...$$

The resolution will be denoted by $\mathcal{G}^*$. If each $\mathcal{G}_i$ are fine, then the resolution is called fine. If each $\mathcal{G}_i$ are torsionless, then so is $\mathcal{G}^*$.

Example 5.1: cochain complex induced by a resolution
Suppose we have the situation as in Definition 5.2. Now let $\mathcal{F}$ be another sheaf. We will form a cochain complex $\mathrm{C}^*_{\mathcal{F}}$ associated to the sheaf $\mathcal{F}$ and the resolution of the sheaf $\mathcal{K}$. First we note that the homomorphism in the sheaf sequence can be extended to homomorphism $\mathcal{G}_i \otimes \mathcal{F} \to \mathcal{G}_{i+1} \otimes \mathcal{F}$, for $i \geq 1$, by the tensor product with $\mathrm{Id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$. We will not get an exact sequence, but we do have that for each x the following holds:

$$\mathrm{Im}\,(g^i \otimes \mathrm{Id}_{\mathcal{F}})_x \subset \ker(g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_x.$$

This is nothing more than an easy computation, for we have that $(g^i \otimes \mathrm{Id}_{\mathcal{F}})_x : (\mathcal{G}^i \otimes \mathcal{F})_x \to (\mathcal{G}^{i+1} \otimes \mathcal{F})_x; r^U_x(g_i \otimes f) \mapsto r^U_x(h^i_U(g_i) \otimes f)$. This means that:

$$(g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_x \circ (g^i \otimes \mathrm{Id}_{\mathcal{F}})_x(r^U_x(g_i \otimes f)) = r^U_x(g^{i+1}(g^i(g_i)) \otimes f) = r^U_x(0 \otimes f) = 0$$

By this computation it is also clear that exactness will not hold in general. Another direct consequence is that we have a sequence of K-modules:

$$0 \longrightarrow (\mathcal{G}^0 \otimes \mathcal{F})(X) \xrightarrow{(g^1 \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{G}^1 \otimes \mathcal{F})(X) \xrightarrow{(g^2 \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{G}^2 \otimes \mathcal{F})(X) \longrightarrow ...$$

with $\mathrm{Im}\,(g^i \otimes \mathrm{Id}_{\mathcal{F}})_X \subset \ker(g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_X$. Here we use that in fact $(\mathcal{G}^i \otimes \mathcal{F})(X) = \Gamma(X, \overline{\mathcal{G}^i \otimes \mathcal{F}})$ and the computation made above. For a $s : X \to \overline{\mathcal{G}^{i+1} \otimes \mathcal{F}} \in \mathrm{Im}\,(g^i \otimes \mathrm{Id}_{\mathcal{F}})_X$ we have that there exists a $t : X \to \overline{\mathcal{G}^i \otimes \mathcal{F}}$ such that $s(x) = (g^i \otimes \mathrm{Id}_{\mathcal{F}})_x(t(x))$ for each x and hence:

$$\Big((g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_X(s)\Big)(x) = (g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_x(s(x)) = (g^{i+1} \otimes \mathrm{Id}_{\mathcal{F}})_x((g^i \otimes \mathrm{Id}_{\mathcal{F}})_x(t(x))) = 0$$

Now we are finally able to set up the cochain complex $\mathrm{C}^*_{\mathcal{F}}$. For each $q < 0$ we define $\mathrm{C}^q_{\mathcal{F}} := 0$ and for $q \geq 0$ we define $\mathrm{C}^q_{\mathcal{F}} := (\mathcal{G}^q \otimes \mathcal{F})(X)$. The homomorphism $d^q_{\mathcal{F}} : \mathrm{C}^q \to \mathrm{C}^{q+1}$ will be given as above: $d^q_{\mathcal{F}} = (g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X$. The discussion above shows that this is actually a cochain complex, which depends on the given resolution of $\mathcal{K}$ and on the sheaf $\mathcal{F}$. When there is no doubt about $\mathcal{K}$ we will denote this cochain complex by either $\mathrm{C}^*_{\mathcal{F}}$ or $(\mathcal{G}^* \otimes \mathcal{F})(X)$.
In the same way a sheaf homomorphism $h : \mathcal{F} \to \mathcal{F}'$ induces a map, when tensored with the identity on $\mathcal{G}^q$. This map is a cochain map, since on presheaf level we have the following commutativity:

$$\begin{aligned} d^q \circ (\mathrm{Id} \otimes h)_X(g \otimes f) &= d^q(g \otimes h(f)) = (g^{q+1}(g) \otimes h(f)) = (\mathrm{Id} \otimes h)_X(g^{q+1}(g) \otimes f) \\ &= (\mathrm{Id} \otimes h)_X \circ d^{q+1}(g \otimes f). \end{aligned}$$

Theorem 4: Let

$$0 \longrightarrow \mathcal{K} \xrightarrow{g^0} \mathcal{G}^0 \xrightarrow{g^1} \mathcal{G}^1 \xrightarrow{g^2} \mathcal{G}^2 \longrightarrow \ldots$$

be a fine torsionless resolution of the constant sheaf $\mathcal{K}$. Then there exists a sheaf cohomology theory associated to it.
Proof: We have to define the K-modules $H^q(X, \mathcal{F})$ for each q and each sheaf $\mathcal{F}$ and homomorphism between modules such that (a) till (f) holds. So let $H^q(X, \mathcal{F}) = H^q((\mathcal{G}^* \otimes \mathcal{F})(X))$. Furthermore, for each homomorphism $h : \mathcal{F} \to \mathcal{F}'$ we get a cochain map α as discussed above, $\alpha^q = (\mathrm{Id} \otimes h)_X$. Then we define $h^q := \overline{\alpha}^q$, the associated homomorphism of cohomology modules. Finally for a short exact sequence

$$0 \longrightarrow \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{h'} \mathcal{F}'' \longrightarrow 0$$

of sheaves we use Theorem 2 to get a short exact sequence of cochain maps

$$0 \longrightarrow (\mathcal{G}^* \otimes \mathcal{F}')(X) \xrightarrow{(\mathrm{Id} \otimes h)_X} (\mathcal{G}^* \otimes \mathcal{F})(X) \xrightarrow{(\mathrm{Id} \otimes h')_X} (\mathcal{G}^* \otimes \mathcal{F}'')(X) \longrightarrow 0$$

Now Theorem 3 gives maps $\partial^q : H^q((\mathcal{G}^* \otimes \mathcal{F}'')(X)) \to H^q((\mathcal{G}^* \otimes \mathcal{F}')(X))$. This leaves us to check that these definitions satisfy (a) till (f)
(a) Since $\mathrm{C}^q = 0$ for all $q < 0$ we immediately get that $H^q(X, \mathcal{F}) = 0$ when $q < 0$. Now let $\mathcal{L}_q = \ker(g^{q+1}) \subset \mathcal{G}^q$. Clearly this gives a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{L}_q \xhookrightarrow{i} \mathcal{G}^q \xrightarrow{g^{q+1}} \mathcal{L}_{q+1} \longrightarrow 0$$

Note that $\mathcal{L}_q$ is a subsheaf of $\mathcal{G}^q$ and therefore torsionless as well. Using Theorem 2 and Lemma 3.1 we get a short exact sequence and an exact sequence:

$$0 \longrightarrow \mathcal{L}_q \otimes \mathcal{F} \xrightarrow{i \otimes \mathrm{Id}_{\mathcal{F}}} \mathcal{G}^q \otimes \mathcal{F} \xrightarrow{g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}}} \mathcal{L}_{q+1} \otimes \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow (\mathcal{L}_q \otimes \mathcal{F})(X) \xrightarrow{(i \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{G}^q \otimes \mathcal{F})(X) \xrightarrow{(g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{L}_{q+1} \otimes \mathcal{F})(X)$$

This second sequence implies that $(i \otimes \mathrm{Id}_{\mathcal{F}})_X : (\mathcal{L}_q \otimes \mathcal{F})(X) \to (\mathcal{G}^q \otimes \mathcal{F})(X)$ is injective for each q. We can now look at the composition:

$$(\mathcal{G}^q \otimes \mathcal{F})(X) \xrightarrow{(g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{L}_{q+1} \otimes \mathcal{F})(X) \hookrightarrow (\mathcal{G}^{q+1} \otimes \mathcal{F})(X)$$

with the composite $(\mathcal{G}^q \otimes \mathcal{F})(X) \xrightarrow{(g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{G}^{q+1} \otimes \mathcal{F})(X)$.

Therefore we conclude that $\ker((h^q \otimes \mathrm{Id}_{\mathcal{F}})_X) = (\mathcal{L}_q \otimes \mathcal{F})_X$ for all $q > 0$. In particular we now know that:

$$H^0(X, \mathcal{F}) = H^0((\mathcal{G}^* \otimes \mathcal{F})(X)) = \ker((g^1 \otimes \mathrm{Id}_{\mathcal{F}})_X) / \{0\} = (\mathcal{L}_0 \otimes \mathcal{F})(X).$$

Since we have a resolution of the sheaf $\mathcal{K}$ we know that $g^0 : \mathcal{K} \to \mathcal{L}_0$ is not only an injection, but a surjection as well. So g^0 is an isomorphism and $\mathcal{K} \equiv \mathcal{L}_0$. For the sheaf $\mathcal{F}$ this now induces an isomorphism $g^0 \otimes \mathrm{Id}_{\mathcal{F}} : \mathcal{K} \otimes \mathcal{F} \to \mathcal{L}_0 \otimes \mathcal{F}$. We can also find an isomorphism $k : \mathcal{K} \otimes \mathcal{F} \to \mathcal{F}$, which is the induced isomorphism of the presheafmap consisting of isomorphisms $k_U : K \otimes \mathcal{F}(U) \to \mathcal{F}(U)$ and each k_U is again induced by the bilinear map $k_U : K \times \mathcal{F}(U) \to K$; $(k, f) \mapsto kf$. This is all uniquely determined and clearly it induces an isomorphism. Finally we conclude that $\mathcal{F} \simeq \mathcal{L}_0 \otimes \mathcal{F}$. In particular we have the isomorphism between $\mathcal{F}(X)$ and $(\mathcal{L}_0 \otimes \mathcal{F})(X)$.
So al together we get that $\mathcal{F}(X) \simeq (\mathcal{L}_0 \otimes \mathcal{F})(X) = H^0(X, \mathcal{F})$. This now induces the commutative diagram, when $h : \mathcal{F} \to \mathcal{F}'$ is a homomorphism:

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{(g^0 \otimes \mathrm{Id}_{\mathcal{F}})_X \circ k_X^{-1}} & H^0(X, \mathcal{F}) \\
\downarrow h_X & & \downarrow (\mathrm{Id} \otimes h)_X \\
\mathcal{F}'(X) & \xrightarrow{(g^0 \otimes \mathrm{Id}_{\mathcal{F}})_X \circ k_X'^{-1}} & H^0(X, \mathcal{F}')
\end{array}$$

The commutativity is a consequence of the fact that on presheaf level we have that:

$$k'_X \circ (\mathrm{Id} \otimes h)_X (k \otimes f) = k'_X (k \otimes h(f)) = k \cdot h(f) = h(k \cdot f) = h_X \circ k_X(f).$$

And with this we have shown that our choices satisfy (a).
(b) Here we assume that $\mathcal{F}$ is a fine sheaf. From (a) and Theorem 2 we get the following short exact sequence.

$$0 \longrightarrow (\mathcal{L}_q \otimes \mathcal{F})(X) \xrightarrow{(i \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{G}^q \otimes \mathcal{F})(X) \xrightarrow{(g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X} (\mathcal{L}_{q+1} \otimes \mathcal{F})(X) \longrightarrow 0$$

Which means that $\mathrm{Im}\,((g^q \otimes \mathrm{Id}_{\mathcal{F}})_X) = (\mathcal{L}_q \otimes \mathcal{F})(X)$. Furthermore we get from (a) that:

$$H^q(X, \mathcal{F}) = H^q((\mathcal{G}^* \otimes \mathcal{F})(X)) = \ker((g^{q+1} \otimes \mathrm{Id}_{\mathcal{F}})_X) / \mathrm{Im}\,((g^q \otimes \mathrm{Id}_{\mathcal{F}})_X) = (\mathcal{L}_q \otimes \mathcal{F})(X) / (\mathcal{L}_q \otimes \mathcal{F})(X) = 0,$$

which is what we needed to fulfill (b).
(c) This is a direct consequence of Theorem 3 and the short exact sequence we have constructed in (III).

$$0 \longrightarrow (\mathcal{G}^* \otimes \mathcal{F}')(X) \xrightarrow{(\mathrm{Id}\otimes h)_X} (\mathcal{G}^* \otimes \mathcal{F})(X) \xrightarrow{(\mathrm{Id}\otimes h')_X} (\mathcal{G}^* \otimes \mathcal{F}'')(X) \longrightarrow 0$$

(d) On presheaf level we have that $\alpha^q = (\mathrm{Id} \otimes h)_X$. Now since h is the identity on $\mathcal{F}$ we get that α^q is the identity on presheaf level. Clearly, by using lemma 2.6 and 2.7, we get that α^q is the identity. This implies that for $\sigma \in H^q((\mathcal{G}^* \otimes \mathcal{F})(X))$:

$$h^q([\sigma]) = \overline{\alpha}^q([\sigma]) = [\alpha^q(\sigma)] = [\sigma].$$

We conclude that h^q is the identity as well.
(e) Let $h : \mathcal{F} \to \mathcal{F}'$, $h' : \mathcal{F} \to \mathcal{F}''$ and $h'' : \mathcal{F}' \to \mathcal{F}''$ be homomorphisms such that $h'' \circ h = h'$. This clearly induces the following commutative diagram at presheaf level:

$$\begin{array}{ccc} (\mathcal{G}^q \otimes \mathcal{F})(X) & \xrightarrow{(\mathrm{Id}\otimes h)_X} & (\mathcal{G}^q \otimes \mathcal{F}')(X) \\ & \searrow^{(\mathrm{Id}\otimes h')_X} & \downarrow (\mathrm{Id}\otimes h'')_X \\ & & (\mathcal{G}^q \otimes \mathcal{F}'')(X) \end{array}$$

for each q. Now using lemma 4.2 and that $H^q(X, \mathcal{F}) = H^q((\mathcal{G}^* \otimes \mathcal{F})(X))$, we get the wanted commutative diagram:

$$\begin{array}{ccc} H^q(X,\mathcal{F}) & \xrightarrow{h^q} & H^q(X,\mathcal{F}') \\ & \searrow^{h'q} & \downarrow h''q \\ & & H^q(X,\mathcal{F}'') \end{array}$$

(f) This is a again a consequence of Theorem 3 and the commutativity of the following diagram, which is clear on presheaflevel:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{G}^q \otimes \mathcal{F}')(X) & \xrightarrow{(\mathrm{Id}\otimes h)_X} & (\mathcal{G}^q \otimes \mathcal{F})(X) & \xrightarrow{(\mathrm{Id}\otimes h')_X} & (\mathcal{G}^q \otimes \mathcal{F}'')(X) & \longrightarrow & 0 \\ & & \downarrow (\mathrm{Id}\otimes g')_X & & \downarrow (\mathrm{Id}\otimes g)_X & & \downarrow (\mathrm{Id}\otimes g'')_X & & \\ 0 & \longrightarrow & (\mathcal{G}^q \otimes \tilde{\mathcal{F}}')(X) & \xrightarrow{(\mathrm{Id}\otimes \tilde{h})_X} & (\mathcal{G}^q \otimes \tilde{\mathcal{F}})(X) & \xrightarrow{(\mathrm{Id}\otimes \tilde{h}')_X} & (\mathcal{G}^q \otimes \tilde{\mathcal{F}}'')(X) & \longrightarrow & 0 \end{array}$$

We conclude that this fine torsionless resolution does indeed induce a sheaf cohomology theory. □

Definition 5.3: cohomology theory homomorphisms
Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two sheaf cohomology theories for X with coefficients in sheaves of $K-$modules over M. Suppose furhter that we have a module homomorphism $f^q_{\mathcal{F}} : H^q(X, \mathcal{F}) \to \tilde{H}^q(X, \mathcal{F})$ for each q and each sheaf $\mathcal{F}$. Then we say that this collection $f = \{f^q_{\mathcal{F}}\}$ is a *homomorphism of the cohomology theory* $\mathcal{H}$ *to the theory* $\tilde{\mathcal{H}}$ if it satisfies the following axioms.

(a) For $q = 0$ we have for each sheaf $\mathcal{F}$ the commutative diagram

$$\begin{array}{ccc} H^0(X,\mathcal{F}) & \xrightarrow{h_{\mathcal{F}}} & \mathcal{F}(X) \\ \downarrow f^0_{\mathcal{F}} & & \downarrow \mathrm{Id} \\ \tilde{H}^0(X,\mathcal{F}) & \xrightarrow{\tilde{h}_{\mathcal{F}}} & \mathcal{F}(X) \end{array}$$

(b) When $h : \mathcal{F} \to \mathcal{F}'$ is a homomorphism, then for each q we have the commutative diagram

$$\begin{array}{ccc} H^q(X,\mathcal{F}) & \xrightarrow{h^q} & H^q(X,\mathcal{F}') \\ \Big\downarrow f^q_{\mathcal{F}} & & \Big\downarrow f^q_{\mathcal{F}'} \\ \tilde{H}^q(X,\mathcal{F}) & \xrightarrow{\tilde{h}^q} & \tilde{H}^q(X,\mathcal{F}') \end{array}$$

(c) Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have the commutative diagram

$$\begin{array}{ccc} H^q(X,\mathcal{F}'') & \xrightarrow{\partial^q} & H^{q+1}(X,\mathcal{F}') \\ \Big\downarrow f^q_{\mathcal{F}''} & & \Big\downarrow f^{q+1}_{\mathcal{F}'} \\ \tilde{H}^q(X,\mathcal{F}'') & \xrightarrow{\tilde{\partial}^q} & \tilde{H}^{q+1}(X,\mathcal{F}') \end{array}$$

An *isomorphism of cohomology theories* is a homomorphism of cohomology theories $f = \{f^q_{\mathcal{F}}\}$ such that each $f^q_{\mathcal{F}}$ is an isomorphism of $K-$modules.

Theorem 5: For each cohomology theories $\mathcal{H}$ and $\tilde{\mathcal{H}}$ there exists an unique homomorphism f between them.
Proof: Let $\mathcal{F}$ be a sheaf and consider $\mathcal{G} = \mathcal{F}_{\tilde{\mathcal{F}}}$ to be the sheaf of discontinuous sections over $\tilde{\mathcal{F}}$. From example 3.1 we know that $\mathcal{G}$ is fine. Furthermore, since $\mathcal{F} \simeq \Gamma_{\tilde{\mathcal{F}}}$ we get that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$. Now example 2.3 gives the short exact sequence:

$$0 \longrightarrow \mathcal{F} \xhookrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{G}/\mathcal{F} \longrightarrow 0$$

First we will show uniqueness of such a homomorphism.
Since $\mathcal{H}$ is a cohomology theory we have the exact sequence

$$\ldots \longrightarrow H^q(X,\mathcal{G}) \xrightarrow{\pi^q} H^q(X,\mathcal{G}/\mathcal{F}) \xrightarrow{\partial^q} H^{q+1}(X,\mathcal{F}) \xrightarrow{i^{q+1}} H^{q+1}(X,\mathcal{G}) \longrightarrow \ldots \tag{4}$$

Furthermore since $\mathcal{G}$ is fine we know that $H^q(X,\mathcal{G}) = 0$ for all $q \geq 1$. The same is true for $\tilde{\mathcal{H}}$ as well. Now let f be a homomorphism between $\mathcal{H}$ and $\tilde{\mathcal{H}}$. First note that $f^0_{\mathcal{F}}$ is completely and uniquely determined by (a) of definition 5.3. Furthermore when combining (4), (a) of definition 5.1 (a) and (c) of definition 5.3 we get that

$$\begin{array}{ccccccccc} & H^0(X,\mathcal{G}) & \xrightarrow{\pi^0} & H^0(X,\mathcal{G}/\mathcal{F}) & & \xrightarrow{\partial^0} & H^1(X,\mathcal{F}) & \xrightarrow{i^1} & H^1(X,\mathcal{G}) = 0 \\ & \Big\downarrow h_{\mathcal{G}} & & \Big\downarrow h_{\mathcal{G}/\mathcal{F}} & & & \Big| & & \\ f^0_{\mathcal{G}} & \mathcal{G}(X) & \xrightarrow{\pi_X} & (\mathcal{G}/\mathcal{F})(X) & f^0_{\mathcal{G}/\mathcal{F}} & & \Big| f^1_{\mathcal{F}} & & \\ & \Big\uparrow \tilde{h}_{\mathcal{G}} & & \Big\uparrow \tilde{h}_{\mathcal{G}/\mathcal{F}} & & & \Big\downarrow & & \\ & \tilde{H}^0(X,\mathcal{G}) & \xrightarrow{\tilde{\pi}^0} & \tilde{H}^0(X,\mathcal{G}/\mathcal{F}) & & \xrightarrow{\tilde{\partial}^0} & \tilde{H}^1(X,\mathcal{F}) & \xrightarrow{\tilde{i}^1} & \tilde{H}^1(X,\mathcal{G}) = 0 \end{array} \tag{5}$$

commutes and has exact rows. This means that ∂^0 and $\tilde{\partial}^0$ are surjective and since $h_{\mathcal{F}}$ is an isomorphism for each sheaf $\mathcal{F}$ we get that $f^1_{\mathcal{F}}$ is uniquely determined as well. For $q \geq 1$ we can use (4) and (c) of definition 5.3 to get

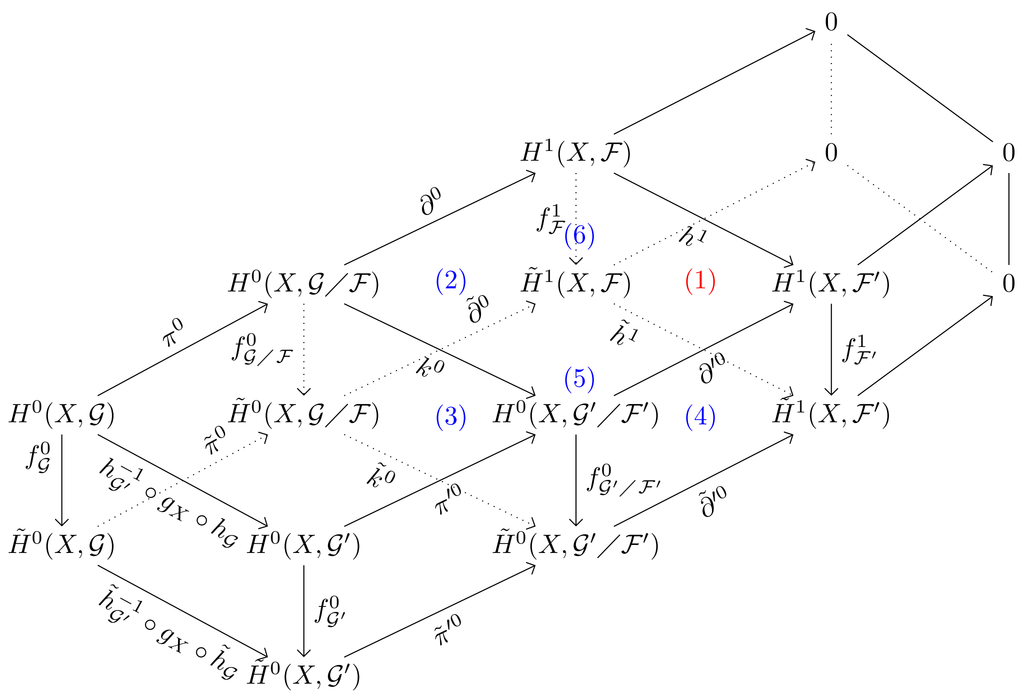
$$\begin{array}{ccccccc} 0 & \xrightarrow{\pi^q} & H^q(X,\mathcal{G}/\mathcal{F}) & \xrightarrow{\partial^q} & H^{q+1}(X,\mathcal{F}) & \xrightarrow{i^{q+1}} & 0 \\ & & \Big\downarrow f^q_{\mathcal{G}/\mathcal{F}} & & \Big\downarrow f^{q+1}_{\mathcal{F}} & & \\ 0 & \xrightarrow{\tilde{\pi}^q} & \tilde{H}^q(X,\mathcal{G}/\mathcal{F}) & \xrightarrow{\tilde{\partial}^q} & \tilde{H}^{q+1}(X,\mathcal{F}) & \xrightarrow{\tilde{i}^{q+1}} & 0 \end{array} \tag{6}$$

This means ∂^q and $\tilde{\partial}^q$ are isomorphisms for $q \geq 1$ and therefore each $f_{\mathcal{F}}$ is uniquely determined out of $f^{q-1}_{\mathcal{G}/\mathcal{F}}$. We conclude with induction that f is unique.

The final part is the existence of such a homomorphism between $\mathcal{H}$ and $\tilde{\mathcal{H}}$. We note that the above proof of uniqueness precisely gives a map f which satisfies (a) and the case that $q = 0$ for (b). So we need to show that this f satisfies (b) for $q \geq 1$ and (c).
(b) : Let $q = 1$ and suppose we have a homomorphism $h : \mathcal{F} \to \mathcal{F}'$. First we note that h induces a homomorphism $g : \mathcal{G} \to \mathcal{G}'$ by letting $(g_U(s))(x) = h_x(s(x))$ for $x \in U$. And this homomorphism induces the commutative square:

$$\begin{array}{ccc} H^0(X,\mathcal{G}) & \xrightarrow{h^{-1}_{\mathcal{G}'} \circ g_X \circ h_{\mathcal{G}}} & H^0(X,\mathcal{G}') \\ \Big\downarrow f^0_{\mathcal{G}} & & \Big\downarrow f^0_{\mathcal{G}'} \\ \tilde{H}^0(X,\mathcal{G}) & \xrightarrow{\tilde{h}^{-1}_{\mathcal{G}'} \circ g_X \circ \tilde{h}_{\mathcal{G}}} & \tilde{H}^0(X,\mathcal{G}') \end{array}$$

where commutativity follows from (a) of the definition of f.
Furthermore we have an induced map $k : \mathcal{G}/\mathcal{F} \to \mathcal{G}'/\mathcal{F}'$. To define this map we let s be a continuous map such that $s(x) \in \mathcal{G}_x/\mathcal{F}_x$ for all $x \in U$, that is $s(x) = [b_x]$ for some $b_x \in \mathcal{G}_x$. Then define $(k_U(s))(x) = [g_x(b_x)]$. Let $k_0 := h^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ h_{\mathcal{G}/\mathcal{F}}$ and $\tilde{k}_0 = \tilde{h}^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ \tilde{h}_{\mathcal{G}/\mathcal{F}}$. Now consider the following lattice:



Here (2) and (4) is a commutative square, since $f^1_{\mathcal{F}}$ and $f^1_{\mathcal{F}'}$ are defined such that (5) commutes. Commutativity of (3) again follows from the fact that f satisfies (a) in definition 5.3, so we have that $h_{\mathcal{G}/\mathcal{F}} = \tilde{h}_{\mathcal{G}/\mathcal{F}} \circ f^0_{\mathcal{G}/\mathcal{F}}$ and the same for $\mathcal{F}'$. This now implies that:

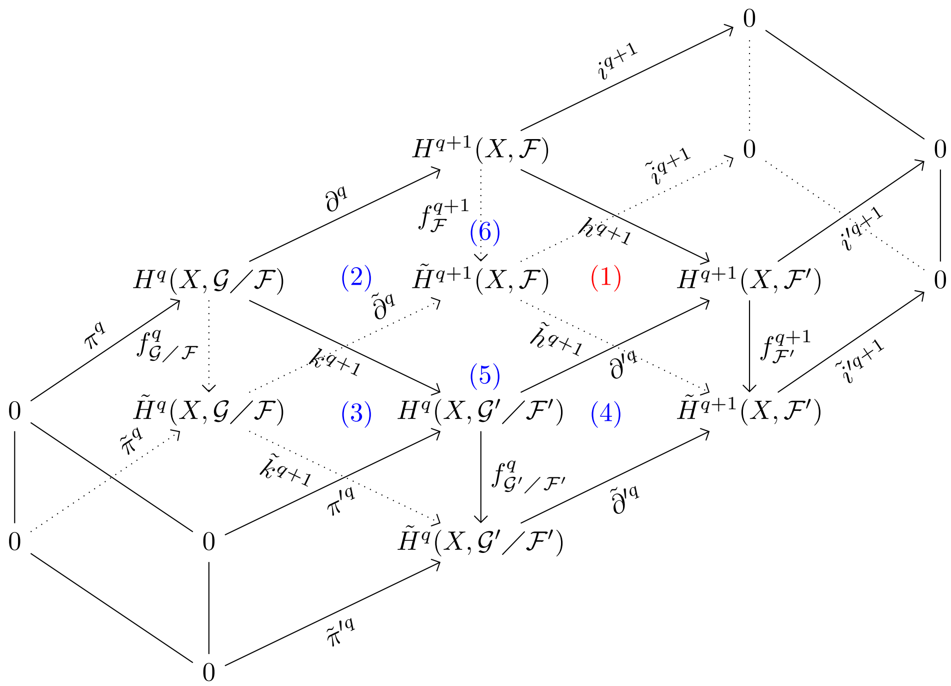
$$\tilde{k}^0 \circ f^0_{\mathcal{G}/\mathcal{F}} = \tilde{h}^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ \tilde{h}_{\mathcal{G}/\mathcal{F}} \circ f^0_{\mathcal{G}/\mathcal{F}} = \tilde{h}^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ h_{\mathcal{G}/\mathcal{F}},$$

$$f^0_{\mathcal{G}'/\mathcal{F}'} \circ k^0 = f^0_{\mathcal{G}'/\mathcal{F}'} \circ h^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ h_{\mathcal{G}/\mathcal{F}} = \tilde{h}^{-1}_{\mathcal{G}'/\mathcal{F}'} \circ k_X \circ h_{\mathcal{G}/\mathcal{F}}.$$

Finally (5) and (6) are commutative as well, which follow from (e) of the definition of $\mathcal{H}$ and $\tilde{\mathcal{H}}$. Now in this lattice we want commutativity of square (1), knowing that all the other squares commute. Let therefore $\alpha \in H^1(X, \mathcal{F})$. Since the rows are exact, we find a $s \in H^0(X, \mathcal{G}/\mathcal{F})$ such that $\partial^0(s) = \alpha$. Using this we find:

$$\begin{aligned} f^1_{\mathcal{F}'}(h^1(\alpha)) &= f^1_{\mathcal{F}'}(h^1(\partial^0(s))) = f^1_{\mathcal{F}}(\partial'^0(k^0(s))) = \tilde{\partial}'^0(f^0_{\mathcal{G}'/\mathcal{F}'}(k^0(s))) \\ &= \tilde{\partial}'^0(\tilde{k}^0(f^0_{\mathcal{G}/\mathcal{F}}(s))) = \tilde{h}^1(\tilde{\partial}^0(f^0_{\mathcal{G}/\mathcal{F}}(s))) = \tilde{h}^1(f^1_{\mathcal{F}}(\partial^0(s))) = \tilde{h}^1(f^1_{\mathcal{F}}(\alpha)). \end{aligned}$$

We conclude that (b) holds for $q = 1$. For $q \geq 1$ we again set up such a lattice:



Again we check commutativity for the squares (2) till (6), to conclude that (1) commutes. First note that (3) is by induction a commutative square, (2) and (4) is again by definition of f and (5) and (6) by definition of $\mathcal{H}$ and $\tilde{\mathcal{H}}$. Then, for each $\alpha \in H^{q+1}(X, \mathcal{F})$ there exists a unique $s \in H^q(X, \mathcal{G}/\mathcal{F})$ such that $\partial^q(s) = \alpha$ and this gives that:

$$\begin{aligned} f^{q+1}_{\mathcal{F}'}(h^{q+1}(\alpha)) &= f^{q+1}_{\mathcal{F}'}(h^{q+1}(\partial^q(s))) = f^{q+1}_{\mathcal{F}'}(\partial'^q(k^{q+1}(s))) = \tilde{\partial}'^q(f^q_{\mathcal{G}'/\mathcal{F}'}(k^{q+1}(s))) \\ &= \tilde{\partial}'^q(\tilde{k}^{q+1}(f^q_{\mathcal{G}/\mathcal{F}}(s))) = \tilde{h}^{q+1}(\tilde{\partial}^q(f^q_{\mathcal{G}/\mathcal{F}}(s))) \\ &= \tilde{h}^{q+1}(f^{q+1}_{\mathcal{F}}(\partial^q(s))) = \tilde{h}^{q+1}(f^{q+1}_{\mathcal{F}}(\alpha)) \end{aligned}$$

We conclude that (b) holds for $q \geq 2$.
Now we want to proof (c), so let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves and define $\mathcal{G}$ and $\mathcal{G}'$ and the map g as before. Since $\mathcal{F}'_x \to \mathcal{F}_x$ and $\mathcal{F}_x \to \mathcal{G}_x$ are both injective we

get the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}' & \xrightarrow{h} & \mathcal{F} & \xrightarrow{h'} & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k_1} & & \\
0 & \longrightarrow & \mathcal{F}' & \xrightarrow{i \circ h} & \mathcal{G} & \xrightarrow{\pi} & \mathcal{G}/\mathcal{F}' & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle k_2} & & \\
0 & \longrightarrow & \mathcal{F}' & \xrightarrow{i'} & \mathcal{G}' & \xrightarrow{\pi'} & \mathcal{G}'/\mathcal{F}' & \longrightarrow & 0
\end{array}
\tag{7}
$$

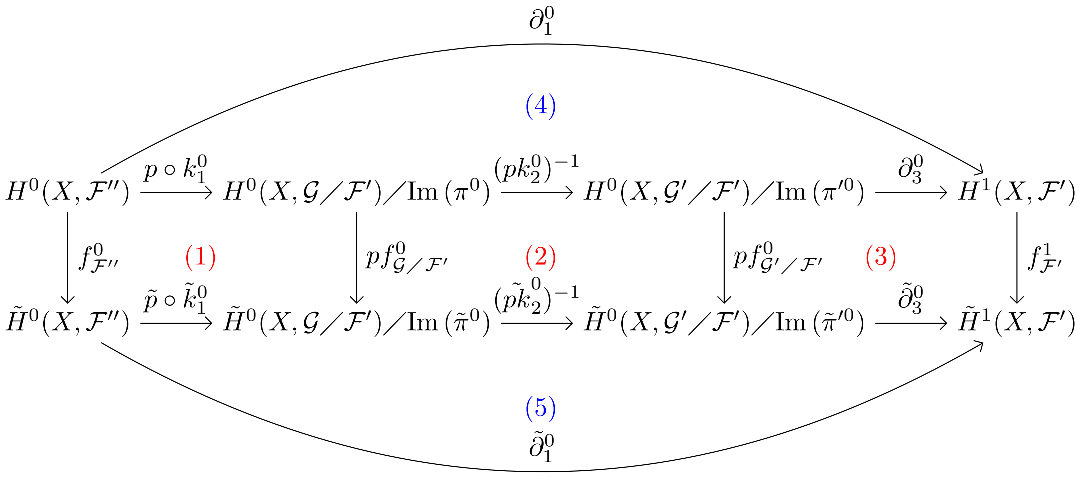
Where the left squares commute trivially and the other two by construction of k_i:
To define k_1 we first define $\psi_1 : \overline{\mathcal{F}''} \to \overline{\mathcal{G}/\mathcal{F}'}$. Let $z \in \mathcal{F}''_x$. Since h'_x is surjective we find a $y \in \mathcal{F}_x$ such that $h'_x(y) = z$. Then we define $\psi_1(z) := \pi_x \circ i_x(y)$ and k_1 the induced sheaf morphism. We have to check whether ψ_1 is well defined, so suppose $h'_x(y_1) = h'_x(y_2) = z$. Then $y_1 - y_2 \in \ker(h'_x) = \mathrm{Im}\,(h_x)$, so we can find a $w \in \mathcal{F}'_x$ such that $h_x(w) = y_1 - y_2$. From this we conclude that $i_x(y_1 - y_2) = i_x(h_x(w)) \in \mathrm{Im}\,(i_x \circ h_x) = \ker(\pi_x)$ and thus ψ_1 is well defined.
In the same manner we define k_2 and ψ_2: for each $z \in \mathcal{G}'_x/\mathcal{F}'_x$ there exists a $y \in \mathcal{G}'_x$ such that $\pi'_x(y) = z$ and then $\psi(z) := \pi_x \circ g_x(y)$, and again we let k_2 to be the induced sheaf morphism. Checking whether ψ_2 is well defined gives y_1 and y_2 such that $\pi'_x(y_i) = z$ and thus $y_1 - y_2 \in \ker(\pi'_x) = \mathrm{Im}\,(i')$. Again we get $y_1 - y_2 = i'_x(w)$ for some $w \in \mathcal{F}'$, and $g(y_1 - y_2) i \circ h(w) \in \mathrm{Im}\,(i_x \circ h_x) = \ker(\pi)$. We conclude that ψ_2 is well defined as well.
Using that $\mathcal{H}$ is a cohomology, so it satisfies (a) (b), (c) and (f) of definition 5.1, we get the following commutative diagram with exact rows, where we let $\partial_1^q := \partial^q(\mathcal{F}', \mathcal{F}, \mathcal{F}'')$, $\partial_2^q := \partial^q(\mathcal{F}', \mathcal{G}, \mathcal{G}/\mathcal{F}')$ and $\partial_3^q := \partial^q(\mathcal{F}', \mathcal{G}', \mathcal{G}'/\mathcal{F}')$, to simplify notation. First for $q = 0$:

$$
\begin{array}{ccccccccc}
\dots & \xrightarrow{h^0} & H^0(X,\mathcal{F}) & \xrightarrow{h'^0} & H^0(X,\mathcal{F}'') & \xrightarrow{\partial_1^0} & H^1(X,\mathcal{F}') & \xrightarrow{h^1} & \dots \\
 & & \downarrow{\scriptstyle i^0} & & \downarrow{\scriptstyle k_1^0} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
\dots & \xrightarrow{(i\circ h)^0} & H^0(X,\mathcal{G}) & \xrightarrow{\pi^0} & H^0\left(X,\mathcal{G}/\mathcal{F}'\right) & \xrightarrow{\partial_2^0} & H^1(X,\mathcal{F}') & \xrightarrow{(i\circ h)^1} & H^1(X,\mathcal{G}) = 0 \\
 & & \uparrow{\scriptstyle g^0} & & \uparrow{\scriptstyle k_2^0} & & \uparrow{\scriptstyle \mathrm{Id}} & & \\
\dots & \xrightarrow{i'^0} & H^0(X,\mathcal{G}') & \xrightarrow{\pi'^0} & H^0\left(X,\mathcal{G}'/\mathcal{F}'\right) & \xrightarrow{\partial_3^0} & H^1(X,\mathcal{F}') & \xrightarrow{i'^1} & H^1(X,\mathcal{G}') = 0
\end{array}
$$

Now let $z \in H^0(X, \mathcal{F}'')$, we want to see what $\partial_1^0(z)$ is. First of all we see that $\partial_1^0(z) = \partial_2^0(k_1^0(z)) \in H^1(X, \mathcal{F}') = \mathrm{Im}\,(\partial_3^0)$. Which means we can find a $y \in H^0(X, \mathcal{G}'/\mathcal{F}')$ such that $\partial_3^0(y) = \partial_2^0(k_1^0(z))$. For any such y we get that $\partial_3^0(y) = \partial_2^0(k_2^0(y))$ and thus that $k_1^0(z) - k_2^0(y) \in \ker(\partial_2^0) = \mathrm{Im}\,(\pi^0)$. This means that for some $w \in H^0(X, \mathcal{G})$: $k_2^0(y) = k_1^0(z) - \pi^0(w) \in [k_1^0(z)]$, which is an element of $H^0(X, \mathcal{G}/\mathcal{F}')/\mathrm{Im}\,(\pi^0)$. Further we note that $\partial_3^0(y_1) = \partial_3^0(y_2) = \partial_2^0(k_1^0(z))$ implies that $y_1 - y_2 \in \ker(\partial_3^0) = \mathrm{Im}\,(\pi'^0)$ and therefore $y_2 \in [y_1]$ which is an element of $H^0(X, \mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$. We conclude that $(k_2^0)^{-1}$ is well defined as a map $(pk_2^0)^{-1} : H^0(X, \mathcal{G}/\mathcal{F}')/\mathrm{Im}\,(\pi^0) \to H^0(X, \mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$. And since ∂_3^0 is constant on classes of $H^0(X, \mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$ we get that, when we let p to be the projection $H^0(X, \mathcal{G}/\mathcal{F}') \to H^0(X, \mathcal{G}/\mathcal{F}')/\mathrm{Im}\,(\pi^0)$:

$$\partial_1^0 = \partial_3^0 \circ (pk_2^0)^{-1} \circ p \circ k_1^0.$$

The same we can get for $\tilde{\mathcal{H}}$ and this gives us the commutative squares (4) and (5) in the following diagram:



Here we define $pf^0_{\mathcal{G}/\mathcal{F}'} : H^0(X,\mathcal{G}/\mathcal{F}')/\mathrm{Im}\,(\pi^0) \to \tilde{H}^0(X,\mathcal{G}/\mathcal{F}')/\mathrm{Im}\,(\tilde{\pi}^0)$ and $pf^0_{\mathcal{G}'/\mathcal{F}'} : H^0(X,\mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0) \to \tilde{H}^0(X,\mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\tilde{\pi}'^0)$ as

$$pf^0_{\mathcal{G}/\mathcal{F}'}(p(\alpha)) := \tilde{p}(f^0_{\mathcal{G}/\mathcal{F}'}(\alpha)),$$

$$pf^0_{\mathcal{G}'/\mathcal{F}'}(p'(\alpha)) := \tilde{p}'(f^0_{\mathcal{G}'/\mathcal{F}'}(\alpha)),$$

with p' the projection into $H^0(X,\mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$. This now means that $pf^0_{\mathcal{G}/\mathcal{F}'}\circ p = \tilde{p}\circ f^0_{\mathcal{G}/\mathcal{F}'}$. Furthermore (b) gives that $f^0_{\mathcal{G}/\mathcal{F}'}\circ k_1^0 = \tilde{k}_0^1 \circ f^0_{\mathcal{F}''}$. Combining these two gives the commutativity at (1). For (2) we use the discussion above and (b) again, which gives us for any $p'(\alpha) \in H^0(X,\mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$:

$$\begin{aligned}\tilde{pk}_2^0(pf^0_{\mathcal{G}'/\mathcal{F}'}(p'(\alpha))) = \tilde{pk}_2^0(\tilde{p}'(f^0_{\mathcal{G}'/\mathcal{F}'}(\alpha))) &= \tilde{p}(\tilde{k}_2^0(f^0_{\mathcal{G}'/\mathcal{F}'}(\alpha))) \\ &\overset{(b)}{=} \tilde{p}(f^0_{\mathcal{G}/\mathcal{F}'}(k_2^0(\alpha))) = pf^0_{\mathcal{G}/\mathcal{F}'}(p(k_2^0(\alpha))) = pf^0_{\mathcal{G}/\mathcal{F}'}(pk_2^0(p'(\alpha))),\end{aligned}$$

so (2) commutes. Finally we use (6), in which we have defined $f^1_{\mathcal{F}'}$, such that for $p'(\alpha) \in H^0(X,\mathcal{G}'/\mathcal{F}')/\mathrm{Im}\,(\pi'^0)$:

$$f^1_{\mathcal{F}'}(\partial_3^0(p'(\alpha))) = f^1_{\mathcal{F}'}(\partial_3^0(\alpha)) \overset{(6)}{=} \tilde{\partial}_3^0(f^0_{\mathcal{G}'/\mathcal{F}'}(\alpha)) = \tilde{\partial}_3^0(\tilde{p}'(f^0_{\mathcal{G}'/\mathcal{F}'}) = \tilde{\partial}_3^0(pf^0_{\mathcal{G}'/\mathcal{F}'}(p'(\alpha))).$$

We conclude that (3) commutes as well and from this diagram we get that (c) of definition 5.3 is fulfilled for $q = 0$.
Secondly for $q \geq 1$ (7) gives, in the same way as before, the commuting diagram (with exact rows):

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{h^q} & H^q(X,\mathcal{F}) & \xrightarrow{h'^q} & H^q(X,\mathcal{F}'') & \xrightarrow{\partial_1^q} & H^{q+1}(X,\mathcal{F}') & \xrightarrow{h^{q+1}} & \dots \\
 & & & & \downarrow k_1^q & & \downarrow \mathrm{Id} & & \\
0 = H^q(X,\mathcal{G}) & & \xrightarrow{\pi^q} & & H^q(X,\mathcal{G}/\mathcal{F}') & \xrightarrow{\partial_2^q} & H^{q+1}(X,\mathcal{F}') & \xrightarrow{(i\circ h)^{q+1}} & H^{q+1}(X,\mathcal{G}) = 0 \\
 & & & & \uparrow k_2^q & & \uparrow \mathrm{Id} & & \\
0 = H^q(X,\mathcal{G}') & & \xrightarrow{\pi'^q} & & H^q(X,\mathcal{G}'/\mathcal{F}') & \xrightarrow{\partial_3^q} & H^{q+1}(X,\mathcal{F}') & \xrightarrow{i'^{q+1}} & H^{q+1}(X,\mathcal{G}') = 0
\end{array}$$

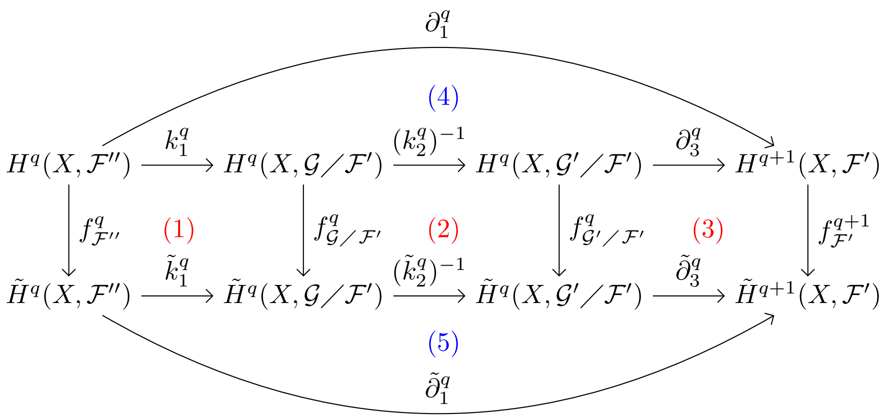
And again we want to find what $\partial_1^q(z)$ is for $z \in H^q(X,\mathcal{F}'')$. We continue like before, although the steps are quite easier: the commutative diagram tells that

$$\partial_1^q(z) = \partial_2^q(k_1^q(z)) \in H^{q+1}(X,\mathcal{F}') = \ker(i'^{q+1}) = \mathrm{Im}\,(\partial_3^q).$$

So there exists a $y \in H^q(X, \mathcal{G}'\diagup\mathcal{F}')$ such that $\partial_3^q(y) = \partial_2^q(k_1^q(z))$. This time however, this y is unique. For let y_1 and y_2 be such that $\partial_3^q(y_i) = \partial_2^q(k_1^q(z))$. Then $\partial_3^q(y_1 - y_2) = 0$ and so $y_1 - y_2 \in \ker(\partial_3^q) = \mathrm{Im}\,(\pi'^q) = \{0\}$, which implies that $y_1 = y_2$. Furthermore we have that $\partial_2^q(k_1^q(z)) = \partial_3^q(y) = \partial_2^q(k_2^q(y))$, so $k_1^q(z) - k_2^q(y) \in \ker(\partial_2^q) = \mathrm{Im}\,(\pi^q) = \{0\}$. We conclude that $k_1^q(z) = k_2^q(y)$. Since y was unique and actually exists $(k_2^q)^{-1}$ is well defined and

$$\partial_1^q = \partial_q^3 \circ (k_2^q)^{-1} \circ k_1^q$$

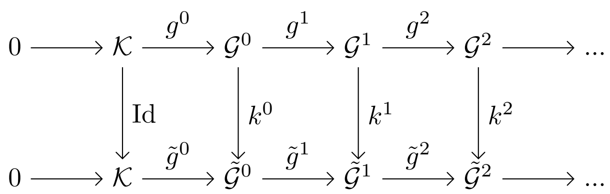
holds. When considering the same for $\tilde{\mathcal{H}}$, we get a diagram:



in which (4) and (5) commute, by the above discussion. Furthermore commutativity of (1) and (2) are consequences of (b) of definition 5.3, which we have already proven to be true for f, and commutativity of (3) follows from (6) in which we have defined $f^{q+1}_{\mathcal{F}'}$. We conclude from this commuting diagram that (c) holds for $q \geq 1$ and therefore f is the unique homomorphism between $\mathcal{H}$ and $\tilde{\mathcal{H}}$. □

Remark: The uniqueness from the homomorphisms between the cohomology theories implies that the homomorphisms are actually isomorphisms. For let $f : \mathcal{H} \to \tilde{\mathcal{H}}$ and $\tilde{f} : \tilde{\mathcal{H}} \to \mathcal{H}$ be the unique homomorphisms. Then by uniqueness $f \circ \tilde{f} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ and $\tilde{f} \circ f : \mathcal{H} \to \mathcal{H}$ have to be the identity homomorphism. So f and $\tilde{f}$ are isomorphisms.

Example 5.2: commutative diagram of resolutions
Let $\mathcal{G}^*$ and $\overline{\mathcal{G}}^*$ be two fine torsionless resolutions of the constant sheaf $\mathcal{K}$ with maps k^i? between them such that the following diagram commutes:



And let $\mathcal{H}$ be the cohomology theory induced by the first row as in Theorem 4 and $\tilde{\mathcal{H}}$ the theory induced by the second row. Using the same notation as in Example 5.1 we see that the above diagram implies that k induces a cochain map $(k^q \otimes \mathrm{Id}_{\mathcal{F}})_X : C^q_{\mathcal{F}} \to \tilde{C}^q_{\mathcal{F}}$. Now define $f^q_{\mathcal{F}} := (k^q \otimes \mathrm{Id}_{\mathcal{F}})_X$. Then f is a homomorphism between $\mathcal{H}$ and $\tilde{\mathcal{H}}$ and by the last Theorem it is the only one and it is an isomorphism as well. To proof f satisfies (a) of Definition 5.3 we consider the diagram:

$$\begin{array}{ccccc} & & h_{\mathcal{F}} & & \\ \mathcal{F}(X) & \xleftarrow{k_X} & (\mathcal{K}\otimes\mathcal{F})(X) & \xrightarrow{(g^0\otimes \mathrm{Id}_{\mathcal{F}})_X} & (\mathcal{L}_0\otimes\mathcal{F})(X) = H^0(X,\mathcal{F}) \\ \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} & & \downarrow f^0_{\mathcal{F}} \\ \mathcal{F}(X) & \xleftarrow{k_X} & (\mathcal{K}\otimes\mathcal{F})(X) & \xrightarrow{(\tilde{g}^0\otimes \mathrm{Id}_{\mathcal{F}})_X} & \left(\tilde{\mathcal{L}}_0\otimes\mathcal{F}\right)(X) = \tilde{H}^0(X,\mathcal{F}) \\ & & \tilde{h}_{\mathcal{F}} & & \end{array}$$

Here $f^0_{\mathcal{F}}$ is, on presheaf level, a map form $(\mathcal{L}_0\otimes\mathcal{F})(X)$ into $\left(\tilde{\mathcal{L}}_0\otimes\mathcal{F}\right)(X)$, since if it holds that $l\in\mathcal{L}_0$ we know that $\tilde{g}^1(k^0(l)) = k^1(g^1(l)) = k^1(0) = 0$, so that $k^0(l)\in\ker\tilde{g}^1 = \tilde{\mathcal{L}}_0$. When considering all the spaces on presheaf level and let $k\otimes f\in\mathcal{K}(X)\otimes\mathcal{F}(X)$ we get that:

$$f^0_{\mathcal{F}}\circ h^{-1}_{\mathcal{F}}\circ k_X(k\otimes f) = f^0_{\mathcal{F}}(g^0(k)\otimes f) = k^0(g^0(k))\otimes f = \tilde{g}^0(k)\otimes f = \tilde{h}^{-1}_{\mathcal{F}}\cdot k_X(k\otimes f).$$

Note that this implies that $\tilde{h}_{\mathcal{F}}\circ f^0_{\mathcal{F}} = h_{\mathcal{F}}$ and therefore (a) is satisfied. For (b) it is again enough to show that the following diagram commutes on presheaf level, bacause the wanted commutative diagram is then a consequence of the fact that this induces commutativity on sheaf level and then we use that all the used maps are cochain maps to conclude (b) holds.

$$\begin{array}{ccc} \mathrm{C}^q_{\mathcal{F}} & \xrightarrow{(\mathrm{Id}_{\mathcal{G}^q}\otimes h)_X} & \mathrm{C}^q_{\mathcal{F}'} \\ \downarrow (k^q\otimes\mathrm{Id}_{\mathcal{F}})_X & & \downarrow (k^q\otimes\mathrm{Id}_{\mathcal{F}'})_X \\ \tilde{\mathrm{C}}^q_{\mathcal{F}} & \xrightarrow[(\mathrm{Id}_{\tilde{\mathcal{G}}^q}\otimes h)_X]{} & \tilde{\mathrm{C}}^q_{\mathcal{F}'} \end{array}$$

That this diagram is commutative on presheaf level is quite clear, since then $C^q_{\mathcal{F}} = \mathcal{G}^q(X)\otimes\mathcal{F}(X)$. Finally we have to prove (c), given a short exact sequence $0\to\mathcal{F}'\overset{h}{\to}\mathcal{F}\overset{h'}{\to}\mathcal{F}''\to 0$, we construct the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}^*_{\mathcal{F}'} & \xrightarrow{(\mathrm{Id}\otimes h)_X} & \mathrm{C}^*_{\mathcal{F}} & \xrightarrow{(\mathrm{Id}\otimes h')_X} & \mathrm{C}^*_{\mathcal{F}''} & \longrightarrow & 0 \\ & & \downarrow (k^q\otimes\mathrm{Id}_{\mathcal{F}'})_X & & \downarrow (k^q\otimes\mathrm{Id}_{\mathcal{F}})_X & & \downarrow (k^q\otimes\mathrm{Id}_{\mathcal{F}''})_X & & \\ 0 & \longrightarrow & \tilde{\mathrm{C}}^*_{\mathcal{F}'} & \xrightarrow[(\mathrm{Id}\otimes h)_X]{} & \tilde{\mathrm{C}}^*_{\mathcal{F}} & \xrightarrow[(\mathrm{Id}\otimes h')_X]{} & \tilde{\mathrm{C}}^*_{\mathcal{F}''} & \longrightarrow & 0 \end{array}$$

Here the rows are exact, because of Theorem 2. Then Theorem 3 gives (c).

The final part in this set up of sheaf cohomology is to look closer at what $H^q(X,\mathcal{F})$ is for a given sheaf $\mathcal{F}$. Here we will use a fine resolution of $\mathcal{F}$.

Lemma 5.1: Let $\mathcal{H}$ be a cohomology theory for X with coefficients in sheaves of $K-$modules over X. Suppose we have a sheaf $\mathcal{F}$ and a fine resolution $\mathcal{G}^*$ of $\mathcal{F}$,

$$0\longrightarrow\mathcal{F}\xrightarrow{g^0}\mathcal{G}^0\xrightarrow{g^1}\mathcal{G}^1\xrightarrow{g^2}\mathcal{G}^2\longrightarrow\ldots \tag{8}$$

Then there exists isomorphism between $H^q(X,\mathcal{F})$ and $H^q(\mathcal{G}^*(X))$ for all q.
Proof: First of all Lemma 3.1 and the remark after it shows us that we have the exact sequence:

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{g_X^0} \mathcal{G}^0(X) \xrightarrow{g_X^1} \mathcal{G}^1(X)$$

In the cochain complex $\mathcal{G}^*(X)$ we have that $d^q = g_X^{q+1}$ for $q \geq 0$ and $d^{-1} : 0 \to \mathcal{G}^0(X)$ is just the trivial map and therefore (8) gives that:

$$H^0(\mathcal{G}^*(X)) = \ker(d^0)/\mathrm{Im}\,(d^{-1}) = \ker(g_X^1)/\{0\} = \mathrm{Im}\,(g_X^0) = \mathcal{F}(X).$$

So the zeroth isomorphism is just given by $h_{\mathcal{F}}$.
For $q \geq 1$ we let $\mathcal{L}_q := \ker(g^{q+1}) \subset \mathcal{G}^q$. This definition gives us the short exact sequence

$$0 \longrightarrow \mathcal{F}_q \xhookrightarrow{g^0} \mathcal{G}^0 \xrightarrow{g^1} \mathcal{L}_1 \longrightarrow 0$$

The exactness at $\mathcal{L}_1$ is because $\mathrm{Im}\,(g^1) = \ker(g^2) = \mathcal{L}_2$ from the exactness in (8). Now since $\mathcal{H}$ is a cohomology theory, it satisfies (b) and (c) of Definition 5.1. Combining this with the fact that $\mathcal{G}^0$ is a fine sheaf gives exactness in the following:

$$\ldots \longrightarrow 0 = H^q(X, \mathcal{G}^0) \xrightarrow{g^{1,q}} H^q(X, \mathcal{L}_1) \xrightarrow{\partial^q} H^{q+1}(X, \mathcal{F}) \xrightarrow{g^{0,q+1}} H^{q+1}(X, \mathcal{G}^0) = 0 \longrightarrow \ldots$$

This means that for each $q \geq 1$ we have the isomorphism $\partial^q : H^q(X, \mathcal{L}_1) \to H^{q+1}(X, \mathcal{F})$. When $q = 0$ we do not get that $H^0(X, \mathcal{G}^0) = 0$, but the long exact sequence does give us that ∂^0 is surjective and hence

$$\partial^0 : H^0(X, \mathcal{L}_1)/\ker(\partial^0) \to H^1(X, \mathcal{F})$$

is an isomorphism. Furthermore we know that $\ker(\partial^0) = \mathrm{Im}\,(g^{1,0})$. So we get:

$$\begin{aligned} H^1(X, \mathcal{F}) &\overset{\partial^0}{\simeq} H^0(X, \mathcal{L}_1)/\ker(\partial^0) = H^0(X, \mathcal{L}_1)/\mathrm{Im}\,(g^{1,0}) \overset{h_{\mathcal{L}_1}}{\simeq} \mathcal{L}_1(X)/\mathrm{Im}\,(g_X^1 \circ h_{\mathcal{G}^0}) \\ &= \mathcal{L}_1(X)/\mathrm{Im}\,(g_X^1) = \ker(g_X^2)/\mathrm{Im}\,(g_X^1) = H^1(\mathcal{G}^*(X)) \end{aligned}$$

Here we have to use (a) of Definition 5.1, with g^1 instead of h, for the second isomorphism. In the equality there after we use that $h_{\mathcal{G}^0}$ is an isomorphism as well. This gives the first isomorphism.
With the definition of $\mathcal{L}_q$ we also get the following short exact sequences, with $q \geq 1$:

$$0 \longrightarrow \mathcal{L}_q \xhookrightarrow{i} \mathcal{G}^q \xrightarrow{g^{q+1}} \mathcal{L}_{q+1} \longrightarrow 0$$

Again we get a long exact sequence of the cohomology theory:

$$\ldots \longrightarrow 0 = H^p(X, \mathcal{G}^q) \xrightarrow{g^{q+1,p}} H^p(X, \mathcal{L}_{q+1}) \xrightarrow{\partial_q^p} H^{p+1}(X, \mathcal{L}_q) \xrightarrow{g^{q+1,p+1}} H^{p+1}(X, \mathcal{G}^q) = 0 \longrightarrow \ldots$$

From this we conclude that $\partial_q^p : H^p(X, \mathcal{L}_{q+1}) \to H^{p+1}(X, \mathcal{L}_q)$ is an isomorphism. Again we do not have that $H^0(X, \mathcal{G}^q) = 0$, but that $\partial_q^0 : H^0(X, \mathcal{L}_{q+1})/\ker(\partial_q^0) \to H^1(X, \mathcal{L}_q)$ is an isomorphism. Using the same calculation as before:

$$\begin{aligned} H^1(X, \mathcal{L}_q) &\overset{\partial_q^0}{\simeq} H^0(X, \mathcal{L}_{q+1})/\ker(\partial_q^0) = H^0(X, \mathcal{L}_{q+1})/\mathrm{Im}\,(g^{q+1,0}) \overset{h_{\mathcal{L}_{q+1}}}{\simeq} \mathcal{L}_{q+1}(X)/\mathrm{Im}\,(g_X^{q+1} \circ h_{\mathcal{G}^q}) \\ &= \mathcal{L}_{q+1}(X)/\mathrm{Im}\,(g_X^{q+1}) = \ker(g_X^{q+2})/\mathrm{Im}\,(g_X^{q+1}) = H^{q+1}(\mathcal{G}^*(X)) \end{aligned}$$

Combining all the above isomorphisms shows the isomorphism for $q > 1$:

$$H^q(X, \mathcal{F}) \overset{(\partial^{q+1})^{-1}}{\simeq} H^{q-1}(X, \mathcal{L}_1) \overset{\partial_1^{q-2}}{\simeq} H^{q-2}(X, \mathcal{L}_2) \overset{\partial_2^{q-3}}{\simeq} \ldots \overset{\partial_{q-2}^1}{\simeq} H^1(X, \mathcal{L}_{q-1}) \simeq H^q(\mathcal{G}^*(X))$$

□

7 The classical cohomology theories

In this section we will discuss some cohomology theories, both classical and as sheaf cohomology theory. For each of the first three of them we will see that the classical version are isomorphic to the sheaf cohomology version. For the sheaf versions we already know that they all are isomorphic and therefore the conclusion of this section is that each of the first three classical theories are isomorphic. The fourth classical cohomology, will be shown directly that it is a cohomology theory as in Definition 5.1. Some of the cohomologies, like the de Rham cohomology, are only defined when X is a differentiable manifold, so we will assume from now on that this is the case. Every time we have to proof that some defined sheaf sequence is a fine torsionless resolution of the constant sheaf $\mathcal{K}$ and we will see that the ideas of the proofs are almost identical each time, although the detail can become quite technical.

7.1 Alexander-Spanier

First we will discuss the sheaf cohomology theory. To do this we will define a particular fine torsionless resolution of $\mathcal{K}$, which gives us cohomology as defined in chapter 5.

Definition 6.1.1: the p-th Alexander-Spanier sheaf $\mathcal{A}^p$

For each open $U \subset X$ and $p \geq 0$ we define $\mathrm{A}^p_B(U) := \{f : U^{p+1} \to B\}$, where $U^{p+1} = \overbrace{U \times ... \times U}^{\text{p+1 times}}$ and B is some $K-$module. We make $\mathrm{A}^p_B(U)$ into a $K-$module by using pointwise addition and $(k \cdot f)(x) := k \cdot f(x)$. Then A^p_B is a presheaf, when using the natural restriction of functions. The sheaf $\mathcal{A}^p_B$ associated to this presheaf is the called the *sheaf of Alexander-Spanier p-cochains with coefficients in B*.

Lemma 6.1.1: For each open $U \subset X$, $\mathrm{A}^*(U)_B$ is a cochain complex:

$$\mathrm{A}^*_B(U) = \qquad ... \longrightarrow 0 \longrightarrow \mathrm{A}^0_B(U) \xrightarrow{d^0} \mathrm{A}^1_B(U) \xrightarrow{d^1} \mathrm{A}^2_B(U) \xrightarrow{d^2} ...$$

Proof: Since A^p_B is a presheaf of $K-$modules for each p, we only need to define presheaf homomorphisms $d^p : \mathrm{A}^p_B \to \mathrm{A}^{p+1}_B$ such that $d^{p+1} \circ d^p = 0$ for each $p \geq 0$. So let $f \in \mathrm{A}^p_B(U)$, then we need that $d^p(f) : U^{p+2} \to B$, so we define:

$$(d^p(f))(x_0, ..., x_{p+1}) := \sum_{i=0}^{p+1} (-1)^i \cdot f(x_0, ..., x_{i-1}, x_{i+1}, ..., x_{p+1}).$$

From now on we will denote $f(x_0, ..., x_{i-1}, x_{i+1}, ..., x_{p+1})$ by $f(x_0, ..., \hat{x_i}, ..., x_{p+1})$. To check whether these d^p are homomorphisms we let $f_1, f_2 \in \mathrm{A}^p_B(U)$ and $k_1, k_2 \in K$. Then we have that:

$$\begin{aligned}(d^p(k_1 \cdot f_1 + k_2 \cdot f_2))(x_0, ..., x_{p+1}) &= \sum_{i=0}^{p+1} (-1)^i (k_1 \cdot f_1 + k_2 \cdot f_2)(x_0, ..., \hat{x_i}, ..., x_{p+1}) \\ &= k_1 \cdot \sum_{i=0}^{p+1} (-1)^i f_1(x_0, ..., \hat{x_i}, ..., x_{p+1}) \\ &+ k_2 \cdot \sum_{i=0}^{p+1} (-1)^i f_2(x_0, ..., \hat{x_i}, ..., x_{p+1}) \\ &= (k_1 \cdot d^p(f_1) + k_2 \cdot d^p(f_2))(x_0, ..., x_{p+1})\end{aligned}$$

They clearly commute with the restriction homomorphisms, so d^p is a presheaf homomorphism for each p. And more importantly we get that:

$$\begin{aligned}(d^{p+1}(d^p(f)))(x_0,...,x_{p+2}) &= \sum_{i=0}^{p+2}(-1)^i \cdot (d^p(f))(x_0,...,\hat{x_i},...,x_{p+2}) \\ &= \sum_{i=0}^{p+2}(-1)^i \sum_{j<i}(-1)^j \cdot f(x_0,...,\hat{x_j},...,\hat{x_i},...,x_{p+2}) \\ &+ \sum_{i=0}^{p+2}(-1)^i \sum_{j>i}(-1)^{j-1} \cdot f(x_0,...,\hat{x_i},...,\hat{x_j},...,x_{p+2}) \\ &= (1-1)\cdot \sum_{i=0}^{p+2}\sum_{j<i}(-1)^{i+j} \cdot f(x_0,...,\hat{x_j},...,\hat{x_i},...,x_{p+2}) = 0\end{aligned}$$

□

The homomorphisms d^p induce sheaf homomorphisms $d^p : \mathcal{A}_B^p \to \mathcal{A}_B^{p+1}$. The following lemma proves that these sheaves and sheaf homomorphisms give a fine resolution of $\mathcal{B}$, the sheaf induced by $X \times B$. The second lemma proves that the resolution is torsionless as well, when $B = K$. A consequence of these lemmas is therefore the existence of cohomology theories, when we use Theorem 4.

Lemma 6.1.2: $\mathcal{A}_B^*$ forms a fine resolution of the constant sheaf $\mathcal{B}$.
Proof: To proof this lemma we need to show that each $\mathcal{A}_B^p$ is fine and that

$$0 \longrightarrow \mathcal{B} \xrightarrow{i} \mathcal{A}_B^0 \xrightarrow{d^0} \mathcal{A}_B^1 \xrightarrow{d^1} \mathcal{A}_B^2 \xrightarrow{d^2} ...$$

is exact.
We will start by showing that $\mathcal{A}_B^p$ is a fine sheaf. Since X is a paracompact Hausdorff space (this is all we need of X) we have a partition of unity $\{\psi_i\}$ subordinated to the cover locally finite cover $\{U_i\}$, for each such cover. We take the partition of unity as in Example 3.1, with a shrinking $\{V_i\}$ of $\{U_i\}$, and we use this partition to define endomorphisms $\tilde{l^i} : \mathrm{A}^p \to \mathrm{A}^p$ by setting:

$$(\tilde{l}_U^i(f))(x_0,...,x_p) := \psi_i(x_0)\cdot f(x_0,...,x_p).$$

We have to check whether these maps are really presheaf endomorphisms. First of all we note that it is clear that they commute with the restriction homomorphisms. Secondly we see that

$$\begin{aligned}(\tilde{l}_U^i(k_1\cdot f_1 + k_2 \cdot f_2))(x_0,...,x_p) &= \psi_i(x_0)(k_1\cdot f_1 + k_2\cdot f_2)(x_0,...,x_p) \\ &= k_1\cdot\psi_i(x_0)f_1(x_0,...,x_p) + k_2\cdot\psi_i(x_0)f_2(x-0,...,x_p) \\ &= k_1\cdot(\tilde{l}_U^i(f_1))(x_0,...,x_p) + k_2\cdot(\tilde{l}_U^i(f_2))(x_0,...,x_p) \\ &= (k_1\cdot(\tilde{l}_U^i(f_1)) + k_2\cdot(\tilde{l}_U^i(f_2)))(x_0,...,x_p),\end{aligned}$$

so $\tilde{l}_U^i$ is indeed an endomorphism. They induce sheaf endomorphisms $l^i : \mathcal{A}_B^p \to \mathcal{A}_B^p$. Now let $s \in \mathcal{A}_B^p(U)$ and $x \in U$. Then $s(x) \in A_{B,x}^p$ implies that there exists some open V and $f \in A_B^p(V)$ such that $x \in V$ and $s(x) = r_x^V(f)$. Furthermore for $(x_0,...,x_p) \in V^{p+1}$ we have that:

$$f(x_0,...,x_p) = \sum_i \psi_i(x_0)f(x_0,...,x_p) = \sum_i(\tilde{l}_V^i(f))(x_0,...,x_p) = \left(\sum_i \tilde{l}_V^i(f)\right)(x_0,...,x_p),$$

Where the first equality holds since for any $x_0 \in V$ there is exactly one i such that $\psi_i(x_0) \neq 0$. Using this we get that:

$$s(x) = r_x^V(f) = r_x^V(\sum_i \tilde{l}_V^i(f)) = \sum_i r_x^V(\tilde{l}_V^i(f)) = \sum_i (l_U^i(s))(x) = \left(\sum_i l_U^i(s)\right)(x),$$

so the endomorphisms l^i satisfy I of Definition 3.2. Secondly of $x \notin \overline{V_i}$, then we find an open W around x such that $W \subset X \setminus \overline{V_i}$, since the last one is open. Now any open U around x, $f \in \mathrm{A}_B^p(U)$ and $(x_0, ..., x_p) \in (W \cap U)^{p+1}$ the following holds:

$$(r_{W\cap U}^U(\tilde{l}_U^i(f)))(x_0, ..., x_p) = \psi_i(x_0) \cdot f(x_0, ..., x_p) = 0.$$

Here we use that $x_0 \in W$ implies that $x_0 \notin V_i$ and therefore it has to be true that $\psi_i(x_0) = 0$. The above result implies that $0 = r_x^U(\tilde{l}_U^i(f)) = l_x^i(r_x^U(f))$. Since U and f were arbitrary we are allowed to conclude that $l_x^i \equiv 0$. Therefore $\overline{\{x \in X | l_x^i \not\equiv 0\}} \subset \overline{V_i} \subset U_i$, which is exactly condition II of Definition 3.2. Therefore the sheaves are fine.

Finally we will show that the sequence is exact. This is a consequence of the exactness at presheaf level and consists of three parts. Starting with the exactness at $\mathcal{B}_x$, we first look at the image of i_x. Since $\mathcal{B}_x = B$ we need to check what the image of $b \in B$ is under i_x. Since $i_U(b) = f_b$, with $f_b(x) = b$ for all $x \in U$, we get that $i_x(b) = r_x^U(f_b)$. Hence an element in $\mathrm{Im}\,(i_x)$ is always of the form $r_x^U(f_b)$. Now suppose that $i_x(b) = 0$, so that $b \in \ker(i_x)$. This implies that $r_x^U(f_b) = 0$, so we can find a $W \subset U$ such that $f_b(x) = 0$ for all $x \in W$. So $b = 0$ and this gives that $\ker(i_x) \subset \{0\}$. The other inclusion is trivial, so we have exactness at $\mathcal{B}_x$.
Secondly, the sequence is exact at $\mathcal{A}_{B,x}^0$. Let f_b be as above, then we have that $(d^0(f_b))(x_0, x_1) = f_b(x_1) - f_b(x_0) = b - b = 0$. Using this we get that

$$d_x^0(r_x^U(f_b)) = r_x^U(d^0(f_b)) = r_x^U(0) = 0.$$

From which we conclude that $\mathrm{Im}\,(i_x) \subset \ker(d_x^0)$. Now supposing that $0 = d_x^0(r_x^U(f))$ implies that $0 = r_x^U(d^0(f))$ and hence we find a $W \subset U$ such that $0 = r_W^U(d^0(f))$. This means that for all $(x_0, x_1) \in W \times W$: $0 = (d^0(f))(x_0, x_1) = f(x_1) - f(x_0)$. So f is constant to some $b \in B$ on W and $r_x^W(f)$ is therefore the image of b under i_x. We conclude that $\ker(d_x^0) \subset \mathrm{Im}\,(i_x)$ which implies equality, so the sequence is exact at $\mathcal{A}_{B,x}^0$ as well.
The final part is the exactness at $\mathcal{A}_{B,x}^p, p \geq 1$. Suppose that $d_x^{p-1}(r_x^U(f)) \in \mathrm{Im}\,(d_x^{p-1})$. Then

$$d_x^p(d_x^{p-1}(r_x^U(f))) = d_x^p(r_x^U(d^{p-1}(f))) = r_x^U(d^p \circ d^{p-1}(f)) = r_x^U(0) = 0.$$

Note that we switch to the presheaf morphism, from which we know that $d^p \circ d^{p-1} = 0$ for each p. From this we conclude that $\mathrm{Im}\,(d_x^{p-1}) \subset \ker(d_x^p)$. To show equality we first look at presheaf level again. Suppose $f \in \mathrm{A}_B^p(W)$ such that $d^p(f) = 0$ and let $(x, x_0, ..., x_p) \in W^{p+2}$. Then we have that:

$$0 = (df)(x, x_0, ..., x_{p+1}) = f(x_0, ..., x_p) + \sum_{i=0}^{p} (-1)^{i+1} f(x, x_0, ..., \hat{x_i}, ..., x_p).$$

Now define $g \in \mathrm{A}_B^{p-1}(W)$ such that $g(x_0, ..., x_{p-1}) = f(x, x_0, ..., x_{p-1})$ for a certain chosen $x \in U$. Now the following calculation shows that $dg = f$, where we need the result of $df = 0$ in the final equal-sign:

$$\begin{aligned}(dg)(x_0, ..., x_p) &= \sum_{i=0}^{p} (-1)^i g(x_0, ..., \hat{x_i}, ..., x_p) = \sum_{i=0}^{p} f(x, x_0, ..., \hat{x_i}, ..., x_p) \\ &= f(x_0, ..., x_p).\end{aligned}$$

Now on sheaf level we suppose that $0 = d^p_x(r^U_x(f))$. Since $d^p_x(r^U_x(f)) = r^U_x(d^p(f))$ we find a $W \subset U$ such that $0 = r^U_W(d^p(f)) = d^p(r^U_W(f))$. Using the above we find a $g \in \mathrm{A}^{p-1}_B(W)$ such that $d^{p-1}g = r^U_W(f)$, which implies that $r^U_x(f) = r^W_x(r^U_W(f)) = r^W_x(d^{p-1}g) = d^{p-1}_x(r^W_x(g))$. And therefore we have that $\ker(d^q_x) \subset \mathrm{Im}\,(d^{q-1}_x)$ and hence equality. We conclude that the whole sequence is exact. □

Lemma 6.1.3: The above fine resolution of $\mathcal{K}$ is torsionless, when $B = K$.
Proof: We will show that for $p \geq 0$ and U an open subset of X, $\mathcal{A}^p_K(U)$ is torsionless so that by Lemma 3.6 each $\mathcal{A}^p_K$ are torsionless sheaves. Let $k \in K \setminus \{0\}$ and $s \in \mathcal{A}^p_K(U)$ such that $k \cdot s = 0$. This implies that for each $x \in U$ $k \cdot s(x) = 0$. By definition of $\mathcal{A}^p_K$ we know that $s(x) \in A_{K,x}$ for all x. So for each x there exists an open V_x and a $f_x \in A^p_K(V_x)$ such that $x \in V_x$ and $s(x) = r^{V_x}_x(f_x)$. Now $k \cdot s(x) = 0$ implies that $r^{V_x}_x(k \cdot f) = 0$, so there exists an open $W_x \subset V_x$ such that $x \in W_x$ and for each $y \in W_x$ we have that $k \cdot f_x(y) = 0$. Since K is a integral domain and $k \neq 0$ we conclude that $r^{V_x}_{W_x}(f) = 0$, which implies that $s(x) = 0$. We have shown this for any $x \in U$ and therefore $s = 0$ and $\mathcal{A}^p_K(U)$ is torsionless. □

The final part about Alexander-Spanier cohomology is defining the classical cohomology modules and then proving that they are isomorphis to the sheaf-version, as defined above.

Definition 6.1.2: classical Alexander-Spanier cohomology modules
Let $\mathrm{A}^p_{B,0} := \{f \in A^p_B(X) | r^X_x(f) = 0\}$. Then the *classical Alexander-Spanier cohomology modules for X with coefficients in B* are defined as:

$$H^q_{A-S}(X; G) := H^q(\mathrm{A}^*_B(X) \diagup \mathrm{A}^*_{B,0}).$$

Lemma 6.1.4: The above definition is well defined.
Proof: There are two thing to check. First of all we need that $\mathrm{A}^p_{B,0}$ is a submodule of $\mathrm{A}^p_B(X)$ for each p such that the quotient $\mathrm{A}^p_B(X) \diagup \mathrm{A}^*_{B,0}$ is well defined. Secondly we need that $\mathrm{A}^*_B(X) \diagup \mathrm{A}^*_{B,0}$ form a cochain complex, so the cohomology modules are defined as in Definition 4.1.
We will start with the submodule part, which is just two calculations. Let $f, g \in \mathrm{A}^p_{B,0}$ and $k \in K$. Then for all $x \in X$

$$r^X_x(f + g) = r^X_x(f) + r^X_x(g) = 0 + 0 = 0,$$

$$r^X_x(k \cdot f) = k \cdot r^X_x(f) = k \cdot 0 = 0,$$

which imply that $f + g, k \cdot f \in \mathrm{A}^p_{B,0}$. So $\mathrm{A}^p_{B,0}$ is indeed a submodule.
For the second part we look at the chain

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{B} \xrightarrow{d^0} \mathrm{A}^0_B(X) \diagup \mathrm{A}^0_{B,0} \xrightarrow{d^1} \mathrm{A}^1_B(X) \diagup \mathrm{A}^1_{B,0} \xrightarrow{d^2} \mathrm{A}^2_B(X) \diagup \mathrm{A}^2_{B,0} \longrightarrow \ldots$$

Here the maps d^p are the maps on the quotients, induced by the usual maps d^p of lemma 6.1.1. These map exist when $d^p(f) \in \mathrm{A}^{p+1}_{B,0}$ for each $f \in \mathrm{A}^p_{B,0}$. So let $f \in \mathrm{A}^p_{B,0}$ and $x \in X$. Then we have that $r^X_x(f) = 0$, so we find a $W \subset X$ such that $x \in W$ and $r^X_W(f) = 0$. Since the restriction homomorphisms are the natural restriction maps of functions we get that this implies that $f(x_0, ..., x_p) = 0$ for each $(x_0, ..., x_p) \in W^{p+1}$. Now we let $(x_0, ..., x_{p+1}) \in W^{p+2}$. Then:

$$(d^p(f))(x_0, ..., x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \cdot f(x_0, ..., \hat{x_i}, ..., x_p) = \sum_{i=0}^{p+1} (-1)^i \cdot 0 = 0,$$

where we use that $(x_0, ..., \hat{x_i}, ..., x_p) \in W^{p+1}$. From this we conclude that $r^X_W(d(f)) = 0$ and so $r^X_x(d(f)) = 0$. Since we can make this computation for all $x \in X$, we get that $d^p(f) \in \mathrm{A}^{p+1}_{B,0}$ holds.

Finally we check whether $d^{p+1} \circ d^p$ still holds, so let $[f] \in \mathrm{A}^p_{B,0}$. Then:

$$d^{p+1} \circ d^p([f]) = d^{p+1}([d^p(f)]) = [d^{p+1} \circ d^p(f)] = [0] = 0.$$

Clearly this is enough to show that Definition 6.1.2 is well-defined. □

Before we can get to the result of Alexander-Spanier cohomology we need one more (technical) lemma.

Lemma 6.1.5: Let $\mathcal{F}$ be a presheaf of $K-$modules on X, which satisfies (S_2). Denote the associated sheaf of $\mathcal{F}$ by $\Gamma_{\mathcal{F}}$ and let $\mathcal{F}_0$ be defined as:

$$\mathcal{F}_0 := \{f \in \mathcal{F}(X) | r^X_x(f) = 0 \, \forall x \in X\}.$$

Furthermore: let $g : \mathcal{F}(X) \to \mathcal{F}'(X)$ be the map that sends $f \in \mathcal{F}(X)$ into s_f such that $s_f(x) = r^X_x(f)$ for all $x \in X$. Then the following is a short exact sequence:

$$0 \longrightarrow \mathcal{F}_0 \overset{i}{\hookrightarrow} \mathcal{F}(X) \overset{g}{\longrightarrow} \Gamma_{\mathcal{F}}(X) \longrightarrow 0$$

Proof: Exactness at $\mathcal{F}_0$ is clear, since $\mathcal{F}_0$ is trivially injected into $\mathcal{F}(X)$. Now g is defined as h_X as in Theorem 1; so we have already seen that this definition makes g into a continuous map. Since $(g(f))(x) = r^X_x(f)$, it is obvious that $f \in \mathcal{F}_0$ if and only of $g(f) = 0$. So exactness at $\mathcal{F}(X)$ holds as well. Finally we will show that the map g in surjective.
Let $t \in \Gamma_{\mathcal{F}}(X)$ and $x \in X$. Then there exists some open U and $f_x \in \mathcal{F}(U)$ such that $r^U_x(f_x) = t(x)$. By definition of the opens in $\bar{\mathcal{F}}$ we know that $\mathrm{O}_{f_x} = \{r^Y_y(f_x) | y \in U\}$ is an open. Furthermore we know that t is continuous, so $V := t^{-1}(\mathrm{O}_{f_x})$ is open as well. For any $y \in V$ we have that $t(y) \in \mathrm{O}_{f_x}$ and therefore $t(y) = r^U_y(f_x)$. Now let $U_x = U \cap V$. With this we compute for $y \in U_x$ that

$$\Big(g\big(r^U_{U_x}(f_x)\big)\Big)(y) = s_{r^U_{U_x}(f_x)}(y) = r^{U_x}_y\big(r^U_{U_x}(f_x)\big) = r^U_y(f_x) = t(y),$$

so that on U_x $t = g\big(r^U_{U_x}(f_x)\big)$. We now have an open cover of X, since $x \in U_x$ for each x. By paracompactness of X are we able to take a locally finite refinement $\{U_i\}$ of this cover. Then it still holds that on U_i: $t = g\big(f_i\big)$, with f_i the restriction of some f_x to U_x. Since X is Hausdorff as well we can take a shrinking $\{V_i\}$ of $\{U_i\}$, so $\{V_i\}$ is a refinement and $\overline{V_i} \subset U_i$ for all i. Let $x \in X$ and denote $I_x := \{i | x \in \overline{V_j}\}$. This is a finite set, since $\{V_i\}$ is locally finite. Now pick an open W_x for each x such that all W_x satisfy these 3 conditions:

(A) $W_x \cap \overline{V_i} = \emptyset$ for all $i \notin I_x$;

(B) W_x lies in the intersection of all the U_i such that $i \in I_x$;

(C) $r^{U_i}_{W_x}\big(f_i\big) = r^{U_j}_{W_x}\big(f_j\big)$ whenever i and j are both elements of I_x .

We are actually able to construct such an open W_x for each x. To do this we use the locally finiteness of $\{U_i\}$ and start with an open neighborhood Z of x such that $Z \cap U_i$ is only finitely many times non-empty. So there are only finitely many i such that $Z \cap U_i \neq \emptyset$ while $i \notin I_x$. Denote this set by J_x and let $Z' := Z \setminus \cup_{i \in J_x} \overline{V_i}$. Note that this is still an open neighborhood of x, since we remove only finitely many closed subsets of Z. Now Z' satisfies (A). Furthermore when we let $Z'' := \cap_{i \in I_x} U_i \cap Z'$, then Z'' is again an open neighborhood of x, and satisfies (A) and (B). Finally, since I_x is finite, we have only finitely many pairs of $(i, j) \in I_x \times I_x$. We will shrink Z'' even further such that it satisfies (C). Note that for $(i, j) \in I_x \times I_x$ we have that:

$$r^{Z''}_x\big(r^{U_i}_{Z''}(f_i)\big) = r^{U_i}_x(f_i) = \big(g(t_i)\big)(x) = t(x) = \big(g(t_j)\big)(x) = r^{U_j}_x(f_j) = r^{Z''}_x\big(r^{U_j}_{Z''}(f_j)\big).$$

This means there exists some Wi, j, which is an open neighborhood of x and a subset of Z'' such that $r^{U_i}_{W_{i,j}}(f_i) = r^{U_j}_{W_{i,j}}(f_j)$. The intersection of all these opens $W_{i,j}$ is a nonempty open neighborhood of x and satisfies (A), (B) and (C). So this is our W_x.
Using (C) we let $f^x = r^{U_i}_{W_x}(f_i) \in \mathcal{F}(W_x)$ for any $i \in_X$. For $x, y \in X$ such that $W_{x,y} := W_x \cap W_y \neq \emptyset$ there exists a $z \in W_{x,y}$. Since $\{V_i\}$ is a cover of X there must exist some i such that $z \in \overline{V_i}$. (A) clearly implies that $i \in I_x \cap I_x$, since $W_x \cap \overline{V_i} \neq \emptyset$ and $W_y \cap \overline{V_i} \neq \emptyset$. Using this i gives that

$$r^{W_x}_{W_{x,y}}(f^x) = r^{W_x}_{W_{x,y}}\left(r^{U_i}_{W_x}(f_i)\right) = r^{U_i}_{W_{x,y}}(f_i) = r^{W_y}_{W_{x,y}}\left(r^{U_i}_{W_y}(f_i)\right) = r^{W_y}_{W_{x,y}}(f^y)$$

Since $\mathcal{F}$ satisfies (S_2) we are able to conclude that there exists a $f \in \mathcal{F}(X)$ such that $r^X_{W_x}(f) = f^x$ for all x. Finally we get that for all x and some $i \in I_x$

$$(g(f))(x) = r^X_x(f) = r^{W_x}_x\left(r^X_{W_x}(f)\right) = r^{W_x}_x(f^x) = r^{W_x}_x\left(r^{U_i}_{W_x}(f_i)\right) = r^{U_i}_x(f_i) = (g(f_i))(x) = t(x),$$

that is to say, g is surjective. □

Theorem 6.1: The classical Alexander-Spanier cohomology modules with coefficients in B, $H^q_{A-S}(X; B)$, are isomorphic to the cohomology modules of the cohomology theory induced by the p-cochain sheaves of Alexander-Spanier.
Proof: First we will proof that the presheaf A^p_B satisfies condition (S_2) of Definition 1.2. So let $U = \cup U_i$ and $f_i : U_i^{p+1} \to B$ such that if $\vec{x} := (x_0, ..., x_p) \in (U_i \cap U_j)^{p+1}$ for some $i \neq j$, then $f_i(\vec{x}) = f_j(\vec{x})$. Now define $f : U^{p+1} \to B$ as $f(\vec{x}) := f_i(\vec{x})$ when $x_k \in U_i$ for all $0 \leq k \leq p$ and $f(\vec{x}) = 0$ if such a i does not exist. This is clearly well defined, since if there exist i and j such that $x_k \in U_i \cap U_j$ then by assumption $f_i(\vec{x}) = f_j(\vec{x})$. We conclude that A^p_B satisfies (S_2).
Lemma 6.1.5 now implies that the following is exact:

$$0 \longrightarrow \mathrm{A}^p_{B,0} \overset{i}{\hookrightarrow} \mathrm{A}^p_B(X) \xrightarrow{\alpha^p} \mathcal{A}^p_B(X) \longrightarrow 0$$

Then $\beta^p : \mathrm{A}^p_B(X) / \mathrm{A}^p_{B,0} \to \mathcal{A}^p_B(X); [f] \mapsto \alpha^p(f)$ is not only surjective, but injective as well. So it is an isomorphism. Furthermore $\beta = \{\beta^p\}$ is a cochain map, because for $[f] \in \mathrm{A}^p(X) / \mathrm{A}^p_{B,0}$:

$$\begin{aligned}(d^p(\beta^p([f])))(x) &= (d^p(\alpha^p(f)))(x) = (d^p(s_f))(x) = r^X_x(d^p(f)) = (\alpha^{p+1}(d^p(f)))(x) \\ &= (\beta^{p+1}([d^p(f)]))(x) = (\beta^{p+1}(d^p([f])))(x),\end{aligned}$$

so the following diagram commutes.

$$\begin{array}{ccc} \mathrm{A}^{p+1}(X) / \mathrm{A}^{p+1}_{B,0} & \xrightarrow{\beta^{p+1}} & \mathcal{A}^{p+1}_B \\ \uparrow d^q & & \uparrow d^q \\ \mathrm{A}^p(X) / \mathrm{A}^p_{B,0} & \xrightarrow{\beta^p} & \mathcal{A}^p_B \end{array}$$

Therefore $\mathrm{A}^*_B(X) / \mathrm{A}^*_{B,0}$ is isomorphic to $\mathcal{A}^*_B(X)$, which implies that $H^q(\mathrm{A}^*_B(X) / \mathrm{A}^*_{B,0}) \simeq H^q(\mathcal{A}^*_B(X))$ for each q. Lemma 5.1 and Lemma 6.1.2 together show that $H^q(X, \mathcal{B})$ is isomorphic to $H^q(\mathcal{A}^*_B(X))$. We conclude that:

$$H^q_{A-S}(X; B) = H^q(\mathrm{A}^*_B(X) / \mathrm{A}^*_{B,0}) \simeq H^q(\mathcal{A}^*_B(X)) \simeq H^q(X, \mathcal{B})$$

□

7.2 De Rham

The set-up of this section is the same as the section before: we first define a fine torsionless resolution, from which we abstract a cohomology theory, and then we compare it with the classical de Rham cohomology modules. This time however we pick K to be equal to $\mathbb{R}$ and here it is important that X is a differentiable manifold. Otherwise the classical version is not defined and the sheaves we will use will also not be defined.

Definitie 6.2.1: the p-th de Rham sheaf Ω^p
Like in Example 1.4 we let $\Omega^p(U)$ to be equal to the module of $p-$forms over the open U.

Lemma 6.2.1: Ω^* is a fine torsionless resolution of the constant sheaf $\mathcal{R}$, where the exterior derivative d^p is the map between Ω^p and Ω^{p+1}.
Proof: First note that d^p is a sheaf homomorphism, which is clear in local coordinates. We need to show that:

$$0 \longrightarrow \mathcal{R} \xrightarrow{\ i\ } \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \xrightarrow{d^2} ...$$

is exact, that each $\Omega^p(U)$ is torsionless and that each Ω^p is fine.
We start with the exactness at $\mathcal{R}_x = \mathbb{R}$. Since $\Omega^0(U) \simeq \{f : U \to \mathbb{R}\}$ we get that for each $r \in \mathbb{R}$ there exists a $f_r \in \Omega^0(U)$ such that $f_r(x) = r$ for all $x \in U$. So the natural injection i makes sure that i_x sends r into $r_x^X(f_r)$. Supposing that $i_x(r) = 0$ means that we find an open $W \subset X$ such that $x \in W$ and $f_r(x) = 0$ for all $x \in W$. We conclude that $r = 0$ and therefore we have exactness at $\mathcal{R}_x$.
For exactness at Ω_x^0 we use that $d^0(f_r) = 0$, since f_r is just the constant map. Since $r_x^U(f) \in \mathrm{Im}\,(i_x)$ means that $r_x^U(f) = r_x^X(f_r)$ for some $r \in \mathbb{R}$ we now have that it lies in the kernel of d_x^0 as well, since $d_x^0(r_x^X(f)) = r_x^X(d^0(f_r)) = r_x^X(0) = 0$ holds. So: $\mathrm{Im}\,(i_x) \subset \ker(d_x^0)$. Moreover, when $d^0(f) = 0$ for some $f : X \to \mathbb{R}$, we know that f has to be constant. So $0 = d_x^0(r_x^U(f))$ implies that $r_x^U(d^0(f)) = 0$ and hence there must exist some $W \subset U$ such that $x \in W$ and $0 = r_W^U(d^0(f)) = d^0(r_W^X(f))$. Now we can conclude that $r_W^U(f) = f_r$ for some $r \in \mathbb{R}$ and this implies that $r_x^U(f) = r_x^W(r_W^U(f)) = r_x^W(f_r) \in \mathrm{Im}\,(i_x)$. Therefore the sequence is exact at Ω_x^0.
For $p \geq 1$ we look at

$$\Omega_x^{p-1} \xrightarrow{d_x^{p-1}} \Omega_x^p \xrightarrow{d_x^p} \Omega_x^{p+1}$$

The first inclusion is trivial:

$$d_x^p(d_x^{p-1}(r_x^U(\alpha))) = d_x^p(r_x^U(d^{p-1}(\alpha))) = r_x^U(d^p \circ d^{p-1}(\alpha)) = r_x^U(0) = 0,$$

for all U and $p-1$ forms α over U. This calculation implies that $\mathrm{Im}\,(d_x^{p-1}) \subset \ker(d_x^p)$ for all $p \geq 1$ and $x \in X$. Next assume that $r_x^U(\alpha) \in \ker(d_x^p)$. Then we know that $0 = d_x^p(r_x^U(\alpha)) = r_x^U(d^p(\alpha))$, which means that there exists an open $W \subset U$ such that $0 = r_W^U(d^p(\alpha)) = d^p(r_W^U(\alpha))$. We can assume that W is homeomorphic to an open ball in $\mathbb{R}^n$ for some n, since X is a differentiable manifold. If this is not the case, we shrink W such that it lies completely in one of the charts of X. Using Poincaré lemma we get that $r_W^U(\alpha) = d^{p-1}(\beta)$ for some $\beta \in \Omega^{p-1}(W)$. So we get that:

$$r_x^U(\alpha) = r_x^W(r_W^U(\alpha)) = r_x^W(d^{p-1}(\beta)) = d_x^{p-1}(r_x^W(\beta)) \in \mathrm{Im}\,(d_x^{p-1}).$$

From this we can conclude that the sequence is exact at Ω^p for $p \geq 1$, so it is a resolution.

To prove that $\Omega^p(U)$ is torsionless, we pick any $\alpha \in \Omega^p(U)$ and $r \in \mathbb{R} \setminus \{0\}$ and we assume that $r \cdot \alpha = 0$. Since α is a $p-$form, this implies that $0 = r \cdot \alpha(x) \in \bigwedge^p T_x^*(U)$ for each x. Using that $T_x^*(U)$ is a vectorspace over $\mathbb{R}$ and that $r \neq 0$ we get that $\alpha(x) = 0$ for each x. Therefore $\alpha = 0$ and $\Omega^p(U)$ is torsionless and thus is Ω^p a torsionless sheaf for each p, by Lemma

3.6.

Finally we will show that Ω^p is fine for each p. Let $\{U_i\}$ be a locally finite cover of X. Since X is a differentiable manifold, we can pick a partition of unity subordinated to it. For this partition ψ_i, we define $l^i_U : \Omega^p(U) \to \Omega^p(U)$ in the following way: for $\alpha \in \Omega^p(U)$ we let $l^i_U(\alpha)$ to be the section which sends $x \in U$ to $\psi_i(x) \cdot \alpha(x)$. We still have to proof that these maps form a partition of unity for Ω^p. First of all l^i_U is a endomorphism of $\Omega^p(U)$, since for $\alpha_1, \alpha_2 \in \Omega^p(U)$ and $r_1, r_2 \in \mathbb{R}$ we have:

$$\begin{aligned}(l^i_U(r_1 \cdot \alpha_1 + r_2 \cdot \alpha_2))(x) &= \psi_i(x) \cdot (r_1 \cdot \alpha_1 + r_2 \cdot \alpha_2)(x) = r_1 \cdot \psi_i(x) \cdot \alpha_1(x) + r_2 \cdot \psi_i(x) \cdot \alpha_2(x) \\ &= r_1 \cdot l^i_U(\alpha_1)(x) + r_2 \cdot l^i_U(\alpha_2)(x) = (r_1 \cdot l^i_U(\alpha_1) + r_2 \cdot l^i_U(\alpha_2))(x).\end{aligned}$$

To see that l^i is really an endomorphism of the sheaf Ω^p, we need that the following is commutative for any open U, V such that $V \subset U$:

$$\begin{array}{ccc} \Omega^p(U) & \xrightarrow{l^i_U} & \Omega^p(V) \\ \big\downarrow r^U_V & & \big\downarrow r^U_V \\ \Omega^p(V) & \xrightarrow{l^i_V} & \Omega^p(V) \end{array}$$

This is however obvious, since the restriction maps are the normal restriction of maps on X. Now let $\alpha \in \Omega^p(U)$ for some open U. Then for $x \in U$:

$$\left(\sum_i l^i_U(\alpha)\right)(x) = \sum_i \left(l^i_U(\alpha)\right)(x) = \sum_i \psi_i(x) \cdot \alpha(x) = \alpha(x) \cdot \sum_i \psi_i(x) = \alpha(x),$$

where we need to use that only a finite number of $\psi_i(x)$ is not zero, which is a consequence of $\{U_i\}$ being locally finite. We conclude that l^i_U satisfies the first condition of Definition 3.2.
To prove the second condition we let $x \notin \text{support}(\psi_i)$. Since $\text{support}(\psi_i)$ is closed, we can find a open V around x such that $V \subset X \setminus \text{support}(\psi_i)$. For any $\alpha \in \Omega^p(U)$, with U an open containing x, and $y \in U \cap V$ we have that $(l^i_{U\cap V}(r^U_{U\cap V}(\alpha)))(y) = \psi_i(y) \cdot \alpha(y) = 0 \cdot \alpha(y) = 0$. This implies that $0 = l^i_{U\cap V}(r^U_{U\cap V}(\alpha)) = r^U_{U\cap V}(l^i_U(\alpha))$, so clearly $l^i_x(r^U_x(\alpha)) = 0$ and therefore that:

$$\overline{\{x \in X | l^i_x \not\equiv 0\}} \subset \text{support}(\psi_i) \subset U_i.$$

This allows us to conclude that l^i_U is indeed a partition of unity of Ω^p subordinated to $\{U_i\}$, and thus is Ω^p a fine sheaf. □

Since any fine torsionless resolution of $\mathcal{R}$ defines a cohomology theory we now have a cohomology theory $H^q(X, \mathcal{F}) = H^q((\Omega^* \otimes \mathcal{F})(X))$ over X with coefficients in sheaves or real vector spaces $\mathcal{F}$. Now we will define the classical de Rham cohomology groups to show that we have an isomorphism between the sheaf version and the classical one.

Definition 6.2.2: classical de Rham cohomology modules
Let Ω^p be the p-th de Rham sheaf. Then the *classical de Rham cohomology modules for X* are defined as:

$$H^q_{deR}(X) := H^q(\Omega^*(X)).$$

Remark: Note that this defintion is allowed, since we already know that $d^{p+1} \circ d^p = 0$.

Theorem 6.2: The classical de Rham cohomology modules H^q_{deR} are isomorphic to the cohomology modules of the cohomology theory induced by the p-th de Rham sheaves with coefficients in $\mathcal{R}$.

Proof: First of all we have that X is connected, so $\mathcal{R}(X) = \mathbb{R}$. This implies that $\phi^p : \Omega^p(X) \otimes \mathcal{R}(X) \to \Omega^p(X); \alpha \otimes r \mapsto r \cdot \alpha$ is well defined. Letting $r = 1$ proves that this homomorphism is surjective. Now suppose $\sum_i \alpha_i \otimes r_i$ is in the kernel of ϕ^p, then $0 = \sum_i r_i \cdot \alpha_i$. Since $\sum_i \alpha_i \otimes r_i = \sum_i r_i \cdot \alpha_i \otimes 1 = 0 \otimes 1 = 0$, we conclude that ϕ is injective as well. Now the induced sheaf map ϕ is an isomorphism as well. ϕ commutes with the coboundary operators, since again on presheaf level we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^{p-1}(X) \otimes \mathbb{R} & \xrightarrow{\phi^{p-1}} & \Omega^{p-1}(X) \\ \downarrow d^{p-1} \otimes \mathrm{Id} & & \downarrow d^{p-1} \\ \Omega^{p}(X) \otimes \mathbb{R} & \xrightarrow{\phi^{p}} & \Omega^{p}(X) \end{array}$$

The commutativity is because of this calculation:

$$\phi^p(d^{p-1} \otimes \mathrm{Id}(\alpha \otimes r)) = \phi^p(d^{p-1}(\alpha) \otimes r) = r \cdot d^{p-1}(\alpha) = d^{p-1}(r \cdot \alpha) = d^{p-1}(\phi(\alpha \otimes r)).$$

This commutativity on presheaf level implies the commutativity on sheaf level. Using this, we get that

$$\overline{\phi} q : H^q((\Omega^* \otimes \mathcal{R})(X)) \to H^q((\Omega^*)(X))$$

is an isomorphism as well. Since $H^q(X, \mathcal{R}) := H^q((\Omega^* \otimes \mathcal{R})(X))$ we have found the isomorphism:

$$H^q(X, \mathcal{R}) \simeq H^q_{deR}(X)$$

□

7.3 Singular

For this section we let K to be any principal ideal and X a differentiable manifold of dimension m.

Definition 6.3.1: simplices
For $p \geq 1$ we define the *standard p-simplex* Δ^p as follows:

$$\Delta^p := \{(x_1, ..., x_p) \in \mathbb{R}^p | \sum_i x_i \leq 1, x_j \geq 0 \forall j\}$$

and the *standard 0-simplex* is defined as $\{0\}$. Furhtermore for a open $U \subset C$ a *singular p-simplex in* U is a continuous map $\sigma : \Delta^p \to U$. For $p \geq 1$ we will call the singular p-simplex σ *differentiable* if it can be extended to a smooth map of a open neighbourhood of Δ^p into U. Furhtermore, each singular $p-$simplex σ has $p+1$ *facets* σ^i for $0 \leq i \leq p$, which are defined as:

$$\sigma^i = \sigma \circ \rho_i^{p-1},$$

with:

$$\rho_i^p(x_1, ..., x_p) = \begin{cases} 1 & \text{when } p = 0, i = 0; \\ 0 & \text{when } p = 0, i = 1; \\ (1 - \sum_i x_i, x_1, ..., x_p) & \text{when } p \geq 1, i = 0; \\ (x_1, ..., x_{i-1}, 0, x_i, ..., x_p) & \text{when } p \geq 1, 1 \leq i \leq p+1. \end{cases}$$

Finally the *boundary operators* are the operators ∂_p which send a singular $p-$simplex σ into $\sum_{i=0}^{p}(-1)^i\sigma^i$. The image of σ under ∂_p is called its *boundary.*

Definition 6.3.2: singular p-(co)chains
For each open $U \subset X$, we let $\mathrm{S}_p(U) := \langle\sigma|\sigma \text{ a singular } p-\text{simplex}\rangle$; the free abelian group generated by singular p-simplices with coefficients in $\mathbb{Z}$ and $\mathrm{S}_{p,\infty} := \langle\sigma|\sigma \text{ a differentiable singular } p-\text{simplex}\rangle$. We will use $\mathrm{S}_{p,(\infty)}$ when we mean any of both. Elements of $\mathrm{S}_{p,(\infty)}(U)$ are called *(differentiable) singular p-chains with integer coefficients on U* and we can extend the boundary operators ∂_p into maps $\partial_p : \mathrm{S}_{p,(\infty)} \to \mathrm{S}_{p-1,(\infty)}$. Furthermore we let $\mathrm{S}^p_{B,(\infty)}(U) := \{f : \mathrm{S}_{p,(\infty)}(U) \to B|f \text{ linear}\}$ for any $K-$module B. This is a $K-$module when we use pointwise addition. Elements of $\mathrm{S}^p_{B,(\infty)}(U)$ are called *(differentiable) singular $p-$cochains on U with coefficients in B.* Clearly $\mathrm{S}^p_{B,(\infty)}$ is a presheaf, with r^U_V the natural restriction to simplices which lie in V.

Lemma 6.3.1: The *coboundary operator* $d^p_{(\infty)} : \mathrm{S}^p_{B,(\infty)} \to \mathrm{S}^{p+1}_{B,(\infty)}$ defined by $d^p_{(\infty)}(f) : \sigma \mapsto f(\partial_{p+1}(\sigma))$ induces a cochain complex $\mathrm{S}^*_{B,(\infty)}(U)$ for each open U:

$$... \longrightarrow 0 \longrightarrow S^0_{(\infty)}(U) \xrightarrow{d^0_{(\infty)}} S^1_{(\infty)}(U) \xrightarrow{d^1_{(\infty)}} S^2_{(\infty)}(U) \xrightarrow{d^2_{(\infty)}} ...$$

Proof: Let $f_1, f_2 \in \mathrm{S}^p_{B,(\infty)}(U)$ and $k_1, k_2 \in K$. Then:

$$\begin{aligned}(d^p_{(\infty)}(k_1 \cdot f_1 + k_2 \cdot f_2))(\sigma) &= (k_1 \cdot f_1 + k_2 \cdot f_2)(\partial_{p+1}(\sigma)) = k_1 \cdot f_1(\partial_{p+1}(\sigma)) + k_2 \cdot f_2(\partial_{p+1}(\sigma)) \\ &= k_1 \cdot (d^p_{(\infty)}(f_1))(\sigma) + k_2 \cdot (d^p_{(\infty)}(f_2))(\sigma) = k_1 \cdot d^p_{(\infty)}(f_1) + k_2 \cdot d^p_{(\infty)}(f_2)(\sigma),\end{aligned}$$

so $d^p_{(\infty)}$ is a homomorphism. Secondly we want that $d^{p+1}_{(\infty)} \circ d^p_{(\infty)} = 0$ for each p. To proof that, we only need to proof that $(d^{p+1}_{(\infty)}(d^p_{(\infty)}(f)))(\sigma) = 0$ for a $f \in \mathrm{S}^p_{B,(\infty)}(U)$ and a $p-$simplex σ, which lies in U. Then by linearity we have the aimed result. We note that:

$$(d^{p+1}_{(\infty)}(d^p_{(\infty)}(f)))(\sigma) = (d^p_{(\infty)}(f))(\partial_{p+2}(\sigma)) = f(\partial_{p+1}(\partial_{p+2}(\sigma))).$$

Clearly it is enough to show that $\partial_{p+1}(\partial_{p+2}(\sigma)) = 0$ for all $(p+2)-$simplices σ, since f is linear. For this we first need the following identity, where $i \leq j$ and $p \geq 0$:

$$\rho^{p+1}_i \circ \rho^p_j = \rho^{p+1}_{j+1} \circ \rho^p_i \tag{9}$$

When $p = 0$ there are only three cases:

$$\rho^1_0(\rho^0_0(0)) = \rho^1_0(1) = (1-1, 1) = (0, 1) = \rho^1_1(1) = \rho^1_1(\rho^0_0(0)) \text{ for } j = 0 = i;$$

$$\rho^1_0(\rho^0_1(0)) = \rho^1_0(0) = (1-0, 0) = (1, 0) = \rho^1_2(1) = \rho^1_2(\rho^0_0(0)) \text{ for } j = 1,\ i = 0;$$

$$\rho^1_1(\rho^0_1(0)) = \rho^1_1(0) = (0, 0) = \rho^1_2(0) = \rho^1_2(\rho^0_1(0)) \text{ for } j = 1 = i.$$

For $p \geq 1$ we can divide the our problem into four cases, namely $1 \leq i < j$, $1 \leq i = j$, $0 = i < j$ and $0 = i = j$.

$$\begin{aligned}1 : \rho^{p+1}_i(\rho^p_j(x_1, ..., x_p)) &= \rho^{p+1}_i(x_1, ..., x_{j-1}, 0, x_j, ..., x_p) = (x_1, ..., x_{i-1}, 0, x_i, ..., x_{j-1}, 0, x_j, ..., x_p) \\ \rho^{p+1}_{j+1}(\rho^p_i(x_1, ..., x_p)) &= \rho^{p+1}_{j+1}(x_1, ..., x_{i-1}, 0, x_i, ..., x_p) = (x_1, ..., x_{i-1}, 0, x_i, ..., x_{j-1}, 0, x_j, ..., x_p);\end{aligned}$$

$$\begin{aligned}2 : \rho^{p+1}_i(\rho^p_j(x_1, ..., x_p)) &= \rho^{p+1}_i(x_1, ..., x_{j-1}, 0, x_j, ..., x_p) = (x_1, ..., x_{j-1}, 0, 0, x_j, ..., x_p) \\ \rho^{p+1}_{j+1}(\rho^p_i(x_1, ..., x_p)) &= \rho^{p+1}_{j+1}(x_1 ..., x_{i-1}, 0, x_i, ..., x_p) = (x_1, ..., x_{i-1}, 0, 0, x_i, ..., x_p);\end{aligned}$$

$$3: \rho_0^{p+1}(\rho_j^p(x_1,...,x_p)) = \rho_0^{p+1}(x_1,...,x_{j-1},0,x_j,...,x_p) = (1-\sum_k x_k - 0, x_1,...,x_{j-1},0,x_j,...,x_p)$$

$$\rho_{j+1}^{p+1}(\rho_0^p(x_1,...,x_p)) = \rho_{j+1}^{p+1}(1-\sum_k x_k, x_1,...,x_p) = (1-\sum_k x_k, x_1,...,x_{j-1},0,x_j,...,x_p);$$

$$4: \rho_0^{p+1}(\rho_0^p(x_1,...,x_p)) = \rho_0^{p+1}(1-\sum x_k, x_1,...,x_p) = (1-(1-\sum_k x_k + \sum_n x_n), 1-\sum_k x_k, x_1,...,x_p)$$

$$\rho_1^{p+1}(\rho_0^p(x_1,...,x_p)) = \rho_1^{p+1}(1-\sum_k x_k, x_1,...,x_p) = (0, 1-\sum_k x_k, x_1,...,x_{j-1},0,x_j,...,x_p);$$

So (9) holds for all the four cases when $p \geq 1$ and it holds separately for $p = 0$. Using (9) we see that for any $(p+2)-$simplex σ:

$$\begin{aligned}
\partial_{p+1}(\partial_{p+2}(\sigma)) &= \sum_{i=0}^{p+2}(-1)^i \partial_{p+1}(\sigma \circ \rho_i^{p+1}) = \sum_{i=0}^{p+2}\sum_{j=0}^{p+1}(-1)^{i+j}\sigma \circ \rho_i^{p+1} \circ \rho_j^p \\
&= \sum_{i=1}^{p+2}\sum_{j<i}(-1)^{i+j}\sigma \circ \rho_i^{p+1} \circ \rho_j^p + \sum_{i=0}^{p+1}(-1)^{i+i}\sigma \circ \rho_i^{p+1} \circ \rho_i^p \\
&+ \sum_{i=0}^{p}\sum_{j>i}(-1)^{i+j}\sigma \circ \rho_i^{p+1} \circ \rho_j^p \stackrel{(9)}{=} \sum_{j=1}^{p+2}\sum_{j>i}(-1)^{i+j}\sigma \circ \rho_j^{p+1} \circ \rho_i^p \\
&+ \sum_{i=0}^{p+1}\sigma \circ \rho_{i+1}^{p+1} \circ \rho_i^p + \sum_{i=0}^{p}\sum_{j>i}(-a)^{i+j}\sigma \circ \rho_{j+1}^{p+1} \circ \rho_i^p \\
&= \sum_{i=0}^{p+1}\sum_{j>i}(-1)^{i+j}\sigma \circ \rho_j^{p+1} \circ \rho_i^p + \sum_{i=0}^{p}\sum_{j>i+1}(-1)^{i+j-1}\sigma \circ \rho_j^{p+1} \circ \rho_i^p \\
&+ \sum_{i=0}^{p}\sigma \circ \rho_{i+1}^{p+1} \circ \rho_i^p + \sigma \circ \rho_{p+2}^{p+1} \circ \rho_{p+1}^p \\
&= \sum_{i=0}^{p}\sum_{j>i}(-1)^{i+j}\sigma \circ \rho_j^{p+1} \circ \rho_i^p + (-1)^{i+j-1}\sigma \circ \rho_j^{p+1} \circ \rho_i^p \\
&+ (-1)^{p+1+p+2}\sigma \circ \rho_{p+2}^{p+1} \circ \rho_{p+1}^p + \sigma \circ \rho_{p+2}^{p+1} \circ \rho_{p+1}^p = 0.
\end{aligned}$$

□

The coboundary operators $d_{(\infty)}^p$ do not only satisfy $d_{(\infty)}^{p+1} \circ d_{(\infty)}^p = 0$, but the commute with the restriction homomorphisms of $\mathrm{S}_{B,(\infty)}^p$ as well, since for $V \subset U$:

$$(r_V^U(d_{(\infty)}^p(f)))(\sigma = (d_{(\infty)}^p(f))(\sigma) = f(\partial_{p+1}(\sigma)) = (r_V^U(f))(\partial_{p+1}(\sigma)) = (d_{(\infty)}^p(r_V^U(f)))(\sigma),$$

where we use that if $\sigma : \Delta^p \to U$, then $\sigma^i : \Delta^{p-1} \to U$ for each U. So $d_{(\infty)}^p$ is a presheaf morphism for each p. From now on we will denote the sheaf induced by $\mathrm{S}_{B,(\infty)}^p$ as $\mathcal{S}_{B,(\infty)}^p$ and the induced sheaf morphisms by $d_{(\infty)}^p$. Like before we now claim that these sheaves with these homomorphisms induce a fine resolution of $\mathcal{B}$, which is torsionless when $\mathcal{B} = \mathcal{K}$, so that it induces a homology theory.

Lemma 6.3.2: The following sequence is a fine resolution of $\mathcal{B}$:

$$0 \longrightarrow \mathcal{B} \xrightarrow{i} \mathcal{S}_{B,(\infty)}^0 \xrightarrow{d_{(\infty)}^0} \mathcal{S}_{B,(\infty)}^1 \xrightarrow{d_{(\infty)}^1} \mathcal{S}_{B,(\infty)}^2 \xrightarrow{d_{(\infty)}^2} ...$$

with i induced by $i_x : B \to \mathcal{S}^0_{B,(\infty),x}; b \mapsto r^X_x(f_b)$ where $f_b(x) = b$ for all $x \in X$.
Proof: We will start with showing that the sheaves $\mathcal{S}^p_{B,(\infty)}$ are fine for each p and B. For each locally finite cover $\{U_i\}$ we can take a shrinking $\{V_j\}$ and a partition of unity $\{\psi_i\}$ for X subordinated to $\{U_i\}$ as in Example 3.1. Exactly like in Lemma 6.1.2 we let $(\tilde{l}^i_U(f))(\sigma) = \psi_i(\sigma(\vec{0})) \cdot f(\sigma)$ and we denote the induced sheaf morphisms by l^i_U. We are able to do this when the defined $\tilde{l}^i$ is indeed a presheaf homomorphism. We first check linearity:

$$\begin{aligned}\left(\tilde{l}^i_U(k_1 \cdot f_1 + k_2 \cdot f_2)\right)(\sigma) = \psi_i(\sigma(\vec{0})) \cdot (k_1 \cdot f_1 + k_2 \cdot f_2)(\sigma) &= k_1 \cdot \psi_i(\sigma(\vec{0})) \cdot f_1(\sigma) \\ + k_2 \cdot \psi_i(\sigma(\vec{0})) \cdot f_2(\sigma) &= k_1 \cdot (\tilde{l}^i_U(f_1))(\sigma) + k_2 \cdot (\tilde{l}^i_U(f_2))(\sigma) \\ &= (k_1 \cdot \tilde{l}^i_U(f_1) + k_2 \cdot \tilde{l}^i_U(f_2))(\sigma).\end{aligned}$$

The following square is clearly commutative, where $V \subset U$:

$$\begin{array}{ccc} \mathrm{S}^p_{B,(\infty)}(U) & \xrightarrow{\tilde{l}^i_U} & \mathrm{S}^p_{B,(\infty)}(U) \\ \downarrow r^U_V & & \downarrow r^U_V \\ \mathrm{S}^p_{B,(\infty)}(V) & \xrightarrow{\tilde{l}^i_V} & \mathrm{S}^p_{B,(\infty)}(V) \end{array}$$

So the $\tilde{l}^i$ form indeed a presheaf morphism for each i.
For I of Definition 3.1 we let $s : U \to \overline{\mathrm{S}}^p_{B,(\infty)}$ be an element of $\mathcal{S}^p_{B,(\infty)}(U)$ for some open U and let $x \in U$. We then know that $s(x) \in \mathrm{S}^p_{B,x,(\infty)}$ so it has to be equal to $r^V_x(f)$ for some open V containing x and a $f \in \mathrm{S}^p_{B,(\infty)}(V)$. For f and any singular p-simplex σ we have that:

$$f(\sigma) = \sum_i \psi_i(\sigma(\vec{0})) \cdot f(\sigma) = \sum_i (\tilde{l}^i_V(f))(\sigma) = \left(\sum_i \tilde{l}^i_V(f)\right)(\sigma),$$

which implies that $f = \sum_i \tilde{l}^i_V(f)$. Note that we use that $\sigma(\vec{0}) \in X$, so that there exists exactly one i such that $\psi(\sigma(\vec{0})) \neq 0$ and in this case $\psi(\sigma(\vec{0})) = 1$. Using this we get the following equality:

$$s(x) = r^V_x(f) = r^V_x\left(\sum_i \tilde{l}^i_V(f)\right) = \sum_i r^V_x(\tilde{l}^i_V(f)) = \sum_i (l^i_U(s))(x) = \left(\sum_i l^i_U(s)\right)(x).$$

This computation can be made for each $x \in U$ and therefore we conclude that $s = \sum_i l^i_U(s)$. s was arbitrary as well and we are therefore allowed to conclude that l^i satisfy I for each i.
For II of Definition 3.2 we assume that $x \notin \overline{V_j}$ and since $X \setminus \overline{V_j}$ is open there exists some open neighborhood W of x such that $W \subset X \setminus \overline{V_j}$. We want to proof that $l^i_x(r^U_x(f)) = 0$ for any open U containing x and $f \in S^p_{B,(\infty)}(U)$, because this implies that $\overline{\{x \in X | l^i_x \not\equiv 0\}} \subset \overline{V_j}$ and the latter is a subset of U_i, so this proves II. Given such a U and f, it is obvious that if $\sigma \in S^p_{B,(\infty)}(U \cap W)$, then $(r^U_{U\cap W}(\tilde{l}^i_U(f)))(\sigma) = (\tilde{l}^i_{U\cap W}(f))(\sigma) = \psi_i(\sigma(\vec{0})) \cdot f(\sigma) = 0$. This holds since $\sigma(\vec{0}) \in W$ implies that $\sigma(\vec{0}) \notin \overline{V_j}$, so $\psi_i(\sigma(\vec{0})) = 0$. We conclude that $l^i_x(r^U_x(f)) = r^U_x(\tilde{l}^i_U(f)) = 0$. With this we have shown that l^i satisfy I and II of Definition 3.2, so the sheaves $\mathcal{S}^p_{B,(\infty)}$ are fine.

Secondly we show exactness of the sequence, so that it is a resolution. Again we divide it in three problems, the first is exactness at $\mathcal{B}_x$, the second is exactness at $\mathcal{S}^0_{x,B,(\infty)}$ and the final one exactness at $\mathcal{S}^p_{x,B,(\infty)}$ with $p \geq 1$. This one is the most difficult and we will need to be very careful for the differentiable case.
But first we let $b \ker(i_x)$. Then $0 = i_x(b) = r^X_x(f_b)$. By definition there has to exist some open U such that $x \in U$ and $r^X_U(f_b) = 0$. We know however that $0 = f_b(x) = b$, so b lies in the image of the zero map. This proves exactness at $\mathcal{B}_x$.

When σ is a singular 1−simplex, then it is just a map $\sigma : [0,1] \to X$ and with the definition of ∂_1 we get that $\partial_1(\sigma) = \sigma(1) - \sigma(0)$. So when $f \in \mathrm{S}^0_{B,(\infty)}(X)$, then $(d^0_{(\infty)}(f))(\sigma) = f(\partial_1(\sigma)) = f(\sigma(1)) - f(\sigma(0)) = 0$, where the final equality holds for every σ when $f = f_b$. Now let $b \in B$. Then:

$$d^0_{x,(\infty)}(i_x(b)) = d^0_{x,(\infty)}(r^X_x(f_b)) = r^X_x(d^0_{(\infty)}(f_b)) = r^X_x(0) = 0.$$

We conclude that $\mathrm{Im}\,(i_x) \subset \ker(d^0_{x,(\infty)})$. For the other inclusion we have to remember that any manifold is locally path connected. This implies that if a subset of X is connected, then it is path connected as well. Now assume that $r^U_x(f) \in \ker(d^0_{x,(\infty)})$. This implies that $0 = d^0_{x,(\infty)}(r^U_x(f)) = r^U_x(d^0_{(\infty)}(f))$. This implies that we can find an open $V \subset U$ around x such that $0 = r^U_V(d^0_{(\infty)}(f))$. If V is not connected we can take the connected component in which x lies, so assume that V is connected and therefore path connected. Now for any two point v_1, v_2 in V we can find a smooth path $\sigma : [0,1] \to V$ such that $\sigma(0) = v_1$ and $\sigma(1) = v_2$, see ......?. Note that these σ are all elements of $\mathrm{S}_{1,(\infty)}$. And therefore $0 = (d^0_{(\infty)}(f))(\sigma) = f(\partial_1(\sigma)) = f(v_2) - f(v_1)$. So f is constant to some $b \in B$ on V: $r^U_V(f) = r^X_V(f_b)$. Finally see conclude that:

$$r^U_x(f) = r^V_x(r^U_V(f)) = r^V_x(r^X_V(f_b)) = r^X_x(f_b) = i_x(b) \in \mathrm{Im}\,(i_x),$$

so we can conclude that $\ker(d^0_{x,(\infty)}) \subset \mathrm{Im}\,(i_x)$ and that the sequence is exact at $\mathcal{S}^0_{B,x,(\infty)}$.

The final part is to show exactness in the following sequence:

$$\mathcal{S}^{p-1}_{B,x,(\infty)} \xrightarrow{d^{p-1}_{x,(\infty)}} \mathcal{S}^{p}_{B,x,(\infty)} \xrightarrow{d^{p}_{x,(\infty)}} \mathcal{S}^{p+1}_{B,x,(\infty)}$$

Like always, the first inclusion is trivial: for any $f \in \mathrm{S}^{p-1}_{B,(\infty)}(U)$ we have:

$$d^p_x(d^{p-1}_x(r^U_x(f))) = d^p_x(r^U_x(d^{p-1}_{(\infty)}(f))) = r^U_x(d^p_{(\infty)}(d^{p-1}_{(\infty)}(f))) = r^U_x(0) = 0,$$

where we use that $d^{p+1}_{(\infty)} \circ d^p_{(\infty)} = 0$ for all p, which we have proven in Lemma 6.3.1. This calculation shows that $\mathrm{Im}\,(d^{p-1}_x) \subset \ker(d^p_x)$.
To prove the other inclusion we let $r^U_x(f) \in \mathcal{S}^p_{B,x,(\infty)}$ and we assume that $r^U_x(f) \in \ker(d^p_{x,(\infty)})$, so: $0 = d^p_{x,(\infty)}(r^U_x(f)) = r^U_x(d^p_{(\infty)}(f))$. This now implies that there exists an open neighborhood $V \subset U$ of x such that $0 = r^U_V(d^p_{(\infty)}(f)) = d^p_{(\infty)}(r^U_V(f))$. We can assume that V is connected and small enough to be homeomorphic to the open unit ball $B_1(0)$ around 0 in $\mathbb{R}^m$. The first assumption is allowed, because we can restrict ourself to the connected component of V which contains x and the second one is allowed because X is a manifold and therefore locally Euclidean. So f sends maps $\sigma : \Delta^p \to B_1(0)$ into elements in B. Denote $r^U_V(f)$ as f_V, then we want to find a $g \in \mathrm{S}^{p-1}(V)$ such that $d^{p-1}_{(\infty)}(g) = f_V$.
To find this g we will define so called homotopy operators $h^p : \mathrm{S}^p_{B,(\infty)}(B_1(0)) \to \mathrm{S}^{p-1}_{B,(\infty)}(B_1(0))$ in the same fashion as $d^p_{(\infty)}$: we first define $h_p : \mathrm{S}_{p,(\infty)}(B_1(0)) \to \mathrm{S}_{p+1,(\infty)}(B_1(0))$ and then we let $(h^p(f))(\sigma) := f(h_{p-1}(\sigma))$. First we let ψ to be the following smoothing function:

$$\psi : \mathbb{R} \to [0,1];\ t \mapsto \begin{cases} 0 & \text{when } t < 0; \\ \frac{e^{-\frac{1}{t}}}{e^{-\frac{1}{t}} + e^{-\frac{1}{1-t}}} & \text{when } 0 \le t \le 1; \\ 1 & \text{when } 1 < t. \end{cases}$$

Here we define $\psi(0)$ and $\psi(1)$ as the limit of t going to 0 and 1 respectively. So $\psi(0) = 0$ and $\psi(1) = 1$ and ψ itself is a smooth function. Now we let

$$(h_p(\sigma))(x_1, ..., x_{p+1}) := \psi\left(\sum_{i=1}^{p+1} x_i\right) \cdot \sigma\left(x_2/\sum_{i=1}^{p+1} x_i, ..., x_{p+1}/\sum_{i=1}^{p+1} x_i\right).$$

Since $\mathrm{Im}\,(\psi) \subset [0,1]$ we get that $h_p(\sigma)$ is a map into $B_1(0)$ as well. If we do not use the smoothing operator we will in general only get a continuous singular $(p+1)-$complex, since at 0 there may be some differentiability issues. So using ψ guarantees that our approach will work for both cases: continuous and differentiable. To see that $h_p(\sigma)$ is actually a differentiable singular $(p+1)-$simplex we need to extend σ to a differentiable map on whole $\mathbb{R}^p$ such that σ and all its derivatives are bounded. Then we can extend $h_p(\sigma)$ to whole $\mathbb{R}^{p+1}$ with the same definition, when we let $(h_p(\sigma))(x_1, ..., x_{p+1}) = 0$ whenever $\sum x_i = 0$. Finally we can conclude that $h_p(\sigma)$ is differentiable, since ψ and its derivatives vanish fast enough around 0 and since ψ and all its derivatives are bounded. Moreover we have that for a singular $p-$simplex σ, $x_1, ..., x_p \in \Delta^p$ and $\overline{x} := \sum_{i=1}^p x_i$ the following two identities:

$$\begin{aligned}(h_{p-1}(\partial_p(\sigma)))(x_1, ..., x_p) &= \psi(\overline{x}) \cdot (\partial_p(\sigma))(x_2/\overline{x}, ..., x_p/\overline{x}) = \psi(\overline{x}) \cdot \sum_{k=0}^{p} (-1)^k \sigma(\rho_k^{p-1}(x_2/\overline{x}, ..., x_p/\overline{x})) \\ &= \psi(\overline{x}) \cdot (-1)^0 \sigma(1 - (x_2 + ... + x_p)/\overline{x}, x_2/\overline{x}, ..., x_p/\overline{x}) \\ &+ \psi(\overline{x}) \cdot \sum_{k=1}^{p} (-1)^k \sigma(x_2/\overline{x}, ..., x_k/\overline{x}, 0, x_{k+1}/\overline{x}, ..., x_p/\overline{x}) \\ &= \psi(\overline{x}) \cdot \sigma(x_1/\overline{x}, ..., x_p/\overline{x}) + \psi(\overline{x}) \cdot \sum_{k=1}^{p} \sigma(x_2/\overline{x}, ..., x_k/\overline{x}, 0, x_{k+1}/\overline{x}, ..., x_p/\overline{x}),\end{aligned}$$

$$\begin{aligned}(\partial_{p+1}(h_p(\sigma)))(x_1, ..., x_p) &= \sum_{j=0}^{p+1} (h_p(\sigma))(\rho_j^p(x_1, ..., x_p)) = (-1)^0 (h_p(\sigma))(1 - \overline{x}, x_1, ..., x_p) \\ &+ (-1)^1 (h_p(\sigma))(0, x_1, ..., x_p) + \sum_{k=1}^{p} (-1)^{k+1} (h_p(\sigma))(x_1, ..., x_k, 0, x_{k+1}, ..., x_p) \\ &\overset{*}{=} \psi(1) \cdot \sigma(x_1, ..., x_p) + \sum_{k=1}^{p} (-1)^{k+1} \psi(\overline{x}) \cdot \psi(x_2/\overline{x}, ..., x_k/\overline{x}, 0, x_{k+1}/\overline{x}, ..., x_p/\overline{x}) \\ &- \psi(\overline{x}) \cdot \sigma(x_1/\overline{x}, ..., x_p/\overline{x}).\end{aligned}$$

At $*$ we use that $1 - \overline{x} + \overline{x} = 1$. When we add these two together, we get that $(h_{p-1} \circ \partial_p + \partial_{p+1} \circ h_p)(\sigma) = \psi(1) \cdot \sigma = \sigma$. Using this identity gives us the same identity for h^p and $d^p_{(\infty)}$:

$$\begin{aligned}\left((h^{p+1} \circ d^p_{(\infty)} + d^{p-1}_{(\infty)} \circ h^p)(f)\right)(\sigma) &= (d^p_{(\infty)}(f))(h_p(\sigma)) + (h^p(f))(\partial_p(\sigma)) \\ &= f(\partial_{p+1}(h_p(\sigma))) + f(h_{p-1}(\partial_p(\sigma))) = f((\partial_{p+1} \circ h_p + h_{p-1} \circ \partial_p)(\sigma)) \\ &= f(\sigma).\end{aligned}$$

Finally using f_V gives us that:

$$f_V = (h^{p+1} \circ d^p_{(\infty)} + d^{p-1}_{(\infty)} \circ h^p)(f_V) = h^{p+1}(d^p_{(\infty)}(f_V)) + d^{p-1}_{(\infty)}(h^p(f_V)) = d^{p-1}_{(\infty)}(h^p(f_V)).$$

So we choose the wanted g to be equal to $h^p(f_V)$ and this implies that:

$$r_x^U(f) = r_x^V(f_V) = r_x^V(d^{p-1}_{(\infty)}(g)) = d^{p-1}_{x,(\infty)}(r_x^V(g)) \in \mathrm{Im}\,(d^{p-1}_{x,(\infty)}).$$

With this we have shown that $\ker(d_x^p) \subset \mathrm{Im}\,(d^{p-1}_{x,(\infty)})$ and we are therefore allowed to conclude that the sequence is exact at $\mathcal{S}^p_{x,B,(\infty)}$ for each $p \geq 1$ as well. □

Lemma 6.3.3: The above fine resolution is torsionless, when $B = K$.

Proof: We will show that $\mathcal{S}^p_{K,(\infty)}(U)$ is torsionless for each open $U \subset X$ and p. Let s be an element of $\mathcal{S}^p_{K,(\infty)}(U)$, so $s : U \to \overline{S^p_{K,(\infty)}}$ such that $s(x) \in S^p_{K,(\infty),x}$ for each $p \in U$. Now for any $k \in K \setminus \{0\}$ we get that $0 = k \cdot s$ implies that for each $x \in U$: $0 = k \cdot s(x)$. $s(x) \in S^p_{K,(\infty),x}$ implies that there exists an open V and an element $f \in S^p_{K,(\infty)}(V)$ such that $x \in V$ and $r^V_x(f) = s(x)$. So we get that $0 = k \cdot s(x) = k \cdot r^V_x(f) = r^V_x(k \cdot f)$. Therefore there exists a open $W \subset V$ such that $x \in W$ and $0 = r^V_W(k \cdot f)$, that is: for each $\sigma \in S_{p,(\infty)}(W)$ we have that $0 = (k \cdot f)(\sigma) = k \cdot f(\sigma)$. Because $f(\sigma) \in K$ and K is integral domain, we are allowed to conclude that $f(\sigma) = 0$ for each σ. This implies that $r^V_W(f) = 0$. Finally we can conclude that $s(x) = 0$, since $s(x) = r^V_x(f) = r^W_x(r^V_W(f)) = r^W_x(0) = 0$. We can do this for all $x \in U$ and therefore s has to be the zero-section. With this we have proven that $\mathcal{S}^p_{K,(\infty)}(U)$ is torsionless and with Lemma 3.6 we see that all the sheaves $\mathcal{S}^p_{K,(\infty)}$ are torsionless.

Like always we will now define the classical versions. After this definition there will be a technical lemma. Then the final part of this section consists of Theorem 6.3, in which we will make use of this lemma to conclude that the classical and the sheaf cohomologies are indeed isomorphic.

Definition 6.3.3: classical (differentiable) singular cohomology modules
Since $d^{p+1}_{(\infty)} \circ d^p_{(\infty)} = 0$ we have the cochain complex $\mathrm{S}^*_{B,(\infty)}(X)$, for each $K-$module B. The *classical (differentiable) singular cohomology modules for X with coefficients in B* are then defined as:

$$H^q_{\Delta,(\infty)}(X; B) := H^q(\mathrm{S}^*_{B,(\infty)}(X)).$$

Definition 6.3.4: singular chains on a cover
Let $\{U_i\}$ be an open cover of X. Then we will call a (differentiable) singular $p-$simplex σ a *(differentiable) singular $p-$simplex on $\{U_i\}$* if there exists an i such that $\mathrm{Im}\,(\sigma) \subset U_i$. Then we define $\tilde{\mathrm{S}}_{p,(\infty)} := \langle \sigma | \sigma$ a (differentiable) singular $p-$simplex on $\{U_i\}\rangle$ and analogously as in Definition 6.3.2 we let $\tilde{\mathrm{S}}^p_{B,(\infty)}$ contain all the linear maps $f : \tilde{\mathrm{S}}_{p,(\infty)} \to B$. Elements of $\tilde{\mathrm{S}}^p_{B,(\infty)}$ are called *(differentiable) singular $p-$cochains on $\{U_i\}$ with coefficients in B.*

Lemma 6.3.4: Let $\{U_i\}$ be an open cover of X and let $\tilde{\mathrm{S}}^p_{B,(\infty)}$ be as in the above definition. Finally let $j^p : \mathrm{S}^p_{B,(\infty)}(X) \to \tilde{\mathrm{S}}^p_{B,(\infty)}$ be the natural restriction homomorphism. Then j induces an isomorphism $\overline{j}^q : H^q_{\Delta,(\infty)} \to H^q(\tilde{\mathrm{S}}^*_{B,(\infty)})$.
Proof: First we note that j^p maps f into $\tilde{f}$ such that $\tilde{f}(\sigma) = f(\sigma)$ for each $\sigma \in \tilde{\mathrm{S}}^p_{B,(\infty)}$. Secondly it is obvious that $\overline{j}^q$ is well defined, since it commutes with the coboundary operators, and sends $[f]$ into $[\tilde{f}]$. We now will define new homotopy operators $h^p : \mathrm{S}^p_{B,(\infty)}(X) \to \mathrm{S}^{p-1}_{B,(\infty)}(X)$ and a cochain map $k : \tilde{\mathrm{S}}^*_{B,(\infty)} \to \mathrm{S}^*_{B,(\infty)}(X)$ such that for all p:

$$j^p \circ k^p = \mathrm{Id} : \tilde{\mathrm{S}}^*_{B,(\infty)} \to \tilde{\mathrm{S}}^*_{B,(\infty)}. \tag{10}$$

And furthermore that

$$h^{p+1} \circ d^p + d^{p-1}_{(\infty)} \circ h^p = \mathrm{Id} - k^p \circ j^p. \tag{11}$$

First we define linear $p-$simplices $(v_0, ..., v_p) : \Delta^p \to \Delta^q$ in Δ^q, with $q \geq 1$, and $v_0, ..., v_p \in \Delta^q$, such that:

$$(v_0, ..., v_p) : (x_1, ..., x_p) \mapsto \left(1 - \sum_i x_i\right) v_0 + x_1 v_1 + ... + x_p v_p.$$

Note that $(0, e_1, ..., e_p)$, with e_i the standard basis in $\mathbb{R}^p$ induces the identity map on Δ^p, which we will denote by ι_p. Let L^q_p be the free abelian group generated by such elements $(v_0, ..., v_p)$ and let $b(v_0, ..., v_p) := \frac{\sum_i v_i}{p+1} = \frac{\bar{v}}{p+1}$ and $w \cdot (v_0, .., v_p) := (w, v_0, ..., v_p)$. We note that

$$\partial(v_0, ..., v_p) = \sum_{i=0}^{p} (-1)^i (v_0, ..., \hat{v_i}, ..., v_p)$$

holds and therefore we get that

$$\partial(w \cdot (v_0, ..., v_p)) = (v_0, ..., v_p) + \sum_{i=0}^{p} (-1)^{i+1} (w, v_0, ..., \hat{v_i}, ..., v_p) = (v_0, .., v_p) - w \cdot \partial(v_0, ..., v_p). \quad (12)$$

The diameter of a linear simplex is defined as $r(v_0, ..., v_p) := \max i, j |v_i - v_j|$ and for each $u, v \in \mathrm{Im}\,(v_0, ..., v_p)$ we have that $|u - v| \leq d(v_0, ..., v_p)$. Before we can construct k and h^p we need to define two more maps $\tilde{T}_p : L_p^q \to L_p^q$ and $\tilde{R}_p : L_p^q \to L_{p+1}^q$, which we will extend later on to maps on $\mathrm{S}_{p,B,(\infty)}$. We demand the maps to be linear and for a linear $p-$simplex $\sigma = (v_0, ..., v_p)$ we define:

$$\tilde{T}_p(\sigma) := \begin{cases} b(\sigma) \cdot \tilde{T}_{p-1}(\partial(\sigma)) & \text{when } p \geq 1; \\ \sigma \text{ when } p = 0. \end{cases}$$

$$\tilde{R}_p(\sigma) := \begin{cases} b(\sigma) \cdot (\sigma - \tilde{T}_p(\sigma) - \tilde{R}_{p-1}(\partial(\sigma))) & \text{when } p \geq 1; \\ 0 & \text{when } p = 0. \end{cases}$$

So $\tilde{T}$ divides a linear simplex in multiple simplices. We can determine the first few to see this happen. It is given that $\tilde{T}_0(v_0) = (v_0)$. For $\sigma = (v_0, v_1)$ we have that

$$\tilde{T}_1(\sigma) = \frac{\bar{v}}{2} \cdot \tilde{T}_0(\partial(\sigma)) = \frac{\bar{v}}{2} \cdot ((v_1) - (v_0)) = \left(\frac{\bar{v}}{2}, v_1\right) - \left(\frac{\bar{v}}{2}, v_0\right).$$

And for a $2-$simplex $\sigma = (v_0, v_1, v_2)$ we have that

$$\begin{aligned} \tilde{T}_2(\sigma) &= \frac{\bar{v}}{2} \cdot \tilde{T}_1(\partial(\sigma)) = \frac{\bar{v}}{2} \cdot \tilde{T}_1((v_1, v_2) - (v_0, v_2) + (v_0, v_1)) \\ &= \left(\frac{\bar{v}}{3}, \frac{v_1 + v_2}{2}, v_2\right) - \left(\frac{\bar{v}}{3}, \frac{v_1 + v_2}{2}, v_1\right) + \left(\frac{\bar{v}}{3}, \frac{v_0 + v_2}{2}, v_0\right) - \left(\frac{\bar{v}}{3}, \frac{v_0 + v_2}{2}, v_2\right) \\ &+ \left(\frac{\bar{v}}{3}, \frac{v_0 + v_1}{2}, v_1\right) - \left(\frac{\bar{v}}{3}, \frac{v_0 + v_1}{2}, v_0\right) \end{aligned}$$

If σ is a linear $0-$simplex we have trivially that $\partial(\tilde{T}_0(\sigma)) = \tilde{T}_{-1}(\partial(\sigma))$. With the inductionstep at $*$ we get this identity for all p, since:

$$\begin{aligned} \partial(\tilde{T}_p(\sigma)) &= \partial(b(\sigma) \cdot \tilde{T}_{p-1}(\partial(\sigma))) \stackrel{(12)}{=} \tilde{T}_{p-1}(\partial(\sigma)) - b(\sigma) \cdot \partial(\tilde{T}_{p-1}(\partial(\sigma))) \\ &= \tilde{T}_{p-1}(\partial(\sigma)) - b(\sigma) \cdot \sum_{i=0}^{p} \left((-1)^i \partial(\tilde{T}_{p-1}(\sigma^i))\right) \stackrel{*}{=} \tilde{T}_{p-1}(\partial(\sigma)) - b(\sigma) \cdot \sum_{i=0}^{p} \left((-1)^i \tilde{T}_{p-2}(\partial(\sigma^i))\right) \\ &= \tilde{T}_{p-1}(\partial(\sigma)) - b(\sigma) \cdot \tilde{T}_{p-2}(\partial^2(\sigma)) = \tilde{T}_{p-1}(\partial(\sigma)) \end{aligned}$$

For a linear $0-$simplex we also have that

$$\begin{aligned} \partial(\tilde{R}_0(\sigma)) + \tilde{R}_{-1}(\partial(\sigma)) = \partial(\tilde{R}_0(\sigma)) &= \partial\left(b(\sigma) \cdot (\sigma) - \tilde{T}_0(\sigma) - \tilde{R}_{-1}(\partial(\sigma))\right) \\ &\stackrel{(12)}{=} \sigma - \tilde{T}_0(\sigma) - b(\sigma) \cdot \partial(\sigma - \tilde{T}_0(\sigma)) = \sigma - \tilde{T}_0(\sigma). \end{aligned}$$

Using induction again, at $*$ gives this identity for all p:

$$\begin{aligned}
\partial(\tilde{R}_p(\sigma)) + \tilde{R}_{p-1}(\partial(\sigma)) &= \partial\left(b(\sigma)\cdot(\sigma - \tilde{(}\sigma) - \tilde{R}_{p-1}(\partial(\sigma)))\right) + \tilde{R}_{p-1}(\partial(\sigma)) \\
&\overset{(12)}{=} \sigma - \tilde{T}_p(\sigma) - \tilde{R}_{p-1}(\partial(\sigma)) - b(\sigma)\cdot\left(\partial(\sigma - \tilde{T}_p(\sigma) - \tilde{R}_{p-1}(\partial(\sigma)))\right) + \tilde{R}_{p-1}(\partial(\sigma)) \\
&= \sigma - \tilde{T}_p(\sigma) - b(\sigma)\cdot(\partial(\sigma) - \tilde{T}_{p-1}(\partial(\sigma))) + b(\sigma)\cdot\partial(\tilde{R}_{p-1}(\partial(\sigma))) \\
&\overset{*}{=} \sigma - \tilde{T}_p(\sigma) - b(\sigma)\cdot(\partial(\tilde{R}_{p-1}(\partial(\sigma))) + \tilde{R}_{p-2}(\partial^2(\sigma))) + b(\sigma)\cdot\partial(\tilde{R}_{p-1}(\partial(\sigma))) \\
&= \sigma - \tilde{T}_p(\sigma)
\end{aligned}$$

Now define $T_p : \mathrm{S}_{p,(\infty)}(U) \to \mathrm{S}_{p,(\infty)}(U)$ and $R_p : \mathrm{S}_{p,(\infty)}(U) \to \mathrm{S}_{p+1,(\infty)}(U)$ as:

$$T_p(\sigma) := \sigma \circ \tilde{T}_p(\iota_p), \qquad R_p(\sigma) := \sigma \circ \tilde{R}_p(\iota_p).$$

With these definitions it is clear that $\partial \circ T_p = T_{p-1} \circ \partial$ and $\partial \circ R_p + R_{p-1} \circ \partial = \mathrm{ID} - T_p$. Let $\sigma = (v_0, ..., v_p)$ be a linear $p-$simplex and let τ be one of the simplices of $\tilde{T}_p(\sigma)$. We want to determine the diameter of τ given the diameter of σ. By construction of $\tilde{T}_p$ we know that given two vertices of τ, at least one is is of the form $\frac{v_{i_0}+...+v_{i_k}}{k+1}$. This implies that

$$\left|\frac{v_{i_0} + ... + v_{i_k}}{k+1} - v_n\right| \leq \frac{1}{k+1}\sum_{j=0}^{k}|v_{i_j} - v_n| \leq \frac{1}{k+1}\sum_{j=0}^{k}\max_{n,m}|v_m - v_n| = \frac{k}{k+1}r(\sigma) \leq \frac{p}{p+1}r(\sigma),$$

from which we conclude that $r(\tau) \leq \frac{p}{p+1}r(\sigma) < r(\sigma)$. This implies that we are able to cut our simplex into smaller ones. Now we will define k^p and h^p and check that the chosen definitions satisfy (10) and (11). Let σ be a (differentiable) singular $p-$simplex. Then we get an open cover $\{\sigma^{-1}(U_i)\}$ of Δ^p which in compact in $\mathbb{R}^p$. By Lebesgue's number lemma there exists a positive number δ such that every subset of Δ^p with diameter less or equal to δ lies inside an element of our cover. This implies that for these subsets, their image under σ lies inside U_i for a unique i. Since $\tilde{T}_p$ shrinks the diameter strictly and since $\tilde{T}_p(\iota_p)$ consists of a finite amount of simplices, the smallest positive real number $s(\sigma)$, such that each simplex of $\tilde{T}_p^{s(\sigma)}(\iota_p)$ lies in exactly one $\sigma^{-1}(U_i)$ must exist. This implies that $T_p^{s(\sigma)}(\sigma) \in U_i$ for exactly one U_i. Therefore we can define k^p and h^p as following:

$$(k^p(f))(\sigma) := f\left(T_p^{s(\sigma)}(\sigma) + \sum_{j=0}^{p}(-1)^j R_{p-1}\left(\sum_{1=s(\sigma^j)}^{s(\sigma)-1} T_{p-1}^i(\sigma^j)\right)\right),$$

$$(h^p(f))(\sigma) := f\left(R_{p-1}\left(\sum_{i=0}^{s(\sigma)-1} T_{p-1}^i(\sigma)\right)\right).$$

Finally we will proof (11):

$$
\begin{aligned}
((h^{p+1}\circ d^p_{(\infty)} + d^{p-1}_{(\infty)}\circ h^p)(f))(\sigma) &= (d^p_{(\infty)}(f))\Big(R_p\Big(\sum_{i=0}^{s(\sigma)-1} T^i(\sigma)\Big)\Big) + (h^p(f))(\partial_p(\sigma))\\
&= f\Big(\partial_{p+1}\circ R_p\Big(\sum_{i=0}^{s(\sigma)-1} T^i(\sigma)\Big) + \sum_{j=0}^{p}(-1)^j R_{p-1}\Big(\sum_{i=0}^{s(\sigma^j)-1} T^i(\sigma^j)\Big)\Big)\\
&= f\Big(\sum_{i=0}^{s(\sigma)-1} \partial_{p+1}\circ R_p(T^i(\sigma)) + \sum_{i=0}^{s(\sigma)-1} R_{p-1}\Big(T^i\Big(\sum_{j=0}^{p}(-1)^j\sigma^j\Big)\Big)\Big)\\
&\quad - f\Big(\sum_{j=0}^{p}(-1)^j \sum_{i=s(\sigma^i)}^{s(\sigma)-1} R_{p-1}(T^i(\sigma^j))\Big)\\
&= f\Big((\partial\circ R_p + R_{p-1}\circ\partial)\Big(\sum_{i=0}^{s(\sigma)-1} T^i(\sigma)\Big) - \sum_{j=0}^{p}(-1)^j \sum_{i=s(\sigma^i)}^{s(\sigma)-1} R_{p-1}(T^i(\sigma^j))\Big)\\
&= f\Big((\mathrm{Id}-T)\Big(\sum_{i=0}^{s(\sigma)-1} T^i(\sigma)\Big) - \sum_{j=0^p}(-1)^j \sum_{i=s(\sigma^i)}^{s(\sigma)-1} R_{p-1}(T^i(\sigma^j))\Big)\\
&= f(T^0(\sigma)) - f\Big(T^{s(\sigma)}(\sigma) + \sum_{j=0}^{p}(-1)^j R_{p-1}\Big(\sum_{i=s(\sigma^j)}^{s(\sigma)-1} T^i(\sigma^j)\Big)\Big)\\
&= f(\sigma) - (k^p(f))(\sigma).
\end{aligned}
$$

(10) follows trivially as well, since for any $f \in \tilde{\mathrm{S}}^p_{B,(\infty)}$ and a (differentiable) singular $p-$simplex σ on $\{U_i\}$ we have that:

$$
\begin{aligned}
((j\circ k)(f))(\sigma) = (k(f))(\sigma) &= f\Big(T_p^{s(\sigma)}(\sigma) + \sum_{j=0}^{p}(-1)^j R_{p-1}\Big(\sum_{i=s(\sigma^j)}^{s(\sigma)-1} T^i_{p-1}(\sigma^j)\Big)\Big)\\
&\overset{(s(\sigma)=0)}{=} f(\sigma) + f\Big(\sum_{j=1}^{p}(-1)^j R_{p-1}(0)\Big) = f(\sigma).
\end{aligned}
$$

The final part we had to proof about the map k is that it is indeed a cochain map, so it has to commute with d. Since $k^p\circ j^p = \mathrm{Id}$ for each p, we get that

$$d^p_{(\infty)}\circ k^p\circ j^p = d^p_{(\infty)} = k^{p+1}\circ j^{p+1}\circ d^p_{(\infty)} = k^{p+1}\circ d^p_{(\infty)}\circ j^p,$$

where the final equality is due to j being a cochain map. Furthermore we have that j^p is clearly surjective for each p and thus we conclude that k is a cochain map as well.
At last we use the constructed maps k and h^p to proof that $\overline{j}^p$ is an isomorphism. Let $[f]\in H^p(\tilde{\mathrm{S}}^*_{B,(\infty)})$ be arbitrary. From (10) we get that:

$$[f] = [\ \mathrm{Id}(f)] = \overline{\mathrm{Id}}([f]) = \overline{j^p\circ k^p}([f]) = \overline{j}^p([k^p(f)]),$$

so $\overline{j}^p$ is surjective. Injectivity follows from (11) since it gives that:

$$[f] - \overline{k^p\circ j^p}([f]) = \overline{h}^{p+1}([d^p_{(\infty)}(f)]) + [d^{p-1}_{(\infty)}(h^p(f))] = [h^{p-1}(0)] + 0 = 0,$$

which implies that $[f] = \overline{k^p\circ j^p}([f])$. So if $[f]$ and $[g]$ are such that $\overline{j}^p([f]) = \overline{j}^p([g])$, then we get that $[f] = \overline{k}^p(\overline{j}^p([f])) = \overline{k}^p(\overline{j}^p([g])) = [g]$ □

Theorem 6.3: The classical singular cohomology modules with coefficients in B, $H^q_{\Delta,(\infty)}(X;B)$, are isomorphic to the cohomology modules of the cohomology theory induced by the singular p-cochain sheaves.

Proof: Let $\mathrm{S}^p_{0,B,(\infty)}$ be defined as $\{f \in \mathrm{S}^p_{B,(\infty)}(X) | r^X_x(f) = 0 \, \forall x \in X\}$. We start with realizing that $\mathrm{S}^p_{B,(\infty)}$ is a presheaf, which satisfies (S_2). This is true, since for each open cover $\{U_i\}$ of an open U and elements $f_i \in \mathrm{S}^p_{B,(\infty)}(U_i)$ such that $r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)$ for all i, j, we can construct f in the following way: for a $\sigma : \Delta^p \to U$ we let

$$f(\sigma) = \begin{cases} f_i(\sigma) & \text{if } \mathrm{Im}\,(\sigma) \subset U_i; \\ 0 & \text{if there exists no } i \text{ such that } \mathrm{Im}\,(\sigma) \subset U_i. \end{cases}$$

Then f is well defined when we extend it linearly to a map from $\mathrm{S}_{p,(\infty)}$ and clearly satisfies the condition that $r^U_{U_i}(f) = f_i$. Furthermore for $p = 0$ we have that $\mathrm{S}^0_{B,(\infty)}$ satisfies condition (S_1) as well and is therefore a sheaf. To proof this we let $U = \cup U_i$ and $f, g \in \mathrm{S}^0_{B,(\infty)}(U)$ such that for each i: $r^U_{U_i}(f) = r^U_{U_i}(g)$. When $\sigma \in \mathrm{S}_{0,(\infty)}(U)$ then it assign a single point $x \in U$. There exists an i such that $x \in U_i$ since $\{U_i\}$ is an open cover. So this means that $f(\sigma) = (r^U_{U_i}(f))(\sigma) = (r^U_{U_i}(g))(\sigma) = g(\sigma)$. We conclude that $f = g$.

The modules $\mathrm{S}^p_{0,B,(\infty)}$ form a cochain complex with cochain map d, since $r^X_x(d^p_{(\infty)}(f)) = d^p_x(r^X_x(f)) = d^p_x(0) = 0$ for every p and $f \in \mathrm{S}^p_{0,B,(\infty)}$. Let $f \in \mathrm{S}^0_{0,B,(\infty)}$. Then for each $x \in X$ $r^X_x(f) = 0$, which implies that for each x there exists an open U_x which contains x such that $r^U_{U_x}(f) = 0$. Since $X = \cup U_x$ and $\mathrm{S}^0_{B,(\infty)}$ is a sheaf we get that $f = 0$. So the cochain complex is as follows.

$$\ldots \longrightarrow 0 \longrightarrow 0 = \mathrm{S}^0_{0,B,(\infty)} \xrightarrow{d^0_{(\infty)}} \mathrm{S}^1_{0,B,(\infty)} \xrightarrow{d^1_{(\infty)}} \mathrm{S}^2_{0,B,(\infty)} \xrightarrow{d^2_{(\infty)}} \ldots$$

So we have that $H^q(\mathrm{S}^*_{0,B,(\infty)}) = 0$ for $q \leq 0$. We will proof that the other cohomology modules are 0 as well, so let $p \geq 1$ and $f \in \mathrm{S}^p_{0,B,(\infty)}$ such that $d^p_{(\infty)}(f) = 0$. For each x we have that $r^X_x(f) = 0$, so let U_x be as above. These opens form an open cover of X, so we get a surjective map j as in Lemma 6.3.4. Let $\mathcal{L}^p_{B,(\infty)} \subset \mathrm{S}^p_{B,(\infty)}(X)$ be the kernel of j^p. $\mathcal{L}^*_{B,(\infty)}$ is a cochain complex, since j commutes with d. This implies that we have the short exact sequences

$$0 \longrightarrow \mathcal{L}^p_{B,(\infty)} \xhookrightarrow{i} \mathrm{S}^p_{B,(\infty)}(X) \xrightarrow{\alpha^p} \tilde{\mathrm{S}}^p_{B,(\infty)} \longrightarrow 0$$

Theorem 3 implies exactness of the associated long sequence

$$\ldots \longrightarrow H^q(\mathcal{L}^*_{B,(\infty)}) \xrightarrow{i^q} H^q(\mathrm{S}^*_{B,(\infty)}(X)) \xrightarrow{\bar{j}^q} H^q(\tilde{\mathrm{S}}^*_{B,(\infty)}) \xrightarrow{\partial^q} H^{q+1}(\mathcal{L}^*_{B,(\infty)}) \longrightarrow \ldots$$

In Lemma 6.3.4 we were able to conclude that $\bar{j}^q$ is an isomorphism for each q and therefore $H^q(\mathcal{L}^*_{B,(\infty)})$ has to be 0. This is proven in two steps. First of all $\ker(\partial^{q-1}) = \mathrm{Im}\,(\bar{j}^{q-1}) = H^{q-1}(\tilde{\mathrm{S}}^*_{B,(\infty)})$, which implies that $\{0\} = \mathrm{Im}\,(\partial^{q-1}) = \ker(i^q)$. Secondly $\mathrm{Im}\,(i^q) = \ker(\bar{j}^q) = \{0\}$. So $H^q(\mathcal{L}^*_{B,(\infty)}) = 0$.

Because we have chosen our open cover nicely, we know that $f \in \mathcal{L}^p_{B,(\infty)}$. We already knew that $0 = d^p_{(\infty)}(f) \in \mathcal{L}^{p+1}_{B,(\infty)}$, so the cohomology being 0 implies that there exists an $g \in \mathcal{L}^{p-1}_{B,(\infty)}$ such that $d^{p-1}_{(\infty)}(g) = f$. Since $\mathcal{L}^{p-1}_{B,(\infty)} \subset \mathrm{S}^{p-1}_{B,(\infty)}(X)$ we get that $g \in \mathrm{S}^{p-1}_{0,B,(\infty)}$ and $d^{p-1}_{(\infty)}(g) = f$. We conclude that the cohomologies $H^q(\mathrm{S}^*_{0,B,(\infty)})$ for $q \geq 1$ are zero as well.

The final steps of the proof are quite short: Lemma 6.1.5 implies that for each p we have a short exact sequence:

$$0 \longrightarrow \mathrm{S}^p_{0,B,(\infty)} \overset{i^p}{\hookrightarrow} \mathrm{S}^p_{B,(\infty)}(X) \xrightarrow{\alpha^p} \mathcal{S}^p_{B,(\infty)}(X) \longrightarrow 0$$

Theorem 3 implies that we have the long exact sequence:

$$... \longrightarrow 0 = H^q(\mathrm{S}^*_{0,B,(\infty)}) \xrightarrow{\overline{i}^q} H^q(\mathrm{S}^*_{B,(\infty)}(X)) \xrightarrow{\overline{\alpha}^q} H^q(\mathcal{S}^*_{B,(\infty)}(X)) \xrightarrow{\partial^q} H^{q+1}(\mathrm{S}^*_{0,B,(\infty)}) = 0 \longrightarrow ...$$

So $\tilde{\alpha}^q$ is actually an isomorphism $H^q(\mathrm{S}^*_{B,(\infty)}(X)) \simeq H^q(\mathcal{S}^*_{B,(\infty)}(X))$. Finally we use Lemma 5.1 and Lemma 6.3.2 to conclude that:

$$H^q_{\Delta,(\infty)}(X;B) = H^q(\mathrm{S}^*_{B,(\infty)}(X)) \simeq H^q(\mathcal{S}^*_{B,(\infty)}(X)) \simeq H^q(X,\mathcal{B})$$

□

7.4 Čech

The final example of cohomology is the Čech cohomology on a paracompact Hausdorff space X. This cohomology will not appear associated to a fine torsionless resolution of the constant sheaf. We will proof that our definition indeed is a cohomology theory directly, so with Definition 5.1. We start with a definition which is rather unfortunate, but standard.

Definition 6.4.1: simplices and cochains
Given an open cover $\{U_i\}_{i\in I}$ of X, an ordered collection of $q+1$ opens $\sigma = (U_0, ..., U_q)$ of the cover is called a *q-simplex of* $\{U_i\}$ when $|\sigma| := U_0 \cap ... \cap U_q \neq \emptyset$. In this case $|\sigma|$ is called the *support of* σ. Each $q-$simplex has $q+1$ *facets* $\sigma^i := (U_0, ..., \hat{U}_i, ..., U_q)$ for $0 \le i \le q$. Denote the set of $q-$simplices as $\tilde{\mathrm{C}}^q_{\{U_i\}}$. For a sheaf of $K-$modules $\mathcal{F}$ we define $\mathrm{C}^q_{\{U_i\},\mathcal{F}}$ as

$$\mathrm{C}^q_{\{U_i\},\mathcal{F}} := \{f : \tilde{\mathrm{C}}^q_{\{U_i\}} \to \mathcal{F} \,|\, f(\sigma) \in \mathcal{F}(|\sigma|)\forall\sigma\}$$

and elements of $\mathrm{C}^q_{\{U_i\},\mathcal{F}}$ are called *q−cochains*. Finally we define a *coboundary homomorphism* $d^q_{\{U_i\},\mathcal{F}} = d^q : \mathrm{C}^q(\{U_i\},\mathcal{F}) \to \mathrm{C}^{q+1}(\{U_i\},\mathcal{F})$ as the map such that for $f \in: \mathrm{C}^q(\{U_i\},\mathcal{F})$ and $\sigma \in \tilde{\mathrm{C}}^{q+1}_{\{U_i\}}$:

$$(d^q(f))(\sigma) := \sum_{i=0}^{q+1} (-1)^i r^{|\sigma^i|}_{|\sigma|}(f(\sigma^i))$$

Lemma 6.4.1: $\mathrm{C}^*_{\{U_i\},\mathcal{F}}$ together with the coboundary homomorphisms form a cochain complex for each open cover $\{U_i\}$ and sheaf $\mathcal{F}$.
Proof: First of all $\mathrm{C}^q_{\{U_i\},\mathcal{F}}$ is a $K-$module for each q when we let $(f+g)(\sigma) := f(\sigma)+g(\sigma)$ and $(k \cdot f)(\sigma) := k \cdot f(\sigma)$. These operations are well defined, since $f(\sigma), g(\sigma) \in \mathcal{F}(|\sigma|)$, which is a $K-$module itself. Secondly we have to check whether $d^{q+1} \circ d^q = 0$. To simplify the notation in the calculation we let $\sigma^{i,j} := \left(\sigma^i\right)^j$. This notation implies that when $\sigma = (U_0, ..., U_q)$, then:

$$\sigma^{i,j} = \begin{cases} (U_0, ..., \hat{U}_i, ..., \hat{U}_{j+1}, ..., U_q) = \sigma^{j+1,i} & \text{if } j \ge i; \\ (U_0, ..., \hat{U}_j, ..., \hat{U}_i, ..., U_q) = \sigma^{j,i-1} & \text{if } j < i. \end{cases}$$

Now let σ be a $(q+2)-$simplex and $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$. Then:

$$
\begin{aligned}
(d^{q+1} \circ d^q(f))(\sigma) &= \sum_{i=0}^{q+2}(-1)^i \Big(r^{|\sigma^i|}_{|\sigma|}(d^q(f))\Big)(\sigma) = \sum_{i=0}^{q+2}(-1)^i r^{|\sigma^i|}_{|\sigma|}\Bigg(\sum_{j=0}^{q+1}(-1)^j r^{|\sigma^{i,j}|}_{|\sigma^i|}(f(\sigma^{i,j}))\Bigg) \\
&= \sum_{i=0}^{q+2}\sum_{j\geq i}(-1)^{i+j} r^{|\sigma^{i,j}|}_{|\sigma|}(f(\sigma^{i,j})) + \sum_{i=0}^{q+2}\sum_{j<i}(-1)^{i+j} r^{|\sigma^{i,j}|}_{|\sigma|}(f(\sigma^{i,j})) \\
&= \sum_{i=0}^{q+2}\sum_{j\geq i} r^{|\sigma^{i,j}|}_{|\sigma|}(f(\sigma^{j+1,i})) + \sum_{i=0}^{q+2}\sum_{j<i}(-1)^{i+j} r^{|\sigma^{i,j}|}_{|\sigma|}(f(\sigma^{i,j})) \\
&= \sum_{j=0}^{q+2}\sum_{i>j}(-1)^{i+j-1} r^{|\sigma^{i,j}|}_{|\sigma|} + \sum_{i=0}^{q+2}\sum_{j<i}(-1)^{i+j} r^{|\sigma^{i,j}|}_{|\sigma|}(f(\sigma^{i,j})) \\
&= \sum_{i=0}^{q+2}\sum_{j<i}(-1+1)\cdot(-1)^{i+j} r^{|\sigma^{i,j}|}_{|\sigma|} = 0,
\end{aligned}
$$

where we have only renamed our summation constants and have switched the order of summation. □

So each cover gives us a cochain complex. We can look at the cohomology modules associated to this cochain complex, but this will still depend on the chosen cover. We will start with this and out of these cohomology modules we will extract a "global" cohomology. We will finish with the theorem that states that these final cohomology modules form a cohomology theory, so that it is isomorphic to all the other (sheaf)cohomology theories.

Definition 6.4.2: Čech cohomology modules associated to $\{U_i\}$
For each sheaf $\mathcal{F}$ and open cover $\{U_i\}$ we define the *$q-$th Čech cohomology module of X associated to $\{U_i\}$ with coefficients in $\mathcal{F}$*

$$\check{H}^q_{\{U_i\}}(X, \mathcal{F}) := H^q(\mathrm{C}^*_{\{U_i\},\mathcal{F}})$$

Its elements will be denoted by $[f]_{\{U_i\}}$ for $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$.

Lemma 6.4.2: Let $\{V_j\}$ be an open refinement of the open cover $\{U_i\}$. Then there exist canonical maps $f^q_{\{U_i\},\{V_j\},\mathcal{F}} : \check{H}^q_{\{U_i\}}(X, \mathcal{F}) \to \check{H}^{q+1}_{\{V_j\}}(X, \mathcal{F})$.
Proof: Since $\{V_j\}$ is a refinement of $\{U_i\}$ we can pick an i_j for each j such that $V_j \subset U_{i_j}$. Let $\mu : \{V_j\} \to \{U_i\}$ be that map, which sends $V_j \mapsto U_{i_j}$. This μ induces maps $\mu^q : \mathrm{C}^q_{\{U_i\},\mathcal{F}} \to \mathrm{C}^q_{\{V_j\},\mathcal{F}}$; $(\mu^q(f))(\sigma)$ when we define for $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$ and $\sigma \in \tilde{\mathrm{C}}^{q+1}_{\{U_i\}}$

$$(\mu^q(f))(\sigma) := r^{|\mu(\sigma)|}_{|\sigma|}(f(\mu(\sigma))).$$

These maps are a cochain map when they commute with the coboundary d, so that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{C}^q_{\{U_i\},\mathcal{F}} & \xrightarrow{\mu^q} & \mathrm{C}^q_{\{V_j\},\mathcal{F}} \\
\Big\downarrow d^q_{\{U_i\}} & & \Big\downarrow d^q_{\{V_j\}} \\
\mathrm{C}^{q+1}_{\{U_i\},\mathcal{F}} & \xrightarrow{\mu^{q+1}} & \mathrm{C}^{q+1}_{\{V_j\},\mathcal{F}}
\end{array}
$$

First we start with $\sigma \in \tilde{\mathrm{C}}^{q+1}_{\{U_i\}}$ and we note that

$$\mu(\sigma^i) = \mu((V_0, ..., \hat{V}_i, ..., V_{q+1})) = (U_0, ..., \hat{U}_i, ..., U_{q+1}) = (U_0, ..., U_{q+1})^i = \mu(\sigma)^i. \tag{13}$$

So let $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$. We will compute $d^q_{\{V_j\}}(\mu^q(f))$ and $\mu^{q+1}(d^q_{\{U_i\}}(f))$ to check whether they are equal:

$$\begin{aligned}\left(d^q_{\{V_j\}}\left(\mu^q(f)\right)\right)(\sigma) &= \sum_{i=0}^{q+1}(-1)^i r^{|\sigma^i|}_{|\sigma|}\left((\mu^q(f))(\sigma^i)\right) = \sum_{i=0}^{q+1}(-1)^i r^{|\sigma^i|}_{|\sigma|}\left(r^{|\mu(\sigma^i)|}_{|\sigma^i|}\left(f(\mu(\sigma^i))\right)\right)\\ &= \sum_{i=0}^{q+1}(-1)^i r^{|\mu(\sigma^i)|}_{|\sigma|}\left(f(\mu(\sigma^i))\right),\end{aligned}$$

$$\begin{aligned}\left(\mu^{q+1}\left(d^q_{\{U_i\}}(f)\right)\right)(\sigma) &= r^{|\mu(\sigma)|}_{|\sigma|}\left((d^q_{\{U_i\}}(f))(\mu(\sigma))\right) = r^{|\mu(\sigma)|}_{|\sigma|}\left(\sum_{i=0}^{q+1}(-1)^i r^{|\mu(\sigma)^i|}_{|\mu(\sigma)|}\left(f(\mu(\sigma)^i)\right)\right)\\ &= \sum_{i=0}^{q+1}(-1)^i r^{|\mu(\sigma)^i|}_{|\sigma|}\left(f(\mu(\sigma)^i)\right).\end{aligned}$$

With (13) we conclude that the above diagram is indeed commutative.
Let $\overline{\mu}^q : \check{H}^q_{\{U_i\}}(X,\mathcal{F}) \to \check{H}^{q+1}_{\{V_j\}}(X,\mathcal{F})$ be as in Chapter 4. We want to define $f^q_{\{U_i\},\{V_j\},\mathcal{F}} = \overline{\mu}^q$, but we do not know whether μ is the unique map with the chosen properties. So let τ be another map such that $V_j \subset \tau(V_j)$ for all j, and we will check whether $\overline{\mu}^q = \overline{\tau}^q$. To do this we define for a $q-$simplex $\sigma = (V_0, ..., V_q) \in \tilde{\mathrm{C}}^q_{\{V_j\}}$:

$$\sigma_j := (\mu(V_0), ..., \mu(V_j), \tau(V_j), ..., \tau(V_q)) \in \tilde{\mathrm{C}}^{q+1}_{\{U_i\}}.$$

This implies that:

$$(\sigma^i)_j = \begin{cases}(\mu(V_0), ..., \mu(V_j), \tau(V_j), ..., \widehat{\tau(V_i)}, ..., \tau(V_q)) = (\sigma_j)^{i+1} & \text{if } j < i;\\ (\mu(V_0), ..., \widehat{\mu(V_i)}, ..., \mu(V_{j+1}), \tau(V_{j+1}), ..., \tau(V_q)) = (\sigma_{j+1})^i & \text{if } j \geq i.\end{cases}$$

Furthermore we note that

$$(\sigma_{i-1})^i = (\mu(V_0), ..., \mu(V_{i-1}), \tau(V_i), ..., \tau(V_q)) = (\sigma_i)^i,$$

$(\sigma_0)^0 = \tau(\sigma)$ and that $(\sigma_q)^{q+1} = \mu(\sigma)$. With this we are able to define new homotopy operators $h^q : \mathrm{C}^q_{\{U_i\},\mathcal{F}} \to \mathrm{C}^{q-1}_{\{V_j\},\mathcal{F}}$ such that

$$(h^q(f))(\sigma) := \sum_{j=0}^{q-1}(-1)^j r^{|\sigma_j|}_{|\sigma|} f(\sigma_j),$$

for a $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$ and $\sigma \in \tilde{\mathrm{C}}^{q-1}_{\{V_j\}}$. Now let us determine $d^{q-1} \circ h^q$, with the above identities for $(\sigma^i)_j$:

$$\begin{aligned}(d^{q-1} \circ h^q(f))(\sigma) &= \sum_{i=0}^{q}\sum_{j=0}^{q-1}(-1)^{i+j} r^{|(\sigma^i)_j|}_{|\sigma|} f((\sigma^i)_j)\\ &= -\sum_{i=0}^{q+1}\sum_{j=0}^{i-2}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i) - \sum_{i=0}^{q+1}\sum_{j=i+1}^{q}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i).\end{aligned}$$

While on the other hand we have that

$$\begin{aligned}(d^{q-1} \circ h^q(f))(\sigma) &= \sum_{i=0}^{q+1}\sum_{j=0}^{q}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i) = \sum_{i=0}^{q+1}\sum_{k=i-1}^{i}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i)\\ &+ \sum_{i=0}^{q+1}\sum_{j=0}^{i-2}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i) + \sum_{i=0}^{q+1}\sum_{j=i+1}^{q}(-1)^{i+j} r^{|(\sigma_j)^i|}_{|\sigma|} f((\sigma_j)^i).\end{aligned}$$

Adding these two gives that

$$
\begin{aligned}
((h^{q+1}\circ d^q + d^{q-1}\circ h^q)(f))(\sigma) &= (\tau^q(f))(\sigma) - (\mu^q(f))(\sigma) + \sum_{i=1}^{q}(r_{|\sigma|}^{|(\sigma_{i-1})^i|}f((\sigma_{i-1})^i) - r_{|\sigma|}^{|(\sigma_i)^i|}f((\sigma_i)^i)) \\
&= (\tau^q(f))(\sigma) - (\mu^q(f))(\sigma).
\end{aligned}
$$

Finally we are able to conclude that τ and μ induce the same map $f^q_{\{U_i\},\{V_j\},\mathcal{F}}$, since for a $[f]_{\{U_i\}} \in \check{H}^q_{\{U_i\}}(X,\mathcal{F})$ we have that:

$$
\begin{aligned}
\overline{\tau}^q([f]_{\{U_i\}}) - \overline{\mu}^q([f]_{\{U_i\}}) &= [(\tau^q - \mu^q)(f)]_{\{U_i\}} = [(h^{q+1}\circ d^q + d^{q-1}\circ h^q)(f)]_{\{U_i\}} \\
&= \overline{h}^{q+1}[d^q(f)]_{\{U_i\}} + [d^{q-1}(h^q(f))]_{\{U_i\}} = 0.
\end{aligned}
$$

□

Definition 6.4.3: Čech cohomology modules
For each sheaf $\mathcal{F}$ of $K-$modules we define an equivalence relation on the disjoint union $\bigsqcup_{\{U_i\}}\check{H}^q_{\{U_i\}}(X,\mathcal{F})$ of the Čech cohomology modules associated to the open covers $\{U_i\}$ of X. Just like in Definition 2.4 we say that $[f]_{\{U_i\}} \sim [g]_{\{V_j\}}$ for a $[f]_{\{U_i\}} \in \check{H}^q_{\{U_i\}}(X,\mathcal{F})$ and $[g]_{\{V_j\}} \in \check{H}^q_{\{V_j\}}(X,\mathcal{F})$ if there exists an open cover $\{W_k\}$ which is a refinement of both $\{U_i\}$ and $\{V_j\}$ such that

$$
f^q_{\{U_i\},\{W_k\},\mathcal{F}}[f]_{\{U_i\}} = f^q_{\{V_j\},\{W_k\},\mathcal{F}}[g]_{\{V_j\}}.
$$

Then the *q-th Čech cohomology module $\check{H}^q(X,\mathcal{F})$ for X with coefficients in the sheaf $\mathcal{F}$* is defined as

$$
\check{H}^q(X,\mathcal{F}) = \bigsqcup_{\{U_i\}}\check{H}^q_{\{U_i\}}(X,\mathcal{F})\big/\sim.
$$

Its elements will be denoted by $\big[[f]_{\{U_i\}}\big]$ for $[f]_{\{U_i\}} \in \check{H}^q_{\{U_i\}}(X,\mathcal{F})$.

Theorem 6.4: The Čech cohomology modules form a cohomology theory $\mathcal{H}$.
Proof: We will use Definition 5.1 to proof this, so we start with defining maps h^q for a sheaf homomorphism h and ∂^q for a short exact sequence of sheaves. After that we check (a)-(f).
When $h : \mathcal{F} \to \mathcal{F}'$ is a sheaf homomorphism, we get a map $\xi_{\{U_i\}} : \mathrm{C}^*_{\{U_i\},\mathcal{F}} \to \mathrm{C}^*_{\{U_i\},\mathcal{F}'}$ by composition with h. It is a cochain map, since it commutes with the coboundary operator:

$$
\begin{aligned}
(d^q\circ\xi^q_{\{U_i\}}(f))(\sigma) &= \sum_{i=0}^{q+1}(-1)^i r_{|\sigma|}^{|\sigma^i|}((\xi^q_{\{U_i\}}(f))(\sigma^i)) = \sum_{i=0}^{q+1}(-1)^i r_{|\sigma|}^{|\sigma^i|}(h_{|\sigma^i|}(f(\sigma^i))) \\
&= h_{|\sigma|}\left(\sum_{i=0}^{q+1}(-1)^i r_{|\sigma|}^{|\sigma^i|}(f(\sigma^i))\right) = \xi^{q+1}_{\{U_i\}}((d^q(f))(\sigma)) = (\xi^{q+1}_{\{U_i\}}\circ d^q(f))(\sigma).
\end{aligned}
$$

Like always we now have maps $\overline{\xi}^q_{\{U_i\}} : \check{H}_{\{U_i\}}(X,\mathcal{F}) \to \check{H}_{\{U_i\}}(X,\mathcal{F}')$. Furthermore we have that for a refinement $\{V_j\}$ with maps μ^q as in Lemma 6.4.2

$$
\begin{aligned}
(\xi^q_{\{V_j\}}\circ\mu^q(f))(\sigma) &= h_{|\sigma|}((\mu^q(f))(\sigma)) = h_{|\sigma|}(r_{|\sigma|}^{|\mu(\sigma)|}(f(\mu(\sigma)))) \\
&= r_{|\sigma|}^{|\mu(\sigma)|}(h_{|\mu(\sigma)|}(f(\mu(\sigma)))) = r_{|\sigma|}^{|\mu(\sigma)|}((\xi^q_{\{U_i\}}(f))(\mu(\sigma))) = (\mu^q\circ\xi^q_{\{U_i\}}(f))(\sigma),
\end{aligned}
$$

which implies that $\overline{\xi}^q_{\{V_j\}}\circ f^q_{\{U_i\},\{V_j\},\mathcal{F}} = f^q_{\{U_i\},\{V_j\},\mathcal{F}'}\circ\overline{\xi}^q_{\{U_i\}}$. So once more we get an induced map on the quotient, and we let h^q to be this map.

Next we will construct the maps ∂^q for each short exact sheaf sequence $0 \to \mathcal{F}' \xrightarrow{h} \mathcal{F} \xrightarrow{h'} \mathcal{F}'' \to 0$. We first start with looking at the following sequence and proof that it is exact.

$$0 \longrightarrow \mathrm{C}^q_{\{U_i\},\mathcal{F}'} \xrightarrow{\xi^q_{\{U_i\}}} \mathrm{C}^q_{\{U_i\},\mathcal{F}} \xrightarrow{\xi'^q_{\{U_i\}}} \mathrm{C}^q_{\{U_i\},\mathcal{F}''}$$

Let $f' \in \mathrm{C}^q_{\{U_i\},\mathcal{F}'}$. Then $\xi^q_{\{U_i\}}(f) = 0$ implies that for all $q-$simplices σ: $0 = (\xi^q_{\{U_i\}}(f))(\sigma) = h_{|\sigma|}(f(\sigma))$. When we let $Z = |\sigma|$ in Lemma 3.1 we get that $f(\sigma) = 0$ and thus we conclude that $f =$ and the sequence is exact at $\mathrm{C}^q_{\{U_i\},\mathcal{F}'}$. Exactness at $\mathrm{C}^q_{\{U_i\},\mathcal{F}}$ uses the same lemma, since we have that $(\xi'^q(\xi^q_{\{U_i\}}(f')))(\sigma) = h'_{|\sigma|}(h_{|\sigma|}(f'(\sigma))) = 0$ for all σ and thus $\mathrm{Im}\,(\xi^q_{\{U_i\}}) \subset \ker(\xi'^q_{\{U_i\}})$. The other way around, when $f \in \ker(\xi'^q_{\{U_i\}})$, then we get for all σ that $0 = (\xi'^q_{\{U_i\}}(f))(\sigma) = h'_{|\sigma|}(f(\sigma))$. So we find an $f'_\sigma \in \mathcal{F}'$ such that $= h_{|\sigma|}(f'_\sigma) = f(\sigma)$. Letting $f'(\sigma) := f'_\sigma$ implies that $\xi^q_{\{U_i\}}(f') = f$, so that we have exactness at $\mathrm{C}^q_{\{U_i\},\mathcal{F}}$ as well.
The above exact sequence gives us a short exact sequence of cochain complexes, when we let $\bar{\mathrm{C}}^q_{\{U_i\},\mathcal{F}''} := \mathrm{Im}\,(\xi'^q_{\{U_i\}}) \subset \mathrm{C}^q_{\{U_i\},\mathcal{F}''}$

$$0 \longrightarrow \mathrm{C}^*_{\{U_i\},\mathcal{F}'} \xrightarrow{\xi_{\{U_i\}}} \mathrm{C}^*_{\{U_i\},\mathcal{F}} \xrightarrow{\xi'_{\{U_i\}}} \bar{\mathrm{C}}^*_{\{U_i\},\mathcal{F}''} \longrightarrow 0$$

We have already seen that μ commutes with d and with ξ and ξ' for a refinement $\{V_j\}$ with refinement map μ. This implies that μ is a homomorphism of the short exact sequences of cochain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}^*_{\{U_i\},\mathcal{F}'} & \xrightarrow{\xi_{\{U_i\}}} & \mathrm{C}^*_{\{U_i\},\mathcal{F}} & \xrightarrow{\xi'_{\{U_i\}}} & \bar{\mathrm{C}}^*_{\{U_i\},\mathcal{F}''} & \longrightarrow & 0 \\ & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \\ 0 & \longrightarrow & \mathrm{C}^*_{\{V_j\},\mathcal{F}'} & \xrightarrow{\xi_{\{V_j\}}} & \mathrm{C}^*_{\{V_j\},\mathcal{F}} & \xrightarrow{\xi'_{\{V_j\}}} & \bar{\mathrm{C}}^*_{\{V_j\},\mathcal{F}''} & \longrightarrow & 0 \end{array}$$

Theorem 3 gives maps $\hat{\partial}^q_{\{U_i\}}$ and another commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \check{H}^q_{\{U_i\}}(X,\mathcal{F}') & \xrightarrow{\bar{\xi}^q_{\{U_i\}}} & \check{H}^q_{\{U_i\}}(X,\mathcal{F}) & \xrightarrow{\bar{\xi}'^q_{\{U_i\}}} & \bar{H}^q_{\{U_i\}}(X,\mathcal{F}'') & \xrightarrow{\hat{\partial}^q_{\{U_i\}}} & \check{H}^{q+1}_{\{U_i\}}(X,\mathcal{F}') & \longrightarrow & \dots \\ & & \downarrow \overline{\mu}^q & & \downarrow \overline{\mu}^q & & \downarrow \overline{\mu}^q & (1) & \downarrow \overline{\mu}^{q+1} & & \\ \dots & \longrightarrow & \check{H}^q_{\{V_j\}}(X,\mathcal{F}') & \xrightarrow{\bar{\xi}^q_{\{V_j\}}} & \check{H}^q_{\{V_j\}}(X,\mathcal{F}) & \xrightarrow{\bar{\xi}'^q_{\{V_j\}}} & \bar{H}^q_{\{V_j\}}(X,\mathcal{F}'') & \xrightarrow{\hat{\partial}^{q-1}_{\{V_j\}}} & \check{H}^{q+1}_{\{V_j\}}(X,\mathcal{F}') & \longrightarrow & \dots \end{array}$$

where $\bar{H}^q_{\{U_i\}}(X,\mathcal{F}'') = H^q(bar\mathrm{C}^*_{\{U_i\},\mathcal{F}''})$ and commutativity of (1) is a consequence of the theorem and the commutativity of the other squares is already shown above. Since (1) is commutative, we get an induced map $\hat{\partial}'^q : \bar{H}^q(X,\mathcal{F}'') \to \check{H}^{q+1}(X,\mathcal{F})$. Al together gives another exact sequence, where the exactness is a direct consequence of the exactness in the diagram above:

$$\dots \longrightarrow \check{H}^q(X,\mathcal{F}') \xrightarrow{h^q} \check{H}^q(X,\mathcal{F}) \xrightarrow{h'^q} \bar{H}^q(X,\mathcal{F}'') \xrightarrow{\hat{\partial}'^q} \check{H}^{q+1}(X,\mathcal{F}') \longrightarrow \dots \tag{14}$$

Let $i : \bar{\mathrm{C}}^q_{\{U_i\},\mathcal{F}''} \to \mathrm{C}^q_{\{U_i\},\mathcal{F}''}$ be the inclusion map and consider the $K-$modules $\tilde{\mathrm{C}}^q_{\{U_i\}} := \mathrm{C}^q_{\{U_i\},\mathcal{F}''}/\bar{\mathrm{C}}^q_{\{U_i\},\mathcal{F}''}$, which form a cochain complex $\tilde{\mathrm{C}}^*_{\{U_i\}}$ in the obvious way. Moreover in gives a short exact sequence of cochains:

$$0 \longrightarrow \bar{\mathrm{C}}^*_{\{U_i\},\mathcal{F}''} \xrightarrow{i} \mathrm{C}^*_{\{U_i\},\mathcal{F}''} \xrightarrow{\pi} \tilde{\mathrm{C}}^*_{\{U_i\}} \longrightarrow 0$$

Theorem 3 gives us a long exact sequence again

$$\ldots \longrightarrow \bar{H}^q_{\{U_i\}}(X,\mathcal{F}'') \xrightarrow{\bar{i}^q} \check{H}^q_{\{U_i\}}(X,\mathcal{F}'') \xrightarrow{\overline{\pi}'^q} \tilde{H}^q_{\{U_i\}}(X,\mathcal{F}'') \xrightarrow{\tilde{\partial}^q} \bar{H}^{q+1}_{\{U_i\}}(X,\mathcal{F}'') \longrightarrow \ldots$$

We will look at $\tilde{H}^q_{\{U_i\}}(X,\mathcal{F}'') := H^q(\tilde{\mathrm{C}}^*_{\{U_i\}})$ a bit more. We start with noting that the restriction maps $f_{\{U_i\},\{V_j\},\mathcal{F}''}$ induces restriction maps for $\tilde{H}^q_{\{U_i\}}(X,\mathcal{F}'')$ as well, since for $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$ we have that $\mu^q(\xi'^q_{\{U_i\}}(f)) = \xi'^q_{\{V_j\}}(\mu^q(f)) \in \bar{\mathrm{C}}^q_{\{U_i\},\mathcal{F}''}$. This means we can define $\tilde{H}(X,\mathcal{F}'')$ in exactly the same way as the other modules. Furthermore exactly like before we get the long exact sequence

$$\ldots \longrightarrow \tilde{H}^{q-1}(X,\mathcal{F}'') \xrightarrow{\tilde{\partial}'^{q-1}} \bar{H}^q(X,\mathcal{F}'') \xrightarrow{\bar{i}'^q} \check{H}^q(X,\mathcal{F}'') \xrightarrow{\overline{\pi}'^q} \tilde{H}^q(X,\mathcal{F}'') \longrightarrow \ldots \tag{15}$$

We will now proof that $\tilde{H}^q(X,\mathcal{F}'') = 0$ for all q. Let $\{U_i\}$ be a locally finite cover, $\{V_i\}$ a shrinking of this cover and $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}''}$. Then pick for each $x \in X$ an open W_x such that these four conditions hold:

(A) $W_x \subset V_i$ for some i;

(B) $W_x \cap V_i \neq \emptyset$ implies that $W_x \subset U_i$;

(C) W_x lies in the intersection of all the U_i which contain x;

(D) When $\sigma = (U_0, ..., U_q)$ is a $q-$simplex of the cover $\{U_i\}$ and $p \in |\sigma|$ then there exists an element $s_{x,\sigma} \in \mathcal{F}(W_x)$ such that $r^{|\sigma|}_{W_X}(f(\sigma)) = h'_{W_x}(s_{x,\sigma})$.

It is indeed possible to achieve this. We can start with the open neighborhood Z of x which intersects only with finitely many of opens of $\{V_i\}$. This is possible, since $\{V_i\}$ is locally finite as well. Now take the intersection with V_i for an i such that $x \in V_i$ and with U_i for all the other i which satisfy $x \in V_i$. Then the resulting Z' already satisfies (A) and (C). Secondly we shrink Z' even further by removing the $\overline{V_i}$ for which $x \notin \overline{V_i}$ and $V_i \cup Z \neq \emptyset$. Since this is just a finite amount of closed sets, we get that the new set Z'' is still an open neighborhood of x and it satisfies (B). Finally we note that there only finitely many U_i which contain x and therefore only finitely many $q-$simplices σ such that $x \in |\sigma|$. Since $\mathcal{F}', \mathcal{F}$ and $\mathcal{F}''$ form an exact sequence of sheaves we have that for every σ there exists an open neighborhood Z_σ of x and an element $r^{Z_\sigma}_x(s_{x,\sigma}))$ such that :

$$r^{Z_\sigma}_x(s_{x,\sigma}) = r^{Z''}_x\Big(r^{|\sigma|}_{Z''}(f(\sigma))\Big).$$

This now implies that we find an even smaller subset Z'_σ of $Z_\sigma \cap Z''$ such that

$$r^{Z_\sigma}_{Z'_\sigma}(s_{x,\sigma}) = r^{Z''}_{Z'_\sigma}\Big(r^{|\sigma|}_{Z''}(f(\sigma))\Big).$$

There are only finitely many such Z'_σ and thus we can let $W_x := Z'' \cap_\sigma Z'_\sigma$.

These opens W_x form a refinement of $\{U_i\}$ and we are able to pick an V_x and U_x for each x such that $W_x \subset V_x \subset U_x$ by (A). We let μ be the refinement map, which sends W_x into U_x. We let $\sigma = (W_{x_0}, ..., W_{x_q})$ be a $q-$simplex of our refinement. We then clearly have that $W_{x_0} \cap V_{x_j}$ for $0 \leq j \leq q$, so (B) implies that $W_{x_0} \subset U_{x_j}$ for all j. This now implies that $W_{x_0} \subset |\mu(\sigma)|$ and (D) now implies that

$$\begin{aligned}(\mu^q(f))(\sigma) &= r^{|\mu(\sigma)|}_{|\sigma|}\big(f(\mu(\sigma))\big) = r^{W_{x_0}}_{|\sigma|}\Big(r^{|\mu(\sigma)|}_{W_{x_0}}\big(f(U_{x_0}, ..., U_{x_p})\big)\Big) \\ &= r^{W_{x_0}}_{|\sigma|}\big(h'_{W_{x_0}}(s_{x_0,\mu(\sigma)})\big) = h'_{|\sigma|}\big(r^{W_{x_0}}_{|\sigma|}(s_{x_0,\mu(\sigma)})\big).\end{aligned}$$

There clearly exists a $f' \in \mathrm{C}^q_{\{W_x\},\mathcal{F}}$ such that $f'(\sigma) = r^{W_{x_0}}_{|\sigma|}(s_{x_0,\mu(\sigma)})$, since this is just an element of $\mathcal{F}(|\sigma|)$. Moreover, we have that

$$(\xi'^q(f'))(\sigma) = h'_{|\sigma|}\big(f'(\sigma)\big) = h'_{|\sigma|}\big(r^{W_{x_0}}_{|\sigma|}(s_{x_0,\mu(\sigma)})\big).$$

We conclude that $\mu^q(f) \in \bar{\mathrm{C}}^q_{\{W_x\},\mathcal{F}''}$. This means that the class of $\mu^q(f)$ is zero in $\tilde{\mathrm{C}}^q_{\{W_x\}}$. Therefore for $\tilde{H}^q(X,\mathcal{F}'')$ we get that

$$\big[[f]_{\{U_i\}}\big] = \big[\overline{\mu}^q[f]_{\{U_i\}}\big] = \big[[\mu^q(f)]_{\{U_i\}}\big] = 0.$$

We are finally able to see that $\tilde{H}^q(X,\mathcal{F}'') = 0$ for all q, since any open cover of X has a locally finite refinement $\{U_i\}$. The long exact sequence (15) changes in exact sequences:

$$0 \longrightarrow \bar{H}^q(X,\mathcal{F}'') \xrightarrow{\bar{i}'^q} \check{H}^q(X,\mathcal{F}'') \longrightarrow 0$$

and hence $\bar{i}'^q$ are isomorphisms of the modules. Now we are able to define $\partial^q : \check{H}^q(X,\mathcal{F}'') \to \check{H}^{q+1}(X,\mathcal{F}')$:

$$\partial^q := \hat{\partial}'^q \circ (\bar{i}'^q)^{-1}.$$

Now we will check (a) till (f):
(a) Let $\mathcal{F}$ be a sheaf. Since $\tilde{\mathrm{C}}^q = \emptyset$ whenever $q < 0$ we get that $\mathrm{C}^q_{\{U_i\},\mathcal{F}} = 0$ for $q < 0$, which implies that $\check{H}^q_{\{U_i\}}(X,\mathcal{F})$ and finally $\check{H}^q(X,\mathcal{F})$ are zero when $q < 0$. Secondly $\check{H}^0_{\{U_i\}}(X,\mathcal{F}) = \ker(d^0)/\mathrm{Im}\,(d^{-1}) = \ker(d^0)$, so when $f \in \mathrm{C}^0_{\{U_i\},\mathcal{F}}$ such that $d^0(f) = 0$, then $[f]_{\{U_i\}} = \{f\}$. Furthermore we get that for all i, j:

$$0 = (d^0(f))(U_i, U_j) = r^{U_j}_{U_i\cap U_j}(f_j) - r^{U_i}_{U_i\cap U_j}(f_i),$$

with $f_i := f\big((U_i)\big)$. $\mathcal{F}$ is a sheaf and therefore (S_2) implies that there exists a $\tilde{f} \in \mathcal{F}(X)$ such that $r^X_{U_i}(\tilde{f}) = f_i$ and (S_1) implies that this $\tilde{f}$ is unique. The other way around we see that for each $\tilde{f} \in \mathcal{F}(X)$ there exists a $f \in \ker(d^0)$ which is defined as $f\big((U_i)\big) := r^X_{U_i}(\tilde{f})$. This map $\alpha^{\mathcal{F}}_{\{U_i\}} : \check{H}^0_{\{U_i\}}(X,\mathcal{F}) \to \mathcal{F}(X); [f]_{\{U_i\}} \mapsto \tilde{f}$ is an isomorphism of the $K-$modules. The above discussion shows that it is bijective and for $[f]_{\{U_i\}}, [g]_{\{U_i\}} \in \check{H}^0_{\{U_i\}}(X,\mathcal{F})$ and $k \in K$ we have for every i that:

$$\begin{aligned} r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}} + [g]_{\{U_i\}})\big) &= (f+g)\big((U_i)\big) = f\big((U_i)\big) + g\big((U_i)\big) = r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})\big) \\ &\quad + r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([g]_{\{U_i\}})\big) = r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}}) + \alpha^{\mathcal{F}}_{\{U_i\}}([g]_{\{U_i\}})\big) \end{aligned}$$

$$\begin{aligned} r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}(k\cdot[f]_{\{U_i\}})\big) &= (k\cdot f)\big((U_i)\big) = k\cdot f\big((U_i)\big) = k\cdot r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})\big) \\ &= r^X_{U_i}\big(k\cdot \alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})\big). \end{aligned}$$

Since $\cup U_i = X$ and $\mathcal{F}$ satisfies (S_1) we can conclude that $\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}}+[g]_{\{U_i\}}) = \alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})+\alpha^{\mathcal{F}}_{\{U_i\}}([g]_{\{U_i\}})$ and $\alpha^{\mathcal{F}}_{\{U_i\}}(k\cdot[f]_{\{U_i\}}) = k\cdot\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})$, which is to say: $\alpha^{\mathcal{F}}_{\{U_i\}}$ is a homomorphism as well. We also see that for the usual refinement(map):

$$\begin{aligned} \Big(r^X_{V_j}\big(\alpha^{\mathcal{F}}_{\{V_j\}} \circ f_{\{U_i\},\{V_j\},\mathcal{F}}\big)\Big)([f]_{\{U_i\}}) &= \Big(\mu^q(f)\Big)\big((V_j)\big) = r^{\mu(V_j)}_{V_j}\big(f\big((\mu(V_j))\big)\big) \\ &= r^{\mu(V_j)}_{V_j}\Big(r^X_{\mu(V_j)}\Big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})\Big)\Big) = r^X_{V_j}\big(\alpha^{\mathcal{F}}_{\{U_i\}}\big)([f]_{\{U_i\}}). \end{aligned}$$

The same argument as before allows us to conclude that $\alpha^{\mathcal{F}}_{\{V_j\}} \circ f_{\{U_i\},\{V_j\},\mathcal{F}} = \alpha^{\mathcal{F}}_{\{U_i\}}$. This implies that we have an induced map $h_{\mathcal{F}} : \check{H}^0(X,\mathcal{F}) \to \mathcal{F}(X)$. To proof (a) we need to show that for a sheaf morphism $h : \mathcal{F} \to \mathcal{F}'$ the defined maps satisfy $h_X \circ h_{\mathcal{F}} = h_{\mathcal{F}'} \circ h^0$. Let $[f]_{\{U_i\}} \in \check{H}^0_{\{U_i\}}(X,\mathcal{F})$ for an open cover $\{U_i\}$. Then we have that:

$$r^X_{U_i}\Big(h_X \circ h_{\mathcal{F}}\big([[f]_{\{U_i\}}]\big)\Big) = h_{U_i}\Big(r^X_{U_i}\big(\alpha^{\mathcal{F}}_{\{U_i\}}([f]_{\{U_i\}})\big)\Big) = h_{U_i}\big(f\big((U_i)\big)\big),$$

$$r^X_{U_i}\big(h_{\mathcal{F}'}\big(h^0\big([[f]_{\{U_i\}}]\big)\big)\big) = r^X_{U_i}\big(h_{\mathcal{F}'}\big([[\xi^0_{\{U_i\}}(f)]_{\{U_i\}}]\big)\big) = (\xi^0_{\{U_i\}}(f))\big((U_i)\big) = h_{U_i}\big(f\big((U_i)\big)\big).$$

From this we conclude that (a) holds.

(b) Here we suppose that $\mathcal{F}$ is a fine sheaf and throughout this part we assume that $q \geq 1$. Let $\{V_j\}$ be a locally finite cover of X and $\{l^j\}$ a partition of unity for $\mathcal{F}$ subordinated to it. For $\sigma = (V_0, ..., V_{q-1})$ and $f \in \mathrm{C}^q_{\{V_j\},\mathcal{F}}$ we get that the support of the section $l^j_{V_j \cap |\sigma|}(f(V_j, V_0, ..., V_{p-1}))$ is equal to $\overline{\{x \in X : l^j_x \not\equiv 0\}} \cap V_j \cap |\sigma| \subset V_j \cap |\sigma|$. And so we can extend it to a continuous section over $|\sigma|$ by letting it be 0 in $\sigma \setminus V_j$. We will denote this extended section by $l^j_\sigma(f(V_j, V_0, ..., V_{p-1}))$ and we define homotopy-like operators $h^q : \mathrm{C}^q_{\{V_j\},\mathcal{F}} \to \mathrm{C}^{q-1}_{\{V_j\},\mathcal{F}}$ as

$$\big(h^p(f)\big)\big((V_0, ..., V_{q-1})\big) := \sum_j l^j_\sigma(f(V_j, V_0, ..., V_{q-1})).$$

With this definition we get for f the following computations, where $\sigma = (V_0, ..., V_q)$:

$$\big(d^{q-1} \circ h^q(f)\big)(\sigma) = \sum_{i=0}^q (-1)^i r^{|\sigma^i|}_{|\sigma|}\Big(\big(h^q(f)\big)(\sigma^i)\Big) = \sum_{i=0}^q \sum_j (-1)^i r^{|\sigma^i|}_{|\sigma|}\Big(l^j_{\sigma^i}\big(f(V_j, V_0, ..., \hat{V}_i, ..., V_q)\big)\Big),$$

$$\begin{aligned}\big(h^{q+1} \circ d^q(f)\big)(\sigma) &= \sum_j l^j_\sigma\Big(\big(d^q(f)\big)(V_j, V_0, ..., V_q)\Big) \\ &= \sum_j l^j_\sigma\Big(-\sum_{i=0}^q (-1)^i r^{V_j \cap |\sigma^i|}_{V_i \cap |\sigma|}\big(f(V_j, V_0, ..., \hat{V}_i, ..., V_q)\big) + r^{|\sigma|}_{V_j \cap |\sigma|}\big(f(\sigma)\big)\Big) \\ &= \sum_j l^j_\sigma(f) - \sum_j \sum_{i=0}^q (-1)^i r^{|\sigma^i|}_{|\sigma|}\Big(l^j_{\sigma^i}\big(f(V_j, V_0, ..., \hat{V}_i, ..., V_q)\big)\Big).\end{aligned}$$

Adding these two gives and using that $\sum_j (l^j(f(\sigma))) = f(\sigma)$ we get the identity

$$d^{q-1} \circ h^q + h^{q+1} \circ d^q = \mathrm{Id}. \tag{16}$$

We will use this to show that $\check{H}^q(X, \mathcal{F}) = 0$. So let $[f]_{\{U_i\}} \in \check{H}^q_{\{U_i\}}(X, \mathcal{F})$ for some open cover $\{U_i\}$ and consider $\big[[f]_{\{U_i\}}\big] \in \check{H}^q(X, \mathcal{F})$. Since X is paracompact we can take a locally finite refinement $\{V_j\}$, with refinement map μ. $\mu^q(f)$ is still an element of $\ker(d^q)$, so with (16) we find that $\mu^q(f) = d^{q-1}\big(h^q(\mu^q(f))\big) \in \mathrm{Im}\,(d^{q-1})$. This implies that $[\mu^q(f)]_{\{}V_j\} = 0$ and therefore we have that

$$\big[[f]_{\{U_i\}}\big] = \big[f_{\{U_i\},\{V_j\},\mathcal{F}}\big([f]_{\{U_i\}}\big)\big] = \big[[\mu^q(f)]_{\{V_j\}}\big] = [0] = 0.$$

All the choices were made arbitrary, so we can conclude that $\check{H}^q(X, \mathcal{F}) = 0$, for the fine sheaf $\mathcal{F}$ and $q \geq 1$.

(c) This is a consequence of the exactness of (14) and the fact that $\bar{i}'^q$ is an isomorphism

(d) Let $h : \mathcal{F} \to \mathcal{F}$ be the identity. For any q we get that $h^q = \mathrm{Id}$, since for any $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$ and σ a $q-$simplex we have that $\big(\xi^q_{\{U_i\}}(f)\big)(\sigma) = h_{|\sigma|}(f(\sigma)) = f(\sigma)$, which implies that:

$$h^q\big([[f]_{\{U_i\}}]\big) = [\overline{\xi}^q_{\{U_i\}}([f]_{\{U_i\}})] = [[\xi^q_{\{U_i\}}(f)]_{\{U_i\}}] = [[f]_{\{U_i\}}].$$

(e) Suppose that

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{h} & \mathcal{F}' \\ & \searrow^{h'} & \downarrow h'' \\ & & \mathcal{F}'' \end{array}$$

commutes. Then for any open cover $\{U_i\}$ and any $f \in \mathrm{C}^q_{\{U_i\},\mathcal{F}}$ and σ a $q-$simplex we get that

$$\Big(\xi''^q_{\{U_i\}} \circ \xi^q_{\{U_i\}}(f)\Big)(\sigma) = h''_{|\sigma|}\bigg(\Big(\xi^q_{\{U_i\}}(f)\Big)(\sigma)\bigg) = h''_{|\sigma|} \circ h_{|\sigma|}(f(\sigma)) = h'_{|\sigma|}(f(\sigma)) = \Big(\xi'^q_{\{U_i\}}(f)\Big)(\sigma).$$

With this is will be easy to show that (e) is fulfilled:

$$\begin{aligned} h''^q \circ h^q\Big(\big[[f]_{\{U_i\}}\big]\Big) &= h''^q\Big(\big[[\xi^q_{\{U_i\}}(f)]_{\{U_i\}}\big]\Big) = \big[[\xi''^q_{\{U_i\}} \circ \xi^q_{\{U_i\}}(f)]_{\{U_i\}}\big] \\ &= \big[[\xi'^q_{\{U_i\}}(f)]_{\{U_i\}}\big] = h'^q\Big(\big[[f]_{\{U_i\}}\big]\Big). \end{aligned}$$

(f) The homomorphism of the short exact sequences of sheaves clearly induces commutativity in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}^*_{\{U_i\},\mathcal{F}'} & \xrightarrow{\xi_{\{U_i\}}} & \mathrm{C}^*_{\{U_i\},\mathcal{F}} & \xrightarrow{\xi'_{\{U_i\}}} & \bar{\mathrm{C}}^*_{\{U_i\},\tilde{\mathcal{F}}''} & \longrightarrow & 0 \\ & & \downarrow \xi_{g',\{U_i\}} & & \downarrow \xi_{g,\{U_i\}} & & \downarrow \xi_{g'',\{U_i\}} & & \\ 0 & \longrightarrow & \mathrm{C}^*_{\{U_i\},\tilde{\mathcal{F}}'} & \xrightarrow{\tilde{\xi}_{\{U_i\}}} & \mathrm{C}^*_{\{U_i\},\tilde{\mathcal{F}}} & \xrightarrow{\tilde{\xi}'_{\{U_i\}}} & \bar{\mathrm{C}}^*_{\{U_i\},\tilde{\mathcal{F}}''} & \longrightarrow & 0 \end{array}$$

Now Theorem 3 implies that

$$\begin{array}{ccc} \bar{H}^q_{\{U_i\}}(X,\mathcal{F}'') & \xrightarrow{\hat{\partial}_{\{U_i\},\mathcal{F}}} & \check{H}^{q+1}_{\{U_i\}}(X,\mathcal{F}') \\ \downarrow \overline{\xi}_{g'',\{U_i\}} & & \downarrow \overline{\xi}_{g',\{U_i\}} \\ \bar{H}^q_{\{U_i\}}(X,\tilde{\mathcal{F}}'') & \xrightarrow{\hat{\partial}_{\{U_i\},\tilde{\mathcal{F}}}} & \check{H}^{q+1}_{\{U_i\}}(X,\tilde{\mathcal{F}}') \end{array}$$

is commutative. Using the isomorphism $\overline{i}'^q$ we get that

$$\begin{aligned} g'^{q+1} \circ \partial^q\Big(\big[[f]_{\{U_i\}}\big]_{\mathcal{F}}\Big) &= g'^{q+1}\Big(\big[\hat{\partial}^q_{\{U_i\}}([f]_{\{U_i\}})\big]_{\mathcal{F}}\Big) = \big[\overline{\xi}^{q+1}_{g'} \circ \hat{\partial}^q_{\{U_i\}}([f]_{\{U_i\}})\big]_{\tilde{\mathcal{F}}} \\ &= \big[\hat{\partial}^q_{\{U_i\}} \circ \overline{\xi}^q_{g''}([f]_{\{U_i\}})\big]_{\tilde{\mathcal{F}}} = \partial^q\Big(\big[(\overline{\xi}^q_{g''}[f]_{\{U_i\}})\big]_{\tilde{\mathcal{F}}}\Big) = \partial^q \circ g''^q\Big(\big[[f]_{\{U_i\}}\big]_{\mathcal{F}}\Big) \end{aligned}$$

and with this we have proven (f) as well. We conclude that the Čech cohomology modules indeed form a cohomology theory □

8 The de Rham theorem

In the last few sections we have shown that there exists a cohomology theory for a paracompact, Hausdorff space X. Furthermore we have shown that when X is a differentiable manifold the de Rham cohomology is isomorphic to the differentiable singular cohomology: $H^q_{deR}(X) \simeq H^p(X,\mathcal{R}) \simeq H^q_{\Delta,\infty}(X;\mathbb{R})$. In this section we will realize this de Rham isomorphism .

Lemma 7.1: Let $k_U^p : \Omega^q(U) \to \mathrm{S}^p_{\mathbb{R},(\infty)}(U)$ be the map such that

$$(k_X^p(\omega))(\sigma) = \int_\sigma \omega,$$

for a p-form ω over the open U and a differentiable singular $p-$chain ω in U. Then k_U is a cochain map.
Proof: We have to show commutativity in the following diagram:

$$\begin{array}{ccc} \Omega^p(X) & \xrightarrow{k^p} & \mathrm{S}^p_{\mathbb{R},(\infty)}(X) \\ \downarrow d^q & & \downarrow d^q_\infty \\ \Omega^{p+1}(X) & \xrightarrow{k^{p+1}} & \mathrm{S}^{p+1}_{\mathbb{R},(\infty)}(X) \end{array}$$

So we have to show that

$$\int_\sigma d^q(\omega) = \int_{\partial_{p+1}(\sigma)} \omega.$$

For each ω and σ. This is however true by Stokes' theorem. □

Definition 7.1: de Rham homomorphism
Let k_X be as in Lemma 7.1. Then the induced maps on the cohomology modules $\overline{k}^q_X : H^q_{deR}(X) \to H^q_{\Delta,\infty}(X;\mathbb{R})$ form the so-called *de Rham homomorphism.*

Theorem 7: de Rham theorem The de Rham homomorphism is the canonical isomorphism between $H^*_{deR}(X)$ and $H^*_{\Delta,\infty}(X;\mathbb{R})$.
Proof: Lemma 7.1 implies that we have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{R} & \xrightarrow{i} & \Omega^0 & \xrightarrow{d^0} & \Omega^1 & \xrightarrow{d^1} & \omega^2 & \xrightarrow{d^2} & ... \\ & & \downarrow \mathrm{Id} & & \downarrow k^0 & & \downarrow k^1 & & \downarrow k^2 & & \\ 0 & \longrightarrow & \mathcal{R} & \xrightarrow{i} & \mathcal{S}^0_{\mathbb{R},\infty} & \xrightarrow{d^0_\infty} & \mathcal{S}^1_{\mathbb{R},\infty} & \xrightarrow{d^1_\infty} & \mathcal{S}^2_{\mathbb{R},\infty} & \xrightarrow{d^2_\infty} & ... \end{array}$$

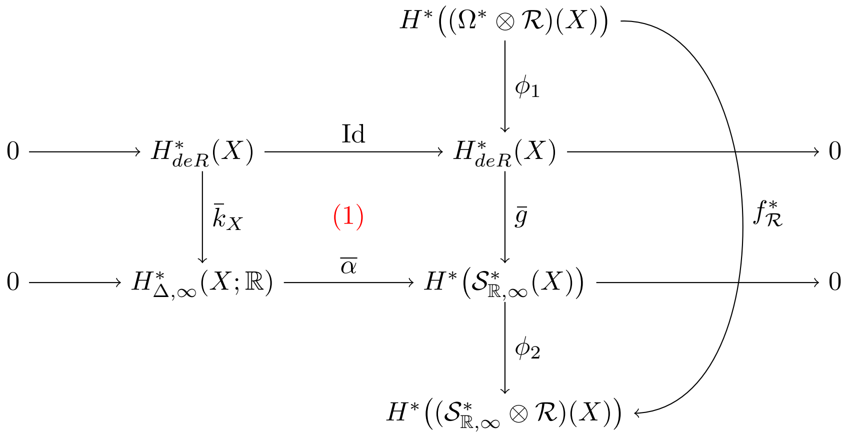
We will use this diagram together with Example 5.2, but we first consider the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \Omega^*(X) & \xrightarrow{\mathrm{Id}} & \Omega^*(X) & \longrightarrow & 0 \\ & & & & \downarrow k_X & & \downarrow g & & \\ 0 & \longrightarrow & \mathrm{S}_{0,\mathbb{R}^*,\infty} & \xhookrightarrow{i} & \mathcal{S}^*(X) & \xrightarrow{\alpha} & \mathcal{S}^*_{\mathbb{R},\infty}(X) & \longrightarrow & 0 \end{array}$$

Here α is as in Theorem 6.3 and g is induced by k so that

$$(g^p(\omega))(x) = r_x^X\left(k_X^p(x)\right) = (\alpha^p(k_X^p(\omega)))(x).$$

Hence the diagram is commutative and furthermore by Lemma 6.1.5 has it exact rows. When going over to the cohomologies of the cochain complexes we get the following diagram:



with ϕ_1 the isomorphism constructed in section 7.2, ϕ_2 the isomorphism given by Lemma 5.1 and $f_{\mathcal{R}}$ by Example 5.2. Since g is a homomorphism, we have that $\overline{g}$ is one as well. This implies that $\phi_1 \circ \overline{h} \circ \phi_2$ is a homomorphism from $H^*\big((\Omega^* \otimes \mathcal{R})(X)\big)$ into $H^*\big((\mathcal{S}^*_{\mathbb{R},\infty} \otimes \mathcal{R})(X)\big)$. We have shown in Theorem 5 and the remark there after that there only exists one unique homomorphism between them and $f_{\mathcal{R}}$ is already such an isomorphism. Therefore we have that $\phi_1 \circ \overline{g} \circ \phi_2 = f_{\mathcal{R}}$. Furthermore we still have commutativity in (1), since:

$$\overline{g}^p\big([\omega]\big) = [g^p(\omega)] = [\alpha^p(k^p_X(\omega))] = \overline{\alpha}^p\big([k^p_X(\omega)]\big) = \overline{\alpha}^p \circ \overline{k_X}^p\big([\omega]\big).$$

Finally we note that by construction the canonical isomorphism between the de Rham and is given by $(\overline{\alpha})^{-1} \circ (\phi_2)^{-1} \circ f_{\mathcal{R}} \circ (\phi_1)^{-1}$. Using the above identities gives therefore that the canonical isomorphism is the same as:

$$(\overline{\alpha})^{-1} \circ (\phi_2)^{-1} \circ f_{\mathcal{R}} \circ (\phi_1)^{-1} = (\overline{\alpha})^{-1} \circ \overline{g} = \overline{k}_X.$$

□

9 The Hodge theorem

Throughout this section we let X to be a compact oriented Riemannian manifold X of dimension n. For X we will proof the Hodge theorem and use this to proof the Poincaré duality for the de Rham cohomology. In this section we will assume two propositions to be true. These can be proven with help of Sobolev spaces and require quite some analysis.

Definition 8.1: Hodge star operator on a inner product space
First we will define the *Hodge star operator* $\star$ *for a* $n-$*dimensional real inner product space* V. We will pick an orientation on V by picking one of the two connected components of $\bigwedge^n V - \{0\}$ and declare this to be positive. Then for each orthonormal basis $\{e_i\}$ of V we let

$$\star(1) = \pm e_1 \wedge ... \wedge e_n, \qquad \star(e_1 \wedge ... \wedge e_n) = \pm 1,$$
$$\star(e_1 \wedge ... \wedge e_p) = \pm e_{p+1} \wedge ... \wedge e_n.$$

Here we use a $+1$ whenever $e_1 \wedge ... \wedge e_n$ lies in the positive component and -1 otherwise. Then we extend this operator to whole $\bigwedge V$ such that it is linear.

Definition 8.2: Inner product on $\bigwedge V$
Let V be as before. We can extend the inner product on V to an *inner product* $(,)$ *on* $\bigwedge V$. When $v = v_1 \wedge ... \wedge v_p$ and $w = w_1 \wedge ... \wedge w_p$ are two decomposable elements of $\bigwedge^p V$, then we set:

$$(v, w) := \det\Big(\big((v_i, w_j)\big)\Big)$$

and we extend this linear. When the elements are not of the same degree, then we let $(v, w) = 0$.

Lemma 8.1: $(,)$ on $\bigwedge V$ is an inner product.
Proof: We have to proof three things: linearity, symmetry and positive-definiteness.
1: Linearity holds by definition.
2: Our original inner product is symmetric. So for the decomposable elements v and w we get symmetry as well:

$$(v,w) = \det\Big(\big((v_i, w_j)\big)\Big) = \det\Big(\big((w_j, v_i)\big)\Big) = \det\Big(\big((w_i, v_j)\big)\Big) = (w, v).$$

And with the linearity we get symmetry in general.
3: Pick an orthonormal basis $\{e_i\}$. Then any element in $\bigwedge^p V$ can be written as $\sum_{|I|=p} c_I e_I$, where I is an ordered subset of $\{1, ..., n\}$ and $e_I = \bigwedge_{i \in I} e_i$ with respect to the ordering. Suppose that $k \in I$ and $k \notin J$. Then $(e_j, e_k) = 0$ for all $k \in K$. Which implies that there is an row with only zeroes in de matrix $\big((e_i, e_j)\big)_{i \in I, j \in J}$. From this we conclude that $(e_I, e_J) = 0$ whenever $J \neq I$. Furthermore when $I = J$ we get that the matrix has only zeroes except on the diagonal. And on the diagonal there is a one in each entry. So:

$$\Big(\sum_{|I|=p} c_I e_I, \sum_{|I|=p} c_I e_I\Big) = \sum_I \sum_J c_I \cdot c_J (e_I, e_J) = \sum_I c_I \cdot c_I (e_I, e_I) = \sum_I (c_I)^2.$$

So it holds that $(v, v) \geq 0$ for all $v \in \bigwedge V$. Furthermore, from the above calculation we can conclude that it is only zero when all the c_I are zero and hence only when $v = 0$. $\square$

Lemma 8.2: The Hodge star satisfies the following identities for $v, w \in \bigwedge^p V$:

$(A):\ \star\star(v) = (-1)^{p(n-p)} v,$ $\qquad (B):\ w \wedge \star(v) = v \wedge \star(w),$

$(C):\ \star(w \wedge \star(v)) = (v, w).$

Proof: Let $v = e_1 \wedge ... \wedge e_p$ for an orthonormal basis $\{e_i\}$. Then:

$$\begin{aligned} e_{p+1} \wedge ... \wedge e_n \wedge e_1 \wedge ... \wedge_p &= (-1)^{n-p} e_1 \wedge e_{p+1} \wedge ... \wedge e_n \wedge e_2 \wedge ... \wedge_p = ... \\ &= (-1)^{p(n-p)} e_1 \wedge ... \wedge e_n. \end{aligned}$$

From this we conclude that if $(-1)^{p(n-p)}$ is positive, then lies $e_{p+1} \wedge ... \wedge e_n \wedge e_1 \wedge ... \wedge_p$ in the same component as $e_1 \wedge ... \wedge e_n$, and otherwise it lies in the other component. This implies that:

$$\begin{aligned} \star\star(e_1 \wedge ... \wedge e_p) &= \pm \star (e_{p+1} \wedge ... \wedge e_n) = (\pm 1) \cdot (\pm 1) \cdot (-1)^{p(n-p)} e_1 \wedge ... \wedge e_p \\ &= (-1)^{p(n-p)} e_1 \wedge ... \wedge e_p. \end{aligned}$$

We know that $\{e_I\}_{|I|=p}$ is a basis of $\bigwedge^p V$, whenever $\{e_i\}$ is a basis of V. Since the above made computation holds for any re-ordening of our orthonormal basis, we conclude with linearity of the Hodge star that (A) holds for any $v \in \bigwedge^p V$.
Let $\{e_i\}$ and v be as before and suppose that w is decomposable in this basis. When $e_j \wedge w \neq 0$ for some $1 \leq j \leq p$, then $e_j \wedge \star(w) = 0$, so $v \wedge \star(w) = 0$. At the same time we get that there must exist a $k > p$ such that $w = e_k \wedge \tilde{w}$. This is due to $\{e_i\}$ being an orthonormal basis. Furthermore it is clear that $e_k \wedge \star(v) = 0$ so that $w \wedge \star(v) = 0$. In this case (B) holds. Now assume w is still decomposable, but that $e_j \wedge w = 0$ for all $1 \leq j \leq p$. So w is up to a constant the same as v. Due to linearity we have that $v \wedge \star(cv) = (cv) \wedge \star(v)$ and so in this case (B) holds as well. Since our orthonormal basis was arbitrary, we get that (B) holds for any re-ordening of a fixed basis. So letting v, w be arbitrary, that is $v = \sum_{|I|=p} c_I e_I$ and $w = \sum_{|J|=p} c_J e_J$, gives that:

$$\begin{aligned} v \wedge \star(w) &= \Big(\sum_I c_I e_I\Big) \wedge \star\Big(\sum_J c_J e_J\Big) = \sum_I \sum_J c_I \cdot c_J e_I \wedge \star(e_J) \\ &= \sum_I \sum_J c_I \cdot c_J e_J \wedge \star(e_I) = \Big(\sum_J \cdot c_J e_J\Big) \wedge \star\Big(\sum_I c_I e_I\Big). \end{aligned}$$

so (B) holds in general.
For (C) we repeat the above strategy: for e_I and e_J with $I \neq J$ we get that $(e_I, e_J) = 0 = \star(0) = \star(e_I \wedge \star(e_J))$. Furthermore for e_I we get that $\star(e_I \wedge \star(e_I)) = \pm\star(e_I \wedge e_{I^c}) = (\pm 1)^2 1 = 1 = (e_I, e_I)$. So C holds for elements e_I and e_J. These elements formed a basis. Now we will use linearity again to see that:

$$\Big(\sum_I c_I e_I, \sum_J c_J e_J\Big) = \sum_I \sum_J c_I \cdot c_J (e_I, e_J) = \sum c_I \cdot c_J \star \big(e_I \wedge \star(e_J)\big)$$
$$= \star\Big(\sum_I c_I e_I \wedge \star\big(\sum_J c_J e_J\big)\Big).$$

Hence (C) holds in general. □

Definition 8.3: Hodge star operator on a manifold
Since our manifolds are Riemannian and oriented we can define a Hodge star operator $\star_x$ on the tangent space of X at x for each $x \in X$. We define the *Hodge star operator* $\star$ *on the space of forms* $\Omega(X)$ such that $(\star(\alpha))(x) = \star_x(\alpha(x))$.

Lemma 8.3: The Hodge star operator on a manifold is well-defined and it satisfies (A) and (B) of Lemma 8.1.
Proof: We need to show that $\star$ takes smooth forms into smooth forms. Pick coordinates, then α is represented by $\sum_I f_I dx_I$ with f_I smooth functions. Then by definition $\star(\alpha) = \sum_I \pm f_I dx_I$ is a smooth form over our open U. It is even a smooth form over whole X, since the orientation of the tangent spaces is smoothly, which implies that the signs change smoothly as well.
First, for (A) we get that:

$$\big(\star\star(\alpha)\big)(x) = \star_x\big((\star(\alpha))(x)\big) \star_x \star_x(\alpha(x)) = (-1)^{p(n-p)}\alpha(x).$$

And for (B) we have that:

$$\big(\alpha \wedge \star(\beta)\big)(x) = \alpha(x) \wedge (\star(\beta))(x) = \alpha(x) \wedge \star_x(\beta(x)) = \beta(x) \wedge \star_x(\alpha(x))$$
$$= \beta(x) \wedge (\star(\alpha))(x) = \big(\beta \wedge \star(\alpha)\big)(x).$$

□

Definition 8.4: δ operator and the Laplacian
We define the *$p-$th $\delta-$operator* $\delta^p : \Omega^p(X) \to \Omega^{p-1}(X)$ such that for a $p-$form α on X we let

$$\delta^p(\alpha) = \begin{cases} (-1)^{m(p+1)+1} \star^{n-p+1} d^{n-p} \star^p (\alpha) & \text{when } p \geq 1; \\ 0 & \text{when } p = 0, \end{cases}$$

where composition is understood. The *$p-$th Laplacian* $\Delta^p : \Omega^p(X) \to \Omega^P(X)$ is then defined as $\Delta^p = \delta^{p+1}d^p + d^{p-1}\delta^p$. From now one we will omit the indices p for all the operators, so that $d(\alpha) = d^p(\alpha)$ whenever α is a $p-$form.

Lemma 8.4: The Laplacian commutes with the Hodge star operator.
Proof: Let α be a $p-$form for $1 \leq p \leq n-1$. Then we want that

$$d\delta \star (\alpha) + \delta d \star (\alpha) = \Delta^{n-p} \star (\alpha) = \star\Delta^p(\alpha) = \star d\delta(\alpha) + \star\delta d^p(\alpha).$$

Note that modulo 2 we have that $n \equiv n^2$ and $-n \equiv n$ for all $n \in \mathbb{Z}$. We will use this in the following computations. The left hand side is equal to:

$$\Delta^{n-p} \star (\alpha) = (-1)^{n(n-p+1)+1} d \star d \star\star(\alpha) + (-1)^{n(n-p+2)+1} \star d \star d \star (\alpha)$$
$$= (-1)^{n(n-p+1)+1} \cdot (-1)^{p(n-p)} d \star d(\alpha) + (-1)^{n(n-p+2)+1} \star d \star d \star (\alpha)$$
$$= (-1)^{p+1} d \star d(\alpha) + (-1)^{n+np+1} \star d \star d \star (\alpha).$$

The right hand side is equal to:

$$\begin{aligned}\star\Delta^p(\alpha) &= (-1)^{n(p+1)+1} \star d \star d \star (\alpha) + (-1)^{n(p+2)+1} \star \star d \star d(\alpha) \\ &= (-1)^{n(p+1)+1} \star d \star d \star (\alpha) + (-1)^{n(p+2)+1} \cdot (-1)^{(n-p)p} d \star d(\alpha) \\ &= (-1)^{n+np+1} \star d \star d \star (\alpha) + (-1)^{p+1} d \star d(\alpha),\end{aligned}$$

and hence we have equality of $\star\Delta^p$ with $\Delta^{n-p}\star$ for $1 \leq p \leq n-1$. For $p = 0$ and $p = n$ we have that $\Delta^0 = \partial^1 d^0$ and $\Delta^n = d^{n-1}\partial^n$. This means that:

$$\Delta^n \star (\alpha) = d^{n-1}\delta^n \star (\alpha) = -d \star d \star \star(\alpha) = -d \star d(\alpha) = - \star \star d \star d(\alpha) = \star\delta^1 d^0(\alpha) = \star\Delta^0(\alpha),$$

$$\Delta^0 \star (\alpha) = \delta^1 d^0 \star (\alpha) = - \star d \star d \star (\alpha) = \star d^{n-1}\delta^n(\alpha) = \star\Delta^n(\alpha).$$

So the Laplacian commutes with the Hodge star for every p. □

Definition 8.5: inner product on $\sum_p \Omega^p(X)$
We are able to define a $p-th$ *inner product associated to the Hodge star* $\langle , \rangle$ on $\Omega^p(X)$ by setting

$$\langle \alpha, \beta \rangle := \int_X \alpha \wedge \star(\beta).$$

We are able to extend this to an inner product on $\sum_p \Omega^p(X)$, when we say that $\langle \alpha, \beta \rangle = 0$ whenever α and β are not both elements of $\Omega^p(X)$ for some p. The induced norm is denoted by $||.||$.
Lemma 8.5: The above introduced inner product is indeed an inner product.
Proof: We will need to proof three things: linearity, symmetry and positive-definiteness.
1: Linearity is obvious, since integrals are always linear in their integrands.
2: This is an easy computation with help of Lemma 8.3:

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \star(\beta) = \int_X \beta \wedge \star(\alpha) = \langle \beta, \alpha \rangle.$$

3: Lemma 8.2 tells us that for each $x \in X$ and all forms α: $(\alpha(x), \alpha(x)) = (\star(\alpha \wedge \star(\alpha)))(x)$ and therefore:

$$(\alpha \wedge \star(\alpha))(x) = (-1)^0 (\alpha \wedge \star(\alpha))(x) = \star\star(\alpha \wedge \star(\alpha))(x) = \star((\alpha(x), \alpha(x))).$$

Since X is orientable, there exists a volume form ω over X, such that $\int_X \omega \geq 1$ and $\star(1) = \omega$. Using this form gives us that:

$$\langle \alpha, \alpha \rangle = \int_X \alpha \wedge \star(\alpha) = \int_X (\alpha(x), \alpha(x))\omega \geq 0.$$

Here we have used $(,)$ is positive-definite. Clearly the above computation shows that $\langle \alpha, \alpha \rangle = 0$ if and only if $\alpha(x) = 0$ for each x and thus only when $\alpha = 0$. We conclude that the $\langle , \rangle$ is positive-definite. □

From now on we will mean this inner product whenever we mention an inner product.

Lemma 8.6: δ is the adjoint of d and the Laplacian is self-adjoint.
Let α be a $p-1$ form and β a $p$form. Since d is an anti-derivation, we have that $d(\alpha \wedge \star(\beta)) = d(\alpha) \wedge \star(\beta) + (-1)^{p-1}\alpha \wedge d \star (\beta)$. Therefore we get that:

$$\begin{aligned}\langle d(\alpha), \beta \rangle &= \int_X d(\alpha) \wedge \star(\beta) = \int_X d(\alpha \wedge \star(\beta)) + (-1)^p \int_X \alpha \wedge d \star (\beta) \\ &= \int_{\partial(X)} \alpha \wedge \star(\beta) + (-1)^p \int_X \alpha \wedge (-1)^{(n-p+1)(p-1)} \star \star d \star (\beta) \\ &= 0 + \int_X \alpha \wedge \star(-1)^{n(p+1)+1} \star d \star (\beta) = \int_X \alpha \wedge \star\delta(\beta) = \langle \alpha, \delta(\beta) \rangle.\end{aligned}$$

Because the inner product is symmetric we indeed have that δ is the adjoint of d. Furthermore we have that

$$\begin{aligned}\langle \Delta(\alpha), \beta \rangle &= \langle \delta d(\alpha), \beta \rangle + \langle d\delta(\alpha), \beta \rangle = \langle d(\alpha), d(\beta) \rangle + \langle \delta(\alpha), \delta(\beta) \rangle \\ &= \langle \alpha, \delta d(\beta) \rangle + \langle \alpha, d\delta(\beta) \rangle = \langle \alpha, \Delta(\beta) \rangle.\end{aligned}$$

So Δ is self-adjoint. □

Definition 8.6: closed and harmonic forms
A $p-$form α is called *closed* when $d(\alpha) = 0$ and *co-closed* when $\delta(\alpha) = 0$. When $\Delta(\alpha) = 0$ we will call it *harmonic*. The space of all harmonic $p-$forms will be denoted by H^p.

Lemma 8.7: A form is harmonic if and only if it is closed and co-closed.
Proof: The first part is trivial: when $d(\alpha) = 0$ and $\delta(\alpha) = 0$ we get that $\Delta(\alpha) = \delta d(\alpha) + d\delta(\alpha) = 0 + 0 = 0$. Now suppose that $\Delta(\alpha) = 0$. Then:

$$0 = \langle 0, \alpha \rangle = \langle \Delta(\alpha), \alpha \rangle = \langle d\delta(\alpha), \alpha \rangle + \langle \delta d(\alpha), \alpha \rangle = \langle \delta(\alpha), \delta(\alpha) \rangle + \langle d(\alpha), d(\alpha) \rangle.$$

Since $\langle , \rangle$ is an inner product, we conclude that $d(\alpha) = 0 = \delta(\alpha)$. So α is closed and co-closed. □

Definition 8.7: weak solutions of $\Delta(\omega) = \alpha$
Given an $\alpha \in \Omega^p(X)$, we say that a bounded linear functional $l : \Omega^p(X) \to \mathbb{R}$ is a *weak solution of* $\Delta(\omega) = \alpha$ when for all $\beta \in \Omega^p(X)$ the following holds:

$$l(\Delta(\beta)) = \langle \alpha, \beta \rangle.$$

The definition for a general partial differential operator L is slightly different. Then we will have to look for a solution l such that $l(L^*(\beta)) = \langle \alpha, \beta \rangle$, with L^* the formal adjoint of L. However $\Delta^* = \Delta$ so this definition implies Definition 8.5. The first proposition we will assume is about this weak solutions. The second one is about Cauchy sequences in $\Omega^p(X)$ with respect to the norm $||.||$. These two proposition will be then used in the Hodge Theorem.

Proposition 8.1: Let $\alpha \in \Omega^p(X)$ and suppose that l is a weak solution of $\Delta(\omega) = \alpha$. Then there exists an $\omega \in \Omega^p(X)$ such that $l(\beta) = \langle \omega, \beta \rangle$ for all $\beta \in \Omega^p(X)$. This ω is then a solution of $\Delta(\omega) = \alpha$, since for all β:

$$\langle \Delta(\omega) - \alpha, \beta \rangle = \langle \omega, \Delta(\beta) \rangle - \langle \alpha, \beta \rangle = l(\Delta(\beta)) - \langle \alpha, \beta \rangle = 0.$$

Proposition 8.2: Let $(\alpha_n)_n$ be a sequence of forms in $\Omega^p(X)$ such that there exists a constant $c > 0$ for which the following holds for all n:

$$||\alpha_n|| \leq c, \qquad ||\Delta(\alpha_n)|| \leq c.$$

Then there exists a subsequence of $(\alpha_n)_n$ which is Cauchy.

Lemma 8.8: There exists a constant $c > 0$ such that for all $\beta \in (H^p)^\perp$ the following inequality holds:

$$||\beta|| \leq ||\Delta(\beta)||.$$

Proof: We will proof this by contradiction. Suppose that for all $c > 0$ there exists a $\beta \in (H^p)^\perp$ such that $||\beta|| > c||\Delta(\beta)||$ holds. Let β_n be the normalized form, corresponding to $c = \frac{1}{n}$. This gives us a sequence $(\beta_n)_n$ such that $||\beta_n|| = 1$ for all n and $||\Delta(\beta_n)|| \to 0$. Proposition 8.2 gives us a subsequence of $(\beta_n)_n$ which is Cauchy. Denote this subsequence with $(\beta_n)_n$ as well. Now let $\psi \in \Omega^p(X)$. Then the sequence $\langle \beta_n, \psi \rangle$ is Cauchy, since for large enough n, m we get with Cauchy-Schwarz that:

$$|\langle \beta_n, \psi \rangle - \langle \beta_m, \psi \rangle| = |\langle \beta_n - \beta_m, \psi \rangle| \overset{C.S.}{\leq} ||\beta_n - \beta_m|| \cdot ||\psi|| < \epsilon.$$

$\mathbb{R}$ is complete and we get that $\lim_n \langle \beta_n, \psi \rangle$ exists. Now we are able to define $l : \Omega^p(X) \to \mathbb{R}$ by setting $l(\psi) = \lim_n \langle \beta_n, \psi \rangle$. l is clearly linear, and bounded as well, since:

$$||l|| = \sup_{||\psi||=1} |l(\psi)| = \sup_{||\psi||=1} |\lim_n \langle \beta_n, \psi \rangle| \leq \sup_{||\psi||=1} \lim_n |\langle \beta_n, \psi \rangle| \overset{C.S.}{\leq} \sup_{||\psi||=1} \lim_n ||\beta_n|| \cdot ||\psi|| = 1.$$

Furthermore:

$$l(\Delta(\psi)) = \lim_n \langle \beta_n, \Delta(\psi) \rangle = \lim_n \langle \Delta(\beta_n), \psi \rangle \overset{C.S.}{\leq} \lim_n ||\Delta(\beta_n)|| \cdot ||\psi|| = 0.$$

Clearly this implies that l is a weak solution of $\Delta(\omega) = 0$ and with Proposition 8.1 we have a $\beta \in \Omega^p(X)$ such that $l(\psi) = \langle \beta, \psi \rangle$. So

$$0 = \lim_n \langle \beta_n, \psi \rangle - \langle \beta, \psi \rangle = \langle \lim_n \beta_n - \beta, \psi \rangle,$$

for all ψ. We conclude that $\lim_n \beta_n = \beta$. Since $||\beta_n|| = 1$ for all n we have that $||\beta|| = 1$ and since $\beta_n \in (H^p)^\perp$ for all n we have that $\beta \in (H^p)^\perp$. Proposition 8.1 implies however that $\beta \in H^p$. This is a contradiction, since $||\beta|| = 1$ implies that $\beta \neq 0$. We conclude that there exists a $c > 0$ such that $||\beta|| \leq ||\Delta(\beta)||$ for all $\beta \in (H^p)^\perp$. $\square$

Theorem 8: Hodge decomposition theorem For each p the space of harmonic forms H^p is finite dimensional. Furthermore we have that an orthogonal direct sum decomposition of $\Omega^p(X)$:

$$\begin{aligned} \Omega^p(X) &= \Delta(\Omega^p) \oplus H^p \\ &= d\delta(\Omega^p) \oplus \delta d(\Omega^p) \oplus H^p \\ &= d(\Omega^{p-1}) \oplus \delta(\Omega^{p+1}) \oplus H^p. \end{aligned}$$

Proof: Assume that the space of harmonic forms in not finite dimensional. Then we can find a sequence (α_n) of forms with length one, which are all orthogonal to each other. Each elements α_n and α_m with $n \neq m$ give us that:

$$||\alpha_n - \alpha_m||^2 = \langle \alpha_n - \alpha_m, \alpha_n - \alpha_m \rangle = \langle \alpha_n, \alpha_n \rangle + \langle \alpha_m, \alpha_m \rangle - 2\langle \alpha_n, \alpha_m \rangle = 2.$$

Clearly no subsequence can be Cauchy and H^p being infinite dimensional is therefore in contradiction with Proposition 8.2.
Now pick an orthonormal basis $\{\omega_i\}_{1 \leq i \leq r}$ of H^p. Then an arbitrary $p-$form α can be written as:

$$\alpha = \Big(\alpha - \sum_{i=1}^r \langle \alpha, \omega_i \rangle \omega_i\Big) + \sum_{i=1}^r \langle \alpha, \omega_i \rangle \omega_i.$$

Note that $(\alpha - \sum_{i=1}^r \langle \alpha, \omega_i \rangle \omega_i)$ is an element of $(H^p)^\perp$. This is true since for all ω_j we have that:

$$\langle \Big(\alpha - \sum_{i=1}^r \langle \alpha, \omega_i \rangle \omega_i\Big), \omega_j \rangle = \langle \alpha, \omega_j \rangle - \sum_{i=1}^r \langle \alpha, \omega_i \rangle \langle \omega_i, \omega_j \rangle = \langle \alpha, \omega_j \rangle - \langle \alpha, \omega_j \rangle = 0.$$

So we get the decomposition $\Omega^p(X) = (H^p)^\perp \oplus H^p$. We will show that $\Delta(\Omega^p) = (H^p)^\perp$, which proves the first decomposition. The first part is trivial: for each $\alpha \in H^p$ and $\beta \in \Omega^p(X)$ the following holds:

$$\langle \Delta(\beta), \alpha \rangle = \langle \beta, \Delta(\alpha) \rangle = \langle \beta, 0 \rangle = 0,$$

so that $\Delta(\Omega) \subset (H^p)^\perp$. For the other inclusion we let $\alpha \in (H^p)^\perp$. Furthermore, let $l : \Delta(\Omega^p) \to \mathbb{R}$ be defined as $l(\Delta(\beta)) := \langle \alpha, \beta \rangle$ for each $\beta \in \Omega^p(X)$. l is linear since Δ and the inner product are. It is well defined since $\Delta(\beta) = \Delta(\phi)$ gives us that $\beta - \phi \in H^p$ and therefore $0 = \langle \alpha, \beta - \phi \rangle = l(\Delta(\beta)) - l(\Delta(\phi))$. Finally we will use Lemma 8.8 to show that l is bounded. Let $\beta \in \Omega^p(X)$. Let $\xi = \beta - \psi$, with ψ the harmonic part of β. Note that $\Delta(\xi) = \Delta(\beta) - \Delta(\psi) = \Delta(\beta)$. So we get that:

$$|l(\Delta(\beta)) = |l(\Delta(\xi))| = |\langle \alpha, \xi \rangle| \overset{C.S.}{\leq} ||\alpha|| \cdot ||\xi|| \leq c||\alpha|| \cdot ||\Delta(\xi)|| = c||\alpha|| \cdot ||\Delta(\beta)||,$$

from which we conclude that $||l|| \leq c||\alpha||$ and is thus bounded. The Hahn-Banach theorem assures us that we can extend l unto whole $\Omega^p(X)$ and this is extension is therefore a weak solution of $\Delta(\omega) = \alpha$. Proposition 8.1 implies that such an ω exists and we conclude that $\alpha \in \Delta(\Omega^p)$. Therefore the second inclusion holds as well and $\Delta(\Omega^p) = (H^p)^\perp$. We now have the first decomposition that we had to proof.
For the second decomposition we use that $\Delta = d\delta + \delta d$. We note that $d\delta(\Omega^p)$ and $\delta d(\Omega^p)$ are orthogonal, since for each $\alpha, \beta \in \Omega^p(X)$ Lemma 8.6 gives us that:

$$\langle d\delta(\alpha), \delta d(\beta) \rangle = \langle d^2\delta(\alpha), d(\beta) \rangle = 0.$$

So the second decomposition holds. The third decomposition is then a direct consequence of Lemma 8.7. □

Definition 8.8: Green's operator
Let $H : \Omega^p(X) \to H^p$ be the orthogonal projection onto the harmonic forms. Then the *Green's operator* $G : \Omega^p(X) \to (H^p)^\perp$ is the map which sends a $p-$form α into ω_α, which is a solution of $\Delta(\omega_\alpha) = \alpha - H(\alpha)$ such that $\omega_\alpha \in \Delta(\Omega^p)$.
Lemma 8.9: The Green's operator is a well-defined bounded self-adjoint operator and it takes bounded sequences into sequences which have a Cauchy subsequence.
Proof: By Theorem 8 we find a unique $\Delta(\omega) \in \Delta(\Omega^p)$ such that $\Delta(\omega) = \alpha - H(\alpha)$. Whenever $\Delta(\tilde{\omega})$ is another solution, then $0 = \Delta(\omega) - \Delta(\tilde{\omega}) = \Delta(\omega - \tilde{\omega})$, and hence $\omega - \tilde{\omega} \in H^p$. Again with Theorem 8 we get that $\omega = \Delta(\xi) - H(\omega)$ for some unique $\Delta(\xi) \in \Delta(\Omega^p)$ and we let $\omega_\alpha = \Delta(\xi)$ for this ξ. So the map G is unique. Furthermore H is a projection and therefore linear. Using this gives us that $G(\alpha + \beta)$ is the unique solution of:

$$\Delta(\omega_{\alpha+\beta}) = (\alpha + \beta) - H(\alpha + \beta) = \alpha - H(\alpha) + \beta - H(\beta) = \Delta(\omega_\alpha) + \Delta(\omega_\beta) = \Delta(\omega_\alpha + \omega_\beta).$$

Since $\omega_\alpha + \omega_\beta \in \Delta(\Omega^p)$ and uniqueness of $\omega_{\alpha+\beta}$ we conclude that $\omega_{\alpha+\beta} = \omega_\alpha + \omega_\beta$ and that G therefore linear is. We will use Lemma 8.8 and the orthonormal basis $\{\omega_i\}_{i=1}^r$ of H^p to proof that G is bounded. We find that:

$$\begin{aligned} ||G|| &= \sup_{||\alpha||=1} ||G(\alpha)|| \leq \sup_{||\alpha||=1} c \cdot ||\Delta(\omega_\alpha)|| = \sup_{||\alpha||=1} c \cdot ||\alpha - H(\alpha)|| \leq \sup_{||\alpha||=1} c(||\alpha|| + ||H(\alpha)||) \\ &= \sup_{||\alpha||=1} c\Big(1 + \Big|\Big|\sum_{j=1}^r \langle \alpha, \omega_j \rangle \omega_j \Big|\Big|\Big) \leq \sup_{||\alpha||=1} c\Big(1 + \sum_{j=1}^r |\langle \alpha, \omega_j \rangle|\Big) \overset{C.S.}{\leq} \sup_{||\alpha||=1} c\Big(1 + \sum_{j=1}^r ||\alpha|| \cdot ||\omega_j||\Big) \\ &= c \cdot (r+1). \end{aligned}$$

So G is indeed bounded. The next thing we will check is whether G self-adjoint is. First of all we note that $G(\alpha) \in (H^p)^\perp$ and is therefore orthogonal to anything in the image of H. We compute that:

$$\langle G(\alpha), \beta \rangle = \langle \omega_\alpha, \Delta(\omega_\beta) + H(\beta) \rangle = \langle \Delta(\omega_\alpha), \omega_\beta \rangle + 0 = \langle \alpha, \omega_\beta \rangle - \langle H(\alpha), \omega_\beta \rangle = \langle \alpha, G(\beta) \rangle.$$

For the final part of this lemma we let $(\alpha_n)_n$ be a bounded sequence of forms, bounded by the constant M. Denote the image of α_n under G by ω^n. Like before: for each α_n we have that:

$$||H(\alpha_n)|| = \Big|\Big|\sum_{j=1}^r \langle \alpha_n, \omega_j \rangle \omega_j \Big|\Big| \leq \sum_{j=1}^r |\langle \alpha_n, \omega_i \rangle| \cdot ||\omega_j|| \overset{C.S.}{\leq} \sum_{j=1}^r ||\alpha_n|| \cdot ||\omega_j|| \leq r \cdot M.$$

Combine this with Lemma 8.8 and we get that

$$||\omega^n|| \leq c \cdot ||\Delta(\omega^n)|| = c \cdot ||\alpha_n - H(\alpha_n)|| \leq c \cdot ||\alpha_n|| + c \cdot ||H(\alpha_n)|| \leq c \cdot M + c \cdot r \cdot M.$$

All these constant are independent of n and therefore are we allowed to use Proposition 8.2, which tells us that the image of $(\alpha_n)_n$ has indeed a subsequence which is Cauchy. □

Theorem 9: Hodge Theorem The following identities hold:

(A) $HG = GH = H\Delta = \Delta H = 0$;

(B) $\mathrm{Id} = H + \Delta G = H + G\Delta$;

(C) $\Delta d = d\Delta$ and $\delta\Delta = \Delta\delta$;

(D) $dG = Gd$ and $\delta G = G\delta$ and $\Delta G = G\Delta$,

and there exists a unique harmonic form for each de Rham cohomology class.
Proof: We will start with proving that not only G but H is self-adjoint as well. Using the orthonormal basis $\{\omega_i\}$ for H^p gives us that for arbitrary $p-$forms α and β:

$$\langle H(\alpha), \beta\rangle = \langle \sum_{j=1}^{r} \langle \alpha, \omega_i\rangle \omega_i, \beta\rangle = \sum_{j=1}^{r} \langle \alpha, \omega_i\rangle \cdot \langle \omega_i, \beta\rangle = \langle \alpha, \sum_{j=1}^{r} \langle \beta, \omega_i\rangle \omega_i\rangle = \langle \alpha, H(\beta)\rangle.$$

Using that $G(\alpha) \in (H^p)^\perp$ and $H(\alpha) \in H^p$ for all α gives that for a fixed α and an arbitrary β:

$$\langle HG(\alpha), \beta\rangle = \langle G(\alpha), H(\beta)\rangle = 0, \qquad \langle GH(\alpha), \beta\rangle = \langle H(\alpha), G(\beta)\rangle = 0.$$

This holds for all β, so we are allowed to conclude that $HG = GH = 0$. In exactly the same manner we compute that:

$$\langle H\Delta(\alpha), \beta\rangle = \langle \Delta(\alpha), H(\beta)\rangle = 0, \qquad \langle \Delta H(\alpha), \beta\rangle = \langle H(\alpha), \Delta(\beta)\rangle = 0,$$

since $\Delta(\alpha) \in (H^p)^\perp$ as well. So again $H\Delta = \Delta H = 0$ and we have proven (A).
The first equality of (B) is by construction of G. From this equality and (A) we conclude that $\Delta(\alpha) = H\Delta(\alpha) + \Delta\omega_{\Delta(\alpha)} = \Delta(\omega_{\Delta(\alpha)})$. Therefore we get that $\alpha - \omega_{\Delta(\alpha)} \in H^p$. By construction we have that $\omega_{\Delta(\alpha)} \in (H^p)^\perp$ and therefore there must exist some $\xi \in H^p$ such that $\alpha = \omega_{\Delta(\alpha)} + \xi$. By Theorem 8 we have that ξ has to be equal to $H(\alpha)$, because of the orthogonal decomposition. With this we have proven the second equality of (B) as well.
δ^2 is up to a sign equal to $\star d \star \star d \star$, which is op to a sign equal to $\star d^2 \star$ because of Lemma 8.3. So we get that $d^2 = 0 = \delta^2$. So $d\Delta = d^2\delta + d\delta d = d\delta d = d\delta d + \delta d^2 = \Delta d$ and $\delta\Delta = \delta^2 d + \delta d\delta = \delta d\delta = \delta d\delta + d\delta^2 = \Delta\delta$. So (C) is easily shown to hold.
Let $\partial \in \{d, \delta\}$. Using the last decomposition of $\Omega^p(X)$ of Theorem 8 we get that for a fixed α and an arbitrary β:

$$\langle H\partial(\alpha), \beta\rangle = \langle \partial(\alpha), H(\beta)\rangle = 0, \qquad \langle \partial H(\alpha), \beta\rangle = \langle H(\alpha), \partial^*(\beta)\rangle = 0.$$

from which we conclude that $H\partial = 0 = \partial H$. This means with (B) and (C) that

$$\Delta\partial G(\alpha) = \partial\Delta(\omega_\alpha) = \partial(\alpha - H(\alpha)) = \partial(\alpha) - \partial H(\alpha) = \partial(\alpha) = \partial(\alpha) - H\partial(\alpha) = \Delta G\partial(\alpha),$$

for any α. Therefore $G\partial(\alpha) - \partial G(\alpha) \in H^p$. However by construction of G we get that $G\partial(\alpha) \in (H^p)^\perp$ and $\partial G(\alpha) \in (H^p)^\perp$ is a consequence of Theorem 8. So $\partial G(\alpha) - G\partial(\alpha) \in H^p \cap (H^p)^\perp = \{0\}$. With this we have proven the first two identities of (D). The third one is a direct consequence of these two:

$$\Delta G = d\delta G + \delta dG = dG\delta + \delta Gd = Gd\delta + G\delta d = G\Delta.$$

Finally we will show that there exists a unique harmonic form in each de Rham class. We start with existence. Let α be a closed $p-$form. Then:

$$\alpha \overset{(B)}{=} H(\alpha) + \Delta G(\alpha) = H(\alpha) + d\delta G(\alpha) + \delta d G(\alpha) \overset{(D)}{=} H(\alpha) + d\delta G(\alpha) + \delta G d(\alpha)$$
$$= H(\alpha) + d\delta G(\alpha).$$

This implies that the harmonic form $H(\alpha)$ lies in the same de Rham class as α and thus has each class a harmonic representative. Now we will check uniqueness. Suppose α and β are both harmonic and in the same de Rham class. Then $\beta = \alpha + d\gamma$ for some $(p-1)-$form γ. Rewriting gives that $0 = d\gamma + (\alpha - \beta)$. Lemma 8.6 together with Lemma 8.7 has a nice consequence now:

$$\langle d\gamma, \alpha - \beta \rangle = \langle \gamma, \delta(\alpha - \beta) \rangle = \langle \gamma, 0 \rangle = 0,$$

so $d\gamma$ is orthogonal to $\alpha - \beta$. we conclude that for $0 = d\gamma + (\alpha - \beta)$ to hold, that both $d\gamma$ and $\alpha - \beta$ have to be zero and therefore is the harmonic representative unique. □

With this we have proven Hodge Theory for the Laplacian. The rest of this section are some consequence of this theorem.

Consequence 8.1: The cohomology modules of a compact, orientable, differentiable manifold are all finite dimensional.
Proof: First of all we can pick a metric on each differentiable manifold, by using the standard one on each chart. Then by this Theorem 9 there exists a bijection between the de Rham cohomology groups and the harmonic forms. By Theorem 8, each cohomology group is finite dimensional, since H^p is. Finally we use section 7 to see that each cohomology theory is isomorphic to de Rham cohomology and therefore finitely dimensional. □

Consequence 8.2: Poincaré duality The function $f : H^p_{deR}(X) \times H^{n-p}_{deR}(X)$ for a compact orientable, differentiable manifold of dimension n, which is defined as

$$f([\alpha], [\beta]) := \int_X \alpha \wedge \beta$$

is well-defined and bilinear. Furthermore it is a non-singular pairing and determines isomorphisms of $H^p_{deR}(X)$ with the dual space of $H^{n-p}_{deR}(X)$.
Proof: To prove that f is well-defined we need to check whether the integral depends on which forms of the classes we take. We will use Stokes' theorem to prove that it does not depend on this choice. Remember that for a $p-$form α we have that $d(\alpha \wedge \gamma) = d(\alpha) \wedge \gamma + (-1)^p \alpha \wedge d(\gamma)$. Now:

$$\int_X \alpha \wedge (\beta + d(\gamma)) = \int_X \alpha \wedge \beta + \int_X \alpha \wedge d(\gamma) = \int_X \alpha \wedge \beta + (-1)^p \int_X d(\alpha \wedge \gamma) - (-1)^p \int_X d(\alpha) \wedge \gamma$$
$$= \int_X \alpha \wedge \beta + (-1)^p \int_{\partial(X)} \alpha \wedge \gamma - (-1)^p \int_X 0 \wedge \gamma = \int_X \alpha \wedge \beta,$$

$$\int_X (\alpha + d(\gamma)) \wedge \beta = \int_X \alpha \wedge \beta + \int_X d(\gamma) \wedge \beta = \int_X \alpha \wedge \beta + \int_X d(\gamma \wedge \beta) - (-1)^{p-1} \int_X \gamma \wedge d(\beta)$$
$$= \int_X \alpha \wedge \beta + \int_{\partial(X)} \gamma \wedge \beta - (-1)^{p-1} \int_X \gamma \wedge 0 = \int_X \alpha \wedge \beta.$$

Bilinearity of f is obvious from the fact that the wedge product is bilinear and integral are linear as well. To prove that f is a non-singular pairing, we need to find a class $[\beta]$ in $H^{n-p}_{deR}(X)$ for each non-zero class $[\alpha]$ in $H^p_{deR}(X)$ such that $f([\alpha], [\beta]) \neq 0$, and the other way around. So let $[\alpha]$ be any class. We can assume that α is harmonic by Theorem 9. And $\alpha \neq 0$, since $[\alpha] \neq 0$. Because of Lemma 8.4 we get that $\star(\alpha)$ is harmonic as well, and it represents a class in $H^{n-p}_{deR}(X)$ because of Lemma 8.7. Furthermore:

$$f([\alpha],[\star(\alpha)]) = \int_X \alpha \wedge \star(\alpha) = ||\alpha||^2 \neq 0.$$

The other way around is obvious now: we let $[\alpha] := [\star(\beta)]$. The isomorphism ψ that f induces is the map which sends $[\alpha] \mapsto f([\alpha], .)$. We still need to proof that ψ is an isomorphism, but it is clearly injective. The map $\tilde{\psi} : H^{n-p}_{deR}(X) \to (H^p_{deR})^*$ is injective as well. Since Consequence 8.1 holds, we get that $\mathrm{Dim}(H^p_{deR}(X)) \leq \mathrm{Dim}(H^{n-p}_{deR}(X)) \leq \mathrm{Dim}(H^p_{deR}(X))$, where I use a standard fact of linear algebra, which states that finite dimensional vector spaces are isomorphic to its dual. SO ψ is an injective linear map between same-dimensional vector spaces. We conclude that ψ is indeed an isomorphism. □

Consequence 8.3: If X is a compact, orientable, differentiable manifold of dimension n, then $H^n_{deR}(X) \simeq \mathbb{R}$.
Proof: Zero-forms are smooth maps from X into $\mathbb{R}$ and only the constant maps are closed. Since there exist no -1-forms, we see that $H^0_{deR}(X) \simeq \mathbb{R}$. The above consequence and the algebra fact stated in the proof of it imply that $H^n_{deR}(X) \simeq \mathbb{R}$. □

10 Bibliography

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