Between generalized complex and Poisson geometry

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Between generalized complex and Poisson geometry

Elliptic symplectic structures, boundary fibrations and homogeneous vector fields

Tussen gegeneraliseerde complexe en Poisson meetkunde

Elliptisch symplectische structuren, randfibraties en homogene vectorvelden

(met een samenvatting in het Nederlands)

Proefschrift

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Introduction

One of the main topics in mathematics is the study of *spaces*. The first class of spaces one encounters are *topological spaces*; in these one can make sense of two points being close to each other. However, to model the physics of the space in which we live, we need notions such as speed and acceleration. *Smooth manifolds* are precisely the topological spaces where we have these notions. To address specific problems one often uses extra data on these manifolds, called *geometric structures*. For instance, if one wants to measure distances one needs a *Riemannian metric* and if one wants to measure areas, one needs an *orientation*. We will study two geometric structures in this thesis; *Poisson structures* and *generalized complex structures*.

Poisson geometry

Poisson geometry can be seen as a combination of three classical mathematical subjects: foliation theory, symplectic geometry and Lie theory.

A *(singular) foliation* on a manifold is a "nice" partition into subspaces of (possibly varying) dimension, called the *leaves*. The best understood foliations are the *regular* ones, which are those for which the leaves all have the same dimension. The existence of regular foliations is well-studied: for instance, a compact manifold admits a regular foliation with leaves of codimension-one if and only if its Euler characteristic is zero.

Symplectic geometry provides the mathematical background for the Lagrangian formulation of classical mechanics – it provides a framework to understand the notion of phase space. A Poisson structure can be thought of as a family of symplectic structures: it defines a singular foliation, together with a symplectic structure on each of the leaves. The leaves of this foliation are therefore called the *symplectic leaves* of the Poisson structure.

Lie algebras are algebraic objects which model (infinitesimal) symmetries of physical systems. From a Poisson geometric point of view they correspond to one of the simplest classes of Poisson manifolds, the ones with linear behaviour.

As in any area of mathematics, the question of constructing and classifying examples of Poisson structures is studied extensively. However, as any manifold admits the trivial Poisson structure, the question of existence has to be refined to finding "nice" Poisson structures. What "nice" means depends on the context;

we can look through the lens of each of the three subjects making up Poisson geometry:

From a foliation theory point of view, the class of nice Poisson structures consists of *regular* Poisson structures, which are those for which the underlying foliation is regular. Examples of these structures are difficult to find. Especially on closed manifolds there is very little known. For instance, such a Poisson structure on S^5 was only discovered recently [65].

From a symplectic point of view, the class of nice Poisson structures consists of those which are very close to being non-degenerate. These are Poisson structures which have a big (open and dense) symplectic leaf and can be interpreted as symplectic structures with singularities. In the past years an entire zoo of such structures has been studied. To name a few: Log symplectic [45], scattering symplectic [53], elliptic symplectic [16] and c-symplectic [64]. The singularities which can occur are best described using *Lie algebroids* – objects which behave like a tangent bundle and capture the singularities of the desired object. A general framework for these structures was put forth in [52]. These structures seem to be more abundant than regular Poisson structures, although that still does not make it easy to construct them. See the above citations, as well as [58, 14].

From a Lie theoretic point of view, the class of nice Poisson structures consists of those which admit an *integrating symplectic groupoid*, an object which plays the same role as a Lie group integrating a Lie algebra. The problem of when a given Poisson manifold admits such an integration has been completely solved in [23]. This sparked the quest of finding Poisson manifolds with well-behaved integrations [25, 27, 26]. Constructing interesting examples of these seems to be even more difficult than constructing regular Poisson structures, see [59, 81].

Normal forms An important tool in the study of Poisson structures are *normal form theorems*. These are results which describe the local behaviour of a Poisson structure around a submanifold and can be very useful in constructing new examples via surgery operations. The first normal form for symplectic structures is *Darboux' theorem* which states that all symplectic structures are locally isomorphic.

General Poisson structures are not all locally isomorphic. However, *Weinstein's splitting theorem* states that around any point $x \in M$ a Poisson manifold is isomorphic to a product $(N, \pi_N) \times (S, \pi_S)$, where π_S is a symplectic structure and π_N is a Poisson structure which vanishes at x.

Because of Darboux' theorem, we are thus left with studying the normal form of Poisson structures vanishing at a point. *Conn's linearisation theorem* [20] states that, under certain Lie algebraic conditions, a Poisson structure is locally isomorphic to its *linearisation* around x. The question of which Poisson structures are linearisable around zeroes is still open and active, see for instance the recent thesis of Zeiser [79] and the references therein.

The story for higher dimensional submanifolds becomes more involved. In symplectic geometry, *Weinstein's Lagrangian neighbourhood theorem* [76] provides, for a Lagrangian submanifold $L \subset (M, \omega)$, a local model on T^*L, to which the original symplectic structure is isomorphic around L. Similarly, let (M, π) be a regular Poisson structure with underlying foliation $\mathcal{F}$ and L a submanifold which intersects every symplectic leaf of π in a Lagrangian submanifold. Then there is a tubular neighbourhood $p : \mathcal{U} \to L$ and a local model on $p^*T^*(\mathcal{F} \cap L)$ to which the original Poisson structure is isomorphic on $\mathcal{U}$.

Many of the above normal form theorems share the same property: The model spaces are vector spaces or vector bundles and the model structure is *homogeneous* with respect to the scalar multiplication. For instance, a multi-vector field $X \in \mathfrak{X}^p(E)$ on a vector bundle E is said to be homogeneous of degree d if $m_\lambda^* X = \lambda^{d-p} X$, where $m_\lambda : E \to E$ denotes the scalar multiplication on the fibres. Therefore, it is interesting to study these homogeneous structures on their own. This is the topic of Chapter 4 of this thesis: We will study homogeneous multi-vector fields and describe the algebraic structure present on the space of these. We will then outline a philosophy on how to use this algebraic structure to better understand the homogeneous objects appearing in normal form results. The study of this filtered Gerstenhaber structure is interesting on its own and may very well have applications outside of normal form results. For instance, as we shall explain in Chapter 4, it allows us to give an explicit description of Lie algebroid cohomology with values in certain representations up to homotopy.

Generalized complex geometry

Generalized complex structures ([47], [41]) are a simultaneous generalisation of symplectic and complex structures. Any generalized complex structure induces a Poisson structure and we can therefore talk about its symplectic leaves. Moreover, it provides a complex structure on the directions normal to the symplectic leaves. Most interesting to study are those generalized complex structures for which the underlying symplectic foliation is non-regular, a phenomenon known as *type change*. The simplest type change behaviour happens for *stable generalized complex structures* ([16, 38]). These are generalized complex structures which have one open and dense symplectic leaf and have type change among a codimension-two embedded submanifold. Although they only display the simplest type-change behaviour, stable generalized complex manifolds have a rich geometry and there are several interesting examples, especially in four-dimensions.

The type change behaviour of stable generalized complex structures is so well-behaved that they are completely determined (up to gauge equivalence) by the underlying Poisson structure. The class of Poisson structures underlying stable generalized complex structures is called *elliptic symplectic*. These can be described as symplectic structures with singularities, just as the ones discussed above. The question of existence of stable generalized complex structures thus becomes a question of existence of certain Poisson structures.

Many stable generalized complex structures were constructed. For instance, on the manifolds $n\#\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}$ ([15]), $n\#(S^2 \times S^2)$ ([74]) and $n\#(S^2 \times S^2)\#(S^1 \times S^3)$ ([39]), under clear conditions on n and m.

Although there exists many examples of stable generalized complex structures no Poisson Fano manifold admits a smooth stable generalized complex structure in real dimension greater than four [44] (including $\mathbb{C}P^n$ with $n > 2$). Another strange feature is that, although many examples are constructed on manifolds which are connected sums, there is no connected sum procedure for stable generalized complex structures. This is one of the reasons why we will consider the larger class of *self-crossing stable* generalized complex structures. These are generalized complex structures where the type change locus is now allowed to intersect itself transversely. This, apparently small, weakening will give us a lot of flexibility. In particular, it will be immediate that $\mathbb{C}P^{2n}$ admits a self-crossing stable generalized complex structure. And thus indeed, the space of manifolds admitting a self-crossing stable generalized complex structure is larger than the space of manifolds admitting a stable generalized complex structure. At the same time, we move closer to algebraic geometry and in particular to log structures [50], where often even no real distinction is made between holomorphic sections transverse to zero and sections transverse to zero with self-crossings.

On the other hand, the weakening is small enough that much of the original theory of [16] carries over. In particular, it holds that these generalized complex structures are completely determined by their underlying Poisson structure (Theorem 1.3.16). In four dimensions we can actually do much better: Theorem 1.4.12 shows that self-crossing stable generalized complex structures can be deformed into ones with embedded type change locus. Consequently, to prove the existence of stable generalized complex structures in four dimensions, it suffices to construct self-crossing stable generalized structures.

This strategy allows us to use one of the quite remarkable flexibility properties self-crossing stable generalized complex structures exhibit: they behave well under taking connected sums (Theorem 1.5.9). This is in stark contrast with symplectic geometry, in which such a construction cannot exist in dimension higher than 2. We will use very simple self-crossing stable generalized complex structures (or slight variations thereof) on $\mathbb{C}P^2, \overline{\mathbb{C}P^2}, S^2 \times S^2$ and $S^1 \times S^3$ and then use connected sums to construct many examples (Theorem 1.6.5), including the ones from [15, 74, 39] appearing above.

Fibrations

A general theme in the world of geometric structures is the interplay between specific geometric structures and particular types of maps. The simplest example arises in the context of fibrations, where one studies whether the presence of a geometric structure on the base and the fibre implies its existence on the total space. However, not many manifolds admit fibrations and for this reason one has to consider fibrations with singularities.

Especially in symplectic geometry, this idea has proven to be useful. The maps of relevance are *Lefschetz fibrations*, which are surjective submersions outside of some singular points. While at the singular points the map has a Lefschetz singularity, that is, there exists complex coordinates $(z_1, \ldots, z_n)$ in which the fibration takes the form $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$. Using these one can prove broad existence results for symplectic structures [29, 35, 36] and similar techniques have been used in Poisson and related geometries. The study of (singular) fibrations to understand the topology of, and geometric structures on, manifolds has proven very successful, see for instance [2, 4, 5, 6, 7, 14, 13, 17, 31, 32, 54]. In these papers, maps with Lefschetz and similar singularities have been used to prove existence results for several different types of geometric structures.

Another way in which singular fibrations arise is from proper group actions. These are particularly well studied for (full) torus actions, where the singularities are so well-behaved that the quotient is a manifold with corners. The singularities which might occur in the quotient map are build up by taking combinations of the basic elliptic singularity $f(x_1, x_2) = x_1^2 + x_2^2$. The coupling of torus actions with geometric structures leads to many fruitful concepts, one of the highlights being toric geometry.

The Lefschetz and the toric pictures come together in semi-toric geometry [69], where maps are allowed to have both elliptic and Lefschetz singularities. In Chapter 2 we will introduce the maps which are hinted at by semi-toric geometry; *self-crossing boundary (Lefschetz) fibrations* (Definition 2.1.12).

These maps are precisely the ones we need to construct self-crossing stable generalized complex structures. Following the general philosophy, Theorem 2.1.23 shows that when a self-crossing boundary Lefschetz fibration admits the correct geometric structures on its base and fibres, then the total space will admit a stable generalized complex structure. The correct geometric structures on the base are *self-crossing log symplectic structures* which have also appeared in [43, 64]. In this way we will generalise the similar result for stable generalized complex structures from [13].

Having established such an existence result, it now makes sense to study self-crossing boundary Lefschetz fibrations on their own. We will show that these are compatible with taking connected sums (Theorem 2.2.3), which is in sheer contrast to toric geometry.

On the other hand, we will also introduce a smoothing procedure (Theorem 1.4.12) which will allow us to trade Lefschetz singularities with elliptic–elliptic singularities. This is precisely the abstract version of the nodal–trade procedure from semi-toric geometry [80, 61, 55], but now without reference to other geometric structures.

Organisation of the thesis

Below we provide a quick summary of the main results. For a more detailed description, see the introductions to each of the chapters, which can be read relatively independently of each other.

Chapter 1 is based on [18] and describes self-crossing stable generalized complex structures. First, we recall the notions of divisors, which will be used to describe the singularities we encounter. We use these to define the elliptic tangent bundle, a Lie algebroid which will describe the singular symplectic structures which are in one-to-one correspondence with self-crossing stable generalized complex structures (Theorem 1.3.16). In Theorem 1.4.12 we show that every self-crossing stable generalized structure can be deformed into one with embedded type change locus. Theorem 1.5.9 then provides the connected sum procedure, which we will use to construct examples of stable generalized complex structures on $n\#\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}\#l(S^2\times S^2)\#k(S^1\times S^3)$, provided $1-b_1+b_2^+$ is even and the Euler characteristic is non-negative (Theorem 1.6.5).

Chapter 2 is based on [19] and describes the fibrations used to construct self-crossing stable generalized complex structures. After studying their properties we will prove a Gompf-Thurston theorem for these (Theorem 2.1.23) and show that they are compatible with connected sums (Theorem 2.2.3). We then explain the relation with (semi-)toric geometry and exhibit examples of self-crossing boundary Lefschetz fibrations.

Chapter 3 is based on [78] and describes the Lie algebroid cohomology of the elliptic tangent bundle. This is of interest to the deformation of elliptic symplectic structures. In Theorem 3.3.1 we show that the cohomology is completely determined by the topological data of the manifold, the degeneracy locus and its normal bundle.

In Chapter 4 we study homogeneous multi-vector fields on vector bundles. We describe these as a higher degree analogue of multi-derivations (Definition Definition 4.2.3) and study the algebraic structure present (Section 4.2.5). We will very explicitly show that the space of homogeneous multi-vector fields can be described as the space of sections of a vector bundle over the base space (Corollary 4.2.51). This statement can be neatly described in terms of the coadjoint representation up to homotopy of a Lie algebroid (Theorem 4.3.7).

In Chapter 5 we describe the procedure to approximate Poisson structures by their homogeneous Taylor approximation. We will use this to study linear normal form results for Lagrangian submanifolds (Theorem 5.1.34), and quadratic normal form results for log and elliptic symplectic structures.

Chapter 1

Self-crossing stable generalized complex structures

In this chapter, which is based on [18] and joint work with Ralph Klaasse and Gil Cavalcanti, we study a class of generalized complex structures. Generalized complex structures ([47, 41]) are a simultaneous generalisation of complex and symplectic structures. Infinitesimally, a generalized complex manifold is equivalent to a product of a symplectic and a complex vector space. The number of complex directions is called the *type* of the generalized complex structure. It is an upper-semi continuous function and can jump only with two complex dimensions at a time. The simplest type change which can therefore occur is when the generalized complex structure has type zero almost everywhere and type two along a submanifold. In this chapter we study such generalized complex structures.

Any generalized complex structure induces a section of the anticanonical bundle of the manifold, which is non-zero precisely when the type is zero. *Smooth stable generalized complex structures* ([16, 38]) are the generalized complex structures for which this section vanishes transversely. These structures display interesting behaviour and many examples, especially in four-dimensions, have been constructed ([15, 74, 39]).

Yet, smooth stable generalized complex structures have a shortcoming: no Poisson Fano manifold can be smooth generalized complex in real dimension greater than four, as follows from the theory in [44]. Also, many of the manifolds constructed in [15, 74, 39] are connected sums, but are not obtained by performing connected sums. Instead, examples of these structures are constructed using surgery techniques and only the diffeomorphism type is determined only afterwards.

In this chapter we overcome these shortcomings by introducing a larger class of generalized complex structures, called *self-crossing stable*. These are generalized complex structures for which the anticanonical section may now vanish transversely with self-crossings. This is analogous to the move, in real Poisson geometry, from log-symplectic to log-symplectic with normal self-crossings [43, 64].

On the one hand, this weakening is small enough that most of the original theory for smooth stable structures still holds, while one the other hand it allows for flexibility and many more examples. While on the other hand, $\mathbb{C}P^{2n}$ with $n > 2$ obtains a self-crossing stable structure in an immediate fashion, it does not admit a smooth one. Although this notion is a genuine weakening, in dimension four we have the following:

Theorem 1.4.12. *Any four-dimensional self-crossing stable generalized complex structure can be deformed into a stable generalized complex structure.*

Consequently, this allows us to construct examples of smooth stable generalized complex structures by first constructing self-crossing ones. We will construct the latter using symplectic techniques: we will define a Lie algebroid called the elliptic tangent bundle and we will show that self-crossing stable generalized complex structures are in one-to-one correspondence with a subclass of Lie algebroid symplectic structures on the elliptic tangent bundle (Theorem 1.3.13).

This subclass of Lie algebroid symplectic structures satisfy some combination of (semi-)global and local conditions. We will also study symplectic structures on the elliptic tangent bundle which fail to satisfy the local conditions at some points. This allows us to endow certain manifolds which cannot admit generalized complex structures (like $\overline{\mathbb{C}P^2}$ and S^4) with such a symplectic structure on this Lie algebroid.

The existence of such a structure on S^4 paves the way for providing a connected sum procedure for these symplectic structures (c.f. Theorem 1.5.9). After describing some (very simple) examples of these structures we are able to use this procedure to directly construct examples including those from [15, 74, 39]:

Theorem 1.6.5. *The manifolds in the following two families admit stable generalized complex structures:*

1. $\#n(S^2 \times S^2)\#\ell(S^1 \times S^3)$, *with* $n, \ell \in \mathbb{N}$;
2. $\#n\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}\#\ell(S^1 \times S^3)$, *with* $n, m, \ell \in \mathbb{N}$,

as long as $1 - b_1 + b_2^+$ *is even and the Euler characteristic is non-negative.*

Notice that if $1 - b_1 + b_2^+$ is odd for a four-manifold M, then M does not admit any generalized complex structure as it is not even almost complex by [46] or [36, Theorem 1.4.13]. The requirement that the Euler characteristic is positive, on the other hand, seems to be more a limitation of our methods.

Organisation of the chapter. In Section 1.1 we introduce self-crossing complex and elliptic divisors: the basic geometric objects that allow us to develop the theory of stable generalized complex structures. In Section 1.2 we will introduce the Lie algebroids induced by these divisors. These are the spaces where stable generalized complex structures become the more amenable symplectic structures. In Section 1.3 we introduce self-crossing stable generalized complex structures and show that they are equivalent to a certain class of elliptic symplectic structures. In Section 1.4 we focus on four-dimensional structures. We prove a normal form theorem for self-intersection points in the divisor and show that a stable structure can be deformed into a smooth one (Theorem 1.4.12). In Section 1.5 we show that one can perform connected sums of stable generalized complex structures (Theorem 1.5.9) and in Section 1.6 we provide concrete examples obtained via connected sum and prove Theorem 1.6.5.

1.1 Self-crossing divisors

This section covers the basic definitions and properties of the singularities we will encounter. We start by recalling the definition of a divisor, before introducing the complex log divisors we are mostly interested in. After discussing their zero sets in some detail, we turn to elliptic divisors. These can be induced from complex log divisors and will play a large role throughout the thesis. Finally we describe the relation between complex log and elliptic divisors in detail.

1.1.1 Divisors

In this section we define the objects which will govern the singularities of the geometric structures that are to come.

Definition 1.1.1. A real/complex **divisor** on M is a locally principal ideal I of $C^\infty(M;\mathbb{R})$ respectively $C^\infty(M;\mathbb{C})$ which is locally generated by functions with nowhere dense zero set. ♢

Divisor can be equivalently described using line bundles with sections:

Proposition 1.1.2. *Let I be a real/complex divisor on M. Then there exists a real/complex line bundle $L \to M$ with section $\sigma \in \Gamma(L)$ such that $\sigma(\Gamma(L^*)) = I$. Conversely, given any complex line bundle with section with nowhere dense zero set (L, σ), the ideal $I_\sigma := \sigma(\Gamma(L^*))$ defines a complex log divisor.*

Proof. Because I is locally principal, there exists a cover $\{U_\alpha\}$ of M such that $I|_{U_\alpha}$ is generated by a function $f_\alpha \in C^\infty(U_\alpha)$. On the overlaps $U_\alpha \cap U_\beta$ we have $f_\alpha = g_{\alpha\beta} f_\beta$, for some nowhere vanishing functions $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta})$. These functions provide the data of a Čech cocycle defining the desired line bundle L. The section σ is defined to be given by f_α on U_α. □

Note that the line bundle L is uniquely determined up to vector bundle isomorphism (covering the identity), and that the section σ is unique up to multiplication by a smooth function. Given a pair (L, σ), we denote the associated divisor by I_σ.

Definition 1.1.3. Let (M, I_M) and (N, I_N) be manifolds with divisors. A smooth map $\varphi\colon M \to N$ is a **morphism of divisors** if $\varphi^* I_N = I_M$, where the left-hand side denotes the ideal generated by all pullbacks. It is called a **diffeomorphism of divisors** if φ is a diffeomorphism. ♢

Definition 1.1.4. A **smooth real/complex log divisor** is a real/complex divisor I locally generated by transverse vanishing functions. ♢

The **vanishing locus** of a real log divisor has codimension one and is denoted by Z. The vanishing locus of a complex log divisor has codimension two and is denoted by D. By locally demanding a divisor to be a product of log divisors we obtain the following:

Definition 1.1.5. A **self-crossing real/complex log divisor** on a manifold M is a divisor I, such that for every point $p \in M$ there exists a neighbourhood U of p such that

$$I_\sigma(U) = I_1 \cdot \ldots \cdot I_j,$$

where $I_1, \ldots, I_j$ are real/complex log divisors with transversely intersecting[1] vanishing loci. ♢

A self-crossing real log divisor is determined by its vanishing locus Z, as its ideal equals the ideal of functions vanishing on Z. In contrast, for a self-crossing complex log divisor, the subspace D does not determine the divisor.

Remark 1.1.6 (Terminology)**.** Our definition of a smooth complex log divisor appears in [16] without the prefix smooth attached. For brevity, we will often write "complex log divisor", which has to be understood to possibly have self-crossings. Whenever we deal with a smooth complex log divisor we will explicitly stress this. ♢

Definition 1.1.7. Let I_D be a real/complex log divisor. For a given point $p \in M$ and a neighbourhood U of p let n_U be the number j as in Definition 1.1.5. The **multiplicity of** p is the minimum of n_U taken over all neighbourhoods of p. The **pointwise multiplicity of the divisor** is the maximum of the multiplicities of its points. ♢

Example 1.1.8. Let $\{I_{D_i}\}$ for $i = 1, \ldots, n$ be a collection of smooth real/complex log divisors with transversely intersecting vanishing loci, for which there exist at least one point in the intersection $\cap_{i=1}^n D_i$. Then their product $I_D := \otimes_{i=1}^n I_{D_i}$ defines a real/complex log divisor with pointwise multiplicity n. We call such a divisor a **strict normal crossing divisor.** △

By definition every real/complex log divisor is locally of this form, which will often be used.

[1] A collection of submanifolds $\{D_i\}$ is said to intersect transversely at a point $x \in M$ if

$$\mathrm{codim}(\bigcap_i T_x D_i) = \sum_i \mathrm{codim}(T_x D_i).$$

Definition 1.1.9. Given a real/complex log divisor I_D, we call a choice of local smooth real/complex log divisors near a point as in Example 1.1.8 a **local normal crossing**. ♢

Note that the only choice in a local normal crossing for a given real/complex log divisor is the ordering of the smooth divisors.

Example 1.1.10. Let $\mathcal{O}(k)$ be the holomorphic line bundle on $\mathbb{C}P^n$ obtained as the k-fold tensor product of the dual of the tautological line bundle. Recall that sections of $\mathcal{O}(k)$ can be identified with homogeneous polynomials of degree k in $n+1$ variables. Under this identification we can view the polynomial $p := z_0 \cdot \ldots \cdot z_n$ as a section of $\mathcal{O}(n+1)$. We conclude that $(\mathcal{O}(n+1), p)$ defines a complex log divisor with pointwise multiplicity n on $\mathbb{C}P^n$. △

Example 1.1.11. Let $E \to M$ be a complex line bundle. Then $\Gamma((E^{1,0})^*) \subset C^\infty(E)$ generates an ideal I on E. Locally, if $U \subset M$ is an open neighbourhood on which $E|_U$ is trivialised, there exists a corresponding fibre coordinate z on U which generates I. We conclude that every complex line bundle carries a canonical smooth complex log divisor whose vanishing locus is the zero section $M \subset E$. △

Example 1.1.12.

- Let $(z_1, \ldots, z_j, x_{2j+1}, \ldots, x_{2j+m})$ be coordinates on $\mathbb{C}^j \times \mathbb{R}^m$ and define smooth complex log divisors $I_{D_i} := \langle z_i \rangle$. Then the ideal $I_D = \otimes_{i=1}^j I_{D_i}$ is called the **standard complex log divisor with pointwise multiplicity** j **on** $\mathbb{C}^j \times \mathbb{R}^m$.
- Let $(x_1, \ldots, x_j, y_i)$ be coordinates on $\mathbb{R}^j \times \mathbb{R}^m$ and define real log divisors $I_{Z_i} := \langle x_i \rangle$. Then the ideal $I_Z = \otimes_{i=1}^j I_{Z_i}$ is called the **standard real log divisor with pointwise multiplicity** j **on** $\mathbb{R}^j \times \mathbb{R}^m$.

△

Lemma 1.1.13.

- *If I_D is a complex log divisor and $p \in M$ is a point of multiplicity j, then I_D is locally diffeomorphic to the standard complex log divisor around p.*
- *If I_Z is a real log divisor and $p \in M$ is a point of multiplicity j, then I_Z is locally diffeomorphic to the standard real log divisor around p.*

Proof. Because the representatives $f_1, \ldots, f_j$ of the ideals of a local normal crossing vanish transversely and their zero sets are transverse, they can be completed to a local coordinate system on an open neighbourhood of p. These coordinates provide the divisor diffeomorphism. □

The vanishing locus of a real/complex log divisor is not an embedded submanifold when its pointwise multiplicity is larger then one, it is however immersed.

Lemma 1.1.14. *Let I_D be a real/complex log divisor on M and let D be its vanishing locus. Then the vanishing locus D is an immersed submanifold.*

Proof. As being immersed is a local property, we need to show that every point in D has a neighbourhood on which D is immersed. By Lemma 1.1.13 it suffices to show that the vanishing locus of the standard real/complex log divisor is an immersed submanifold. The obvious map of inclusions of the different coordinate planes $\bigcup_{i=1}^{j} \mathbb{C}^{j-1} \times \mathbb{R}^m \to \mathbb{C}^j \times \mathbb{R}^m$ provides this immersion. □

Intuitively, the vanishing locus of a real/complex log divisor is the immersion of a manifold, $\widetilde{D}$, obtained from D by duplicating the intersection locus and separating the strands whenever self-crossings occur. Here we need to introduce some subtle language variation to distinguish between different meanings of the word *component* a **connected component** of D is just that, a connected component of D as a subspace of M, while an **irreducible component** of D is the image of a connected component of $\widetilde{D}$.

The degeneracy locus is not only an immersed submanifold but it is also stratified by embedded smooth submanifolds.

Definition 1.1.15. Let I_D be a real/complex log divisor with pointwise multiplicity n on a manifold M. Given $1 \leq j \leq n$, the set of points with multiplicity *at least* j will be denoted by $D(j)$. These sets induce a filtration on M, namely

$$M = D(0) \supset D = D(1) \supset D(2) \supset \cdots \supset D(n).$$

The strata of this stratification are denoted by

$$D[i] := D(i) \backslash D(i+1),$$

and consist of the points with multiplicity *exactly* i. ♢

The following is immediate and will be used without further mention. We will call this stratification the **multiplicity stratification** of M induced by I_D.

Lemma 1.1.16. *Let I_D be a real/complex log divisor on a manifold M with pointwise multiplicity at least i. Then $I_D|_{M\backslash D(i+1)}$ is a real/complex log divisor on $M\backslash D(i+1)$ with pointwise multiplicity i.*

With this in mind we can verify that the definition of the stratification makes sense.

Lemma 1.1.17. *The filtration from Definition 1.1.15 defines a smooth stratification on M.*

Proof. We first note that the highest codimension stratum $D(n)$ is a smooth submanifold by the regular value theorem. Next, the subset $D[i] = D(i)\backslash D(i+1)$ is the highest codimension stratum of the restricted divisor to $M\backslash D(i+1)$ and is therefore smooth. Finally the filtration induces a stratification precisely because we have the local form as described in Lemma 1.1.13. □

1.1.2 Elliptic divisors

We now introduce self-crossing elliptic divisors. These divisors arise as the real part of complex log divsors, but can be defined independently.

Definition 1.1.18 ([16]). A **smooth elliptic divisor** is a real divisor $I_{|D|}$ locally generated by definite Morse–Bott functions with codimension-two critical set D. ♢

In fact, the existence of a local generating function of $I_{|D|}$ implies that there exists a global generator $f \in I_{|D|}$. Note that a given codimension-two submanifold may carry multiple smooth elliptic divisor structures. To proceed we again consider divisors which are local normal crossings of smooth elliptic divisors.

Definition 1.1.19. A **self-crossing elliptic divisor** on a manifold M is a real divisor $I_{|D|}$ such that for every $p \in M$ there exists an open neighbourhood U of p such that

$$I_{|D|}(U) = I_{|D_1|} \cdot \ldots \cdot I_{|D_j|}.$$

Here the $I_{|D_i|}$ are smooth elliptic divisors on U whose zero loci D_i intersect transversely. ♢

As discussed in Remark 1.1.6 we will often omit the prefix "self-crossing" and instead add "smooth" when referring to an elliptic divisor in the sense of [16].

Remark 1.1.20. The condition that the loci of the smooth elliptic divisors are transverse is equivalent to the following statement: for all local generators f_i of $I_{|D_i|}$ we have that

$$\ker \mathrm{Hess}_p(f_{i_1}) \cap \cdots \cap \ker \mathrm{Hess}_p(f_{i_k}) \qquad \text{for all } p \in \cap_{l=1}^k f_{i_l}^{-1}(\{0\})$$

has minimal dimension for all multi-indices $(i_1, \ldots, i_k)$ with distinct elements and of length smaller or equal than k. ♢

Many of the notions we defined for complex log divisors with self-crossings can also be defined for elliptic divisors with self-crossings. In particular they have a multiplicity and an induced stratification, which will be denoted in the same manner as in the complex log case.

An important class of elliptic divisors arises from complex log divisors:

Example 1.1.21. If I_D is a complex log divisor, then $I_D \cdot \overline{I}_D$ is invariant under conjugation. Therefore, there exists a real ideal $I_{|D|}$ such that $I_{|D|} \otimes \mathbb{C} = I_D \cdot \overline{I}_D$. By definition, locally $I_D = I_{D_1} \cdot \ldots \cdot I_{D_n}$, where the I_{D_i} are smooth complex log divisors with transverse zero loci. Therefore we see that

$$I_{|D|}(U) \otimes \mathbb{C} = (I_{D_1} \cdot \overline{I}_{D_1}) \cdot \ldots \cdot (I_{D_n} \cdot \overline{I}_{D_n}),$$

hence $I_{|D|}(U)$ is given as the product of smooth elliptic divisors with transverse vanishing loci. We conclude that $I_{|D|}$ is an elliptic divisor and call it **the elliptic divisor induced by a complex log divisor**. △

The following is analogous to the complex log setting (Example 1.1.8):

Example 1.1.22. Given smooth elliptic divisors $I_{|D_i|}$ for which the vanishing loci D_i are transverse, we have that $I_{|D|} := \otimes_{i=1}^n I_{|D_i|}$ defines an elliptic divisor with pointwise multiplicity n. We call this a **strict normal crossing** elliptic divisor. $\triangle$

By definition, every elliptic divisor is locally of the above form.

Definition 1.1.23. Given an elliptic divisor $I_{|D|}$, we call a choice of local smooth complex divisors near a point as in Definition 1.1.19 a **local normal crossing**. $\diamondsuit$

Example 1.1.24. Let $(x_1, y_1, \ldots, x_j, y_j, x_{j+1}, \ldots, x_m)$ be coordinates on $\mathbb{R}^{2j} \times \mathbb{R}^m$ and define smooth elliptic divisors $I_{|D_i|} = \langle x_i^2 + y_i^2 \rangle$. We call $I_{|D|} := \otimes_{i=1}^n I_{|D_i|}$ the **standard elliptic divisor of pointwise multiplicity** j **on** $\mathbb{R}^{2j} \times \mathbb{R}^m$. $\triangle$

Using the Morse–Bott lemma we can locally put an elliptic divisor in standard form.

Lemma 1.1.25. *Let $I_{|D|}$ be an elliptic divisor on a manifold M and let $p \in M$ have multiplicity j. Then $I_{|D|}$ is locally diffeomorphic to the standard elliptic divisor of pointwise multiplicity j on $\mathbb{R}^{2j} \times \mathbb{R}^m$, where $\dim M = 2j + m$.*

Proof. To simplify notation we will consider $j = 2$ as the proof in the general case is identical. Let U be an open neighbourhood of a point p of multiplicity 2 and let $I_{|D_1|}, I_{|D_2|}$ be a local normal crossing. Let f_1, f_2 be representatives of $I_{|D_1|}, I_{|D_2|}$ respectively and apply the Morse–Bott lemma to obtain coordinates $(x_1, y_1, x_2, y_2, z_3, \ldots)$, $(\widetilde{x}_1, \widetilde{y}_1, \widetilde{x}_2, \widetilde{y}_2, \widetilde{z}_3, \ldots)$ on neighbourhoods U_1, U_2 of p respectively such that $f_1 = x_1^2 + y_1^2$ and $f_2 = \widetilde{x}_2^2 + \widetilde{y}_2^2$. Consider

$$\Phi = (x_1, y_1, \widetilde{x}_2, \widetilde{y}_2, z_3, \ldots) \colon U_1 \cap U_2 \to \mathbb{R}^n,$$

and

$$F \colon \mathbb{R}^n \to \mathbb{R} \quad (x_1, \ldots, x_n) \mapsto x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Then $f_1 + f_2 = F \circ \Phi$. Note that the normal Hessian of F is non-degenerate and that the normal Hessian of $f_1 + f_2$ is non-degenerate by the independence condition in Remark 1.1.20. Because

$$\mathrm{Hess}_{f_1+f_2} = \mathrm{Hess}_{F\circ\Phi} = \mathrm{Hess}_F(\Phi_* \cdot, \Phi_* \cdot),$$

we conclude that Φ_* must be injective on $N(D_1 \cap D_2)$. Therefore, using the inverse function theorem and possibly shrinking the domain of definition, we conclude that $(x_1, y_1, \widetilde{x}_2, \widetilde{y}_2)$ provide coordinates in the directions normal to $D_1 \cap D_2$. Complementing with coordinates on $D_1 \cap D_2$ gives the required coordinate system. $\square$

It follows from this lemma that, just as for complex divisors, the vanishing locus of an elliptic divisor, $I_{|D|}$, is an immersed submanifold, D, with transverse self crossings. Furthermore, if we let (R, q) denote the line bundle with section associated to $I_{|D|}$, since q is a trivialization of R over $M \backslash D$ and D has codimension two, R is also trivializable and q determines a preferred orientation for R. Therefore, one can define an elliptic divisor alternatively as the ideal generated by a function $f\colon M \to \mathbb{R}_+$ whose zeros are locally of the form

$$f(x_1, y_1, \dots, x_k, y_k, x_{k+1}, \dots, x_n) = (x_1^2 + y_1^2) \dots (x_k^2 + y_k^2).$$

Examples A class of examples of elliptic divisors arises from toric geometry.

Example 1.1.26. Let $\mu\colon M \to \mathbb{R}^n$ be a toric manifold with moment polytope Δ and let $\lambda_i \in \mathbb{R}^n$ be vectors transverse to its faces, of which we assume it has k. If we denote $f_{\lambda_i}\colon \mathbb{R}^n \to \mathbb{R}, x \mapsto \langle x, \lambda_i \rangle$, let $c_i \in \mathbb{R}$ be constants such that $I = \langle (f_{\lambda_1} - c_1) \cdot \dots \cdot (f_{\lambda_k} - c_k) \rangle$, when restricted to Δ, defines the vanishing ideal of $\partial\Delta$. Then the ideal

$$f^* I := \langle (f_{\lambda_1} \circ \mu - c_1) \cdot \dots \cdot (f_{\lambda_k} \circ \mu - c_k) \rangle \tag{1.1.1}$$

defines an elliptic divisor on M with pointwise multiplicity n and vanishing locus $\mu^{-1}(\partial\Delta)$. Namely, the functions f_{λ_i} are linear and have the faces of the moment polytope as zero sets. Because M is toric, the components of the moment map are definite Morse–Bott functions, hence so are the compositions $f_{\lambda_i} \circ \mu$. We conclude that the ideal I defines an elliptic divisor. △

An explicit case of the setting in the above example occurs on the manifold $\mathbb{C}P^n$.

Example 1.1.27. Consider $\mathbb{C}P^n$ with the moment map

$$\mu\colon \mathbb{C}P^n \to \Delta \qquad [z_0 : z_1 : \dots : z_n] \mapsto \frac{(|z_0|^2, |z_1|^2, \dots, |z_{n-1}|^2)}{|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}. \tag{1.1.2}$$

Here $\Delta = \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} : \sum_i x_i \leq 1\}$ denotes the moment polytope. Proceeding as in Example 1.1.26 endows $\mathbb{C}P^n$ with the structure of an elliptic divisor. Note that this is the elliptic divisor induced by the complex log divisor of Example 1.1.10. △

1.1.3 Elliptic versus complex log divisors

As we have seen in Example 1.1.21, a complex log divisor (L, σ) induces an elliptic divisor. The complex log divisor also induces a complex structure on the normal bundle of $D[1]$ via the isomorphism[2]

$$d^\nu \sigma|_{D[1]} : ND[1] \to L|_{D[1]}.$$

If we were to pick another section $f\sigma$, for some $f \in C^\infty(M)$, then

$$d^\nu (f\sigma)|_{D[1]} = f|_{C(1)} \, d^\nu(\sigma)|_{D[1]},$$

[2] Because σ vanishes transversely on $D[1]$, the normal derivative is an isomorphism.

and thus the complex structure induced using $d^\nu(f\sigma)|_{D[1]}$ is isomorphic to the complex structure induced using $d^\nu(\sigma)|_{D[1]}$. In particular, the orientation on $ND[1]$ induced is independent of the choice of section. These two pieces of information, the elliptic divisor and the co-orientation, completely determine the complex log divisor. This statement was already mentioned in the smooth case in [16, Section 1.2], but appeared there without proof.

Proposition 1.1.28. *Let M be a manifold. The association*

$$(L, \sigma) \mapsto ((R, q), \mathfrak{o})$$

which sends a complex log divisor on M to its associated elliptic divisor, together with the induced co-orientation of $D[1]$, induces a bijection of complex log divisors and elliptic divisors with chosen co-orientation of $D[1]$.

Proof. Let I_1, I_2 be two complex log divisors. Assume that I_1 and I_2 both induce the same elliptic ideal and the same co-orientation on the normal bundle to $D[1]$. We have to prove that $I_1 = I_2$. We will proceed via several steps.

Injectivity for smooth divisors: We first prove injectivity for smooth divisors. Let z, w be complex coordinates such that $I_1 = \langle z\rangle$ and $I_2 = \langle w\rangle$. By assumption we have $I_1 \otimes \overline{I_1} = I_2 \otimes \overline{I_2}$ and thus there exists a nowhere vanishing function $g \in C^\infty(M, \mathbb{R})$ such that $gz\overline{z} = w\overline{w}$. The fact that I_1 and I_2 induce the same co-orientation ensures the existence of some strictly positive function $h \in C^\infty(D; \mathbb{R})$ such that $hd^\nu z \wedge d^\nu\overline{z} = d^\nu w \wedge d^\nu\overline{w}$.

Claim. *We have that $h = g|_D$.*

Proof of claim. By taking the derivative of $gz\overline{z} = w\overline{w}$ with respect to $\overline{w}$, we obtain that

$$\frac{\partial g}{\partial\overline{w}}z\overline{z} + g\frac{\partial z}{\partial\overline{w}}\overline{z} + gz\frac{\partial\overline{z}}{\partial\overline{w}} = w.$$

By taking the derivative of this equation with respect to z, we get

$$\frac{\partial^2 g}{\partial z\partial\overline{w}}z\overline{z} + \frac{\partial g}{\partial\overline{w}}\overline{z} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial\overline{w}}\overline{z} + g\frac{\partial}{\partial z}\frac{\partial z}{\partial\overline{w}}\overline{z} + \frac{\partial g}{\partial z}z\frac{\partial\overline{z}}{\partial\overline{w}} + g\frac{\partial\overline{z}}{\partial\overline{w}} = \frac{\partial w}{\partial z}.$$

In particular we find $g|_D\,\frac{\partial\overline{z}}{\partial\overline{w}}\big|_D = \frac{\partial w}{\partial z}\big|_D$. Noting that

$$\langle d^\nu w \wedge d^\nu\overline{w}, \partial_z \wedge \partial_{\overline{w}}\rangle = \frac{\partial w}{\partial z} = h\frac{\partial\overline{z}}{\partial\overline{w}},$$

we can combine these facts to conclude that $g|_D = h$. □

Continuing our main line of reasoning, in order to show that $w \in I_1$ we are going to invoke Malgrange's Theorem, [57, Theorem 1.1], which states that $w \in I_1$ if and only if its formal power series with respect to z and $\overline{z}$ is divisible by z. We expand w as a power series in z and $\overline{z}$:

$$w = a_{10}z + a_{01}\overline{z} + \sum_{i+j\geq 2} a_{ij}z^i\overline{z}^j.$$

Claim. *We have that* $|a_{10}|^2 = |a_{01}|^2 + h$.

Proof of claim. Using $hd^\nu z \wedge d^\nu \overline{z} = d^\nu w \wedge d^\nu \overline{w}$, we find

$$\langle d^\nu w \wedge d^\nu \overline{w}, \partial_z \wedge \partial_{\overline{z}} \rangle = h,$$

but also on D we have

$$\begin{aligned}\langle d^\nu w \wedge d^\nu \overline{w}, \partial_z \wedge \partial_{\overline{z}} \rangle &= \frac{\partial w}{\partial z}\frac{\partial \overline{w}}{\partial \overline{z}} - \frac{\partial \overline{w}}{\partial z}\frac{\partial w}{\partial \overline{z}} \\ &= \left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial \overline{w}}{\partial z}\right|^2 \\ &= |a_{10}|^2 - |a_{01}|^2,\end{aligned}$$

from which we conclude the desired expression. □

Knowing this, we can express the product $w\overline{w}$ as

$$\begin{aligned} w\overline{w} = (|a_{10}|^2 + |a_{01}|^2)z\overline{z} + a_{10}\overline{a}_{01}z^2 + a_{10}\sum_{i+j\geq 2}\overline{a_{ij}z}^i z^{j+1} \\ + a_{01}\overline{a}_{10}\overline{z}^2 + a_{01}\sum_{i+j\geq 2}\overline{a}_{ij}\overline{z}^{i+1}z^j + \overline{a}_{10}\sum_{i+j\geq 2}a_{ij}z^i\overline{z}^{j+1} \\ + \overline{a}_{01}\sum_{i+j\geq 2}a_{ij}z^{i+1}\overline{z}^j + \sum_{\substack{i+j\geq 2\\ k+l\geq 2}}a_{ij}\overline{a}_{kl}z^{i+k}\overline{z}^{j+l}, \end{aligned}$$

and because $|a_{10}|^2 = |a_{01}|^2 + h$ and $w\overline{w} = gz\overline{z}$, we see that

$$\begin{aligned} 0 = (h - g + 2|a_{01}|^2)z\overline{z} + a_{10}\overline{a}_{01}z^2 + a_{10}\sum_{i+j\geq 2}\overline{a_{ij}z}^i z^{j+1} \\ + a_{01}\overline{a}_{10}\overline{z}^2 + a_{01}\sum_{i+j\geq 2}\overline{a}_{ij}\overline{z}^{i+1}z^j + \overline{a}_{10}\sum_{i+j\geq 2}a_{ij}z^i\overline{z}^{j+1} \\ + \overline{a}_{01}\sum_{i+j\geq 2}a_{ij}z^{i+1}\overline{z}^j + \sum_{\substack{i+j\geq 2\\ k+l\geq 2}}a_{ij}\overline{a}_{kl}z^{i+k}\overline{z}^{j+l}. \end{aligned}$$

By expanding $h - g$ as a power series and because $g|_D = h$, we see that $h - g = \sum_{i+j\geq 1} b_{ij}z^i\overline{z}^j$. In conclusion we have obtained the following equality of power series:

$$\begin{aligned} 0 = \left(\sum_{i+j\geq 1} b_{ij}z^i\overline{z}^j + 2|a_{01}|^2\right)z\overline{z} + a_{10}\overline{a}_{01}z^2 + a_{10}\sum_{i+j\geq 2}\overline{a_{ij}z}^i z^{j+1} \\ + a_{01}\overline{a}_{10}\overline{z}^2 + a_{01}\sum_{i+j\geq 2}\overline{a}_{ij}\overline{z}^{i+1}z^j + \overline{a}_{10}\sum_{i+j\geq 2}a_{ij}z^i\overline{z}^{j+1} \\ + \overline{a}_{01}\sum_{i+j\geq 2}a_{ij}z^{i+1}\overline{z}^j + \sum_{\substack{i+j\geq 2\\ k+l\geq 2}}a_{ij}\overline{a}_{kl}z^{i+k}\overline{z}^{j+l}. \end{aligned}$$

Therefore all the coefficients of this power series need to vanish. The term of degree 1 in z and degree 1 in $\overline{z}$ is given by $2\,|a_{01}|\,z\overline{z}$, hence we conclude that $a_{01} = 0$. The degree-n term in $\overline{z}$ is given by $\overline{a}_{10}a_{0,n-1}\overline{z}^n$. Hence we can conclude that $\overline{a}_{0,i} = 0$ for all $i \geq 0$. Therefore the formal power series of w is divisible by z and we conclude that $w \in \langle z\rangle$. By symmetry we conclude that $\langle w\rangle = \langle z\rangle$, from which we conclude that $I_1 = I_2$ which finishes this part of the proof.

Injectivity for strict normal crossings: Suppose that $I_{|D|} = \otimes_i I_{|D_i|}$ is a strict normal crossing elliptic divisor. Let I_{D^1} and I_{D^2} be two complex log divisors which induce $I_{|D|}$, which both need to be strict normal crossing divisors because $I_{|D|}$ is. Moreover, assume that they induce the same co-orientation on $D[1]$. This implies that they induce the same co-orientation on each of the zero loci D_i. After reordering, we may assume that

$$I_{\left|D_i^1\right|} = I_{|D_i|} = I_{\left|D_i^2\right|}.$$

Thus for each i, we have that $I_{\left|D_i^1\right|}$ and $I_{\left|D_i^2\right|}$ induce the same smooth elliptic divisor and the same co-orientation and therefore $I_{\left|D_i^1\right|} = I_{\left|D_i^2\right|}$ by the above, from which the result follows.

Injectivity for general divisors: Because $I_{|D|}$ is locally a normal crossing of elliptic divisors, around every point $x \in D$ we can find an open neighbourhood U such that $I_{|D|}\big|_U$ is a normal crossing of elliptic divisors. Therefore, if two general complex log divisors induce the same elliptic ideal and same co-orientation, we can use the injectivity for strict normal crossing divisors and argue locally to prove that the complex log divisors must be equal.

Surjectivity for smooth divisors: Let $I_{|D|}$ be a smooth elliptic divisor and $\mathfrak{o}$ an orientation of ND. Given a choice of representative $f \in I_{|D|}$, we can view $\mathrm{Hess}^\nu f \in \Gamma(S^2N^*D)$ as a metric on N^*D. We use this metric to transport the orientation of ND to an orientation on N^*D. The orientation together with the metric induces a complex structure on ND and hence a complex log divisor structure on ND by Example 1.1.11. This complex log divisor induces $I_{|D|}$, which proves surjectivity for smooth divisors.

Surjectivity for strict normal crossings: Let $I_{|D|} = \otimes I_{|D_i|}$ be a strict normal crossing elliptic divisor. Because $ND[1]|_{D_i\setminus D(2)} = ND_i|_{D_i\setminus D(2)}$ and $D(2)$ is codimension two in D_i we conclude that each ND_i is orientable. Therefore, by the above, there exist complex log divisors I_{D_i} such that $I_{|D_i|} = I_{D_i} \otimes I_{\overline{D}_i}$. We conclude that

$$I_{|D|} \otimes \mathbb{C} = \otimes_i(I_{D_i} \otimes I_{\overline{D_i}}) = (\otimes_i I_{D_i}) \otimes \overline{(\otimes_i I_{D_i})}.$$

Surjectivity for general divisors: Because $I_{|D|}$ is locally a strict normal crossing of elliptic divisors, for every point $x \in D$ we can find an open neighbourhood U such that $I_{|D|}\big|_U$ is a strict normal crossing. Therefore, by the previous part there exists a complex log divisor I_{D_U} which induces $I_{|D|}\big|_U$. Let $\mathcal{U}$ be an open cover of M, such that $I_{|D|}$ is a strict normal crossing on each open in the cover and construct complex log divisors inducing $I_{|D|}$ on each of these opens. Let

$U, U' \in \mathcal{U}$ and let I_U and $I_{U'}$ be complex log divisors inducing $I_{|D|}\big|_U$ and $I_{|D|}\big|_{U}$. On the overlap $U \cap U'$ both $I_{U'}|_{U\cap U'}$ and $I_U|_{U\cap U'}$ induce the same elliptic ideal, by the above we therefore have $I_U|_{U\cap U'} = I_{U'}|_{U\cap U'}$. We conclude that the local complex log divisors glue to a strict complex log divisor which induces $I_{|D|}\big|_{U'}$, which finishes the proof. □

Motivated by this result we define the following.

Definition 1.1.29. An elliptic divisor $I_{|D|}$ is **co-orientable** if $D[1]$ is co-orientable. ♢

1.2 Lie algebroids associated to self-crossing divisors

In this section we introduce the Lie algebroids associated to self-crossing complex log and elliptic divisors. Much of this section follows along the same lines as [16]. The Lie algebroids will be defined by imposing that their sections interact appropriately with the divisors, in that they must preserve the divisor ideals.

1.2.1 Complex log tangent bundle

Lemma 1.2.1. *Let I_D be a complex log divisor on a manifold M. The complex vector fields preserving the complex ideal I_D are sections of a complex Lie algebroid $\mathcal{A}_D$.*

Proof. By Lemma 1.1.13, at every point there exist coordinates $(z_1, \ldots, z_n, x_i)$ such that locally $I_D = \langle z_1 \cdot \ldots \cdot z_n \rangle$. One can readily check that in these coordinates the vector fields preserving I_D are generated by

$$\{z_1\partial_{z_1}, \partial_{\bar{z}_1} \ldots, z_n\partial_{z_n}, \partial_{\bar{z}_n}, \partial_{x_i}\}$$

and therefore form a locally free sheaf which, by the Serre–Swan theorem, correspond to sections of a vector bundle, $\mathcal{A}_D$. The Lie bracket of vector fields induces a bracket on the sections of $\mathcal{A}_D$ making it into a Lie algebroid. □

Definition 1.2.2. The Lie algebroid $\mathcal{A}_D$ is the **complex log tangent bundle** induced by I_D. ♢

There is a local description of the complex log tangent bundle, whose proof is immediate.

Lemma 1.2.3. *Let I_D be a complex log divisor on a manifold M and let $I_{D_1}, \ldots, I_{D_n}$ be a choice of local normal crossing on some open U. Then $\mathcal{A}_D$ is given by a repeated fiber product:*

$$\mathcal{A}_D|_U \simeq \mathcal{A}_{D_1} \times_{T_{\mathbb{C}}M} \times \cdots \times_{T_{\mathbb{C}}M} \mathcal{A}_{D_n}.$$

The story for real log divisors is completely similar:

Definition 1.2.4. Let I_Z be a real log divisor on M. The real vector fields preserving the ideal I_Z are sections of a real Lie algebroid $\mathcal{A}_Z$. ♢

We call this Lie algebroid the **real log tangent bundlereal log tangent bundle** induced by I_Z.

One can prove Darboux-type normal form theorems for symplectic Lie algebroids using a thorough understanding of their Lie algebroid cohomology, by a straightforward adaptation of the Moser lemma. However, in the above cases this cohomology is generally locally non-trivial, so that there is no unique local model. For future reference we remark:

Lemma 1.2.5. *Let I_Z be a real log divisor on Σ^2 and let $\omega \in \Omega^2(\mathcal{A}_Z)$ be a log-symplectic form. For each point $p \in Z[2]$ there are coordinates (x_1, x_2) centered at p and $\lambda \in \mathbb{R}_{>0}$ such that*

$$\omega = \lambda d \log x_1 \wedge d \log x_2.$$

Since in two dimensions every nowhere zero two-form is closed and nondegenerate we have the following source of examples of log-symplectic manifolds:

Lemma 1.2.6. *Let Σ^2 be a compact oriented surface with corners. Then $(\Sigma, I_{\partial\Sigma})$ admits a log-symplectic structure.*

Proof. The ideal $I_{\partial\Sigma}$ defines a real log divisor. Because Σ is oriented, let $h \in C^\infty(M)$ be a defining function for ∂M, so that $I_{\partial\Sigma} = \langle h \rangle$ and let $\omega \in \Omega^2(\Sigma)$ be a volume form. Then $h^{-1}\omega \in \Omega^2(\mathcal{A}_{\partial M})$ is a nondegenerate log two-form that is closed for dimensional reasons. □

1.2.2 Elliptic tangent bundle

There is also a Lie algebroid associated to an elliptic divisor.

Lemma 1.2.7. *Let $I_{|D|}$ be an elliptic divisor on a manifold M. The vector fields preserving the ideal $I_{|D|}$ are sections of a Lie algebroid $\mathcal{A}_{|D|}$.*

Proof. By Lemma 1.1.25 there exists coordinates $(r_1, \theta_1, \ldots, r_n, \theta_n, x_i)$ such that locally we have $I_{|D|} = \langle r_1^2 \cdot \ldots \cdot r_n^2 \rangle$ and in which the vector fields preserving $I_{|D|}$ are generated by

$$\{r_1\partial_{r_1}, \partial_{\theta_1}, \ldots, r_n\partial_{r_n}, \partial_{\theta_n}, \partial_{x_i}\}.$$

This collection forms a locally free sheaf, hence there exists a Lie algebroid $\mathcal{A}_{|D|}$ whose sections are the vector fields preserving $I_{|D|}$. □

Definition 1.2.8. The Lie algebroid $\mathcal{A}_{|D|}$ is the **elliptic tangent bundle** associated to $I_{|D|}$. ♢

Similar to Lemma 1.2.3 we have the following local description of the elliptic tangent bundle.

Lemma 1.2.9. *Let $I_{|D|}$ be an elliptic divisor on a manifold M and let $I_{|D_1|}, \ldots, I_{|D_n|}$ be a choice of local normal crossing on some open U. Then $\mathcal{A}_{|D|}$ is given by a repeated fiber product:*

$$\mathcal{A}_{|D|}\big|_U \simeq \mathcal{A}_{|D_1|} \times_{TM} \times \cdots \times_{TM} \mathcal{A}_{|D_n|}.$$

Above we argued that the ideal $I_{|D|}$ determines the elliptic tangent bundle. The converse is also true. Namely, suppose we are given any Lie algebroid $L \to M$ whose rank agrees with the dimension of M. The anchor map, ρ, induces a bundle map $\det_\rho \colon \wedge^n L \to \wedge^n TM$, which can be regarded as a section of the real line bundle $\wedge^n L^* \otimes \wedge^n TM$. That is, L determines the real divisor $(\wedge^n L^* \otimes \wedge^n TM, \det_\rho)$. Given the local expression for generators of $\mathcal{A}_{|D|}$ we have:

Lemma 1.2.10. *The elliptic tangent bundle determines its underlying ideal.*

When an elliptic divisor is induced from a complex log divisor we can relate the Lie algebroids.

Proposition 1.2.11. *Let I_D be a complex log divisor and let $I_{|D|}$ be the induced elliptic divisor. Then*

$$\mathcal{A}_D \times_{T_{\mathbb{C}}M} \mathcal{A}_{\overline{D}} = \mathcal{A}_{|D|} \otimes \mathbb{C}.$$

Proof. Using local coordinates as in Lemma 1.1.13, we see that the intersection $\rho(\mathcal{A}_D) \cap \rho(\mathcal{A}_{\overline{D}})$ has constant rank and hence we can form their fibre product. The anchor maps, mapping to $\Gamma(T_{\mathbb{C}}M)$, satisfy

$$\rho(\Gamma(\mathcal{A}_D \times_{T_{\mathbb{C}}M} \mathcal{A}_{\overline{D}})) = \rho(\Gamma(\mathcal{A}_D)) \cap \rho(\Gamma(\mathcal{A}_{\overline{D}})).$$

The right-hand side consists of the vector fields preserving $I_D \otimes \overline{I}_D = I_{|D|} \otimes \mathbb{C}$. Therefore $\mathcal{A}_D \times_{T_{\mathbb{C}}M} \mathcal{A}_{\overline{D}}$ is isomorphic to the complexification of the Lie algebroid from Definition 1.2.8. □

The forms of the complex log tangent bundle are particularly easy to work with from the differential geometric point of view, as they relate on forms defined on the complement.

Lemma 1.2.12. *Let I_D be a complex log divisor on M, and let $\alpha \in \Omega^k(M\backslash D)$ be a differential form. Then α extends to a smooth complex log form $\widetilde{\alpha} \in \Omega^k(\mathcal{A}_D)$ if and only if for any local generator f of I_D both $f\alpha$ and $f d\alpha$ extend as smooth forms over D.*

The proof is the same as in the holomorphic setting, and thus omitted.

The cohmology of the complex log tangent bundle also relates neatly to the complement. To compute it we need the following topological result regarding divisors with normal crossings.

Lemma 1.2.13. *If D is the standard normal crossing complex log (or elliptic) divisor with pointwise multiplicity n, then $D[i]$ is homotopic to the disjoint union of $\binom{n}{i}$ copies of T^{n-i}.*

Proof. Let $I_{D_1}, \ldots, I_{D_n}$ be the complex log divisors corresponding to $z_1, \ldots, z_n$ respectively. Then

$$D[i] = D(i)\backslash D(i+1) = \bigsqcup_{(j_1,\ldots,j_i)\subset(1,\ldots,n)} (D_{j_1} \cap \cdots \cap D_{j_i})\backslash D(i+1).$$

All the summands in $D[i]$ are diffeomorphic. For notational clarity we will thus consider the summand corresponding to the ordered multi-index $I := (1, 2, \dots, i)$. We have

$$D_I := D_1 \cap \cdots \cap D_i = \underbrace{\{0\} \times \cdots \times \{0\}}_{i-\text{times}} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{(n-i)-\text{times}}.$$

To proceed we consider the intersection of D_I with $D(i+1)$. To write this down, for $k \in \mathbb{N}$ let

$$\mathbf{0}_k := \underbrace{\{0\} \times \cdots \times \{0\}}_{k-\text{times}}, \qquad \mathbb{C}_k := \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{k-\text{times}}.$$

With this notation in hand, one readily verifies that

$$D_I \cap D(i+1) = \mathbf{0}_i \times \{0\} \times \mathbb{C}_{n-i-1} \cup \mathbf{0}_i \times \mathbb{C} \times \{0\} \times \mathbb{C}_{n-i-2} \cup \cdots \cup \mathbf{0}_i \times \mathbb{C}_{n-i-1} \times \{0\}.$$

From this we immediately see that

$$D_I \backslash D(i+1) = \underbrace{\{0\} \times \cdots \times \{0\}}_{i-\text{times}} \times \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{(n-i)-\text{times}},$$

which is homotopic to T^{n-i}. Therefore, all components of $D[i]$ are homotopic to T^{n-i} and as there are $\binom{n}{i}$ of these this finishes the proof. □

the following result is the self-crossing analogue of [16, Theorem 1.3], and appears in [40, Theorem 1.2] in the algebraic context.

Theorem 1.2.14. *Let $D = (L, \sigma)$ be a complex log divisor on a manifold M. Then the inclusion $\iota \colon M \backslash D \hookrightarrow M$ induces an isomorphism*

$$H^k(M, \mathcal{A}_D) \simeq H^k(M \backslash D, \mathbb{C}).$$

Proof. We give an argument in the same spirit as [40, Theorem 1.2], using that $\mathcal{A}_D$ defines a soft sheaf. As is shown there it suffices to show that ι induces an isomorphism on the level of sheaf cohomology. Below we will implicitly identify the sheaf $\Omega^\bullet(M\backslash D)$ with its push-forward $\iota_*(\Omega^\bullet(M\backslash D))$. Given a point $p \in M\backslash D$ and a contractible neighbourhood U of p which is disjoint from D we have that $\mathcal{A}_D = TD$ and hence $H^k(U, \mathcal{A}_D) = H^k(U\backslash D)$. Next, for any j less than or equal to the pointwise multiplicity of D, let $p \in D[j]$. Let $z_1, \dots, z_j$ be coordinates on an open neighbourhood U of p as in Lemma 1.1.13. In these coordinates we see that $H^\bullet(U, \mathcal{A}_D)$ is the free algebra generated by $\{1, d\log z_1, \dots, d\log z_j\}$. By Lemma 1.2.13 we have that $U\backslash D$ is homotopic to $\mathbb{T}^j$. Under these identifications, the cochain morphism ι^* takes the generators of $H^\bullet(U, \mathcal{A}_D)$ to the generators of $H^\bullet(U\backslash D)$. Therefore we conclude that ι^* is a local isomorphism and hence also globally, which finishes the proof. □

1.2.3 Residue maps

Let $(I_{|D|}, \mathfrak{o})$ be a co-oriented smooth elliptic divisor on a manifold M. The restriction of the smooth elliptic tangent bundle to D fits into a sequence of Lie algebroids

$$0 \to \ker \rho|_D \to \mathcal{A}_{|D|}\big|_D \to TD \to 0.$$

In [16] it is explained that this sequence induces a cochain map

$$\mathrm{Res}_q \colon \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-2}(TD),$$

called the elliptic residue. In local Morse–Bott coordinates for the divisor, the elliptic residue map is given by

$$\mathrm{Res}_q(\alpha) = \iota_D^*(\iota_{r\partial_r}\iota_{\partial_\theta}\alpha), \qquad \alpha \in \Omega^\bullet(\mathcal{A}_{|D|}).$$

Definition 1.2.15. Let $(I_{|D|}, \mathfrak{o})$ be a co-oriented elliptic divisor and let an elliptic form $\alpha \in \Omega^\bullet(\mathcal{A}_{|D|})$ be given. Its **elliptic residue** is defined by

$$\mathrm{Res}_q(\alpha) := \mathrm{Res}_q(\iota_{M\setminus D(2)}^*\alpha),$$

where the right-hand side is the elliptic residue for the smooth elliptic divisor $I_{|D|}\big|_{M\setminus D(2)}$. ♢

We will also need the *radial residue map*, which picks out all the $d\log r_i$ generators. We will study its construction in more detail in Section 3.2, for now we have:

Definition 1.2.16. Let $(I_{|D|}, \mathfrak{o})$ be a co-oriented elliptic divisor and let an elliptic form $\alpha \in \Omega^\bullet(\mathcal{A}_{|D|})$ be given. Its **radial residue** at $p \in D[i]$ is defined by

$$\mathrm{Res}_r(\alpha)(p) := \iota_D^*(\iota_{r_1\partial_{r_1}} \cdots \iota_{r_i\partial_{r_i}}\alpha)(p),$$

where $r_1, \ldots, r_i$ are the radial coordinates in local Morse–Bott coordinates around p. ♢

The radial residue map takes values in the Lie algebroid forms of some quotient Lie algebroid. We will describe this Lie algebroid in more detail in Section 3.2, but as we are interested in the space of forms for which this residue vanish the precise description does not matter right now.

In later constructions, the cohomology of the complex of forms with vanishing radial residue will play a role:

Lemma 1.2.17. *Let $I_{|D|}$ be a self-crossing elliptic divisor on a manifold M and let $\Omega^\bullet_{0,0}(\mathcal{A}_{|D|}) \subset \Omega^\bullet(\mathcal{A}_{|D|})$ be the subcomplex defined as the kernel of the map Res_r. Then the inclusion map $i\colon M\setminus D \to M$ of the complement of D induces a quasi-isomorphism $i^*\colon \Omega^\bullet_{0,0}(\mathcal{A}_{|D|}) \to \Omega^\bullet(M\setminus D)$.*

Proof. The argument uses the observation from [40] (and the fact that $\mathcal{A}_{|D|}$ defines a soft sheaf) that it suffices to show that ι^* induces an isomorphism on the level of sheaf cohomology. Below we will implicitly identify the sheaf $\Omega^\bullet(M\backslash D)$ with its push-forward $\iota_*(\Omega^\bullet(M\backslash D))$. For all points $p \in M\backslash D$ there exists a contractible open neighbourhood U of p disjoint from D. On this open $\mathcal{A}_{|D|} = TM$ and $\mathrm{Res}_r \equiv 0$ and therefore ι_* is simply the identity. Let j be any integer less than or equal to the pointwise multiplicity of D, take $p \in D[j]$ and let U be a contractible open around p as in Lemma 1.1.13. In those coordinates, $H^\bullet_{0,0}(U, \mathcal{A}_{|D|})$ is the free algebra generated by $\{1, d\theta_1, \ldots, d\theta_j\}$. By an elementary argument $U\backslash D$ is homotopic to $\mathbb{T}^j$ and using this homotopy ι^* takes the generators of $H^\bullet_{0,0}(\mathcal{A}_{|D|})$ to the generators of $H^\bullet(U\backslash D)$. Therefore we conclude that ι^* is a local isomorphism and consequently a global isomorphism. □

We will need a couple more residue maps:

Definition 1.2.18. Let $(I_{|D|}, \mathfrak{o})$ be a co-oriented elliptic divisor and let an elliptic two-form $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be given. Consider oriented coordinates around a point $p \in D(k)$ with $k \geq 2$ as provided by Lemma 1.1.25 and define three types of residues:

$$\mathrm{Res}_{r_i r_j}\, \omega(p) := \omega_p(r_i\partial_{r_i}, r_j\partial_{r_j}),\ \mathrm{Res}_{r_i\theta_j}\, \omega(p) := \omega_p(r_i\partial_{r_i}, \partial_{\theta_j}),$$
$$\mathrm{Res}_{\theta_i\theta_j}\, \omega(p) := \omega_p(\partial_{\theta_i}, \partial_{\theta_j}). \qquad \diamondsuit$$

These expressions do not depend on the chosen coordinates. They do depend on the choice of co-orientation and the particular ordering of the divisors, but only up to signs.

1.3 Self-crossing stable structures

In this section we will start our discussion of generalized complex geometry and self-crossing stable generalized complex structures in particular. We will first recall the basics of generalized complex geometry, before defining the (self-crossing) stable condition, using the divisors of Section 1.1. We then show how these can be described using symplectic-like forms in the complex log tangent bundle (Theorem 1.3.13), in analogy with [16]. Next we discuss that these are in fact symplectic structures in its associated elliptic tangent bundle, satisfying certain additional cohomological conditions (Theorem 1.3.16). From that point onwards we can proceed to study self-crossing stable generalized complex structures using symplectic techniques.

1.3.1 Generalized complex structures

Generalized geometry refers to the study of geometric structures on $\mathbb{T}M := TM \oplus T^*M$, for a manifold M. We briefly recall the notions from generalized complex geometry which are needed in this thesis. For a more in depth discussion see [41].

Definition 1.3.1. A **generalized complex structure** on a manifold M is a pair $(\mathbb{J}, H)$, where $H \in \Omega^3(M)$ is a closed three-form and $\mathbb{J}$ is an endomorphism of $\mathbb{T}M$ for which $\mathbb{J}^2 = -\mathrm{Id}$ and the $+i$-eigenbundle $L \subset (\mathbb{T}M) \otimes \mathbb{C}$ is involutive with respect to the Dorfman bracket:

$$[\![X+\xi, Y+\eta]\!]_H := [X,Y]+\mathcal{L}_X\eta-\iota_Y d\xi+\iota_X\iota_Y H, \qquad X+\xi, Y+\eta \in \Gamma(\mathbb{T}M). \ \diamondsuit$$

Two generalized complex structures $(\mathbb{J}, H)$ and $(\mathbb{J}', H')$ are **gauge equivalent** if there exists $B \in \Omega^2(M)$ such that $H' = H + dB$ and, using the associated map $B^\flat \colon TM \to T^*M$, we have

$$\mathbb{J}' = \begin{pmatrix} 1 & B^\flat \\ 0 & 1 \end{pmatrix} \mathbb{J} \begin{pmatrix} 1 & -B^\flat \\ 0 & 1 \end{pmatrix}.$$

Given an element $X + \xi \in \mathbb{T}M$, let $(X + \xi) \cdot \rho := \iota_X \rho + \xi \wedge \rho$ denote the Clifford action of $\mathbb{T}M$ on elements $\rho \in \wedge^\bullet T^*M$, which are called **spinors**. Moreover, define transposition of an element in $\Gamma(\wedge^\bullet T^*M)$ on decomposable degree k-forms by

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)^T := \alpha_k \wedge \cdots \wedge \alpha_1, \qquad \alpha_i \in \Omega^1(M).$$

The **Chevalley pairing** on spinors, $(\cdot,\cdot)_{\mathrm{Ch}} \colon \wedge^\bullet T^*M \times \wedge^\bullet T^*M \to \wedge^{\mathrm{top}} T^*M$, is defined as

$$(\gamma, \rho)_{\mathrm{Ch}} := (\gamma \wedge \rho^T)_{\mathrm{top}}, \quad \gamma, \rho \in \Gamma(\wedge^\bullet T^*M). \tag{1.3.1}$$

Generalized complex structures can be equivalently described using the following:

Lemma 1.3.2 ([41]). *There is a one-to-one correspondence between generalized complex structures $(\mathbb{J}, H)$ and complex line subbundles $K \subset \wedge^\bullet T^*_{\mathbb{C}}M$, satisfying the following properties:*

- *For all $x \in M$ the vector space K_x is generated over $\mathbb{C}$ by spinors of the form*

$$\left(e^{B+i\omega} \wedge \Omega\right)_x, \quad B, \omega \in \Omega^2(M), \quad \Omega \in \Omega^k(M),$$

  where Ω is a decomposable form;

- *For every nonvanishing local section $\rho \in \Gamma(K)$, there exists $u \in \Gamma(\mathbb{T}_{\mathbb{C}}M)$ such that $d\rho + H \wedge \rho = u \cdot \rho$;*

- *For all non-zero $\rho_x \in K_x$, we have $(\rho_x, \overline{\rho}_x)_{Ch} \neq 0$.*

The line bundle K is called the **canonical line bundle** of $\mathbb{J}$. It can be defined in terms of the generalized complex structure by the relation

$$L = \{u \in \mathbb{T}_{\mathbb{C}}M : u \cdot K = 0\},$$

where L is the $+i$-eigenbundle of $\mathbb{J}$. Note here that $\mathbb{T}_{\mathbb{C}}M = \mathbb{T}M \otimes \mathbb{C}$.

Definition 1.3.3. Let $(\mathbb{J}, H)$ be a generalized complex structure and let K be its canonical line bundle. The map $s \colon K \to \wedge^0 T^*_{\mathbb{C}}M = \mathbb{C}$ defined by $\rho \mapsto \rho_0$, sending a spinor to its degree-zero part, defines a section $s \in \Gamma(K^*)$ called the **anticanonical section** of $\mathbb{J}$. $\diamondsuit$

Example 1.3.4. Given a complex structure, J, or a symplectic structure, ω, on a manifold M, we can endow M with a generalized complex structure (for $H = 0$) given by

$$\mathbb{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \qquad \mathbb{J}_\omega := \begin{pmatrix} 0 & -(\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix}.$$

More interestingly, let (M, J, π) be a holomorphic Poisson manifold with $\pi = \pi_R + i\pi_I$ and denote by $\pi_I^\sharp \colon T^*M \to TM$ the map associated to π_I. Then

$$\mathbb{J}_{J,\pi} := \begin{pmatrix} -J & 4\pi_I^\sharp \\ 0 & J^* \end{pmatrix}$$

is also a generalized complex structure.

The corresponding canonical line bundles and anti-canonical sections are given by:

$$\begin{aligned} &K_\omega = \langle e^{i\omega} \rangle, && K_J = \wedge^{n,0} T^*M, && K_{J,\pi} = e^{\pi}(\wedge^{n,0} T^*M), \\ &s_\omega \equiv 1, && s_J \equiv 0, && s_{J,\pi} = \iota_{\pi^n} \Omega_{\text{vol}}. \end{aligned}$$ △

There is an interesting relation between generalized complex geometry and Poisson geometry obtained in [22]. Given a generalized complex structure $\mathbb{J}$, denote by $\pi_{\mathbb{J}}^\sharp \colon T^*M \to TM$ the associated bundle map obtained by $\pi_{\mathbb{J}}^\sharp = \text{pr}_{TM} \circ \mathbb{J}|_{T^*M}$. This map is skew-symmetric and:

Lemma 1.3.5. *Let $\mathbb{J}$ be a generalized complex structure. Then $\pi_{\mathbb{J}} \in \mathfrak{X}^2(M)$ is a Poisson structure. Moreover, if $\mathbb{J}, \mathbb{J}'$ are gauge-equivalent generalized complex structures, then $\pi_{\mathbb{J}} = \pi_{\mathbb{J}'}$.*

1.3.2 Stable generalized complex structures

In this section we will extend the notion of stable generalized complex structures to allow for the degeneracy locus to have self-crossing singularities. The main reason to allow for normal crossing singularities is that it gives much more flexibility: for example, this class is now closed under taking products. This makes it easier to provide examples (see Section 1.6) and there are also more constructions available, such as the connect sum procedure of Section 1.5. Moreover, in four dimensions these structures can be used to construct smooth stable generalized complex structures, as explained in Section 1.4.

Definition 1.3.6. A **(self-crossing) stable generalized complex structure** is a generalized complex structure such that its anticanonical divisor $D = (K^*, s)$ defines a complex log divisor. We call it **smooth stable** when $D = (K^*, s)$ defines a smooth complex log divisor. ♢

We can immediately see that the class of such structures is closed under products.

Example 1.3.7. Let (L_i, σ_i) be two complex log divisors with multiplicities n_i on two manifolds M_i for $i = 1, 2$. Then the pair $(\pi_1^* L_1 \otimes \pi_2^* L_2, \pi_1^* \sigma_1 \otimes \pi_2^* \sigma_2)$ on the product manifold $M = M_1 \times M_2$, with projection maps $p_i \colon M \to M_i$, is a complex log divisor with pointwise multiplicity $n_1 + n_2$. This shows that the product $(M, \pi_1^* H_1 + \pi_2^* H_2, \mathbb{J}_1 \oplus \mathbb{J}_2)$ of two stable generalized complex manifolds $(M_i, H_i, \mathbb{J}_i)$ is endowed with a stable generalized complex structure. △

Stable generalized complex structures can come about via holomorphic Poisson structures.

Example 1.3.8. Let $\pi \in \mathfrak{X}^2(M^{2n}; \mathbb{C})$ be a holomorphic Poisson structure such that the pair $(\wedge^{n,0} TM, \wedge^n \pi)$ is a complex log divisor. Then $\mathbb{J}_{J,\pi}$ is a stable generalized complex structure. △

An important gain from allowing self-crossings is that deformations of higher dimensional Fano manifolds by holomorphic Poisson bivectors may provide examples of these structures, but are never smoothly stable [44, Theorem 7.13].

Example 1.3.9. One can readily construct a holomorphic Poisson structure on $\mathbb{C}P^{2n}$ for which $(\wedge^n TM, \wedge^n \pi)$ is a complex log divisor, making $\mathbb{C}P^{2n}$ into a stable generalized complex manifold. Indeed, using the standard coordinates $(z_1, \ldots, z_{2n+1})$ on $\mathbb{C}^{2n+1}$, let

$$\widetilde{\pi} := z_1 z_2 \partial_{z_1} \wedge \partial_{z_2} + \cdots z_{2n-1} z_{2n} \partial_{z_{2n-1}} \wedge \partial_{z_{2n}}$$

be a holomorphic Poisson structure on $\mathbb{C}^{2n+1}$. This bivector is scaling invariant and therefore descends to a bivector π on $\mathbb{C}P^{2n}$. By naturality of the Schouten bracket it follows that π is Poisson,and by direct computation we see that $(\wedge^n T_{\mathbb{C}} M, \wedge^n \pi)$ is a complex log divisor. △

1.3.3 Complex log symplectic structures

The next sections aim to prove that stable generalized complex structures are equivalent to a certain type of symplectic structures on an associated elliptic tangent bundle. Before we do so, we first prove, in this section, that they are equivalent to an auxiliary structure. This will be a symplectic-like structure for the complex log tangent bundle.

Given a complex log divisor I_D and its induced elliptic divisor $I_{|D|}$ we consider the Lie algebroid morphism $\iota \colon \mathcal{A}_{|D|} \otimes \mathbb{C} \to \mathcal{A}_D$ obtained from Proposition 1.2.11. If we compose the pullback ι^* with taking the imaginary part of a form we obtain a cochain morphism $\Im^* \colon \Omega^\bullet(\mathcal{A}_D) \to \Omega^\bullet(\mathcal{A}_{|D|})$.

Definition 1.3.10. A form $\sigma \in \Omega^2(\mathcal{A}_D)$ is called **complex log symplectic** if $d\sigma = H \in \Omega^3(M; \mathbb{R})$ and $\Im^* \sigma \in \Omega^2(\mathcal{A}_{|D|})$ is non-degenerate. Two complex log symplectic forms $\sigma, \sigma' \in \Omega^2(\mathcal{A}_D)$ are said to be **gauge equivalent** if there exists a two-form $B \in \Omega^2(M, \mathbb{R})$ such that $\sigma' = \sigma + B$. ♢

Since the symplectic structure given by the imaginary part of a complex log symplectic form clearly contains important information, it is useful to give it a name as well.

Definition 1.3.11. Let M be a manifold with an elliptic divisor $I_{|D|}$. A **(self-crossing) elliptic symplectic form** is an elliptic two-form $\omega \in \Omega^2(\mathcal{A}_{|D|})$ that is closed and non-degenerate. ♢

Remark 1.3.12. An elliptic symplectic structure $\omega \in \Omega^2(\mathcal{A}_{|D|})$ always induces an orientation because ω^n defines a volume form outside a codimension-two subset. This is in contrast with log symplectic manifolds, which need not be orientable. ♢

The following theorem is the self-crossing generalisation of [16, Theorem 3.2].

Theorem 1.3.13. *Let M be a manifold. There is a one-to-one correspondence between stable generalized complex structures with self-crossings on M and complex log divisors endowed with complex log symplectic forms with self-crossings. Moreover, this correspondence preserves gauge equivalences.*

Explicitly, the correspondence is given by the map

$$\left\{ \begin{array}{c} (\mathbb{J}, H)\colon \\ \mathbb{J} \textit{ is a stable GCS} \end{array} \right\} \to \left\{ \begin{array}{c} (I_D, \sigma)\colon \\ I_D \textit{ is a complex log divisor and} \\ \sigma \in \Omega(\mathcal{A}_D) \textit{ is a complex log-symplectic form.} \end{array} \right\}$$

where I_D is the divisor induced by the anticanonical section and $\sigma = \rho_2/\rho_0$ where ρ is any local spinor for $\mathbb{J}$.

Proof. We will first consider the direct implication. Let $\rho \in \Omega^\bullet(M)$ be a local pure spinor for the stable generalized complex structure, defined on an open set $U \subset M$. On $U \backslash D$ the anticanonical section is non-vanishing and therefore we have that $\rho = ce^{\varepsilon}$ on $U \backslash D$ for some $c \in C^\infty(M;\mathbb{C})$ and $\varepsilon \in \Omega^2(M;\mathbb{C})$. Looking at the degree-zero part of this equation we obtain $c = \rho_0$ and looking at the degree-two part we obtain $\varepsilon = \rho_2/\rho_0$. By continuity we conclude that $\rho = \rho_0 e^{\rho_2/\rho_0}$ on the entirety of U.

Claim. *The two-form $\sigma := \rho_2/\rho_0$ defines a global smooth complex log form, with $d\sigma = -H$.*

Proof of claim. Assume that the local form, σ, defined using a local canonical section is indeed a local complex log form and let $\widetilde{\sigma}$ be another local form obtained from another local canonical section, $\widetilde{\rho}$. Then, from the previous argument we have $\widetilde{\rho} = \widetilde{\rho}_0 e^{\widetilde{\rho}_2/\widetilde{\rho}_0}$. In particular we see that as complex log forms we have

$$e^{\sigma} = \frac{\rho}{\rho_0} = \frac{\widetilde{\rho}}{\widetilde{\rho}_0} = e^{\widetilde{\sigma}},$$

where the middle equality follows from the fact that $\frac{\rho}{\rho_0}$ and $\frac{\widetilde{\rho}}{\widetilde{\rho}_0}$ are sections of the canonical bundle with the same degree-zero component. Therefore σ is a global complex log form.

Hence to conclude the result we must prove that σ is a local log form. By the integrability of the generalized complex structure there exist $X + \xi \in \Gamma(\mathbb{T}M)$ such that

$$\begin{aligned} d\rho_0 &= \iota_X \rho_2 + \xi \wedge \rho_0, \\ d\rho_2 &= \iota_X \rho_4 + \xi \wedge \rho_2 - \rho_0 H. \end{aligned}$$

Because $\rho_4 = \frac{1}{2\rho_0}\rho_2 \wedge \rho_2$ on $U\backslash D$, we find that

$$d\rho_2 = \frac{\iota_X(\rho_2 \wedge \rho_2)}{2\rho_0} + \xi \wedge \rho_2 - \rho_0 H.$$

On $U\backslash D$ we thus have:

$$\begin{aligned} d\left(\frac{\rho_2}{\rho_0}\right) &= \frac{d\rho_2}{\rho_0} - \frac{\rho_2 \wedge d\rho_0}{\rho_0^2} \\ &= \frac{\iota_X \frac{\rho_2^2}{2\rho_0} + \xi \wedge \rho_2}{\rho_0} - H - \frac{\rho_2 \wedge (\iota_X \rho_2 + \xi \wedge \rho_0)}{\rho_0^2} \\ &= -H. \end{aligned}$$

Now we can apply Lemma 1.2.12 to $\sigma = \rho_2/\rho_0$ to conclude that σ defines a well-defined complex log form. □

The algebraic condition $(\rho, \overline{\rho})_{\text{Ch}} \neq 0$ results in

$$|\rho_0|^2 (\sigma - \overline{\sigma})^n \neq 0.$$

Therefore we conclude that the elliptic form $\sigma - \overline{\sigma}$ is non-degenerate and hence σ is a complex log symplectic form.

Next we consider the converse implication. Let I_D denote the complex log divisor and let $\sigma \in \Omega^2(\mathcal{A}_D)$ be a complex log symplectic form for this ideal. Let $\langle e^\sigma \rangle_{\mathbb{C}} \subset \Omega^\bullet(\mathcal{A}_D)$ denote the complex line generated by e^σ. Then the product $I_D \otimes \langle e^\sigma \rangle_{\mathbb{C}} \subset \Omega^\bullet(\mathcal{A}_D)$ is in fact smooth, that is, $I_D \langle e^\sigma \rangle_{\mathbb{C}} \subset \Omega^\bullet(M)$. We will show that this line bundle defines a stable generalized complex structure. In local coordinates as in Lemma 1.1.13 we have that $\rho = z_1 \cdot \ldots \cdot z_k e^\sigma$ is a local trivialisation of the line bundle. Now

$$(\rho, \overline{\rho})_{\text{Ch}} = \overline{z_1}^2 \cdot \ldots \cdot \overline{z_k}^2 (\Im^* \sigma)^n,$$

defines a volume form because $\Im^* \sigma$ is a nondegenerate elliptic form with self-crossings. We are left to prove integrability of ρ. Since integrability is a closed condition, it is enough check it in $M\backslash D$, but by construction in this region the structure is just a B-field transform of a symplectic structure. □

1.3.4 Equivalence with elliptic symplectic

As we just saw, a stable generalized complex structures is closely related to an elliptic symplectic form. Yet, the elliptic tangent bundle, $\mathcal{A}_{|D|}$, which appears in the context of generalized complex structure arises from a complex log tangent bundle, $\mathcal{A}_D$ and the elliptic symplectic form arises as imaginary part of complex log symplectic form. Therefore our next step is to pinpoint precisely which elliptic symplectic forms arise in this way. That is we are interested in describing the image of 'taking the imaginary part': $\Im^* : \Omega^\bullet(\mathcal{A}_D) \to \Omega^\bullet(\mathcal{A}_{|D|})$.

From now on we denote this image by $\Omega^\bullet_\Im(\mathcal{A}_{|D|}) \subset \Omega^\bullet(\mathcal{A}_{|D|})$. Describing elements in $\Omega^\bullet_\Im(\mathcal{A}_{|D|})$ of arbitrary degree for a general complex divisor is a little involved, so we will focus on our object of interest: two-forms. To describe elements in $\Omega^2_\Im(\mathcal{A}_{|D|})$ we need to use the residue maps for points in $D[2]$ introduced in Section 1.2.3, such as, $\mathrm{Res}_{r_i r_j}$ and $\mathrm{Res}_{r_i\theta_j}$. Recall that these residues depend on an ordering of the coordinates and a choice of co-orientation, the kernels of some combinations of these residues only depends on the co-orientation of the divisor. To be precise, given a co-oriented elliptic divisor $(I_{|D|}, \mathfrak{o})$, the spaces $\ker(\mathrm{Res}_q)$, $\ker(\mathrm{Res}_{\theta_i r_j} - \mathrm{Res}_{r_i\theta_j})$ and $\ker(\mathrm{Res}_{r_i r_j} + \mathrm{Res}_{\theta_i\theta_j})$ do not depend on the order of the divisors.

Lemma 1.3.14. *Let I_D be a complex log divisor and let $(I_{|D|}, \mathfrak{o})$ be its associated co-oriented elliptic divisor. Then*

$$\Omega^2_\Im(\mathcal{A}_{|D|}) = \ker(\mathrm{Res}_q) \cap (\ker(\mathrm{Res}_{\theta_i r_j} - \mathrm{Res}_{r_i\theta_j})) \cap (\ker(\mathrm{Res}_{r_i r_j} + \mathrm{Res}_{\theta_i\theta_j})).$$

Proof. Choose local coordinates as in Lemma 1.1.25. For all pairs i, j and $\beta \in \Omega^1(M;\mathbb{R})$ we have:

$$\begin{aligned}
\Im^*(d\log z_i \wedge d\log z_j) &= d\theta_i \wedge d\log r_j + d\log r_i \wedge d\theta_j,\\
\Im^*(id\log z_i \wedge d\log z_j) &= d\log r_i \wedge d\log r_j - d\theta_i \wedge d\theta_j,\\
\Im^*(d\log z_i \wedge \beta) &= d\theta_i \wedge \beta,\\
\Im^*(id\log z_i \wedge \beta) &= d\log r_i \wedge \beta.
\end{aligned}$$

Therefore we see that $\Omega^2_\Im(\mathcal{A}_{|D|})$ lies in the intersection of the kernels. Conversely, if α is in the intersection of the kernels then locally it must be a linear combination of the above forms together with smooth forms. Therefore by the above computation, we see that $\alpha \in \Omega^2_\Im(\mathcal{A}_{|D|})$ which concludes the proof. $\square$

Since $\Im^*$ is a map of complexes, $\Omega^\bullet_\Im(\mathcal{A}_{|D|}) \subset \Omega^\bullet(\mathcal{A}_{|D|})$ is a subcomplex and we can compute its cohomology, as we will do in Section 1.3.6. Next we see that the imaginary part of a complex two-form determines it up to gauge equivalence:

Proposition 1.3.15. *Let M be a manifold with a complex log divisor I_D and let $(I_{|D|}, \mathfrak{o})$ be its induced co-oriented elliptic divisor. Then the following sequence of cochain complexes is exact:*

$$0 \to \Omega^\bullet(M;\mathbb{R}) \to \Omega^\bullet(\mathcal{A}_D) \stackrel{\Im^*}{\to} \Omega^\bullet_\Im(\mathcal{A}_{|D|}) \to 0.$$

Proof. We can check exactness of the above sequence by checking exactness of the corresponding sheaf sequence. By Lemma 1.1.13 we may assume we are in the setting of Example 1.1.8. The map $\Im^*$ is surjective by definition and it is clear that that $\Omega^\bullet(M;\mathbb{R})$ lies in the kernel of $\Im^*$. Therefore we are left to show that if a general complex log form has vanishing imaginary part, then it must be smooth. Consider coordinates $(z_1,\ldots,z_k,\overline{z}_1,\ldots,\overline{z}_k,x_i,\ldots,x_l)$ and for multi-indices $I=(i_1,\ldots,i_l)$ make use of the following shorthand notation:

$$\begin{aligned} z_I &:= z_{i_1}\cdots z_{i_l}, \quad (d\log z)_I := d\log z_{i_1}\wedge\cdots\wedge d\log z_{i_l}, \\ (d\overline{z})_I &:= d\overline{z}_{i_1}\wedge\cdots\wedge d\overline{z}_{i_l}. \end{aligned}$$

Using these, a general complex log form $\rho\in\Omega^\bullet(\mathcal{A}_D)$ may be written locally as

$$\rho=\sum_{I,J,K}\alpha_{IJK}(d\log z)_I\wedge(d\overline{z})_J\wedge(dx)_K,$$

where the $\alpha_{IJK}\in C^\infty(M;\mathbb{C})$ are smooth and the sum ranges over all multi-indices. The vanishing of the imaginary part of ρ implies

$$\begin{aligned} 0=&\sum_{I,J,K}\alpha_{IJK}(d\log z)_I\wedge(d\overline{z})_J\wedge(dx)_K \\ &-\sum_{I,J,K}\overline{\alpha}_{IJK}(d\log\overline{z})_I\wedge(dz)_J\wedge(dx)_K, \end{aligned}$$

which using $z_I(d\log z)_I=(dz)_I$ gives:

$$\begin{aligned} 0=&\sum_{I,J,K}\overline{z}_J\alpha_{IJK}(d\log z)_I\wedge(d\log\overline{z})_J\wedge(dx)_K \\ &-\sum_{I,J,K}z_J\overline{\alpha}_{IJK}(d\log\overline{z})_I\wedge(d\log z)_J\wedge(dx)_K. \end{aligned}$$

By linear independence, each of the terms involving $(d\log z)_I\wedge(d\log\overline{z})_J$ and their conjugates needs to vanish independently. That is, for all multi-indices I,J,K we have

$$0=(\overline{z}_J\alpha_{IJK}(d\log z)_I\wedge(d\log\overline{z})_J-z_I\overline{\alpha}_{JIK}(d\log\overline{z})_J\wedge(d\log z)_I)\wedge(dx)_K,$$

and hence

$$(\overline{z}_J\alpha_{IJK}-(-1)^{|I||J|}z_I\overline{\alpha}_{JIK})=0,$$

from this we conclude that α_{IJK} is divisible by z_I. This proves that ρ is a smooth form. □

Putting Theorem 1.3.13, Proposition 1.1.28 and Proposition 1.3.15 together we have the following.

Theorem 1.3.16. *Let M be a manifold. There is a correspondence between gauge equivalence classes of stable generalized complex structures with self-crossings on M and co-oriented elliptic divisors, $(I_{|D|}, \mathfrak{o})$, endowed with an elliptic symplectic form $\omega \in \Omega^2_{\Im}(\mathcal{A}_{|D|})$.*

Explicitly, this equivalence is induced by the map

$$\left\{ \begin{array}{c} (\mathbb{J}, H): \\ \mathbb{J} \text{ is a Stable GCS} \end{array} \right\} \to \left\{ \begin{array}{c} (I_{|D|}, \mathfrak{o}, \omega): \\ (I_{|D|}, \mathfrak{o}) \text{ is a co-oriented elliptic divisor and} \\ \omega \in \Omega_{\Im}(\mathcal{A}_{|D|}) \text{ is a symplectic form.} \end{array} \right\}$$

which assigns to each stable generalized complex structure on M the co-oriented elliptic divisor determined by its anticanonical section and the imaginary part of its corresponding complex log symplectic form.

Remark 1.3.17. Under the equivalence above, $[H] = \delta[\omega]$, where δ is the connecting morphism coming from Proposition 1.3.15. ♢

1.3.5 Equivalence with nondegenerate elliptic Poisson

A consequence of Theorem 1.3.16 is that nearly all the information of a stable generalized complex structure is already encoded in its underlying Poisson structure.

Theorem 1.3.18. *The gauge equivalence class of a stable generalized complex structure $(\mathbb{J}, H)$ is fully determined by its underlying Poisson structure.*

Proof. In the symplectic locus the Poisson structure is given by ω^{-1}. By smooth continuation ω^{-1} on $M \backslash D$ determines ω on $\mathcal{A}_{|D|}$, which in turn determines $\mathbb{J}$ up to gauge equivalence by Theorem 1.3.16. The only point that needs attention in this argument is that we extended ω^{-1} from $M \backslash D$ to $\mathcal{A}_{|D|}$ but we did not argue yet that the Poisson structure itself determines $\mathcal{A}_{|D|}$. This is indeed the case, as shown by Lemma 1.3.19 below. □

Lemma 1.3.19. *Let $I_{|D|}$ be an elliptic divisor. Given $\omega \in \Omega^2(\mathcal{A}_{|D|})$ an elliptic symplectic form, let $\pi = \rho(\omega^{-1})$ be its associate Poisson bivector on M, where $\rho\colon \mathcal{A}_{|D|} \to TM$ is the anchor map. Then $(\wedge^n TM, \wedge^n \pi)$ defines the elliptic divisor $I_{|D|}$.*

Conversely, if $(\wedge^n TM, \wedge^n \pi)$ defines an elliptic divisor $I_{|D|}$, then π admits a nondegenerate lift to $\mathcal{A}_{|D|}$ and hence defines an elliptic symplectic structure.

Proof. In local coordinates expressing $I_{|D|}$ as a normal crossing, ω^n is a volume form, hence there is a nonvanishing function, f, for which

$$\omega^n = f d\log r_1 \wedge d\theta_1 \wedge \cdots \wedge d\log r_k \wedge d\theta_k \wedge dx_{2k+1} \wedge \dots dx_{2n}.$$

Hence

$$\pi^n = \rho(\omega^{-n}) = f^{-1} r_1 \partial_{r_1} \wedge \partial_{\theta_1} \wedge \cdots \wedge r_k \partial_{r_k} \wedge \partial_{\theta_k} \wedge \partial_{x_{2k+1}} \wedge \dots \partial_{x_{2n}},$$

which defines the divisor $I_{|D|}$.

Conversely, assume that $(\wedge^n TM, \wedge^n \pi)$ defines an elliptic divisor $I_{|D|}$. Due to [52, Theorem A], to prove that π lifts to to $\mathcal{A}_{|D|}$ it suffices to show that $\mathcal{A}^*_{|D|}$ is locally generated by closed one-forms. From the local description in Lemma 1.1.25 this is immediate.

Since, by Lemma 1.2.10, the ideal defined by $\mathcal{A}_{|D|}$ is $I_{|D|}$, the lift above is non-degenerate, by [52, Theorem A]. □

1.3.6 Cohomology

Due to Theorem 1.3.16, in order to study stable generalized complex structures, we can turn our attention to symplectic forms in the associated elliptic tangent bundle that lie in the subcomplex given by $\Omega^\bullet_{\Im}(\mathcal{A}_{|D|})$. Therefore, the cohomology that is relevant to the study of these symplectic structures is not the elliptic cohomology but, $H^2_{\Im}(\mathcal{A}_{|D|})$, the cohomology of the subcomplex $\Omega^\bullet_{\Im}(\mathcal{A}_{|D|})$. We study this cohomology next.

Proposition 1.3.20. *Let M be a manifold and let $I_{|D|}$ be a co-oriented elliptic divisor. Then*

$$H^i_{\Im}(\mathcal{A}_{|D|}) \simeq H^{i+1}(M, M\backslash D) \oplus H^i(M\backslash D).$$

Proof. We have the following morphism of cochain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^\bullet(M,\mathbb{R}) & \longrightarrow & \Omega^\bullet(\mathcal{A}_D) & \longrightarrow & \Omega^\bullet_{\Im}(\mathcal{A}_{|D|}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^\bullet(M,\mathbb{R}) & \xrightarrow{\iota^*_{M\backslash D}} & \Omega^\bullet(M\backslash D,\mathbb{C}) & \longrightarrow & \Omega^\bullet(M\backslash D,\mathbb{C})/\Omega^\bullet(M,\mathbb{R}) & \longrightarrow & 0
\end{array}$$

where $\iota^*_{M\backslash D}$ is the natural inclusion, the middle vertical arrow corresponds to restriction to $M\backslash D$ and the rightmost vertical arrow restriction composed with the quotient map. Let $C^\bullet$ denote the quotient complex, then we have the corresponding commutative diagram in cohomology:

$$\begin{array}{ccccccccccc}
H^\bullet(M) & \longrightarrow & H^\bullet(\mathcal{A}_D) & \longrightarrow & H^\bullet_{\Im}(\mathcal{A}_D) & \rightarrow & H^{\bullet+1}(M) & \longrightarrow & H^{\bullet+1}(\mathcal{A}_D) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^\bullet(M) & \rightarrow & H^\bullet(M\backslash D,\mathbb{C}) & \longrightarrow & H^\bullet(C^\bullet) & \longrightarrow & H^{\bullet+1}(M) & \rightarrow & H^{\bullet+1}(M\backslash D,\mathbb{C}) & \rightarrow & \dots
\end{array}$$

Here we let $C^\bullet$ denote the cohomology of the quotient complex. By Theorem 1.2.14 and the Five Lemma we conclude that $H^\bullet_{\Im}(\mathcal{A}_{|D|}) \simeq H^\bullet(C^\bullet)$. The quotient complex splits:

$$C^\bullet = \Omega^\bullet(M\backslash D,\mathbb{R})/\Omega^\bullet(M,\mathbb{R}) \oplus i\Omega^\bullet(M\backslash D,\mathbb{R}).$$

Let $E^\bullet = \Omega^\bullet(M\backslash D,\mathbb{R})/\Omega^\bullet(M,\mathbb{R})$, as we have that $H^\bullet(C^\bullet) \cong H^\bullet(M\backslash D) \oplus H^\bullet(E^\bullet)$. We are left to show that $E^\bullet$ computes the relative cohomology, for which we will make use the relative de Rham cohomology (see for instance [9]). Let

$$\Omega^q(i_{M\backslash D}) = \Omega^q(M) \oplus \Omega^{q-1}(M\backslash D), \quad d(\omega,\theta) = (d\omega, \iota^*_{M\backslash D}\omega - d\theta),$$

this cohomology computes the relative cohomology $H^\bullet(M, M\backslash D)$, and fits in the long exact sequence

$$\cdots \to H^{k-1}(M\backslash D) \to H^k(i_{M\backslash D}) \to H^k(M) \overset{\iota^*_{M\backslash D}}{\to} H^k(M\backslash D) \to \cdots$$

But because $E^\bullet$ is defined as a quotient, its cohomology is also part of a long exact sequence

$$\cdots \to H^k(M\backslash D) \to H^k(E^\bullet) \to H^{k+1}(M) \overset{\iota^*_{M\backslash D}}{\to} H^{k+1}(M\backslash D) \to \cdots .$$

Therefore we conclude that $H^k(E^\bullet) \simeq H^{k+1}(M, M\backslash D)$, which finishes the proof. □

1.4 Self-crossing stable structures in dimension four

In the remaining of this chapter we will focus on stable generalized complex structures in four dimensions. From a practical point of view, dimension four is special because the way different components of the divisor intersect is very restricted and at these intersections points the symplectic structure behaves as a meromorphic volume form. In this dimension it is particularly simple to state and prove a local normal form for a neighbourhood of a point (Theorem 1.4.3). Simple as this local form is, it has interesting consequences. Firstly, it can be used to show that every self-crossing stable generalized complex structure can be changed into a smooth stable structure (Theorem 1.4.12). Secondly it allows us to introduce a connected sum operation for stable generalized complex structures (Theorem 1.5.9).

1.4.1 Normal forms

Having related stable generalized complex structures to symplectic structures on a Lie algebroid, we have a wealth of symplectic techniques available to deal with them. One of the most basic tools from symplectic geometry, the Moser Lemma, carries through to Lie algebroids with some modifications. This allows us to tackle deformations and corresponding neighbourhood theorems. As for regular symplectic structures, the (local) deformations are governed by the Lie algebroid cohomology in degree two. One new feature, however, is that the local cohomology of the elliptic tangent bundle is non-trivial and hence the local model must depend on parameters.

The next lemma is a direct adaptation of the Moser lemma.

Lemma 1.4.1. *Let $I_{|D|}$ be an elliptic divisor on a manifold M and let $\omega_1, \omega_2 \in \Omega^2(\mathcal{A}_{|D|})$ be two elliptic symplectic forms defined on a neighbourhood U of a closed irreducible component of $D(i)$. If $[\omega_1] = [\omega_2] \in H^2(\mathcal{A}_{|D|})$ and $t\omega_1 + (1-t)\omega_2$ is non-degenerate for all $t \in [0,1]$, then there exists neighbourhoods $U_1, U_2 \subset U$ of $D(i)$ and a Lie algebroid isomorphism $\widetilde{\varphi}\colon \mathcal{A}_{|D|}\big|_{U_2} \to \mathcal{A}_{|D|}\big|_{U_1}$ such that $\widetilde{\varphi}^*\omega_1 = \omega_2$.*

Proof. The proof is nearly identical to the usual proof of the Moser Lemma: let $\alpha \in \Omega^1_{\Im}(\mathcal{A}_{|D|})$ be such that $\omega_1 - \omega_2 = d\alpha$ and define $X_t \in \Gamma(\mathcal{A}_{|D|})$ by $(t\omega_1 + (1-t)\omega_2)(X_t) = \alpha$. Then the time-one flow, $\widetilde{\varphi}$, of X_t on $\mathcal{A}_{|D|}$ (over the flow, φ, of $\rho(X_t)$ on U) will satisfy $\widetilde{\varphi}^*\omega_1 = \omega_2$. Since $X_t \in \Gamma(\mathcal{A}_{|D|})$, $\rho(X_t)$ is tangent to $D(i)$ and hence if we take a possibly smaller neighbourhood U_2 of $D(i)$ we have $\varphi(U_2) \subset U$ and taking $U_1 = \varphi(U_2)$ we obtain the result. □

A Darboux Theorem for points in $D(1)$, the smooth locus of the divisor, was already obtained in [16]. In four dimensions the only extra stratum available is $D[2]$ and we present the version of the Darboux Theorem for those points now.

Proposition 1.4.2. *Let $(I_{|D|}, \mathfrak{o})$ be a co-oriented elliptic divisor on M^4. Let $\omega \in \Omega^2_{\Im}(M)$ be an elliptic symplectic form and let $p \in D[2]$. Choose coordinates that express $I_{|D|}$ as the standard elliptic divisor in a neighbourhood of p and denote its residues by*

$$\lambda_1 = \mathrm{Res}_{r_1\theta_2}\,\omega(p) = \mathrm{Res}_{\theta_1 r_2}\,\omega(p), \qquad \lambda_2 = \mathrm{Res}_{r_1 r_2}\,\omega(p) = -\,\mathrm{Res}_{\theta_1\theta_2}\,\omega(p).$$

Then $\lambda_1^2 + \lambda_2^2 \neq 0$ and there exists coordinates $(r_1, \theta_1, r_2, \theta_2)$ on a neighbourhood of p on which $I_{|D|}$ is the standard elliptic divisor and ω is given by:

$$\lambda_1(d\log r_1 \wedge d\theta_2 + d\theta_1 \wedge d\log r_2) + \lambda_2(d\log r_1 \wedge d\log r_2 - d\theta_1 \wedge d\theta_2).$$

Proof. Indeed, $\{d\log r_1, d\theta_1, d\log r_2, d\theta_2\}$ is a frame for $\Omega^1(\mathcal{A}_{|D|})$ in a neighbourhood of p. Hence we can write ω in terms of wedge products of these generators with functions as coefficients. At p all these coefficients are residues and due to Lemma 1.3.14 we have

$$\omega(p) = \lambda_1(d\log r_1 \wedge d\theta_2 + d\theta_1 \wedge d\log r_2) + \lambda_2(d\log r_1 \wedge d\log r_2 - d\theta_1 \wedge d\theta_2).$$

Since $\omega^2(p) \neq 0$, we have $\lambda_1^2 + \lambda_2^2 \neq 0$.

Now consider

$$\omega_0 = \lambda_1(d\log r_1 \wedge d\theta_2 + d\theta_1 \wedge d\log r_2) + \lambda_2(d\log r_1 \wedge d\log r_2 - d\theta_1 \wedge d\theta_2) \in \Omega^2_{\Im}(\mathcal{A}_{|D|}).$$

Since $\omega(p) = \omega_0(p)$, the convex combination $t\omega + (1-t)\omega_0$ is symplectic in a neighbourhood of p for all $t \in [0,1]$ and Proposition 1.3.20 implies that $[\omega] = [\omega_0] \in H^2_{\Im}(\mathcal{A}_{|D|})$. Since $\{p\}$ is a connected component of $D[2]$, the Moser Lemma gives us the desired diffeomorphism between ω and ω_0. □

A direct consequence is a normal form for stable generalized complex structures.

Theorem 1.4.3. *Let M^4 be a stable generalized complex manifold. Then for every point $p \in D[2]$ there exists complex coordinates (z_1, z_2) around p where the complex log divisor is the standard one and such that a local trivialisation of the canonical line bundle is given by the pure spinor*

$$\rho = e^B(\lambda z_1 z_2 + dz_1 \wedge dz_2),$$

for some non-zero $\lambda \in \mathbb{C}$ and $B \in \Omega^2(M; \mathbb{R})$.

Proof. By Theorem 1.3.16, gauge equivalence classes of stable generalized complex structures are in equivalence with elliptic symplectic forms $\omega \in \Omega^2_{\Im}(M)$. By Proposition 1.4.2, any such symplectic form is given, in appropriate coordinates, by

$$\begin{aligned}\omega &= \lambda_1(d\log r_1 \wedge d\theta_2 + d\theta_1 \wedge d\log r_2) + \lambda_2(d\log r_1 \wedge d\log r_2 - d\theta_1 \wedge d\theta_2)\\ &= \Im^*((\lambda_1 + i\lambda_2)(d\log r_1 + id\theta_1) \wedge (d\log r_2 + id\theta_2)).\end{aligned}$$

Hence, up to the action of 2-forms, the stable generalized complex structure is given by

$$(\lambda_1 + i\lambda_2)^{-1} z_1 z_2 + dz_1 dz_2,$$

with $z_j = r_j e^{i\theta_j}$ and the stated normal form follows. □

1.4.2 Locally complex elliptic symplectic structures

Theorem 1.3.16 gives an equivalence between stable generalized complex structures and certain elliptic symplectic forms together with a co-orientation of the corresponding elliptic divisor. Of course, the main use of that result is to work on the symplectic side to conclude properties of the generalized complex structure. Here there is a minor difficulty: we must co-orient an elliptic divisor in the hopes of getting the desired residue relations, but there is no preferred way to do that. That is, if we are interested in constructing a generalized complex structure on a given manifold M with an elliptic divisor $I_{|D|}$, we must choose a co-orientation for D but must also keep in mind that we may have started with the wrong choice. We find it fruitful to introduce a notion that is independent of the choice of co-orientation and will allow us to get a better grip on the problem.

Definition 1.4.4. Let $I_{|D|}$ be an elliptic divisor and let $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be an elliptic symplectic form. We say that ω is **locally complex** if around every point there exists an open neighbourhood U and a complex log divisor I_D on U inducing the restricted divisor $I_{|D|}|_U$ together with a complex log symplectic form $\sigma \in \Omega^2(U; \mathcal{A}_D)$ such that $\Im^*\sigma = \omega$. ♢

Concretely, ω is locally complex if and only if its elliptic residue vanishes over $D(1)$ and over $D[2]$, for a choice of co-orientation of D, one of the following two possibilities holds:

$$(\mathrm{Res}_{\theta_i r_j}\,\omega - \mathrm{Res}_{r_i\theta_j}\,\omega = 0 \quad \text{and} \quad \mathrm{Res}_{r_i r_j}\,\omega + \mathrm{Res}_{\theta_i\theta_j}\,\omega = 0), \quad \text{or} \tag{1.4.1}$$

$$(\mathrm{Res}_{\theta_i r_j}\,\omega + \mathrm{Res}_{r_i\theta_j}\,\omega = 0 \quad \text{and} \quad \mathrm{Res}_{r_i r_j}\,\omega - \mathrm{Res}_{\theta_i\theta_j}\,\omega = 0), \tag{1.4.2}$$

with the second possibility indicating that the chosen co-orientation for one of the components of D is not compatible with ω.

In four dimensions, the existence of a locally complex elliptic symplectic structure forces the degeneracy locus to be of a very specific form.

Proposition 1.4.5. *Let M^4 be a four-dimensional compact locally complex elliptic symplectic manifold with respect to a co-orientable elliptic divisor $I_{|D|}$. Then the connected components of D are tori, spheres which intersect themselves in one point, or necklaces of spheres.*

Proof. It follows from the normal form from [16], that over $D(1)$ the modular vector field of the underlying Poisson structure is nowhere-vanishing. In particular, if a connected component D' of D is smooth and co-orientable, then it is orientable and has a nowhere vanishing vector field, hence is diffeomorphic to a torus.

Assume that a connected component D' is immersed but not embedded and let $\widetilde{D}' \to D'$ denote the immersion. Because $D' \backslash D[2]$ is a Poisson submanifold, the modular vector field of the Poisson structure is tangent to $D' \backslash D(2)$ and is nowhere-vanishing on $D' \backslash D(2)$. Moreover, in local coordinates as in Proposition 1.4.2 the modular vector field takes the form $-r_1\partial_{r_1} - r_2\partial_{r_2}$, which lifts to a vector field on $\widetilde{D}'$ with positive zeros at the pre-image of $D(2)$. Therefore each irreducible component of D' is an orientable surface with positive Euler characteristic (equal to the number of points in $\widetilde{D}'$ that map to $D(2)$. That is, each irreducible component of D' is a sphere with two points in $D(2)$. Thus either the irreducible component of D' intersects itself in one point, or it intersects another irreducible component(s) at two points. That irreducible component can in turn intersect the previous irreducible component or some other irreducible component. By compactness there are only finitely many irreducible components and thus these spheres have to form a necklace. □

Now we address the question of when a locally complex elliptic symplectic structure is actually induced by a complex log symplectic form. To do so we introduce the following notion.

Definition 1.4.6. Let M^4 be an oriented manifold with a co-oriented elliptic divisor $(I_{|D|}, \mathfrak{o})$. For each $p \in D[2]$ let D_1, D_2 be the corresponding local normal crossing divisors.

- We say that the **intersection index of** p **is** 1 if the isomorphism $T_pM \simeq N_pD_1 \oplus N_pD_2$ is orientation-preserving;
- We say that the **intersection index of** p **is** -1 otherwise. ♢

Since ND_1 and ND_2 are both two-dimensional their choice of ordering does not effect the orientation of the resulting direct sum $ND_1 \oplus ND_2$.

An elliptic symplectic structure always provides a preferred orientation for M. If the structure is also locally complex, we can verify whether an intersection point in $D[2]$ is positive or not by considering the values of the residues at that point.

Lemma 1.4.7. *Let $(M^4, I_{|D|}, \mathfrak{o})$ be a manifold endowed with a co-oriented elliptic divisor and let ω be a locally complex elliptic symplectic form. Then $p \in D[2]$ is positive with respect to the orientation induced by ω if and only if* (1.4.1) *holds.*

This can be rephrased in more global terms.

Lemma 1.4.8. *Let M^4 be a manifold endowed with a co-oriented elliptic divisor $(I_{|D|}, \mathfrak{o})$ and an elliptic symplectic form, ω. The triple $(I_{|D|}, \mathfrak{o}, \omega)$ is in the image of the map in Theorem 1.3.16 if and only if ω is locally complex and each point in $D[2]$ has positive index.*

Proof. By Lemma 1.3.14, $\omega \in \Omega^2_{\Im}(M)$ if and only if (1.4.1) holds, which by the previous Lemma corresponds to all intersection points being positive. □

Whenever a locally complex elliptic symplectic structure has a divisor with a few negative intersection points, we may try to fix this by flipping the co-orientation of one of the components arriving at that point. The problem with this is that such a change of co-orientation will also change the sign at 'the other' intersection point of that component. Reflecting on this for a moment we see that the parity of the number of negative points is the relevant piece of data.

Definition 1.4.9. Let $(M, I_{|D|}, \mathfrak{o})$ be an oriented four-manifold with co-oriented elliptic divisor. If we denote by ϵ_p the index of $p \in D[2]$, the **parity** of a connected component D' of D is given by

$$\varepsilon_{D'} = \Pi_{p \in D'[2]} \epsilon_p. \qquad \diamondsuit$$

We extend the definition of parity to a smooth irreducible component D' of D by declaring $\epsilon_{D'}$ to be $+1$ if D' is co-orientable and -1 if not. If $D'[2]$ has n points, a change of orientation of M changes the parity of D' by $(-1)^n$.

Lemma 1.4.10. *Let $(M, I_{|D|}, \mathfrak{o})$ be a four-manifold with co-oriented elliptic divisor and let $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be a locally complex elliptic symplectic form. Orient M by the orientation induced by ω, then:*

- *For each connected component D' of D, the parity, $\varepsilon_{D'}$, does not depend on the choice of co-orientation;*
- *We have $\varepsilon_{D'} = 1$ for all connected components D' of D if and only if there is a co-orientation $\mathfrak{o}'$ of D for which $(I_{|D|}, \mathfrak{o}', \omega)$ is in the image of the map in Theorem 1.3.16.*

Proof. The proof relies on Proposition 1.4.5, which describes what a connected component of D looks like. The case when D is smooth and co-orientable was treated in [16] so we need to consider the cases when D has self-intersections.

If D' has only one irreducible component which intersects itself in one point, p if we change the co-orientation of D' we change the co-orientation of both strands arriving at p and hence the index of p does not change. So $\varepsilon_{D'} = 1$ if and only if the index of p is 1 which, by Lemma 1.4.8, happens if and only if $(I_{|D|}, \mathfrak{o}, \omega)$ is in the image of the map in Theorem 1.3.16 in a neighbourhood of D'.

In general, since each irreducible component has two intersection points, changing its co-orientation changes two signs and hence the parity, $\varepsilon_{D'}$, remains unchanged by this operation. The proof that this is the only relevant invariant can be done intuitively by breaking the necklace of spheres, D', at one intersection point, so that we get an array of spheres and then fixing the co-orientation of 'the first' irreducible component, D_1, of this array. Then we inductively keep or change the co-orientations of the next irreducible components one by one depending on whether the index of the next intersection point is positive or negative until we get to the last sphere, D_n. Since D' is a necklace, not an array, there is one last index to be computed, namely the one between D_n and D_1. Since the parity of the number of negative indexed points is fixed, this last index is positive if the parity is positive. Hence, again by Lemma 1.4.8, the parity is positive if and only if $(I_{|D|}, \mathfrak{o}, \omega)$ is in the image of the map in Theorem 1.3.16 in a neighbourhood of D'.

It is clear that $(I_{|D|}, \mathfrak{o}, \omega)$ is in the image of the map in Theorem 1.3.16 if and only if for each irreducible component D' of D, $(I_{|D|}, \mathfrak{o}, \omega)$ is in the image of the map in Theorem 1.3.16 in a neighbourhood of D'. □

1.4.3 Smoothening self-crossing stable structures

In this section we will show that if a four-dimensional manifold admits a stable generalized complex structure, then it also admits a smooth stable generalized complex structure.

Given $\varepsilon > 0$ consider the following two stable generalized complex structures on $\mathbb{C}^2$:

$$\rho_0 = \lambda z_1 z_2 + dz_1 \wedge dz_2, \quad \rho_1 = (\lambda z_1 z_2 + \varepsilon) + dz_1 \wedge dz_2, \tag{1.4.3}$$

which determine, respectively, the complex log divisors:

$$I_{D_0} = \langle z_1 z_2 \rangle, \quad I_{D_1} = \langle \lambda z_1 z_2 + \varepsilon \rangle,$$

and corresponding complex log symplectic forms

$$\sigma_0 = \frac{dz_1 \wedge dz_2}{\lambda z_1 z_2} \qquad \text{and} \qquad \sigma_1 = \frac{dz_1 \wedge dz_2}{\lambda z_1 z_2 + \varepsilon}. \tag{1.4.4}$$

Lemma 1.4.11. *Let σ_0 and σ_1 be the complex log symplectic forms in* (1.4.4). *Then there are annuli $A_0, A_1 \subset \mathbb{C}^2$ and a diffeomorphism $\Phi\colon A_0 \to A_1 \subset \mathbb{C}^2$ which is a morphism of divisors between I_{D_1} and I_{D_0} and satisfies $\Im^*\sigma_0 = \Phi^*\Im^*\sigma_1$. Moreover, the map Φ is ambient isotopic to the natural inclusion $\iota\colon A_0 \to \mathbb{C}^2 \backslash \{0\}$.*

Proof. The proof relies on a version of the Moser argument: We will find annuli A_0 and A_1 together with a diffeomorphism, $\varphi\colon A_0 \to A_1$, with the following properties

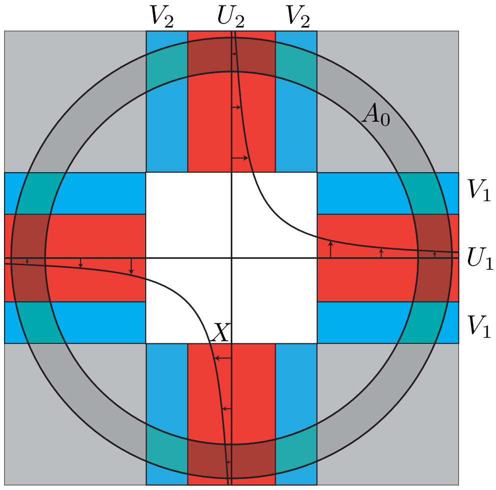


Figure 1.1: The divisor associated to ρ_0 are the coordinate axes while the divisor associated to ρ_1 is the hyperbola. In the complement of the polydisc of radius $2\varepsilon/|\lambda|$ both of these lie in U. The vector field X flows the coordinate axes to the intersection of the hyperbola with A and vanishes in the grey area.

- φ is a morphism of divisors between I_{D_1} and I_{D_0},
- $\varphi^*\Im^*\sigma_1$ lies in the same cohomology class of $\Im^*\sigma_0$ and
- the line connecting $\varphi^*\Im^*\sigma_1$ and $\Im^*\sigma_0$ is made of symplectic forms.

Once we have found such a φ, it is clear that the result follows from the Moser argument.

We start by considering $A \subset \mathbb{C}^2$, the complement of the polydisc of radius $2\varepsilon/|\lambda|$ on $\mathbb{C}^2$, that is

$$A = \{(z_1, z_2) \colon |z_1| > 2\sqrt{\varepsilon/|\lambda|} \text{ or } |z_2| > 2\sqrt{\varepsilon/|\lambda|}\}.$$

On A, we let

$$U = \{(z_1, z_2) \in A \colon |z_1| < \sqrt{\varepsilon/|\lambda|} \text{ or } |z_2| < \sqrt{\varepsilon/|\lambda|}\},$$
$$V = \{(z_1, z_2) \in A \colon |z_1| < 2\sqrt{\varepsilon/|\lambda|} \text{ or } |z_2| < 2\sqrt{\varepsilon/|\lambda|}\}.$$

Notice that both U and V consist of two components, namely U_1, V_1, which are neighbourhoods of $\{z_2 = 0\} \cap A$ and U_2, V_2, which are neighbourhoods of $\{z_1 = 0\} \cap A$. Also, notice that the zeros of $\lambda z_1 z_2 + \varepsilon$ in A also lie inside U, that is U is also a neighbourhood of the zero locus of the divisor associated to ρ_1 (See Figure 1.1).

Let $\psi\colon [0,\infty) \to [0,1]$ be a monotone bump function which is 1 on $[0, \frac{3}{2}\sqrt{\varepsilon/|\lambda|}]$ and zero on $[2\sqrt{\varepsilon/|\lambda|}, \infty)$ and consider the following complex vector field:

$$X := \begin{cases} -\frac{\psi(|z_2|)\varepsilon}{\lambda z_1}\partial_{z_2} & \text{on } V_1, \\ -\frac{\psi(|z_1|)\varepsilon}{\lambda z_2}\partial_{z_1} & \text{on } V_2, \\ 0 & \text{otherwise.} \end{cases}$$

The time-one flow of the real part of this vector field defines a diffeomorphism $\varphi\colon A \to A$.

Since in U_1, $X = -\frac{\varepsilon}{\lambda z_1}\partial_{z_2}$, the flow of its real part is the shear transformation

$$\varphi_t(z_1, z_2) = (z_1, z_2 - t\frac{\varepsilon}{\lambda z_1}) \tag{1.4.5}$$

as long as the flow remains in U_1. Hence the time-one flow satisfies

$$\varphi^*(\lambda z_1 z_2 + \varepsilon) = \lambda z_1(z_2 - \frac{\varepsilon}{\lambda z_1}) + \varepsilon = \lambda z_1 z_2. \tag{1.4.6}$$

A similar computation holds in U_2 and hence $\varphi^* I_{D_1} = I_{D_0}$.

Since $\Im^*$ is a cochain map, to prove that $[\Im^*\varphi^*\sigma_1] = [\Im^*\sigma_0]$ it is enough to prove that $[\varphi^*\sigma_1] = [\sigma_0] \in H^2(\mathcal{A}_{D_0})$. By Theorem 1.2.14 we have that $H^2(A, \mathcal{A}_{D_0}) \simeq H^2(A\backslash D_0)$. Because $A\backslash D_0$ deformation retracts onto a torus, for example, $T^2 = \{|z_1| = R, |z_2| = R\}$ with $R > 2\sqrt{\varepsilon/|\lambda|}$, we see that $H^2(A\backslash D_0) \simeq \mathbb{C}$ and we are left to show that $\int_{T^2}\sigma_0 = \int_{T^2}\varphi^*\sigma_1$. This torus is disjoint from V, hence φ is the identity near T^2 and we compute

$$\begin{aligned} \int_{T^2}\varphi^*\sigma_1 &= \int_{|z_1|=R}\int_{|z_2|=R}\frac{dz_1\wedge dz_2}{\lambda z_1 z_2 + \varepsilon} \\ &= \int_{|z_1|=R}\int_{|z_2|=R}\frac{1}{\lambda z_1}\frac{dz_1\wedge dz_2}{z_2 + \frac{\varepsilon}{\lambda z_1}} \\ &= 2\pi i\int_{|z_1|=R}\frac{1}{\lambda}\frac{dz_1}{z_1} = \frac{-4\pi^2}{\lambda}, \end{aligned}$$

which coincides with the integral of σ_0 over the same torus. Therefore we conclude that indeed $[\sigma_0] = [\varphi^*\sigma_1] \in H^2(A, \mathcal{A}_{D_0})$ and consequently that $[\Im^*\sigma_0] = [\varphi^*\Im^*\sigma_1] \in H^2(A, \mathcal{A}_{|D_0|})$.

Next we will find a radius r such that on $A\backslash B_r$ (the complement of the ball of radius r) the form $\sigma_t := t\varphi^*\sigma_1 + (1-t)\sigma_0$ is complex log symplectic for all values of t. To find r, we will study separately the behaviour of σ_t in three regions: U, $A\backslash V$ and $V\backslash U$.

Let us start with the region U. If $r > 0$ is large enough, the vector field X becomes small on $A\backslash B_r$ and its time-one flow starting at U_1 is the shear transformation (1.4.5), hence we have $\varphi^*\sigma_1 = \sigma_0$. The same argument holds at U_2 and hence, for r large enough, on $U\backslash B_r$, $\sigma_t = \sigma_0$ is symplectic for all t.

Next we consider $A\backslash V$. In this region, $\varphi = \mathrm{id}$ and therefore

$$\sigma_t = t\frac{dz_1 \wedge dz_2}{\lambda z_1 z_2 + \varepsilon} + (1-t)\frac{dz_1 \wedge dz_2}{\lambda z_1 z_2}$$
$$= \frac{\lambda z_1 z_2 + (1-t)\varepsilon}{(\lambda z_1 z_2 + \varepsilon)(\lambda z_1 z_2)} dz_1 \wedge dz_2.$$

Since in this region $|z_1| > 2\sqrt{\varepsilon/|\lambda|}$ and $|z_2| > 2\sqrt{\varepsilon/|\lambda|}$, the form above does not vanish and hence its imaginary part is symplectic.

Finally, we deal with $V\backslash(U \cup B_r)$. We will focus on $V_1\backslash U_1$. Apply the Mean Value Theorem to the map $t \mapsto \Im^*(\varphi_X^t)^*\sigma_1(p)$, we obtain the estimate

$$|\Im^*\sigma_1(p) - \Im^*\varphi^*\sigma_1(p)| < |\Im^*\mathcal{L}_X\sigma_1(p')|.$$

for p' in line segment between p and $\varphi(p)$. Using the explicit forms of X and σ_1 we find

$$|\Im^*\sigma_1 - \Im^*\varphi^*\sigma_1| < O(1/r^2).$$

We can also estimate the difference

$$|\Im^*\sigma_0 - \Im^*\sigma_1| = \left|\Im^*\left(\frac{\varepsilon dz_0 dz_1}{\lambda z_0 z_1(\lambda z_0 z_1 + \varepsilon)}\right)\right| = O(1/r^2).$$

Therefore, on $V_1\backslash(U_1 \cup B_r)$ we can estimate

$$\Im^*\sigma_t = \Im^*\sigma_0 + t(\Im^*\varphi^*\sigma_1 - \Im^*\sigma_0)$$
$$= \Im^*\sigma_0 - t((\Im^*\varphi^*\sigma_1 - \Im^*\sigma_1) + (\Im^*\sigma_1 - \Im^*\sigma_0)) = \Im^*\sigma_0 - tO(1/r^2).$$

Since $\Im^*\sigma_0 = O(1/r)$ the above is symplectic as long as r is large enough.

Therefore, by picking r as above and $R > r$ we can apply the Moser Lemma to the annulus $A_0 = B_R\backslash B_r$ to find a diffeomorphism $\widetilde{\Phi}\colon A_0 \to \widetilde{\Phi}(A_0)$ such that $\widetilde{\Phi}^*(\varphi^*\Im^*\sigma_1) = \Im^*\sigma_0$. That is, $\Phi = \varphi \circ \widetilde{\Phi}\colon A_0 \to (\varphi \circ \widetilde{\Phi})(A_0)$ is the diffeomorphism we were looking for. Since Φ is the composition of flows of vector fields it is ambient isotopic to the inclusion $A_0 \to \mathbb{C}^2\backslash\{0\}$. □

Using this lemma we can now prove that every locally complex elliptic symplectic structure can be changed into a smooth one by performing a surgery.

Theorem 1.4.12. *Let $(M^4, I_{|D|})$ be a manifold endowed with a co-oriented elliptic divisor and let $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be elliptic symplectic. Then:*

- *If ω is locally complex, then it can be changed into a smooth elliptic symplectic form with zero elliptic residue $\widetilde{\omega}$;*
- *The resulting structure will be induced by a stable generalized complex structure if and only if the original structure was.*

Proof. Since $D[2]$ has codimension four and is compact, it is a finite collection of points. Because ω is locally complex, given $p \in D[2]$ there is a neighbourhood of p in which ω is the imaginary part of some complex log symplectic form σ. By Lemma 1.4.3 there exists complex coordinates (z_1, z_2) on a ball, B, around p on which σ is (B-field equivalent) to $\lambda d \log z_1 \wedge d \log z_2$. Note that as σ is scaling-invariant in both the z_1- and z_2-direction, we can ensure that this ball is as large as necessary. Therefore we can apply Lemma 1.4.11 to find an annulus $A_0 \subset B$ together with its accompanying diffeomorphism $\Phi\colon A_0 \to A_1$. We denote by $B' \subset B$ the inner ball enclosed by A_0. Let $\widetilde{B} \subset \mathbb{C}^2$ be the ball enclosed by the outer boundary of the annulus A_1. Because Φ is ambient isotopic to the inclusion of A_0 in $D^4 \backslash \{0\}$, we have $M \simeq M \backslash B' \cup_{A,\varphi} \widetilde{B}$. If we endow $\widetilde{B}$ with the smooth stable structure ρ_1 in Equation (1.4.3), we see by Lemma 1.4.11 that the map Φ is an elliptic symplectomorphism. We conclude that M obtains an elliptic symplectic structure with $D[2]$ consisting of one point less. Moreover, as the surgery is purely local in nature we can ensure that the structure does not change around the remaining points in $D[2]$. By performing this procedure for all points in $D[2]$ we conclude that M admits a smooth elliptic symplectic structure with zero elliptic residue.

For the second part of the theorem, if ω was induced by a stable structure, then the local coordinates from Theorem 1.4.3 would be orientation-preserving. As all maps used in the surgery are orientation-preserving we conclude that the resulting divisor is co-orientable. We conclude that M admits a smooth stable generalized complex structure. If ω is not induced by a stable structure, then there is at least one set of coordinates obtained from Theorem 1.4.3 which is orientation-reversing. Therefore the resulting divisor will not be co-orientable, which finishes the proof. □

1.5 Connected sums

In this section we will introduce a connected sum operation for elliptic symplectic structures in four dimensions. To do so we will make use of normal form results obtained in Section 1.4. The operation will be phrased in terms of locally complex symplectic forms (Definition 1.4.4) and it will be useful for us to keep track of the index of points as we perform the connected sum.

1.5.1 Glueing divisors

We will perform connected sums on elliptic symplectic four-manifolds at points which lie in their respective sets $D[2]$. In arbitrary dimensions it is possible to take connected sums of elliptic divisors at points with the same multiplicity.

Lemma 1.5.1. *Let $(M^n, I_{|D_M|})$, $(N^n, I_{|D_N|})$ be two oriented manifolds endowed with elliptic divisors and let $p \in D_M[k]$ and $q \in D_N[k]$ for $k \in \mathbb{N}$. Then $M \#_{p,q} N$ admits an elliptic divisor $I_{|D|}$, for which the inclusions $M \backslash \{p\}, N \backslash \{q\} \to (M \#_{p,q} N)$ are morphisms of divisors.*

Proof. Using Lemma 1.1.25 there exist coordinates around p and q such that both divisors are precisely given by the ideal $I = \langle r_1^2 \cdot \ldots \cdot r_k^2 \rangle$. To take the connected sum of M and N at p and q, we need to use an orientation-preserving diffeomorphism, F, from an annulus around p to an annulus around q, which reverses the co-orientation of the sphere and for which $F^*I = I$. Here we consider oriented charts defined in neighbourhoods of p and q which map to the unit ball in $\mathbb{R}^n$ and we will use the diffeomorphism given in spherical coordinates by

$$F : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}, \quad (r, \varphi_1, \ldots, \varphi_n) \mapsto (r^{-1}, \varphi_1, \ldots, -\varphi_n). \tag{1.5.1}$$

To verify that $F^*I = I$ we write $r_i^2 = r^2 \psi_i(\varphi_1, \ldots, \varphi_n)$, where ψ_i is a function which only depends on the angular coordinates and satisfies $\psi_i(\varphi_1, \ldots, -\varphi_n) = \psi_i(\varphi_1, \ldots, \varphi_n)$. We have

$$F^*(r_i^2) = \frac{1}{r^2}\psi_i(\varphi_1, \ldots, \varphi_n), \text{ and thus } r^4 F^*(r_i^2) = r_i^2.$$

Hence $r^{4k} F^*(r_1^2 \cdot \ldots \cdot r_k^2) = r_1^2 \cdot \ldots \cdot r_k^2$ and as r^{2k} is a non-zero function on $\mathbb{R}^n \backslash \{0\}$ we conclude that $F^*I = I$. □

Similarly we can perform a self-connected sum, which when M^m is connected corresponds to attaching a 1-handle and hence the diffeomorphism type of the resulting space is $M \#(S^1 \times S^{m-1})$.

Lemma 1.5.2. *Let M^m be an oriented connected manifold endowed with an elliptic divisor $I_{|D|}$ and let $p_1, p_2 \in D[k]$ be distinct points. Then $M\#(S^1 \times S^{m-1})$ and admits an elliptic divisor $I_{|\widetilde{D}|}$ for which the natural inclusion $(M \backslash \{p_1, p_2\}, I_{|D|}) \to (M\#(S^1 \times S^{m-1}), I_{|\widetilde{D}|})$ is a morphism of divisors.*

There are a couple of points about this construction that we should stress.

Remark 1.5.3. There is some freedom in the glueing of elliptic divisors. Given a choice of local coordinates $(z_1, \ldots, z_m)$ around p and q we can furthermore compose the map F by a permutation of the first k coordinates. Note that this does not change $M\#_{p,q}N$, but it could change the topology of the zero locus of the divisor. Because of this ambiguity in ordering, there are potentially $k!$ different topological types for the vanishing locus of the divisor on the connected sum $M\#_{p,q}N$. ♢

Remark 1.5.4 (Connected components)**.** Although there is some freedom in the choices we can still distinguish the number of connected components on the divisor on the connected sum:

1. When p_1 and p_2 lie in different connected components of the divisor, be that either in the connected sum of two manifolds or in a self-connected sum, the connected components containing p_1 and p_2 will combine into a single connected component of $\widetilde{D}$.

2. When p_1 and p_2 are in the same connected component, D, a case that can only happen in a self-connected sum, the resulting divisor, $\widetilde{D} \subset M\#(S^1 \times S^{m-1})$, may have one or two connected components originating from D. $\diamondsuit$

Remark 1.5.5. The reason for flipping the sign of φ_n in the map in (1.5.1) is that we assumed that the coordinates we chose are compatible with the orientations of M and N. If for some reason only one of the chosen coordinates was compatible with the orientations of M and N, we should not flip the sign of φ_n. $\diamondsuit$

1.5.2 Glueing symplectic structures

We will introduce a connected sum operation for elliptic symplectic structures in dimension four. The existence of such an operation contrasts starkly with ordinary symplectic geometry, where connected sums are not possible above dimension two. We first note that F introduced in (1.5.1) is a local symplectomorphism for a specific order of the coordinates.

Lemma 1.5.6. *Let (z_1, z_2) be complex coordinates on $M = \mathbb{C}^2 \backslash \{0\}$ and consider the elliptic symplectic structure on M given by*

$$\omega = \Im^*(i\, d\log z_1 \wedge d\log z_2).$$

Using polar coordinates

$$(z_1, z_2) = (r\cos\varphi_1 + ir\sin\varphi_1\cos\varphi_2, r\sin\varphi_1\sin\varphi_2\cos\varphi_3 + ir\sin\varphi_1\sin\varphi_2\sin\varphi_3),$$

the map defined in (1.5.1) *satisfies*

$$F^*\omega = -\omega.$$

Proof. The proof is a direct computation using that we can express F in complex coordinates as

$$F(z_1, z_2) = \frac{1}{r^2}(z_1, \overline{z}_2). \qquad \square$$

Definition 1.5.7. Let $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be a locally complex elliptic symplectic form. We say that ω has **imaginary parameter** at a point $p \in D[2]$ if

$$\mathrm{Res}_{r_1\theta_2}\,\omega(p) = \mathrm{Res}_{r_2\theta_1}\,\omega(p) = 0. \qquad \diamondsuit$$

Note that this definition does not depend on a choice of co-orientation.

Remark 1.5.8. By Proposition 1.4.2, an elliptic symplectic form with imaginary parameter ω is locally isomorphic to

$$\lambda\Im^*(id\log z_1 \wedge d\log z_2), \tag{1.5.2}$$

for an appropriate choice of complex coordinates, where $\lambda = \mathrm{Res}_{r_1 r_2}(\omega)$. This justifies the terminology. It is also immediate that these complex coordinates are compatible with the orientation defined by the complex structure.

Notice however that if the divisor is co-oriented, the complex coordinates above may not be compatible with the co-orientations. If we require compatibility between co-orientation and complex coordinates, ω will be isomorphic to either the form above or

$$\lambda \Im^*(id \log z_1 \wedge d \log \overline{z}_2).$$

In the latter case, the complex coordinates and symplectic structure induce opposite orientations. $\diamondsuit$

We can now turn to the main result of this section, namely the connected sum operation in dimension four for elliptic symplectic structures with imaginary parameter at points in $D(2)$ whose imaginary parameters match in absolute value.

Theorem 1.5.9. *Let $(M, I_{|D_M|})$ and $(N, I_{|D_N|})$ be four-manifolds with elliptic divisors. Let ω_1, ω_2 be locally complex elliptic symplectic forms with imaginary parameters at $p \in D_M(2)$ and $q \in D_N(2)$ and denote by D'_M and D'_N the connected components of the divisor containing p and q. If $|\mathrm{Res}_{r_1 r_2}\, \omega_1(p)| = |\mathrm{Res}_{r_1 r_2}\, \omega_2(q)|$, then:*

- *The connected sum $M\#_{p,q}N$ admits a locally complex elliptic symplectic form, ω, for which the inclusions $(M\backslash\{p\}, \omega_1), (N\backslash\{q\}, \omega_2) \hookrightarrow (M\#_{p,q}N, \omega)$ are elliptic symplectic maps.*
- *If either $D'_M(2)$ or $D'_N(2)$ has more than one point, the parity of $D'_{M\#N}$ is given by*
$$\varepsilon_{D'_{M\#N}} = -\varepsilon_{D'_M}\varepsilon_{D'_N}.$$
- *If $D'_M(2) = \{p\}$ and $D_N(2) = \{q\}$, then $D'_{M\#N}$ is co-orientable if and only if p and q have opposite parities.*

Proof. We will prove the claims of the theorem in turn. The tools needed are the normal form for locally complex elliptic symplectic structures with imaginary parameter from Remark 1.5.8 and the local symplectomorphisms from Lemma 1.5.6.

To prove the first claim we choose complex coordinates in a neighbourhood of p which render the symplectic structure in the form (1.5.2) and do the same for q, but reverse their order so that $\mathrm{Res}_{r_1 r_2}\, \omega_1(p) = -\mathrm{Res}_{r_1 r_2}\, \omega_2(q)$.

For this choice of coordinates, if we use F to perform the connected sum, Lemma 1.5.6 implies that the structures on $M\backslash\{p\}$ and $N\backslash\{q\}$ agree on their overlap on $M\#_{p,q}N$ which therefore inherits an elliptic symplectic structure.

Now we move to the second claim. Choose co-orientations for D'_M and D'_N. If p and q have the same index and, say, $D'_M(2)$ has more than one point, we can change the choice of co-orientation of one of the irreducible components arriving at p which causes the index of p to change sign. So we may assume without loss of generality that the indices of p and q are opposite. We assume that p has negative index and q has positive index as the other case is analogous.

In this case, the complex coordinates used in a neighbourhood of p in the first claim only give the correct co-orientation of one of the irreducible components of D'_M passing through p, say, the one given by $[z_2 = 0]$. That is the co-orientation of $[z_1 = 0]$ is determined by $d\overline{z_2}$ and the co-orientation of $[z_2 = 0]$ by dz_1. Since q has positive index, we may assume that the complex coordinates chosen for q are compatible with co-orientations. Finally we observe that the map F from Lemma 1.5.6 sends the line dz_1 over $[z_2 = 0]$ to itself and sends the line dz_2 over $[z_1 = 0]$ to $d\overline{z_2}$. Therefore, the co-orientations of D'_M and D'_N are mapped to each other under F and hence $D'_{M\#N}$ inherits a natural co-orientation from those of D'_M and D'_N and we can compute its index:

$$\begin{aligned}\varepsilon_{D'_{M\#N}} &= \Pi_{r\in D'_M(2), r\neq p}\epsilon_r \Pi_{s\in D'_N(2), s\neq q}\epsilon_s\\ &= -\epsilon_p\epsilon_q \Pi_{r\in D'_M(2), r\neq p}\epsilon_r \Pi_{s\in D'_N(2), s\neq q}\epsilon_s = -\varepsilon_{D'_M}\varepsilon_{D'_N}.\end{aligned}$$

Finally we prove the last claim. If the indices of p and q are opposite, the previous argument shows that a choice of co-orientation for $D'_M(2)$ and $D'_N(2)$ induces a co-orientation for $D'_{M\#N}$, which is therefore co-orientable. If the indices of p and q agree, then the argument above shows that map F matches co-orientations of one of the strands arriving at p and q and reverses the other. Since D'_M and D'_N are both connected, this implies that $D'_{M\#N}$ is not co-orientable. □

Instead of performing connected sums of two manifolds we can also perform a connected sum of a manifold with itself (self-connected sum), that is we can glue neighbourhoods of points p and $q \in M$ by an inversion on the annulus. In this case, if M is connected, this operation corresponds to attaching a 1-handle and hence the diffeomorphism type of the resulting space is always $M\#(S^1\times S^3)$. In this context, Theorem 1.5.9 becomes:

Corollary 1.5.10. *Let $(M^4, I_{|D|})$ be a four-manifold with an elliptic divisor and let ω be a locally complex elliptic symplectic form with imaginary parameters at $\{p, q\} \in D[2]$ with $p \neq q$. Denote by D'_p and D'_q the connected components of the divisor containing p and q, respectively. If $|\mathrm{Res}_{r_1r_2}\,\omega(p)| = |\mathrm{Res}_{r_1r_2}\,\omega(q)|$, then:*

- *$M^4\#(S^1\times S^3)$ admits a locally complex elliptic symplectic structure for which the inclusion $M^4\backslash\{p, q\} \hookrightarrow (M^4\#(S^1\times S^3))$ is an elliptic symplectic map.*

- *If $D'_p \neq D'_q$, then the corresponding connected component $\widetilde{D}'$ of $\widetilde{D}$ satisfies:*

$$\varepsilon_{\widetilde{D}'} = -\varepsilon_{D'_p}\varepsilon_{D'_q};$$

- *If $D'_p = D'_q$, then the corresponding connected components $\widetilde{D}'_p, \widetilde{D}'_q$ of $\widetilde{D}$ satisfy:*

$$\varepsilon_{\widetilde{D}'_p}\varepsilon_{\widetilde{D}'_q} = -\varepsilon_{D'_p}\varepsilon_{D'_q}.$$

Remark 1.5.11. It might be more desirable to arrive at the conclusion of this corollary by producing a stable generalized complex structure on $S^1 \times S^3$ for which $D[2] \neq \emptyset$ and then using Theorem 1.5.9 to perform the connected sum. Unfortunately, presently we do not know if $S^1 \times S^3$ has such a structure. ♢

1.6 Examples

In this section we will use the connected sum procedure of Theorem 1.5.9 to create several examples of elliptic symplectic structures and stable generalized complex structures. In order to do so we will start by constructing several building blocks, which are then combined via connect sum. Throughout, we will visualise the examples making use of certain singular torus fibrations. These fibrations and their relation to stable generalized complex structures is studied in more detail in the next chapter.

1.6.1 Simple examples

As we showed in Example 1.3.9, $\mathbb{C}P^2$ admits a stable generalized complex structure whose divisor is given by three lines. The next few examples show that $S^2 \times S^2$ has such a structure as well, while $\overline{\mathbb{C}P^2}$ and S^4 do not, but do have locally complex elliptic symplectic structures.

Example 1.6.1 ($\overline{\mathbb{C}P^2}$)**.** Let $\overline{\mathbb{C}P^2}$ denote the oriented manifold which is $\mathbb{C}P^2$ endowed with the orientation opposite to the one coming from the complex structure. In this example we construct an elliptic symplectic form with imaginary parameter on $\overline{\mathbb{C}P^2}$ for the elliptic divisor induced by $(\mathcal{O}(3), z_0z_1z_2)$. This elliptic divisor cannot be the imaginary part of a complex log symplectic form, since $\overline{\mathbb{C}P^2}$ does not admit generalized complex structures. Indeed, any generalized complex manifold is almost complex and $\overline{\mathbb{C}P^2}$ does not admit an almost complex structure compatible with the orientation.

Let $\mathcal{O}(1) \to \mathbb{C}P^2$ be the dual of the tautological line bundle and for $i = 0, 1, 2$ let $z_i \in \Gamma(\mathcal{O}(1))$ be the section induced by the homogeneous polynomial z_i on $\mathbb{C}^3$. Consider the three smooth complex log divisors, $D_i = (\mathcal{O}(1), z_i)$, let $D = (\mathcal{O}(3), z_0z_1z_2)$ be their product and let $|D|$ be the corresponding elliptic divisor. Using the underlying affine coordinates,

$$u_1 = \frac{z_1}{z_0}, \qquad u_2 = \frac{z_2}{z_0}, \qquad v_0 = \frac{z_0}{z_1}, \qquad v_2 = \frac{z_2}{z_1}, \qquad w_0 = \frac{z_0}{z_2}, \qquad w_1 = \frac{z_1}{z_2},$$

we define the following global elliptic two-form

$$\omega := \begin{cases} \Im^*(id\log\overline{u_1} \wedge d\log u_2) & \text{if } z_0 \neq 0, \\ \Im^*(-id\log\overline{v_0} \wedge d\log v_2) & \text{if } z_1 \neq 0, \\ \Im^*(-id\log\overline{w_1} \wedge d\log w_0) & \text{if } z_2 \neq 0. \end{cases}$$

It is immediate from the expression above that ω is locally complex with imaginary parameter. Moreover we see that it induces the orientation opposite from the usual complex structure on $\mathbb{C}P^2$, hence it is a locally complex elliptic symplectic structure with imaginary parameter for every point in $D[2]$ on $\overline{\mathbb{C}P^2}$.

Finally, if we consider the co-orientation of the elliptic divisors induced by the complex log divisor in Example 1.1.10 we have that the points $D_0 \cap D_2$ and $D_0 \cap D_1$ have positive index, while the point in $D_1 \cap D_2$ has negative index. By Lemma 1.4.10 we conclude that ω cannot be the imaginary part of a complex log symplectic form. Further, we observe that if we were to choose the opposite co-orientation for D_0, all intersection indices would be -1.

We can provide a simple picture to illustrate this and all the other examples in this section. Recall that $\mathbb{C}P^2$ admits a singular torus fibration whose fibers are the orbits of the standard torus action on $\mathbb{C}P^2$. The quotient space $\mathbb{C}P^2/T^2$ is a triangle and the elliptic symplectic structure constructed above is invariant under this action (with symplectic fibers). The zero locus of the divisor is the pre-image of the edges of the triangle and points in $D[2]$ are the pre-images of the vertices. With this in mind, we use a triangle to represent $\mathbb{C}P^2$ (or $\overline{\mathbb{C}P^2}$) and decorate each vertex of the triangle with the intersection index of the corresponding point in $D[2]$ (see Figure 1.2). $\triangle$

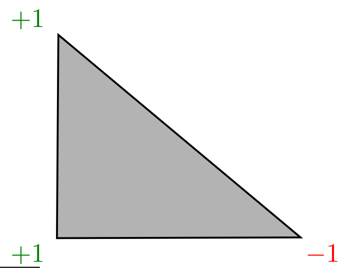


Figure 1.2: We visualise $\overline{\mathbb{C}P^2}$ as its image under the moment map and each vertex in the triangle corresponds to a point in $D[2]$. We label the vertices with ± 1 according to the intersection index of the corresponding point in $\overline{\mathbb{C}P^2}$.

Example 1.6.2 ($S^2 \times S^2$). The manifold $S^2 \times S^2$ admits a complex log symplectic structure σ for which $D[2]$ consists of four points. The imaginary part of σ is an elliptic symplectic form with imaginary parameter. Indeed, identifying S^2 with the extended complex plane, the vector field $z\partial_z$ vanishes transversely at 0 and ∞ and hence in $S^2 \times S^2$ (with complex coordinates z and w), the bivector field $\pi = -izw\partial_z\partial_w$ is Poisson and determines a complex log divisor. Therefore we can use π to deform the complex structure of $S^2 \times S^2$ into a stable generalized complex structure (as in Example 1.3.8). A direct check shows that this structure has imaginary parameter at all points in $D[2]$. Since this stable generalized complex structure is obtained from a holomorphic Poisson structure, the natural co-orientation of each irreducible component of the divisor (induced by the complex structure) makes all intersection indices positive. Yet, by changing co-orientations, we can arrange that any pair or all four points in $D[2]$ have negative index.

Just as for $\overline{\mathbb{C}P^2}$, we can provide an illustration for this structure using the toric description of $S^2 \times S^2$ (see Figure 1.3). $\triangle$

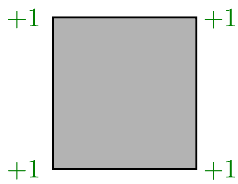


Figure 1.3: We visualise $S^2 \times S^2$ as the image under the map given by the two height functions. We label the vertices with ± 1 according to the intersection index of the corresponding points in $S^2 \times S^2$. Different choices of co-orientations yield different sign combinations at the vertices.

Example 1.6.3 (S^4)**.** The manifold S^4 admits an elliptic symplectic form with imaginary parameter with divisor consisting of two copies of S^2 intersecting each other at the north and south pole.

Consider two copies of D^4 and endow one copy with the two-form $\omega := \Im^*(id \log z_1 \wedge d \log z_2)$ and the other copy with $-\omega$. Using the map F, as in Lemma 1.5.6, we can glue an annulus in one of the disks to an annulus in the other disk while preserving the elliptic symplectic structures. The resulting manifold is diffeomorphic to S^4 and the divisors intersect at the points $(z_1, z_2) = (0, 0)$ in both copies of D^4, which correspond to the north and south pole of the sphere. Because F involves a complex conjugation, a choice of co-orientations for which one point in $D[2]$ has positive index causes the other point to have negative index. As in the previous examples, S^4 admits a natural torus action which rotates each complex coordinate in D^4 for which the hyperplanes $[z_i = 0]$ have S^1 isotropy and the north and south poles are fixed points. This allows us to produce a two-dimensional illustration of this structure (see Figure 1.4). $\triangle$

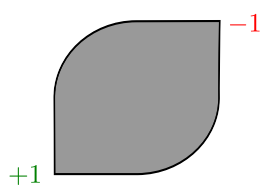


Figure 1.4: We visualise S^4 as its quotient by the standard torus action: the edges correspond to the hyperplanes $[z_i = 0]$ and the corners to the north and south poles.

Example 1.6.4. Because S^4 admits an elliptic symplectic form with imaginary parameter, by Example 1.6.3, we can use Theorem 1.4.12 to obtain a smooth elliptic symplectic structure on S^4. However the degeneracy locus is non co-orientable, because the smoothing process did not preserve co-orientations. Because S^4 is orientable, we conclude that the degeneracy locus has to be non-orientable. Since the modular vector field is tangent to the degeneracy locus and nowhere zero, we see that the degeneracy locus is diffeomorphic to a Klein bottle. △

1.6.2 Main class of examples

In this section we will combine all of our four-dimensional results to create new examples of elliptic symplectic and stable generalized complex structures on a large class of four-manifolds exhibited as connected sums.

Theorem 1.6.5. *The manifolds in the following two families admit stable generalized complex structures:*

1. $X_{n,\ell} := \#n(S^2 \times S^2)\#\ell(S^1 \times S^3)$, *with* $n, \ell \in \mathbb{N}$;

2. $\widehat{X}_{n,m,\ell} := \#n\mathbb{C}P^2\#m\overline{\mathbb{C}P^2}\#\ell(S^1 \times S^3)$, *with* $n, m, \ell \in \mathbb{N}$,

as long as $1 - b_1 + b_2^+$ *is even and the Euler characteristic is non-negative.*

Notice that if $1 - b_1 + b_2^+$ is odd for a four-manifold M, then M does not admit any generalized complex structure as it is not even almost complex by [46] or [36, Theorem 1.4.13]. The requirement that the Euler characteristic is positive, on the other hand, seems to be more of a limitation of our methods.

Proof. We will first prove that the manifolds in the list above admit elliptic symplectic structures with imaginary parameters as long as the Euler characteristic is non-negative and then show that these elliptic symplectic structures come from a generalized complex one if $1 - b_1 + b_2^+$ is even.

In Examples 1.3.9, 1.6.1 and 1.6.2 we produced elliptic symplectic structures on $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ with 3, 3 and 4 points in $D[2]$ respectively. These are all locally complex with imaginary parameter, so that by applying Theorem 1.5.9 inductively we obtain elliptic symplectic structures on $X_{n,0}$ and $\widehat{X}_{n,m,0}$ for all values of n, m, including the case $n = m = 0$ by Example 1.6.3. The number of points in $D[2]$ in these manifolds is, respectively, $n + m + 2$ and $2n + 2$. By Corollary 1.5.10 we can self-connect sum these spaces up to $\lfloor \frac{n+m+2}{2} \rfloor$ and $n + 1$ times, respectively, to obtain elliptic symplectic structures with imaginary parameters on the spaces of the list with non-negative Euler characteristic.

To prove that the elliptic symplectic structures constructed above are induced by stable generalized complex structures, it suffices to show that the parity is 1, by Lemma 1.4.10. Due to Theorem 1.5.9 and Corollary 1.5.10, the parity of the symplectic structure for both families is $(-1)^{n-1+\ell}$, which is positive if and only if $n - 1 + \ell$ is even, that is, $1 - b_1 + b_2^+$ is even. □

Remark 1.6.6. Several of the manifolds in Theorem 1.6.5, although not all, have already appeared before. The family $\widehat{X}_{n,m,0}$ with $n > 0$ and $m \geq 0$ can be found in [15]. The manifolds $X_{n,0}$, $X_{n,1}$ and $\widehat{X}_{n,m,1}$) appeared in [74]. The examples with $\ell > 1$ have not appeared before. The main advantage of our approach is that it is direct. We construct geometric structures on manifolds which are connected sums by showing that the connected sum operation is compatible with the structures in question. This is in contrast with those references, which instead construct manifolds with the desired structure via surgeries and then determine the resulting diffeomorphism type at a later stage. $\diamondsuit$

Remark 1.6.7. The manifolds in both families in Theorem 1.6.5 do not admit complex or symplectic structures if $n > 1$. Indeed, for $n > 1$, the manifolds are connected sums of manifolds with $b_2^+ > 0$ and by results from Seiberg–Witten theory due to Taubes (see [72]) no such manifold admits a symplectic structure.

To prove the non-existence of complex structures requires a slightly longer argument. For ℓ even, if the manifolds admitted complex structures they would be Kähler and hence also symplectic, but we ruled out this possibility already. The argument for ℓ odd comes from a paper by Belgun [8] and goes as follows. It follows from the Kodaira classification of surfaces that if one of these manifolds were complex, call it X, then its Kodaira dimension would be 1 and X would be an elliptic surface, possibly with multiple fibers, $X \to B$. But in this case there is a finite cover, $\widetilde{X}$ of X, corresponding to a branched cover $\widetilde{B}$ of B which is a genuine fibration: $\widetilde{X} \to \widetilde{B}$. By a result of Mehara [56], any such $\widetilde{X}$ is a quotient of $\mathbb{C}^2$ by a discrete group. In particular, we conclude that $\widetilde{X}$ and (hence also X) would be aspherical, which is not the case for our manifolds. We are thankful to Ornea and Vuletescu for pointing us towards this argument. $\diamondsuit$

Chapter 2

Fibrations in semi-toric and generalized complex geometry

In this chapter, which is based on [19] and joint work with Ralph Klaasse and Gil Cavalcanti, we study a class of singular fibrations related to both semi-toric geometry and generalized complex geometry.

We define the class of singular fibrations called *self-crossing boundary (Lefschetz) fibrations*. These are maps of pairs $f : (M, D) \to (N, Z)$, where D is a codimension two and Z a codimension one immersed submanifold with normal self-crossings such that f is an honest fibration away from Z and degenerates in a quadratic fashion along D.

Consider a self-crossing boundary Lefschetz fibration $f\colon (M, D) \to (N, Z)$. The map f has singularities precisely such that it induces an elliptic divisor $I_{|D|}$, as in Definition 1.1.19, and a Lie algebroid morphism

$$(\varphi, f)\colon (\mathcal{A}_{|D|}, M) \to (\mathcal{A}_Z, N),$$

between the elliptic tangent bundle (Definition 1.2.8) and the real log tangent bundle (Definition 1.2.4). Moreover, φ is a Lie algebroid version of a Lefschetz fibration. We will use these fibrations to construct self-crossing elliptic symplectic structures $\omega \in \Omega^2(\mathcal{A}_{|D|})$ on their total space. The relevant geometric structure on the base of this fibration is a symplectic structure on $\mathcal{A}_Z$, also known as a *self-crossing log-symplectic structure* ([43, 64]).

Following the strategy of constructing symplectic structures out of fibrations, we prove a Gompf–Thurston theorem for self-crossing stable generalized complex structures. This result is the generalisation of a similar result for stable generalized complex structures with embedded type-change locus appearing in [13], but to apply it in our setting we require several adaptations of that argument. We say that a map f is *homologically essential* if its generic fibre is non-trivial in homology.

Theorem 2.1.23. *Let $f\colon (M^4, D^2) \to (N^2, \partial N)$ be a self-crossing boundary Lefschetz fibration. If f is homologically essential then M^4 admits an elliptic symplectic structure, which induces a self-crossing stable generalized complex structure if the locus D is co-orientable and its index is equal to* 1.

Construction Having established that boundary Lefschetz fibrations supply self-crossing stable generalized complex structures, we decouple the map from the geometric structure and study them separately. Seeing that in Chapter 1 we have proven that one can take connected sums of stable generalized complex structures, we expect the same to hold for the boundary Lefschetz fibrations:

Theorem 2.2.3. *Let $f_i\colon (M_i^4, D_i^2) \to (N_i^2, \partial N_i)$ for $i = 1, 2$ be boundary Lefschetz fibrations and let $p_i \in M_i$ be such that $q_i = f_i(p_i)$ are corners of the manifolds N_i. Then there exists a boundary Lefschetz fibration on their connected sum*

$$f_1 \# f_2 \colon (M_1 \#_{p_1,p_2} M_2, D_1 \# D_2) \to (N_1 \#_{q_1,q_2} N_2, \partial(N_1 \# N_2)),$$

which is compatible with the inclusion $M_i \backslash \{p_i\} \hookrightarrow M_1 \# M_2$. Moreover, the map $f_1 \# f_2$ is homologically essential if and only if f_1 and f_2 are.

This result is in sheer contrast with the situation in toric geometry. There is no symplectic connected sum procedure and most of the manifolds obtained using the above proposition will have no toric structure. This difference in rigidity between the generalized complex and toric worlds is already apparent on the base of these fibrations. Namely, for generalized complex structures the base carries a self-crossing log-symplectic structure, which is quite flexible. On the other hand, in toric geometry the base carries an integral affine structure, which is very rigid. In other words, although toric manifolds do not behave well with respect to connected sums, the underlying torus actions and abstract quotient maps do.

Singularity trades The nodal trade procedure in semi-toric geometry exchanges elliptic–elliptic singularities of the moment map for focus–focus singularities [80, 61] and vice versa [55]. These procedures rely heavily on the existence of a singular integral affine structure on the base. Following our general strategy, decoupling the geometric structure from the maps allows us to prove an abstract statement for boundary Lefschetz fibrations:

Theorem 2.3.3. *Let $f\colon (M^4, D^2) \to (N^2, \partial N)$ be a boundary Lefschetz fibration and let $p \in M$ be an elliptic–elliptic singularity. Then there exists a boundary Lefschetz fibration*

$$\widetilde{f}\colon (M, \widetilde{D}) \to (\widetilde{N}, \partial\widetilde{N}),$$

agreeing with f outside a neighbourhood of p and such that the elliptic–elliptic singularity is traded for a Lefschetz singularity. The map $\widetilde{f}$ is homologically essential if and only if f is.

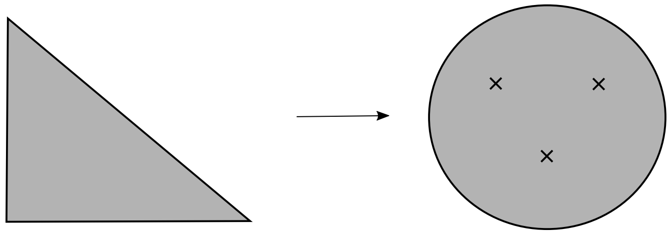


Figure 2.1: The picture on the left presents $\mathbb{C}P^2$ using the usual moment map, which is the prototypical example of a self-crossing boundary fibration. Theorem 2.3.3 tells us that we can slightly modify this fibration to obtain a boundary Lefschetz fibration, with three Lefschetz singularities.

The proof of this result, and its converse, Theorem 2.3.4, relies on the connected sum procedure and the existence of a particular boundary Lefschetz fibration on S^4.

Examples Using simple manifolds as building blocks, the connected sum procedure allows us to construct many examples of boundary Lefschetz fibrations. For instance, we can show that the stable generalized complex manifolds from Theorem 1.6.5 admit boundary Lefschetz fibrations:

Theorem 2.4.12. *The manifolds in the following two families*

- $X_{n,\ell} := \#n(S^2 \times S^2)\#\ell(S^1 \times S^3)$, *with* $n, \ell \in \mathbb{N}$;
- $Y_{n,m,\ell} := \#n\mathbb{C}P^2\#m\overline{\mathbb{C}P}^2\#\ell(S^1 \times S^3)$, *with* $n, m, \ell \in \mathbb{N}$,

admit homologically essential boundary fibrations whenever their Euler characteristic is non-negative.

Combining this result with Theorem 2.3.3 we conclude that the above manifolds also admit boundary Lefschetz fibrations with embedded degeneracy locus.

Organisation of the chapter

This chapter is organised as follows. In Section 2.1 we extend the notion of boundary (Lefshetz) fibration from [13] to allow for self-crossing of the degeneracy locus. Moreover, we prove a Gompf–Thurston result, Theorem 2.1.23. In Section 2.2 we show that taking connected sums is allowed fo boundary Lefschetz fibrations and prove Theorem 2.2.3. In Section 2.3 we prove the singularity trade results, namely Theorem 2.3.3 and Theorem 2.3.4. Finally in Section 2.4 we show that torus actions give rise to boundary fibrations and exhibit several examples, including Theorem 2.4.12.

2.1 Boundary maps and Lefschetz fibrations

We will single out a class of maps that admits enough singularities to make them interesting, while also giving us enough control on the singular behaviour so that we can use these maps to perform geometric constructions. The main point of this section is to extend the notion of boundary Lefschetz fibration defined in [13] for manifolds with smooth divisors to manifolds with self-crossing divisors. This extension allows for maps to have one extra type of singularity: elliptic–elliptic type. This change allows us to get a much better grasp on many generalized complex manifolds as those can be easily described as fibrations with elliptic–elliptic singularities.

2.1.1 Boundary maps

Our first step is to single out a very general class of maps which is compatible with the Lie algebroids introduced in Section 1.2. These are the *boundary maps* which already illustrate how singular behaviour of maps can be coupled with Lie algebroids.

We start with some basic terminology. A **pair**, (M, D), is a manifold M together with a (possibly) immersed submanifold $D \subseteq M$. A **map of pairs** $f\colon (M, D) \to (N, Z)$ is a map $f\colon M \to N$ such that $f(D) \subseteq Z$. A **strong map of pairs** furthermore satisfies $f^{-1}(Z) = D$. Finally, (N, Z) is a **log pair** if the vanishing ideal I_Z is a log divisor ideal on N.

Definition 2.1.1. Let $f\colon (M, D) \to (N, Z)$ be a strong map of pairs onto a real log pair. Then f is a **boundary map** if $I_{|D|} := f^* I_Z$ defines an elliptic divisor ideal. ♢

Example 2.1.2. The basic example to have in mind for boundary map is

$$f_1\colon (\mathbb{C}^2, D) \to (\mathbb{R}^2, Z), \qquad f_1(z_1, z_2) = (|z_1|^2, |z_2|^2),$$

where $D \subset \mathbb{C}^2$ and $Z \subset \mathbb{R}^2$ are the two coordinate axes.

There are other examples of boundary maps that we will eventually exclude by imposing further requirements, but which are also interesting to keep in mind for now:

$$f_2\colon (\mathbb{C}^2, D) \to (\mathbb{R}, \{0\}), \qquad f_2(z_1, z_2) = |z_1|^2 |z_2|^2,$$

where $D \subset \mathbb{C}^2$ is again the two coordinate axes and

$$f_3\colon (S^2, \{p_N, p_S\}) \to (S^1, \{-1\}), \qquad f_3(x, y, z) = \exp(\pi i z),$$

where p_N, p_S are the north and south poles of the unit sphere and we regard S^1 as the complex numbers of length 1. △

Notice that in the first two examples above, the image of the maps considered are manifolds with corners, and for all intents and purposes we could have considered them as maps into their image with the divisor being determined by the boundary. This is in line with the original idea behind log geometry (also known as b-geometry) developed by Mazzeo and Melrose [63]. The third map shows that sometimes the image may be a genuine manifold (without boundary). Note that f_3 factors through the height map $\widetilde{f}_3\colon (S^2, \{p_N, p_S\}) \to (I, \partial I)$, $\widetilde{f}_3(x, y, z) = z$,

$$\begin{array}{ccc} & & (I, \partial I) \\ & \overset{\widetilde{f}_3}{\nearrow} & \big\downarrow \exp(\pi i \cdot) \\ (S^2, \{p_N, p_S\}) & \xrightarrow{f_3} & (S^1, \{-1\}), \end{array}$$

which has image a manifold with boundary, and boundary as divisor. This is a specific example of a more general construction, namely that we can "cut N open along Z". Next we will describe this procedure, which justifies the name *boundary map*.

Lemma 2.1.3. *Let (N, Z) be a real log pair with N a manifold without boundary. Then there is a manifold with corners $\widetilde{N}$ and a map $p\colon (\widetilde{N}, \partial\widetilde{N}) \to (N, Z)$ such that*

- *p is a map of divisors,*
- *$p\colon \widetilde{N}\backslash\partial\widetilde{N} \to N\backslash Z$ is a bijection,*
- *every point $x \in \widetilde{N}$ has a neighbourhood U such that $p\colon U \to p(U)$ is a diffeomorphism.*

Further, if $p'\colon (N', \partial N') \to (N, Z)$ is another manifold and map satisfying the properties above then there is a unique diffeomorphism $\Psi\colon (N', \partial N') \to (\widetilde{N}, \partial\widetilde{N})$ for which $p' = p \circ \Psi$.

Proof. We start with a local construction. Denoting by $\mathbb{R}^n_\ell$ the manifold $\mathbb{R}^n$ with divisor given by the hyperplanes determined by the equation $x_1 \cdots x_\ell = 0$, we let $\widetilde{\mathbb{R}}^n_\ell$ be given by

$$\widetilde{\mathbb{R}}^n_\ell = \dot{\bigcup_{K\in\{-1,1\}^\ell}} \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon k_i x_i \geq 0, \text{ where } K = (k_1, \ldots k_\ell)\},$$

and we let $p\colon \widetilde{\mathbb{R}}^n_\ell \to \mathbb{R}^n_\ell$ be the natural inclusion: $p(x) = x$. Figure 2.2 shows this construction for $\mathbb{R}^2$ with the two coordinate axes as its real log divisor. We call each connected component of $\widetilde{\mathbb{R}}^n_\ell$ defined above a *quadrant*.

Notice that $p\colon \widetilde{\mathbb{R}}^n_\ell \to \mathbb{R}^n_\ell$ is a map of divisors, and if a smooth map $f\colon M \to \mathbb{R}^n$ has its image in a quadrant, then it admits a smooth lift to $\widetilde{\mathbb{R}}^n_\ell$. Further, if M is connected and the image of f has points which are not in the hyperplanes determined by $x_1 \cdots x_\ell = 0$, then this lift is unique.

For the global construction, we observe that charts in N provide a way to glue the local construction above to produce a manifold with corners. Indeed,

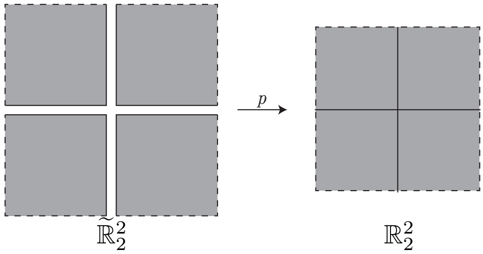


Figure 2.2: Boundarification of $\mathbb{R}^2$ with two coordinate axes.

given two charts that render the divisor in standard form, in their overlap the change of coordinates gives a diffeomorphism $\Phi\colon \mathbb{R}^n_\ell \to \mathbb{R}^n_\ell$, for some ℓ. Since the charts are adapted to the divisor, Φ also induces a diffeomorphism of quadrants, that is, it lifts to a diffeomorphism $\widetilde{\Phi}\colon (\widetilde{\mathbb{R}}^n_\ell, \partial\widetilde{\mathbb{R}}^n_\ell) \to (\widetilde{\mathbb{R}}^n_\ell, \partial\widetilde{\mathbb{R}}^n_\ell)$.

Since the changes of coordinates arising from an atlas for N give rise to a Čech cocycle of diffeomorphisms, the same holds for their lifts, so the procedure can be used to produce a manifold with corners $\widetilde{N}$. Further, the natural local maps ,"p", defined in coordinate charts above patch together to give a global map of divisors $p\colon (\widetilde{N}, \partial\widetilde{N}) \to (N, Z)$. By construction, $p\colon \widetilde{N}\backslash\partial\widetilde{N} \to N\backslash Z$ is a bijection away from the divisors and a local diffeomorphism onto its local image.

Finally, if $p'\colon (N', \partial N') \to (N, Z)$ is a map of divisors with the two properties above then we show that p' has a unique lift $\Psi\colon (N', \partial N') \to (\widetilde{N}, \partial\widetilde{N})$:

$$\begin{array}{ccc} & & (\widetilde{N}, \partial\widetilde{N}) \\ & \overset{\Psi}{\nearrow} & \downarrow p \\ (N', \partial N') & \xrightarrow{p'} & (N, Z). \end{array}$$

Indeed, in this case, $p^{-1} \circ p'\colon N'\backslash\partial N' \to \widetilde{N}\backslash\partial\widetilde{N}$ is a diffeomorphism and by the third property any point $x \in \partial N'$ has a connected neighbourhood $U \subset N'$ that maps diffeomorphically onto its image. Hence, taking U small enough, since U is connected, $p'(U)$ lies in a quadrant for a coordinate chart in N and hence p' has a unique (local) lift to $\widetilde{N}$. Patching these local lifts together we obtain the map Ψ. Since Ψ is a diffeomorphism in the interior of N' and by construction also a local diffeomorphism for points in the boundary of N' it is a global diffeomorphism. □

Definition 2.1.4. The **boundarification** of a manifold without boundary together with a real divisor, (N, Z), is a manifold with corners $(\widetilde{N}, \partial\widetilde{N})$ together with a map $p\colon (\widetilde{N}, \partial\widetilde{N}) \to (N, Z)$ satisfying the properties of Lemma 2.1.3. ♢

Example 2.1.5. If we take N to be the two-dimensional torus and Z to be an embedded circle which represents a primitive homology class, the boundarification of N is a cylinder and the map p identifies the two ends of the cylinder. If we take

Z to be a pair of embedded circles intersecting transversely and which represent a basis for the homology of the torus, then the boundarification is a rectangle and the quotient map identifies opposite sides in the usual fashion. △

Proposition 2.1.6. *Let $f\colon (M, D) \to (N, Z)$ be a boundary map onto a manifold without boundary equipped with a real log divisor. Then there exists a unique boundary map $\widetilde{f}\colon (M, D) \to (\widetilde{N}, \partial\widetilde{N})$ to its boundarification that is a lift of f, i.e. which satisfies $f = p \circ \widetilde{f}$ for $p\colon (\widetilde{N}, \partial\widetilde{N}) \to (N, Z)$.*

Proof. All we need to prove is that every point $x \in M$ has a neighbourhood U such that $f|_U\colon U \to N$ admits a unique lift $\widetilde{f}|_U\colon U \to \widetilde{N}$. Indeed, if this is the case, then any two such local lifts will agree in their overlap by uniqueness and hence the local lifts patch together to give a unique global map.

Because D has codimension two in M every point $x \in M$ has a neighbourhood U such that $U \backslash D$ is connected, it follows that $f(U \backslash D)$ lies in a connected component of $N \backslash Z$. By taking U small enough, we have that $f(U \backslash D)$ lies in a connected component of the complement of Z in a coordinate patch $V \subset N$, that is, $f(U \backslash D)$ lies in a quadrant and, by continuity, so does $f(U)$. As such there is a unique lift to a map $\widetilde{f}\colon U \to \widetilde{N}$. □

We intend to use boundary maps to construct geometric structures on their total space. Thus we can, without loss of generality, assume that the target of a boundary map is $(N, \partial N)$, a manifold with corners whose real log divisor is determined by its boundary. This also explains the terminology "boundary map".

2.1.2 Boundary Lefschetz fibrations

The notion of a boundary map f is still too general to give us enough information about the singularities of the map. To get a good grasp on f we need to ensure that its singularities are well controlled and this is what we do next. There are two ways to constrain the singularities of f: we can either impose that they display a specific behaviour with respect to the ideals (and Lie algebroids) present, or we can impose that singularities disjoint from the vanishing loci of those ideals acquire a specific normal form. We will follow both routes here.

Note that a boundary map is by definition a map of pairs, so that it satisfies $f(D) \subseteq Z$. The first restriction we impose is that the map moreover respects the stratifications present on both D and Z.

Definition 2.1.7. A **fibrating boundary map** is a boundary map $f\colon (M, D) \to (N, Z)$ such that for each $k \geq 1$ we have that:

- $f\colon (M, D[k]) \to (N, Z[k])$ is a strong map of pairs;
- each restriction $f|_{D[k]} : D[k] \to Z[k]$ is a submersion. ♢

In Example 2.1.2, f_1 and f_3 are fibrating boundary maps, while f_2 is not as it does not satisfy the first condition.

For a fibrating boundary map, f, we can use the ideals on M and N to control the singular behaviour of f in a neighbourhood of their corresponding divisors. Concretely, we have a pointwise normal form for the map.

Lemma 2.1.8. *Let $f\colon (M^n, D^{n-2}) \to (N^m, Z^{m-1})$ be a fibrating boundary map and let $x \in D[k]$. Then there exist coordinates $(x_1, \ldots, x_n)$ around x and $(z_1, \ldots, z_k, y_i)$ around $f(x)$ such that*

- *Z is the standard log divisor with pointwise multiplicity k on $\mathbb{R}^k \times \mathbb{R}^{m-k}$*
- *D is the standard elliptic divisor with pointwise multiplicity k on $\mathbb{R}^{2k} \times \mathbb{R}^{n-2k}$, and*
- *in these coordinates, the map f takes the form*

$$f(x_1, \ldots, x_n) = (x_1^2 + x_2^2, \ldots, x_{2k-1}^2 + x_{2k}^2, x_{n-m+k}, \ldots, x_n).$$

Conversely, if for every point in D the map f is given in standard coordinates for the divisors by the expression above, then it is a fibrating boundary map.

Proof. Choose a tubular neighbourhood $\mathcal{V}$ of $Z[k]$ and denote by $\mathrm{pr}_{Z[k]}\colon NZ[k] \to Z[k]$ the projection. Let $V' \subset \mathcal{V}$ be an open neighbourhood of $f(x)$ on which $Z[k]$ is the standard log divisor and write $V' \cap Z[k] = \{z_1 \cdot \ldots \cdot z_k = 0\}$. Choose a coordinate system $(y_{k+1}, \ldots, y_m)$ on $V' \cap Z[k]$, so that the set

$$\{z_1, \ldots, z_k, \mathrm{pr}^*_{Z[k]} y_{k+1}, \ldots, \mathrm{pr}^*_{Z[k]} y_m\}$$

forms a coordinate system on V' which is possible because $\mathrm{pr}_{Z[k]}$ is a submersion. Because f is a morphism of divisors, $f^*(z_1 \cdot \ldots \cdot z_k)$ generates an elliptic divisor ideal on $U := f^{-1}(V') \subseteq M$. Using that f is fibrating, after possibly shrinking U around x, let $(x_1, \ldots, x_n)$ be coordinates on U in which this is the standard elliptic divisor, and such that $f^*(z_j) = x_{2j-1}^2 + x_{2j}^2$. Because the restriction $f|_{Z[k]}$ is a submersion we see that

$$\{x_1, \ldots, x_{2k}, f^*\mathrm{pr}^*_{Z[k]} y_{k+1}, \ldots, f^*\mathrm{pr}^*_{Z[k]} y_m\}$$

forms a functionally independent set. We can complete this to a coordinate system on M and relabel these as $(x_1, \ldots, x_n)$. If we use the coordinate system $(z_1, \ldots, z_k, \mathrm{pr}^*_{Z[k]} y_{k+1}, \ldots, \mathrm{pr}^*_{Z[k]} y_m)$ on N and the above coordinates on M, then f takes the required form.

The converse follows immediately from the local expression for f. □

Remark 2.1.9. Even if M and N are oriented manifolds and we require the use of coordinate charts compatible with orientations, we can still arrange that the local expression for f is given by the expression in Lemma 2.1.8. Indeed, using complex conjugation on the domain and permutation of the coordinates on both domain and codomain we can change a coordinate chart which is not compatible with the given orientations into one that is.

In four dimensions, if $D[2]$ is nonempty, Lemma 2.1.8 implies that N is two-dimensional. Moreover, in a neighbourhood of a point $p \in D[2]$, orientations of M and N in fact dictate which one is "the first" strand of D arriving at p and which one is "the second", as this information is determined by the orientation of N. ♢

Lemma 2.1.10. *Let $f\colon (M, D) \to (N, Z = \partial N)$ be a fibrating boundary map whose fibres near D are connected. Then the fibres of $f|_{D[k]} : D[k] \to Z[k]$ are connected for all $k \geq 1$.*

Proof. The proof goes by induction over the strata. Note that $Z[k+1]$ is a hypersurface in $Z[k]$, and therefore

$$f|_{D[k]} : (D[k], D[k+1]) \to (Z[k], Z[k+1])$$

is a fibrating boundary map for all $k \geq 0$. Applying [13, Proposition 5.25] to $f|_{M\backslash D(2)}$ tells us that the fibres of $f|_{D[1]}$ are connected. Thus we can apply the same result to $f|_{D[1]}$ to conclude that the fibres of $f|_{D[2]}$ are connected. Continuing inductively we arrive at the desired result. □

The conditions imposed on the maps have, up to this point, been on behaviour near D. Next we impose the conditions away from D:

Definition 2.1.11. A **boundary fibration** is a fibrating boundary map

$$f\colon (M, D) \to (N, Z)$$

such that $f|_{M\backslash D} : M\backslash D \to N\backslash Z$ is a surjective submersion. ♢

Definition 2.1.12. A **boundary Lefschetz fibration** is a fibrating boundary map $f\colon (M^{2n}, D) \to (\Sigma^2, Z)$ between oriented manifolds such that $f|_{M\backslash D} : M\backslash D \to \Sigma\backslash Z$ is a Lefschetz fibration. That is, the map $f\colon M \to N$ is proper, $f|_{M\backslash D}$ is injective on critical points and for each critical point $p \in M\backslash D$ there exist orientation-preserving complex coordinate charts centered at p and $f(p)$ in which f takes the form

$$f\colon \mathbb{C}^n \to \mathbb{C}, \qquad f(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2. \qquad ♢$$

If M is four dimensional, the definition above allows for three different types of singularities. It is worth giving them names:

Definition 2.1.13. Let $f\colon M^4 \to \Sigma^2$ be a smooth map.

- An **elliptic singularity** of f is a point p for which f has the local expression

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2, x_4), \qquad x_i \in \mathbb{R};$$

- An **elliptic–elliptic singularity** of f is a point p for which f has the local expression
$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2, x_3^2 + x_4^2), \qquad x_i \in \mathbb{R};$$
- A **Lefschetz singularity** of f is a point p for which f has the local expression
$$f(z_1, z_2) = z_1^2 + z_2^2, \qquad z_i \in \mathbb{C}. \qquad \diamondsuit$$

The level sets associated to these singularities are, respectively, an **elliptic, elliptic–elliptic and Lefschetz fibre**.

The first two singularities above happen at the different strata of the divisor whereas the Lefschetz singularities do not interact with the divisor. In dimension four the geometry of these fibrations can be understood.

Proposition 2.1.14. *Let $f\colon (M^4, D^2) \to (\Sigma^2, Z)$ be a boundary Lefschetz fibration with connected fibres, and let D' be a connected component of D. Then*

- *The generic fibres of f near D are tori;*
- *When $D'[2]$ has $k \geq 1$ points, then D' is a union of k pairwise transversely intersecting spheres.*

In particular, if $D'[2] \neq \emptyset$, then D' is co-orientable.

Proof. The first point follows immediately from [13, Corollary 5.18].

For the second, assume that $D'[2]$ has at least one point. Then $f|_{D[1]} : D[1] \to Z[1]$ is a surjective submersion by assumption, which by Lemma 2.1.10 has connected fibres. The corresponding locus $Z'[1]$ is a disjoint union of k open intervals, and as the fibres of $f|_{D[1]}$ are connected they must be circles. Therefore $D'[1]$ has to be a disjoint union of cylinders. The immersed submanifold D' is obtained from $D'[1]$ by replacing the boundary circles by points and pairwise glueing these points, which implies it is as described above.

Finally, because each component of D' is an immersed sphere and thus automatically co-orientable, each component of $D'[1]$ is also co-orientable. □

To construct stable generalized complex structures using Lemma 1.4.10, the condition of co-orientability of D is satisfied as long as $D'[2]$ is nonempty for every component D' of D. For smooth components D' of D however, i.e. when $D'[2] = \emptyset$, co-orientability is not guaranteed.

2.1.3 Boundary maps and Lie algebroids

Given that the ideals I_Z and $I_{|D|}$ of a boundary map determine Lie algebroids, one should expect that boundary maps (and their further specializations) are compatible with them. This is indeed the case.

Lemma 2.1.15. *Let $f\colon (M,D) \to (N,Z)$ be a boundary map. Then there is a Lie algebroid morphism $(\varphi, f)\colon \mathcal{A}_{|D|} \to \mathcal{A}_Z$ such that $\varphi \equiv df$ on sections.*

Proof. To prove that df induces a Lie algebroid morphism φ, by [13, Proposition 3.14] it suffices to show that f^* extends to an algebra morphism $\varphi^*\colon \Omega^\bullet(\mathcal{A}_Z) \to \Omega^\bullet(\mathcal{A}_{|D|})$. This can be done locally, so given $p \in D$ and $f(p) \in Z$ consider coordinates adapted to the divisors as in Example 1.1.22:

$$(X_1, Y_1, \dots, X_s, Y_s, X_{2s+1}, \dots, X_n) \text{ around } p,$$
$$(x_1, \dots, x_j, y_{j+1}, \dots, y_m) \text{ around } f(p).$$

In these coordinates we have

$$\Omega^\bullet(\mathcal{A}_{|D|}) = \langle d\log r_1, d\theta_1, \dots, d\log r_s, d\theta_s, dX_{2s+1}, \dots, dX_n\rangle\,,$$
$$\Omega^\bullet(\mathcal{A}_Z) = \langle d\log x_1, \dots, d\log x_j, dy_{j+1}, \dots, dy_m\rangle\,.$$

We must verify that $f^*(d\log x_i)$ defines an elliptic form. Because f is a morphism of divisors and the ideals are locally principal, it sends generators to generators thus there must exist a nowhere-vanishing function g such that $f^*(x_1 \cdot \ldots \cdot x_j) = g r_1^2 \cdot \ldots \cdot r_s^2$. Consequently, by functional indivisibility of the r_i^2 we conclude that $f^*(x_i) = h r_{i_1}^2 \cdot \ldots \cdot r_{i_\ell}^2$ for some nowhere vanishing function h and (possibly empty) subset $\{i_1, \dots, i_\ell\} \subseteq \{1, \dots, s\}$. We find that

$$f^*(d\log x_i) = d\log f^*(x_i) = d\log h + 2d\log r_{i_1} + \ldots + 2d\log r_{i_l},$$

which is an elliptic form as desired, so that φ is a Lie algebroid morphism. □

The conditions imposed on boundary maps have a direct counterpart in Lie algebroid language. Given a Lie algebroid $\rho_{\mathcal{A}}\colon \mathcal{A} \to M$, let $M_{\mathcal{A}}$ be the open subset where the anchor map is an isomorphism.

Definition 2.1.16 ([13]). A Lie algebroid morphism $(\varphi, f)\colon (\mathcal{A}, M) \to (\mathcal{A}', N)$ is said to be a:

- **Lie algebroid fibration** if the induced morphism $\varphi^!\colon \mathcal{A} \to f^*\mathcal{A}'$ is surjective;
- **Lie algebroid Lefschetz fibration** if $M_{\mathcal{A}}$ is dense, $f^{-1}(N_{\mathcal{A}'}) = M_{\mathcal{A}}$ and there exists a discrete set $\Delta \subset M_{\mathcal{A}}$ such that
  - $f|_{M_{\mathcal{A}}}\colon M_{\mathcal{A}} \to N_{\mathcal{A}'}$ is a Lefschetz fibration with $\mathrm{Crit}(f|_{M_{\mathcal{A}}}) = \Delta$;
  - $(\varphi, f)\colon (\mathcal{A}, M\backslash f^{-1}(f(\Delta))) \to (\mathcal{A}', N\backslash f(\Delta))$ is a Lie algebroid fibration.

Note that the Lefschetz condition forces that $\mathrm{rank}(\mathcal{A}) = 2n$ and $\mathrm{rank}(\mathcal{A}') = 2$. ♢

The following lemmas follow immediately from the definition, combined with Lemma 2.1.15.

Lemma 2.1.17. *Let $f\colon (M,D)\to(N,Z)$ be a boundary fibration. Then there is a Lie algebroid fibration $(\varphi, f)\colon (\mathcal{A}_{|D|}, M)\to(\mathcal{A}_Z, N)$ such that $\varphi\equiv df$ on sections.*

Lemma 2.1.18. *Let $f\colon (M^4,D^2)\to(N^2,Z^1)$ be a boundary Lefschetz fibration. Then there is a Lie algebroid Lefschetz fibration $(\varphi, f)\colon (\mathcal{A}_{|D|}, M^4)\to(\mathcal{A}_Z, N^2)$ such that $\varphi\equiv df$ on sections.*

We summarise these statements and the relationship between the different concepts in the table below:

Boundary (Lefschetz) fibration	$\Rightarrow$	Lie algebroid (Lefschetz) fibration
$\Downarrow$		$\Downarrow$
Fibrating boundary map	$\Rightarrow$	Lie algebroid map *submersive* over the singular locus
$\Downarrow$		$\Downarrow$
Boundary map	$\Rightarrow$	Lie algebroid map

2.1.4 Construction of self-crossing stable generalized complex structures

With the desired notion of boundary Lefschetz fibration in hand, we are set to prove our first result relating them to stable generalized complex structures.

From now on we will adopt the following convention: given a boundary Lefschetz fibration $f\colon (M,D)\to(N,Z)$, we will orient the fibres of $f\colon M\backslash D\to N\backslash Z$ by declaring that the orientation of the fibre together with the orientation of the base yield the orientation of M, so that each fibre determines a homology class on $M\backslash D$. With this convention, integration over the fibre is a well-defined operation which induces the natural pairing between homology and cohomology.

Definition 2.1.19. A boundary Lefschetz fibration, $f\colon (M^4,D)\to(N^2,Z)$, is **homologically essential** if the homology class $[F]$ of a fibre of $f\colon M\backslash D\to N\backslash Z$ is non-trivial in $H_2(M\backslash D;\mathbb{R})$ or, equivalently, if there is a class $c\in H^2(M\backslash D;\mathbb{R})$ such that $\langle c,[F]\rangle\neq 0$. ♢

Definition 2.1.20. A boundary Lefschetz fibration, $f\colon (M^4,D)\to(N^2,Z)$, and an elliptic symplectic form $\omega\in\Omega^2(\mathcal{A}_{|D|})$ are **compatible** if $\ker\varphi\subseteq\mathcal{A}_{|D|}$ consists of symplectic vector spaces, where $\varphi\colon\mathcal{A}_{|D|}\to\mathcal{A}_Z$ is the induced map of Lie algebroids. ♢

In what follows we will have two ongoing simplifying assumptions:

1. We will assume that the target manifold is $(N,\partial N)$. This is not a restriction since by Proposition 2.1.6 we can lift f to a boundary Lefschetz fibration over the boundarification of (N,Z);

2. We will assume that the level sets of f are connected. This also is not restriction since by [13, Proposition 5.24] we may assume that the generic fibres of f are connected and Lemma 2.1.10 then implies that the level sets over $Z[1]$ and $Z[2]$ are connected as well.

Before we continue it is worth to stop and take stock of where we stand and place our quest into context. The case when the elliptic divisor is smooth was already treated in [13]. Even though there the authors only dealt with the compact case, the following is an immediate generalisation for a proper map:

Theorem 2.1.21 ([13], Theorem 7.1). *Let $(M^4, I_{|D|})$ be an oriented manifold with a smooth elliptic divisor and let $f\colon (M, D) \to (N^2, Z, \omega_N)$ be a homologically essential, proper, Lefschetz fibration with connected fibres over a possibly open log-symplectic surface. Denote by $\varphi\colon \mathcal{A}_{|D|} \to \mathcal{A}_Z$ the induced map of Lie algebroids. Let $c \in H^2(M\backslash D) = H^2_{0,0}(M, \mathcal{A}_{|D|})$ be a cohomology class such that $\langle c, [F]\rangle > 0$, where F is a regular fibre of f. Then there exists a closed two-form $\eta \in \Omega^2_{0,0}(M, \mathcal{A}_{|D|})$ with $[\eta] = c$ and a positive function $\rho_0 \in C^\infty(N)$ such that:*

- *η is fibrewise nondegenerate, that is, for every $p \in M$, η is nondegenerate in $\ker(\varphi_p)$,*
- *The form $\omega = \eta + f^*(\rho\omega_\Sigma)$ is symplectic with zero elliptic residue on $\mathcal{A}_{|D|}$ for every $\rho \in C^\infty(N)$ as long as $\rho \geq \rho_0$.*

Apart from the theorem above, [13] also includes a general Gompf–Thurston result for Lie algebroid Lefschetz fibrations: under similar conditions on a Lie algebroid Lefschetz fibration one can construct a Lie algebroid symplectic form on the domain by adding a form which is symplectic on the fibres to a large multiple of the pull back of a symplectic form on the base.

Neither result can be directly applied to our case: Theorem 2.1.21 does not work because our divisor is not smooth, while the failure of the general result on Lie algebroid fibrations to yield stable generalized complex structures can already be seen in the simplest example.

Example 2.1.22. Consider the boundary fibration:

$$f_1\colon (\mathbb{C}^2, D) \to (\mathbb{R}^2, Z), \qquad f_1(z_1, z_2) = (|z_1|^2, |z_2|^2),$$

where D and Z are the coodinate axes on $\mathbb{C}^2$ and $\mathbb{R}^2$ respectively, as in Example 2.1.2.

We can endow $\mathbb{R}^2$ with the log-symplectic structure $d\log x_1 \wedge d\log x_2$, and consider on $\mathbb{C}^2$ the closed elliptic form

$$\eta = -d\theta_1 \wedge d\theta_2 + d\log r_1 \wedge d\theta_2 + d\log r_2 \wedge d\theta_1,$$

which is non-degenerate on the fibres of f_1. The Gompf–Thurston theorem then provides us with a 1-parameter family of forms

$$\omega_t = \eta + tf^*(d\log x_1 \wedge d\log x_2),$$

which is symplectic for $t > 1$. This poses a problem: although this defines a legitimate elliptic symplectic form, there is no value of t for which it corresponds to a stable generalized complex structure, since $|\mathrm{Res}_{r_1 r_2}\,\omega_t| \neq |\mathrm{Res}_{\theta_1\theta_2}\,\omega_t|$ for $t > 1$. We conclude that the process of scaling up the symplectic structure on the base to achieve nondegeneracy is incompatible with the residue conditions. △

What we do next is to adapt Theorem 2.1.21 for the self-crossing case.

Theorem 2.1.23. *Let $f\colon (M^4, D^2) \to (N^2, Z = \partial N)$ be a homologically essential boundary Lefschetz fibration with connected fibres between compact connected oriented manifolds. Denote by $\varphi\colon \mathcal{A}_{|D|} \to \mathcal{A}_Z$ the induced map of Lie algebroids. Then $(M, I_{|D|})$ admits an elliptic symplectic structure with zero elliptic residue and imaginary parameter which is compatible with f.*

If D is co-orientable and the index of each connected component of D is 1 this elliptic symplectic structure induces a stable generalized complex structure.

Proof. Fix a log-symplectic structure $\omega_N \in \Omega^2(N, \mathcal{A}_Z)$. First we consider the map $f\colon M\backslash D[2] \to N\backslash Z[2]$. This is a homologically essential, proper, boundary Lefschetz fibration with smooth elliptic divisor, hence, by Theorem 2.1.21, there is a form $\eta \in \Omega^2_{0,0}(M\backslash D[2]; \log|D\backslash D[2]|)$ and a function $\rho_0 \in \Omega^0(N\backslash Z[2])$ (recall, Lemma 1.2.17) such that $\omega = \eta + f^*(\rho\omega_N)$ is a zero elliptic residue symplectic form for any function $\rho \in \Omega^0(N\backslash Z[2])$ with $\rho \geq \rho_0$.

Now we show how to change this construction so that the form it yields extends over $D[2]$, is elliptic symplectic with zero elliptic residue, and has imaginary parameter.

For each point $p \in D[2]$, fix open neighbouhoods $U_1 \Subset U_2 \Subset U_3$ and oriented coordinates charts defined on U_3 and $f(U_3)$ in which f has the form

$$f(z_1, z_2) = (|z_1|^2, |z_2|^2).$$

As usual, we express the complex coordinates in U_3 in polar form, $z_i = r_i e^{i\theta_i}$, and denote by (x_1, x_2) the coordinates on the base, so $f^*x_i = r_i^2$.

The strategy will be to change the symplectic form ω described above in a very precise way:

- in the complement of U_3, ω remains unchanged except for a further constant scaling of the symplectic form ω_N,
- in $U_3\backslash U_2$ we change η into a multiple of $d\theta_2 \wedge d\theta_1$ and we preserve nondegeneracy by rescaling the symplectic form ω_N by a constant,
- in $U_2\backslash U_1$ we interpolate the possibly large $f^*\omega_N$ to $f^*(d\log x_1 \wedge d\log x_2)$ and observe that this interpolation does not spoil the symplectic condition,
- in U_1 we extend the symplectic form as $d\theta_2 \wedge d\theta_1 + d\log r_1 \wedge d\log r_2$, which clearly has the desired properties at p.

Now we carry out his plan explicitly. Fix $\rho \geq \rho_0$. On U_3 we have by Lemma 1.2.17 that

$$[\eta] \in H^2_{0,0}(U_3 \backslash D[2], \mathcal{A}_{|D\backslash D[2]|}) \cong H^2(U_3\backslash D) = \mathbb{R},$$

and the generator of this cohomology pairs nonzero with the torus given in coordinates by $F = f^{-1}(r_1, r_2)$, where r_1 and r_2 are any two small positive numbers. Let $\lambda = \int_F \eta$ where integration is with respect to the fibre orientation of F, hence $\lambda > 0$. On U_3 consider the elliptic form $\widetilde{\eta} = \frac{\lambda}{4\pi^2} d\theta_2 \wedge d\theta_1$. Then $\widetilde{\eta}$ is closed in U_3 and also integrates to λ over F. Therefore $[\eta] = [\widetilde{\eta}] \in H^2(U_3\backslash\{p\}, \mathcal{A}_{D\backslash\{p\}})$ and there is a one-form $\alpha \in \Omega^1(U_3\backslash\{p\}, \mathcal{A}_{|D\backslash\{p\}|})$ such that $\widetilde{\eta} = \eta + d\alpha$.

Let $k \geq 1$, let ψ_1 and ψ_2 be positive functions on $f(U_3)$ such that ψ_1 is equal to 1 in neighbourhood of $f(U_1)$ and has support in $f(U_2)$ and ψ_2 is equal to 1 in neighbourhood of $f(U_2)$ and has support in $f(U_3)$. Then consider the form

$$\widetilde{\omega} := \begin{cases} \frac{\lambda}{4\pi^2}(d\theta_2 \wedge d\theta_1 + d\log r_1 \wedge d\log r_2) & \text{in } U_1, \\ \widetilde{\eta} + f^*((1-\psi_1)k\rho\omega_N + \psi_1 \frac{\lambda}{4\pi^2} d\log x_1 \wedge d\log x_2) & \text{in } U_2\backslash U_1, \\ \eta + d((f^*\psi_2)\alpha) + f^*(k\rho\omega_N) & \text{in } U_3\backslash U_2, \\ \eta + f^*(k\rho\omega_N) & \text{in } M\backslash U_3. \end{cases}$$

Because of our choice of bump functions, this form is smooth. Also, it is clearly closed. Since $k \geq 1$, we have $k\rho \geq \rho \geq \rho_0$ and hence $\widetilde{\omega}$ is symplectic in $M\backslash U_3$ for all possible values of k.

On $\overline{U_3\backslash U_2}$ we observe that the form $\eta + d((f^*\psi)\alpha)$ is fibrewise symplectic. Indeed, its restriction to each fibre it is given by

$$\eta + (f^*\psi_2)d\alpha = (f^*\psi_2)(\eta + d\alpha) + (1-(f^*\psi_2))\eta = (f^*\psi_2)\widetilde{\eta} + (1-(f^*\psi_2))\eta,$$

hence it is a convex combination of η and $\widetilde{\eta}$ and these are both symplectic and determine the same orientation on each fibre. Since $\eta + d((f^*\psi)\alpha)$ is fibrewise symplectic and $\rho\omega_N$ is symplectic on N, the combination $\eta + d((f^*\psi)\alpha) + f^*(k\rho\omega_N)$ is symplectic on the compact set $\overline{U_3\backslash U_2}$ as long as k is large enough.

On $U_2\backslash U_1$, the form $\widetilde{\eta}$ is given by $\frac{\lambda}{4\pi^2} d\theta_2 \wedge d\theta_1$, while the summand $f^*((1-\psi_1)k\rho\omega_N + \psi_1 d\log x_1 \wedge d\log x_2)$ is a convex combination of two log-symplectic structures on N which determine the same orientation, that is

$$f^*((1-\psi_1)k\rho\omega_N + \psi_1 \frac{\lambda}{4\pi^2} d\log x_1 \wedge d\log x_2) = f^*(\kappa d\log x_1 \wedge d\log x_2),$$

for some positive function κ and hence on $U_2\backslash U_1$

$$\widetilde{\omega} = \frac{\lambda}{4\pi^2} d\theta_2 \wedge d\theta_1 + (f^*\kappa) d\log r_1 \wedge d\log r_2,$$

which is clearly (zero residue) elliptic symplectic.

Finally, on U_1 we have $\omega = \mathrm{Im}(i\frac{\lambda}{4\pi^2} d\log z_1 \wedge d\log z_2)$, showing that it has the desired properties. $\square$

2.2 Connected sums of boundary Lefschetz fibrations

In this section we describe a connected sum procedure for boundary Lefschetz fibrations along zero-dimensional strata of their elliptic divisors. This procedure will allow us to construct elaborate examples out of basic ones. For simplicity, we immediately restrict ourselves to dimension four, but we note that since the connected sum takes place at points of the divisor, this procedure can also be carried out for boundary fibrations in higher dimensions.

Before we start taking connected sums of boundary Lefschetz fibrations, first recall from Lemma 1.5.1 that if $(M^n, I_{|D_M|})$ and $(N^n, I_{|D_N|})$ are manifolds endowed with elliptic divisors, then for $p \in D_M[k]$ and $q \in D_N[k]$ we have that $M\#_{p,q}N$ inherits an elliptic divisors, for which the inclusions of $M\backslash\{p\}$ and $N\backslash\{q\}$ are morphisms of divisors.

Similarly, by Lemma 1.5.2 we have that if $p_1, p_2 \in D[k]$ are distinct we can perform a self-connected sum resulting in an elliptic divisor on $M^n\#(S^1 \times S^{n-1})$.

We also refer back to the discussions in Remarks 1.5.3 and 1.5.4 that a priori there are multiple options for the connected sum procedure, which all result in diffeomorphic total spaces but for which the zero loci of the divisors have different topological types. Next we show that the connected sum operation is also compatible with boundary (Lefschetz) fibrations. To describe how the connected sum procedure interacts with the base of the fibration we first consider what happens in the local model:

Lemma 2.2.1. *Let $\Delta_r \subseteq \mathbb{R}^2$ be the triangle bounded by the axes and the line $x + y = r$, and let (x, y) be oriented coordinates on Δ_r and (z_1, z_2) be complex coordinates on $\mathbb{D}^4_r$, the disc of radius r. Consider the following maps:*

- $p\colon (\mathbb{D}^4_2\backslash\mathbb{D}^4_{1/2}) \to (\Delta_2\backslash\Delta_{1/2})$, *given by* $(z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$;
- $\Phi\colon (\mathbb{D}^4_2\backslash\mathbb{D}^4_{1/2}) \to (\mathbb{D}^4_2\backslash\mathbb{D}^4_{1/2})$, *given by* $(z_1, z_2) \mapsto \frac{1}{|z_1|^2+|z_2|^2}(z_2, \overline{z}_1)$;
- $\Psi\colon (\Delta_2\backslash\Delta_{1/2}) \to (\Delta_2\backslash\Delta_{1/2})$ *given by* $(x, y) \mapsto \frac{(y,x)}{(x+y)^2}$.

Then the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{D}^4_2\backslash\mathbb{D}^4_{1/2}) & \xrightarrow{\Phi} & (\mathbb{D}^4_2\backslash\mathbb{D}^4_{1/2}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ (\Delta_2\backslash\Delta_{1/2}) & \xrightarrow{\Psi} & (\Delta_2\backslash\Delta_{1/2}). \end{array}$$

The proof of this lemma is a simple verification. Just as we used the map Φ to perform a connected sum compatible with elliptic divisors, we want to use the map Ψ to define a sort of connected sum operation of the base:

Definition 2.2.2. Let Σ_1, Σ_2 be oriented surfaces with corners, and let q_1, q_2 be corners of Σ_1, Σ_2 respectively. The **oriented corner connected sum** of Σ_1 and

Σ_2 is defined by identifying a trapezoid neighbourhood of q_1 to a trapezoid neighbourhood of q_2 via Ψ. The oriented corner connected sum is an oriented surface with corners denoted by $\Sigma_1 \#_{q_1,q_2} \Sigma_2$ (see Figure 2.3). ♢

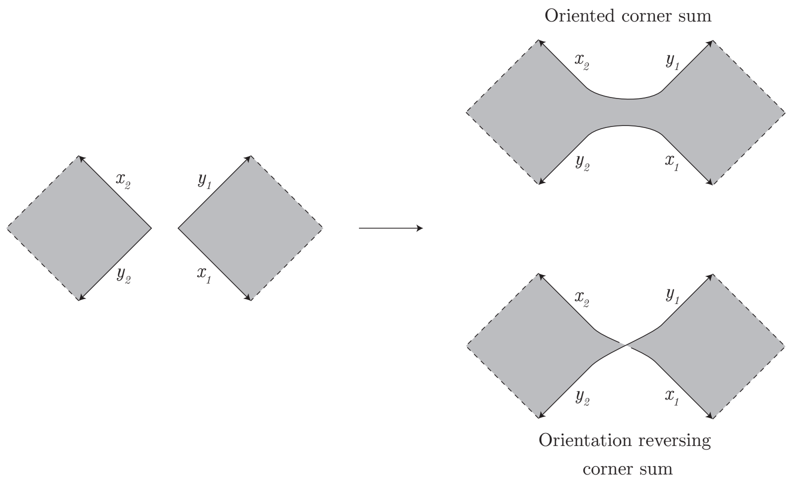


Figure 2.3: Local oriented and orientation-reversing corner connected sums.

The oriented corner connected sum is naturally oriented and does not depend on the neighbourhoods chosen. Together with the local normal form for fibrating boundary maps, we can now prove the following:

Theorem 2.2.3. *Let $f_i\colon (M_i^4, D_i) \to (N_i^2, \partial N_i)$ be boundary (Lefschetz) fibrations with connected fibres between oriented manifolds for $i = 1, 2$, let $p_i \in D_i[2]$ and $q_i = f(p_i)$. Then there exists a boundary (Lefschetz) fibration on one of the two possible connected sums $M_1 \#_{p_1,p_2} M_2$ whose base is the oriented corner sum $N_1 \#_{q_1,q_2} N_2$:*

$$(f_1 \# f_2)\colon (M_1 \#_{p_1,p_2} M_2, \widetilde{D}) \to (N_1 \#_{f(p_1),f(p_2)} N_2, \partial N_1 \#_{q_1,q_2} \partial N_2),$$

which is compatible with the (orientation-preserving) inclusions $M_i \backslash \{p_i\} \hookrightarrow M_1 \# M_2$.

Furthermore let D_1', D_2' denote the connected components of the zero locus of the divisor D containing p_1, p_2 respectively. Then the parities satisfy:

$$\varepsilon_{\widetilde{D}} = -\varepsilon_{D_1'} \varepsilon_{D_2'}.$$

Finally $f_1 \# f_2$ is homologically essential if and only if f_1 and f_2 are.

Proof. By Lemma 2.1.8 there exists neighbourhoods U_1, U_2 of $f(p_1), f(p_2)$ respectively which provide coordinates as in the setting of Lemma 2.2.1. We perform the connected sum procedure using the maps described there. Because these maps are compatible with the fibrations on M_1 and M_2 we conclude that $M_1 \#_{p_1,p_2} M_2$ admits a boundary fibration. The computation of the parity is already given in Theorem 1.5.9. □

Recall from Remark 1.5.3 that there are a priori two possible topological types for the elliptic divisor, depending on the ordering of the local coordinates. However, when we are presented with fibrations between oriented manifolds $f\colon (M_i, D_i) \to (N_i, \partial N_i)$ the orientation on the base determines an order for the strands of D for every point $p_i \in D_i[2]$ (c.f. Remark 2.1.9). The gluing of fibrations which is compatible with orientations on M_i and N_i is the one that flips the first and second strands arriving the points where the sum is performed. In particular, from the possible divisors discussed in Remark 1.5.4, (2), only the one with two connected components occurs.

Remark 2.2.4 (Non-orientable case)**.** If we were to allow the map Ψ used in the corner sum to be orientation reversing, we would still be able to define a corner connected sum and obtain a boundary fibration. When taking the connected sum of two manifolds this does not cause a qualitative change in the outcome. However, if we use the orientation-reversing corner sum on the base for a self-connected sum, we see that the resulting base manifold is not orientable as a Möbius band appears. $\diamondsuit$

Now that we understand precisely what happens to the connected components of the divisor on the self-connected sum, we can state the following:

Corollary 2.2.5. *Let $f\colon (M^4, D^2) \to (N^2, \partial N)$ be a boundary (Lefschetz) fibration with connected fibres between oriented manifolds, and let $p_1, p_2 \in D[2]$ be distinct. Then $M\#(S^1 \times S^3)$ admits a boundary (Lefschetz) fibration $\widetilde{f}$ which is compatible with the inclusion $M\backslash\{p_1, p_2\} \hookrightarrow M\#_{p_1,p_2}(S^1 \times S^3)$, and for which $\widetilde{D}[2] = D[2]\backslash\{p_1, p_2\}$.*

Moreover let D'_{p_1}, D'_{p_2} denote the connected components of D containing p_1, p_2 respectively.

- *If $p_i \in D'_{p_i}[2]$ and $D'_{p_1} \neq D'_{p_2}$, then the corresponding connected component $\widetilde{D}'$ of $\widetilde{D}$ satisfies:*

$$\varepsilon_{\widetilde{D}'} = -\varepsilon_{D'_1}\varepsilon_{D'_2};$$

- *If $p_i \in D'_{p_i}[2]$ and $D'_{p_1} = D'_{p_2}$, then the corresponding connected components $\widetilde{D}'_1, \widetilde{D}'_2$ of $\widetilde{D}$ satisfy:*

$$\varepsilon_{\widetilde{D}'_1}\varepsilon_{\widetilde{D}'_2} = -\varepsilon_{D'_{p_1}}\varepsilon_{D'_{p_2}}.$$

Finally $\widetilde{f}$ is homologically essential if and only if f is.

2.3 Singularity trades

The goal of this section is to prove two theorems which allow one to trade Lefschetz for elliptic–elliptic singularities and vice-versa. To formulate these results, we need to recall the notion of vanishing cycle for both Lefschetz and elliptic singularities.

Given a boundary Lefschetz fibration $f\colon (M^4, D^2) \to (N^2, \partial N)$ and an elliptic or a Lefschetz singularity $p_1 \in M$, let $q_1 = f(p_1)$ be the corresponding singular value. We fix $q \in N$, a reference regular point of f and $\gamma\colon [0,1] \to N$, a simple path connecting q to q_1 which goes through no critical values of f except for q_1 at time 1. We can consider $F_q = f^{-1}(q)$, $F_\gamma = f^{-1}(\gamma([0,1]))$ and the natural inclusion $\iota\colon F_q \to F_\gamma$. Then F_q is a two-torus and $H_1(F_\gamma)$ is one-dimensional:

- In the case of a Lefschetz singularity the inclusion $H_1(F_q) \to H_1(F_\gamma)$ has kernel given by the Lefschetz vanishing cycle, which corresponds to the boundary of a Lefschetz thimble emanating from the singularity.
- In the case of an elliptic singularity F_γ is the product of circle and a solid torus with F_q as boundary, hence $\iota_*\colon H_1(F_q) \to H_1(F_\gamma)$ also has one-dimensional kernel given by the cycle in F_q which becomes a boundary in F_γ.

In both cases the kernel of ι_* is generated by one primitive element in $H_1(F_q;\mathbb{Z})$ which depends only on the homotopy class of γ in $N\backslash\mathrm{Crit}(f)$.

Definition 2.3.1. In the situation above, the **vanishing cycle** associated to the singular value q_1 and the homotopy class of the path γ is either of the primitive elements in $H_1(F_q;\mathbb{Z})$ which generates the kernel of $H_1(F_q;\mathbb{Z}) \to H_1(F_\gamma;\mathbb{Z})$. ♢

Definition 2.3.2. Let $f\colon (M^4, D^2) \to (N^2, \partial N)$ be a boundary Lefschetz fibration, let F_{q_0} and F_{q_1} be Lefschetz or elliptic fibres. We say that the vanishing cycles at F_{q_0} and F_{q_1} are a **dual pair** if there is a simple path $\gamma\colon [0,1] \to N$ such that:

- $\gamma(0) = q_0$ and $\gamma(1) = q_1$,
- $\gamma((0,1))$ only contains regular values of f,
- the vanishing cycles on both ends of γ together generate the integral homology of the regular torus fibre, say $F_{\gamma(1/2)}$. ♢

With these notions at hand, we can give the precise statements of our singularity trade theorems.

Theorem 2.3.3 (Elliptic–elliptic trade)**.** *Let $f\colon (M^4, D) \to (N^2, \partial N)$ be a boundary Lefschetz fibration with connected fibres, and let $p \in D[2]$. Then M admits a boundary Lefschetz fibration $\widetilde{f}\colon (M^4, \widetilde{D}) \to (\widetilde{N}^2, \partial\widetilde{N})$ such that:*

- *$\widetilde{N}$ is obtained from N by smoothing out the corner $f(p)$,*
- *$\widetilde{f}$ and $\widetilde{D}$ agree with f and D outside a small ball centered at p,*
- *$\widetilde{D}[2] = D[2]\backslash\{p\}$, i.e. $\widetilde{f}$ has one elliptic–elliptic singularity less than f,*
- *$\widetilde{D}$ and D have the same parity,*

- *$\widetilde{f}$ has one Lefschetz singularity more than f,*
- *$\widetilde{f}$ has an elliptic singularity whose vanishing cycle forms a dual pair with the new Lefschetz vanishing cycle,*
- *$\widetilde{f}$ is homologically essential if and only if f is.*

By induction, any manifold which admits a boundary Lefschetz fibration admits one with a smooth embedded divisor.

The converse trade is given by the next theorem.

Theorem 2.3.4 (Lefschetz trade). *Let $\widetilde{f}\colon (\widetilde{M}^4, \widetilde{D}) \to (\widetilde{N}^2, \partial\widetilde{N})$ be a boundary Lefschetz fibration with connected fibres, and assume that the vanishing cycles at a Lefschetz fibre, F_{q_0}, and at an elliptic fibre, F_{q_1} form a dual pair. Then there is a boundary Lefschetz fibration, $f\colon (M^4, D) \to (N^2, \partial N)$, such that:*

- *N is obtained from $\widetilde{N}$ by adding a corner at q_1,*
- *f and D agree with $\widetilde{f}$ and $\widetilde{D}$ outside $\widetilde{f}^{-1}(V_2)$, where V_2 is a neighbourhood of the path that expresses a vanishing cycles as a dual pair,*
- *$D[2] = \widetilde{D}[2] \cup \{p\}$ and hence f has one elliptic–elliptic singularity more than $\widetilde{f}$,*
- *D and $\widetilde{D}$ have the same parity,*
- *f has one Lefschetz singularity less than $\widetilde{f}$,*
- *f is homologically essential if and only if $\widetilde{f}$ is.*

The proofs of these theorems rely on the existence of specific boundary Lefschetz fibrations on S^4 and on the open disc $\mathbb{D}^4$.

Lemma 2.3.5. *There exists a homologically essential boundary Lefschetz fibration with connected fibres, $f_{S^4}\colon (S^4, D^2) \to (N, \partial N)$, with the following properties (see Figure 2.4):*

- *$D[2]$ has only one point, which has index -1,*
- *N is the disk with one corner,*
- *f_{S^4} has only one Lefschetz singularity,*
- *the vanishing cycles of the Lefschetz fibre and any elliptic fibre form a dual pair.*

The proof of this lemma is somewhat long, so we will postpone it to this end of this section.

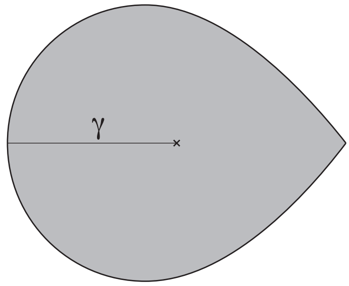


Figure 2.4: The base of the boundary Lefschetz fibration on S^4 together with a path expressing the Lefschetz and elliptic singularities as a dual pair.

Lemma 2.3.6. *Let $(\mathbb{D}^4, D)$ be the open disc in $\mathbb{C}^2$ with divisor I_{z_1} and let $\mathbb{D}^2_+ \subset \mathbb{R}^2$ be the open half disc with boundary in the real axis*

$$\mathbb{D}^2_+ = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1 \text{ and } x \geq 0\}.$$

Then there is a proper boundary Lefschetz fibration with connected fibres, $f_{\mathbb{D}^4} \colon (\mathbb{D}^4, D) \to (\mathbb{D}^2_+, \partial\mathbb{D}^2_+)$, such that:

- *$f_{\mathbb{D}^4}$ has a single Lefschetz fibre,*
- *the vanishing cycles of the Lefschetz fibre and the elliptic fibre form a dual pair.*

Further, if $f \colon (M, D) \to (\mathbb{D}^2_+, \partial\mathbb{D}^2_+)$ is a proper boundary Lefschetz fibration with connected fibres with the two properties above, then f is equivalent to $f_{\mathbb{D}^4}$, that is, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{D}^4 & \longrightarrow & M \\ \downarrow f_{\mathbb{D}^4} & & \downarrow f \\ (\mathbb{D}^2_+, \partial\mathbb{D}^2_+) & \longrightarrow & (\mathbb{D}^2_+, \partial\mathbb{D}^2_+) \end{array}$$

where the horizontal maps are diffeomorphisms.

Proof. The existence of the fibration $f_{\mathbb{D}^4}$ follows from Lemma 2.3.5. Indeed, we split the base of f_{S^4} in two parts, V_1, a neighbourhood of the vertex and V_2, the rest of the base plus a small overlap with V_1, as indicated in Figure 2.5. Then, due to Lemma 2.1.8, on $f^{-1}(V_1)$, in appropriate coordinates, we have

$$V_1 = \{(x,y) \in \mathbb{R}^2 \colon x + y < 1, x \geq 0, y \geq 0\}$$

and the fibration is given by

$$f_{S^4}(z_1, z_2) = (|z_1|^2, |z_2|^2).$$

Hence, $f^{-1}_{S^4}(V_1)$ is a disc and its complement $f^{-1}_{S_4}(V_2)$ is also a disc. But

$$f_{S_4}|_{V_2} \colon V_2 \to f_{S^4}(V_2)$$

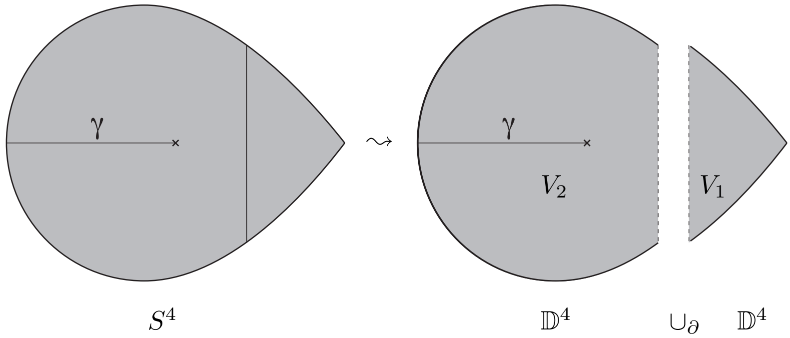


Figure 2.5: The base of the boundary Lefschetz fibration on S^4 split in two halfs, each half being a fibration of $\mathbb{D}^4$.

has all the properties required in the lemma after we choose a diffeomorphism between V_2 and $\mathbb{D}^2_+$. Therefore we have existence.

To prove the uniqueness part we study all possible ways such a fibration may arise. Let $f\colon M \to (\mathbb{D}^2_+, \partial\mathbb{D}^2_+)$ be a boundary Lefschetz fibration satisfying the assumptions of the lemma. Without loss of generality, we assume that the image of the Lefschetz singularity is $(2/3, 0)$ and we split $\mathbb{D}^2_+$ in two parts:

$$U_1 = \{(x, y) \in \mathbb{D}^2_+ : x \leq 1/2\},$$
$$U_2 = \{(x, y) \in \mathbb{D}^2_+ : x \geq 1/2\}.$$

The set $f^{-1}(U_2)$ is a neighbouhood of the Lefschetz fibre and hence its differentiable type as a fibration is fully determined [36]. Similarly, the set $f^{-1}(U_1)$ is a neighbourhood of an elliptic fibre hence its differentiable type as a fibration is also fully determined:

$$f^{-1}(U_1) = \mathbb{D}^2 \times S^1 \times (-1, 1), \quad f(re^{i\theta}, \psi, t) = (r^2, t).$$

Therefore all possible different fibrations with the desired properties are determined by the different ways these two pieces can be glued together modulo the action of the isomorphism group of each half of the fibration.

Since the gluing takes place over a regular fibration over an interval, the isotopy class of the gluing map is determined by the isotopy class of the map it induces at a single fibre. Since the fibres are tori, this is in turn determined by the corresponding map in homology. Since the vanishing cycles form a dual pair there is, modulo the action of the isomorphism group of the fibration over V_1, a unique way to glue these together. □

Next we show how to use Lemmas 2.3.5 and 2.3.6 to prove both singularity trade theorems:

Proof of Theorem 2.3.3. Applying Theorem 2.2.3 to the boundary Lefschetz fibration on M and on S^4 gives rise to a boundary Lefschetz fibration on $M\#S^4 \simeq M$, for which the inclusion $M\backslash\{p\} \hookrightarrow M$ preserves fibrations, in particular, we see that the new fibration on M only changes in the small ball around p used for the connected sum procedure. Since the divisor in S^4 has only one point in the top stratum, the new divisor satisfies $\widetilde{D}[2] = D[2]\backslash\{p\}$ and $\widetilde{D}$ and D have the same index. Given the way the fibrations are glued, we see that the effect on the base is to smooth out the corner corresponding to $f(p)$. Continuing inductively gives rise to a boundary Lefschetz fibration with embedded divisor. □

Proof of Theorem 2.3.4. Under the conditions of the theorem, γ has a neighbourhood, V_2, diffeomorphic to $\mathbb{D}^2_+$ in which the fibration has only one Lefschetz singularity whose vanishing cycle forms a dual pair with the elliptic singularity. Hence, by Lemma 2.3.6, $f^{-1}(V_2)$ is diffeomorphic to $\mathbb{D}^4$ and f is equivalent to the fibration of Lemma 2.3.6. Since the fibration on S^4 splits as two discs, one fibreing over $\mathbb{D}^2_+$ and the other fibreing over a neighbourhood, V_1, of the origin in $(\mathbb{R}_+)^2$ (see Figure 2.5), we can realise $M\#S^4$ as follows: remove the disc $f^{-1}(V_2)$ and glue back, by the natural identification of the boundary, $f_{S^4}^{-1}(V_1)$.

Since this procedure corresponds to performing connected sum with S^4, the final manifold is still diffeomorphic to M and the fibration only changes in the part that has been surgered in, which includes the removal of the Lefschetz singularity from $f^{-1}(V_2)$ and the inclusion of the elliptic–elliptic singularity of $f_{S^4}^{-1}(V_1)$. Finally, notice that the process of filling the boundary of $f^{-1}(V_2)$ with $f_{S^4}^{-1}(V_1)$ is not compatible with the given orientations of these spaces since they both appear at opposite sides of a boundary in S^4. That is, the orientation of M is compatible with the opposite orientation of $f_{S^4}^{-1}(V_1)$. Since the elliptic–elliptic singularity for the fibration in S^4 had index -1 and the orientation of S^4 was reversed in the connected sum process, the intersection index of the new elliptic–elliptic singularity on M is $+1$ and hence the overall parity of the divisor is unchanged. □

To finish the proof of the trade theorems we must establish Lemma 2.3.5, which we do next.

Proof of Lemma 2.3.5. The proof is done in two steps. In the first step we show that if M is the total space of a boundary Lefschetz fibration whose singularities are as stated in Lemma 2.3.5, then $M = S^4$. In the second step we show that such a fibration exists.

Step 1. We observe once again that M is made of two fibrations glued together, as illustrated in Figure 2.5: one fibration with an elliptic–elliptic singularity over V_1 and one with a Lefschetz singularity over V_2. The fibration over V_1 is a copy of $\mathbb{D}^4$ added along its S^3 boundary, that is, $M = f^{-1}(V_2) \cup$ 4-handle. The space $f^{-1}(V_2)$ itself can be readily described as a handlebody: we start with a neighbourhood of a regular fibre, then add a -1-framed 2-handle along the vanishing cycle of the Lefschetz singularity to obtain a neighbourhood of the Lefschetz singular fibre and a 0-framed 2-handle along the vanishing cycle of the elliptic singularity. Therefore the Kirby diagram of M is the one depicted in Figure 2.6 (a). We can then slide the 2-handle that goes around both 1-handles to

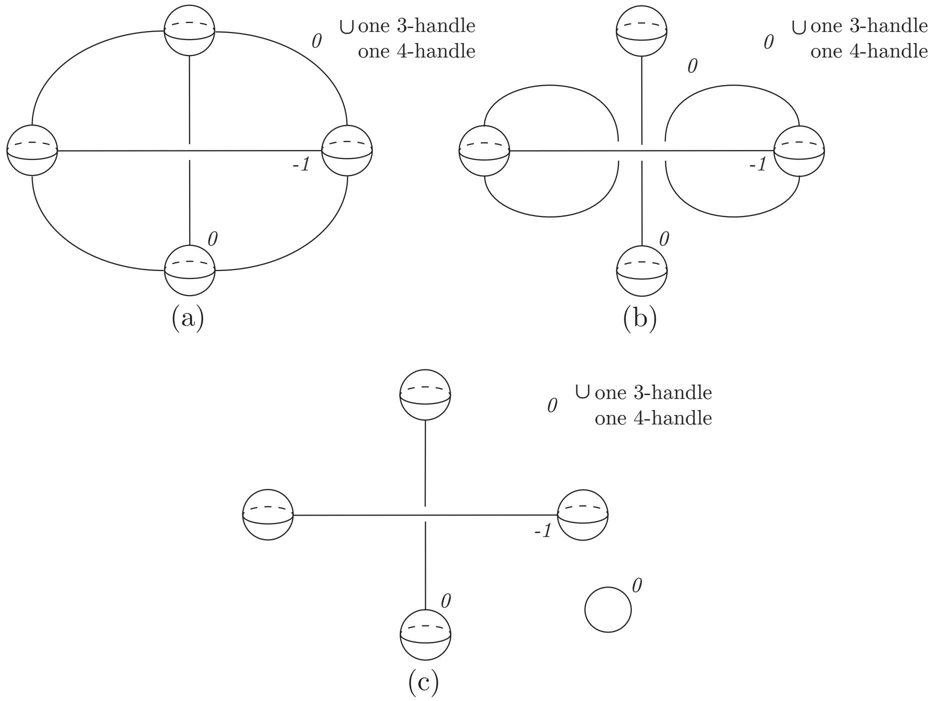


Figure 2.6: Kirby diagram for the total space of the fibration described in Lemma 2.3.6.

obtain Figure 2.6 (b) and see that the resulting 2-handle separates as a 0-framed 2-handle from the rest of the diagram and hence cancels with the 3-handle. The remaining pairs of 1- and 2-handles clearly cancel each other (Figure 2.6 (c)) leaving us with the empty diagram, which corresponds to S^4.

Step 2. To construct the desired fibration we will use a plumbing construction applied to the disc bundle of $\mathcal{O}(2) \to \mathbb{C}P^1$ in a way that is compatible with the natural torus fibration of that space. Throughout we will use fixed parametrizations $\varphi_1, \varphi_2 \colon \mathbb{C}^2 \to \mathcal{O}(2)$ for which the change of coordinates is given by

$$\varphi_2^{-1} \circ \varphi_1(z, w) = (z^{-1}, z^{-2}w).$$

We will refer to φ_1 as parametrizing a trivialization of $\mathcal{O}(2)$ with the south pole removed and similarly φ_2 does not cover the fibre over the north pole.

Rotation on both coordinates in the parametrization φ_2 give rise to a torus action on $\mathcal{O}(2)$ which, in the parametrizations above, is given by

$$\begin{aligned} (e^{i\theta_1}, e^{i\theta_2}) \cdot \varphi_1(z, w) &= \varphi_1(e^{-i\theta_1} z, e^{i(-2\theta_1 + \theta_2)} w), \\ (e^{i\theta_1}, e^{i\theta_2}) \cdot \varphi_2(z, w) &= \varphi_2(e^{i\theta_1} z, e^{i\theta_2} w). \end{aligned} \tag{2.3.1}$$

To describe the quotient of $\mathcal{O}(2)$ by this torus action, we will also want to consider $[-1,1]\times\mathbb{R}_+$. Of course this space can be parametrised by a single, rather obvious, chart, but it will be convenient to parametrise it by two charts instead. We consider the parametrizations

$$\psi_1\colon \mathbb{R}_+\times\mathbb{R}_+ \to (-1,1]\times\mathbb{R}_+ \subset [-1,1]\times\mathbb{R}_+ \quad \psi_1(x_1,y_1) = \left(\frac{1-x_1}{1+x_1}, \frac{y_1}{(1+x_1)^2}\right),$$

$$\psi_2\colon \mathbb{R}_+\times\mathbb{R}_+ \to [-1,1)\times\mathbb{R}_+ \subset [-1,1]\times\mathbb{R}_+ \quad \psi_2(x_2,y_2) = \left(-\frac{1-x_2}{1+x_2}, \frac{y_2}{(1+x_2)^2}\right),$$

and keep in mind that these parametrizations induce opposite orientations, with ψ_2 agreeing with the natural orientation of $[-1,1]\times\mathbb{R}_+$.

Lemma 2.3.7. *If we let $h\colon S^2\to\mathbb{R}$ be the height function and $g\colon \mathrm{Sym}^2\mathcal{O}(2)\to\mathbb{R}$ be the Fubini-Study metric, then*

$$\begin{aligned} f\colon \mathcal{O}(2) &\to [-1,1]\times\mathbb{R}_+ \\ (z,w) &\mapsto (h(z), g_z(w,w)), \end{aligned}$$

defines a quotient map for the torus action on $\mathcal{O}(2)$. Further, f is a proper boundary fibration with elliptic divisor induced by the holomorphic log divisor consisting of the zero-section and fibres over the north and south pole.

Proof. In the parametrizations φ_i, the height and distance function take the form:

$$\begin{aligned} h\circ\varphi_1(z,w) &= \frac{1-|z|^2}{1+|z|^2}, \quad & g\circ\varphi_1(z,w) &= \frac{|w|^2}{(1+|z|^2)^2}, \\ h\circ\varphi_2(z,w) &= -\frac{1-|z|^2}{1+|z|^2}, \quad & g\circ\varphi_2(z,w) &= \frac{|w|^2}{(1+|z|^2)^2}, \end{aligned}$$

which are clearly invariant under the T^2-action in Equation (2.3.1). Further, for $i=1,2$, the image of $f\circ\varphi_i$ lands in the image of the parametrization ψ_i and we can compute the expression for f in these parametrizations:

$$f_i(z,w) := \psi_i^{-1}\circ f\circ\varphi_i(z,w) = (|z|^2, |w|^2), \tag{2.3.2}$$

which shows clearly that f not only is the quotient map but also a boundary fibration. □

Now we perform a plumbing on $\mathcal{O}(2)$.

Definition 2.3.8. Let $\pi\colon M^{2n}\to N^n$ be a $\mathbb{D}^n$-bundle, and let $\mathbb{D}_1,\mathbb{D}_2$ be disjoint disks in N over which π is trivialisable. A **self-plumbing** of π at $\mathbb{D}_1$ and $\mathbb{D}_2$ is obtained by identifying $\pi^{-1}(\mathbb{D}_1)\simeq\mathbb{D}_1\times\mathbb{D}^n$ and $\pi^{-1}(\mathbb{D}_2)\simeq\mathbb{D}_2\times\mathbb{D}^n$ using a map which preserves the product structure but reverses the factors. ◇

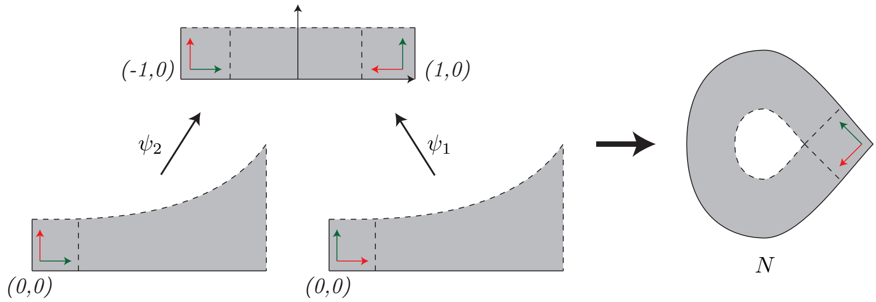


Figure 2.7: The base of the boundary fibration constructed in Lemma 2.3.9.

For the case at hand, let $\mathbb{D}^2\mathcal{O}(2)$ be the open ε-disk bundle with respect to the Fubini–Study metric. By restricting f to $\mathbb{D}^2\mathcal{O}(2)$, we obtain a proper boundary fibration $f\colon \mathbb{D}^2\mathcal{O}(2) \to [-1,1]\times[0,\varepsilon)$.

Further, we observe that φ_1 and φ_2 provide trivializations of $\mathbb{D}^2\mathcal{O}(2)$, hence we can use them to perform a self-plumbing of $\mathbb{D}^2\mathcal{O}(2)$ at the north and south poles. Let M be defined as the self-plumbing of $\mathbb{D}^2\mathcal{O}(2)$ via the trivializations φ_i and the map

$$\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2 : (z,w) \mapsto (\overline{w},\overline{z}),$$

that is, $\varphi_1(z,w)$ is identified with $\varphi_2(\overline{w},\overline{z})$.

Since the map used for the plumbing preserves elliptic ideals and identifies the north and south pole, M is endowed with an elliptic divisor with a single point in $D[2]$. Since the map Φ does not match co-orientations, the elliptic divisor in M has intersection index -1.

To endow M with a boundary fibration we only need to take a quotient of the base, $[-1,1]\times[0,\varepsilon)$, by the equivalence relation that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{D}^2\mathcal{O}(2) & \xrightarrow{\sim_\Phi} & \mathbb{D}^2\mathcal{O}(2) \\ \downarrow f & & \downarrow f \\ [-1,1]\times[0,\varepsilon) & \xrightarrow{\sim_\Psi} & [-1,1]\times[0,\varepsilon) \end{array}$$

Since f is surjective, there is a unique identification, $\sim_\Psi$, that gives rise to such a diagram. In fact, we can easily compute it in the parametrizations ψ_i, where it is induced by the map $\Psi(x,y) = (1-y, x+1)$. That is, the point $\psi_1(x,y)$ is identified with the point $\psi_2(y,x)$. Since ψ_1 and ψ_2 induce opposite orientations, this identification preserves the natural orientation of $[-1,1]\times[0,\varepsilon)$ and the quotient is an oriented half-open cylinder with one corner (see Figure 2.7).

Lemma 2.3.9. *The map $f\colon \mathbb{D}^2\mathcal{O}(2) \to [-1,1]\times[0,\varepsilon)$ descends to a boundary fibration $\widehat{f}\colon M \to N$.*

Next we compute its monodromy along a generator of $\pi_1(N)$.

Lemma 2.3.10. *Let $\widehat{f}\colon M \to N$ be the boundary fibration from Lemma 2.3.9. Then the monodromy of $\widehat{f}$ around a loop around the hole is a positive Dehn twist.*

Proof. This is a direct computation using the given change of coordinates and the plumbing map Φ. Indeed, all we need to to is to track what happens with the torus action as we move along from the chart covered by φ_2 to the chart covered by φ_1 and then back to φ_2 via Ψ:

$$\begin{aligned}(e^{i\theta_1}, e^{i\theta_2}) \cdot \varphi_2(z, w) = \varphi_2(e^{i\theta_1} z, e^{i\theta_2} w) &= \varphi_1(e^{-i\theta_1} z^{-1}, e^{i(-2\theta_1+\theta_2)} z^{-2} w) \\ &\sim_\Phi \varphi_2(e^{i(2\theta_1-\theta_2)} \overline{z}^{-2} \overline{w}, e^{i\theta_1} \overline{z}^{-1}) \\ &= (e^{i(2\theta_1-\theta_2)}, e^{i\theta_1}) \varphi_2(\overline{z}^{-2} \overline{w}, \overline{z}^{-1}).\end{aligned}$$

Therefore we see that, in the basis $\{e_{\theta_1}, e_{\theta_2}\}$ for $H^1(F)$ corresponding to the generators of the action, the monodromy transformation is given by the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

Notice that using the complex orientation of $\mathcal{O}(2)$ and the standard orientation of $\mathbb{R}^2$, $\{e_{\theta_1}, e_{\theta_2}\}$ is a negative basis for the homology of the fibre. Using this, we see that the transformation above is a positive Dehn twist on the cycle $e_{\theta_1} + e_{\theta_2}$. □

Now we can complete M to a closed manifold by glueing a neighbourhood of a single Lefschetz fibre with vanishing cycle $e_{\theta_1} + e_{\theta_2}$ in the hole of the annulus. Finally we observe that this vanishing cycle forms a dual pair with either of the two vanishing cycles of the elliptic singularity, which in the parametrization φ_2 are given by either of the cycles e_{θ_1} or e_{θ_2}. □

Remark 2.3.11. Simply drawing a base diagram for a boundary Lefschetz fibration does not guarantee the existence of a fibration that realises it. For example, there is no manifold whose base diagram is that of Figure 2.4, but for which the elliptic–elliptic singularity has intersection index 1. In the construction above this would manifest itself in the fact that without using complex conjugation the monodromy of the plumbing would be a negative Dehn twist. This highlights that the long second step in the proof above is indeed necessary. ♢

Remark 2.3.12. Note that the existence of the boundary Lefschetz fibration in Lemma 2.3.5 also provides an alternative proof of Theorem 1.4.12. Indeed, because by Theorem 2.1.23 S^4 admits a stable generalized complex structure with one point in $D[2]$, which has negative index. Then by consecutively taking connected sums with this S^4 and the original stable generalized complex manifold, we obtain a stable generalized complex structure with embedded type change locus. ♢

2.4 Examples

In this section we give several concrete examples of boundary fibrations. We will first show that they arise naturally as the quotient maps of effective torus actions and that our framework fits particularly well with the theory of integrable systems. This connection provides us immediately with a wealth of examples of both boundary fibrations and stable generalized complex structures. We will further illustrate our constructions by showing how starting with simple examples (of manifolds with torus actions) we can use the connected sum procedure to obtain many more examples of boundary fibrations.

2.4.1 Torus actions

We show that quotient maps of torus actions provide boundary fibrations.

Proposition 2.4.1. *Let T^n act effectively on a smooth manifold M^{2n}, with connected isotropy groups. Then:*

- *$N := M^{2n}/T^n$ is a manifold with corners;*
- *the quotient map defines a boundary fibration $f\colon (M,D) \to (N,\partial N)$ with connected fibres;*
- *the multiplicity stratification of the elliptic ideal coincides with the stratification by orbit types on M;*
- *$ND[1]$ is co-orientable;*
- *if M is oriented, then so is N;*
- *if M is four-dimensional and the action is not free, f is homologically essential.*

Proof. Let $p \in M$, and let G_p denote the isotropy group of p and let $\mathcal{O}_p$ denote the orbit of p. By assumption, G_p is connected and therefore isomomorphic to T^ℓ for some $\ell \leq n$. By the slice theorem, there exists a neighbourhood of $\mathcal{O}_p$ which is equivariantly diffeomorphic to a neighbourhood of the zero section in

$$G \times_{G_p} N_p\mathcal{O}_p,$$

where G_p acts linearly on $N_p\mathcal{O}_p$ by the differentiated action. Because all groups in consideration are Abelian and connected, this implies that there is a neighbourhood U around p of the form

$$U = T^{n-\ell} \times (\mathbb{R}^{n-\ell} \times \mathbb{C}^\ell).$$

The $T^n = (T^{n-\ell} \times T^\ell)$-action of U decomposes as $T^{n-\ell}$ acting by multiplication on $T^{n-\ell}$ and T^ℓ acting linearly on $\mathbb{C}^\ell$. Since the irreducible representations of T^ℓ are one-dimensional, we may without loss of generality assume that each coordinate line in $\mathbb{C}^\ell$ is preserved by the action. Therefore, if we let $\mathfrak{t}$ denote the

Lie algebra of T^ℓ, let $\mathfrak{l}$ denote the kernel of $\exp\colon \mathfrak{t} \to T^\ell$, with minimal generating set $\{\xi_1, \ldots, \xi_\ell\}$ and choose $\{\alpha_1, \ldots, \alpha_\ell\} \in \mathfrak{l}^*$ the dual basis for the dual lattice, then the action on each irreducible representation has the form

$$\exp(\Theta) \cdot z_j = e^{2\pi i \langle \Theta, n_j \alpha_j \rangle} z_j, \qquad \Theta \in \mathfrak{t}.$$

Since the action is effective we have that $n_j \neq 0$, and because the isotropy groups are connected we must furthermore have $n_j = \pm 1$. Hence, after appropriately changing the signs of some of the α_j, the T^ℓ-action is given by

$$(\exp(\theta_1 \xi_1 \cdot \ldots \cdot \theta_\ell \xi_\ell)) \cdot (z_1, \ldots, z_\ell) = (e^{2\pi\theta_1 i} z_1, \ldots e^{2\pi\theta_\ell i} z_\ell).$$

This normal form for the action has the following consequences:

- The quotient manifold is endowed with charts of the form $\mathbb{R}^{n-\ell} \times (\mathbb{C}^\ell)/T^\ell \simeq \mathbb{R}^n_\ell$, and is therefore a manifold with corners;
- The quotient map $f\colon M \to N$ in the above local coordinates is given by

$$\begin{aligned} f\colon T^{n-\ell} \times (\mathbb{R}^{n-\ell} \times \mathbb{C}^\ell) &\to \mathbb{R}^n_\ell \\ (q, x, z_1, \ldots, z_\ell) &\mapsto (x, |z_1|^2, \ldots, |z_\ell|^2). \end{aligned}$$

  By Lemma 2.1.8 we see that f is a boundary fibration with respect to the log divisor ∂N;
- Because the vanishing locus of the induced elliptic ideal is given by $f^{-1}(\partial N)$ it follows that the multiplicity stratification coincides with the orbit type stratification;
- At points $p \in D[1]$, the isotropy group is given by S^1 and therefore $N_p\mathcal{O}_p$ inherits an S^1-action and consequently admits an orientation. We conclude that $D[1]$ is co-orientable;
- When M is oriented, a choice of orientation for T^n gives rise to an orientation for N by observing that $M \backslash D \to N \backslash \partial N$ is a principal T^n-bundle;
- When M is four-dimensional and the action is not free it is shown in [68] that f admits a section. As a generic fibre and the image of this section intersect only once, it follows that the intersection pairing of the fibre with the image of this section is non-zero, and therefore f is homologically essential.

□

The group actions underlying toric manifolds satisfy the conditions of this proposition, leading to the following result:

Corollary 2.4.2. *Let (M^{2n}, ω) be a toric manifold and let $f\colon M^{2n} \to \Delta^n$ denote the quotient map. Then f is a boundary fibration.*

In four dimensions Proposition 2.4.1 provides us with fibrations that satisfy nearly all the assumptions required to apply Theorem 2.1.23. However, the torus action does not guarantee that the parity of the elliptic divisor is one. To proceed we must add hypotheses to ensure that this is the case.

Proposition 2.4.3. *Let $f\colon (M^4,\omega) \to \mathbb{R}^2$ be a toric manifold. Then the parity of the elliptic divisor obtained from Proposition 2.4.1 is 1, and therefore M admits a stable generalized complex structure compatible with f.*

Proof. By Proposition 2.4.1 we have that f is a boundary fibration, and therefore by Theorem 2.1.23 the manifold M admits an elliptic symplectic structure. As each of the preimages of the faces of the moment polytope is a symplectic submanifold of (M,ω), the symplectic structure provides each component of the elliptic divisor with a natural co-orientation for which the intersections have positive index. It follows that the parity of the elliptic divisor is 1. □

2.4.2 Simple examples

We give examples of boundary fibrations obtained from torus actions which will serve as the building blocks for the connected sum procedure. The existence of stable generalized complex/elliptic symplectic structures in all examples below was known previously, and most of them are discussed in Section 1.6 as well. The new input is that these spaces admit fibrations compatible with the structure in question.

Example 2.4.4 ($\mathbb{C}P^2$)**.** Consider the standard toric structure on $\mathbb{C}P^2$. Proposition 2.4.3 implies that f is a homologically essential boundary fibration and that $\mathbb{C}P^2$ admits an elliptic divisor with parity 1 (three lines intersecting at different points). Therefore $\mathbb{C}P^2$ admits a stable generalized complex structure compatible with its moment map. △

Example 2.4.5 ($\overline{\mathbb{C}P}^2$)**.** We consider $\overline{\mathbb{C}P}^2$, i.e. $\mathbb{C}P^2$ with the orientation opposite to the standard complex structure. As an oriented manifold this is not a toric manifold, but there is still a T^2-action with connected isotropies present. Therefore 2.4.1 implies that the quotient map is a homologically essential boundary fibration. Consequently, by Theorem 2.1.23 there exists a compatible elliptic symplectic structure with imaginary parameter on $\overline{\mathbb{C}P}^2$. The parity of the elliptic divisor is -1 so this symplectic structure does not induce a stable generalized complex structure. As $\overline{\mathbb{C}P}^2$ is not almost complex it can not have a stable generalized complex structure, hence this problem can not be remedied. △

Example 2.4.6 ($S^2 \times S^2$)**.** Let $(S^2 \times S^2)$ be given its standard toric structure, i.e. the symplectic form is the product of the standard area forms and T^2 acts on rotation by S^1 one each of the factors. Proposition 2.4.3 implies that the quotient map is a homologically essential boundary fibration and that $S^2 \times S^2$ admits a compatible stable generalized complex structure. △

Example 2.4.7 (S^4)**.** Consider $S^4 \subset \mathbb{C}^2 \times \mathbb{R}$ and let T^2 act in the standard way on $\mathbb{C}^2$. This provides an effective T^2-action on S^4 with connected isotropies. Therefore by Proposition 2.4.1 we find that the quotient map is a homologically essential boundary fibration. Consquently Theorem 2.1.23 implies the existence of a compatible elliptic symplectic structure with imaginary parameter on S^4. The parity of the divisor is -1. Just as $\overline{\mathbb{C}P^2}$, S^4 is not almost-complex so the index can not be fixed by making different choices of divisor or orientations. △

The following example of a boundary fibration appears also in [13]:

Example 2.4.8 ($S^3 \times S^1$)**.** There are two interesting T^2-actions on $S^3 \times S^1$. First, consider $S^3 \subset \mathbb{C}^2$ as the unit sphere and restrict the natural T^2-action on $\mathbb{C}^2$ to S^3. This provides an effective T^2-action on S^3 with S^1 isotropy at all points in the intersection with the coordinate hyperplanes. Extending the T^2-action trivially to the S^1-factor provides an effective T^2-action on $S^3 \times S^1$ with only S^1 isotropy groups. The quotient map

$$f_1 \colon (S^3 \times S^1, D_1) \to (I \times S^1, \{0, 1\} \times S^1),$$

then becomes a homologically essential boundary fibration by Proposition 2.4.1. Note that D_1 is given by the union of two disjoint tori.

Another T^2-action on $S^3 \times S^1$ is obtained by letting one S^1 act by rotation on one of the coordinates of $S^3 \subset \mathbb{C}^2$ and let the other act by multiplication on S^1. The quotient map

$$f_2 \colon (S^3 \times S^1, D_2) \to (\mathbb{D}^2, \partial\mathbb{D}^2),$$

then again becomes a homologically essential boundary fibration by Proposition 2.4.1. In this case D_2 is a single torus. In both cases Theorem 2.1.23 implies the existence of a compatible elliptic symplectic structure with zero elliptic residue. Moreover, as the vanishing locus of the elliptic divisor is smooth and co-orientable, we obtain two stable generalized complex structures on $S^3 \times S^1$. △

The example we consider next is more elaborate than the previous ones. The existence of stable generalized complex structures on these spaces is a consequence of the more general Theorem 2 from [75].

Example 2.4.9 ($(\#nS^1 \times S^2) \times S^1$)**.** In [68], it is shown that for $2g + h > 1$, the manifold $M = (\#(2g + h - 1)S^1 \times S^2) \times S^1$ admits an effective T^2-action with connected isotropy groups over a base, B, which is a surface of genus g with h small open discs removed. In fact, part of the action is just rotation of the last S^1-factor, so this action has no fixed points (a fact that also follows from the Euler characteristic of M being 0).

By Proposition 2.4.1 we conclude that there exists a homologically essential boundary fibration

$$f \colon ((\#(2g + h - 1)S^1 \times S^2) \times S^1, D) \to (B, \partial B).$$

The degeneracy locus consist of h disjoint tori – precisely the number of boundary components of B – and is in particular co-orientable. Consequently by Theorem 2.1.23 there exists a compatible (smooth) stable generalized complex structure on M whose type change locus has h connected components. △

To illustrate the elliptic-elliptic trade theorem we give some examples:

Example 2.4.10 ($\mathbb{C}P^2$). Applying Theorem 2.3.3 to Example 2.4.4 yields several boundary Lefschetz fibrations $f\colon (\mathbb{C}P^2, D) \to (N, \partial N)$. The number of elliptic–elliptic and Lefschetz singularities adds up to three, but any combination is possible. See also Remark 2.4.13. △

Example 2.4.11 (S^4). Applying Theorem 2.3.3 to Example 2.4.7 yields a boundary Lefschetz fibration $f\colon (S^4, \widetilde{D}) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ with two Lefschetz singularities. Because the parity of the original divisor on S^4 is -1, the new divisor $\widetilde{D}$ will be non-co-orientable. Therefore it is non-orientable and as it admits an S^1-fibration it must then be a Klein bottle. △

2.4.3 Main class of examples

Using the above examples as building blocks we can now construct many more examples:

Theorem 2.4.12. *The manifolds in the following two families admit homologically essential boundary fibrations:*

- $X_{n,\ell} := \#n(S^2 \times S^2)\#\ell(S^1 \times S^3)$*, with* $n, \ell \in \mathbb{N}$*;*
- $Y_{n,m,\ell} := \#n\mathbb{C}P^2\#m\overline{\mathbb{C}P}^2\#\ell(S^1 \times S^3)$*, with* $n, m, \ell \in \mathbb{N}$*,*

whenever their Euler characteristic is non-negative. Therefore, each of these manifolds admits a compatible elliptic symplectic structure, which induces a stable generalized complex structure if $1 - b_1 + b_2^+$ *is even.*

Proof. In the previous section we exhibited boundary fibrations on $\mathbb{C}P^2, \overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ with $3, 3, 4$ points in $D[2]$ respectively. Therefore we may apply Theorem 2.2.3 inductively to obtain homologically essential boundary fibrations on $X_{n,0}$ and $Y_{n,m,0}$ for all possible values of n and m, including $n = m = 0$ by Example 2.4.7. The number of points in $D[2]$ for these manifolds is $2n + 2$ and $n + m + 2$ respectively. Therefore we can apply Corollary 2.2.5 respectively $n + 1$ and $\lfloor \frac{n+m+2}{2} \rfloor$-times to obtain homologically essential boundary fibrations on $X_{n,l}$ and $Y_{n,m,\ell}$, for $\ell \leq n + 1, \lfloor \frac{n+m+2}{2} \rfloor$ respectively. A simple computation of the Euler characteristic of these manifolds shows that this is precisely when their Euler characteristic is non-negative. The parity of the divisor in $\mathbb{C}P^2, \overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ is $1, -1, 1$ respectively. Therefore Theorem 2.2.3 gives us that the parity of $X_{n,0}$ and $Y_{n,m,0}$ is $(-1)^{n-1}$. Corollary 2.2.5 gives us that the parity of the divisor in $X_{n,\ell}$ and $Y_{n,m,\ell}$ is $(-1)^{n-1+\ell}$. By Theorem 2.1.23 these manifolds admit compatible elliptic symplectic structures. These induce stable generalized complex structures when $(-1)^{n-1+\ell} = 1$, which is to say that $1 - b_1 + b_2^+$ is even. □

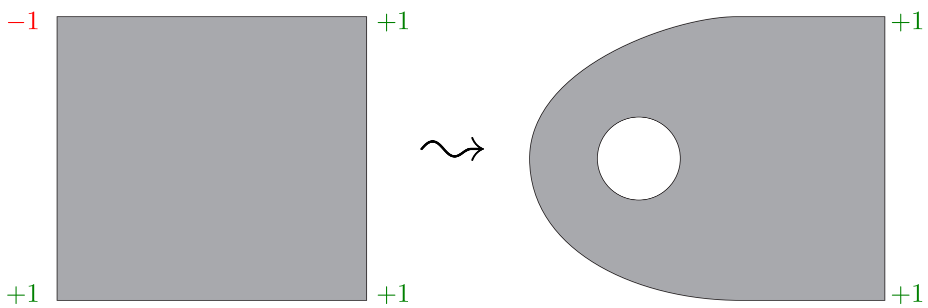


Figure 2.8: We obtain a boundary fibration on $\#2\mathbb{C}P^2\#(S^1 \times S^3)$ from $\#2\mathbb{C}P^2$ via self-connected sum.

The following remarks elaborate on the assumptions on the n, m and ℓ in the above theorem.

Remark 2.4.13 (Euler characteristic)**.** The condition on the Euler characteristic is necessary. Indeed a simple application of Mayer–Vietoris shows that if $f\colon M^4 \to \Sigma^2$ is a boundary Lefschetz fibration over a surface with k corners, and ℓ Lefschetz singular fibres then $\chi(M) = k + \ell$. In particular, we find that the Euler characteristic of a manifold admitting a boundary Lefschetz fibration is necessarily non-negative. Therefore we conclude that we found all members of the families appearing in Theorem 2.4.12 that admit boundary fibrations.

This observation has another consequence. One can perform connected sum of manifolds with elliptic divisors (M_i, D_i) at points in $p_i \in D_i[1]$, but, differently from the case of points in $D_i[2]$, connected sum at $D_i[1]$ is not compatible with the existence of boundary Lefschetz fibrations. Indeed, if each M_i had such a fibration which induced one in $M_1\#_{p_1,p_2}M_2$ by some identification of neighbourhoods of p_1 and p_2, the number of corners and Lefschetz singularities would be no less than the sum of corners and singularities for each M_i, but the Euler characteristic of $M_1\#M_2$ equals the sum of the Euler characteristics of M_1 and M_2 minus two. ♢

Remark 2.4.14 (Betti numbers)**.** The existence of a generalized complex structure on a manifold implies the existence of an almost-complex structure. Such a structure cannot exist when $1 - b_1 + b_2^+$ is odd, which explains that we found all members of the families appearing in Theorem 2.4.12 that admit stable generalized complex structures arising from boundary fibrations. ♢

Remark 2.4.15 (Torus actions)**.** Torus actions persist under taking connected sums of disjoint manifolds at fixed points [68]. In fact, [68] provides a classification of simply connected four-manifolds with effective torus actions and connected isotropy groups. The manifolds admitting such actions are precisely the manifolds $X_{n,0}$, $Y_{n,m,0}$, and S^4 appearing in Theorem 2.4.12. Whenever such a T^2-action is present it is possible to ensure that the elliptic symplectic structure arising from Theorem 2.4.12 is T^2-invariant, hence we obtained all such simply connected four-manifolds admitting T^2-invariant stable generalized complex structures.

In the non-simply connected case, [68] also provides a classification of effective non-free torus actions with only S^1-isotropy groups on compact oriented connected four-manifolds. It is proven that any of these manifolds is of the form as described in Example 2.4.9, hence we have obtained all manifolds with such actions and T^2-invariant stable generalized complex structures. ♢

2.4.4 Relation to semi-toric geometry

We finish by relating our results to semi-toric geometry. Recall that a **focus–focus** singularity of a completely integrable system (M, ω, f) is a point $p \in M$ where there are Darboux coordinates (x_1, y_1, x_2, y_2) for ω in which f takes the form

$$(x_1, y_1, x_2, y_2) \overset{f}{\longmapsto} (x_1 y_2 - x_2 y_1, x_1 x_2 + y_1 y_2).$$

Semi-toric manifolds ([69]) are generalisations of four-dimensional toric manifolds where the moment map, besides elliptic and elliptic–elliptic singularities, may also have focus–focus singularities. If we use the above Darboux coordinates to define complex coordinates

$$(w_1, w_2) = \frac{1}{4}(x_1 + y_2 + i(x_1 - y_2), x_1 - y_2 + i(x_1 + y_2)),$$

we see that the point p becomes a Lefschetz singularity of the moment map f.

Proposition 2.4.16. *Moment maps of semi-toric manifolds are boundary Lefschetz fibrations. Consequently, semi-toric manifolds admit compatible stable generalized complex structures, for which the elliptic divisor is the pre-image of the boundary of the moment map image.*

Proof. In light of Theorem 2.1.23 and the toric case (Proposition 2.4.3) we need only argue that the map is homologically essential. This follows because the homotopy type of $M \backslash D$ is obtained from a regular fibre by adding 2-cells along the vanishing cycles corresponding to each Lefschetz singularity. □

Remark 2.4.17. Theorem 2.3.3 trades an elliptic–elliptic singularity for a Lefschetz singularity in the context of a fibration without further geometric structures. This is reminiscent of the nodal trade/Hamiltonian Hopf bifurcation from semi-toric geometry [80, 61] in the context of Lagrangian fibrations. In the Hamiltonian

Hopf bifurcation, elliptic–elliptic singularities are traded for focus-focus singularities, which by the above are equivalent to Lefschetz singularities. However these maps interact differently with the underlying geometric structure. Notably, in the semi-toric version, the base of the fibration has a singular integral affine structure which helps with the extension of the fibration beyond a neighbourhood of the singularities involved.

The converse trade for semi-toric geometry, similar to our Theorem 2.3.4, appeared in [55]. There, the authors also make use of the singular integral affine structure structure present in such integrable systems. ◇

Chapter 3

The cohomology of the elliptic tangent bundle

In this chapter, based on [78], we will make the first step into understanding the deformation theory of the self-crossing stable and elliptic symplectic structures defined in Chapter 1.

In [16] deformations of stable generalized complex structures with embedded type change locus and the corresponding elliptic symplectic structures are described. It is shown there that deformations are completely controlled by the cohomology of the relevant Lie algebroid.[1]

If we allow the stable generalized crossings to have a type change locus with self-crossings this will no longer be the case, still the computation of the Lie algebroid cohomology is the first step in understanding the deformation theory. We will show that the cohomology of the elliptic tangent bundle is completely determined by the data of the manifold, of the type change locus and of the normal bundle.

Organisation: In Section 3.1 we describe the geometric structure a self-crossing elliptic divisor induces on the directions normal to its degeneracy locus. In Section 3.2 we will use this geometric structure to define residue maps, which we will use in Section 3.3 to compute the Lie algebroid cohomology of the elliptic tangent bundle. In Section 3.4 we compute this cohomology in some explicit examples. We end this section by giving an outlook on the deformation theory of self-crossing stable generalized complex structures.

3.1 Geometric structure on the normal bundle

In this section we study the geometric structure present on the normal bundles of the strata of the vanishing locus of an elliptic divisor. In the next section, this

[1] The description of deformations of stable generalized complex structures was obtained independently in [38].

geometric structure will be used to describe the Lie algebroid cohomology of the elliptic tangent bundle.

Remark 3.1.1 (Metrics)**.** Let $I_{|D|}$ be a smooth elliptic divisor with vanishing locus D, and let f be any global function generating $I_{|D|}$. Because f is nowhere vanishing on $M \backslash D$, and D has codimenion-two in M the sign of f is constant on the entirety of M. Consequently, we can always choose a non-negative representative of the elliptic ideal. Therefore the normal Hessian of f, $\mathrm{Hess}^\nu f \in \Gamma(\mathrm{Sym}^2 N^*D)$ is positive definite, and thus defines a metric. ♢

Remark 3.1.1 explains how a smooth elliptic divisor $I_{|D|}$ induces a metric on the normal bundle to D. The metric depends on the choice of particular trivialisation of R, but the conformal class does not. Phrased in another way, $\mathrm{Fr}(ND)$ inherits a canonical reduction of its structure group to $O(2) \times \mathbb{R}_+$. For self-crossing elliptic divisors we have the following result for the stratification from Remark 1.1.15:

Proposition 3.1.2. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n. Then $\mathrm{Fr}(ND[n])$ admits a canonical structure group reduction to $(O(2) \times \mathbb{R}_+)^n \rtimes S_n$.*

Proof. For a point $x \in D[n]$, let $I_{|D_1|}, \ldots, I_{|D_n|}$ be any choice of local normal crossing divisors and let the f_i be any representatives of the $I_{|D_i|}$, which we may choose to be all non-negative. Because the D_i all intersect transversely, we have

$$N_x(D[n]) \simeq \bigoplus_{i=1}^{n} N_x(D_i)|_{D[n]} .$$

If $u_i = (u_{i_1}, u_{i_2})$ forms a local frame for $ND_i|_{D[n]}$, then $(u_{1_1}, u_{1_2}, \ldots, u_{n_1}, u_{n_2})$ forms a local frame for $N(D[n])$. We consider all such local frames for which u_i is orthonormal with respect to $\mathrm{Hess}^\nu(f_i)$. There are two important points to remark. Firstly, the choice of representatives f_i is not canonical. Secondly, the ordering of the local divisors is not well-defined, and instead there is an S_n-symmetry present which permutes the divisors. Therefore we consider all the frames which are orthonormal with respect to any of these possible choices of representatives f_i and any choice of orderings for the local divisors. This provides a reduction of the structure group to $(O(2) \times \mathbb{R}_+)^n \rtimes S_n$. □

Corollary 3.1.3. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n, then*

- *$\mathrm{Fr}(ND[n])$ admits a (non-canonical) structure group reduction to $O(2)^n \ltimes S_n$.*
- *If $I_{|D|}$ is a global normal crossing divisor, then a choice of non-negative representatives $f_1, \ldots, f_n$ of the smooth elliptic divisor ideals induces a further structure group reduction to $O(2)^n$.*
- *If $D[1]$ is furthermore co-orientable, then a further choice of co-orientation of $D[1]$ induces a structure group reduction to $U(1)^n$.*

Proof.

- Recall that if P is a principal G-bundle, and H a normal subgroup, then P admits a structure group reduction to H if the quotient G/H is contractible. The quotient of $(O(2) \times \mathbb{R}_+)^n \rtimes S_n$ by $O(2)^n \ltimes S_n$ is $(\mathbb{R}_+)^n$, and thus contractible. Therefore the required structure group reduction exists.

- Let $I_{|D_1|}, \ldots, I_{|D_n|}$ be global smooth elliptic divisors for $I_{|D|}$, and let the functions $f_1, \ldots, f_n$ be generators of these ideals. Because $ND[n]$ is now globally isomorphic to $\oplus_{i=1}^n ND_i|_{D[n]}$, we can consider frames $(u_1, \ldots, u_n)$ of $ND[n]$, where u_i is a local frame of $ND_i|_{D[n]}$. Considering those frames for which u_i is orthonormal with $\mathrm{Hess}^\nu(f_i)$ provides the required $O(2)^n$-reduction.

- Recall, that if a line bundle is triviallisable away from a subset of codimension at least two, then it is in fact everywhere triviallisable. Therefore, the co-orientation on $D[1]$ induces a co-orientation on each D_i away from a codimension-two submanifold, and therefore a co-orientation on the entirety of D_i. Proceeding as in the previous point with oriented frames provides a structure group reduction to $U(1)^n$. □

Note that for smooth elliptic divisors, passing from the canonical structure group reduction of $\mathrm{Fr}(ND)$ from $O(2) \times \mathbb{R}_+$ to $O(2)$ corresponds precisely to choosing a particular representative of the conformal class of metrics on ND. For self-crossing elliptic divisors, we know that such a structure group reduction exists, but we cannot make the choices involved more precise.

Remark 3.1.4. Let $I_{|D|}$ be an elliptic divisor, and $i < n$. The restriction of $I_{|D|}$ to $M \backslash D(i+1)$ is an elliptic divisor of pointwise multiplicity i. The highest stratum of this divisor is precisely $D[i]$, and consequently we can apply the results in this section to obtain structure group reduction of $\mathrm{Fr}(ND[i])$ for each $1 \leq i \leq n$. ◊

3.2 Residue maps

In this section we describe residue maps for elliptic divisors. These are maps which pick out the coefficient of a singular generator of a Lie algebroid form, much akin to the residue of a meromorphic differential form. In [52] a general theory of residue maps of Lie algebroids is described. We will first recall the general definition and then specialise to the case of elliptic divisors.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of vector spaces with $l = \dim A$. A splitting of this sequence induces an isomorphism $B \simeq A \oplus C$. Consequently $\wedge^k B^*$ decomposes as a direct sum $\oplus_i (\wedge^{k-i} C^* \otimes \wedge^k A^*)$. The projections from $\wedge^k B^*$ to each of these factors depend on the particular splitting; however, the projection to $\wedge^{k-l} C^* \otimes \wedge^l A^* = \wedge^{k-l} C^* \otimes \det A^*$ does not. We define the **residue map** to be this projection:

$$\mathrm{Res} : \wedge^k B^* \to \wedge^{k-l} C^* \otimes \det A^*.$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of Lie algebroids with $\text{rk}(\mathcal{A}) = l$. Applying the residue map fibre-wise gives a map $\text{Res} : \Omega^k(\mathcal{B}) \to \Omega^{k-l}(\mathcal{C}; \det(\mathcal{A}^*))$. To endow $\Omega^\bullet(\mathcal{C}; \det(\mathcal{A}^*))$ with a differential, one needs a flat $\mathcal{C}$-connection on $\det(\mathcal{A}^*)$. We now describe a situation in which such a connection exists and the residue map is a cochain morphism with respect to this differential.

Proposition 3.2.1. *Let $\mathcal{B} \to M$ be a Lie algebroid and let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be an abelian ideal subalgebroid of rank l. Then there is a canonical flat $(\mathcal{B}/\mathcal{A})$-connection on $\det(\mathcal{A}^*)$.*

Furthermore assume that for every point $x \in M$, there exists a neighbourhood U of p and closed sections $\beta_1, \ldots, \beta_l \in \Omega^1(\mathcal{B}|_U)$ such that $i^*\beta_1, \ldots, i^*\beta_l$ generate $\Omega^1(\mathcal{A}|_U)$. Then $\text{Res} : \Omega^k(\mathcal{B}) \to \Omega^{k-l}(\mathcal{B}/\mathcal{A}; \det(\mathcal{A}^*))$ is a cochain morphism.

Proof. Let $\sigma : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ be any splitting and define:

$$\nabla_c(a) = [\sigma(c), a], \quad c \in \Gamma(\mathcal{B}/\mathcal{A}), a \in \Gamma(\mathcal{A}).$$

It is straightforward to verify that this is connection is flat and does not depend on the choice of splitting. If we let $\beta_1, \ldots, \beta_l \in \Omega^1(\mathcal{B})$ be local closed forms with the property that their restriction to $\mathcal{A}$ defines a local frame for $\Omega^1(\mathcal{A})$, then

$$\text{Res}(\beta_1 \wedge \ldots \wedge \beta_l \wedge \gamma) = \gamma \otimes (\beta_1 \wedge \ldots \wedge \beta_l),$$

and one easily verifies that Res is a cochain morphism. □

If $I_{|D|}$ is an elliptic ideal with pointwise multiplicity n then the restriction of the anchor of the elliptic tangent bundle to $D[n]$ has image inside $TD[n]$, and hence $\mathcal{A}_{|D|}\big|_{D[n]}$ is again a Lie algebroid. We will describe this Lie algebroid using the language of Atiyah algebroids, which we recall in Appendix 3.5. The explicit description of this Lie algebroid will allow us to construct residue maps which we will use in the computation of the Lie algberoid cohomolog of $\mathcal{A}_{|D|}$ in Theorem 3.3.1.

To describe $\mathcal{A}_{|D|}\big|_{D[n]}$, we will use a quadratic approximation of of the elliptic divisor $I_{|D|}$ on $\text{tot}(ND[n])$. Let $I_1, \ldots, I_n$ be local normal crossing elliptic divisors for $I_{|D|}$ and let $f_1, \ldots, f_n$ by generators of these ideals. Let $Q_{f_i} \in C^\infty(\text{tot}(ND_i))$ denote the quadratic approximations of these functions, and define the ideal $I_{|D|}^{ND[n]} = \langle Q_{f_1} \cdots Q_{f_n} \rangle$ on $\text{tot}(ND[n])$. Although the functions Q_{f_i} are neither unique nor global this ideal is and thus defines an elliptic divisor on $ND[n]$. Let $\mathcal{A}_{|D|}^{ND[n]}$ denote the elliptic tangent bundle associated to this ideal, and let $\Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\text{lin}}$ denote the submodule of its sections which under the anchor are send to linear vector fields on $\text{tot}(ND[n])$.

Lemma 3.2.2. *Let $I_{|D|}$ be an elliptic divisor on M. Given $X \in \Gamma(\mathcal{A}_{|D|}\big|_{D[n]})$ that there exists a unique linear vector field $\widetilde{X} \in \Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\text{lin}}$ with $\widetilde{X}\Big|_{D[n]} = X$. Moreover, the map*

$$\varphi \colon \Gamma(\mathcal{A}_{|D|}\big|_{D[n]}) \to \Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\text{lin}}, \qquad X \mapsto \widetilde{X}. \tag{3.2.1}$$

is a bracket preserving bijection.

Proof. Let $p : \mathcal{U} \to D[n]$ be a tubular neighbourhood of $D[n]$ and let $X' \in \Gamma(\mathcal{A}_{|D|}\big|_{\mathcal{U}})$ be any extension of X. If a vector field Z preserves the ideal generated by one f_i, then one readily verifies that the linearisation $\nu(Z)$ preserves the ideal generated by the quadratic approximation Q_{f_i}. Using this we find that because X' preserves the ideal $I_{|D|}$, its linearisation $\widetilde{X} := \nu(X')$ preserves the ideal $I_{|D|}^{ND[n]}$. Consequently we can define $\varphi(X) = \widetilde{X} \in \Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\mathrm{lin}}$. To show that this does not depend on the choice of extension X', we will use the local coordinates from Lemma 1.1.25. In these coordinates we have

$$X = \sum_i f_i \cdot r_i \partial_{r_i}|_{D[n]} + g_i \cdot \partial_{\theta_i}|_{D[n]} + Y,$$

with $f_i, g_i \in C^\infty(D[n])$ and $Y \in \mathfrak{X}^1(D[n])$. Any extension of X, will thus be of the form

$$X' = \sum_i \widetilde{f_i} \cdot r_i \partial_{r_i} + \widetilde{g_i} \cdot \partial_{\theta_i} + \widetilde{Y},$$

with $f_i, g_i \in C^\infty(\mathrm{tot}(ND[n])), \widetilde{Y} \in \Gamma(p^*TD[n])$. Its linearisation will be given by

$$\widetilde{X} = \sum_i f_i \cdot r_i \partial_{r_i} + g_i \cdot \partial_{\theta_i} + Y,$$

and we see that this does not depend on the choice of X'.

If $X_1, X_2 \in \Gamma(\mathcal{A}_{|D|}\big|_{D[n]})$ have equal linearisations $\widetilde{X}_1, \widetilde{X}_2$, then in particular $X_1 = \widetilde{X_1}|_{D[n]} = \widetilde{X_2}|_{D[n]} = X_2$, which shows that φ is injective.

If $Z \in \Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\mathrm{lin}}$, then $Z|_{D[n]} \in \Gamma(\mathcal{A}_{|D|}\big|_{D[n]})$ and $\varphi(Z|_{D[n]}) = Z$, which shows that φ is surjective. □

Proposition 3.2.3. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n and let P denote the $(O(2) \times \mathbb{R}_+)^n \rtimes S_n$-structure group reduction of $\mathrm{Fr}(ND[n])$ from Proposition 3.1.2. Then $\mathcal{A}_{|D|}\big|_{D[n]} \simeq \mathrm{At}(P)$.*

Proof. By Lemma 3.2.2 we have that $\Gamma(\mathcal{A}_{|D|}\big|_{D[n]})$ is isomorphic to $\Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\mathrm{lin}}$, and we are thus left to show that this coincides with sections of $\mathrm{At}(P)$. Let F be a generator of $I_{|D|}^{ND[n]}$, then

$$\Gamma(\mathcal{A}_{|D|}^{ND[n]})_{\mathrm{lin}} = \{X \in \mathfrak{X}^1(\mathrm{tot}(ND[n]))_{\mathrm{lin}} : \mathcal{L}_X(F) = \lambda F \text{ for some } \lambda \in C^\infty(D[n])\}.$$

Note that an element of $\Gamma(\mathcal{A}_{|D|}^{ND[n]})$ we have $\mathcal{L}_X(F) = \lambda F$ for a function $\lambda \in C^\infty(\mathrm{tot}(ND[n]))$, however because X is linear we must have $\lambda \in C^\infty(D[n])$.

As F is locally of the form $Q_{f_1} \cdots Q_{f_n}$ and all these functions are functionally indivisible, we must have that for all i there exists precisely one j such that

$\mathcal{L}_X(Q_{f_i}) = \lambda^i_j Q_{f_j}$ for some function $\lambda^i_j \in C^\infty(D[n])$. On the other side, using Lemma 3.5.5 one can show that

$$\mathrm{At}(P) = \{X \in \mathfrak{X}(\mathrm{tot}(ND[n]))_{\mathrm{lin}} : \forall\, i\ \exists!\, j \text{ s.t. } \mathcal{L}_X(\mathrm{Hess}^\nu(f_i)) = \lambda^i_j \,\mathrm{Hess}^\nu(f_j)\},$$

with the $\lambda^i_j \in C^\infty(D[n])$. Using the fact that for a linear vector field X and definite Morse–Bott functions f, g we have $\mathcal{L}_X(\mathrm{Hess}^\nu f) = g$ if and only if $\mathcal{L}_X(Q_f) = Q_g$ we see that indeed $\Gamma(\mathcal{A}^{ND[n]}_{|D|})_{\mathrm{lin}} = \Gamma(\mathrm{At}(P))$, which finishes the proof. □

3.2.1 Residues for smooth elliptic divisors

In this section we describe the radial residue map for a smooth elliptic divisor. The radial residue map was already used in [16, Definition 1.7] to compute the Lie algebroid cohomology of the elliptic tangent bundle of a smooth elliptic divisor. We recall the construction and add some details which are especially important in the general case. To do this, we first have to study the isotropy of the restricted Lie algebroid $\mathcal{A}_{|D|}\big|_D$:

Lemma 3.2.4. *Let $I_{|D|}$ be a smooth elliptic divisor on a manifold M, and let $\rho : \mathcal{A}_{|D|} \to TM$ be the corresponding elliptic tangent bundle. There exists a unique nowhere vanishing section $\mathcal{E}_D$ in $\ker \rho|_D$ with the property that every extension is an Euler like vector field[2].*

Proof. Let $\mathcal{U}$ be any tubular neighbourhood of D, and let $\mathcal{E}$ be the corresponding Euler vector field. Let $f \in I_{|D|}$ be a local representative of the ideal, and pick local Morse–Bott coordinates in which $f = r^2$. In these coordinates $\mathcal{E}$ is given by $r\partial_r$. Therefore, $\mathcal{E}$ defines a section of the elliptic tangent bundle and its restriction to D defines a section $\mathcal{E}_D$ of $\ker \rho|_D$. It is readily verified that $\mathcal{E}_D$ is the unique vector field with the given property. □

Because $L_r = \mathbb{R} \cdot \mathcal{E}_D$ is an ideal of $\mathcal{A}_{|D|}\big|_D$ we can consider the quotient Lie algebroid. In the co-orientable case this quotient Lie algebroid is described in [16, Equation 1.20].

Lemma 3.2.5. *Let $I_{|D|}$ be a smooth elliptic divisor with non-negative generator $f \in I_{|D|}$. Let $Q = O(2)ND$ denote the induced structure group reduction from Corollary 3.1.3. Then*

$$\begin{aligned}\psi : \mathcal{A}_{|D|}\big|_D / L_r &\to \mathrm{At}(Q)\\ [X] &\mapsto \varphi(X) - \frac{\varphi(X)(Q_f)}{Q_f}\mathcal{E}\end{aligned}$$

is an isomorphism of Lie algebroids, here $\varphi(X)$ is as in Lemma 3.2.2.

[2] That is, a vector field which vanishes along D and has linearisation the Euler vector field.

Proof. To check that this map is well-defined note that $\mathcal{E}(Q_f) = Q_f$, and thus $\psi([\mathcal{E}|_D]) = 0$. Moreover, using the description of $\mathrm{At}(Q)$ as in Lemma 3.5.4 we see that ψ has the desired target.

Let $X_1, X_2 \in \Gamma(\mathcal{A}_{|D|}\big|_D/L_r)$, with $\varphi(X_1) = \varphi(X_2)$. Let $g \in C^\infty(D)$ be any function, then $\varphi(X_1)(p^*g) = p^*X_1(g)$ and similarly for X_2. As this holds for all functions we conclude that $X_1 = X_2$, and thus that ψ is injective. Let $Y \in \Gamma(\mathrm{At}(Q))$, then $X = Y|_D \in \Gamma(\mathcal{A}_{|D|}\big|_D)$ and $Y = \psi([X])$, and thus ψ is surjective. Proving that ψ is bracket-preserving is a straightforward, but slightly tedious, computation and thus omitted. □

Because L_r is an abelian ideal of $\mathcal{A}_{|D|}\big|_D$ we may apply Proposition 3.2.1 to obtain a residue map $\Omega^\bullet(\mathcal{A}_{|D|}\big|_D) \to \Omega^{\bullet-1}(\mathrm{At}(O(2)ND); L_r^*)$. By Lemma 1.1.25 $\Omega^1(\mathcal{A}_{|D|})$ is locally generated by closed forms and therefore Proposition 3.2.1 implies that the residue map is a cochain morphism. After pre-composing with the restriction to D to obtain the the **radial residue map**:

$$\mathrm{Res}_r : \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-1}(\mathrm{At}(O(2)ND); L_r^*) = \Omega^{\bullet-1}(\mathrm{At}(O(2)ND)),$$

where we used that L_r^* is trivial as an $\mathrm{At}(O(2)ND)$-representation. Note that this residue map depends on the choice of representative $f \in I_{|D|}$, but any two different choices result in isomorphic Atiyah algebroids. In the local coordinates $(x_1, y_1, w_3, \ldots, w_n)$ of Lemma 1.1.25 the radial residue map is given by:

$$\mathrm{Res}_r(\alpha) = (\iota_{r_1\partial_{r_1}}\alpha)|_D, \qquad \alpha \in \Omega^\bullet(\mathcal{A}_{|D|}).$$

3.2.2 Residues for self-crossing elliptic divisors

We will now describe residue maps for the self-crossing elliptic tangent bundle. As in the case for smooth elliptic divisors, the (local) Euler vector fields generate an abelian ideal:

Lemma 3.2.6. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n. Then there exists a canonical abelian ideal subalgebroid L_r of $\mathcal{A}_{|D|}\big|_{D[n]}$. In local coordinates of Lemma 1.1.25 we have*

$$\Gamma(L_r) \simeq \left\langle r_1\partial_{r_1}|_{D[n]}, \ldots, r_n\partial_{r_n}|_{D[n]} \right\rangle.$$

Proof. Let $x \in D[n]$, and let $I_{|D_1|}, \ldots, I_{|D_n|}$ be a local normal crossing divisor on a neighbourhood U of x and let $\rho_i : \mathcal{A}_{|D_i|} \to TU$ be the corresponding elliptic tangent bundles. By transverality of the loci D_i we obtain the following splitting:

$$\ker \rho|_{D[n]\cap U} \simeq \ker \rho_1|_{D[n]\cap U} \oplus \cdots \oplus \ker \rho_n|_{D[n]\cap U}.$$

By Lemma 3.2.4 there exist canonical vector subbundles L_{r_i} of $\ker \rho_i|_{D(n)}$ with generators $\mathcal{E}_{D_i}$. Define $L_r|_U$ as the direct sum of these vector subbundles. Although each of the summands L_{r_i} is not globally defined and neither is their order, L_r does define a global vector bundle. Alternatively, we can define L_r as the sheaf of $C^\infty(D[n])$-modules which assigns to any open V of $D[n]$ such that $D[n] = D_1 \cap \cdots \cap D_n$ the module

$$\{ X|_{D[n]} \in \Gamma(V, \mathcal{A}_{|D|}|_{D[n]}) : \rho(X) \text{ is Euler-like for some } D_i\}.$$

Finally, in the local coordinates of Lemma 1.1.25 we have that generators of this module are of the form $r_i\partial_{r_i}$ as desired. □

Because L_r is an ideal we can consider the quotient Lie algebroid. As in the smooth case, we can again describe this as the Atiyah algebroid of a principal bundle, although this description is slightly more involved:

Lemma 3.2.7. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n. There is an $(O(2)^n \rtimes S_n)$-principal bundle Q such that $\mathrm{At}(Q) = \mathcal{A}_{|D|}|_{D[n]}/L_r$.*

Proof. Let $P = (O(2) \times \mathbb{R}_+)^n \rtimes S_n$ be the structure group reduction from Proposition 3.1.2. As locally the sections of L_r are generated by the Euler vector fields of the normal bundles to the local loci D_i we have that L_r generates an $(\mathbb{R}_+)^n$-action on P. If $x \in M$ and $(u_1, \ldots, u_n) \in P_x$ is a frame with u_i a frame of N_xD_i, then this action is given by $(\lambda_1, \ldots, \lambda_n) \cdot (u_1, \ldots, u_n) = (\lambda_1 u_1, \ldots, \lambda_n u_n)$. We can consider the quotient bundle $Q := P/(\mathbb{R}_+)^n$, which is an $(O(2)^n \rtimes S_n)$-principal bundle. Consequently $\mathcal{A}_{|D|}|_{D[n]}/L_r = \mathrm{At}(Q)$ which finishes the proof. □

As in the smooth case we apply Proposition 3.2.1 to obtain a residue map, which we compose with the restriction to $D[n]$ to obtain the **total radial residue map**:

$$\mathrm{Res}_r^n : \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-n}(\mathrm{At}(Q); \det(L_r^*)),$$

In the local coordinates of Lemma 1.1.25 we have that the total radial residue map is given by

$$\mathrm{Res}_r^n(\alpha) = (\iota_{r_1\partial_{r_1}\wedge\cdots\wedge r_n\partial_{r_n}}\alpha)|_{D[n]}.$$

When the elliptic divisor is a global normal crossing divisor the situation simplifies significantly:

Lemma 3.2.8. *Let $I_{|D|} = I_{|D_1|} \cdot \ldots \cdot I_{|D_n|}$ be a global normal crossing elliptic divisor. Let $Q = O(2)^n ND[n]$ denote the structure group reduction from Corollary 3.1.3. Then $\mathcal{A}_{|D|}|_{D[n]}/L_r \simeq \mathrm{At}(Q)$.*

Proof. Let $f_1, \ldots, f_n$ be non-negative generators of the smooth elliptic ideals. Let $L_r = L_{r_1} \oplus \cdots \oplus L_{r_n}$ be the ideal subalgebroid from Lemma 3.2.6 and let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ denote the Euler-vector fields of $ND_1, \ldots, ND_n$ respectively. Let $f_1, \ldots, f_n$ be non-negative generators of $I_{|D_1|}, \ldots, I_{|D_n|}$. Then

$$\mathcal{A}_{|D|}|_{D[n]}/L_r \to \mathrm{At}(Q),$$
$$[X] \mapsto \varphi(X) - \frac{\mathcal{L}_X(Q_{f_1})}{Q_{f_1}}\mathcal{E}_1 - \ldots - \frac{\mathcal{L}_X(Q_{f_n})}{Q_{f_n}}\mathcal{E}_n,$$

is the desired isomorphism, with φ as in Lemma 3.2.2. □

In this case $\det(L_r^*)$ is trivial as an $\mathrm{At}(O(2)^n)$-representation and we conclude that the total radial residue map becomes a map:

$$\mathrm{Res}_r^n : \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-n}(\mathrm{At}(O(2)^n ND[n])).$$

3.3 Cohomology of the elliptic tangent bundle

Using the total radial residue map introduced in the previous section, we will compute the Lie algebroid cohomology of the elliptic tangent bundle.

Let $I_{|D|}$ be an elliptic divisor of pointwise multiplicity n, in this section we will write $\mathcal{A}^n_{|D|}$ to keep track of this number and to simpify notion we write:

$$\Omega^\bullet_{\text{res}}(\mathcal{A}^n_{|D|}) := \Omega^\bullet(\mathrm{At}(Q)); \det(L_r^*)),$$

and denote its cohomology by $H^\bullet_{\text{res}}(\mathcal{A}^n_{|D|})$.

Recall that if $i \leq n$, then the restriction of $I_{|D|}$ to $M\backslash D(i+1)$ defines an elliptic divisor of pointwise multiplicity i, which we denote by $I_{|D\backslash D(i+1)|}$. Therefore the inclusion $\iota_i : M\backslash D(i+1) \hookrightarrow M$ induces a cochain map:

$$\iota_i^* : \Omega^\bullet(\mathcal{A}^n_{|D|}) \to \Omega^\bullet(\mathcal{A}^i_{|D\backslash D(i)|}).$$

Consequently, we can consider the radial residue map for the divisor $I_{|D\backslash D(i+1)|}$ and the composition:

$$\mathrm{Res}_r^i \circ \iota_i^* : \Omega^\bullet(\mathcal{A}^n_{|D|}) \to \Omega^{\bullet-i}_{\text{res}}(\mathcal{A}^i_{|D\backslash D(i+1)|}),$$

for each $1 \leq i \leq n-1$. Combining these maps will give us the desired isomorphism in computing the cohomology of the elliptic tangent bundle:

Theorem 3.3.1. *Let $I_{|D|}$ be an elliptic divisor with pointwise multiplicity n. Then*

$$H^k(\mathcal{A}^n_{|D|}) \to H^k(M\backslash D) \oplus \bigoplus_{i=1}^n H^{k-i}_{\text{res}}(\mathcal{A}^i_{|D\backslash D(i+1)|}) \tag{3.3.1}$$
$$[\alpha] \mapsto (\iota_0^*[\alpha], \mathrm{Res}_r^1 \circ \iota_1^*[\alpha], \ldots, \mathrm{Res}_r^{n-1} \circ \iota_{n-1}^*[\alpha], \mathrm{Res}_r^n[\alpha])$$

is an isomorphism.

Proof. The argument uses the following observation, used by Grothendieck in [40, Theorem 2] to do a similar computation for the complex of meromorphic forms with poles along a divisor. This observation was also used in [16, Theorem 1.3]. Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$ be soft sheaves of cochain complexes (e.g. Lie algebroid differential forms), and let $\varphi : \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet$ be a cochain morphism. Let $\mathcal{H}_{\mathcal{F}_i} : U \to H^\bullet(\mathcal{F}_i^\bullet(U))$ denote the cohomology presheaves and assume that φ induces a presheaf isomorphism $\mathcal{H}_{\mathcal{F}_1} \to \mathcal{H}_{\mathcal{F}_2}$. Then φ in fact induces a global isomorphism $H^\bullet(\mathcal{F}_1^\bullet(M)) \simeq H^\bullet(\mathcal{F}_2^\bullet(M))$. The proof of this observation goes by passing through the Cech-de Rham complex $\check{C}^p(M, \mathcal{F}_i^q)$, and using that by softness of $\mathcal{F}_i^\bullet$ the cohomology of this complex vanishes whenever $p > 0$.

We will apply this observation with $\mathcal{F}_1^\bullet = \Omega^\bullet(\mathcal{A}_{|D|})$ the sheaf of elliptic differential forms and $\mathcal{F}_2^\bullet = (\iota_0)_*\Omega^\bullet(M\backslash D) \oplus \bigoplus_{i=1}^n (\iota_j)_*\Omega_{\text{res}}^{\bullet-j}(\mathcal{A}^i_{|D\backslash D(i+1)|})$, where we push-forward the sheaves using the inclusion maps ι_j to sheaves on M.

Consequently, we are left to show that Equation 3.3.1 induces an isomorphism on the cohomology presheaves. This can be achieved by proving that it induces an isomorphism on the level of stalks. Therefore, for every $x \in M$ we are left to study the local picture, and may make use of the local model of elliptic divisors.

Let $x \in D[l]$ and let U be a neighbourhood of x as in Lemma 1.1.25, that is $U \cap D[l+1] = \emptyset$, $U = \mathbb{R}^{2l} \times \mathbb{R}^m$ and $I_{|D|}$ is the standard elliptic divisor of pointwise multiplicity l; $I_{|D_1|} \cdot \ldots \cdot I_{|D_l|}$.

Below we will implicitly push-forward sheaves using the inclusion maps ι_j whenever required. The left hand side of equation (3.3.1) can be easily described as the following free algebra:

$$H^\bullet(U, \mathcal{A}^n_{|D|}) = \langle 1, d\log r_1, \ldots, d\log r_l, d\theta_1, \ldots, d\theta_l \rangle .$$

Now let $i \leq l$. By Lemma 3.2.8 $\Omega^\bullet_{\text{res}}(\mathcal{A}^i_{|D\backslash D(i+1)|}) = \Omega^\bullet(\text{At}(O(2)^i ND[i]))$. Because $D[i]$ is orientable, Lemma 3.5.6 furthermore implies $\Omega^\bullet_{\text{res}}(\mathcal{A}^i_{|D\backslash D(i+1)|}) = \Omega^\bullet(\text{At}(U(1)^i ND[i]))$. Recall that the latter can be viewed as $U(1)^i$-invariant differential forms on $\text{tot}(U(1)^i ND[i])$, whose cohomology is given by $H^\bullet_{\text{dR}}(U(1)^i ND[i])$. We will thus view $\text{Res}^i_r \circ \iota_i^*$ as a map into the latter cohomology.

First, let us consider the map ι_0^*. By an elementary argument $U\backslash D$ is homotopic to $U(1)^l$, and ι_0^* sends the forms α, with $\alpha \in \langle 1, d\theta_1, \ldots, d\theta_l \rangle$ precisely to the generators of $H^\bullet(U(1)^l)$, proving that ι_0^* is surjective. For $i > l$, $\iota_i^* : H^\bullet(U, \mathcal{A}^n_{|D|}) \to H^\bullet_{\text{res}}(U, \mathcal{A}^i_{|D\backslash D(i+1)|})$ is the zero map as $\mathcal{A}^i_{|D\backslash D(i+1)|} = TU$ on U and U is contractible.

We will prove that all maps $\iota_i^* \circ \text{Res}^i_r$ with $1 \leq i \leq l$ are surjective and have disjoint kernels.

First we need to establish the homotopy type of $D[i]$; remark that

$$D[i] = \dot{\bigcup_{(j_1,\ldots,j_i)}} (D_{j_1} \cap \cdots \cap D_{j_i})\backslash D(i+1),$$

where the sum runs over all multi-indices of length i. By an elementary argument, $(D_{j_1} \cap \cdots \cap D_{j_i})\backslash D(i+1)$ is homotopic to $U(1)^{l-i}$. Moreover, $ND[i]$ decomposes as a direct sum, and thus so does the associated $U(1)^i$-bundle:

$$U(1)^i ND[i] = \bigoplus_{(j_1,\ldots,j_i)} S^1 ND_{j_1}|_{D[i]} \oplus \cdots \oplus S^1 ND_{j_i}|_{D[i]}$$

Because each D_{j_k} is contractible, the above is a sum of trivial circle bundles, and thus $U(1)^i ND[i] = U(1)^i \times D[i]$. Combining this with the description of the homotopy type of $D[i]$ we conclude that:

$$H^\bullet_{\text{res}}(U, \mathcal{A}^i_{|D\backslash D(i+1)|}) \simeq \bigoplus_{(j_1,\ldots,j_i)} H^\bullet(U(1)^l). \tag{3.3.2}$$

Given $\alpha \in \Omega^k(\mathcal{A}^i_{|D\backslash D(i+1)|})$, then $\mathrm{Res}^i_r(\alpha)$ is only non-zero if α contains a term of the form $d\log r_{j_1} \wedge \cdots \wedge d\log r_{j_i} \wedge \beta$, with $\beta \in \langle 1, d\theta_1, \ldots d\theta_l\rangle$. Moreover, Res^i_r takes forms of this form precisely to the generators of the $H^\bullet(T^n)$ in (3.3.2) corresponding to the multi-index $(j_1, \ldots, j_i)$. This shows that $\iota_i^* \circ \mathrm{Res}^i_r$ is surjective, and that the kernels of these maps for different values of i are all disjoint. We conclude that $(\iota_0^*, \iota_1^* \circ \mathrm{Res}^1_r, \ldots, \iota^*_{n-1} \circ \mathrm{Res}^{n-1}_r, \mathrm{Res}^n_r)$ is an isomorphism. □

When the elliptic divisor is a co-orientable normal crossing divisor the cohomology becomes easier to describe:

Corollary 3.3.2. *If $I_{|D|}$ is a global normal crossing elliptic divisor for which $D[1]$ is co-orientable. Then*

$$H^k(\mathcal{A}_{|D|}) \simeq H^k(M\backslash D) \oplus \bigoplus_{i=1}^n H^{k-i}(\mathrm{tot}(U(1)^i N(D[i]))).$$

Proof. By Corollary 3.1.3 $\mathrm{Fr}(ND[i])$ admits a $U(1)^i$-structure group reduction, and using Lemma 3.5.6 we see that $\Omega^\bullet_{\mathrm{Res}}(\mathcal{A}^i_{|D\backslash D(i+1)|}) = \Omega^\bullet(\mathrm{At}(U(1)^i ND[i]))$. Therefore by Theorem 3.3.1 we obtain that the cohomology is isomorphic to

$$H^k(\mathcal{A}_{|D|}) \simeq H^k(M\backslash D) \oplus \bigoplus_{i=1}^n H^{k-i}(\mathrm{At}(U(1)^i N(D[i])))$$

Recall that we can view elements of $\Omega(\mathrm{At}(U(1)^i N(D[i])))$ precisely as the $U(1)^i$-invariant differential forms on $\mathrm{tot}(U(1)^i N(D[i]))$. Because $U(1)^i$ is compact and connected the cohomology of $U(1)^i$-invariant forms on $\mathrm{tot}(U(1)^i N(D[i]))$ coincides the the ordinary de Rham cohomology of $\mathrm{tot}(U(1)^i N(D[i]))$, and therefore we arrive at the required result. □

In applications we are often interested in the case when the manifold is four-dimensional, and hence $D[2]$ consists of a collection of points. In this case the cohomology also simplifies for general divisors:

Corollary 3.3.3. *Let $I_{|D|}$ be an elliptic divisor on M^4 for which $D[1]$ is co-orientable. Then*

$$H^k(\mathcal{A}_{|D|}) \simeq H^k(M\backslash D) \oplus H^{k-1}(\mathrm{tot}(S^1 ND[1])) \oplus H^{k-2}(U(1)^2)^{(\#D[2])}.$$

3.4 Examples and outlook

In this section we give some explicit examples of elliptic divisors and compute their associated cohomology. The following example plays an important role in constructing examples of stable generalized complex structures in [18, Example 2.9, 2.24]:

Example 3.4.1. Let $M = \mathbb{C}P^2$ and let D be a normal crossings divisor consisting of three lines intersecting transversely, for instance $D = \{z_0 z_1 z_2 = 0\}$. The associated holomorphic line bundle with section (L, σ) is a complex log divisor, and let (R, q) be the associated elliptic divisor. We will now compute the cohomology of the associated elliptic tangent bundle:

- The complement $M \backslash D = (\mathbb{C}^*)^2$ and thus homotopic to T^2.
- The stratum $D[1]$ consists of three disjoint cylinders, and therefore every complex line bundle on it is trivial. Consequently $S^1 N D[1]$ is homotopic to three disjoint tori.
- The stratum $D[2]$ consists of three points.

Corollary 3.3.3 now gives us the following description of the elliptic cohomology:

k	0	1	2	3	4	otherwise
$H^k(\mathcal{A}_{\lvert D \rvert})$	$\mathbb{R}$	$\mathbb{R}^5$	$\mathbb{R}^{10}$	$\mathbb{R}^9$	$\mathbb{R}^3$	0

The comohology of the complex log tangent bundle is much simpler and given by:

k	0	1	2	otherwise
$H^k(\mathcal{A}_D)$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$	0

△

The following example plays an important role in the construction of codimension-one symplectic foliations in work in progress of Cavalcanti and Crainic:

Example 3.4.2. Let $M = \mathbb{C}P^2$ and let D be the smooth complex divisor given by the zero-set of a smooth cubic, for instance $D = \{z_0^3 + z_1^3 + z_2^3 = 0\}$. By the genus-degree formula we know that D is isomorphic to a torus. The cohomology of the complement can be computed using the long exact sequence associated to the inclusion $D \hookrightarrow M$:

$$0 \to \mathbb{R} \xrightarrow{\text{id}} \mathbb{R} \to H^0(\mathbb{C}P^2, D) \to 0 \to \mathbb{R}^2 \to$$
$$H^1(\mathbb{C}P^2, D) \to \mathbb{R} \xrightarrow{f} \mathbb{R} \to H^2(\mathbb{C}P^2, D) \to 0.$$

Because D is a complex submanifold of a Kähler manifold it is also a symplectic submanifold with respect to ω_{FS}. Therefore the map $H^2(\mathbb{C}P^2) \to H^2(D)$ is an isomorphism, and the map f becomes the identity, and we can easily read of the cohomology of $M\backslash D$ form the above sequence. To compute the cohomology of $P = S^1ND$ we employ the Thom-Gysin sequence, which takes the following form:

$$0 \to \mathbb{R} \to H^0(P) \to 0 \to \mathbb{R}^2 \to H^1(P) \to \mathbb{R} \overset{\cup e}{\to} \mathbb{R} \to H^2(P) \to \mathbb{R}^2 \to 0.$$

We have $ND = \mathcal{O}(3)|_D$, because $\mathcal{O}(3)$ is non-trivial and $H^2(\mathbb{C}P^2) \to H^2(D)$ is an isomorphism, we have that the Euler-class of P is non-trivial. Therefore the map $\cup e$ is an isomorphism, and we can easily read of the cohomology of P from the sequence. Corollary 3.3.3 now gives us the following description of the elliptic cohomology:

k	0	1	2	3	otherwise
$H^k(\mathcal{A}_{\vert D\vert})$	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^2$	$\mathbb{R}^2$	0

Again, this is quite different from the complex log tangent bundle:

k	0	1	otherwise
$H^k(\mathcal{A}_D)$	$\mathbb{C}$	$\mathbb{C}^2$	0

△

Example 3.4.3. Let $f : M^4 \to \Sigma^2$ be a Lefschetz fibration with singular value p and corresponding singular fibre $F = f^{-1}(p)$. Let (E, s) be the complex divisor on Σ corresponding to the divisor $\{p\} \subset \Sigma$, and define a self-crossing complex log divisor on M by (f^*E, f^*s). This divisor has as vanishing locus precisely the singular fibre F, and $D[2]$ consist precisely of the critical points on F.

Let n be the number of critical points of F. Then $D[1]$ consist of a disjoint union of n cylinders, and consequently $S^1ND[1]$ is homotopic to n tori.

Corollary 3.3.3 now gives us the following description of the elliptic cohomology:

$H^k(\mathcal{A}_{|D|}) = H^k(M\backslash F) \oplus$

k	1	2	3	4	otherwise
	$\mathbb{R}^n$	$\mathbb{R}^{3n}$	$\mathbb{R}^{3n}$	$\mathbb{R}^n$	0

△

3.4.1 Outlook: Deformation theory

We end by given an outlook on studying the deformation theory of self-crossing stable generalized complex structures and self-crossing elliptic symplectic structures.

Already the deformation theory of all smooth elliptic symplectic structures is more difficult than that of smooth stable generalized complex structures. The reason is that smooth complex log divisors are very rigid: if one deforms a complex function which vanishes transversely a bit, than one again obtains a function which vanishes transversely. This is used in [16, Lemma 1.16] to show that any two smooth complex log divisors which are deformations of each other are in fact isomorphic. Therefore, deformations of smooth stable generalized complex structures can be reduced to deformations on a fixed Lie algebroid, simplifying the discussion significantly. In the end, this ensures that deformations of smooth stable generalized complex structures are completely controlled by the cohomology of the Lie algebroid.

Smooth elliptic divisors are much less rigid than smooth complex log divisors. If $I_{|D|}$ is a smooth elliptic ideal with non-negative generator f then for all $\epsilon > 0$, the function $f + \varepsilon$ will be nowhere vanishing. Therefore small deformations of elliptic divisors will not necessarily be isomorphic to the original divisor, which significantly complicates the deformation theory of general elliptic symplectic structures.

Allowing for self-crossings allows for even more flexibility, so that in this case also complex log divisors are no longer rigid. The reason is that the self-crossings can be smoothend. Take for instance the complex divisor $I_D = \langle z_1 z_2 \rangle$ on $\mathbb{C}^2$, which has as vanishing locus the coordinate hyperplanes. The divisor $I_{D_t} = \langle z_1 z_2 + t \rangle$ is a smooth complex log divisor for all $t \in [0, 1]$ and provides a deformation between I_D and a complex log divisor with embedded vanishing locus. This shows that for self-crossing complex log divisors, deformation equivalent divisors need not be isomorphic.

One might restrict the discussion to deformations for which the divisors are isomorphic. We have good reasons to believe that in that case the results of [16, Theorem 3.14] can be generalized and the deformations will be controlled by the cohomology. However, deformations as the one above are very important to the theory. The particular deformation I_{D_t} is used in [18, Theorem 5.12] to show that in four dimensions any self-crossing stable generalized complex structure can be deformed into one with embedded degeneracy locus.

Therefore it is most natural to study general deformations, which would thus combine the deformation theory of the Lie algebroid with that of the elliptic symplectic structures. To study this one will have to combine the cohomology governing the deformations of the Lie algebroid, which is the deformation cohomology of Crainic-Moerdijk ([66]) together with the Lie algebroid cohomology of the elliptic tangent bundle.

3.5 Atiyah algebroids

This section recalls some results on Atiyah algebroids needed in this thesis. These results are all classical and can be found in the literature, see for instance [23].

Definition 3.5.1. Let P be a principal G-bundle over M. The **Atiyah algebroid** of P, is defined as $\mathrm{At}(P) = TP/G$. The anchor is induced by the differential of the projection $P \to M$. The bracket is obtained by viewing sections of $\mathrm{At}(P)$ as G-invariant vector fields on P. ♢

The Atiyah algebroid is a transitive Lie algebroid (i.e., the anchor maps is surjective), and defines a short exact sequence:

$$0 \to P \times_G \mathfrak{g} \to \mathrm{At}(P) \to TS \to 0, \tag{3.5.1}$$

called the Atiyah sequence.

If $\rho : A \to M$ is an arbitrary transitive Lie algebroid, then we call the short exact sequence

$$0 \to K \to A \xrightarrow{\rho} TS \to 0 \tag{3.5.2}$$

an **abstract Atiyah sequence.** We call it integrable if $A = \mathrm{At}(P)$ for a principal G-bundle P. Splittings of the Atiyah sequence play an important role:

Proposition 3.5.2. *Let $P \to M$ be a principal G-bundle with Atiyah sequence*

$$0 \to K \to \mathrm{At}(P) \to TM \to 0.$$

There is a one-to-one correspondence between:

- *Connection one-forms $\theta \in \Omega^1(P, \mathfrak{g})$ on P and*
- *Vector bundle splittings $\sigma : \mathrm{At}(P) \to K$ of the Atiyah sequence of $\mathrm{At}(P)$.*

Given a vector bundle E, we can consider its frame bundle $\mathrm{Fr}(E)$ and its associated Atiyah algebroid, which we will denote by $\mathrm{At}(E)$. We can describe its sections as fibre-wise linear vector fields:

Lemma 3.5.3. *Let $E \to M$ be a vector bundle. Then $\Gamma(\mathrm{At}(E)) \simeq \mathfrak{X}^1(E)_{\mathrm{lin}} = \{X \in \mathfrak{X}^1(E) : [X, \mathcal{E}] = 0\}$.*

Consequently, its Atiyah sequence is given by:

$$0 \to \mathrm{End}(E) \to \mathrm{At}(E) \to TM \to 0$$

Proposition 3.5.2 therefore tells us that splittings of this sequence are the same as linear connections on E.

We are mainly interested in the case of metrics, for which we have the following descriptions:

Lemma 3.5.4. *Given a metric g on a vector bundle E let Q denote the corresponding $O(n)$ structure group reduction of $\mathrm{Fr}(E)$. We have*

$$\Gamma(\mathrm{At}(Q)) \simeq \{\widetilde{X} \in \mathfrak{X}^1(E)_{\mathrm{lin}} : \mathcal{L}_{\widetilde{X}} g = 0\}.$$

Lemma 3.5.5. *Given a conformal class of metrics $[g]$ let Q denote the corresponding $O(n) \times \mathbb{R}_+$ structure group reduction of $\mathrm{Fr}(E)$. We have*

$$\Gamma(\mathrm{At}(Q)) \simeq \{\widetilde{X} \in \mathfrak{X}^1(E)_{\mathrm{lin}} : \mathcal{L}_{\widetilde{X}} g = \lambda g, \textit{ for some } \lambda \in C^\infty(M)\}.$$

The Atiyah algebroid cannot detect discrete parts of the structure group:

Lemma 3.5.6. *If P is a principal G-bundle and Q is a structure group reduction to H, and $\mathfrak{g} \simeq \mathfrak{h}$ then $\mathrm{At}(P) \simeq \mathrm{At}(Q)$.*

Chapter 4

Homogeneous multivector fields

Introduction

In the coming chapters we study multivector fields, and in particular Poisson structures, on vector bundles. We will study those multivector fields which are compatible with the vector bundle structure in the sense that they are fibrewise polynomial. Although these objects are simple to define, the space of polynomial multivector fields has a very rich algebraic structure. We will study this algebraic structure in this chapter, and indicate how to use it to study normal form results for Poisson structures, as we will do in the next chapter. In this introduction we will gather the main definitions and results of this chapter.

In the first section we start by introducing the necessary terminology.

Throughout, we denote by $E \to M$ a real vector bundle with finite rank r and let E^* denote its dual. We will describe multivector fields on E^* by how they act on polynomial functions. Recall that fibrewise homogeneous polynomial functions $f \in C^\infty(E^*)$ of degree d can be described as sections $\Gamma(S^d E)$; $\Gamma(E)$ in particular describes fibrewise linear functions.

For $X \in \mathfrak{X}^{p+1}(E^*)$, one can restrict the associated multi-derivation $\mathcal{L}_X$ acting on $C^\infty(E^*)$ to fibrewise linear functions:

$$\mathcal{L}_X : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1)-\text{times}} \to C^\infty(E^*).$$

This restriction determines the multivector field X uniquely and we will refer to $\mathcal{L}_X$ as the **core** of X. We will give an abstract definition of such objects:

Definition 4.2.3. *We define $\mathscr{V}^p(E)$ to be the space of skew-symmetric maps*

$$D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1)-\text{times}} \to C^\infty(E^*),$$

with the property that there exists a map, called the ***symbol*** *of* D*,*

$$\sigma_D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{p-\text{times}} \to \Gamma(\pi^* TM),$$

such that for all $s_0, \ldots, s_p \in \Gamma(E)$ *and* $f \in C^\infty(M)$

$$D(s_0, \ldots, s_{p-1}, f s_p) = f D(s_0, \ldots, s_{p-1}, s_p) + s_p \cdot \langle \sigma_D(s_0, \ldots, s_{p-1}), df \rangle \, .$$

Here $s_p \cdot$ *denotes the product as smooth functions on* E^* *and* $\pi^* TM$ *is the pullback to* E^* *via the projection* $\pi : E^* \to M$*.*

Therefore we conclude:

Lemma 4.2.4. *The map sending* X *to its core* $\mathcal{L}_X$ *is a linear isomorphism:*

$$\mathfrak{X}^{p+1}(E^*) \cong \mathscr{V}^p(E).$$

We are in particular interested in studying vector fields that are compatible with the vector bundle structure: We call a vector field $X \in \mathfrak{X}^{p+1}(E^*)$ **homogeneous of degree** d if it satisfies $m_\lambda^* X = \lambda^{d-p-1} X$ and denote the space of such vector fields by $\mathfrak{X}^{p+1}(E^*)_{d-\text{pol}}$. To describe these vector fields in terms of their cores, let $\mathscr{V}^p(E)_{d-\text{pol}}$ denote the subset of $\mathscr{V}^p(E)$ consisting of elements that map into $\Gamma(S^d E) \subset C^\infty(E^*)$.

Lemma 4.2.13. *The isomorphism from Lemma 4.2.4 restricts to a linear isomorphism:*

$$\mathfrak{X}^{p+1}(E^*)_{d-\text{pol}} \cong \mathscr{V}^p(E)_{d-\text{pol}}.$$

The symbols of elements of $\mathscr{V}^p(E)$ again have symbols of their own. In fact, we define the spaces $\mathscr{V}^p_l(E)$ consisting of multilinear maps

$$D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l+1)-\text{times}} \to \Gamma(\pi^* \Lambda^l TM)$$

with the property that there exists a map

$$\sigma_D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l)-\text{times}} \to \Gamma(\pi^* \Lambda^{l+1} TM)$$

satisfying the Leibniz type identity (4.2.4). Note that $\mathscr{V}^p_0(E) = \mathscr{V}^p(E)$. We will show that if $D \in \mathscr{V}^p_l(E)$, then $\sigma_D \in \mathscr{V}^p_{l+1}(E)$, and thus we obtain an entire sequence:

$$\mathscr{V}^p_0(E) \xrightarrow{\sigma} \mathscr{V}^p_1(E) \xrightarrow{\sigma} \mathscr{V}^p_2(E) \xrightarrow{\sigma} \ldots, \tag{4.2.5}$$

We will call this sequence **the symbol tower**. Similarly we will define polynomial analogues, $\mathscr{V}^p_l(E)_{d-\text{pol}}$, and also obtain a sequence:

$$\mathscr{V}^p_0(E)_{d-\text{pol}} \xrightarrow{\sigma} \mathscr{V}^p_1(E)_{d-\text{pol}} \xrightarrow{\sigma} \mathscr{V}^p_2(E)_{d-\text{pol}} \xrightarrow{\sigma} \ldots \tag{4.2.7}$$

Using the isomorphisms from Lemma 4.2.4 and 4.2.13 we endow $\mathscr{V}^\bullet(E)$ and $\mathscr{V}^\bullet(E)_{\bullet-\mathrm{pol}}$ with the structure of a Gerstenhaber algebra. The towers (4.2.5) and (4.2.7) are part of a filtration on this Gerstenhaber structure. We will describe this filtration in detail in Section 4.2.5.

Given a connection ∇ on E^* one can use the induces isomorphism $TE^* \simeq \pi^*(E \oplus TM)$ to obtain a description of $\mathscr{V}_l^p(E)$. This description fits into the Gerstenhaber picture, by saying that it leads to a splitting of the sequences (4.2.5) and (4.2.7), and therefore:

Corollary 4.2.49. *A connection on E induces an isomorphism*

$$\mathscr{V}_l^p(E) \simeq \bigoplus_{i=0}^{p-l+1} \Gamma(\Lambda^{p-l+1-i}E^* \otimes \pi^*\Lambda^{l+i}TM).$$

If we endow $\Gamma(\Lambda^\bullet E^* \otimes \pi^*\Lambda^\bullet TM)$ with the associative algebra structure induced by the wedge products, this isomorphism induces an isomorphism of associative algebras

$$\mathscr{V}_\bullet^\bullet(E) \simeq \Gamma(\Lambda^\bullet E^* \otimes \pi^*\Lambda^\bullet TM).$$

And in the polynomial case:

Corollary 4.2.51. *Each $\mathscr{V}_l^p(E)_{d-\mathrm{pol}}$ is the space of sections of a vector bundle over M, $\mathbb{V}_l^p(E)_{d-\mathrm{pol}}$, the symbol map is part of a short exact sequences of vector bundles*

$$0 \to \Lambda^{p-l+1}E^* \otimes S^{d-l}E \otimes \Lambda^l TM \longrightarrow \mathbb{V}_l^p(E)_{d-\mathrm{pol}} \xrightarrow{\sigma} \mathbb{V}_{l+1}^p(E)_{d-\mathrm{pol}} \to 0.$$

Moreover, any connection ∇ on E gives rise to an isomorphism of vector bundles

$$\mathbb{V}_l^p(E)_{d-\mathrm{pol}} \simeq \bigoplus_{i=0}^{p-l+1} \left(\Lambda^{p-l+1-i}E^* \otimes S^{d-l-i}E \otimes \Lambda^{l+i}TM\right).$$

If we endow $\Gamma(\Lambda^\bullet E^* \otimes S^\bullet E \otimes \Lambda^\bullet TM)$ with the associative algebra structure obtained from combining the wedge products and symmetric products, we obtain an isomorphism of associate algebras:

$$\mathscr{V}_\bullet^\bullet(E)_{\bullet-\mathrm{pol}} \simeq \Gamma(\Lambda^\bullet E^* \otimes S^\bullet E \otimes \Lambda^\bullet TM).$$

In the case when $E = A$ is a Lie algebroid, we furthermore describe a relation with representations up to homotopy:

Theorem 4.3.7. *Let $A \to M$ be a Lie algebroid. A connection ∇ on A induces an isomorphism of differentially graded algebras:*

$$(\mathscr{V}^\bullet(A)_{d-\mathrm{pol}}, d_{\pi_1}) \simeq \Omega(A; S^d \operatorname{ad}_\nabla)^{p+1}.$$

Here $\pi_1 \in \mathfrak{X}^2(A^*)$ is the linear Poisson structure corresponding to the algebroid structure on A and $S^d \operatorname{ad}_\nabla$ is the symmetric power of the adjoint representation up to homotopy.

The idea is to use the isomorphism from Corollary 4.2.51 to decompose a homogeneous Poisson structure $\pi \in \mathfrak{X}^2(E^*)_{d-\mathrm{pol}}$ into pieces. This will shine some light upon certain normal form results.

4.1 Poisson geometry

In this section we recall some of the basic notions of Poisson geometry, foliation theory, and Lie algebroids needed in the coming chapters.

Definition 4.1.1. A **Poisson bracket** $\{\cdot,\cdot\}$ is a Lie bracket on $C^\infty(M)$ which acts by derivations. ♢

Spelled out, this means it is an $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which satisfies:

- Skew symmetry: $\{f, g\} = -\{g, f\}$
- Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$,
- Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,

for all $f, g, h \in C^\infty(M)$.

Equivalently it can be described using a bivector field:

Definition 4.1.2. A **Poisson structure** on M is a bivector field $\pi \in \mathfrak{X}^2(M)$ for which the Schouten-Nijenhuis bracket with itself vanishes: $[\pi, \pi] = 0$. ♢

The bracket and bivector are related through the condition $\{f, g\} = \pi(df, dg)$ for all $f, g \in C^\infty(M)$. A Poisson bracket allows one to associate to each function $f \in C^\infty(M)$ a vector field, called the **Hamiltonian vector field** X_f of f which is defined via the relation:

$$X_f(g) := \{f, g\} \quad \text{for all } g \in C^\infty(M).$$

Given a Poisson structure we have an associated map $\pi^\sharp : T^*M \to TM$, defined by the relation $\pi^\sharp_x(\alpha_x)(\beta_x) = \pi_x(\alpha_x, \beta_x)$ for all $\alpha_x, \beta_x \in T^*_xM$. Note that $X_f = \pi^\sharp(df)$.

Example 4.1.3. For any manifold M, the zero-bivector $\pi = 0$ defines a Poisson structure $\pi = 0$. △

Example 4.1.4. Let M be a smooth manifold. A **symplectic structure** is a two-form $\omega \in \Omega^2(M)$ which is closed ($d\omega = 0$) and non-degenerate, that is the map $\omega^\flat : TM \to T^*M$ defined by $\omega^\flat(X)(Y) = \omega(X, Y)$ is an isomorphism.

Given a symplectic structure one defines a Poisson bracket via the relation

$$\{f, g\} := -\omega((\omega^\flat)^{-1}(df), (\omega^\flat)^{-1}(dg)).$$

Conversely, any Poisson structure for which $\pi^\sharp$ is an isomorphism is induced by a symplectic form ω with $\omega^\flat = (\pi^\sharp)^{-1}$. △

Poisson geometry is deeply related to *foliation theory*.

Definition 4.1.5. A **(regular) foliation** $\mathcal{F}$ on M^n is a partition of $M = \cup_x L_x$ into disjoint connected immersed submanifolds (called the **leaves**) such that for all $x \in M$ there is a neighbourhood U of x and coordinates $(x_1, \dots, x_n)$ on M such that

$$U \cap L_y = \{x_{k+1} = \text{const}, \dots, x_n = \text{const}\},$$

for all $y \in U$. The number k is called the **codimension** of the foliation. ♢

Although a regular foliation is a very geometric object, in practice it is easier to work with the infinitesimal counterpart, which are the tangent spaces to the leaves, $T\mathcal{F}$. These form a subbundle of TM, defined by $T_x\mathcal{F} = T_xL$, where L is the leaf passing through $x \in M$.

This subbundle has the following special property:

Definition 4.1.6. A subbundle $E \subset TM$ is called **involutive** if $[\Gamma(E), \Gamma(E)] \subset \Gamma(E)$. ♢

For any subbundle $E \subset TM$ a submanifold $S \subset M$ is called an **integral submanifold** if $T_xS = E_x$ for all $x \in S$. An integral submanifold is called **maximal** if it is not contained in any other integral submanifold.

Frobenius theorem ensures the existence of (maximal) integerable submanifolds for involutive subbundles of TM:

Theorem 4.1.7 (Frobenius Theorem)**.** *There is a one-to-one correspondence between regular foliations and involutive subbundles of TM:*

- *Given a foliation $\mathcal{F}$, the tangent spaces to its leaves $T\mathcal{F}$ form an involutive distribution.*
- *Given an involutive subbundle of TM, the maximal integral submanifolds form the leaves of a foliation.*

Associated to a regular foliation is the complex of leafwise differential forms:

$$\Omega^\bullet(\mathcal{F}) := \Gamma(\Lambda^\bullet T^*\mathcal{F}),$$

endowed with the leafwise de Rham differential $d_\mathcal{F}$:

$$\begin{aligned}(d_\mathcal{F}\alpha)(X_1, \dots, X_{k+1}) := &\sum_{i=1}^{k+1} (-1)^{i+1} \mathcal{L}_{X_i} \alpha(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}),\end{aligned}$$

for $\alpha \in \Omega^k(\mathcal{F})$ and $X_1, \dots, X_{k+1} \in \Gamma(T\mathcal{F})$.

If (M, π) is a **regular** Poisson structure, i.e. the rank of $\pi^\sharp_x$ does not depend on $x \in M$, then $\pi^\sharp(T^*M)$ defines an involutive subbundle. Therefore, by Frobenius' theorem it corresponds to a regular folation which will be denoted by $\mathcal{F}_\pi$. Locally, the distribution $\pi^\sharp(T^*M)$ is spanned by the Hamiltonian vector fields and therefore we can define a foliated two-form $\omega_{\mathcal{F}_\pi} \in \Omega^2(\mathcal{F}_\pi)$ via the relation:

$$\omega_{\mathcal{F}_\pi}(X_f, X_g) = -\{f, g\} \quad \text{for all } f, g \in C^\infty(M).$$

In this way, we arrive at the following definition:

Definition 4.1.8. A **(regular) symplectic foliation** on M is a foliation $\mathcal{F}$ together with a leafwise two-form $\omega_\mathcal{F} \in \Omega^2(\mathcal{F})$ which is leafwise closed, i.e. $d_\mathcal{F}\omega_\mathcal{F} = 0$, and leafwise non-degenerate, i.e. $\omega_\mathcal{F}^\flat : T_xL \to T_xL^*$ is an isomorphism for all leaves L of $\mathcal{F}$. ♢

For a Poisson manifold (M, π) the pair $(\mathcal{F}_\pi, \omega_{\mathcal{F}_\pi})$ indeed defines a symplectic foliation.

Proposition 4.1.9. *There is a one-to-one correspondence between regular Poisson structures and regular symplectic foliations.*

We will now recall some very special classes of regular Poisson structures:

Example 4.1.10. A **cosymplectic structure** on M^{2n+1} is a pair of closed forms $(\theta, \omega) \in \Omega^1_{\text{cl}}(M) \times \Omega^2_{\text{cl}}(M)$ satisfying $\theta \wedge \omega^n \neq 0$. In this case $\mathcal{F} := \ker\theta$ defines a foliation and $\omega|_\mathcal{F}$ a leafwise symplectic form. △

Definition 4.1.11. An **almost k-cosymplectic structure** on M^{k+2l} is a $(k+1)$-tuple $(\theta_1, \dots, \theta_k, \omega) \in \Omega^1(M) \times \cdots \Omega^1(M) \times \Omega^2(M)$ satisfying the relation $\theta_1 \wedge \cdots \theta_k \wedge \omega^l \neq 0$.

It is called **k-cosymplectic** if all forms are closed. ♢

Note that in particular a 1-cosymplectic structure is a cosymplectic structure. Also note that given a k-cosymplectic structure that $\mathcal{F} := \ker\theta_1 \cap \cdots \cap \ker\theta_k$ defines a foliation and $\omega|_\mathcal{F}$ a leafwise symplectic form.

Definition 4.1.12. An **almost k-Poisson structure** on M^{k+2l} is a $(k+1)$-tuple $(X_1, \dots, X_k, \pi) \in \mathfrak{X}^1(M) \times \cdots \times \mathfrak{X}^1(M) \times \mathfrak{X}^2(M)$ satisfying the relation $X_1 \wedge \cdots X_k \wedge \pi^l \neq 0$.

It is called **k-Poisson** if moreover π is Poisson, the X_i pairwise commute and are Poisson vector fields, i.e. $[X_i, \pi] = 0$. ♢

Just as non-degenerate Poisson structures correspond to symplectic forms, we can relate k-Poisson and k-cosymplectic structures:

Proposition 4.1.13 ([73])**.** *There is a one-to-one correspondence between almost k-Poisson structures and almost k-cosymplectic structures. This correspondence restricts to a correspondence between k-Poisson and k-cosymplectic structures.*

4.1.1 Poisson cohomology and contravariant connections

Every Poisson manifold has an associated cohomology theory:

Definition 4.1.14. Let (M, π) be a Poisson manifold. Then

$$d_\pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M) : \quad X \mapsto [\pi, X]$$

is a differential, i.e. $d_\pi^2 = 0$. The associated cohomology is denoted $H_\pi^\bullet(M)$ and called the **Poisson cohomology** of (M, π). One-cocycles, i.e. vector fields $X \in \mathfrak{X}^1(M)$ with $d_\pi(X) = 0$, are called **Poisson vector fields**. $\diamondsuit$

Let (M, π) be a Poisson manifold and let

$$\wedge^\bullet \pi^\sharp : \Lambda^\bullet T^*M \to \Lambda^\bullet TM$$

denote the unique algebra extension of $\pi^\sharp$, that is $\wedge^\bullet \pi^\sharp(\alpha_1 \wedge \cdots \alpha_k) = \pi^\sharp(\alpha_1) \wedge \cdots \wedge \pi^\sharp(\alpha_k)$.

Lemma 4.1.15. *Let (M, π) be a Poisson manifold. Then*

$$\wedge^\bullet \pi^\sharp : (\Omega^\bullet(M), d) \to (\mathfrak{X}^\bullet(M), d_\pi),$$

is a cochain map.

An important invariant of a Poisson manifold is the modular class: If (M, π) is a Poisson manifold and $\Omega \in \Omega^{\text{top}}(M)$ a volume form, then for any $f \in C^\infty(M)$ the divergence of X_f is defined by

$$\mathcal{L}_{X_f}(\Omega) = \text{div}(X_f)\Omega.$$

The association $f \mapsto \text{div}(X_f)$ is the **modular vector field** associated to Ω, and denoted X_{mod}. Although this vector field depends on the particular choice of volume form, the Poisson cohmology class, $[X_{\text{mod}}] \in H_\pi^1(M)$, does not and is called the **modular class** of (M, π).

Definition 4.1.16. Let (M, π) be a Poisson manifold and let $E \to M$ be a vector bundle. A **contravariant connection**[1] on E is an $\mathbb{R}$-bilinear map:

$$\begin{aligned} \nabla^\pi : \Gamma(T^*M) \times \Gamma(E) &\to \Gamma(E), \\ (\alpha, e) &\mapsto \nabla^\pi_\alpha(e), \end{aligned}$$

which is $C^\infty(M)$ linear in the first entry and satisfies the Leibinz identity in the second entry:

$$\nabla^\pi_\alpha(fe) = f\nabla^\pi_\alpha(e) + \mathcal{L}_{\pi^\sharp(\alpha)}(f) \cdot e,$$

for all $\alpha \in \Gamma(T^*M)$, $f \in C^\infty(M)$ and $e \in \Gamma(E)$. $\diamondsuit$

[1] Also known as Poisson connection.

Definition 4.1.17. Let ∇^π be a contravariant connection on E. Its **curvature** is the endomorphism valued vector field $R_{\nabla^\pi} \in \mathfrak{X}^2(M; \mathrm{End}(E))$ defined by the relation

$$R_{\nabla^\pi}(\alpha, \beta) = \nabla^\pi_\alpha(\nabla^\pi_\beta(e)) - \nabla^\pi_\beta(\nabla^\pi_\alpha(e)) - \nabla^\pi_{[\alpha,\beta]_\pi}(e)$$

for all $\alpha, \beta \in \Omega^1(M)$ and $e \in \Gamma(E)$. Here $[\alpha, \beta]_\pi$ is as in Equation 4.1.2. A contravariant connection is called **flat** if its curvature vanishes. ♢

Contravariant connections form an affine space:

Lemma 4.1.18. *Let (M, π) be a Poisson manifold and let $E \to M$ be a vector bundle. Let ∇^π and $(\nabla^\pi)'$ be two contravariant connections. Then their difference defines an endomorphism valued vector field $\nabla^\pi - (\nabla^\pi)' \in \mathfrak{X}^1(M; \mathrm{End}(E))$.*

Example 4.1.19. Let (M, π) be a Poisson manifold and let $E \to M$ be a vector bundle. Given an ordinary connection ∇ on E, one defines an induced contravariant connection via:

$$\overline{\nabla}^\pi_\alpha(e) = \nabla_{\pi^\sharp(\alpha)}(e).$$

If $R_\nabla \in \Omega^2(M; \mathrm{End}(E))$ is the curvature of ∇, then

$$R_{\overline{\nabla}^\pi}(\alpha, \beta) = R_\nabla(\pi^\sharp(\alpha), \pi^\sharp(\beta)), \quad \text{for all } \alpha, \beta \in \Omega^1(M). \qquad \triangle$$

Out of a contravariant connection ∇^π on a line bundle L we can define a Poisson vector field in the following way: First assume that L is trivialisable and let $e \in \Gamma(L)$ be a nowhere vanishing section. Then necessarily $\nabla_\alpha(e) = f_{\alpha,e} \cdot e$ for some function $f_{\alpha,e} \in C^\infty(M)$. It follows that the association $X_e : \alpha \mapsto f_{\alpha,e}$ defines a vector field. Moreover, if the contravariant connection is flat, then X_e will be d_π-closed. If $e' = ge$ is another non-vanishing section for $g \in C^\infty(M)$, then $X_e - X_{e'} = \pi^\sharp(d \log g)$ and we conclude that X_e defines a well-defined class $c(L, \nabla) := H^1_\pi(M)$.

When L is not trivialisable, $L \otimes L$ still is and one can endow it with the product contravariant connection:

$$\widetilde{\nabla}^\pi_\alpha(s_1 \otimes s_2) = \nabla^\pi_\alpha(s_1) \otimes s_2 + s_1 \otimes \nabla^\pi_\alpha(s_2)$$

and define $c(L, \nabla) = \frac{1}{2} c(L \otimes L, \widetilde{\nabla}^\pi) \in H^1_\pi(M)$.

Definition 4.1.20. Let $L \to M$ be a real line bundle and let ∇^π be a flat contravariant connection. The Poisson cohomology class $c(L, \nabla^\pi) \in H^1_\pi(M)$ is called the **characteristic class** of ∇^π. ♢

Example 4.1.21. Let (M, π) be a Poisson manifold. Then there is a canonical flat-connection on $\wedge^{\text{top}} T^*M$:

$$\nabla^\pi : \Gamma(T^*M) \times \Gamma(\wedge^{\text{top}} T^*M) \to \Gamma(\wedge^{\text{top}} T^*M) : \quad (\alpha, \Omega) \mapsto \mathcal{L}_{\pi^\sharp(\alpha)}\Omega + \langle d\alpha, \pi \rangle \, \Omega.$$

We denote $\wedge^{\text{top}} T^*M$ endowed with this flat connection with K^*_π.

When M is orientable, then the characteristic class of ∇^π is precisely the modular class of (M, π). We can use this flat connection also on non-orientable manifolds to define:

The **modular class** of a Poisson manifold is defined as

$$[X_{\text{mod}}] = c(\Lambda^{\text{top}} T^* M, \nabla^\pi). \qquad \triangle$$

For a regular Poisson manifold we can also consider a slightly weaker invariant. Consider the short exact sequence:

$$0 \to \nu^*_{\mathcal{F}_\pi} \hookrightarrow T^* M \stackrel{\pi^\sharp}{\to} \mathcal{F}_\pi \to 0, \tag{4.1.1}$$

where $\to \nu^*_{\mathcal{F}_\pi} = \ker \pi^\sharp$. Given a contravariant connection ∇^π on a line bundle $L \to M$, we can restrict the connection to obtain a map:

$$\Gamma(\nu^*_{\mathcal{F}_\pi}) \times \Gamma(L) \to \Gamma(L).$$

Because $\nabla^\pi_\alpha(fe) = f\nabla^\pi_\alpha(e) + L_{\rho(\alpha)}(f)e = f\nabla^\pi_\alpha(e)$ for all $\alpha \in \Gamma(\nu^*_{\mathcal{F}_\pi})$ we have that the restriction of ∇^π to $\Gamma(\nu^*_{\mathcal{F}_\pi})$ is $C^\infty(M)$-linear in both components, so it induces a vector bundle map

$$\begin{aligned} \nu^*_{\mathcal{F}_\pi} &\to \text{Hom}(L, L) \\ \alpha &\mapsto \nabla^\pi_\alpha. \end{aligned}$$

Because L is a line bundle, it can be identified with a section $\varphi \in \Gamma(\nu_{\mathcal{F}_\pi})$. This section is called the **isotropy character**[2] of the contravariant connection.

When $X \in \mathfrak{X}^1(M)$ is a representative of the modular class of a regular Poisson manifold (M, π) X mod $\mathcal{F}_\pi$ coincides with the isotropy character of the canonical contravariant connection on $\wedge^{\text{top}} T^* M$.

Remark 4.1.22. There is a slightly different way of defining the characteristic class of a flat contravariant connection ∇^π on an arbitrary line bundle. A first attempt is to see that, as L has rank one, $\text{End}(E) \simeq \underline{\mathbb{R}} \cdot \text{id}$ and thus by Lemma 4.1.18 the difference between any two flat contravariant connections is a d_π-closed vector field. Because L is rank one there exists a flat connection ∇ on L. Consider the vector field $X_\nabla \in \mathfrak{X}^1(M)$ defined by the relation $X_\nabla \otimes \text{id} := \nabla^\pi - \overline{\nabla}^\pi$.

If we had chosen another flat connection ∇', then we would see that $\nabla - \nabla' = \alpha \in \Omega^1(M)$, for some closed one-form α. Therefore, $X_\nabla - X_{\nabla'} = -\pi^\sharp(\alpha)$ and we get a well-defined object in $\Gamma(\nu_{\mathcal{F}})$. This is precisely the isotropy character of the contravariant connection.

To get the characteristic class, we have to restrict the class of connections we work with. We do this in the following way: Let $g \in \Gamma(\text{Sym}^2 L^*)$ be a metric and let ∇^g be the associated unique (flat) connection compatible with g. For the (unique up to sign) local orthonormal frame e of L we have $\nabla^g_X(fe) = X(f)e$. If g' is another orthonormal metric and $g' = hg$ for some nowhere vanishing function $h \in C^\infty(M)$, then $\nabla^g - \nabla^{g'} = d\log h \otimes \text{id}$.

[2] Also known as normal/transverse Higgs field.

Consequently, if we only use connections compatible with a metric to construct the vector field X_∇, we see that we do get a well-defined class in $H^1_\pi(M)$. This class indeed coincides with the characteristic class of ∇^π. ♢

Lie algebroids

Definition 4.1.23. A **Lie algebroid** $\mathcal{A}$ on M consists of a vector bundle $\mathcal{A} \to M$, a vector bundle map $\rho : \Gamma(\mathcal{A}) \to TM$, called the **anchor**, and a Lie bracket on the space of sections $\Gamma(\mathcal{A})$ satisfying the Leibniz identity:

$$[v, fw] = f[v, w] + \mathcal{L}_{\rho(v)}(f)w,$$

for all $v, w \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$. ♢

Example 4.1.24. If $\mathcal{F}$ is a foliation on M, then $T\mathcal{F}$ is involutive and the space of section $\Gamma(T\mathcal{F})$ thus inherits the Lie bracket of vector fields. Together the the inclusion $T\mathcal{F} \hookrightarrow TM$ this defines a Lie algebroid structure on $T\mathcal{F}$. △

Lie algebroids play an important role in Poisson geometry:

Example 4.1.25. Let (M, π) be a Poisson manifold. Then T^*M naturally inherits the structure of a Lie algebroid as follows: The anchor is defined to be $\pi^\sharp : T^*M \to TM$ and the bracket is defined as follows:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta), \tag{4.1.2}$$

for all $\alpha, \beta \in \Omega^1(M)$. This Lie algebroid is referred to as the **cotangent Lie algebroid**. △

Given a Lie algebroid $\mathcal{A}$, one can consider the Lie algebroid forms $\Omega^\bullet(\mathcal{A}) := \Gamma(\wedge^\bullet \mathcal{A})$. The Lie bracket determines a differential on $\Omega^\bullet(\mathcal{A})$ using the Koszul formula: Given $\alpha \in \Omega^k(\mathcal{A})$ define $d_{\mathcal{A}}\alpha \in \Omega^{k+1}(\mathcal{A})$ by:

$$\begin{aligned} d_{\mathcal{A}}(\alpha)(v_1, \ldots, v_{k+1}) = &\sum_{i=1}^{k+1} (-1)^i \mathcal{L}_{\rho(v_i)} \alpha(v_1, \ldots, \widehat{v}_i, \ldots, v_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([v_i, v_j], \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_{k+1}). \end{aligned}$$

Example 4.1.26. Let $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ be the cotangent Lie algebroid. Then the Lie algebroid complex $(\Omega^\bullet(T^*M), d_{T^*M})$ coincides precisely with $(\mathfrak{X}^\bullet(M), d_\pi)$. △

Definition 4.1.27. Let $\mathcal{A} \to M$ be a Lie algebroid and $E \to M$ a vector bundle. An $\mathcal{A}$-**connection** on E is an $\mathbb{R}$-bilinear map:

$$\Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E) : \quad (v, e) \mapsto \nabla_v(e),$$

which is $C^\infty(M)$-linear in the first entry and satisfies the Leibiniz identity in the second entry:

$$\nabla_v(fe) = f\nabla_v(e) + \mathcal{L}_{\rho(v)}(f) \cdot e,$$

for all $v \in \Gamma(\mathcal{A})$, $f \in C^\infty(M)$ and $e \in \Gamma(E)$.

Its **curvature** is the endomorphism valued two-form $R_\nabla \in \Omega^2(\mathcal{A}; \mathrm{End}(E))$ defined by the relation

$$R_\nabla(s_1, s_2) = \nabla_{s_1}(\nabla_{s_2}(e)) - \nabla_{s_2}(\nabla_{s_1}(e)) - \nabla_{[s_1,s_2]}(e)$$

for all $X, Y \in \Gamma(\mathcal{A})$ and $e \in \Gamma(E)$. ♢

Example 4.1.28. The tangent bundle is naturally a Lie algebroid, with anchor the identity map and Lie bracket the bracket of vector fields. TM-connections are the same as ordinary connections. For a Poisson manifold (M, π), T^*M-connections are the same as contravariant connections. △

4.1.2 Cohomology classes for symplectic foliations

We recall some cohomology classes for symplectic foliations which are used in this thesis. Throughout let $(\mathcal{F}, \omega_\mathcal{F})$ be a symplectic foliation and let $\Omega^\bullet_\mathcal{F}(M)$ denote the kernel of the restriction map $\iota^* : \Omega^\bullet(M) \to \Omega^\bullet(\mathcal{F})$ of forms defined on M to $\mathcal{F}$. We have a short exact sequence of cochain complexes:

$$0 \to \Omega^\bullet_\mathcal{F}(M) \to \Omega^\bullet(M) \to \Omega^\bullet(\mathcal{F}) \to 0$$

Here $\Omega^\bullet(\mathcal{F})$ is endowed with the foliated de Rham differential and $\Omega^\bullet(\mathcal{F}, \nu^*)$ is endowed with a differential induced by the Bott-connection. There are a couple of maps relating these complexes. The first is

$$p : \Omega^{k+1}_\mathcal{F}(M) \to \Omega^k(\mathcal{F}, \nu^*),$$
$$p(\alpha)(X_1, \ldots, X_k)(\overline{N}) = \alpha(X_1, \ldots, X_k, N),$$

where $X_1, \ldots, X_k \in \Gamma(T\mathcal{F})$ and for $N \in \mathfrak{X}(M)$, $\overline{N}$ denotes the corresponding section of the normal bundle. Another map, which we simply denote by d, is given by:

$$d : H^k(\mathcal{F}) \to H^{k+1}_\mathcal{F}(M)$$
$$[\alpha] \mapsto [d\widetilde{\alpha}],$$

where $\widetilde{\alpha} \in \Omega^\bullet(M)$ is any extension of α. Combining these two we obtain the variation map $d_\nu = p \circ d : H^k(\mathcal{F}) \to H^k(\mathcal{F}, \nu^*)$.

Definition 4.1.29. Let $(\mathcal{F}, \omega_\mathcal{F})$ be a symplectic foliation. The **symplectic variation** of $(\mathcal{F}, \omega_\mathcal{F})$ is the class $d_\nu[\omega_\mathcal{F}] \in H^2(\mathcal{F}, \nu^*)$. ♢

This class is used frequently in the literature. However from its construction we see that we could also have defined the following class:

Definition 4.1.30. Let $(\mathcal{F}, \omega_{\mathcal{F}})$ be a symplectic foliation. The class $d[\omega_{\mathcal{F}}] \in H^3_{\mathcal{F}}(M)$ is called the **characteristic class of** $(\mathcal{F}, \omega_{\mathcal{F}})$ and denoted $\mathcal{C}(\mathcal{F}, \omega_{\mathcal{F}})$. $\diamondsuit$

Another map relating the above complexes is given by:

$$q : \Omega^k(M, \nu^*) \to \Omega^{k+1}_{\mathcal{F}}(M),$$
$$q(\alpha)(X_1, \ldots, X_{k+1}) = \left\langle \alpha(X_1, \ldots, X_k), \overline{X}_{k+1} \right\rangle + \text{cycl. perm.}$$

A straightforward check shows that the following diagram commutes:

$$\begin{array}{ccc} \Omega^{k+1}_{\mathcal{F}}(M) & \xrightarrow{\ p\ } & \Omega^k(\mathcal{F}, \nu^*) \\ {\scriptstyle q}\uparrow & \nearrow_{\text{rest}} & \\ \Omega^k(M, \nu^*) & & \end{array}$$

Concluding, we obtain the following commutative diagram in cohomology:

$$\begin{array}{ccc} H^k(\mathcal{F}) & & \\ \downarrow{\scriptstyle d} & \searrow^{d_\nu} & \\ H^{k+1}_{\mathcal{F}}(M) & \xrightarrow{\ p\ } & H^k(\mathcal{F}, \nu^*) \\ {\scriptstyle q}\uparrow & \nearrow_{\text{rest}} & \\ H^k(M, \nu^*) & & \end{array}$$

When the normal bundle to the foliation is trivial as a representation, the map q can be expressed explicitly:

Lemma 4.1.31. *Let $\alpha \in \Omega^k(M, \nu^*)$ and suppose that $\nu^* = \langle \theta_1, \ldots, \theta_n \rangle$ for closed one-forms $\theta_1, \ldots, \theta_n$. Then if α corresponds to $(\alpha_1, \ldots, \alpha_n) \in \Omega^k(M, \mathbb{R}^n)$ we have*

$$q(\alpha) = \alpha_1 \wedge \theta_1 + \cdots + \alpha_n \wedge \theta_n.$$

4.2 Multivector fields on vector bundles

4.2.1 Multivector fields on manifolds

We will recall the Gerstenhaber algebra structure on the multivector fields. Let $\mathfrak{X}^k(N)$ denotes the space of k-multivector fields on N:

$$\mathfrak{X}^k(N) := \Gamma(\Lambda^k TN).$$

The wedge product

$$\mathfrak{X}^k(N) \times \mathfrak{X}^l(N) \to \mathfrak{X}^{k+l}(N) \quad (X_1, X_2) \mapsto X_1 \wedge X_2$$

endows $\mathfrak{X}^\bullet(N)$ with the structure of a graded algebra.

Moreover, the Schouten-Nijenhuis bracket

$$[\cdot,\cdot]_{SN} : \mathfrak{X}^{p+1}(N) \times \mathfrak{X}^{q+1}(N) \to \mathfrak{X}^{p+q+1}(N),$$

endows $\mathfrak{X}^\bullet(N)$, up to a shift, with the structure of a graded Lie algebra. The wedge product and Schouten-Nijenhuis bracket are compatible with each other, and axiomatising their interaction gives rise to the notion of Gerstenhaber algebra.

To describe the Schouten-Nijenhuis bracket and wedge product explicitly it is useful to describe multivector fields as multi-deriviations – every $X \in \mathfrak{X}^{p+1}(N)$ gives rise to a multilinear skew-symmetric map

$$\mathcal{L}_X : \underbrace{C^\infty(N) \times \cdots \times C^\infty(N)}_{(p+1)-\text{times}} \to C^\infty(N), \tag{4.2.1}$$
$$\mathcal{L}_X(f_0, \ldots, f_p) = \langle df_0 \wedge \ldots \wedge df_p, X\rangle.$$

Conversely, any $(p+1)$-multilinear skew-symmetric map (4.2.1) which is a derivation in each argument arises in this way.

Definition 4.2.1. The **Schouten-Nijenhuis bracket** of the multivector fields $X \in \mathfrak{X}^{p+1}(N)$ and $Y \in \mathfrak{X}^{q+1}(N)$ is the unique multivector field $[X, Y] \in \mathfrak{X}^{p+q+1}(N)$ satisfying:

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - (-1)^{pq}\mathcal{L}_Y \circ \mathcal{L}_X, \tag{4.2.2}$$

where

$$\mathcal{L}_X \circ \mathcal{L}_Y(f_0, \ldots, f_{p+q}) :=$$
$$= \sum_{\sigma \in S_{p,q+1}} (-1)^{|\sigma|}\mathcal{L}_X(f_{\sigma(0)}, \ldots, f_{\sigma(p-1)}, \mathcal{L}_Y(f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q)}));$$

with sum over all $(p, q+1)$-shuffles. ♢

Definition 4.2.2. The wedge product of the multivector fields $X \in \mathfrak{X}^p(N)$ and $Y \in \mathfrak{X}^q(N)$ is the unique multivector field $X \wedge Y \in \mathfrak{X}^{p+q}(N)$ satisfying:

$$\mathcal{L}_{X\wedge Y}(f_1, \ldots, f_{p+q}) = \sum_{\sigma \in S_{p,q}} \text{sign}(\sigma)\mathcal{L}_X(f_{\sigma(1)}, \ldots, f_{\sigma(p)})\mathcal{L}_Y(f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q)});$$

with sum over all (p, q)-shuffles. ♢

4.2.2 Cores of multivector fields

We will now proceed to study multivector fields on vector bundles. The fibrewise linear structure allows us to talk about fibrewise linear functions, and we can restrict the derivation associated to a multivector field to these functions. This will define an object which captures the multivector field completely, which we call its *core*. We will make an independent study of these cores, and exhibit a rich algebraic structure on the space of cores.

Throughout, let $E \to M$ denote a real vector bundle with finite rank r. Fibrewise linear functions on E^* correspond to sections of E via the inclusion

$$\Gamma(E) \subset C^\infty(E^*).$$

In particular, for $X \in \mathfrak{X}^{p+1}(E^*)$, one can restrict the derivation $\mathcal{L}_X$ from (4.2.1) to $\Gamma(E) \subset C^\infty(E^*)$; the resulting map is

$$L_X : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1)-\text{times}} \to C^\infty(E^*). \tag{4.2.3}$$

This restriction determines the multivector field X uniquely and we will refer to $\mathcal{L}_X$ as the **core** of X. We give an abstract definition of such objects:

Definition 4.2.3. We define $\mathscr{V}^p(E)$ to be the space of skew-symmetric maps

$$D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1)-\text{times}} \to C^\infty(E^*),$$

with the property that there exists a map, called the **symbol** of D,

$$\sigma_D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{p-\text{times}} \to \Gamma(\pi^* TM),$$

such that

$$D(s_0, \dots, s_{p-1}, f s_p) = f D(s_0, \dots, s_{p-1}, s_p) + s_p \cdot \langle \sigma_D(s_0, \dots, s_{p-1}), df \rangle .$$

Here "$s_p \cdot$" denotes the product as smooth functions on E^* and $\pi^* TM$ is the pullback to E^* via the projection $\pi : E^* \to M$. ♢

Where the core is defined as the restriction of $\mathcal{L}_X$ to fibrewise linear functions, we could also have restricted $\mathcal{L}_X$ to some fibrewise constant functions, i.e. those arising from the inclusion

$$\pi^* : C^\infty(M) \hookrightarrow C^\infty(E^*).$$

The symbol of the core of $\mathcal{L}_X$, corresponds precisely to the restriction of $\mathcal{L}_X$ to

$$\underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{p-\text{times}} \times C^\infty(M).$$

Therefore we conclude:

Lemma 4.2.4. *The map sending X to its core $\mathcal{L}_X$ (4.2.3) is a linear isomorphism*

$$\mathfrak{X}^{p+1}(E^*) \cong \mathscr{V}^p(E).$$

Restring more components to fibrewise constant entries gives rise to "higher order" symbols of $\mathcal{L}_X$. This can be done purely algebraic, starting from the definition of $\mathscr{V}^p(E)$:

Lemma 4.2.5. *Let $E \to M$ be a vector bundle, and let $D \in \mathscr{V}^p(E)$. Its first symbol σ_D is unique, and there is a sequence $(D, \sigma_D^1, \sigma_D^2, \ldots, \sigma_D^k)$ of skew-symmetric maps,*

$$
\begin{aligned}
\sigma_D^0 = D : \underbrace{\Gamma(E) \times \Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1)-\text{times}} &\to C^\infty(E^*), \\
\sigma_D^1 = \sigma_D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{p-\text{times}} &\to \Gamma(\pi^* TM), \\
\sigma_D^2 : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p-1)-\text{times}} &\to \Gamma(\pi^* \Lambda^2 TM), \\
&\cdots, \\
\sigma_D^p : \Gamma(E) &\to \Gamma(\pi^* \Lambda^p TM), \\
\sigma_D^{p+1} &\in \Gamma(\pi^* \Lambda^{p+1} TM),
\end{aligned}
$$

each two consecutive ones being related by the Leibniz identity:

$$
\begin{aligned}
\sigma_D^l(s_0, \ldots, s_{p-l-1}, f s_{p-l}) = {} & f \sigma_D^l(s_0, \ldots, s_{p-l-1}, s_{p-l}) \\
& + s_{p-l} \cdot i_{df} \left(\sigma_D^{l+1}(s_0, \ldots, s_{p-l-1}) \right).
\end{aligned} \tag{4.2.4}
$$

Proving this lemma can be done inductively by introducing the spaces $\mathscr{V}_l^p(E)$ consisting of multilinear maps

$$D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l+1)-\text{times}} \to \Gamma(\pi^* \Lambda^l TM)$$

with the property that there exists a map

$$\sigma_D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l)-\text{times}} \to \Gamma(\pi^* \Lambda^{l+1} TM)$$

satisfying the Leibniz type identity (4.2.4). Note that $\mathscr{V}_0^p(E) = \mathscr{V}^p(E)$. Lemma 4.2.5 now follows by applying the following statement inductively:

Lemma 4.2.6. *If $D \in \mathscr{V}_l^p(E)$, then $\sigma_D \in \mathscr{V}_{l+1}^p(E)$.*

Proof. To prove this statement we have to show that there exists a map σ_D^2 such that σ_D also satisfies the Leibniz type identity (4.2.4). This follows from a straightforward, but careful, application of the Leibniz identity for D twice, in both possible orders, to the expression

$$D(s_0, \ldots, s_{p-l-2}, f s_{p-l-1}, g s_{p-l}).$$ □

We conclude that there is an entire sequence:

$$\mathscr{V}_0^p(E) \xrightarrow{\sigma} \mathscr{V}_1^p(E) \xrightarrow{\sigma} \mathscr{V}_2^p(E) \xrightarrow{\sigma} \dots, \tag{4.2.5}$$

which will be called **the symbol tower**. When we want to view σ as a map, we write for $D \in \mathscr{V}_l^p(E)$ that $\sigma(D) := \sigma_D \in \mathscr{V}_{l+1}^p(E)$. One may even want to think of elements $D \in \mathscr{V}^p(E)$ as entire sequences

$$(\sigma_D^0, \sigma_D^1, \dots, \sigma_D^{p+1}),$$

starting with D and σ_D, although each element determines the next one.

Note that the isomorphism from Lemma 4.2.4 does restrict to the intermediate spaces as follows:

Lemma 4.2.7. *There is a linear isomorphism*

$$\mathscr{V}_l^p(E) \simeq \mathfrak{X}^{p+1-l}(E^*; \pi^*\Lambda^l TM).$$

4.2.3 Polynomial mutivector fields

We want to describe multivector fields (on vector bundles) which are fibrewise polynomial. These can be described as sums of multivector fields which are homogeneous with respect to the fibrewise multiplication by scalars $m_\lambda : E^* \to E^*$. One can use the core of a multivector field to describe this behaviour very nicely. Ample examples will be given towards the end of the section.

The rescaling operators m_λ allow us to classify those functions on E^* which are fibrewise polynomial:

Definition 4.2.8. The graded algebra of **fibrewise homogeneous polynomials** $\mathrm{Pol}^\bullet(E^*) \subset C^\infty(E^*)$, consists of those functions $p \in \mathrm{Pol}^d(E^*)$ with the property that $m_\lambda^*(p) = \lambda^d p$. ♢

Lemma 4.2.9. *The association which sends $p \in \Gamma(S^d E)$ to the polynomial of degree d*

$$\widehat{p} : E^* \to \mathbb{R}, \quad e \mapsto p(e, \dots, e).$$

defines an isomorphism of graded algebras $\Gamma(S^\bullet E) \simeq \mathrm{Pol}^\bullet(E^*)$.

Remark 4.2.10. The infinitesimal counterpart of the rescaling operators m_λ is the Euler vector field $\mathcal{E} \in \mathfrak{X}(E^*)$ whose flow at time t is m_{e^t}:

$$\mathcal{E}_v := \left.\frac{d}{dt}\right|_{t=0} e^t v \in T_v E^*.$$

Or, with respect to coordinates (x, y) for E^* arising from coordinates x on the base M and y on the fiber induced by a trivialization of the vector bundle E^*, one has

$$\mathcal{E} = \sum_{i=1}^r y_i \frac{\partial}{\partial y_i}.$$

Because the flow of the Euler vector field is the rescaling operator, we can describe Definition 4.2.8 infinitesimally:

$$g \in \mathrm{Pol}^d(E^*) \quad \Longleftrightarrow \quad \mathcal{L}_{\mathcal{E}}(g) = d \cdot g.$$

Lemma 4.2.11. *Given $X \in \mathfrak{X}^{p+1}(E^*)$ and $d \geq 0$ integer, the following are equivalent:*

(i) $m_\lambda^* X = \lambda^{d-p-1} X$.

(ii) $[\mathcal{E}, X] = (d-p-1) \cdot X$.

(iii) The core (4.2.3) of X takes values in the space of fiberwise polynomial function of degree d.

A multivector field satisfying the conditions of the above lemma is called **homogeneous polynomial** of degree d and the space of such multivector fields is denoted by $\mathfrak{X}^{p+1}(E^*)_{d-\mathrm{pol}}$. We give the abstract definition of the cores of these multivector fields:

Definition 4.2.12. Let $E \to M$ be a vector bundle. We denote by $\mathscr{V}^p(E)_{d-\mathrm{pol}}$ the space of skew-symmetric multilinear maps

$$D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(p+1) \text{ times}} \to \Gamma(S^d E)$$

with the property that there exists a skew-symmetric map

$$\sigma_D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{p \text{ times}} \to \Gamma(TM \otimes S^{d-1}E),$$

satisfying

$$D(s_0, \ldots, s_{p-1}, f s_p) = f \cdot D(s_0, \ldots, s_{p-1}, s_p) + s_p \cdot \langle \sigma_D(s_0, \ldots, s_{p-1}), df \rangle . \quad (4.2.6)$$

♢

The following is now easy to remark:

Lemma 4.2.13. *The isomorphism from Lemma 4.2.4 restricts to a linear isomorphism*

$$\mathfrak{X}^{p+1}(E^*)_{d-\mathrm{pol}} \cong \mathscr{V}^p(E)_{d-\mathrm{pol}}.$$

And, as before, we define polynomial versions of the intermediate spaces: $\mathscr{V}_l^p(E)_{d-\mathrm{pol}}$ consists of the skew-symmetric multilinear maps

$$D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l+1)-\text{times}} \to \Gamma(\Lambda^l TM \otimes S^{d-l}E).$$

with the property that there exists a skew-symmetric multilinear map

$$\sigma_D : \underbrace{\Gamma(E) \times \Gamma(E) \times \cdots \times \Gamma(E)}_{(p-l)-\text{times}} \to \Gamma(\Lambda^{l+1} TM \otimes S^{d-l-1}E).$$

satisfying Leibniz-type condition (4.2.6). Lemma 4.2.6 now immediately implies the following:

Lemma 4.2.14. *$\mathscr{V}^p(E)_{d-\mathrm{pol}}$ coincides with the subspace of $\mathscr{V}^p(E)$ consisting of those D taking values in $\Gamma(S^dE) \subset C^\infty(E^*)$. Furthermore, for $D \in \mathscr{V}^p(E)_{d-\mathrm{pol}}$ there is an entire sequence $(\sigma_D = \sigma_D^1, \sigma_D^2, \ldots, \sigma_D^k)$, with $\sigma_D^i \in \mathscr{V}_i^p(E)_{d-\mathrm{pol}}$.*

Therefore we also obtain a polynomial version of the symbol tower (4.2.5):

$$\mathscr{V}_0^p(E)_{d-\mathrm{pol}} \xrightarrow{\sigma} \mathscr{V}_1^p(E)_{d-\mathrm{pol}} \xrightarrow{\sigma} \mathscr{V}_2^p(E)_{d-\mathrm{pol}} \xrightarrow{\sigma} \ldots, \tag{4.2.7}$$

We can also describe the intermediate spaces again:

Lemma 4.2.15. *The isomorphism from Lemma 4.2.7 restricts to an isomorphism*

$$\mathscr{V}_l^p(E)_{d-\mathrm{pol}} \simeq \mathfrak{X}^{p+1-l}(E^*; \Lambda^l TM)_{(d-l)-\mathrm{pol}}.$$

Here $\mathfrak{X}^{p+1-l}(E^*; \Lambda^l TM)_{(d-l)-\mathrm{pol}}$ consist of linear combinations of elements $\vartheta \otimes X$, with $\vartheta \in \mathfrak{X}^{p+1-l}(E^*)_{(d-l)-\mathrm{pol}}$ and $X \in \wedge^l TM$. We end this section with some important and instructive examples:

Example 4.2.16 (0-polynomial)**.** When $d = 0$ all symbols necessarily vanish and we have

$$\mathscr{V}^p(E)_{0-\mathrm{pol}} = \Gamma(\Lambda^{p+1}E^*).$$

For any $\xi_x \in E_x^*$ one has a short exact sequence of vector spaces

$$0 \to E_x^* \xrightarrow{\iota_{\xi_x}} T_{\xi_x}E^* \longrightarrow T_xM \to 0;$$

where the first map is given by

$$\iota_{\xi_x}(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\xi_x + tv_x).$$

and the second map is the differential of the projection $E^* \to M$. The map ι induces an isomorphism

$$\iota_* : \Gamma(\Lambda^\bullet E^*) \to \mathfrak{X}^\bullet(E^*)_{0-\mathrm{pol}},$$

inverse to the isomorphism in Lemma 4.2.13. △

Example 4.2.17 (1-polynomial vector fields)**.** $\mathscr{V}^0(E)_{1-\mathrm{pol}}$ consists of derivations. That is, maps $D : \Gamma(E) \to \Gamma(E)$ together with a vector field X on M such that:

$$D(fs) = fD(s) + \mathcal{L}_X(f)s$$

for all $s \in \Gamma(E)$, $f \in C^\infty(M)$. We see that Lemma 4.2.13 defines an isomorphism between derivations on E and fibrewise linear vector fields $\mathfrak{X}^1(E^*)_{1-\mathrm{pol}}$. Moreover, if $A \in \mathrm{End}(E^*)$, then we can consider the vector field $a(A) \in \mathfrak{X}^1(E^*)$ given by

$$a(A)_{\xi_x} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\xi_x + tA(\xi_x)),$$

for $\xi_x \in E^*_x$. This provides a map $a : \mathrm{End}(E^*) \to \mathfrak{X}^1(E^*)_{1-\mathrm{pol}}$ and its image corresponds precisely to the vector fields for which the core has zero symbol. We can also describe this construction dually. Let $B \in \mathrm{End}(E)$, and consider the core $\widetilde{a}(B) \in \mathscr{V}^0(E)_{1-\mathrm{pol}}$ defined by

$$\widetilde{a}(B)(s) = B(s),$$

for $s \in \Gamma(E)$. Of course, the two are related: The core corresponding to $a(A)$ is precisely $\widetilde{a}(A^*)$. △

Example 4.2.18 (1-polynomial bivector fields and Lie algebroids)**.** Pre-Lie algebroid brackets $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ (i.e., not necessarily satisfying Jacobi) correspond precisely to elements of $\mathscr{V}^1(E)_{1-\mathrm{pol}}$. The symbol of $[\cdot,\cdot]$ is precisely precisely the anchor map $\rho : \Gamma(E) \to \Gamma(TM)$ (to be continued in Example 4.2.24). △

Example 4.2.19 (1-polynomial multivector fields)**.** For general multivector fields the space

$$\mathscr{V}^p(E)_{1-\mathrm{pol}} = \mathrm{Der}^p(E),$$

is studied in [66] under the name of multi-derivations. △

Example 4.2.20 (Contravariant connections + Warning!)**.** The space $\mathscr{V}^1_1(E)_{2-\mathrm{pol}}$ consists of maps

$$D : \Gamma(E) \to \Gamma(E \otimes TM),$$

such that there exists a $Q \in \mathfrak{X}^2(M)$ such that

$$D(fs) = fD(s) + Q^\sharp(df) \otimes s.$$

Viewing D instead as a map from $\Gamma(E) \times \Gamma(T^*M)$ to $\Gamma(E)$, we see that it almost defines a contravariant connection in the sense of Definition 4.1.16, except that the bivector Q need not be Poisson.

Moreover, there is an unfortunate clash in conventions. If ∇^π is a contravariant connection, then the corresponding element of $\mathscr{V}^1_1(E)$ will satisfy

$$D(fs) = fD(s) - \pi^\sharp(df) \otimes s,$$

and consequently its symbol is $-\pi$. △

4.2.4 Schouten bracket and wedge product

In this section we will explicitly describe the Gerstenhaber algebra structure on $\mathscr{V}^p(E)$ and $\mathscr{V}^p(E)_{d-\mathrm{pol}}$, and in particular the interaction of the Schouten-Nijenhuis bracket and wedge product with the (higher order) symbols.

The wedge product can be transported via the isomorphism from Lemma 4.2.4 to a product structure

$$\wedge : \mathscr{V}^p(E) \otimes \mathscr{V}^q(E) \to \mathscr{V}^{p+q+1}(E) \tag{4.2.8}$$

$$\wedge : \mathscr{V}^p(E)_{d-\mathrm{pol}} \otimes \mathscr{V}^q(E)_{d'-\mathrm{pol}} \to \mathscr{V}^{p+q+1}(E)_{(d+d')-\mathrm{pol}}.$$

Explicitly $(D_1 \wedge D_2)(s_0, \ldots, s_{p+q+1})$ is defined as:

$$\sum_{\tau \in S_{p+1,q-1}} \operatorname{sign}(\tau) D_1(s_{\tau(0)}, \ldots, s_{\tau(p)}) \cdot D_2(s_{\tau(p+1)}, \ldots, s_{\tau(p+q+1)}),$$

where the sum runs over all $(p+1, q+1)$-shuffles.

The Schouten-Nijenhuis bracket can be transported via the isomorphism from Lemma 4.2.4 to a bracket

$$[\cdot,\cdot] : \mathscr{V}^p(E) \otimes \mathscr{V}^q(E) \longrightarrow \mathscr{V}^{p+q}(E). \tag{4.2.9}$$

And using the characterization of polynomiality given by (i) of Lemma 4.2.11, and the naturallity of the rescaling operator, also to a polynomial bracket:

$$[\cdot,\cdot] : \mathscr{V}^p(E)_{d-\mathrm{pol}} \times \mathscr{V}^q(E)_{d'-\mathrm{pol}} \to \mathscr{V}^{p+q}(E)_{(d+d'-1)-\mathrm{pol}}.$$

We can also give an explicit description of this bracket, similar to Definition 4.2.1:

Definition 4.2.21. Let $D_1 \in \mathscr{V}^p(E)$, $D_2 \in \mathscr{V}^q(E)$, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{pq} D_2 \circ D_1,$$

where $(D_1 \circ D_2)(s_0, \ldots, s_{p+q})$ is given by

$$\sum_{\tau \in S_{p,q+1}} \operatorname{sign}(\tau) D_1(s_{\tau(0)}, \ldots, s_{\tau(p-1)}, D_2(s_{\tau(p)}, \ldots, s_{\tau(p+q)})), \tag{4.2.10}$$

with the sum taken over all $(p, q+1)$-shuffles. ♢

Remark 4.2.22 (Warning!)**.** In the expression (4.2.10) we used that D_1 can be applied to arbitrary functions on E. One should read the expression as

$$L_{X_1}(s_{\tau(0)}, \ldots, s_{\tau(p-1)}, D_2(s_{\tau(p)}, \ldots, s_{\tau(p+q)})).$$

When describing the polynomial bracket we can make this explicit without passing to vector fields: Let $D_1 \in \mathscr{V}^p(E)_{d-\mathrm{pol}}, D_2 \in \mathscr{V}^q(E)_{d'-\mathrm{pol}}$, and define

$$\widetilde{D}_1 : \Gamma(E) \times \cdots \times \Gamma(E) \times \Gamma(S^{d'} E) \to \Gamma(S^{d+d'-1} E),$$

by extending D_1 as an algebra morphism in the last slot. In this case the expression in (4.2.10) should read $\widetilde{D_1}(s_{\tau(0)}, \ldots, s_{\tau(p-1)}, D_2(s_{\tau(p)}, \ldots, s_{\tau(p+q)}))$. But we have to be careful when $d' = 0$, because then we are dealing with the symbol of D rather than D itself. In that case the expression in (4.2.10) should read

$$\sigma_{\widetilde{D}_1}(s_{\tau(0)}, \ldots, s_{\tau(p-1)}, D_2(s_{\tau(p)}, \ldots, s_{\tau(p+q)})). \quad ♢$$

Example 4.2.23. The resulting Lie bracket on $\mathscr{V}^\bullet(E)_{0-\mathrm{pol}} \cong \Gamma(\Lambda^{\bullet+1} E^*)$ (see Example 4.2.16) vanishes. On $\mathscr{V}^0(E)_{1-\mathrm{pol}}$ (see Example 4.2.17) the bracket is the ordinary commutator bracket. For $d = 1$ and p arbitrary (see Example 4.2.19) this is the bracket introduced directly in [66]. △

Example 4.2.24. We continue the discussion from Example 4.2.18 on pre-algebroid structures $[\cdot,\cdot]$ on E interpreted as elements $D \in \mathscr{V}^1(E)_{1-\mathrm{pol}}$. Computing $[D, D]$ one finds precisely half the Jacobiator

$$\mathrm{Jac}_{[\cdot,\cdot]} : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E).$$

Hence the Jacobiator becomes an element in $\mathscr{V}^2(E)_{1-\mathrm{pol}}$ with symbol

$$\begin{aligned} &\Gamma(E) \times \Gamma(E) \to \Gamma(TM) \\ &\qquad (s_0, s_1) \mapsto \rho([s_0, s_1]) - [\rho(s_0), \rho(s_1)]. \end{aligned}$$

Therefore the Jacobi identity (and consequently the fact that the anchor is bracket preserving) is equivalent to the Poisson condition on the corresponding bivector on E^*. △

The previous example culminates into the well-known result:

Corollary 4.2.25. *There is a one-to-one correspondence between Poisson structures $\pi \in \mathfrak{X}^2(E^*)_{1-\mathrm{pol}}$ on E^* and Lie algebroid structures on E.*

Example 4.2.26. When M is a point $E = V$ is a finite-dimensional vector space, hence

$$\mathscr{V}^p(V)_{d-\mathrm{pol}} = \Lambda^{p+1}V^* \otimes S^d V.$$

The resulting bracket is uniquely determined by its value on $S_1, S_2 \in \wedge^1 V^*$, $P_1, P_2 \in S^1 V$, given by:

$$\begin{aligned} [S_1, S_2] &= [P_1, P_2] = 0, \\ [S_1 \otimes P_1, S_2 \otimes P_2] &= \langle S_1, P_2\rangle\, P_1 \otimes S_2 - \langle P_1, S_2\rangle\, P_2 \otimes S_1. \end{aligned}$$

△

Example 4.2.27. For a general vector bundle $E \to M$, elements $D \in \mathscr{V}^p(E)_{d-\mathrm{pol}}$ which are $C^\infty(M)$-multilinear, i.e. satisfy $\sigma_D = 0$, define a subspace which is given by

$$\Gamma(\Lambda^{p+1}E^* \otimes S^d E) \subset \mathscr{V}^p(E)_{d-\mathrm{pol}},$$

which is closed under the bracket. This bracket can be described by the same formulae as in Example 4.2.26. △

Symbols of the bracket and wedge product We will now describe the interaction with the bracket (4.2.9) and the (higher) order symbols. This is somewhat subtle because there are no natural brackets on the spaces $\mathscr{V}^p_l(E)$, in which the symbols live. We will elaborate on this in Remark 4.2.35. Nevertheless, in this section we will prove:

Proposition 4.2.28. *For any $D_1 \in \mathscr{V}^{p_1}(E)$, $D_2 \in \mathscr{V}^{p_2}(E)$ the symbol of $[D_1, D_2]$ is given by*

$$\sigma_{[D_1,D_2]} = [D_1, \sigma_{D_2}] + (-1)^{p_1}[\sigma_{D_1}, D_2], \tag{4.2.11}$$

where the right hand side is defined below.

Moreover, for the higher order symbols we have:

Lemma 4.2.29. *Let $D_1 \in \mathscr{V}^{p_1}(E)$ and $D_2 \in \mathscr{V}^{p_2}(E)$, then*

$$\sigma^k_{[D_1,D_2]} = \sum_{i=0}^{k}(-1)^{ip_2}\binom{k}{i}[\sigma^i_{D_1}, \sigma^{k-i}_{D_2}]. \tag{4.2.12}$$

The wedge product interacts with the symbols as follows:

Lemma 4.2.30. *Let $D_1 \in \mathscr{V}^{p_1}(E)$ and $D_2 \in \mathscr{V}^{p_2}(E)$, then*

$$\sigma^k_{D_1\wedge D_2} = \sum_{i=0}^{k}(-1)^{i(p_2+1)}\binom{k}{i}\sigma^i_{D_1} \wedge \sigma^{k-i}_{D_2}, \tag{4.2.13}$$

where the right hand side is defined below.

The rest of this section is dedicated to defining the above expressions and proving them. However, for the rest of the chapter it is not very important to know the precise descriptions of these formulae.

The problems with (4.2.11), (4.2.12) and (4.2.13) is that, while they are equalities of elements in $\mathscr{V}^\bullet_\bullet(E)$, the various terms entering in the right hand side live in larger spaces. We will now define these spaces:

Definition 4.2.31. We define

$$\widetilde{\mathscr{V}^p_l}(E) := \mathrm{Alt}^l(C^\infty(M), \mathscr{V}^{p-l}(E)),$$

that is maps

$$D : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{l-\text{times}} \to \mathscr{V}^{p-l}(E),$$

which are $\mathbb{R}$-multilinear and skew-symmetric. The set of elements of $\widetilde{\mathscr{V}^p_l}(E)$ which take values in $\mathscr{V}^{p-l}(E)_{d-\mathrm{pol}}$ are denoted by $\widetilde{\mathscr{V}^p_l}(E)_{d-\mathrm{pol}}$. ♢

Given $\sigma \in \mathscr{V}^p_l(E)$ we view it as an element of $\widetilde{\mathscr{V}^p_l}(E)$ via

$$\sigma(f_1, \ldots, f_l)(s_0, \ldots, s_{p-l}) = \langle \sigma(s_0, \ldots, s_{p-l}), df_1 \wedge \cdots \wedge df_l\rangle .$$

Moreover, $\mathscr{V}^p_l(E)$ sits inside $\widetilde{\mathscr{V}^p_l}(E)$ precisely as those maps which are derivations in each of the entries. Using these spaces we can define the expressions appearing in (4.2.11), (4.2.12) and (4.2.13):

Definition 4.2.32. Given $D_1 \in \widetilde{\mathscr{V}^{p_1}_{l_1}}(E)$ and $D_2 \in \widetilde{\mathscr{V}^{p_2}_{l_2}}(E)$ one defines the wedge product $(D_1 \wedge D_2)(f_1, \ldots, f_{l_1+l_2})$ by:

$$\sum_{\tau\in S_{l_1,l_2}} \mathrm{sgn}(\tau)D_1(f_{\tau(1)}, \ldots, f_{\tau(l_1)}) \wedge D_2(f_{\tau(l_1+1)}, \ldots, f_{\tau(l_1+l_2)}).$$

with sum over (l_1, l_2)-shuffles. ♢

Definition 4.2.33. Given $D_1 \in \widetilde{\mathscr{V}}^{p_1}_{l_1}(E)$ and $D_2 \in \widetilde{\mathscr{V}}^{p_2}_{l_2}(E)$ one defines their bracket $[D_1, D_2](f_1, \ldots, f_{l_1+l_2})$ by:

$$\sum_{\tau \in S_{l_1,l_2}} \operatorname{sgn}(\tau)[D_1(f_{\tau(1)}, \ldots, f_{\tau(l_1)}), D_2(f_{\tau(l_1+1)}, \ldots, f_{\tau(l_1+l_2)})].$$

with sum over (l_1, l_2)-shuffles and the last bracket the one on $\mathscr{V}^\bullet(E)$.

There are two exceptions: If $D_2 \in \widetilde{\mathscr{V}}^{p_2}_{p_2+1}(E)_{(p_2+1)-\mathrm{pol}} = \mathrm{Alt}^{p_2+1}(C^\infty(M))$, and $D_1 \notin \widetilde{\mathscr{V}}^{p_1}_{p_1+1}(E)_{(p_1+1)-\mathrm{pol}}$ (or vice versa) then $[D_1, D_2](f_1, \ldots, f_{l_1+p_2})$ is given by

$$\sum_{\tau \in S_{l_1-1,p_2+1}} \operatorname{sgn}(\tau)\sigma_{D_1}(f_{\tau(1)}, \ldots, f_{\tau(l_1-1)}, D_2(f_{\tau(l_1)}, \ldots, f_{\tau(l_1+p_2)})).$$

If both $D_1 \in \mathrm{Alt}^{p_1+1}(C^\infty(M))$ and $D_2 \in \mathrm{Alt}^{p_2+1}(C^\infty(M))$, then $[D_1, D_2] \in \mathrm{Alt}^{p_1+p_2+1}(C^\infty(M))$ is the Schouten-Nijenhuis bracket. ♢

Remark 4.2.34. Remark that if $D_1 \in \mathscr{V}^{p_1}_{l_1}(E)$ and $D_2 \in \mathfrak{X}^{p_2+1}(M)$, then $[D_1, D_2]$ does not depend on the whole D_1, but only on its symbol σ_{D_1}. ♢

One unfortunate fact is that the brackets on $\mathscr{V}^p(E)$ do not descend to (natural) brackets on $\mathscr{V}^p_l(E)$:

Remark 4.2.35. We will now explain why there can not be brackets on the spaces $\mathscr{V}^p_l(E)$ which are compatible with the symbol maps. The appropriate notion of compatibility at the first stage would be the commutativity of the following diagram:

$$\begin{array}{ccc} \mathscr{V}^{p+q}(E) & \xrightarrow{\sigma^2} & \mathscr{V}^{p+q}_2(E) \\ \Big\uparrow{\scriptstyle [\cdot,\cdot]} & & \Big\uparrow{\scriptstyle [\cdot,\cdot]} \\ \mathscr{V}^p(E) \times \mathscr{V}^q(E) & \xrightarrow{\sigma\times\sigma} & \mathscr{V}^p_1(E) \times \mathscr{V}^p_1(E) \end{array}$$

However, the formula for $\sigma^2[D_1, D_2]$ as described in (4.2.12) depends not only on σ_{D_1} and σ_{D_2} but also on D_1 and D_2 themselves. This shows that there cannot be a bracket on $\mathscr{V}^p_1(E)$ for which the above diagram commutes. In the exceptional case

$$\mathscr{V}^1(E)_{2-\mathrm{pol}} \times \mathscr{V}^1(E)_{2-\mathrm{pol}},$$

we have $\sigma^2[D_1, D_2] = [\sigma^2(D_1), D_2] - 2[\sigma(D_1), \sigma(D_2)] + [D_1, \sigma^2(D_2)]$. However, the expressions $[\sigma^2(D_1), D_2]$ and $[D_1, \sigma^2(D_2)]$ actually do not depend on D_1 and D_2 but only on their symbols. So in this case we can explicitly define

$$\begin{aligned}[\sigma_1, \sigma_2]_{\mathscr{V}^1_1(E)_{2-\mathrm{pol}}}(e; f, g) = \sigma_1(e; \sigma(\sigma_2)(f, g)) - 2[\sigma_1, \sigma_2](e; f, g) \\ + \sigma_2(e; \sigma(\sigma_1)(f, g)),\end{aligned}$$

for all $\sigma_1, \sigma_2 \in \mathscr{V}^1(E)_{2-\mathrm{pol}}$, and the above diagram will commute.

Recall from Example 4.2.20 that a contravariant connection ∇^π defines an element of $\mathscr{V}_1^1(E)_{2-\mathrm{pol}}$. Moreover we have that

$$[\nabla^\pi, \nabla^\pi]_{\mathscr{V}_1^1(E)_{2-\mathrm{pol}}} = -2K_{\nabla^\pi}. \qquad \diamondsuit$$

Remark 4.2.36. Compatibility with the higher order symbols also rarely happens. However $\mathscr{V}_{p+1}^p(E)_{(p+1)-\mathrm{pol}} = \Gamma(\wedge^{p+1}TM)$ can be endowed with Schouten-Nijenhuis bracket and in this case the following diagram commutes, up to a constant:

$$\begin{array}{ccc}
\mathscr{V}^{p+q}(E)_{(p+q-1)-\mathrm{pol}} & \xrightarrow{\sigma^{p+q+1}} & \mathfrak{X}^{p+q+1}(M) \\
\uparrow{\scriptstyle [\cdot,\cdot]} & & \uparrow{\scriptstyle [\cdot,\cdot]} \\
\mathscr{V}^p(E)_{p-\mathrm{pol}} \times \mathscr{V}^q(E)_{q-\mathrm{pol}} & \xrightarrow{\sigma^{p+1}\times\sigma^{q+1}} & \mathfrak{X}^{p+1}(M) \times \mathfrak{X}^{q+1}(M)
\end{array}$$

Again, the commutativity will follow from $[\sigma^p_{D_1}, \sigma^{q+1}_{D_2}] = [\sigma^{p+1}_{D_1}, \sigma^{q+1}_{D_2}]$. $\diamondsuit$

We will now prove equation (4.2.11), the proof of (4.2.12) is more tedious but the idea is the same. The proof of (4.2.13) is completely similar.

Proof of Proposition 4.2.28. To compute the bracket we will consider the vector fields $X_1 \in \mathfrak{X}^{p_1+1}(E^*)$ and $X_2 \in \mathfrak{X}^{p_2+1}(E^*)$ corresponding to D_1 respectively D_2. The symbol of $[D_1, D_2]$ can then be computed by considering the expression:

$$\begin{aligned}
&[X_1, X_2](s_0, \dots, s_{p_1+p_2-1}, f) = \\
&\qquad \sum_{\tau\in T_1} \mathrm{sign}(\tau) X_1(s_{\tau(0)}, \dots, s_{\tau(p_1-2)}, f, X_2(s_{\tau(p_1)}, \dots, s_{\tau(p_1+p_2)})) \\
&\qquad + \sum_{\tau\in T_2} \mathrm{sign}(\tau) X_1(s_{\tau(0)}, \dots, s_{\tau(p_1-1)}, X_2(s_{\tau(p_1)}, \dots, s_{\tau(p_1+p_2-1)}, f),
\end{aligned}$$

where we decomposed $S_{p_1,p_2+1} = T_1 \cup T_2$, with

$$\begin{aligned}
T_1 &= \{\tau \in S_{p_1,p_2+1} : \tau(p_1+p_2) = p_1 - 1\}, \\
T_2 &= \{\tau \in S_{p_1,p_2+1} : \tau(p_1+p_2) = p_1 + p_2\}.
\end{aligned}$$

Given $\tau \in T_1$ we define $\rho \in S_{p_1-1,p_2+1}$ by

$$\rho(i) = \begin{cases} \tau(i) & \text{if } 0 \le i \le p_1 - 2 \\ \tau(i+1) & \text{if } p_1 \le i \le p_1 + p_2 - 1, \end{cases}$$

where $\mathrm{sign}(\rho) = (-1)^{p_2+1}\,\mathrm{sign}(\tau)$. Similarly, given $\tau \in T_2$, define $\eta \in S_{p_1,p_2}$ by $\eta(i) = \tau(i)$ for all $0 \le i \le p_1 + p_2 - 1$, so that $\mathrm{sign}(\eta) = \mathrm{sign}(\tau)$. In conclusion we find

$$\begin{aligned}
&[X_1, X_2](s_0, \dots, s_{p_1+p_2-1}, f) \\
&= \sum_{\rho\in S_{p_1-1,q_1+1}} (-1)^{p_2+1}\,\mathrm{sign}(\rho) X_1(s_{\rho(0)}, \dots, s_{\rho(p_1-2)}, f, X_2(s_{\rho(p_1-1)}, \dots, s_{\rho(p_1+p_2-1)})) \\
&+ \sum_{\eta\in S_{p_1,p_2}} \mathrm{sign}(\eta) X_1(s_{\eta(0)}, \dots, s_{\eta(p_1-1)}, X_2(s_{\eta(p_1)}, \dots, s_{\eta(p_1+p_2-1)}, f)).
\end{aligned}$$

From this expression we conclude the desired statement. □

4.2.5 The underlying filtered Gerstenhaber structure

In this section we will study the algebraic structure present on the symbol tower (4.2.5) and its polynomial version (4.2.7). In particular we will describe a filtration on $\mathscr{V}^p(E)$ and $\mathscr{V}^p(E)_{d-\mathrm{pol}}$ compatible with the Gerstenhaber structure.

Definition 4.2.37. A **filtered Gerstenhaber algebra** is a Gerstenhaber algebra $(L^\bullet, \cdot, [\cdot,\cdot])$ together with a filtration of type

$$0 = F_{-1}L^\bullet \subset F_0L^\bullet \subset F_1L^\bullet \subset \ldots \subset \bigcup_l F_kL^\bullet = L^\bullet$$

such that $[F_l, F_{l'}] \subset F_{l+l'}$ and $F_l \cdot F_{l'} \subset F_{l+l'}$. ♢

Let $\mathscr{V}^\bullet(E)[-1]$ denote the algebra obtained from shifting the degree by one: $\mathscr{V}^p(E)[-1] = \mathscr{V}^{p-1}(E)$. The bracket (4.2.9) and wedge product (4.2.8) endow $\mathscr{V}^\bullet(E)[-1]$ with the structure of a Gerstenhaber algebra. The compatibility of this bracket and wedge product with the symbols on $\mathscr{V}^p(E)$ allows us to show:

Lemma 4.2.38. *The sets*

$$F_l\mathscr{V}^p(E) := \{D \in \mathscr{V}^p(E) : \sigma_D^{l+1} = 0\},$$

endow the Gerstenhaber algebra $\mathscr{V}^\bullet(E)[-1]$ with the structure of filtered Gerstenhaber algebra.

Proof. We have to prove that $[F_l, F_{l'}] \subset F_{l+l'}$, that is, if $\sigma_D^{l+1} = 0$ and $\sigma_D^{l'+1} = 0$, then $\sigma_{[D,D']}^{l+l'+1} = 0$. This follows immediately from Lemma 4.2.29. Similarly $F_l \cdot F_{l'} \subset F_{l+l'}$ follows from Lemma 4.2.30. □

A filtered Gerstenhaber algebra has very rich algebraic structure, which we will now recall. We would like to warn the reader however that, very often in the literature, filtered Lie algebras are endowed with decreasing filtrations- a situation that is fundamentally different. For a filtered Gerstenhaber algebra $L^\bullet$ one has the following induced objects:

- the associated tower:

$$L^\bullet = L_0^\bullet \xrightarrow{\sigma} L_1^\bullet \xrightarrow{\sigma} L_2^\bullet \xrightarrow{\sigma} \ldots$$

  where $L_i^\bullet := L^\bullet/F_{i-1}L^\bullet$ and σ is the canonical projection. This is a tower of graded vector spaces and σ is pointwise nilpotent (for each $x \in L^\bullet$ there exists k such that $\sigma^k(x) = 0$).

- the graded Gerstenhaber algebra associated to the filtered Gerstenhaber algebra $L^\bullet$:

$$\mathrm{gr}^l(L^\bullet) = \mathrm{Ker}(L_l^\bullet \to L_{l-1}^\bullet) = F_lL^\bullet/F_{l-1}L^\bullet,$$

endowed with the Lie bracket

$$[\cdot,\cdot]:\mathrm{gr}^l(L^\bullet)\times\mathrm{gr}^{l'}(L^\bullet)\to\mathrm{gr}^{l+l'}(L^\bullet)$$

and product

$$\cdot:\mathrm{gr}^l(L^\bullet)\times\mathrm{gr}^{l'}(L^\bullet)\to\mathrm{gr}^{l+l'}(L^\bullet),$$

induced from the ones of $L^\bullet$. It gives rise to a graded Gerstenhaber algebra

$$\mathrm{gr}(L^\bullet)=\oplus\mathrm{gr}^l(L^\bullet).$$

Example 4.2.39. For the filtered Gerstenhaber structure described in Lemma 4.2.38 we have that $L_i^\bullet = L^\bullet/F_{i-1}L^\bullet \simeq \mathscr{V}_i^\bullet(E)$, where the last map is given by $[D]\mapsto\sigma_D^i$. This map is well-defined and injective, however proving surjectivity will have to be postponed Remark 4.2.50.

Consequently, for $\mathscr{V}^\bullet(E)$:

- the associated tower is precisely the symbol tower (4.2.5);
- each gr-space can be identified as

$$\mathrm{gr}_l\mathscr{V}^p(E)=\mathrm{Ker}\left(\sigma:\mathscr{V}_l^p(E)\to\mathscr{V}_{l+1}^p(E)\right)\cong\Gamma\left(\Lambda^{p-l+1}E^*\otimes\pi^*\Lambda^lTM\right),$$

where the product on the right-hand side is the one induced by the wedge product on $\Lambda^\bullet E^*$ and $\Lambda^\bullet TM$. △

Example 4.2.40. Inside Example 4.2.39 one has the polynomial counterpart:

$$\mathscr{V}^\bullet(E)_{\mathrm{pol}}=\oplus_d\mathscr{V}^\bullet(E)_{d-\mathrm{pol}},$$

with $F_l\mathscr{V}^p(E)_{\mathrm{pol}}$ again defined as the kernel of σ^{l+1}. Note that this filtration does not interact with the polynomial degree at all. Consequently, the associated graded Gerstenhaber algebra inherits yet another grading, coming from the polynomality: $\mathrm{gr}^l\mathscr{V}^p(E)_{\mathrm{pol}}=\oplus_d\mathrm{gr}^l\mathscr{V}^p(E)_{d-\mathrm{pol}}$. Therefore,

- the tower corresponding to $\mathscr{V}^\bullet(E)_{d-\mathrm{pol}}$ is the polynomial symbol tower (4.2.7);
- each gr-space can be identified as

$$\mathrm{gr}_l\mathscr{V}^p(E)_{d-\mathrm{pol}}\cong\Gamma(\Lambda^{p-l+1}E^*\otimes S^{d-l}E\otimes\Lambda^lTM).$$

We can explicitly describe the bracket on the graded Gerstenhaber-algebra as follows: If $Q_1\in\mathrm{gr}_{l_1}\mathscr{V}^{p_1}(E)_{d_1-\mathrm{pol}}$ and $Q_2\in\mathrm{gr}_{l_2}\mathscr{V}^{p_2}(E)_{d_2-\mathrm{pol}}$, such that

$$Q_1=S_1\otimes P_1\otimes X_1, Q_2=S_2\otimes P_2\otimes X_2,$$

the bracket is given by the following formula:

$$[S_1\otimes P_1\otimes X_1,S_2\otimes P_2\otimes X_2]=(\langle S_1,P_2\rangle P_1\otimes S_2-\langle P_1,S_2\rangle P_2\otimes S_1)\otimes(X_1\wedge X_2).\ \triangle$$

In some sense,

$$\mathrm{gr}(L^\bullet) := \oplus \mathrm{gr}^l(L^\bullet)$$

is a "first order" approximation of the original filtered graded Lie algebra $L^\bullet$; it comes itself with a filtration

$$F_l\,\mathrm{gr}(L^\bullet) = \mathrm{gr}^0 L^\bullet \oplus \ldots \oplus \mathrm{gr}^l L^\bullet$$

and $\mathrm{gr}(\mathrm{gr}(L^\bullet)) = \mathrm{gr}(L^\bullet)$. In general, $\mathrm{gr}(L^\bullet)$ is much simpler than the original L.

Linear isomorphisms Although in general quite different as Gerstenhaber algebras, $L^\bullet$ and $\mathrm{gr}(L^\bullet)$ have a good chance of being isomorphic as associative algebras.

We will first search for a linear isomorphism between $L^\bullet$ and $\mathrm{gr}(L^\bullet)$. To do so one has to split the corresponding tower, i.e. find linear maps S going backwards in the tower,

$$L_0^\bullet \underset{\sigma}{\longrightarrow} L_1^\bullet \underset{\sigma}{\longrightarrow} L_2^\bullet \underset{\sigma}{\longrightarrow} \cdots\cdots \qquad (S : L_{k+1}^\bullet \dashrightarrow L_k^\bullet)$$

with the property that $\sigma \circ S = \mathrm{Id}$. In that case each S can be viewed as a splitting of the short exact sequence

$$0 \longrightarrow \mathrm{gr}_k(L^\bullet) \xrightarrow{\ i\ } L_k^\bullet \xrightarrow{\ \sigma\ } L_{k+1}^\bullet \longrightarrow 0.$$

Hence there are induced left splittings R as in the diagram:

$$0 \longrightarrow \mathrm{gr}_k(L^\bullet) \longrightarrow L_k^\bullet \underset{\sigma}{\longrightarrow} L_{k+1}^\bullet \longrightarrow 0, \qquad (R : L_k^\bullet \dashrightarrow \mathrm{gr}_k(L^\bullet),\ S : L_{k+1}^\bullet \dashrightarrow L_k^\bullet)$$

uniquely characterised by $R \circ i = \mathrm{id}$, $i \circ R + S \circ \sigma = \mathrm{id}$. The resulting isomorphisms

$$L_k^\bullet \cong L_{k+1}^\bullet \oplus \mathrm{gr}_k(L^\bullet), x \mapsto (\sigma(x), x - \sigma(x)),$$

applied inductively give rise to a linear isomorphism

$$L^\bullet \cong \mathrm{gr}(L^\bullet) \quad x \mapsto (R(x), R\sigma(x), R\sigma^2(x), \ldots) \tag{4.2.14}$$

with inverse

$$i(x_0) + S(i(x_1)) + S^2(i(x_2)) + \ldots \hookleftarrow \quad (x_0, x_1, x_2, \ldots). \tag{4.2.15}$$

And, instead of going all the way to $L_0^\bullet = L^\bullet$, we also have intermediate isomorphisms

$$L_k^\bullet \simeq \bigoplus_{i=k}^{\infty} \mathrm{gr}_i(L^\bullet).$$

In our examples splittings will not be available right away. Instead, we encounter the following interesting situation that we state in full generality.

Definition 4.2.41. Let $L^\bullet$ be a filtered graded Lie algebra. An **h-splitting** consists of linear maps

$$L^\bullet_0 \underset{\sigma}{\overset{h}{\rightleftarrows}} L^\bullet_1 \underset{\sigma}{\overset{h}{\rightleftarrows}} L^\bullet_2 \underset{\sigma}{\overset{h}{\rightleftarrows}} \cdots\cdots$$

satisfying the homotopy-like equations

$$\sigma \circ h - h \circ \sigma = \mathrm{id}\,. \qquad \diamondsuit$$

Lemma 4.2.42. *Let $L^\bullet$ be a filtered graded Lie algebra together with an **h**-splitting h. Then h gives rise to a splitting of the tower given by:*

$$S := \sum_{i\geq 1} \frac{(-1)^{i+1}}{i!} h^i \circ \sigma^{i-1} : L^\bullet_{l+1} \to L^\bullet_l. \tag{4.2.16}$$

Consequently $L^\bullet$ is linearly isomorphic to $\mathrm{gr}(L^\bullet)$.

Proof. Because for each $x \in L^\bullet_{l+1}$ there exists a k such that $\sigma^k(x) = 0$, the sum defining S is finite. Inductively one shows that $\sigma \circ h^i = ih^{i-1} + h^i \circ \sigma$. Therefore

$$\begin{aligned}\sigma \circ S &= \sum_{i\geq 1} \frac{(-1)^{i+1}}{i!} (ih^{i-1} + h^i \circ \sigma) \circ \sigma^{i-1} \\ &= \sum_{i\geq 1} \frac{(-1)^{i+1}}{(i-1)!} h^{i-1} \circ \sigma^{i-1} + \sum_{i\geq 1} \frac{(-1)^{i+1}}{i!} h^i \circ \sigma^i.\end{aligned}$$

This is a telescoping series, so only the first and last term remain. The first term is precisely id, the last term vanishes because for each $x \in L^\bullet_{l+1}$ there exists a k such that $\sigma^k(x) = 0$. ☐

Although the linear isomorphism (4.2.14) induced by an **h**-splitting is in general pretty complicated, the inverse as in (4.2.15) is much easier to describe:

Lemma 4.2.43. *Given an **h**-splitting h of a filtered graded Lie algebra, then for*

$$D \in \ker \sigma : L^\bullet_i \to L^\bullet_{i-1}$$

and $q \in \mathbb{N}$ we have:

$$S^q(D) = \frac{1}{q!} h^q(D),$$

where S is as in Lemma 4.2.42. Consequently (4.2.15) *becomes*

$$\begin{aligned}\mathrm{gr}(L^\bullet) &\to L^\bullet \\ D &\mapsto e^h D.\end{aligned}$$

The proof is a straightforward but somewhat lengthy computation and thus omitted. To upgrade the linear isomorphism (4.2.15) to one which also respects the associative algebra structure we ask for compatibility of the **h**-splitting with the algebra structure:

Lemma 4.2.44. *Let $L^\bullet$ be a filtered Gerstenhaber algebra, and let h be an $\boldsymbol{h}$-splitting which furthermore satisfies:*

$$h^{l_1+l_2}(D_1 \cdot D_2) = \binom{l_1+l_2}{l_1} h^{l_1}(D_1) \cdot h^{l_2}(D_2), \tag{4.2.17}$$

for all $(D_1, D_2) \in \mathrm{gr}^{i_1}(L^\bullet) \times \mathrm{gr}^{i_2}(L^\bullet)$. Then the linear isomorphism (4.2.15) *is furthermore an isomorphism of associative algebras.*

Proof. This follows immediately from Lemma 4.2.43. □

We will now see how Lemma 4.2.42 is relevant in our setting. At the initial stage the question becomes: given $\sigma \in \mathscr{V}_1^\bullet(E)$, find $D \in \mathscr{V}^\bullet(E)$ such that $\sigma(D) = \sigma$. This is straightforward in the polynomial case when $d = 1$:

Example 4.2.45. When $X \in \mathscr{V}_1^0(E)_{1-\mathrm{pol}} = \mathfrak{X}^1(M)$, we want to construct an element $D \in \mathscr{V}^0(E)_{1-\mathrm{pol}}$ with $\sigma_D = X$, that is a derivation $D : \Gamma(E) \to \Gamma(E)$ whose symbol is σ. A connection ∇ induces such a derivation: $D = \nabla_X$.

If we take the vector field $Y \in \mathfrak{X}^1(E^*)_{1-\mathrm{pol}}$ corresponding to D, we see that $Y = \mathrm{hor}^{\nabla^*}(X)$. △

The previous example shows that using a connection to construct the splitting S is the natural way to proceed. Let us give one more example before we continue with the general case:

Example 4.2.46. Let $\pi \in \mathscr{V}_2^1(E)_{2-\mathrm{pol}} = \mathfrak{X}^2(M)$, and let ∇ be a connection on M. We define $D \in \mathscr{V}_1^1(E)_{2-\mathrm{pol}}$ by $D(s_0)(\alpha) = \nabla_{-\pi^\sharp(\alpha)}(s_0)$. Remark that when π is Poisson this coincides with the construction of a contravariant connection from an ordinary one as in Example 4.1.19. Also recall the need of the minus sign from the discussion in Example 4.2.20. △

With the previous examples in mind, there is a clear first attempt:
for $D \in \mathscr{V}_{l+1}^p(E)$ arbitrary, construct $h_\nabla(D) \in \mathscr{V}_l^p(E)$ by:

$$h_\nabla(D)(s_0, \ldots, s_{p-l}) = (-1)^{p-l} \sum_i (-1)^i \langle D(s_0, \ldots, \widehat{s_i}, \ldots, s_{p-l}), \nabla(s_i) \rangle. \tag{4.2.18}$$

Here the pairing in the right-hand-side is the pairing between $\pi^* \wedge^{l-1} TM$ and T^*M. The only problem is that, in general, the symbol of $h_\nabla(D)$ is not quite D. Instead, we find ourselves in the situation described in Lemma 4.2.42:

Lemma 4.2.47. *For any connection ∇ on E,* (4.2.18) *defines an $\boldsymbol{h}$-splitting for $\mathscr{V}^\bullet(E)$, i.e. satisfies $\sigma \circ h_\nabla - h_\nabla \circ \sigma = \mathrm{id}$.*

Proof. Let $D \in \mathscr{V}^p_{l+1}(E)$ we compute $\sigma(h_\nabla(D))(s_0, \ldots, s_{p-l-1}; f)$:

$$
\begin{aligned}
&(-1)^{p-l}(-1)^{p-l} \langle D(s_0, \ldots, s_{p-l-1}), \nabla(f s_{p-l}) - f\nabla(s_{p-l})\rangle \\
&+ (-1)^{p-l} \sum_{i=0}^{p-l-1} (-1)^i \langle D(s_0, \ldots, \widehat{s}_i, \ldots, f s_{p-l}) - f D(s_0, \ldots, \widehat{s}_i, \ldots, s_{p-l}), \nabla(s_i)\rangle \\
&= \langle D(s_0, \ldots, s_{p-l-1}), df \otimes s_{p-l}\rangle \\
&+ (-1)^{p-l} \sum_{i=0}^{p-l-1} (-1)^i \langle s_{p-l} \cdot \sigma_D(s_0, \ldots, \widehat{s}_i, \ldots, s_{p-l})(f), \nabla(s_i)\rangle \\
&+ s_{p-l} \cdot \langle D(s_0, \ldots, s_{p-l}), df\rangle \\
&- (-1)^{p-l} \left\langle \sum_{i=0}^{p} (-1)^i \langle s_{p-l} \cdot \sigma_D(s_0, \ldots, \widehat{s}_i, \ldots, s_{p-l}), \nabla(s_i)\rangle, df \right\rangle \\
&= s_{p-l} \cdot \langle D(s_0, \ldots, s_p), df\rangle \\
&+ (-1)^{p-l-1} s_{p-l} \cdot \left\langle \sum_{i=0}^{p} (-1)^i \langle \sigma_D(s_0, \ldots, \widehat{s}_i, \ldots, s_{p-l}), \nabla(s_i)\rangle, df \right\rangle .
\end{aligned}
$$

Consequently $h_\nabla(D)$ defines an element of $\mathscr{V}^p_l(E)$ with symbol $D + h_\nabla(\sigma_D)$, which proves the desired identity ☐

To prove that the **h**-splitting (4.2.18) is compatible with the algebra structure, we will describe it explicitly. To do so, let

$$\mathrm{hor}^{\nabla^*}_p : \Gamma(\pi^*\Lambda^i TM) \to \mathfrak{X}^p(E^*; \pi^*\Lambda^{i-p}TM),$$

be the (partial) horizontal lift which is defined on decomposable pull-back sections as

$$
\begin{aligned}
&\mathrm{hor}^{\nabla^*}_p(\pi^*X_1 \wedge \cdots \wedge \pi^*X_i) = \\
&\qquad \sum_\tau \mathrm{sign}(\tau)\, \mathrm{hor}^{\nabla^*}(X_{\tau(1)}) \wedge \cdots \wedge \mathrm{hor}^{\nabla^*}(X_{\tau(p)}) \wedge \pi^*X_{\tau(p+1)} \wedge \cdots \wedge \pi^*X_{\tau(i)},
\end{aligned}
$$

where the sum runs over all $(p, i-p)$-shuffles.

Lemma 4.2.48. *We have that*

$$h^j_\nabla : \mathrm{gr}_{l+1}(\mathscr{V}^p(E)) \to \mathfrak{X}^{p+j-l}(E^*; \pi^*\Lambda^{l+1-j}TM),$$

is given by

$$h^j_\nabla(\vartheta \otimes X) = j!\, \iota_*(\vartheta)\, \mathrm{hor}^{\nabla^*}_j(X),$$

for all $\vartheta \otimes X \in \Gamma(\wedge^{p-l}E^* \otimes \pi^*\Lambda^{l+1}TM)$*. Consequently the* ***h****-splitting satisfies* (4.2.17).

Proof. We have

$$\begin{aligned} h_\nabla(\vartheta\otimes X)(s_0,\dots,s_{p-l}) &= (-1)^{p-l}\sum_i(-1)^i\left\langle(\vartheta\otimes X)(s_0,\dots,\widehat{s}_i,\dots,s_{p-1}),\nabla(s_i)\right\rangle \\ &= (-1)^{p-l}\sum_i(-1)^i\vartheta((s_0,\dots,\widehat{s}_i,\dots,s_{p-1})\otimes\langle X,\nabla(s_i)\rangle \\ &= ((\iota_*\vartheta)\wedge h_\nabla(X))(s_0,\dots,s_{p-l}), \end{aligned}$$

where we used that, as $X\in \mathrm{gr}_{l+1}(\mathscr{V}^l(E))$ we have $h_\nabla(X)(s)=\langle X,\nabla(s)\rangle$. Continuing inductively we find that $h^j_\nabla(\vartheta\otimes X)=\iota_*(\vartheta)\cdot h^j_\nabla(X)$.

If $X=fY$, with $f\in C^\infty(E^*)$ and $Y\in\mathfrak{X}^l(M)$ then $h^j_\nabla(X)=fh^j_\nabla(Y)$. A straightforward computation, where extra care with respect to the signs has to be taken, results in

$$h^j_\nabla(Y)(s_0,\dots,s_{l-1})=\sum_\tau\left\langle Y,\nabla(s_{\tau(0)})\wedge\nabla(s_{\tau(1)})\wedge\cdots\wedge\nabla(s_{\tau(l-1)})\right\rangle,$$

where the sum runs over all permutations. Consequently

$$h^j_\nabla(Y)(s_0,\dots,s_{l-1})=j!\left\langle Y,\nabla(s_0)\wedge\cdots\wedge\nabla(s_{\tau(l-1)})\right\rangle.$$

To finish the proof one has to show that the right-hand side coincides with the horizontal lift. This follows from a straightforward but careful computation. Because the horizontal lift is an algebra morphism it will follows that h_∇ satisfies the conditions of Lemma 4.2.44. □

Given a connection ∇ on E, one obtains an isomorphism $TE^*\simeq\pi^*(E^*\oplus TM)$, which in turn can be used to describe the vector fields on E^*. This description of the vector fields on E^* also directly follows from the existence of the **h**-splitting for $\mathscr{V}^\bullet(E)$:

Corollary 4.2.49. *A connection on E induces an isomorphism*

$$\mathscr{V}^p_l(E)\simeq\bigoplus_{i=0}^{p-l+1}\Gamma(\Lambda^{p-l+1-i}E^*\otimes\pi^*\Lambda^{l+i}TM).$$

If we endow $\Gamma(\Lambda^\bullet E^*\otimes\pi^*\Lambda^\bullet TM)$ with the associative algebra structure induced by the wedge products, this isomorphism induces an isomorphism of associative algebras

$$\mathscr{V}^\bullet_\bullet(E)\simeq\Gamma(\Lambda^\bullet E^*\otimes\pi^*\Lambda^\bullet TM).$$

Proof. By Lemma 4.2.47 a connection gives rise to an **h**-splitting for $\mathscr{V}^\bullet(E)$, which by Lemma 4.2.48 is compatible with the algebra structure. Therefore, Equation (4.2.14) provides an algebra isomorphism between $\mathscr{V}^\bullet_l(E)$ and $\oplus_{i=l}\mathrm{gr}_i(\mathscr{V}^\bullet(E))$. By Example 4.2.39 we see that $\oplus_{i=l}\mathrm{gr}_i(\mathscr{V}^\bullet(E))$ is as desired □

Remark 4.2.50. In particular we find that the map $\mathscr{V}^\bullet(E) \to \mathscr{V}_i^\bullet(E) : D \mapsto \sigma_D^i$ is surjective, as announced in Example 4.2.39. Note that it might seem that we already used the fact that the graded spaces of $\mathscr{V}^\bullet(E)$ coincide with $\mathscr{V}_i^\bullet(E)$, because we defined an **h**-splitting as maps in the associated tower. Of course h_∇ defines maps of the symbol tower (4.2.5) satisfying the homotopy like conditions, and S as in Equation (4.2.16) then defines splittings of the symbol tower. Therefore, each of the maps in the symbol tower is surjective and consequently $\mathscr{V}^\bullet(E) \to \mathscr{V}_i^\bullet(E) : D \mapsto \sigma_D^i$, from which we conclude that indeed the graded spaces of $\mathscr{V}^\bullet(E)$ coincide with $\mathscr{V}_i^\bullet(E)$. ♢

It is immediate that the **h**-splitting h_∇ from Lemma 4.2.47 restricts to an **h**-splitting $\mathscr{V}_l^\bullet(E)_{d-\mathrm{pol}} \to \mathscr{V}_{l-1}^\bullet(E)_{d-\mathrm{pol}}$, from which we conclude:

Corollary 4.2.51. *Each $\mathscr{V}_l^p(E)_{d-\mathrm{pol}}$ is the space of sections of a vector bundle over M, $\mathbb{V}_l^p(E)_{d-\mathrm{pol}}$, the symbol map is part of a short exact sequences of vector bundles*

$$0 \to \Lambda^{p-l+1}E^* \otimes S^{d-l}E \otimes \Lambda^l TM \longrightarrow \mathbb{V}_l^p(E)_{d-\mathrm{pol}} \xrightarrow{\sigma} \mathbb{V}_{l+1}^p(E)_{d-\mathrm{pol}} \to 0.$$

Moreover, any connection ∇ on E gives rise to an isomorphism of vector bundles

$$\mathbb{V}_l^p(E)_{d-\mathrm{pol}} \simeq \bigoplus_{i=0}^{p-l+1} \left(\Lambda^{p-l+1-i}E^* \otimes S^{d-l-i}E \otimes \Lambda^{l+i}TM\right).$$

If we endow $\Gamma(\Lambda^\bullet E^* \otimes S^\bullet E \otimes \Lambda^\bullet TM)$ with the associative algebra structure obtained from combining the wedge products and symmetric products we obtain an isomorphism of associate algebras:

$$\mathscr{V}_\bullet^\bullet(E)_{\bullet-\mathrm{pol}} \simeq \Gamma(\Lambda^\bullet E^* \otimes S^\bullet E \otimes \Lambda^\bullet TM).$$

Moreover:

Proposition 4.2.52. *Let ι_* be as in Example 4.2.16. The map*

$$\begin{aligned}\psi : \Gamma(\wedge^{p-l+1-i}E^* \otimes S^{d-l-i}E \otimes \Lambda^{l+i}TM) &\to \mathfrak{X}^{p+1-l}(E^*; \wedge^l TM)_{(d-l)-\mathrm{pol}} \\ \vartheta \otimes P \otimes Q &\mapsto \widehat{P}\iota_*(\vartheta) \wedge \mathrm{hor}_i^{\nabla^*}(Q),\end{aligned}$$

induces a map which is inverse to the isomorphism in Corollary 4.2.51.

Proof. This follows from combining Lemma 4.2.48 and Lemma 4.2.43. □

Remark 4.2.53. Because of the explicit nature of the inverse in Proposition 4.2.52 we can use it to study elements $X \in \mathfrak{X}^{p+1}(E^*)_{d-\mathrm{pol}}$ in the following way:

- Consider $D \in \mathscr{V}^p(E)_{d-\mathrm{pol}}$ associated to X, and use the isomorphism from Corollary 4.2.51 to decompose

$$\varphi(D) = (S_0, \dots, S_{p+1}) \in \bigoplus_{i=0}^{p+1} \left(\Lambda^{p+1-i}E^* \otimes S^{d-i}E \otimes \Lambda^i TM\right).$$

- Use the isomorphism in Proposition 4.2.52 to obtain the equality

$$X = \psi(S_0) + \cdots + \psi(S_{p+1}).$$

In this way, working with the multivector fields X on the vector bundle E^* becomes much more concrete in terms of tensors over M. We will apply this philosophy to Poisson structures in the next chapter. ♢

As remarked in the digression on filtered Lie algebras, $\mathrm{gr}(L^\bullet)$ can be thought of as linear approximation of $L^\bullet$. We will illustrate this for $\mathscr{V}^\bullet(E)_{\mathrm{pol}}$. Having established the linear isomorphism in Proposition 4.2.51 we can transport the Lie bracket on $\mathscr{V}^\bullet(E)_{\mathrm{pol}}$ to $\mathrm{gr}(\mathscr{V}^\bullet(E)_{\mathrm{pol}})$ and compare with the canonical bracket there.

Lemma 4.2.54. *Let ∇ be a connection on $E \to M$, let $K^* \in \Omega^2(M; \mathrm{End}(E^*))$ denote the curvature of the dual connection ∇^* and let $[\cdot,\cdot]_{SN}$ denote the Schouten-Nijenhuis bracket of multivector fields on M.*

Consider the bracket on $\Gamma(\wedge^\bullet E^* \otimes S^\bullet E \otimes \wedge^\bullet TM)$ uniquely determined by its value on $\theta_1, \theta_2 \in \Gamma(\wedge^1 E^*), P_1, P_2 \in \Gamma(S^1 E), Q_1, Q_2 \in \Gamma(TM)$:

$$\begin{aligned} [\theta_1,\theta_2] &= 0, \quad [\theta_1, P_2] = \langle P_2, \theta_1\rangle, \quad [\theta_1, Q_2] = \nabla_{Q_2}(\theta_1) \\ [P_1,P_2] &= 0, \quad [P_1, Q_2] = \nabla^*_{Q_2}(P_1), \quad [Q_1,Q_2] = [Q_1,Q_2]_{SN} + K^*(Q_1,Q_2), \end{aligned}$$

and extended using the Leibniz identity

$$[\eta_1, \eta_2 \cdot \eta_3] = [\eta_1,\eta_2]\cdot\eta_3 + (-1)^{(r_1+t_1-1)(r_2+t_2)}\eta_2 \cdot [\eta_1,\eta_3],$$

for $\eta_i \in \Gamma(\wedge^{r_i} E^* \otimes S^{s_i} E \otimes \wedge^{t_i} TM)$.

Endowed with this bracket the linear isomorphism in Corollary 4.2.51 becomes a Lie algebra isomorphism.

Proof. This follows readily from using the map ψ in Proposition 4.2.52 to compare the brackets in low degrees. Together with the expression

$$[\mathrm{hor}^{\nabla^*}(X), \mathrm{hor}^{\nabla^*}(Y)](s) - \mathrm{hor}^{\nabla^*}([X,Y])(s) = K(X,Y)(s) = a(K(X,Y)^*)(s),$$

for all $X, Y \in \mathfrak{X}^1(M)$ and $s \in \Gamma(E)$ and a is as in Example 4.2.17. □

Considering the bracket in Lemma 4.2.54, we can make a couple of remarks. First, as was to be expected, on $\Gamma(\wedge^\bullet E^* \otimes S^\bullet E)$ it coincides with the bracket on $\mathrm{gr}(\mathscr{V}^\bullet(E)_{\mathrm{pol}})$ as described in Example 4.2.40. However, we also conclude that $\mathscr{V}^\bullet(E)_{d-\mathrm{pol}}$ is isomorphic to $\mathrm{gr}(\mathscr{V}^\bullet(E)_{\mathrm{pol}})$ as Gerstenhaber algebras if and only if M is a point.

4.2.6 Dual picture

While the spaces $\mathscr{V}^p(E)_{d-\mathrm{pol}}$ focus on the vector bundle E, there is a dual version of the same spaces which focuses on the dual vector bundle $\mathcal{N} = E^*$. The best illustration of this duality is by looking at $\mathscr{V}^0(E)_{1-\mathrm{pol}}$, i.e. at derivations $D : \Gamma(E) \to \Gamma(E)$ with symbol X. Then D induces a dual derivation on E^* (still with symbol X):

$$D^* : \Gamma(E^*) \to \Gamma(E^*), \quad \langle D^*(\xi), s\rangle := \langle \xi, D(s)\rangle + \mathcal{L}_X(\langle \xi, s\rangle),$$

for $\xi \in \Gamma(E^*)$, $s \in \Gamma(E)$.

On the other hand, the previous illustration may be misleading as one may expect some duality that allows one to pass from $\mathscr{V}^{p+1}(E)_{d-\mathrm{pol}}$ to $\mathscr{V}^{p+1}(E^*)_{d-\mathrm{pol}}$; but the constant case $d = 0$ ($\Gamma(\wedge^p E)$ v.s. $\Gamma(\wedge^p E^*)$) there cannot be such a canonical duality.

In this section we will introduce the objects dual to elements of $\mathscr{V}^p(E)_{d-\mathrm{pol}}$ and obtain the desired duality. It must be said that, although we can give explicit formulae for such a duality, we are currently unable to find a deeper underlying reason explaining them.

Definition 4.2.55. Let $\mathcal{N} \to M$ be a vector bundle. We denote by $\mathcal{F}^d(\mathcal{N})_p$ the collection of sequences $(G = \sigma_G^0, \sigma_G^1, \ldots, \sigma_G^{d-l})$ of *symmetric* $\mathbb{R}$-multilinear maps

$$G : \underbrace{\Gamma(\mathcal{N}) \times \cdots \times \Gamma(\mathcal{N})}_{d-\text{times}} \to \Gamma(\Lambda^{p+1}\mathcal{N}),$$

$$\sigma_G : \underbrace{\Gamma(\mathcal{N}) \times \cdots \times \Gamma(\mathcal{N})}_{(d-1)-\text{times}} \to \Gamma(\Lambda^{p}\mathcal{N} \otimes TM),$$

$$\sigma_G^2 : \underbrace{\Gamma(\mathcal{N}) \times \cdots \times \Gamma(\mathcal{N})}_{(d-2)-\text{times}} \to \Gamma(\Lambda^{p-1}\mathcal{N} \otimes \Lambda^2 TM),$$

$$\cdots,$$

$$\sigma_G^d \in \Gamma(\Lambda^{p-d}\mathcal{N} \otimes \Lambda^d TM),$$

and each two consecutive ones related via the Leibniz-type identity:

$$\sigma_G^i(s_1, \ldots, f s_{d-l-i}) = f G(s_1, \ldots, s_{d-l-i}) + \iota_{df}(\sigma_G^{i+1}(s_1, \ldots, s_{d-l-i-1})) \wedge s_{d-l-i}. \tag{4.2.19}$$

♢

Remark 4.2.56. It is important to remark that while an element $D \in \mathscr{V}^p(E)_{d-\mathrm{pol}}$ determines all its higher order symbols, this is not the case for the first element of a sequence in $\mathcal{F}^d(\mathcal{N})_p$. In Remark 5.1.59 we will see an example of an element $(G, \sigma_G, \sigma_G^2) \in \mathcal{F}^2(\mathcal{N})_1$ on a rank one vector bundle. Because $\mathcal{N}$ has rank one G has to be identically zero, but σ_G and σ_G^2 need not be. ♢

As before we can define the intermediate spaces $\mathcal{F}_l^d(\mathcal{N})_p$ consisting of linear maps

$$G : \underbrace{\Gamma(\mathcal{N}) \times \cdots \times \Gamma(\mathcal{N})}_{(d-l)-\text{times}} \to \Gamma(\Lambda^{p+1-l}\mathcal{N} \otimes \Lambda^l TM),$$

such that there exists a map

$$\sigma_G : \underbrace{\Gamma(\mathcal{N}) \times \cdots \times \Gamma(\mathcal{N})}_{(d-l-1)-\text{times}} \to \Gamma(\Lambda^{p-l}\mathcal{N} \otimes \Lambda^{l+1}TM),$$

such that the Leibniz identity (4.2.19) is satisfied.

From the definition it it immediate that if $(G, \sigma_G^1 = \sigma_G, \ldots, \sigma_G^{d-l}) \in \mathcal{F}_l^d(\mathcal{N})_p$, then $(\sigma_G^1, \ldots, \sigma_G^{d-l}) \in \mathcal{F}_{l+1}^d(\mathcal{N})_p$ and therefore we again obtain a symbol-sequence:

$$\mathcal{F}^d(\mathcal{N})_p \xrightarrow{\sigma} \mathcal{F}_1^d(\mathcal{N})_p \xrightarrow{\sigma} \mathcal{F}_2^d(\mathcal{N})_p \xrightarrow{\sigma} \ldots$$

We also want to endow $\mathcal{F}^\bullet(\mathcal{N})_\bullet$ with the structure of a bi-graded algebra:

Definition 4.2.57. Given $G_1, G_2 \in \mathcal{F}^{d_1}(\mathcal{N})_{p_1}, \mathcal{F}^{d_2}(\mathcal{N})_{p_2}$, we define

$$G_1 \cdot G_2 \in \mathcal{F}^{d_1+d_2}(\mathcal{N})_{p_1+p_2+1},$$

by

$$(G_1 \cdot G_2)(s_1, \cdots, s_{d_1+d_2}) = \sum_{\tau \in S_{d_1,d_2}} G_1(s_{\tau(1)}, \cdots, s_{\tau(d_1)}) \otimes G_2(s_{\tau(d_1+1)}, \cdots, s_{\tau(d_2)})$$

and $\sigma^k_{G_1 \cdot G_2}(s_1, \ldots, s_{d_1+d_2-k})$ is defined by

$$\sum_{j=0}^{k} \sum_{\tau} \sigma_{G_1}^j(s_{\tau(1)}, \cdots, s_{\tau(d_1-j)}) \otimes \sigma_{G_2}^{k-j}(s_{\tau(1)}, \cdots, s_{\tau(d_2-k+j)})$$

where the first sum runs over (d_1, d_2)-shuffles and the third over (d_1-j, d_2-k+j)-shuffles. ♢

For $\mathcal{N} = E^*$ we will go through some examples which shine some light upon the duality between $\mathscr{V}^\bullet(E)_{\bullet-\mathrm{pol}}$ and $\mathcal{F}^\bullet(E^*)_\bullet$:

Example 4.2.58 ($d = 0$)**.** The set $\mathscr{V}^p(E)_{0-\mathrm{pol}}$ consists of $C^\infty(M)$-linear skew-symmetric maps

$$D : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{p+1-\text{times}} \to C^\infty(M).$$

These correspond precisely to elements of $\Gamma(\Lambda^{p+1}E^*) = \mathcal{F}^0(E^*)_p$. △

In the introduction of this section we already described the duality between derivations on E and derivations on E^*. Using the notation from this section this can be interpreted as an isomorphism between $\mathscr{V}^0(E)_{1-\mathrm{pol}}$ and $\mathcal{F}^1(E^*)_0$.

Example 4.2.59 ($d = 1, p = 1$)**.** Elements of $\mathscr{V}^1(E)_{1-\mathrm{pol}}$ are pre-Lie algebroid structures (see Example 4.2.18 and 4.2.24),

$$L : \Gamma(E) \times \Gamma(E) \to \Gamma(E),$$

with symbol $\sigma_L : E \to TM$. The dual operator is the differential together with the dual of the anchor $(d_E, \sigma_L) \in \mathcal{F}^1(E^*)_{1-\mathrm{pol}}$:

$$\begin{aligned} L^* = d_E : \Gamma(E^*) \to \Gamma(\Lambda^2 E^*), \\ \sigma_L \in \Gamma(E^* \otimes TM) \end{aligned}$$

defined by the usual relation

$$\langle L^*(s), e_1 \wedge e_2 \rangle = \langle e_1, \sigma_L^* d(\langle s, e_2 \rangle) \rangle - \langle e_2, \sigma_L^* d(\langle s, e_1 \rangle) \rangle - \langle s, L(e_1, e_2) \rangle .$$

Moreover, the Jacobiator of L, $\mathrm{Jac}_L : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, an element of $\mathscr{V}^2(E)_{1-\mathrm{pol}}$, is dual to $d_E^2 : \Gamma(E^*) \to \Gamma(\wedge^3 E^*)$, an element of $\mathcal{F}^1(E^*)_2$. △

Example 4.2.60 ($d = 1$)**.** Elements of $\mathcal{F}^1(\mathcal{N})_p$ correspond to the the almost $(p+1)$-differentials of [48], i.e. pairs (δ_1, δ_0) consisting of linear maps

$$\delta_1 : \Gamma(\mathcal{N}) \to \Gamma(\Lambda^{p+1}\mathcal{N}), \quad \delta_0 : C^\infty(M) \to \Gamma(\Lambda^p \mathcal{N})$$

satisfying the relations:

- $\delta_0(fg) = f\delta_0(g) + g\delta_0(f)$, for all $f, g \in C^\infty(M)$
- $\delta_1(fs) = f\delta_1(s) + \delta_0(f) \wedge s$, for all $f \in C^\infty(M), s \in \Gamma(\mathcal{N})$.

Proposition 3.7 of [48] shows that such pairs $(\delta_1, \delta_0) \in \mathcal{F}^1(\mathcal{N})_p$ are in 1-1 correspondence with degree one $(p+1)$-multivector fields on $\mathcal{N}$ and thus with $\mathscr{V}^p(\mathcal{N}^*)_{1-\mathrm{pol}}$. △

The above examples indicate a duality between $\mathscr{V}^p(\mathcal{N}^*)_{d-\mathrm{pol}}$ and $\mathcal{F}^d(\mathcal{N})_p$. Before we continue let us recall some notation: For $e_x \in \mathcal{N}_x$ let

$$\iota_{e_x} : \mathcal{N}_x \to T_{e_x}\mathcal{N}_x,$$

be as in Example 4.2.16. For $s \in \Gamma(\mathcal{N})$ let

$$\wedge^i d_x s : \Lambda^i T_x M \to \Lambda^i T_{s(x)}\mathcal{N},$$

be the algebra morphism induced by $d_x s : T_x M \to T_{s(x)}\mathcal{N}$.

Proposition 4.2.61. *The following defines an algebra morphism:*

$$\varphi : \mathcal{F}^u(\mathcal{N})_p \to \mathfrak{X}^{p+1}(\mathcal{N})_{u-\mathrm{pol}}, \quad G \mapsto X \tag{4.2.20}$$

where for $e_x \in \mathcal{N}_x$ we define:

$$\begin{aligned} X_{e_x} = (u!)^{-1}[\iota_{e_x}(G(s, \dots, s)) - u(\iota_{e_x} \otimes d_x s)(\sigma_G(s, \dots, s)) \\ + \dots + (-1)^u u!(\iota_{e_x} \otimes \wedge^u d_x s)(\sigma_G^u)], \end{aligned} \tag{4.2.21}$$

where s is any section of E with $s(x) = e_x$.

Proof. The argument depends on the following two Leibniz-identities, which follow from relatively short straightforward computations:

Claim 4.2.62. Given $\sigma_G^i \in \mathcal{F}_i^u(\mathcal{N})_p$ and $f \in C^\infty(M)$ and $s \in \Gamma(\mathcal{N})$ we have

- $\sigma_G^i(fs, \ldots, fs) = f^{u-i}\sigma_G^i(s, \ldots, s) + (u-i)f^{u-i-1}\left\langle \sigma_G^{i+1}(s, \ldots, s), df \right\rangle \wedge s.$
- $\wedge^i d(fs)(Y) = f^{i-1}\iota_*(s) \wedge (\wedge^{i-1} ds(\iota_{df} Y)) + f^i \wedge^i ds(Y)$, for all $Y \in \mathfrak{X}^i(M)$.

$\diamondsuit$

We need to show that X_{e_x} does not depend on the choice of section s. So let $f \in C^\infty(M)$ be a function with $f(x) = 1$ then

$$\begin{aligned}\iota_{e_x}(G(fs, \ldots, fs)) &= \iota_{e_x}(f^l G(s, \ldots, s) + l f^{l-1} \langle \sigma_G(s, \ldots, s), df \rangle \wedge s) \\ &= \iota_{e_x}(G(s, \ldots, s) + l \langle \sigma_G(s, \ldots, s), df \rangle \wedge s),\end{aligned}$$

and

$$\begin{aligned}(\iota_{e_x} \otimes d_x(fs))(\sigma_G(fs, \ldots, fs)) = (\iota_{e_x} &\otimes d_x(fs))(f^{l-1}\sigma_G(s, \ldots, s)) \\ &+ (l-1)f^{l-2}\left\langle \sigma_G^2(s, \ldots, s), df \right\rangle \wedge s).\end{aligned}$$

Let us first focus on the first term of the left-hand-side of the previous equation; we find

$$\begin{aligned}(\iota_{e_x} \otimes d_x(fs))(f^{l-1}\sigma_G(s, \ldots, s)) &= \iota_{e_x}(f^{l-1} \langle \sigma_G(s, \ldots, s), df \rangle \wedge s) \\ &\quad + \iota_{e_x}(f d_x s(f^{l-1}\sigma_G(s, \ldots, s)) \\ &= \iota_{e_x}(\langle \sigma_G(s, \ldots, s), df \rangle \wedge s) \\ &\quad + (\iota_{e_x} \otimes d_x s)(\sigma_G(s, \ldots, s)).\end{aligned}$$

We find that the right hand side of (4.2.21) using the section fs is given by:

$$\begin{aligned}&\iota_{e_x}(G(s, \ldots, s)) + l \langle \sigma_G(s, \ldots, s), df \rangle \wedge s - l\iota_{e_x}(\langle \sigma_G(s, \ldots, s), df \rangle \wedge s) \\ &- l(\iota_{e_x} \otimes d_x s)(\sigma_G(s, \ldots, s)) + \ldots = \\ &\iota_{e_x}(G(s, \ldots, s)) - l(\iota_{e_x} \otimes d_x s)(\sigma_G(s, \ldots, s)) + \ldots,\end{aligned}$$

where the terms on the dots only depends on the higher order symbols of G. Continuing in this way we will conclude that that X_{e_x} is well defined, hence we obtain $X \in \mathfrak{X}^{p+1}(\mathcal{N})$. Because X is of degree d in s, it follows that $X \in \mathfrak{X}^{p+1}(\mathcal{N})_{d-\mathrm{pol}}$.

We will now prove that φ is an algebra morphism. Let $G_1 \in \mathcal{F}^{d_1}(\mathcal{N})_{p_1}$ and $G_2 \in \mathcal{F}^{d_2}(\mathcal{N})_{p_2}$. From definition of $\sigma^k_{G_1 \cdot G_2}$ it follows

$$\sigma^k_{G_1 \cdot G_2}(s, \ldots, s) = \sum_{j=0}^{k} \binom{d_1 + d_2 - k}{d_1 - j} \sigma^j_{G_1}(s, \ldots, s) \otimes \sigma^{k-j}_{G_2}(s, \ldots, s).$$

Therefore

$$(d_1+d_2)!\varphi(G_1\cdot G_2) = \sum_k (-1)^k \frac{(d_1+d_2)!}{(d_1+d_2-k)!} \sum_{j=0}^{k} \binom{d_1+d_2-k}{d_1-j} \cdot$$
$$(\iota_{e_x}\otimes \wedge^j d_x s)\sigma^j_{G_1}(s,\dots,s)\otimes(\iota_{e_x}\otimes\wedge^{k-j}d_x s)\sigma^{k-j}_{G_2}(s,\dots,s)$$
$$= \sum_{i,j}(-1)^{i+j}\frac{(d_1+d_2)!}{(d_1-i)!(d_2-j)!}\cdot$$
$$(\iota_{e_x}\otimes\wedge^i d_x s)(\sigma^i_{G_1}(s,\dots,s))\wedge(\iota_{e_x}\otimes\wedge^j d_x s)(\sigma^j_{G_2}(s,\dots,s)).$$

And from

$$\varphi(G_1)\wedge\varphi(G_2) = \frac{1}{(d_1)!(d_2)!}\sum_{i,j}(-1)^{i+1+j+1}\frac{d_1!}{(d_1-i)!}\frac{d_2!}{(d_2-j)!}\cdot$$
$$(\iota_{e_x}\otimes\wedge^i d_x s)(\sigma^i_{G_1}(s,\dots,s))\wedge(\iota_{e_x}\otimes\wedge^j d_x s)(\sigma^j_{G_2}(s,\dots,s))$$

we conclude the desired equality □

Instead of proving that (4.2.20) is an isomorphism directly, we will compose it with the isomorphism $\mathfrak{X}^{p+1}(E^*)_d \simeq \mathscr{V}^p(E)_{d-\mathrm{pol}}$.

To give an explicit formula for this composition we need to introduce a bit of notation: let $I=(i_1,\dots,i_l)$ be a multi-index with $0\le i_j\le p$ and let $\widehat{I}$ be the complementary multi-index, i.e. $I\cup\widehat{I}=\{1,\dots,p\}$. For $s\in\Gamma(E^*)$ and $e_i\in\Gamma(E)$ we denote:

$$\langle ds, e_I\rangle = d\langle s,e_{i_1}\rangle\wedge\cdots\wedge d\langle s,e_{i_l}\rangle \in \Omega^l(M),$$
$$e_I = e_{i_1}\wedge\cdots\wedge e_{i_l}.$$

Lemma 4.2.63. *The composition of* (4.2.20) *with* $\mathfrak{X}^{p+1}(E^*)_{d-\mathrm{pol}}\simeq\mathscr{V}^p(E)_{d-\mathrm{pol}}$*,*

$$\psi:\mathcal{F}^d(E^*)_p\to\mathscr{V}^p(E)_{d-\mathrm{pol}},\quad G\mapsto G^*,$$

is an isomorphism. Explicitly G^* *is given by:*

$$\langle G^*(e_0,\dots,e_p), s\cdot\ldots\cdot s\rangle = \langle G(s,\dots,s), e_0\wedge\cdots\wedge e_p\rangle$$
$$+\sum_{i=1}^{d-l}\sum_{|I|=i}(-1)^i\frac{d!}{(d-i)!}\left\langle\left\langle\sigma^i_G(s,\dots,s),\langle s,e_I\rangle\right\rangle, e_{\widehat{I}}\right\rangle,$$

with $e_0,\dots,e_p\in\Gamma(E)$*, and* $s\in\Gamma(E^*)$ *and the second sum runs over increasing multi-indices.*

To prove the above statement we will have to explicitly compute the symbols of G^*:

Lemma 4.2.64. *Let G^* be as above, then*

$$\langle \sigma^j_{G^*}(e_0,\dots,e_{p-j}), s\cdot\ldots\cdot s\rangle = (-1)^j\frac{d!}{(d-j)!}\left\langle \sigma^j_G(s,\dots,s), e_0\wedge\ldots\wedge e_{p-j}\right\rangle + \sum_{i=j+1}^{d-l}\sum_{|I|=i}(-1)^i\frac{d!}{(d-i)!}\left\langle\left\langle\sigma^i_G(s,\dots,s),\langle s,e_I\rangle\right\rangle, e_{\widehat{I}}\right\rangle.$$

Proof. We consider the identity

$$\langle G^*(e_0,\dots,fe_p), s\cdot\ldots\cdot s\rangle - f\langle G^*(e_0,\dots,e_p), s\cdot\ldots\cdot s\rangle = \sum_{i=1}^{d}\sum_{|I|=i,p\in I}(-1)^i\frac{d!}{(d-i)!}\left\langle\left\langle\sigma^i_G(s,\dots,s),\left\langle ds, e_{I\setminus\{p\}}\right\rangle\wedge df\right\rangle, e_{\widehat{I}}\right\rangle\cdot\langle s,e_p\rangle$$
$$= \langle s,e_p\rangle\,\iota_{df}\sum_{i\geq 1}\sum_{|I|=i,p\in I}(-1)^i\frac{d!}{(d-i)!}\left\langle\left\langle\sigma^i_G(s,\dots,s),\left\langle s,e_{I\setminus p}\right\rangle\right\rangle, e_{\widehat{I}}\right\rangle$$

Carefully continuing inductively one can compute the higher order symbols in a similar fashion, although this is a bit tedious. □

Proof of Lemma 4.2.63. The given expression for G^* follows from carefully pairing (4.2.20) with $e_0\wedge\cdots\wedge e_p$. To prove that the map is surjective, because it is an algebra morphism it is sufficient to show that the generators of $\mathscr{V}^\bullet(E)_{\bullet-\mathrm{pol}}$ are in its image. The generators consist of $\mathscr{V}^{-1}(E)_{1-\mathrm{pol}}$, $\mathscr{V}^0(E)_{0-\mathrm{pol}}$ and $\mathscr{V}^0(E)_{1-\mathrm{pol}}$, which are indeed in the image by Example 4.2.58, and the discussion in the introduction of the section.

Finally, we will show that the association $G\mapsto G^*$ is injective. Assume that $G\in\mathcal{F}^d(E^*)_p$, has $G^*=0$. Then also $\sigma^{p+1}_{G^*}=0$ and as

$$\left\langle(\sigma^{p+1}_G)^*, s\cdot\ldots\cdot s\right\rangle = (-1)^{p+1}\frac{d!}{(d-p)!}\sigma^{p+1}_G(s,\dots,s)\in\Gamma(\wedge^{p+1}TM),$$

we find that $\sigma^{p+1}_G=0$. Next,

$$0=\langle(\sigma^p_G)^*(e), s\cdot\ldots s\rangle = \langle\sigma^p_G(s,\dots,s),e\rangle + \left\langle\sigma^{p+1}_G(s,\dots,s), d\langle e,s\rangle\right\rangle$$

and because $\sigma^{p+1}_G=0$ it will follow that $\sigma^p_G=0$. Carefully continuing inductively one can argue that $\sigma^i_G=0$ for all i. □

There is another, non-canonical way of identifying $\mathcal{F}^d(E^*)_p$ and $\mathscr{V}^p(E)_{d-\mathrm{pol}}$ via the symbol sequence:

$$\mathcal{F}^d_0(E^*)_p\xrightarrow{\sigma}\mathcal{F}^d_1(E^*)_p\xrightarrow{\sigma}\mathcal{F}^d_m(E^*)_p\xrightarrow{\sigma}\dots \qquad (4.2.22)$$

Lemma 4.2.65. *Let ∇ be a connection on E^*, then*

$$h_\nabla : \mathcal{F}^d_{l-1}(E^*)_p \to \mathcal{F}^d_l(E^*)_p,$$
$$h_\nabla(G)(s_0,\ldots,s_{d-l}) \mapsto (-1)^p \sum_i (-1)^i \langle G(s_0,\ldots,\widehat{s_i},\ldots,s_{d-l}), \nabla(s_i)\rangle,$$

*defines an **h**-splitting of* (4.2.22)*. Consequently $\mathcal{F}^d_l(E^*)_n = \Gamma(\mathbb{F}^d_l(E^*)_p)$ for some vector bundle $\mathbb{F}^d_l(E^*)_p$ which is isomorphic to*

$$\mathbb{F}^d_l(E^*)_p \simeq \bigoplus_{i=0}^{p-l+1} \Lambda^{p-l+1-i}E^* \oplus S^{d-l-i}E \oplus \Lambda^{l+i}TM. \tag{4.2.23}$$

Comparing (4.2.23) with Corollary 4.2.51 gives the desired duality between $\mathcal{F}^d(E^*)_p$ and $\mathscr{V}^p(E)_{d-\mathrm{pol}}$

4.3 Representations up to homotopy

In this section we explain how homogeneous multivector fields can be used to give an intrinsic description of Lie algebroid cohomology with coefficients in the symmetric powers of the adjoint representation. At the same time it will explain the appearance of the particular vector bundle in Corollary 4.2.51.

This will generalise a result of Crainic-Moerdijk which relates the deformation complex with the adjoint representation. Moreover, we will comment on the similarities with the description of the Weil algebra of a Lie algebroid using algebroid forms in the symmetric powers of the co-adjoint representation. In this way we will also investigate a possible duality of the latter with homogeneous differential forms.

We will quickly recall the definition of a representation up to homotopy, and the definition of the (co)adjoint representation up to homotopy. For more details we will refer to [1]. Let $A \to M$ be a Lie algebroid and $E \to M$ a graded vector bundle. Then $\Omega(A;E)$ becomes a bigraded algebra, where $\Omega(A;E)^k$ will denote the forms with total degree k, that is:

$$\Omega(A;E)^k = \bigoplus_{i+j=k} \Omega^i(A;E_j).$$

Definition 4.3.1. A **representation up to homotopy** of a Lie algebroid $A \to M$ consists of a graded vector bundle $E \to M$, and an operator, called the structure operator,

$$D : \Omega(A;E)^k \to \Omega(A;E)^{k+1},$$

which increases the total degree by 1 and satisfies $D^2 = 0$ and the graded Leibniz identity

$$D(\omega\eta) = d_A(\omega)\eta + (-1)^k \omega D(\eta),$$

for all $\omega \in \Omega^k(A)$ and $\eta \in \Omega(A; E)$. Here $d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ denotes the Lie algebroid differential. ♢

To recall the definition of the adjoint representation up to homotopy, we first need to recall the necessary building blocks:

Definition 4.3.2. Let A be a Lie algebroid over M and let ∇ be a connection on A, we define

- The **basic A-connection** induced by ∇ on A:

$$\nabla^{\text{bas}}_\alpha(\beta) = \nabla_{\rho(\beta)}(\alpha) + [\alpha, \beta] \quad \text{for } \alpha, \beta \in \Gamma(A).$$

- The **basic A-connection** on TM:

$$\nabla^{\text{bas}}_\alpha(X) = \rho(\nabla_X(\alpha)) + [\rho(\alpha), X] \quad \text{for } \alpha \in \Gamma(A), X \in \Gamma(TM).$$

- The **basic curvature** $R^{\text{bas}}_\nabla \in \Omega^2(A; \text{Hom}(TM; A))$ given by

$$\begin{aligned} R^{\text{bas}}_\nabla(\alpha, \beta)(X) := {} & \nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)] \\ & - \nabla_{\nabla^{\text{bas}}_\beta(X)}(\alpha) + \nabla_{\nabla^{\text{bas}}_\alpha(X)}(\beta). \end{aligned}$$

♢

Definition 4.3.3. Let ∇ be a connection on A. The **adjoint representation up to homotopy** is a representation up to homotopy with graded vector bundle $E_0 = A, E_1 = TM$ and zero otherwise and structure operator

$$D_\nabla : \Omega(A; E)^k \to \Omega(A; E)^{k+1},$$

consisting of:

- The De Rham-like operator associated to the A-connection ∇^{bas}_A on A:

$$d_{\nabla^{\text{bas}}_A, A} : \Omega^k(A; A) \to \Omega^{k+1}(A; A),$$

- The map

$$\rho : \Omega^k(A; A) \to \Omega^k(A; TM),$$

  obtained by applying the anchor $\rho : A \to TM$ to the coefficient.

- The De Rham-like operator associated to the A-connection ∇^{bas}_{TM} on TM:

$$d_{\nabla^{\text{bas}}_{TM}, A} : \Omega^{k-1}(A; TM) \to \Omega^k(A; TM).$$

- The map

$$R^{\text{bas}}_\nabla : \Omega^{k-1}(A; TM) \to \Omega^{k+1}(A; A),$$

  obtained by applying $R^{\text{bas}}_\nabla \in \Omega^2(A; \text{End}(TM, A))$ to the coefficient and then taking the wedge product.

We will denote this representation up to homotopy by ad_∇. ♢

Definition 4.3.4. Let ∇ be a connection on A. The **coadjoint representation up to homotopy** is a representation up to homotopy with graded vector bundle $E_{-1} = T^*M, E_0 = A^*$ and zero otherwise and structure operator

$$D_\nabla : \Omega(A; E)^k \to \Omega(A; E)^{k+1},$$

consisting of:

- The De Rham-like operator associated to the A-connection $(\nabla_A^{\mathrm{bas}})^*$ on A^*:
$$d_A^{(\nabla_A^{\mathrm{bas}})^*} : \Omega^k(A; A^*) \to \Omega^{k+1}(A; A^*)$$
- The map
$$(\rho)^* : \Omega^{k+1}(A; T^*M) \to \Omega^{k+1}(A; A^*),$$
obtained by applying the dual of the anchor $\rho^* : T^*M \to A^*$ to the coefficient.
- The De Rham-like operator associated to the TM-connection $(\nabla_T^{\mathrm{bas}} M)^*$ on T^*M:
$$d_A^{(\nabla_{TM}^{\mathrm{bas}})^*} : \Omega^{k+1}(A; T^*M) \to \Omega^{k+2}(A; T^*M)$$
- The map
$$-(R_\nabla^{\mathrm{bas}})^* : \Omega^k(A; A^*) \to \Omega^{k+2}(A; T^*M),$$
obtained by applying the dual coefficient of $R_\nabla^{\mathrm{bas}} \in \Omega^2(A; \mathrm{End}(TM; A))$ to the coefficient and then taking the wedge product.

We will denote this representation up to homotopy by ad_∇^*. ♢

One of the issues of these definitions is that one needs to use a connection on A to describe the structure operator, although the resulting representation up to homotopy is canonical up to isomorphism. To circumvent the use of a connection one can introduce:

Definition 4.3.5 ([66]). Let A be a Lie algebroid. The **deformation complex** $C^\bullet_{\mathrm{def}}(A)$ is the graded Lie algebra $\mathscr{V}^\bullet(A)_{1-\mathrm{pol}}$ together with the differential δ given by $\delta(D) = [\pi, D]$ where $\pi \in C^1_{\mathrm{def}}(A)$ is the Poisson structure corresponding to the Lie algebroid structure on A. ♢

Proposition 4.3.6 ([1]). *Let A be a Lie algebroid and ∇ a connection on A. Then there exists a canonical isomorphism of cochain complexes:*

$$(C^\bullet_{\mathrm{def}}(A), \delta) \simeq (\Omega(A; \mathrm{ad}_\nabla), D_\nabla).$$

We will now give a similar statement but using coefficients in the symmetric powers of the adjoint representation. As a graded vector bundle this is given by

$$S^d \,\mathrm{ad}_\nabla = \underbrace{(S^d A)}_{\text{degree } 0} \oplus \underbrace{(S^{d-1} A \otimes TM)}_{\text{degree } 1} \oplus \cdots \oplus \underbrace{(A \otimes \wedge^d TM)}_{\text{degree } d-1} \oplus \underbrace{(\wedge^d TM)}_{\text{degree } d}.$$

Theorem 4.3.7. *Let $A \to M$ be a Lie algebroid. A connection ∇ on A induces an isomorphism of differentially graded algebras:*

$$(\mathscr{V}^p(A)_{d-\mathrm{pol}}, d_{\pi_1}) \simeq (\Omega(A; S^d \operatorname{ad}_\nabla)^{p+1}, D_\nabla).$$

Proof. First we remark that

$$\Omega(A; S^d \operatorname{ad}_\nabla)^{p+1} = \bigoplus_{i=0}^{p+1} \Gamma(\Lambda^{p-i+1} A^* \otimes S^{d-i} A \otimes \wedge^i TM)$$

Therefore the isomorphism from Proposition 4.2.51 provides an isomorphism of $C^\infty(M)$-modules between $\Omega(A; S^d \operatorname{ad}_\nabla)^{p+1}$ and $\mathscr{V}^p(A)_{d-\mathrm{pol}}$. We will however work with its inverse $\psi : \Omega(A; S^d \operatorname{ad}_\nabla)^{p+1} \to \mathscr{V}^p(A)_{d-\mathrm{pol}}$ from Proposition 4.2.52. But, for this isomorphism to intertwine the differentials we need to add some signs. We define $\widetilde{\psi}$, which on $\Gamma(\Lambda^{p-i+1} A^* \otimes S^{d-i} A \otimes \wedge^i TM)$ is given by $\widetilde{\psi} = (-1)^i \psi$. As this is still an isomorphism of algebras, we only need to prove it intertwines the differentials in low degrees, that is on $\wedge^1 A^*, S^1 A$ and TM.
On $\wedge^1 A^*$: In this case we need to show that the following diagram commutes

$$\begin{array}{ccc} \Omega^1(A; S^0 A) = \Gamma(\wedge^1 A^*) & \xrightarrow{\iota_*} & \mathscr{V}^0(A)_{0-\mathrm{pol}} \\ \Big\downarrow d_A & & \Big\downarrow d_{\pi_1} \\ \Omega^2(A; S^0 A) = \Gamma(\wedge^2 A^*) & \xrightarrow{\iota_*} & \mathscr{V}^1(A)_{0-\mathrm{pol}} \end{array}$$

which is immediate because π_1 is the fibrewise linear Poisson structure corresponding to the algebroid structure on A.
On $S^1 A$: We have to show that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(S^1 A) & \xrightarrow{\widehat{\ }} & \mathscr{V}^{-1}(A)_{1-\mathrm{pol}} \\ \Big\downarrow D_\nabla & & \Big\downarrow d_{\pi_1} \\ \Gamma(\wedge^1 A^* \otimes S^1 A \oplus TM) & \xrightarrow{\widetilde{\psi}} & \mathscr{V}^0(A)_{1-\mathrm{pol}} \end{array}$$

Let $s \in \Gamma(A)$, we have $D_\nabla(s) = (\nabla^{\mathrm{bas}}_{\cdot}(s), \rho(s))$. Moreover, for any $e \in \Gamma(A)$ we have

$$\begin{aligned} \left\langle \widetilde{\psi}(D_\nabla(s)), e \right\rangle &= \left\langle \widetilde{\psi}(\nabla^{\mathrm{bas}}_{\cdot}(s)), e \right\rangle + \left\langle \widetilde{\psi}(\rho(s)), e \right\rangle \\ &= \widehat{\nabla^{\mathrm{bas}}_e(s)} - \widehat{\nabla_{\rho(s)}(e)} \\ &= \widehat{\nabla_{\rho(s)}(e)} + \widehat{[e,s]} - \widehat{\nabla_{\rho(s)}(e)}, \end{aligned}$$

whereas $\langle d_{\pi_1}(s), e\rangle = \pi(d\widehat{e}, d\widehat{s}) = \widehat{[e,s]}$ from which we conclude that the diagram commutes.
On TM: We have to show that the following diagram commutes

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{\quad\widetilde{\psi}\quad} & \mathscr{V}^0(A)_{1-\mathrm{pol}} \\ \Big\downarrow D_\nabla & & \Big\downarrow d_{\pi_1} \\ \Gamma(\wedge^2 A^* \otimes S^1 A \oplus \wedge^1 A^* \otimes TM) & \xrightarrow{\quad\widetilde{\psi}\quad} & \mathscr{V}^1(A)_{1-\mathrm{pol}} \end{array}$$

We have $D_\nabla(X) = R_\nabla^{\mathrm{bas}}(\cdot,\cdot)(X) + \nabla_\cdot^{\mathrm{bas}}(X)$, and

$$\begin{aligned}\left\langle \widetilde{\psi}(R_\nabla^{\mathrm{bas}}(\cdot,\cdot)(X)), e_1 \wedge e_2) \right\rangle &= \nabla_X([e_1,e_2]) - [\nabla_X(e_1), e_2] - [e_1, \nabla_X(e_2)] \\ &\quad - \nabla_{\nabla_{e_2}^{\mathrm{bas}}(X)}(e_1) + \nabla_{\nabla_{e_1}^{\mathrm{bas}}(X)}(e_2)\end{aligned}$$

so

$$\begin{aligned}\widetilde{\psi}(\nabla_\cdot^{\mathrm{bas}}(X))(e_1,e_2) &= \left\langle -\operatorname{hor}^{\nabla^*}(\nabla_{e_1}^{\mathrm{bas}}(X)), e_2 \right\rangle - \left\langle -\operatorname{hor}^{\nabla^*}(\nabla_{e_2}^{\mathrm{bas}}(X)), e_1 \right\rangle \\ &= -\nabla_{\nabla_{e_1}^{\mathrm{bas}}(X)}(e_2) + \nabla_{\nabla_{e_2}^{\mathrm{bas}}(X)}(e_1)\end{aligned}$$

and consequently

$$\left\langle \widetilde{\psi}(D_\nabla(X)), e_1 \wedge e_2 \right\rangle = \nabla_X([e_1,e_2]) - [\nabla_X(e_1), e_2] - [e_1, \nabla_X(e_2)].$$

On the other hand

$$\begin{aligned}d_{\pi_1}(\widetilde{\psi}(X))(e_1,e_2) &= -\pi_1(d\widehat{e}_1, d\widehat{\nabla_X(e_2)}) + \pi_1(d\widehat{e}_2, d\widehat{\nabla_X(e_1)}) + \nabla_X(\pi_1(d\widehat{e}_1, d\widehat{e}_2)) \\ &= -[e_1, \nabla_X(e_2)] + [e_2, \nabla_X(e_1)] + \nabla_X([e_1,e_2]).\end{aligned}$$

We thus conclude that $\widetilde{\psi}(D_\nabla(X)) = d_{\pi_1}(\widetilde{\psi}(X))$.

□

Remark 4.3.8. Note that when $E \to M$ is just a vector bundle, and does not carry an algebroid structure one can still consider the graded vector bundle $E_0 = E, E_1 = TM$, and the associated graded symmetric power, again denoted by $S^d \operatorname{ad}_\nabla$. In this case Proposition 4.2.51 still provides an isomorphism between $\mathscr{V}^p(A)_{d-\mathrm{pol}}$ and $\Omega(E; S^d \operatorname{ad}_\nabla)^{p+1}$ but neither side carries a differential. ♢

4.3.1 Homogeneous forms and the Weil algebra

We would like to end this chapter by discussing a possible duality between homogeneous differential forms and the Weil algebra.

When one replaces the symmetric powers of the adjoint representation by symmetric powers of the coadjoint representation in Theorem 4.3.7 one could wonder what the object it is isomorphic to is. This turns out to be the Weil algebra:

Definition 4.3.9 ([1])**.** Let $A \to M$ be a Lie algebroid. The **Weil algebra** of A, $W^{p,q}(A)$, consists of sequences of operators $c = (c_0, c_1, \ldots)$

$$c_i : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{p-i \text{ times}} \to \Gamma(\wedge^{q-i} T^*M \otimes S^i A^*)$$

satisfying

$$c_i(\alpha_1, \ldots, f\alpha_{p-i}) = fc_i(\alpha_1, \ldots, \alpha_{p-i}) - df \wedge \partial_{\alpha_{p-i}}(c_{i+1}(\alpha_1, \ldots, \alpha_{p-i-1})),$$

for all $\alpha_j \in \Gamma(A)$ and $f \in C^\infty(M)$. Here $\partial_\alpha : S^k(A^*) \to S^{k-1}(A^*)$ is given by:

$$\partial_\alpha(P)(\alpha_0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P(\alpha_0 + t\alpha).$$

if one views $P \in S^k(A^*)$ as a function on A. ♢

Moreover, $W^{p,q}(A)$ can be equipped with the structure of differential graded algebra. With differential $d_W = d^v + d^h$, where the vertical differential d^v raises p by one and the horizontal differential d^h raises q by one.

Proposition 4.3.10 ([1]). *Let $A \to M$ be a vector bundle. A connection ∇ induces an isomorphism of differentially graded algebras:*

$$W^{p,q}(A) \simeq \Omega(A; S^q \operatorname{ad}^*_\nabla)^{p-q} \simeq \bigoplus_k \Gamma(\wedge^{q-k} T^*M \oplus S^k A^* \oplus \wedge^{p-k} A^*). \quad (4.3.1)$$

Considering how Theorem 4.3.7 induces a relation between algebroid forms with values in the adjoint representation and homogeneous vector fields on the dual of the algebroid, one is lead to wonder whether a similar relation between algebroid forms with values in the *coadjoint* representation and homogeneous differential *forms* exists:

Definition 4.3.11. Let $E \to M$ be a vector bundle, define the space of **homogeneous differential forms** by:

$$\Omega^k_l(E) := \{\omega \in \Omega^k(E) : m^*_\lambda \omega = \lambda^l \omega\}.$$

Elements of the space $\Omega^k_1(E)$ are often **linear** differential forms. ♢

A relation between linear differential forms and the Weil algebra is described in [11]:

Proposition 4.3.12 (Proposition 3 in [11]). *Let E be a vector bundle, then there is a canonical isomorphism of differential graded algebras $(\Omega^\bullet_1(E), d) \simeq (W^{1,\bullet}(E), d_v)$*

There is a very naive guess for a possible extension of this result, namely $\Omega^\bullet_l(E) \simeq W^{l,\bullet}(E)$. However this is clearly wrong as $W^{\bullet,0}(E) = \Gamma(\wedge^\bullet E^*)$ and $\Omega^0_l(E) = \Gamma(S^\bullet E^*)$. So let us proceed with a bit more care: Given $\omega \in \Omega^q_p(E)$, we may view it as a skew-symmetric $C^\infty(E)$-linear map

$$\omega : \underbrace{\mathfrak{X}^1(E) \times \cdots \times \mathfrak{X}^1(E)}_{k-\text{times}} \to C^\infty(E).$$

However, while we could view multivector fields as multi-derivations on functions, we cannot do the same for differential forms. Still we can restrict ω to

$\Gamma(E)$ by the inclusion $\iota_* : \Gamma(E) \hookrightarrow \mathfrak{X}^1(E)$. However, this alone does not fully determine ω; one also needs to use an auxiliary connection ∇ on E to obtain a horizontal lift $\mathfrak{X}^1(M) \hookrightarrow \mathfrak{X}^1(E)$. In this way ω induces a sequence of operators:

$$\underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{q-\text{times}} \to \Gamma(S^{p-q}E^*)$$
$$\underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{(q-1)-\text{times}} \times \mathfrak{X}^1(M) \to \Gamma(S^{p-q+1}E^*)$$
$$\cdots$$
$$\Gamma(E) \times \underbrace{\mathfrak{X}^1(M) \times \cdots \times \mathfrak{X}^1(M)}_{(q-1)-\text{times}} \to \Gamma(S^{p-1}E^*)$$
$$\underbrace{\mathfrak{X}^1(M) \times \cdots \times \mathfrak{X}^1(M)}_{q-\text{times}} \to \Gamma(S^pE^*)$$

Note that all of these maps are $C^\infty(M)$-linear and skew-symmetric, and therefore ω induces an element of

$$\bigoplus_{k=0}^{q} \Gamma(\wedge^{q-k}T^*M \otimes S^{p-k}E^* \otimes \wedge^k E^*).$$

As this process can be reversed we conclude the following:

Lemma 4.3.13. *A connection on E induces an isomorphism*

$$\Omega^q_p(E) \simeq \bigoplus_{k=0}^{q} \Gamma(\wedge^{q-k}T^*M \otimes S^{p-k}E^* \otimes \wedge^k E^*). \tag{4.3.2}$$

This has a striking similarity with the description of the Weil algebra from Proposition 4.3.10 and indeed we see that

$$W^{1,q}(E) = \Gamma(\Lambda^q T^*M \otimes \Lambda^1 E^*) \oplus \Gamma(\Lambda^{q-1}T^*M \oplus S^1E^*),$$

and

$$\Omega^q_1(E) = \Gamma(\Lambda^q T^*M \otimes S^1 E^*) \oplus \Gamma(\Lambda^{q-1}T^*M \otimes \Lambda^1 E^*).$$

This gives a direct isomorphism between $W^{1,q}(E)$, and $\Omega^q_1(E)$, and thus proving Proposition 4.3.12. However, note that from this description it is not clear at all that this isomorphism is canonical, whereas from the description in [11] it is. We also see that this low-degree isomorphism worked so easily because $\wedge^1 E^* = S^1 E^*$. But if we pass to higher values of p, we find by comparing (4.3.1) and (4.3.2) that for any p, q with $p > 1$ there exists no l, m such that $\Omega^q_p(E)$ is isomorphic to $W^{l,m}(E)$.

However, the entire spaces $W^{\bullet,\bullet}(E)$ and $\Omega^\bullet_\bullet(E)$ are isomorphic. It would be very interesting to study the relation between these two spaces. For instance, whereas the left-hand side has two differentials (d^h and d^v), a priori the right-hand side has only one (d_{DR}).

Chapter 5

Homogeneous normal form results in Poisson geometry

In this chapter we will study homogeneous normal form results in Poisson geometry. We will consider Taylor expansions of Poisson structures in terms of homogeneous bivector fields. The algebraic description of homogeneous polynomial bivector fields obtained in the previous chapter will help us understand the desired normal forms. In this introduction we collect the main definitions and results from this chapter.

For a Poisson manifold (N, π) we will consider Taylor expansions of π, around a submanifold $M \subset N$, of the form $\sum_d \pi_d$, with each $\pi_d \in \mathfrak{X}^2(\nu_M)$ of degree d. The first non-zero term of this Taylor expansion always defines a Poisson structure on the normal bundle $\pi_d \in \mathfrak{X}^2(\nu_M)_{d-\mathrm{pol}}$. One could hope that π_d provides a local form theorem for π around M. We will study when this is the case and focus our attention in particular on the linear and quadratic case.
Linear: For the linear approximation π_1 to be Poisson we will need the constant part π_0 to vanish. This condition is equivalent to $M \subset (N, \pi)$ being a *co-isotropic* submanifold. However, we will see that π_1 will have a chance to provide a normal form only if $M \subset (N, \pi)$ is also a *Lagrangian* submanifold. We will describe precisely when this is the case in terms of cohomological conditions. Because π_1 has degree 1, it induces cochain complexes $(\mathfrak{X}^\bullet(\nu_M)_{k-\mathrm{pol}}, d_{\pi_1})$. If we let $F^k\mathfrak{X}^\bullet(\nu_M)_M$ denote germs of multivector fields X for which $X_{i-\mathrm{pol}} = 0$ for all $i < k$ we obtain a cochain complex $(F^k\mathfrak{X}^\bullet(\nu_M)_M, d_\pi)$. Using these we can detect precisely when a Lagrangian is linearisable:

Definition 5.1.32. *Let $M \subset (N, \pi)$ be a Lagrangian submanifold and let X be an Euler-like vector field for M. The **linearisation class** of M is defined to be*

$$\Lambda(\pi, M) := [\pi - d_\pi(X)] \in H^2_\pi(F^2\mathfrak{X}^\bullet(\nu_M)_M).$$

Proposition 5.1.33. *A Lagrangian submanifold $M \subset (N, \pi)$ is linearisable precisely when $\Lambda(\pi, M) = 0$.*

Directly computing the linearisation class is not easy. Yet, we can use the algebraic structure on the homogeneous multivector fields to give sufficient conditions for when it vanishes:

Theorem 5.1.34. *Let (N, π) be a Poisson manifold, and $M \subset (N, \pi)$ a Lagrangian submanifold. If $H^1(\nu_M)_{0-\mathrm{pol}} = 0, H^1(\nu_M)_{1-\mathrm{pol}} = 0$ and $[\pi] = 0 \in H^2(N)_M$, then the linearisation class vanishes.*

As we have seen in Theorem 4.3.7, $H^1(\nu_M)_{0-\mathrm{pol}}$ is the Lie algebroid cohomology for the Lie algebroid associated to π_1 and $H^1(\nu_M)_{1-\mathrm{pol}}$ is the Lie algebroid cohomology with coeffients in the adjoint representation.

We conclude our discussion of the linear case by commenting on possible ways in which integrablity of the Poisson manifold ensures that these cohomology groups vanish, although we obtained no clear applicable result yet.

In higher degrees we will quickly discuss a similar result to Proposition 5.1.33 using a d-th order approximation class defined in Definition 5.1.36.

Quadratic: For the quadratic approximation π_2 to be Poisson we need the linear part π_1 to vanish. This condition will be satisfied for a particular class of *Poisson submanifolds*. After given a digression of what is known on normal form results around general Poisson submanifolds, we will describe a class of Poisson structures for which π_2 gives a normal form result. These are the *log symplectic* and *elliptic symplectic* structures. Most of the normal form results for these structures have appeared before. We will collect explicit formulae for these normal forms, and explain what choices are needed to describe them using the theory of the previous chapter.

Elliptic symplectic structures fall in two categories, depending on whether the elliptic residue vanishes. The ones with vanishing elliptic residue are studied in Chapter 1 because of their relation with generalized complex structures. The ones with non-vanishing elliptic residue had not been studied in detail yet.

Proposition 5.2.25. *Let (N, π) be an elliptic symplectic manifold with degeneracy locus D. If the the elliptic residue of π is $\tau \neq 0$ then there exists a canonical flat connection ∇ on ν_D^* such that*

$$\pi_2 = \tau^{-1}\partial_\theta \wedge r\partial_r + \mathrm{hor}^{\nabla^*}(\pi_D) \in \mathfrak{X}^2(\nu_D)$$

is Poisson diffeomorphic to π on a tubular neighbourhood of D.

With this explicit formula at hand we can prove the following:

Lemma 5.2.38. *Let (M^{2n}, ω) be a symplectic manifold and let $S^{2n-2} \subset M$ be a compact symplectic submanifold with flat normal bundle. Then there exists an elliptic symplectic structure $\widetilde{\omega}$ with non-zero elliptic residue on M with degeneracy locus precisely S.*

Under (nearly) identical assumptions one can prove existence of log symplectic structures ([14]). In the same spirit, there is a folklore result which states that the existence of a log symplectic structure on a symplectic manifold can be deduced from the existence of a cosymplectic submanifold. However, the process can not be reversed: there are log symplectic manifolds which do not admit symplectic structures. In the non-zero elliptic residue case this procedure can however be easily reversed:

Lemma 5.2.41. *Let ω be an elliptic symplectic structure on M with non-zero elliptic residue and degeneracy locus D. Then there exists a symplectic structure on M which has D as symplectic submanifold.*

We conclude that if one's goal is to construct singular symplectic structures on manifolds which are not symplectic, like we did in Chapter 1, one can disregard elliptic symplectic structures with non-zero elliptic residue.

5.1 Submanifolds in Poisson Geometry

5.1.1 Taylor expansions of vector fields

A multivector field $X \in \mathfrak{X}^{p+1}(\mathcal{N})$ is called **(fiberwise) polynomial** if it is a finite sum of homogeneous polynomial multivector fields (of various degrees), that is

$$X \in \mathfrak{X}^{\bullet}(\mathcal{N})_{\mathrm{pol}} = \oplus_d \mathfrak{X}^{\bullet}(\mathcal{N})_{d-\mathrm{pol}}.$$

The completion of this space consists of formal infinite sums of homogeneous polynomial multivector fields, that is

$$\mathfrak{X}^{\bullet}(\mathcal{N})_{\mathrm{form}} = \prod_d \mathfrak{X}^{\bullet}(\mathcal{N})_{d-\mathrm{pol}} = \left\{ \sum_{d=0}^{\infty} X_d : X_d \in \mathfrak{X}^{\bullet}(\mathcal{N})_{d-\mathrm{pol}} \right\}.$$

Its elements are called **formal multivector fields** on $\mathcal{N}$.

Because the rescaling operators $m_\lambda : \mathcal{N} \to \mathcal{N}$ are bracket-preserving, the Lie bracket preserves polynomality and there is an induced Lie bracket:

$$[\cdot,\cdot] : \mathfrak{X}^{\bullet}(\mathcal{N})_{\mathrm{form}} \times \mathfrak{X}^{\bullet}(\mathcal{N})_{\mathrm{form}} \to \mathfrak{X}^{\bullet}(\mathcal{N})_{\mathrm{form}}. \tag{5.1.1}$$

The multiplication operators m_λ on $\mathcal{N}$ also allow us to define a Taylor approximation of multivector fields $X \in \mathfrak{X}^{p+1}(\mathcal{N})$ by considering expansions of $m_\lambda^* X$ of type

$$\sum_{d \geq 0} \lambda^{d-p-1} X_{d-\mathrm{pol}}, \quad \text{with } X_{d-\mathrm{pol}} \in \mathfrak{X}^{p+1}(\mathcal{N}). \tag{5.1.2}$$

The components $X_{d-\mathrm{pol}}$ can be explicitly computed by the following set of equations:

$$X_{0-\mathrm{pol}} = \lim_{\lambda \to 0} \lambda^{p+1} \cdot m_\lambda^* X,$$

$$X_{k-\mathrm{pol}} = \lim_{\lambda \to 0} \frac{1}{\lambda^k} \left(\lambda^{p+1} \cdot m_\lambda^* X - \sum_{i=0}^{k-1} \lambda^i \cdot X_{i-\mathrm{pol}} \right).$$

Lemma 5.1.1. *Taking the infinite jet is a Lie algebra map:*

$$j^\infty : \mathfrak{X}^\bullet(\mathcal{N}) \to \mathfrak{X}^\bullet(\mathcal{N})_{\mathrm{form}}, \quad X \mapsto \sum_d X_{d-\mathrm{pol}}$$

In other words:

1. *For any* $X \in \mathfrak{X}^{p+1}(\mathcal{N})$*,* $X_{d-\mathrm{pol}}$ *is polynomial of degree* d*, i.e.* $m_\lambda^* X_{d-\mathrm{pol}} = \lambda^{d-p-1} X_{d-\mathrm{pol}}$.
2. *For any multivector fields* X *and* Y*,*

$$[X, Y]_{d-\mathrm{pol}} = \sum_{d_1+d_2=d+1} [X_{d_1-\mathrm{pol}}, Y_{d_2-\mathrm{pol}}].$$

All these constructions are functorial with respect to isomorphisms of vector bundles over M. However, as we are interested in tubular neighbourhoods, it is important to look at more general diffeomorphisms

$$\Phi : \mathcal{N} \to \mathcal{N}', \quad \text{with } \Phi|_M = \mathrm{Id}_M, \tag{5.1.3}$$

between two vector bundles $\mathcal{N}, \mathcal{N}'$ over M. It is true that such a map induces a vector bundle isomorphism $d^{\mathrm{vert}}\Phi : \mathcal{N} \to \mathcal{N}'$, but this contains only the linear information of Φ (along M) and that is not enough. On the other hand, Φ also induces an isomorphism

$$\Phi_* : \mathfrak{X}^\bullet(\mathcal{N}) \to \mathfrak{X}^\bullet(\mathcal{N}')$$

but this does not preserve polynomiality. Instead, we need the "Taylor expansion of Φ" (around M), which we define as the map

$$j^\infty\Phi_* : \mathfrak{X}^\bullet(\mathcal{N})_{\mathrm{form}} \to \mathfrak{X}^\bullet(\mathcal{N}')_{\mathrm{form}}, \quad j^\infty X \mapsto j^\infty\Phi_*(X). \tag{5.1.4}$$

To see that this is well defined, one just remarks that if X_d is homogeneous polynomial of degree d than $\Phi_*(X)_k = 0$ for $k < d$. We deduce:

Lemma 5.1.2. *Every diffeomorphism* Φ *as above (5.1.3) induces a Lie algebra isomorphism between the corresponding spaces of formal power series (5.1.4)*

$$j^\infty\Phi_* : \mathfrak{X}^\bullet(\mathcal{N})_{\mathrm{form}} \xrightarrow{\sim} \mathfrak{X}^\bullet(\mathcal{N}')_{\mathrm{form}}.$$

It is also convenient to describe the coefficients of the Taylor expansion as in Equation (5.1.2) in terms of their cores:

Lemma 5.1.3. *Let* $X \in \mathfrak{X}^{p+1}(E^*)$ *and let* $D \in \mathscr{V}^p(E)$ *denote its core. Then the core of* $X_{k-\mathrm{pol}} \in \mathfrak{X}^{p+1}(E^*)_{k-\mathrm{pol}}$ *is* $D_{k-\mathrm{pol}} \in \mathscr{V}^p(E)_{k-\mathrm{pol}}$*, which is given by:*

$$D_{k-\mathrm{pol}}(s_0, \ldots, s_p) := (D(s_0, \ldots, s_p))_{k-\mathrm{pol}},$$

for $s_0, \ldots, s_p \in \Gamma(E)$*. Moreover,*

$$\sigma^l_{D_{k-\mathrm{pol}}}(f_1, \ldots, f_l; s_0, \ldots, s_{p-l}) = \sigma_D(f_1, \ldots, f_l; s_0, \ldots, s_{p-l})_{(k-l)-\mathrm{pol}},$$

for $f_1, \ldots, f_l \in C^\infty(M)$*.*

We now move to our motivating setting, which is understanding multivector fields (and in particular Poisson structures) around submanifolds. Let $M \subset N$ be a submanifold, and $X \in \mathfrak{X}^{p+1}(N)$ a multivector field. Using a tubular neighbourhood, we can transport X to a multivector field $\widetilde{X} \in \mathfrak{X}^{p+1}(\nu_M)$, and take its k-th order approximations $\widetilde{X}_{k-\mathrm{pol}} \in \mathfrak{X}^{p+1}(\nu_M)_{k-\mathrm{pol}}$. However, in general these will depend on the tubular neighbourhood. However, as is common when working with jets, the first non-zero term does not depend on the choice of tubular neighbourhood.

To prove this, we will first consider the simplest case, namely $p = -1$, that is, functions on N. For functions $f \in C^\infty(N)$ taking homogeneous approximations around a submanifold $M \subset N$ is a well-known construction. Starting with the first order approximation, the normal derivative, and the second order approximation, the normal Hessian. We recall this construction for the sake of completeness:

Definition 5.1.4. Let $M \subset N$ be a submanifold and let $f \in C^\infty(N)$ be a function. Suppose that for all vector fields $X_1, \dots, X_{k-1}$ in a neighbourhood of M we have $\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_{k-1}}(f)(x) = 0$ for all $x \in M$. Then we define

$$d^{k,\nu_M} f \in \Gamma(S^k \nu_M^*),$$

by

$$d^{k,\nu_M} f(v_1, \dots, v_k)(p) = \mathcal{L}_{X_1}\mathcal{L}_{X_2} \cdots \mathcal{L}_{X_k}(f)(p),$$

where $p \in M$, $v_1, \dots, v_k \in \Gamma(\nu_M)$ and $X_1, \dots, X_k$ are vector fields in a neighbourhood of M with $v_i|_M \mod TM = X_i$. ♢

We prove this is well-defined:

Proof. Let $X_1, \widetilde{X}_1$ be two vector fields whose normal parts coincide with v_1. Because $d^{k-1,\nu_M} f = 0$ we have that $\mathcal{L}_{X_2} \cdots \mathcal{L}_{X_k}(f)$ vanishes at M, but as $X_1 - \widetilde{X}_1$ is tangent to M we have

$$\mathcal{L}_{X_1 - \widetilde{X}_1}(\mathcal{L}_{X_2} \cdots \mathcal{L}_{X_k}(f)) = 0.$$

We conclude that $d^{k,\nu_M} f$ does not depend on the choice of vector field representing v_1. Continuing inductively we conclude that $d^{k,\nu_M} f$ does not depend on any choices. To check that $d^{k,\nu_M} f$ is symmetric, we note that

$$\mathcal{L}_{X_1}\mathcal{L}_{X_2} \cdots \mathcal{L}_{X_k}(f)(p) - \mathcal{L}_{X_2}\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_k}(f)(p) = \mathcal{L}_{[X_1,X_2]}\mathcal{L}_{X_3} \cdots \mathcal{L}_{X_k}(f)(p).$$

But as $d^{k-1,\nu_M} f$ vanishes, the right-hand side is zero. This proves that $d^{k,\nu_M} f$ is indeed symmetric. □

The following is now readily verified by taking local coordinates:

Lemma 5.1.5. *Let $M \subset N$ be a submanifold and $f \in C^\infty(N)$ a function. If for some tubular neighbourhood the approximations $f_{i-\mathrm{pol}} \in C^\infty(\nu_M)$ vanish for $0 \leq i \leq k-1$, then*

$$f_{k-\mathrm{pol}} = \widehat{d^{k,\nu_M} f}.$$

In particular, $f_{k-\mathrm{pol}} \in C^\infty(\nu_M)_{k-\mathrm{pol}}$ is independent of the tubular neighbourhood chosen.

Proposition 5.1.6. *Let $M \subset N$ be a submanifold and $X \in \mathfrak{X}^{p+1}(N)$ a multivector field. If for some tubular neighbourhood the approximations $X_{i-\mathrm{pol}} \in \mathfrak{X}^{p+1}_{i-\mathrm{pol}}(\nu_M)$ vanish for $0 \leq i \leq k-1$, then $X_{k-\mathrm{pol}} \in \mathfrak{X}^{p+1}(\nu_M)_{k-\mathrm{pol}}$ is independent of the tubular neighbourhood chosen.*

Proof. By Lemma 5.1.3 we have that the core of $X_{i-\mathrm{pol}}$ is given by $D_{i-\mathrm{pol}}$. Therefore $D_{i-\mathrm{pol}}$ vanishes for all $0 \leq i \leq k-1$, and thus by definition

$$D_{i-\mathrm{pol}}(s_0, \dots, s_p) = (D(s_0, \dots, s_p))_{i-\mathrm{pol}},$$

vanishes for all $0 \leq i \leq k-1$. Therefore, by Lemma 5.1.5 $(D(s_0, \dots, s_p))_{k-\mathrm{pol}}$ does not depend on the chosen tubular neighbourhood and thus neither does $X_{k-\mathrm{pol}}$. □

Jets of multivector fields There is another way of taking jets of multivector fields which we will now explain. Let $M \subset N$ be a submanifold, and let $I_M \subset C^\infty(N)$ denote the ideal of functions vanishing on M. This ideal induces a filtration F on $\mathfrak{X}^\bullet(N)$:

$$\mathfrak{X}^\bullet(N) = F^\bullet_{-1} \supset \mathcal{F}^\bullet_0 \supset \mathcal{F}^\bullet_1 \supset \cdots$$
$$\mathcal{F}^\bullet_k = I^{k+1}_M \mathfrak{X}^\bullet(N), k \geq 0.$$

This filtration satisfies

$$[\mathcal{F}^\bullet_k, \mathcal{F}^\bullet_l] \subset \mathcal{F}^\bullet_{k+l}, \quad \mathcal{F}_k \wedge \mathcal{F}_l \subset \mathcal{F}_{k+l+1},$$

and therefore the graded spaces

$$J^k_M(\mathfrak{X}^\bullet(N)) := \mathfrak{X}^\bullet(N)/\mathcal{F}^\bullet_k$$

inherit the structure of Gerstenhaber algebra.

Intuitively, elements of $J^k_M(\mathfrak{X}^\bullet(N))$ are Taylor expansions where for each coefficient function the Taylor expansion up to order k is taken. However $J^k_M(\mathfrak{X}^\bullet(N))$ is an abstract quotient space with no, a priori, explicit description.

Nevertheless, for a given $X \in \mathfrak{X}^{k+1}(N)$ we want to compare $J^k_M X$ with the fibrewise approximation $X_{i-\mathrm{pol}}$. As discussed before, the approximations $X_{i-\mathrm{pol}}$ are not canonical in general and depend on the tubular neighbourhood chosen. So we first restrict ourselves to the case of a vector bundle:

Proposition 5.1.7. *There is an isomorphism of graded vector spaces induced by*

$$\varphi : J^k_M \mathfrak{X}^{p+1}(E^*) \to \bigoplus_{i=0}^{p+1} \mathscr{V}^p_i(E)_{(k+i)-\mathrm{pol}}$$
$$J^k_M W \mapsto \Big((D_W)_{k-\mathrm{pol}}, (\sigma_{D_W})_{(k+1)-\mathrm{pol}}, \dots, (\sigma^{p+1}_{D_W})_{(k+p+1)-\mathrm{pol}} \Big).$$

Proof. We first check that the map is well-defined. By Lemma 5.1.3 we have that

$$(\sigma^i_{D_W})_{(k+i)-\mathrm{pol}}(f_1, \dots, f_i; s_0, \dots, s_{p-i}) = \sigma^i_W(f_1, \dots, f_i, s_0, \dots, s_{p-i})_{k-\mathrm{pol}},$$

only depends on the k-th jet of W. Similarly, we see that if $[W], [\widetilde{W}] \in J^k_M \mathfrak{X}^{p+1}(E^*)$ and $\varphi(J^k_M W) = \varphi(J^k_M \widetilde{W})$ then the k-jets of W and $\widetilde{W}$ coincide and thus φ is injective. By Remark 4.2.50 we have that $\sigma^i : \mathscr{V}^p(E) \to \mathscr{V}^p_i(E)$ is surjective. Thus given $(a_0, \dots, a_{p+1}) \in \oplus_{i=0} \mathscr{V}^p_i(E)_{(k+i)-\mathrm{pol}}$ let $D_j \in \mathscr{V}^j(E)$ with $\sigma^j(D_j) = a_j$. If we let $X_j \in \mathfrak{X}^{p+1}(E^*)$ denote the vector field with core D_j, then

$$\varphi(J^k_M(X_0 + \dots + X_{p+1})) = (a_0, \dots, a_{p+1}),$$

which proves that ψ is surjective. □

Given a submanifold $M \subset N$, we can choose a tubular neighbourhood and associate to an element of $J^k_M \mathfrak{X}^{p+1}(N)$ an element of $\oplus_{i=0}^{p+1} \mathscr{V}^p_i(\nu^*_M)_{(k+i)-\mathrm{pol}}$ using the above proposition. However, in general this does depend on the particular tubular neighbourhood. But as in Proposition 5.1.6 we have:

Proposition 5.1.8. *Let $M \subset N$ be a submanifold and let $W \in \mathfrak{X}^{p+1}(N)$ be such that $j^{k-1}_M W = 0$. Then $j^k_M W$ corresponds to a canonical element of $\oplus_{i=0}^{p+1} \mathscr{V}^p_i(\nu^*_M)_{(k+i)-\mathrm{pol}}$.*

Taylor expansions of Poisson structures The Nijenhuis-Schouten bracket (5.1.1) on formal vector fields allows one to talk about formal Poisson structures:

Definition 5.1.9. A **formal Poisson structure** on the vector bundle $\mathcal{N} \to M$ is any formal bivector field $\pi \in \mathfrak{X}^2(\mathcal{N})_{\mathrm{form}}$ satisfying $[\pi, \pi] = 0$. ◇

Corollary 5.1.10. *On any vector bundle $\mathcal{N} \to M$:*

1. *If $\pi \in \mathfrak{X}^2(\mathcal{N})$ is a Poisson bivector, than $j^\infty \pi \in \mathfrak{X}^2(\mathcal{N})_{\mathrm{form}}$ is a formal Poisson bivector.*
2. *If $\Pi = \sum_d \Pi_d \in \mathfrak{X}^2(\mathcal{N})_{\mathrm{form}}$ is a formal Poisson bivector, then the first non-zero homogeneous component of Π is a homogeneous Poisson bivector on $\mathcal{N}$.*

Proof. By Lemma 5.1.1 the map j^∞ is a Lie algebra map. Therefore

$$[j^\infty \pi, j^\infty \pi]_{d-\mathrm{pol}} = [\pi, \pi]_{d-\mathrm{pol}}$$

which implies that $j^\infty \pi$ is a formal Poisson structure. Let k be the first non-zero integer for which $\pi_k \neq 0$, by the same lemma we have $[\pi, \pi]_{(2k-1)-\mathrm{pol}} = \sum_{d_1+d_2=2k} [\pi_{d_1-\mathrm{pol}}, \pi_{d_2-\mathrm{pol}}]$. But as $\pi_{i-\mathrm{pol}} = 0$ for all $i < k$, the only term in this sum which is non-zero is $[\pi_{k-\mathrm{pol}}, \pi_{k-\mathrm{pol}}]$. As π is a formal Poisson structure we thus conclude that $\pi_{k-\mathrm{pol}}$ is a Poisson bivector of degree k on $\mathcal{N}$. □

Let
$$M \subset (N, \pi)$$
be a submanifold of a Poisson manifold. We would like to understand π around M. Using a tubular neighborhood we may transport π to ν_M. Our general theory gives approximations of π arising from the expansion
$$m_\lambda^*\pi \sim \lambda^{-2}\pi_0 + \lambda^{-1}\pi_1 + \pi_2 + \dots$$
with $\pi_i \in \mathfrak{X}^2(\nu_M)_{i-\mathrm{pol}}$. Combining Corollary 5.1.10 and Proposition 5.1.6 we conclude:

Proposition 5.1.11. *Let $M \subset (N, \pi)$ be a submanifold of a Poisson submanifold. If $\pi_{i-\mathrm{pol}} = 0$ for all $0 \leq i \leq k-1$, then $\pi_{k-\mathrm{pol}} \in \mathfrak{X}^2(\nu_M)$ is Poisson and does not depend on the tubular neighbourhood chosen.*

Consequently, in this setting we have that π_k gives a canonical first candidate for a normal form result around M. Although it is rather naive to expect that only this first non-zero term is enough to capture the Poisson structure around the submanifold, in the next sections we will see that surprisingly often this is the case. We will focus ourselves mostly on the linear, π_1, and quadratic, π_2, case.

Remark 5.1.12. It is instructive to write down the expression for $m_\lambda^*\pi$ in local coordinates. Let (x_i, y_i) be coordinates on $\mathcal{N}$, with the x_i in the direction of M and the y_i in the fibre directions. In these coordinates
$$\pi = \sum_{i,j} A_{i,j}(x,y)\partial_{x_i} \wedge \partial_{x_j} + B_{i,j}(x,y)\partial_{x_i} \wedge \partial_{y_j} + C_{i,j}(x,y)\partial_{y_i} \wedge \partial_{y_j},$$
and
$$m_\lambda^*\pi = \sum_{i,j} A_{i,j}(x,\lambda y)\partial_{x_i} \wedge \partial_{x_j} + \frac{B_{i,j}(x,\lambda y)}{\lambda}\partial_{x_i} \wedge \partial_{y_j} + \frac{C_{i,j}(x,y)}{\lambda^2}\partial_{y_i} \wedge \partial_{y_j}. \quad \diamondsuit$$

5.1.2 The fiberwise constant case

Recall from Example 4.2.23 that the Lie bracket on $\mathfrak{X}^2(\mathcal{N})_{0-\mathrm{pol}}$ vanishes and consequently:

Lemma 5.1.13. *There is a one-to-one correspondence between Poisson structures $\pi \in \mathfrak{X}^2(\mathcal{N})_{0-\mathrm{pol}}$ and sections $\pi_0 \in \Gamma(\Lambda^2\mathcal{N})$.*

If $M = \{x_0\}$ is a point then $\pi_0 = \pi_{x_0}$ is the (constant bivector) on $T_{x_0}N$. When $M = \{x_0\}$, by the Darboux-Weinstein theorem there is a neighbourhood of x_0 on which π is isomorphic to π_0 precisely when x_0 is a regular point of π. For higher dimensional submanifolds π_0 is seldom a good approximation of π.

5.1.3 The fiberwise linear case

The next homogeneous approximation, π_1, is Poisson when $\pi_0 = 0$ which happens when the submanifold is coisotropic:

Definition 5.1.14. Let (N, π) be a Poisson manifold. A submanifold $M \subset N$ is called **coisotropic** if the π-orthogonal of the tangent spaces of TM,

$$TM^{\perp,\pi} := \pi^\sharp(\nu_M^*)$$

is inside TM. ♢

The following equivalence is well-known, see for instance [77]:

Lemma 5.1.15. *Let (N, π) be a Poisson manifold. An embedded submanifold $M \subset N$ is coisotropic if and only if one of the following equivalent conditions hold:*

1. *The vanishing ideal $\mathcal{I}_M \subset C^\infty(N)$ of M is closed under the Poisson bracket.*
2. *For every point $x \in M$ and symplectic leaf S through x we have $T_xM \cap T_xS \subset T_xS$ is a coisotropic subspace of the symplectic vector space $(T_xS, \omega_{S,x})$*

Lemma 5.1.16. *Let $M \subset (N, \pi)$ be an embedded submanifold of a Poisson manifold. Then $\pi_0 = 0$ if and only if M is a coisotropic submanifold of (N, π),*

Proof. Using a tubular neighbourhood we may assume that $N = \nu_M$, the normal bundle of M. The vanishing ideal of M is generated by $\Gamma(\nu_M^*)$. Because

$$\pi_0(d\widehat{s}_1, d\widehat{s}_2) = \pi(d\widehat{s}_1, d\widehat{s}_2)|_M \in C^\infty(M), \quad \text{for all } s_1, s_2 \in \Gamma(\nu_M^*)$$

we see that π_0 vanishes precisely when the vanishing ideal is closed under the bracket. □

Example 5.1.17. When $M = \{x_0\}$ is a point, then the coisotropic condition corresponds to x_0 being a fixed point of π, that is $\pi_{x_0} = 0$. The resulting structure is $\mathfrak{g} := T_{x_0}N$, the isotropy Lie algebra of π at x_0, and the corresponding approximation of M near x_0 will be $\mathfrak{g}^*$ endowed with the linear Poisson structure. △

The most famous result for a normal form theorem using π_1 is Conn's linearisation theorem:

Theorem 5.1.18 ([20])**.** *Let (N, π) be a Poisson manifold and $x \in N$ a fixed point of π. If the isotropy Lie algebra $\mathfrak{g}$ is semi-simple and of compact type, then there exists a neighbourhood of x on which π and π_1 are Poisson diffeomorphic.*

Example 5.1.19. When M is a collection of zeros of (N, π). Then the constant approximation $\pi_0 \in \mathfrak{X}^2(\nu_M)$, vanishes and we can consider the linear $\pi_1 \in \mathfrak{X}^2(\nu_M)$. The corresponding Lie algebroid structure on the dual will have vanishing anchor; i.e., it is a bundle of Lie algebras. △

When $M \subset (N, \pi)$ is coisotropic it follows that ν_M^* becomes a Lie subalgebroid of the cotangent algebroid $(T^*N, [\cdot, \cdot]_\pi)$.

Lemma 5.1.20. *Let $M \subset (N, \pi)$ be a coisotropic submanifold. The following two Lie algebroid structures on ν_M^* coincide:*

- *ν_M^* viewed as Lie subalgebroid of $(T^*N, [\cdot,\cdot]_\pi)$.*
- *The Lie algebroid structure dual to the fibrewise linear Poisson structure π_1, through Lemma 5.1.13.*

We would like to study normal form theorems for coisotropic submanifolds using the fibrewise linear Poisson structure π_1. A necessary condition for such a result to hold is that the submanifold is Lagrangian:

Definition 5.1.21. Let $M \subset (N, \pi)$ be an embedded submanifold of a Poisson manifold. Then M is called **Lagrangian** if one of the following equivalent conditions hold:

- The π-orthogonal coincides with the intersection with the tangent space to the leaf:
$$T_pM^{\perp,\pi} = T_pM \cap T_pS,$$
for all symplectic leaves S of (N, π)
- For every point $x \in M$ and symplectic leaf S through x the intersection $T_xM \cap T_xS \subset (T_xS, \omega_{S,x})$ is a Lagrangian subspace. ♢

Lemma 5.1.22. *Let $A \to M$ be a Lie algebroid, the zero section $M \hookrightarrow (A^*, \pi_1)$ is a* Lagrangian *submanifold.*

Proof. We have to show that for all $p \in M$ we have $\pi_{1,p}^\sharp(\nu_{M,p}^*) = T_pS \cap T_pM$, where S is the symplectic leaf through p. Because M is the zero-section of A^*, we have that $\nu_M^* = A$. Consequently we are left to show that $\pi_1^\sharp(T_pA^*) \cap T_pM = \pi_1^\sharp(A_p)$. Because $T_pA^* \simeq T_pM^* \oplus A_p^*$, we can thus consider both $\pi_1^\sharp(T_pM^*)$ and $\pi_1^\sharp(A_p)$. Recall that for $s \in \Gamma(A)$ and $\alpha \in \Omega^1(M)$ we have

$$\pi_1^\sharp(d\widehat{s}) = (\rho(s), [s,\cdot]), \quad \pi_1^\sharp(\alpha) = (0, \rho^*\alpha) \in T_pA^* \simeq T_pM \oplus A_p^*,$$

consequently $\pi_1^\sharp(T_pA^*) \cap T_pM$ coincides with the image of the anchor, which is precisely $\pi_1^\sharp(A_p)$, which finishes the proof. □

Definition 5.1.23. Let (N, π) be a Poisson manifold, and let $M \subset N$ be a Lagrangian submanifold. If N has a neighbourhood on which it is Poisson diffeomorphic to π_1, then we call M **linearisable**. ♢

We see that in the local model (ν_M^*, π_1), the zero section is not only a coisotropic submanifold but also a Lagrangian submanifold. Therefore we can only expect π to be linearisable around M if M was already Lagrangian in (N, π).

Example 5.1.24. A neighbourhood theorem for general coisotropic submanifolds is proven in the symplectic category: This result is known as Gotay's theorem [37] and the normal form combines both the fibrewise linear part π_1 as well as the fibrewise quadratic part π_2. Gotay's theorem has been extended to several settings in Poisson geometry: To regular Poisson manifolds by Cattaneo and Zambon [12] and to log symplectic manifolds by Geudens and Zambon [33]. △

A Lagrangian neighbourhood theorem We will require several cohomology groups to study the linearisablity of Poisson structures around Lagrangian submanifolds. The first is a localised Poisson cohomology:

Definition 5.1.25. Let (N, π) be a Poisson manifold, and $M \subset N$ a submanifold. Let

$$\mathfrak{X}^\bullet(N)_M := \lim_{M \subset U} \mathfrak{X}^\bullet(U),$$

with the direct limit taken over all opens U containing M. We denote by $H^\bullet_\pi(N)_M$ the resulting localised Poisson cohomology groups. ♢

The rescaling operators of the normal bundle $\mathcal{N}$ induce a filtration of $\mathfrak{X}^\bullet(\mathcal{N})_M$.[1]

Lemma 5.1.26. *For $\mathcal{N} \to M$ a vector bundle, the sets*

$$F_k\mathfrak{X}^\bullet(\mathcal{N})_M := \{X \in \mathfrak{X}^\bullet(\mathcal{N})_M : X_{i-\mathrm{pol}} = 0 \textit{ for all } i \leq k-1\}$$

define a filtration of Gerstenhaber algebras,

$$F_0\mathfrak{X}^\bullet(\mathcal{N})_M = \mathfrak{X}^\bullet(\mathcal{N})_M \supset F_1\mathfrak{X}^\bullet(\mathcal{N})_M \supset F_2\mathfrak{X}^\bullet(\mathcal{N})_M \supset \cdots$$

of $\mathfrak{X}^\bullet(\mathcal{N})_M$ in the sense that:

$$[F_i\mathfrak{X}^\bullet(\mathcal{N})_M, F_j\mathfrak{X}^\bullet(\mathcal{N})_M] \subset F_{i+j-1}\mathfrak{X}^\bullet(\mathcal{N})_M$$
$$F_i\mathfrak{X}^\bullet(\mathcal{N})_M \wedge F_j\mathfrak{X}^\bullet(\mathcal{N})_M \subset F_{i+j}\mathfrak{X}^\bullet(\mathcal{N})_M.$$

The associated graded Gerstenhaber algebra $\mathrm{gr}^d(\mathfrak{X}^\bullet(\mathcal{N})_M)$ is precisely $\mathfrak{X}^\bullet(\mathcal{N})_{d-\mathrm{pol}}$.

Proof. Let $X \in F_i\mathfrak{X}^\bullet(\mathcal{N})_M$ and $Y \in F_j\mathfrak{X}^\bullet(\mathcal{N})_M$, then by Lemma 5.1.1 we have

$$[X, Y]_{(i+j-2)-\mathrm{pol}} = [X_{i-1}, Y_j] + [X_i, Y_{j-1}] + \cdots,$$

vanishes and thus $[X, Y] \in F_{i+j-1}\mathfrak{X}^\bullet(\mathcal{N})_M$. Moreover

$$(X \wedge Y)_{(i+j-1)-\mathrm{pol}} = X_i \wedge X_{j-1} + X_{i+1} \wedge X_{j-2} + \cdots$$

vanishes and thus $X \wedge Y \in F_{i+j}\mathfrak{X}^\bullet(\mathcal{N})_M$. The quotient F_i/F_{i+1} consists of germs of vector fields of degree i, but as these are completely determined by their germs we conclude the quotient is precisely $\mathfrak{X}^\bullet(\mathcal{N})_{i-\mathrm{pol}}$. Because $[X, Y]_{i+j-1-\mathrm{pol}} = [X_i, Y_j]$ and $(X \wedge Y)_{i-\mathrm{pol}} = X_i \wedge Y_i$ we see that the Gerstenhaber structure on $\mathrm{gr}^d(\mathfrak{X}^\bullet(\mathcal{N})_M)$ also coincides with the one on $\mathfrak{X}^\bullet(\mathcal{N})_{d-\mathrm{pol}}$. □

Because $F_k\mathfrak{X}^\bullet(\mathcal{N})_M$ defines a filtration of Gerstenhaber algebras, it also induce a filtration of cochain complexes:

Lemma 5.1.27. *Let $M \subset (N, \pi)$ be a co-isotropic submanifold. Then*

[1] And also of $\mathfrak{X}^\bullet(\mathcal{N})$, but we will need the germs in our application.

- $(F_k\mathfrak{X}^\bullet(\nu_M), d_\pi)$ *and*
- $(\mathfrak{X}^\bullet(\nu_M)_{k-\mathrm{pol}}, d_{\pi_1})$

are cochain complexes, and $\mathrm{gr}^k((\mathfrak{X}^\bullet(\nu_M), d_\pi))$ *is precisely* $(\mathfrak{X}^\bullet(\nu_M)_{k-\mathrm{pol}}, d_{\pi_1})$. *Explicitly, we have a short exact sequence:*

$$0 \to (F_{k+1}\mathfrak{X}^\bullet(\nu_M)_M, d_\pi) \xrightarrow{i} (F_k\mathfrak{X}^\bullet(\nu_M)_M, d_\pi) \xrightarrow{r} (\mathfrak{X}^\bullet(\nu_M)_{k-\mathrm{pol}}, d_{\pi_1}) \to 0, \tag{5.1.5}$$

The last map defined by sending a multivector to its degree k-part. For the corresponding long exact sequence in cohomology the connecting morphism

$$H^i_{\pi_1}(\nu_M)_{k-\mathrm{pol}} \to H^{i+1}_\pi(F_{k+1}\mathfrak{X}^\bullet(\nu_M)_M),$$

coincides with d_π.

Proof. Because $M \subset (N, \pi)$ is co-isotropic, we have that $\pi \in F_1\mathfrak{X}^\bullet(\nu_M)_M$. Therefore $d_{\pi_1}(F_k\mathfrak{X}^\bullet(\nu_M)_M) \subset F_k\mathfrak{X}^\bullet(\nu_M)_M$. Because $\mathrm{gr}^k(\mathfrak{X}^\bullet(\nu_M)_M) = \mathfrak{X}^\bullet(\nu_M)_M$ as Gerstenhaber algebras, they also coincide as cochain complexes. It is straightforward to show that the connection morphism of (5.1.5) coincides with d_π. □

We will prove a normal form theorem using a strategy described in [62], making use of the following concept:

Definition 5.1.28 ([10])**.** Let $M \subset N$. A vector field $X \in \mathfrak{X}^1(N)$ is **Euler-like** for M if it is tangent to N, and the linear approximation $X_{1-\mathrm{pol}} = \mathcal{E}$, the Euler vector field of the normal bundle.[2] ♢

Definition 5.1.29. Let $O \subset \nu_M$ be a star-shaped open neighbourhood of the zero-section. A morphism $\varphi : (O, M) \to (N, M)$, is a **tubular neighbourhood embedding** if φ is an embedding and the linear approximation $\nu(\varphi)$ is the identity. ♢

Theorem 5.1.30 ([10])**.** *An Euler-like vector field determines a unique tubular neighbourhood embedding*

$$\nu_M \supset O \xrightarrow{\varphi} N,$$

such that $\varphi^* X = \mathcal{E}$.

The nice idea of [62] is to find Euler-like vector fields adapted to a particular geometric structure, and use it to prove normal form results. In our setting we have:

Lemma 5.1.31. *Let M be a Lagrangian submanifold of a Poisson manifold (N, π), and let π_1 denote its first order approximation. Assume there exists an Euler-like vector field for M such that $[X, \pi] = -\pi$. Then there exists a tubular neighbourhood of $\mathcal{U}$ of M on which π and π_1 are Poisson diffeomorphic via a diffeomorphism which is the identity on M.*

[2]Note that we have a different degree convention than used in [62].

Proof. Let φ denote the tubular neighbourhood embedding associated to X. Then

$$[\mathcal{E}, \varphi^*\pi] = \varphi^*[X, \pi] = -\varphi^*\pi.$$

Consequently, $\varphi^*\pi$ is degree 1 and thus $\pi_1 = \lim_{\lambda\to 0} \lambda^{-1} m_\lambda^* \varphi^*\pi = \varphi^*\pi$. □

To solve the linearisation problem, we thus have to determine when such an Euler-like vector field exists. To do so we introduce the following notion:

Definition 5.1.32. Let $\pi \in \mathfrak{X}^2(N)$, and $M \subset N$ Lagrangian. Let X be any Euler-like vector field for M, and define the **linearisation class** of M to be

$$\Lambda(\pi, M) := [\pi - d_\pi(X)] \in H^2_\pi(F_2\mathfrak{X}^\bullet(\nu_M)_M) \qquad \diamondsuit$$

Because $M \subset (N, \pi)$ is Lagrangian, we have that $\pi_{0-\mathrm{pol}} = 0$. Therefore

$$(\pi - d_\pi(X))_{0-\mathrm{pol}} = \pi_0 - d_{\pi_0}(X_0) = 0$$

and

$$(\pi - d_\pi(X))_{1-\mathrm{pol}} = \pi_1 - d_{\pi_1}(X_1) = 0$$

and thus $\pi - d_\pi(X)$ is indeed an element of $F_2\mathfrak{X}^2(\nu_M)_M$. Moreover, if $\widetilde{X}$ is another Euler-like vector field then $(X - X')_{1-\mathrm{pol}} = \mathcal{E} - \mathcal{E} = 0$ and thus $X - X' \in F_2\mathfrak{X}^\bullet(\nu_M)_M$. Consequently, $[d_\pi(X)] = [d_\pi(X')] \in H^2_\pi(F_2\mathfrak{X}^\bullet(\nu_M)_M)$. Therefore, the linearisation class does not depend on the choice of Euler-like vector field.

Proposition 5.1.33. *A Lagrangian submanifold $M \subset (N, \pi)$ is linearisable precisely when $\Lambda(\pi, M) = 0$.*

Proof. If L is linearisable, then pushing forward the Euler-vector field by the linearising Poisson diffeomorphism $\varphi : \nu_M \to N$ provides an Euler-like vector field satisfying the assumptions of Lemma 5.1.31, and therefore the linearisation class vanishes.

If $\Lambda(\pi, M) = 0$, then there exists a vector field $Y \in F_2\mathfrak{X}^1(\nu_M)_M$ such that $\pi - d_\pi(X) = d_\pi(Y)$. Because $Y \in F_2\mathfrak{X}^1(\nu_M)_M$ the vector field $\widetilde{X} := X + Y$ is still Euler-like and satisfies $\pi = d_\pi(\widetilde{X})$. Therefore, M is linearisable by Lemma 5.1.31. □

Unfortunately, directly computing the linearisation class is very difficult. Instead, we give sufficient conditions for when it vanishes:

Theorem 5.1.34. *Let M be a Lagrangian submanifold of a Poisson manifold (N, π), and let π_1 denote its first order approximation. If $[\pi] = 0 \in H^2_\pi(N)_M$ and the morphisms*

$$r^* : H^1_\pi(\nu_M)_M \to H^1_{\pi_1}(\nu_M)_{0-\mathrm{pol}}$$
$$r^* : H^1_\pi(F_1\mathfrak{X}^\bullet(\nu_M)_M) \to H^1_{\pi_1}(\nu_M)_{1-\mathrm{pol}}$$

are surjective then $\Lambda(\pi, M) = 0$.

Proof. Because $[\pi] = 0 \in H^2_\pi(\nu_M)_M$ we also have $[\pi - d_\pi(X)] = 0 \in H^2_\pi(\nu_M)_M$.
Let us consider the long exact sequence induced by (5.1.5):

$$\cdots \to H^1(\nu_M)_M \xrightarrow{r^*} H^1(\nu_M)_{0-\mathrm{pol}} \xrightarrow{\delta} H^2_\pi(F_1\mathfrak{X}^\bullet(\nu_M)_M) \xrightarrow{i_*} H^2(\nu_M)_M \to \cdots$$

Because r^* is surjective, the connecting morphism δ vanishes and hence i_* is injective.
As $[\pi - d_\pi(X)] = 0 \in H^2_\pi(\nu_M)_M$, we thus have

$$i_*[\pi - d_\pi(X)] = [\pi - d_\pi(X)] = 0 \in H^2(\nu_M)_M,$$

and consequently $[\pi - d_\pi(X)] = 0 \in H^2_\pi(F_1\mathfrak{X}^\bullet(\nu_M))$. The next connecting morphism vanishes by the same reasoning and therefore we have that

$$[\pi - d_\pi(X)] = 0 \in H^2_\pi(F_2\mathfrak{X}^\bullet(\nu_M)).$$

This proves that the linearisation class vanishes. □

The cohomology groups $H^1(\nu_M)_{0-\mathrm{pol}}$ and $H^1(\nu_M)_{1-\mathrm{pol}}$ appearing in Theorem 5.1.34 have the following description: Using Theorem 4.3.7 we have that $H^1(\nu_M)_{0-\mathrm{pol}} = H^1(\nu_M^*)$, the Lie algebroid cohomology of the conormal Lie algebroid ν_M^*, and $H^1(\nu_M)_{1-\mathrm{pol}} = H^1(\nu_M^*, \mathrm{ad})$, the Lie algebroid cohomology with coefficients in the the adjoint representation up to homotopy.

A higher degree neighbourhood theorem Suppose $\pi \in \mathfrak{X}^2(N)$ and $M \subset N$ with $\pi_i \in \mathfrak{X}^2(\nu_M) = 0$ for all $i < d$. We would again like to determine when π_d provides a normal form for π around N. We will use the same strategy as before and look for an Euler-like vector field X with the property that $[X, \pi] = (d-2)\pi$. If we let $\varphi : \nu_M \to \mathcal{U}$ denote a tubular neighbourhood embedding associated to X, then we have

$$\mathcal{L}_\mathcal{E}\varphi^*\pi = \varphi^*\mathcal{L}_X\pi = (d-2)\varphi^*\pi,$$

which shows that $\varphi^*\pi$ is of degree d, and thus coincides with π_d.
We again want to define a cohomology class which determines when such an Euler-like vector field exists. Note that, in contrary to the linear case, $F_k\mathfrak{X}^\bullet(\nu_M)_M$ is not closed under d_π. Instead we have that

$$d_\pi(F^s\mathfrak{X}^\bullet(\nu_M)_M) \subset F^{s+d-1}\mathfrak{X}^{k+1}(\nu_M)_M.$$

We thus define:

Definition 5.1.35. Let $\pi \in \mathfrak{X}^2(N)$ and $M \subset N$ is given with $\pi_i \in \mathfrak{X}^2(\nu_M) = 0$ for all $i < d$. The cohomology of

$$\to F^{s-d+1}\mathfrak{X}^{k-1}(\nu_M)_M \xrightarrow{d_\pi} F^s\mathfrak{X}^k(\nu_M)_M \xrightarrow{d_\pi} F^{s+d-1}\mathfrak{X}^{k+1}(\nu_M)_M \to \quad (5.1.6)$$

is denoted by $F^sH^k_\pi(\nu_M)$. ♢

The graded of the cochain complex (5.1.6) is

$$\to \mathfrak{X}^{k-1}(\nu_M)_{(s-d+1)-\mathrm{pol}} \xrightarrow{d_{\pi_d}} \mathfrak{X}^{k}(\nu_M)_{s-\mathrm{pol}} \xrightarrow{d_{\pi_d}} \mathfrak{X}^{k+1}(\nu_M)_{(s+d-1)-\mathrm{pol}} \to \quad (5.1.7)$$

and its cohomology is denoted by $H^k_{\pi_d}(\nu_M)_{s-\mathrm{pol}}$.

Definition 5.1.36. Let $M \subset (N,\pi)$ be such that $\pi_i = 0 \in \mathfrak{X}^2(\nu_M)$ for all $i < d$, and let $X \in \mathfrak{X}^1(N)$ be an Euler-like vector field for M. Define the **d-th order approximation class** to be the class

$$\Lambda^d(\pi, M) := [(d-2)\pi - [X,\pi]] \in F^{d+1}H^2_\pi(\nu_M)_M. \qquad \diamondsuit$$

Just as in the linear case this class is independent of the choice of Euler-like vector field.

Lemma 5.1.37. *Let $M \subset (N,\pi)$ be such that $\pi_i = 0 \in \mathfrak{X}^2(\nu_M)$ for all $i < d$. There exists a tubular neighbourhood on which π is Poisson diffeomorphic to π_d if and only if $\Lambda^d(\pi, M) = 0 \in F^{d+1}H^2_\pi(\mathcal{N})$.*

Proof. The class $\Lambda^d(\pi, M) = 0 \in F^{d+1}H^2_\pi(\nu_M)$ if and only if there exists a vector field $Y \in F^2\mathfrak{X}^1(\nu_M)$ such that $(d-2)\pi - [X,\pi] = [Y,\pi]$. We see that $\widetilde{X} := X + Y$ defines an Euler like vector field with $(d-2)\pi = [\widetilde{X},\pi]$. Therefore the desired tubular neighbourhood is induced by the Euler-like vector field $\widetilde{X}$. ☐

Computing the cohomology $F^{d+1}H^2(\nu_M)_M$ in practice is not very viable. As in the linear case we can control the vanishing of the d-th order approximation class:

Proposition 5.1.38. *Let $M \subset (N,\pi)$ be such that $\pi_i = 0 \in \mathfrak{X}^2(\nu_M)$ for all $i < d$. If $[\pi] = 0 \in H^2_\pi(N)_M$ and for all $0 \leq i \leq d$*

$$r^* : F^iH^1_\pi(\nu_M)_M \to H^1_{\pi_d}(\nu_M)_{i-pol}$$

is surjective, then $\Lambda^d(\pi, M) = 0$.

Proof. The proof is as in the linear case, making use of the long exact sequences induced by the short exact sequences

$$0 \to (F^{s+1}\mathfrak{X}^k(\nu_M)_M, d_\pi) \to (F^s\mathfrak{X}^k(\nu_M)_M, d_\pi) \to (\mathfrak{X}^k(\nu_M)_{s-\mathrm{pol}}, d_{\pi_d}) \to 0 \quad ☐$$

The complex (5.1.7), for ν_M a vector space, appears in work of Dufour and Wade [30] and is used to study stability of higher order singularities at points of Poisson structures.

Using integrability for linearisation The cohomology conditions appearing in Theorem 5.1.34 are quite hard to deal with. A common way of controlling the vanishing of Lie algebroid cohomology classes is by imposing the existence of a particular groupoid integrating the Lie algebroid. Unfortunately, as of yet we have been unable to obtain satisfactory integrability assumptions, instead we prove:

Proposition 5.1.39. *Let M be a Lagrangian submanifold of (N, π) and assume there exists a connected neighbourhood U of M such that $(U, \pi|_U)$ is integrable by a Hausdorff source 2-connected proper groupoid. Moreover, assume that ν_M^* is also integerable by a Hausdorff source 1-connected proper groupoid. Then the conditions of Theorem 5.1.34 are satisfied.*

Proof. Let $(\mathcal{G}, \Omega)$ denote the given symplectic groupoid over U. Let $\mathcal{G}_M$ denote the given integration of ν_M^*. Because $\mathcal{G}_M$ is proper we have by [21] that $H^k_{\text{diff}}(\mathcal{G}_M) = \{0\}$ for all $k \geq 1$. Because the source fibres of $\mathcal{G}_M$ are 1-connected, the Van Est map [21] $H^1_{\text{diff}}(\mathcal{G}_M) \to H^1(\nu_M^*)$ is an isomorphism. Therefore we find that the Lie algebroid cohomology $H^1(\nu_M^*)$ vanishes as required. Similarly, using the van Est map and vanishing of Lie groupoid cohomology with coefficients in a representation up to homotopy, it follows from [3] that $H^1(\nu_M^*; \mathrm{ad})$ vanishes.
Because $\mathcal{G}$ is source 2-connected the Van Est map $\Phi : H^2(\mathcal{G})_{\text{diff}} \to H^2_{\pi|_U}(U)$ is an isomorphism. Because $\mathcal{G}$ is proper, we have that $H^2(\mathcal{G})_{\text{diff}}$ vanishes and we thus conclude that $H^2_{\pi|_U}(U) = 0$. In particular $[\pi|_U] = 0 \in H^2_{\pi|_U}(U)$, and consequently $[\pi] = 0 \in H^2(U)_M$. □

This statement should be seen as one step in solving a version of the linearisation problem. One case in which the above can be applied works is when M is a point. In this case the result reduces to the second step of the geometric proof of Conn's linearisation theorem by Crainic-Fernandes [24], whereas Theorem 5.1.34 reduces to the first step. Note that when M is a point it is also a symplectic leaf. A generalisation of Conn's theorem around symplectic leaves is proven in [28].

Remark 5.1.40. In both the case of fixed points and symplectic leaves, the behaviour of the groupoid integrating the algebroid around M is well-understood because M is an orbit of the groupoid. When M is a general Lagrangian, this is no longer the case which complicates the situation significantly. This behaviour manifests in that (global) integrablity of the Poisson manifold by the desired groupoid, does not imply that an integration around M with the desired properties exists, as the next example will show. Note that the Lagrangian is, however, linearisable. ♢

Example 5.1.41. Let (M, ω) be a simply connected symplectic manifold, and let $L \subset M$ be a Lagrangian submanifold which is not simply connected (for instance $M = S^2$ and $L = S^1$ is the equator). The pair groupoid $\mathcal{G} = M \times M$ is a proper s-1-connected integration. However if we let U be a tubular neighbourhood of L, the restriction of $\mathcal{G}$ to U is simply the pair groupoid $U \times U$, which does not have 1-connected source fibres as U is homotopic to L, and L is assumed to be non simply connected. △

Examples We end our discussion of the linear case by listing some of the known classes of Lagrangian submanifolds in Poisson manifolds which are known to be linearisable. We expect that all these examples should be treatable using Theorem 5.2.4.

Example 5.1.42. Any Lagrangian submanifold $M \subset (N, \omega)$ of a symplectic manifold is linearisable, as proven by Weinstein in [76]. △

Example 5.1.43. If (M, π) is a regular Poisson manifold, and L is a Lagrangian which is transverse to the leaves than it is linearisable ([12]), see also [34]). △

Example 5.1.44. If (M, π) is a log symplectic manifold, and L is a Lagrangian which is transverse to the degeneracy locus than it is linearisable ([51]). Similarly if (M, π) is elliptic symplectic and the Lagrangian is transverse to the degeneracy locus than it is linearisable ([16]).

Besides Lagrangian submanifolds which are transverse to the degeneracy locus, one can also consider those which are contained in it. These are studied by Geudens and Zambon in [34]. They prove a normal form result for these Lagrangians, however this normal form is not fibrewise linear, instead combining a linear and a quadratic part. △

Example 5.1.45. All the above examples are examples of Lie algebroid symplectic manifolds. A general linearisation result of Lagrangian submanifolds of Lie algebroid symplectic structures is proven in [71]. Under the assumption that the Lagrangian L is transverse to the anchor $\rho : \mathcal{A} \to N$ and rk $\mathcal{A} = \dim N - \dim M$ linearisability is proven. △

We will also consider an example of a Lagrangian which is not linearisable:

Example 5.1.46. Let $\mathfrak{su}(2)$ be the Lie algebra of trace-less skew Hermitian matrices, and let π be the associated linear Poisson structure on $\mathfrak{su}(2)^*$. If we identify $\mathfrak{su}(2)^*$ with $(\mathbb{R}^3, u_1, u_2, u_3)$ the Poisson bracket is defined by the relations:

- $\{u_1, u_2\} = 2u_3$
- $\{u_2, u_3\} = 2u_1$
- $\{u_3, u_1\} = 2u_2$.

The function $f = u_1^2 + u_2^2 + u_3^2$ is a Casimir, and thus the symplectic leaves of π are given by the concentric spheres together with the origin.

Consider the Lagrangian submanifold $L = \{u_3\} = 0$. The associated fibrewise linear Poisson structure on $\nu_L \simeq \mathbb{R}^3$ is defined by the relations:

- $\{u_1, u_2\}_1 = 0$
- $\{u_2, u_3\}_1 = 2u_1$
- $\{u_3, u_1\}_1 = 2u_2$.

The function $f_1 = u_1^2 + u_2^2$ is a Casimir, and thus the symplectic leaves are given by the concentric cylinders around the u_3-axis and the u_3-axis consists of zero-dimensional leaves.

Any neighbourhood of $L \subset (\mathbb{R}^3, \{\cdot, \cdot\})$ contains both a zero-dimensional leaf, as well as infinitely many spherical leaves. Therefore, there is no neighbourhood of L which is Poisson diffeomorphic to the linearisation $(\mathbb{R}^3, \{\cdot, \cdot\}_1)$.

Note that $\mathfrak{su}(2)$ is integrable by S^3 and thus globally all assumptions of Theorem 5.1.39 are satisfied. However when localising around L this is no longer the case. △

5.1.4 Around Poisson submanifolds: The sesquilinear case

We will now continue our discussion of normal form results using the next approximation, the quadratic part π_2. Instead of directly assuming that π_1 vanishes, we will make a slightly weaker assumption. Namely, we will assume that the submanifold is a Poisson submanifold:

Definition 5.1.47. Let (N, π) be a Poisson manifold, a submanifold $M \subset (N, \pi)$ is called a **Poisson submanifold** if there is a Poisson structure π_M on M such that the inclusion $\iota : M \to N$ is a Poisson map. ♢

Equivalently:

Proposition 5.1.48. *Let (N, π) be a Poisson manifold. An embedded submanifold $M \subset (N, \pi)$ is a Poisson submanifold if and only if one of the following equivalent conditions hold:*

- $\operatorname{Im} \pi_x^\sharp \subset T_xM$ *for all* $x \in M$,
- $TM^{\perp,\pi} = 0$,
- *the vanishing ideal I_M of M is an ideal with respect to the Poisson bracket.*

A Poisson submanifold is in particular a coisotropic submanifold. Because $TM^{\perp,\pi}$ coincides with the image of the anchor of the conormal Lie algebroid ν_M^*, Poisson submanifolds are precisely the coisotropic submanifolds for which the conormal Lie algebroid has vanishing anchor.

Following our narrative, we thus do not ask that π_1 vanishes but only that its symbol does.

Much work has been done in understanding normal form results around Poisson submanifolds. The point of view taken in the literature is that of jets around S. The definition of local data is as follows:

Definition 5.1.49. Let $M \subset N$ be a submanifold and let $\pi_M \in \mathfrak{X}^2(M)$ be a Poisson structure. A **first jet** of a Poisson structure is an element $\tau \in J^1_M(\mathfrak{X}^2_M(N))$, satisfying

$$\tau|_M = \pi_M, \quad \text{and} \quad [\tau, \tau] = 0.$$

The space of such element is denoted by $J^1_{(M,\pi_M)} \operatorname{Poiss}(N)$. ♢

Using Proposition 5.1.7 we will describe the space of first jets of Poisson structures in homogeneous terms. Let us first consider the case of vector bundles:

Definition 5.1.50 ([28],[49]). Let $E \to (M, \pi_M)$ be a vector bundle over a Poisson manifold. A triple $([\cdot,\cdot], \nabla^{\pi_M}, R)$ consisting of a

- A Lie algebroid stucture $[\cdot,\cdot]$ on E
- A contravariant connection ∇^{π_M}, with curvature $K_{\nabla^{\pi_M}} \in \mathfrak{X}^2(M; \mathrm{End}(E))$
- An E-valued bivector $R \in \mathfrak{X}^2(M; E)$

is called a **Poisson triple** if

$$\nabla^{\pi_M}_\alpha([e_1, e_2]) - [e_1, \nabla^{\pi_M}_\alpha(e_2)] = [\nabla^{\pi_M}_\alpha(e_1), e_2] \tag{5.1.8}$$

$$K_{\nabla^{\pi_M}}(\alpha, \beta)(e_1) = [R(\alpha, \beta), e_1] \tag{5.1.9}$$

$$\nabla^{\pi_M}_\alpha(R(\beta, \gamma)) + R(\alpha, [\beta, \gamma]_{\pi_M}) + \text{cycl. perm.} = 0, \tag{5.1.10}$$

for all $e_1, e_2 \in \Gamma(E)$ and $\alpha, \beta, \gamma \in \Omega^1(M)$. ♢

Lemma 5.1.51 ([28, 49]). *Let $E \to (M, \pi_M)$ be a vector bundle over a Poisson manifold. There is a one-to-one connection between:*

- *Poisson triples on E and*
- *First jets of Poisson structures on E.*

Proof. By Proposition 5.1.7 we have an isomorphism

$$J^1_M \mathfrak{X}^2(E) \simeq \mathscr{V}^1(E)_{1-\mathrm{pol}} \oplus \mathscr{V}^1_1(E)_{2-\mathrm{pol}} \oplus \mathscr{V}^1_2(E)_{3-\mathrm{pol}}. \tag{5.1.11}$$

Identifying the elements of the right-hand side (see Example 4.2.24, 4.2.20) we find they are precisely the objects appearing in a Poisson triple, we only need to check the differential conditions.

Given $\tau \in J^1_{(M,\pi_M)} \mathrm{Poiss}(E)$ let $W \in \mathfrak{X}^2(E)$ be such that $J^1_M W = \tau$. By definition $[\tau, \tau] = J^1_M[W, W] = 0$ and using Proposition 5.1.7 again we see that

$$J^1_M[W, W] \sim ([W, W]_{1-\mathrm{pol}}, \sigma([W, W]_{2-\mathrm{pol}}), \sigma^2([W, W]_{3-\mathrm{pol}}), \sigma^3([W, W]_{4-\mathrm{pol}}))$$

From this we conclude:

- $[W, W]_{1-\mathrm{pol}}$ vanishing is equivalent to Jacobi for $[\cdot,\cdot]$,
- $\sigma([W, W]_{2-\mathrm{pol}})$ vanishing is equivalent to (5.1.8),
- $\sigma^2([W, W]_{3-\mathrm{pol}})$ vanishing is equivalent to (5.1.9),
- $\sigma^3([W, W]_{4-\mathrm{pol}})$ vanishing is equivalent to (5.1.10). □

Given a Poisson manifold (N, π) and a Poisson submanifold M, the above Lemma implies that a tubular neighbourhood induces a Poisson triple on ν_M. The Lie algebroid structure on ν_M is independent on the tubular neighbourhood (and equal to the one from Lemma 5.1.20), and so is the contravariant connection. Unfortunately, the bivector R does depend on the choice of tubular neighbourhood.

Remark 5.1.52. There is a rather interesting phenomena in this theory. Namely that given a first jet of a Poisson structure $\tau \in J^1_M\mathfrak{X}^2(N)$ there need not be a Poisson structure π around M with $J^1\pi = \tau$ ([28]). In other words, there are cases where local models need not exist.

However, there are some conditions known under which local models do exists. One particular case is when M is a symplectic leaf, where it also provides a normal form theorem.

It would be very interesting to see whether our framework can say anything on the existence of local models. Viewing a Poisson triple again as an element of

$$([\cdot,\cdot],\nabla^\pi,R) \in \mathscr{V}^1(\nu_M)_{1-\mathrm{pol}} \oplus \mathscr{V}^1_1(\nu_M)_{2-\mathrm{pol}} \oplus \mathscr{V}^1_2(\nu_M)_{3-\mathrm{pol}}.$$

We can choose a connection and apply the splittings of the symbol sequence (see Proposition4.2.51) $S : \mathscr{V}^1_1(\nu_M)_{2-\mathrm{pol}} \to \mathscr{V}^1(\nu_M)_{2-\mathrm{pol}}$ and $S : \mathscr{V}^1_2(\nu_M)_{3-\mathrm{pol}} \to \mathscr{V}^1(\nu_M)_{3-\mathrm{pol}}$ and consider the bivector $\Pi := \pi_1 + S(\nabla^\pi) + S(R)$. It would be very interesting to see under which conditions this is Poisson and how it relates to the known local models. Most likely, the existence of connections adapted to the situation is important. When M is a symplectic leaf we have a canonical condition, namely the one induced by the contravariant connection. $\diamondsuit$

The contravariant connection plays an important role in our discussion so we will describe it explicitly:

Lemma 5.1.53. *Let (N,π) be a Poisson manifold and let $M \subset N$ be a Poisson submanifold. Then the normal bundle $\nu^*_M \to M$ carries an induced contravariant connection: Transport π to ν_M using a tubular neighbourhood and define*

$$(\nabla^{\pi_M})^*_{df}(s) = \{\widehat{f},\widehat{s}\}_{1-\mathrm{pol}}.$$

Moreover, $(\nabla^{\pi_M})^*$ does not depend on the choice of tubular neighbourhood.

Proof. Proving that this is a contravariant connection follows readily from the Jacobi identity for π. By Lemma 5.1.5 $\{\widehat{f},\widehat{s}\}_{1-\mathrm{pol}}$ does not depend on the choice of tubular neighbourhood. $\square$

5.1.5 The fibrewise quadratic case

We now consider the quadratic homogeneous approximation, π_2. This is Poisson when $\pi_0 = \pi_1 = 0$ which happens under the following conditions:

Lemma 5.1.54. *Let (N,π) be a Poisson manifold, and let $M \subset N$ be a submanifold. Then the constant and linear approximation of π around N, π_0 and π_1, vanish if and only if M is a Poisson submanifold whose conormal Lie algebroid is abelian.*

Spelling out the isomorphism $\mathfrak{X}^2(E^*)_{2-\mathrm{pol}} \simeq \mathscr{V}^1(E)_{2-\mathrm{pol}}$ we find:

Proposition 5.1.55. *There is a 1-1 correspondence between fiberwise quadratic Poisson structures $\pi \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ and triples (π_M,∇^π,Q) with:*

- *a Poisson structure π_M on M.*
- *a flat contravariant (w.r.t. π_M) connection $\nabla^\pi : \Gamma(T^*M) \times \Gamma(E) \to \Gamma(E)$.*
- *a bilinear antisymmetric map $Q : \Gamma(E) \times \Gamma(E) \to \Gamma(S^2E)$ satisfying the derivation property*

$$Q(s_1, fs_2) = fQ(s_1, s_2) + \nabla^\pi_{df}(s_1) \cdot s_2 \quad (s_1, s_1 \in \Gamma(E), f \in C^\infty(M))$$

and the Jacoby-type condition

$$Q(s_1, Q(s_2, s_3)) + Q(s_2, Q(s_3, s_1)) + Q(s_3, Q(s_1, s_2)) = 0$$

(where, Q is extended to the entire $\Gamma(S^\bullet E)$ by requiring it to be a derivation in each entry).

We have encountered the contravariant connection ∇^π already before, although it is moreover flat in this setting. The real new piece under consideration is a bilinear antisymmetric map $Q : \Gamma(E) \times \Gamma(E) \to \Gamma(S^2E)$.

Remark 5.1.56. The thesis of Matviichuk [60] studies fibrewise quadratic Poisson structures on holomorphic vector bundles E, under the assumption that the fibres of E are coisotropic submanifolds, or equivalently that the induced Poisson structure on the base vanishes. Therefore, the induced contravariant connection will be $C^\infty(M)$-linear in both-entries and define an element $\varphi \in \mathrm{End}(E; E \otimes TM)$. These elements called **co-Higgs fields** are the main object of study in [60]. ♢

Remark 5.1.57. For a triple (π_M, ∇^π, Q) as in Proposition 5.1.55, the Poisson condition $[\pi_M, \pi_M] = 0$ as well as the flatness of ∇^π actually follow from the Jacoby-type identity for Q. Indeed, this follows from considering the symbols of $[Q, Q]$ using Lemma 4.2.29. Besides these conditions, this identity also implies that Q is compatible with ∇ in the sense that

$$\nabla^\pi_\alpha(Q(s_1, s_2)) = Q(\nabla^\pi_\alpha s_1, s_2) + Q(s_1, \nabla^\pi_\alpha s_2) + \langle d\alpha, \nabla^\pi(s_1) \wedge \nabla^\pi(s_2)\rangle,$$

for all $\alpha \in \Omega^1(M)$. ♢

In the case of line bundles the map Q is already determined by the other pieces: Because E has rank one, there is only one linearly independent fibrewise linear function on E^*. Consequently, a Poisson bracket on E^* is determined by its value on $C^\infty(M) \times \Gamma(E)$, and thus

$$\{\widehat{f}, \widehat{s}\} = \widehat{\nabla^{\pi_M}_{df}(s)}, \quad f \in C^\infty(M), s \in \Gamma(E), \tag{5.1.12}$$

induces a bracket on E^*. The Jacobi identity follows readily from the flattness of ∇^{π_M}. We conclude:

Corollary 5.1.58. *For any line bundle $E \to M$ there is a 1-1 correspondence between fiberwise quadratic Poisson structures $\pi \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ and pairs (π_M, ∇^π) consisting of a Poisson structure π_M on M and a flat contravariant connection ∇^π on E.*

The above result appears in the algebraic/holomorphic setting in work of Polischuk [70]. A thorough study of line bundles with flat contravariant connections is performed in [44].

We want to obtain an explicit formula for the bivector associated to the Poisson bracket in (5.1.12). Using Corollary 4.2.51 we have an isomorphism:

$$\varphi^{\nabla} : \mathscr{V}^1(E)_{2-\mathrm{pol}} \simeq \mathfrak{X}^2(E^*)_{2-\mathrm{pol}} \to \Gamma(\Lambda^2 E^* \otimes S^2 E \oplus E^* \otimes E \otimes TM \oplus \wedge^2 TM), \tag{5.1.13}$$

Given $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ with core Q we write $\varphi^{\nabla}(\pi_2) = (S_0, S_1, S_2)$. To not force the reader to go back to Section 4.2.5, we will recall this isomorphism explicitly. Because we will use this construction in general in a moment, we will do so for general vector bundles:[3]

- $S_2 = \pi_M$ is precisely the induced Poisson structure on the zero section of E^*.
- $S_1 = Y$, where $Y \in \mathfrak{X}^1(M; \mathrm{End}(E))$ is the difference $(\nabla^{\pi_M} - \overline{\nabla}^{\pi_M})$.[4]
- $S_0 = Q - h_{\nabla}(\nabla^{\pi}) + \frac{1}{2}h^2_{\nabla}(\pi_M)$, with for $s_0, s_1 \in \Gamma(E)$

  $$h_{\nabla}(\nabla^{\pi})(s_0, s_1) = \langle \nabla^{\pi}(s_1), \nabla^{\pi}(s_0)\rangle - \langle \nabla^{\pi}(s_0), \nabla^{\pi}(s_1)\rangle$$

  and $\frac{1}{2}h^2_{\nabla}(\pi_M)$ is the core of $\mathrm{hor}^{\nabla^*}(\pi_M) \in \mathfrak{X}^2(E^*)$.

The procedure inverse to Equation (5.1.13) is given by:

$$\pi_2 = \iota_*(S_0) + (\mathrm{hor}^{\nabla^*} \otimes a)(Y) + \mathrm{hor}^{\nabla^*}(\pi_M), \tag{5.1.14}$$

where

- $\iota_*(S_0)$ is obtained by applying ι_* (see Example 4.2.16) to the $\Lambda^2 E^*$ component and viewing the $S^2 E$-component as quadratic function on E^*.
- $(\mathrm{hor}^{\nabla^*} \otimes a)(Y)$ is obtained taking the horizontal lift of the vector field and applying a (see Example 4.2.17) to the coefficient.

For line bundles, the situation simplifies significantly: $S_0 = 0$ and $Y = X \otimes \mathrm{id}$. Therefore, Equation 5.1.14 now simply reads:

$$\pi_2 = \mathrm{hor}^{\nabla^*}(X) \wedge \mathcal{E} + \mathrm{hor}^{\nabla *}(\pi_M). \tag{5.1.15}$$

This gives the explicit bivector corresponding to the bracket in (5.1.12).

Remark 5.1.59. There is another way to argue that the map Q disappears in the case of line bundles. By Lemma 4.2.63 we know that the triple (Q, ∇^{π}, π) corresponds to an element of $(Q^*, (\nabla^{\pi})^*, \pi^*) \in \mathcal{F}^2(E^*)_1$. The map Q^* takes values in $\Gamma(\Lambda^2 E^*)$ and must therefore necessarily vanish. Moreover $(\nabla^{\pi})^*$ is the dual contravariant connection and $\pi^* = \pi$. ♢

[3] Recall from Example 4.2.20 the unfortunate fact that the symbol of ∇^{π_M}, when viewed as element in $\mathscr{V}^1_1(E)_{2-\mathrm{pol}}$, is $-\pi_M$ rather than π_M. To make the formulae more natural, we have thus changed the signs slightly compared to Corollary 4.2.51.

[4] With $\overline{\nabla}^{\pi_M}$ the induced contravariant connection as in Example 4.1.19.

5.1.6 Complex homogeneity

When a vector bundle E furthermore carries a complex structure, we can also study Poisson structures which are compatible with it. We could consider complex Poisson structures and repeat the entire story above. However, we are mainly interested in real Poisson geometry so we will focus on the underlying real geometry. We now furthermore have the complex rescaling operators $m_\lambda : E \to E$ for $\lambda \in \mathbb{C}$ at our disposal. Suppose $\pi \in \mathfrak{X}^2(E^*)$ satisfies $m_\lambda^* \pi = \lambda^{d-2}\pi$. Because the left-hand side is real, we thus see that only $d = 2$ is possible and thus we define:

Definition 5.1.60. Let $E^* \to M$ be a complex vector bundle. We say that $\pi \in \mathfrak{X}^2(E^*)$ is **complex homogeneous of degree 2** if $m_\lambda^* \pi = \pi$ for all $\lambda \in \mathbb{C}$. ♢

If a Poisson structure is complex homogeneous of degree 2, then it is real homogeneous of degree 2 and thus defines an element of $\mathscr{V}^1(E)_{2-\mathrm{pol}}$. However, as is to be expected this element is furthermore compatible with the complex structure:

Proposition 5.1.61. *Let $E \to M$ be a complex vector bundle, with almost-complex structure J. There is a one-to-one correspondence between:*

- *Poisson structures $\pi \in \mathfrak{X}^2(E^*)$ which are complex homogeneous of degree 2.*
- *Triples (Q, ∇^π, π_M) as in Proposition 5.1.55 satisfying*
  - $Q(Js_0, Js_1) = J(Q(s_0, s_1))$ *for all $s_0, s_1 \in \Gamma(E)$.*[5]
  - $\nabla^\pi(Js) = J(\nabla^\pi(s))$ *for all $s \in \Gamma(E)$.*

Note that the condition on the contravariant connection actually follows from the condition on Q.

Given $\pi \in \mathfrak{X}^2(E^*)$ for which $\pi_0 = \pi_1 = 0$. Whereas $\pi_2 = \lim_{\lambda\to 0} m_\lambda^* \pi$ exists when the limit is taken over the real numbers, the limit might not exists when taken over the complex numbers. However if the complex limit exists then Proposition 5.1.61 tells us that π_2 satisfies extra conditions.

Example 5.1.62. The Poisson structure $x_1 x_2 \partial_{x_1} \wedge \partial_{x_2}$ on $\mathbb{R}^2$ is real homogeneous of degree 2, but not complex homogeneous. △

5.2 Log and elliptic symplectic structures

In the previous sections we studied homogeneous Poisson structures and how to obtain such structures from an arbitrary Poisson structure on a vector bundle. We will now study two cases in which this procedure gives an honest local form result; namely log and elliptic symplectic structures. We will show that these are, in a sense, among the easiest examples of Poisson structures for which their quadratic approximation provides a local model.

[5] Here J on the right-hand side denotes the involution on S^2E defined by $J(s_0 \cdot s_1) = Js_0 \cdot Js_1$.

Log symplectic We quickly recall some notation regarding log symplectic structures. A *log symplectic structure* is a Poisson structure $\pi \in \mathfrak{X}^2(N^{2n})$ such that $\wedge^n \pi$ vanishes transversely along a hypersurface Z. The *logarithmic tangent bundle* associated to a hypersurface $Z \subset N$, denoted $\mathcal{A}_Z$, is the Lie algebroid with sections vector fields tangent to Z. A log symplectic structure is equivalently described by a *log symplectic form* $\omega \in \Omega^2(\mathcal{A}_Z)$, which is a Lie algebroid symplectic form for the logarithmic tangent bundle.

In a sense, Poisson structures on the logarithmic tangent bundle (not necessarily non-degenerate) are precisely those which admit a quadratic approximation:

Lemma 5.2.1. *Let $E \to Z$ be a vector bundle, and let $\pi \in \mathfrak{X}^2(E)$. Then $\pi_0 = \pi_1 = 0$ if and only if $\pi \in \mathfrak{X}^2_Z(E)$, i.e. it is tangent to Z.*

Proof. This follows from a straightforward application of the local expression of $m_\lambda^* \pi$ from Remark 5.1.12. □

So, in the case E is a line bundle, the above lemma states that the constant and linear approximation of π vanish if and only if π admits a lift to the logarithmic tangent bundle.

Elliptic symplectic We quickly recall some notation about elliptic symplectic structures from Chapter 1. Let N be a manifold. An *elliptic divisor* on N is an ideal $I_{|D|} \subset C^\infty(N)$ which is generated by a single definite Morse-Bott function with codimension two critical set D. The *elliptic tangent bundle*, $\mathcal{A}_{|D|}$, is the Lie algebroid which has as sections vector fields preserving the ideal $I_{|D|}$. An *elliptic symplectic form* $\omega \in \Omega^2(\mathcal{A}_{|D|})$ is a Lie algebroid symplectic structure for the elliptic tangent bundle.

An elliptic symplectic structure $\omega \in \Omega^2(\mathcal{A}_{|D|})$ is equivalently described by certain Poisson structures: Let N^{2n} be an oriented manifold with co-volume $Q \in \Gamma(\wedge^{2n} TN)$, then for every Poisson structure $\pi \in \mathfrak{X}^2(N^{2n})$ we have $\pi^n = fQ$, for some function $f \in C^\infty(N)$. A Poisson structure is called *elliptic symplectic* if $\langle f \rangle \subset C^\infty(N)$ defines an elliptic divisor.

In what follows we will use the slight abuse of terminology to call both π and ω "the" elliptic structure.

If $I_{|D|}$ is an elliptic divisor for which D is co-oriented the *elliptic residue map* is the cochain morphism:

$$\mathrm{Res}_q : \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-2}(D).$$

In local Morse–Bott coordinates for f, in which $f = x^2 + y^2$, the elliptic residue is given by $\mathrm{Res}_q(\alpha) = \iota_D^*(\iota_{r\partial_r \wedge \partial_\theta} \alpha)$ where $r\partial_r = x\partial_x + y\partial_y$ and $\partial_\theta = y\partial_x - y\partial_x$.

The critical set $D \subset N$ is a Poisson submanifold, and the rank is determined by the value of the residue:

Proposition 5.2.2 ([52]). *Let $\omega \in \Omega^2(\mathcal{A}_{|D|})$ be an elliptic symplectic structure. Then $\mathrm{Res}_q(\omega)$ is constant and:*

- $\mathrm{Res}_q(\omega) = 0$ *if and only if* π_D *has corank 2.*
- $\mathrm{Res}_q(\omega) \neq 0$ *if and only if* π_D *is non-degenerate.*

If $E \to M$ is a complex line bundle then $\Gamma((E^{1,0})^*) \subset C^\infty(E)$ generates a complex ideal I on E. The real ideal defined by the relation $I_{|D|} \otimes \mathbb{C} = I \otimes \overline{I}$ is an elliptic divisor called the *elliptic divisor associated to* E.

When $I_{|D|}$ is an elliptic divisor, then ν_D inherits a natural divisor in the following way: Let $f \in I_{|D|}$ denote a generator, and let $\mathrm{Hess}^\nu(f) \in \Gamma(S^2\nu_D^*)$ be the normal Hessian. The quadratic function $Q_f \in C^\infty(\nu_D)$ associated to $\mathrm{Hess}^\nu(f)$ generates an elliptic ideal $I_{|D|}^{\nu_D}$ on ν_D.

Elliptic vector fields $X \in \Gamma(\mathcal{A}_{|D|})$ are tangent to the degeneracy locus D. Consequently, an elliptic symplectic structure π is tangent to D, i.e. $\pi \in \mathfrak{X}^2_D(N)$. Therefore by Lemma 5.2.1 the constant and linear approximation of π vanish, i.e. $\pi_0 = \pi_1 = 0$. Elliptic bivectors distinguish themselves by the fact that the complex quadratic approximation always exists:

Lemma 5.2.3. *Let* $E \to M$ *be a complex line bundle and let* $\pi \in \mathfrak{X}^2(E)$*. Then the limit* $\lim_{\lambda\to 0} m_\lambda^*\pi$*, with limit taken over* $\lambda \in \mathbb{C}$*, exists if and only if* π *admits a lift to* $\in \Gamma(\wedge^2\mathcal{A}_{|D|})$*, where* $I_{|D|}$ *is the elliptic divisor associated to* E*.*

Proof. Use coordinates (x_i, z), with the x_i coordinates along the base and z a complex fibre coordinate, to write

$$\pi = \sum_{j,k} A_{jk}(x,z)\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} + \sum_j B_j(x,z)\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial z}$$
$$+ \sum_j \overline{B_j(x,z)}\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial \overline{z}} + iC(x,z)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}.$$

Using the complex variant of the local expression of $m_\lambda^*\pi$ as in Remark 5.1.12 we see that $\lim_{\lambda\to 0} m_\lambda^*\pi$ exists if and only if

$$B_j = z\widetilde{B}_j, \quad C = z\overline{z}\widetilde{C}.$$

Which happens precisely when π can be lifted to a section of $\wedge^2\mathcal{A}_{|D|}$. □

5.2.1 Proving the normal form

We will recall how to, abstractly, prove that for log and elliptic symplectic structures there exists a neighbourhood around their degeneracy locus which is Poisson diffeomorphic to the quadratic approximation π_2. The first ingredient is the following Moser result:

Theorem 5.2.4 ([52])**.** *Let* $\mathcal{A} \to M$ *be a Lie algebroid with dense isomorphism locus. Let* $\omega, \omega' \in \Omega^2(M)$ *be two* $\mathcal{A}$*-symplectic forms such that:*

- $Z_{\mathcal{A}}$ *is smooth and* $(\rho_{\mathcal{A}}^{-1})^*(\omega' - \omega)$ *extends smoothly by 0 over* $Z_{\mathcal{A}}$*.*

- *$[\omega] = [\omega'] \in H^2(\mathcal{A})$ and there is a path of $\mathcal{A}$-symplectic forms connecting ω and ω'.*

Then there exists a Lie algebroid isomorphism $\varphi : \mathcal{A}|_U \to \mathcal{A}|_U$ such that $\varphi^*\omega' = \omega$.

Remark 5.2.5. It is important to remark that we do not only ask that ω' coincides with ω on $Z_{\mathcal{A}}$, but also that the difference extends smoothly. Indeed when $\mathcal{A}$ is the elliptic tangent bundle, the form $xd\log r$ vanishes on $Z_{\mathcal{A}}$ as an elliptic form but is not smooth. For the logarithmic tangent bundle this problem clearly does not arise. ♢

Lemma 5.2.6. *Let $\pi \in \mathfrak{X}^2(N)$ be log symplectic, then so is $\pi_2 \in \mathfrak{X}^2(\nu_Z)$. Let $\pi \in \mathfrak{X}^2(N)$ be elliptic symplectic with respect to $I_{|D|}$, then so is $\pi_2 \in \mathfrak{X}^2(\nu_D)$ with respect to $I^{\nu_D}_{|D|}$.*

Moreover, $\pi_p = \pi_{p,2}$ as log respectively elliptic bivectors for all p in the degeneracy locus.

Proof. We will give the argument in the log symplectic case, the elliptic case is analogous. If in local coordinates

$$\pi = \sum_{i,j} A_{i,j}(x,y)\partial_{x_i} \wedge \partial_{x_j} + \sum_i B_i(x,y)\partial_{x_i} \wedge \partial_y,$$

then

$$\pi_2 = \sum_{i,j} A_{i,j}(x,0)\partial_{x_i} \wedge \partial_{x_j} + \sum_i y\frac{\partial B_i}{\partial y}(x,0)\partial_{x_i} \wedge \partial_y.$$

From the expression of π_2 it is clear that it admits a lift to the logarithmic tangent bundle $\widetilde{\pi}_2 \in \Gamma(\mathcal{A}_Z)$. Because π is log symplectic we have that the B_i vanish transversely and therefore $B_i(x,y)$ and $y\frac{\partial B_i}{\partial y}(x,0)$ have the same first order jet. Consequently the lifts $\widetilde{\pi}$ and $\widetilde{\pi}_2$ coincide along Z. Therefore $\widetilde{\pi}_{2,p} : \mathcal{A}^*_{Z,p} \to \mathcal{A}_{Z,p}$ is an isomorphism for all $p \in Z$. Because π_2 is homogeneous it follows that $\widetilde{\pi}_{2,p}$ is an isomorphism for all $p \in Z$. □

This will allow us to prove the abstract normal form theorem for log symplectic Poisson structures.

Corollary 5.2.7 ([45]). *Let $Z \subset N$ be a hypersurface and let π be log symplectic with $(\wedge^n\pi)^{-1}(\{0\}) = Z$. Then there exists a neighbourhood $\mathcal{U}$ around Z on which π is Poisson diffeomorphic to π_2 via a Poisson diffeomorphism which is the identity on Z.*

Proof. By Lemma 5.2.6 we have that π and π_2 coincide on Z as logarithmic bivectors. Therefore there inverses, $\omega, \omega_2 \in \Omega^2(\mathcal{A}_{|D|})$ also coincide on the degeneracy locus. Because of the nature of the logarithmic tangent bundle $\omega - \omega_2$ extends smoothly to zero over the degeneracy locus. Therefore by Theorem 5.2.4 we have that ω an ω_2 are logarithmic symplectomorphic on a tubular neighbourhood. Hence we also conclude that π is Poisson diffeomorphic to π_2 on a tubular neighbourhood. □

For the elliptic case, we need one more ingredient:

Lemma 5.2.8 ([16]). *If a closed elliptic form vanishes along the degeneracy locus, then it is trivial in the elliptic de Rham cohomology.*

Combined with this lemma, we also get the result for the elliptic symplectic structures:

Corollary 5.2.9 ([16]). *Let $\pi \in \mathfrak{X}^2(N)$ be an elliptic symplectic Poisson structure with $D = (\wedge^n \pi)^{-1}(\{0\})$. Then there exists a neighbourhood $\mathcal{U}$ around D on which π is Poisson diffeomorphic to π_2, via a diffeomorphism which is the identity on D.*

Proof. First by Lemma 5.2.6 we have that π and π_2 coincide on D as elliptic bivectors. Therefore there inverses, ω, ω_2 also coincide on D. Therefore the linear interpolation $\omega_t = t\omega_2 + (1-t)\omega$ is non-degenerate on a neighbourhood of D, and by Lemma 5.2.8 we have that $[\omega] = [\omega_2] \in H^2(\mathcal{A}_{|D|})$. Consequently, Theorem 5.2.4 implies that ω an ω_2 are elliptic symplectomorphic on a tubular neighbourhood. Hence we conclude that π is Poisson diffeomorphic to π_2 on a tubular neighbourhood. □

5.2.2 Explicit normal forms

In the previous section we abstractly described a normal form result for log and elliptic symplectic structures. We now use the strategy outlined in Remark 4.2.53 to obtain explicit formulae for the local model π_2.

Log symplectic We first need to relate the contravariant connection on the normal bundle of Z induced by the Poisson structure, with the restriction of the canonical contravariant connection:

Lemma 5.2.10. *Let (N, π) be a log symplectic manifold, with degeneracy locus Z. Then ν_Z^* endowed with the contravariant connection from Lemma 5.1.53 and $\wedge^{2n} T^*N\big|_Z$ endowed with the canonical contravariant connection are isomorphic as vector bundles endowed with a contravariant connection.*

Proof. We will establish this by showing that the connection matrices coincide. Because π^n vanishes transversely along Z, it induces an isomorphism

$$d^\nu \pi^n : \nu_Z \to \Lambda^{2n} TN\big|_Z .$$

If we let z denote a defining function for Z, then it induces a local frame for ν_Z as well as a local orientation $\rho \in \Gamma(\wedge^{2n} TN\big|_Z)$ via $\rho := z^{-1}\pi^n$. The isomorphism $d^\nu \pi^n$ sends these local frames to each other.

Because $\nabla^{\text{can}}(\pi^n) = 0$, we find $\nabla^{\text{can}}(z\rho) = 0$ from which we conclude that $z\nabla^{\text{can}}(\rho) = -\pi^\sharp(dz) \otimes \rho$. We conclude that the connection matrix of ∇^{can} is given by $-z^{-1}\pi^\sharp(dz)$.

On the other had, the contravariant connection on ν_Z induced by Lemma 5.1.53 has, after inspection of the definition, the same connection matrix. □

In Equation (5.1.15) we have already obtained an explicit formula for any fibrewise quadratic Poisson structure on a vector bundle E^*. We now consider under which conditions this is log-symplectic:

Proposition 5.2.11. *Let $E \to Z^{2n-1}$ be a vector bundle and let $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ be a quadratic log-symplectic structure and let ∇ be a connection on E with curvature $K = \widetilde{K} \otimes \mathrm{id} \in \Omega^2(Z; \mathrm{End}(E))$. Then using* (5.1.13) *we have $\varphi^\nabla(\pi_2) = (0, X \otimes \mathrm{id}, \pi_Z)$ with*

- $[\pi_Z, X] = \pi^\sharp(\widetilde{K})$,
- $[\pi_Z, \pi_Z] = 0$,
- $X \wedge \pi_Z^{n-1} \neq 0$.

Therefore,

$$\pi_2 = \mathrm{hor}^{\nabla^*}(X) \wedge \mathcal{E} + \mathrm{hor}^{\nabla^*}(\pi_Z), \tag{5.2.1}$$

with $\mathcal{E}$ the Euler vector field of E^*.

Proof. Using Lemma 4.2.54 to compute $[\pi_2, \pi_2] = 0$, or doing directly, we find:

$$\begin{aligned}
[X \otimes \mathrm{id} + \pi_Z, X \otimes \mathrm{id} + \pi_M] &= 2[X, \pi_Z] \otimes \mathrm{id} + [\pi_Z, \pi_Z] \\
&= 2([X, \pi_Z] + K^*(X, \pi_Z)) \otimes \mathrm{id} \\
&\quad + [\pi_Z, \pi_Z]_{SN} + K^*(\pi_Z, \pi_Z) \\
&= (2[X, \pi_Z]_{SN} + K^*(\pi_Z, \pi_Z)) \otimes \mathrm{id} + [\pi_Z, \pi_Z]_{SN}.
\end{aligned}$$

One can readily show that $K^*(\pi_Z, \pi_Z) = 2\pi_M^\sharp(\widetilde{K}) \otimes \mathrm{id}$, which proves that for π_2 to be Poisson $[\pi_Z, X] = \pi^\sharp(\widetilde{K})$ and $[\pi_Z, \pi_Z] = 0$. The condition $X \wedge \pi_Z^{n-1}$ must hold for π_2 to define a log symplectic structure. ☐

Combining this description with the abstract normal form from the previous section we conclude:

Corollary 5.2.12. *Let $\pi \in \mathfrak{X}^2_Z(N)$ be a log-symplectic structure on a manifold N. Then there is a neighbourhood of Z on which π is Poisson diffeomorphic to $\pi_2 \in \mathfrak{X}^2(\nu_Z)$ in Equation* (5.2.1) *via a Poisson diffeomorphism which is the identity on Z. If the connection used is induced by a metric, then X is the restriction of a modular vector field of (N, π) to Z.*

Proof. By Corollary 5.2.7 we have π is Poisson diffeomorphic to π_2. In Remark 4.1.22 we explained how to use a metric to obtain the vector field X, which then represents the modular class of the contravariant connection from 5.1.53. By Lemma 5.2.10 this contravariant connection is isomorphic to the restriction of the canonical contravariant connection, and consequently X represents the characteristic class of the restriction of the canonical contravariant connection. Consequently, X is the restriction of a modular vector field of (N, π). ☐

Proposition 5.2.11 also tells us that if (X, π_Z) is a pair satisfying the said equations than Equation (5.2.1) defines a quadratic Poisson structure. However, this discussion depends on the connection. The non-degeneracy of the Poisson structure can also be detected from the (canonical) contravariant connection:

Proposition 5.2.13 ([42])**.** *Let $E \to Z$ be a real line bundle. There is a 1-1 correspondence between quadratic log symplectic structures $\pi_2 \in \mathfrak{X}^2(E^*)$ and*

- *Corank 1 Poisson structures π_Z on Z.*
- *contravariant connections ∇^{π_Z} for which the character of the isotropy representation[6] (as in Definition 4.1.1) is nowhere vanishing.*

Proof. By Proposition 5.1.55 we have that there is a one-to-one correspondence between quadratic Poisson structures and contravariant connections. Therefore, we only have to show that π_2 is log symplectic if and only if said conditions hold.

Proposition 5.2.11 tells us that for a quadratic Poisson structure π_2 we have

$$\wedge^n \pi_2 = -\mathcal{E} \wedge \mathrm{hor}^{\nabla^*}(X) \wedge \mathrm{hor}^{\nabla^*}(\pi_Z)^{n-1}.$$

Because X mod $\mathcal{F}_{\pi_Z}$ is precisely the character of the isotropy representation of the connection ∇^{π_Z}, we see that $\wedge^n \pi_2$ vanishes transversely if and only if $(\pi_Z)^{n-1} \neq 0$ and the character of the isotropy representation is nowhere vanishing. □

Remark 5.2.14. In [42, 44], the conditions of the above proposition are described using the notion of the **residue** of a line bundle endowed with a flat contravariant connection. This residue is, in the notation of (5.2.1), precisely $X \wedge \pi_Z^{n-1}$. Note that although X itself depends on the choice of the connection, this residue does not. ♢

Remark 5.2.15 (Regularisation)**.** Given any Poisson manifold (N^n, π), we have the canonical flat contravariant connection on $K = \wedge^n TM$ from Example 4.1.21. Consequently, by Corollary 5.1.58 we have that the total space of K^* itself inherits a Poisson structure.

Now let (N, π) be log-symplectic with degeneracy locus Z. Dualising the flat contravariant connection on ν_Z^* to one on ν_Z we obtain a Poisson structure on ν_Z^*. Because of Lemma 5.2.10 we have that $\nu_Z^* \subset K^*$ is a Poisson submanifold.

This construction might seem harmless, but variants of it have popped up in many places. For log symplectic structures, the Poisson structure on K^* will have constant rank away from the zero section, and thus defines a codimension-one symplectic foliation on $K^* \backslash \mathcal{N}$. Because this Poisson structure is fibrewise homogeneous, this space can be compactified to a circle bundle over N. This Poisson manifold is known as the **regularisation** of (N, π) and appears in [73] for orientable N.

A completely analogous construction appears in [16] for stable generalized complex structures. ♢

[6] Also known as normal/transverse Higgs field.

Log symplectic form We can also explicitly obtain the normal form for the log symplectic form inverse to π_2. To do this we need the following construction:

Let $\mathcal{N} \to Z$ be a real line bundle, and let $\mathcal{N}^\times := \mathrm{Fr}(\mathcal{N}) = \mathcal{N}\backslash Z$ denote the associated principal $\mathbb{R}^*$-bundle. A connection ∇ on $\mathcal{N}$ induces a principal connection on $\mathcal{N}^\times$. Let $\rho \in \Omega^1(\mathcal{N}^\times; \mathbb{R})$ denote the corresponding connection one-form. The curvature of this one-form $d\rho \in \Omega^2(\mathcal{N}^\times; \mathbb{R})_{\mathrm{bas}} \simeq \Omega^2(Z)$ coincides precisely with $\widetilde{K} \in \Omega^2(Z)$ where $\widetilde{K} \otimes \mathrm{id} \in \Omega^2(Z; \mathrm{End}(\mathcal{N}))$ is the curvature or ∇.

Lemma 5.2.16. *Let $\mathcal{N} \to Z$ be a real line bundle with connection ∇. Then the connection one-form $\rho \in \Omega^1(\mathcal{N}^\times; \mathbb{R})$ can be extended to a smooth logarithmic form $\rho \in \Omega^1(\mathcal{A}_Z)$ on $\mathcal{N}$.*

Proof. For ρ to extend to a logarithmic form we only need to check that $\rho(\mathcal{E})$ extends over the zero-section, as this is the coefficient of the singular part. But this is immediate as $\rho(\mathcal{E}) = 1$ because ρ is a connection one-form. □

Using this form we can describe the elliptic symplectic form inverse to π_2:

Lemma 5.2.17. *Let $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ be a quadratic log symplectic structure. Then the elliptic symplectic form $\omega_2 \in \Omega^2(E^*, \mathcal{A}_Z)$ inverse to it is given by:*

$$\omega_2 = \rho \wedge p^*\theta + p^*\beta, \tag{5.2.2}$$

where (θ, β) is the almost 1-cosymplectic structure (see Definition 4.1.11) inverse to the almost 1-cosymplectic structure (X, π_Z).

Proof. To prove that the two are inverses, we will compute $\pi_2^\sharp(\omega_2^\flat(W))$, for W equal to $\mathcal{E}, \mathrm{hor}^{\nabla^*}(X)$ and $\mathrm{hor}^{\nabla^*}(V)$, for V tangent to the foliation on Z. Because these vector fields span TE at every point the desired result follows. What is left is just a computation, making use of the fact that ρ annihilates horizontal vector fields:

$$\begin{aligned}
\pi_2^\sharp(\omega_2^\flat(\mathrm{hor}^{\nabla^*}(X)) &= \pi_2^\sharp(-\rho) = \mathrm{hor}^{\nabla^*}(X) \\
\pi_2^\sharp(\omega_2^\flat(\mathcal{E})) &= \pi_2^\sharp(p^*\theta) = \mathcal{E} \\
\pi_2^\sharp(\omega_2^\flat(V)) &= \pi_2^\sharp(p^*\beta^\flat(V)) = \mathrm{hor}^{\nabla^*}(\pi_Z)^\sharp(p^*\beta(V)) = \mathrm{hor}^{\nabla^*}(V).
\end{aligned}$$ □

Remark 5.2.18. Because line bundles always admit flat connections we have that given an elliptic symplectic structure π we can always associate an actual 1-Poisson structure (X, π_Z) or cosymplectic structure (α, β).

In the literature, it is more common to define the singular form ρ define using a Riemannian metric on E^*, with associated distance function $|\cdot| : E^* \to \mathbb{R}$, and then consider $d\log|x|$. If the connection used to define ρ is however compatible with the Riemannian metric then $\rho = d\log|x|$.

The one form θ, is called the residue of the log symplectic structure $\theta = \mathrm{Res}(\omega)$. ♢

Remark 5.2.19. The study of log symplectic structures was initiated in [67] and continued in [45]. Here a Moser argument is proven to prove that ω and ω_2 are symplectomorphic is given. The normal form result is also discussed in [14], where, a variant of, the explicit formula (5.2.2) is given. There it is also explained that the induced geometric structure is the contravariant connection on the normal bundle. The normal form is also discussed in [42], where the explicit formula (5.2.1) appears in the proof op Proposition 1.9 loc. cit. ♢

Elliptic symplectic For elliptic symplectic structures the story is a bit more complicated. To obtain explicit formulae for the local model π_2 we have to make the additional assumption that the normal bundle to the degeneracy locus is orientable, and can thus be endowed with a complex structure.

The contravariant connection on the normal bundle of the degeneracy locus, and the canonical contravariant connection are related:

Lemma 5.2.20. *Let (N, π) be an elliptic symplectic manifold with co-orientable degeneracy locus, and let (E, ∇^π) denote the contravariant connection induced by Proposition 5.1.61. Consider the Poisson line bundle defined $(R, \widetilde{\nabla}^\pi)$ via:*

$$(R, \widetilde{\nabla}^\pi) \otimes \mathbb{C} := (E, \nabla^\pi) \otimes (E^*, (\nabla^\pi)^*).$$

Then $(R, \widetilde{\nabla}^\pi)$ is isomorphic to the restriction of the canonical representation $\wedge^{\mathrm{top}} T^*N$ to D.

Proof. We again do this by comparing the connection matrices, as we did in Lemma 5.2.10. Note that both R and $\wedge^{\mathrm{top}} T^*N$ are trivialisable. Let $f \in I_{|D|}$ be a function defining the elliptic ideal. Then π^n/f defines a volume form and the connection matrix of ∇^{can} with respect to this volume form is $-\pi^\sharp(d\log f)$. Let z denote a local fibre coordinate of E, such that $|z|^2 = f$ and let $e \in \Gamma(E)$ denote the associated local frame.

Because the contravariant connection ∇^π is complex linear the associated connection matrix is given by $-\pi^\sharp(d\log z)$. Therefore

$$\begin{aligned}
(\nabla^\pi \otimes \overline{\nabla}^\pi)(e \otimes \overline{e}) &= \nabla^\pi(e) \otimes \overline{e} + e \otimes \nabla^\pi(\overline{e}) \\
&= -\pi^\sharp(d\log z) e \otimes \overline{e} + e \otimes \overline{\pi^\sharp(d\log z)}\overline{e} \\
&= -2\pi^\sharp(d\log|z|) e \otimes \overline{e},
\end{aligned}$$

and thus the connection matrix is $-\pi^\sharp(d\log|z|^2)$, which finishes the proof. □

If E^* is a complex line bundle, we may consider the complex Euler vector field. We denote its real and imaginary part by $r\partial_r$ and ∂_θ. If we view E^* as a real vector bundle endowed with a complex structure $J \in \mathrm{End}(E^*)$, we have $r\partial_r = a(\mathrm{id})$ and $\partial_\theta = a(J)$.

For elliptic symplectic structures, we now obtain:

Proposition 5.2.21. *Let $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ be a quadratic elliptic symplectic structure and let ∇ be a complex linear connection on E with curvature*

$$K = K_1 \otimes \mathrm{id} + K_2 \otimes J \in \Omega^2(M; \mathrm{End}(E)).$$

Then for (5.1.13) *we have $\varphi^\nabla(\pi_2) = (g\,\mathrm{id} \cdot J, X \otimes \mathrm{id} + Y \otimes J, \pi_D)$, for some $g \in C^\infty(D)$, $X, Y \in \mathfrak{X}^1(D)$. Therefore*

$$\pi_2 = gr\partial_r \wedge \partial_\theta + \mathrm{hor}^{\nabla^*}(X) \wedge r\partial_r + \mathrm{hor}^{\nabla^*}(Y) \wedge \partial_\theta + \mathrm{hor}^{\nabla^*}(\pi_D). \tag{5.2.3}$$

Proof. Let $\varphi^\nabla(\pi_2) = (S_0, S_1, S_2)$. We apply the discussion around Equation (5.1.14). Because ∇^π is complex linear we have that S_1 is an endomorphism valued vector field $Z \in \mathfrak{X}^1(D; \mathrm{End}(E^*))$, which takes values in the endomorphism which commute with J. Because these endomorphism are spanned by id and J, because E has rank 2, we find that there exists vector fields $X, Y \in \mathfrak{X}^1(D)$ such that $Z = X \otimes \mathrm{id} + Y \otimes J$.

Recall that if we let $Q \in \mathscr{V}^1(E)_{2-\mathrm{pol}}$ denote the core of π_2, $S_0 = Q - h_\nabla(\nabla^\pi) + \frac{1}{2}h^2_\nabla(\pi_D)$, with h_∇ as in Equation 4.2.18. By Proposition 5.1.61 we have that Q is compatible with J in the sense $Q(Js_0, Js_1) = J(Q(s_0, s_1))$. Inspecting the definition of h_∇, we see that also $h_\nabla(\nabla^\pi)$ and $h^2_\nabla(\pi_D)$ satisfy this relation, and therefore so does S_0.

Claim 5.2.22. Let $S \in \Gamma(\wedge^2 E^* \otimes S^2 E)$ satisfy $S(Js_0, Js_1) = JS(s_0, s_1)$, then $S = g\mathrm{id}\widehat{\otimes}J$, for some function $g \in C^\infty(D)$. ♢

Proof of claim. Let s_0, s_1 be a local frame on some open $U \subset D$ adapted to the complex structure, i.e. $Js_0 = s_1$. The local frame provides a trivialisation $S^2E|_U \simeq \underline{\mathbb{R}}^3$, and thus we have

$$S(s_0, s_1) = f_1 s_0 \cdot s_0 + f_2 s_0 \cdot s_1 + f_3 s_1 \cdot s_1,$$

for some functions $f_1, f_2, f_3 \in C^\infty(U)$. But also

$$S(Js_0, Js_1) = f_1 s_1 \cdot s_1 - f_2 s_1 \cdot s_0 + f_3 s_0 \cdot s_0,$$

and hence $f_1 = f_3$ and $f_2 = 0$. Consequently $S(s_0, s_1) = f_1(s_0 \cdot s_0 + s_1 \cdot s_1) = -f_1(s_0 \cdot Js_1 - Js_0 \cdot s_1)$. One readily verifies that the function f_1 does not depend on the local frame and thus gives rise to a global function $g \in C^\infty(D)$. ☐

Combining the above proposition with Corollary 5.2.9 and Lemma 5.2.20 we arrive at:

Corollary 5.2.23. *Let $\pi \in \mathfrak{X}^2(N)$ be an elliptic symplectic structure with co-orientable degeneracy locus. Then there exists a tubular neighbourhood of D on which π is Poisson diffeomorphic to π_2 as in Equation* (5.2.3)*, via a Poisson diffeomorphism which is the identity on D.*

When the connection used is compatible with a metric, X is the restriction of a modular vector field to D.

To simplify the formula for π_2 further we have to differentiate between the zero and non-zero elliptic residue cases.

Non-zero residue We first remark that the assumption of co-orientablity of the normal bundle is automatically verified for non-zero residue elliptic symplectic structures:

Lemma 5.2.24. *Let $\omega \in \Omega^2(N, \mathcal{A}_{|D|})$ be an elliptic symplectic structure with non-zero elliptic residue. Then ν_D is oriented.*

Proof. Remark that the elliptic symplectic structure induces an orientation on the manifold N. By Lemma 5.2.2, we have that D is a symplecitc manifold, and hence an oriented submanifold of an oriented manifold. Therefore ND is oriented. □

We inspect Proposition 5.2.21 in more detail for non-zero residue elliptic symplectic structures. This case is particularly nice because in this case the normal bundle becomes canonically endowed with a flat connection:

Proposition 5.2.25. *There is a one-to-one correspondence between:*

- *Quadratic non-zero residue elliptic symplectic structures $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$.*
- *Triples (λ, ∇, π_D) where λ is a non-zero constant, ∇ is a flat connection on E and π_D is a non-degenerate Poisson structure on D.*

Given such a triple (λ, ∇, π_D) the associated elliptic symplectic structure is given by

$$\pi_2 = \lambda \partial_\theta \wedge r\partial_r + \mathrm{hor}^{\nabla^*}(\pi_D). \tag{5.2.4}$$

Conversely, given $\pi \in \mathfrak{X}^2(N)$ with non-zero elliptic residue the associated π_2 corresponds to $\pi_D = \pi_2|_D$, $\lambda = \mathrm{Res}_q(\pi_2)^{-1}$ and ∇ is the Bott-connection of the symplectic leaf D.

Proof. Recall from Proposition 5.1.61 that the quadratic Poisson structure π_2 is complex linear. Moreover by Lemma 5.2.2 π_D is non-degenerate, and the contravariant connection ∇^{π_D} therefore induces an ordinary connection via:

$$\nabla_X(s) := \nabla_{(\pi_D^\sharp)^{-1}(X)}(s).$$

We therefore see that if we construct X and Y as in Proposition 5.2.21 using this connection they vanish.

We are thus left to compute the function g from Proposition 5.2.21. Because $\pi_2 = gr\partial_r \wedge \partial_\theta + \mathrm{hor}^{\nabla^*}(\pi_D)$, and $[\pi_2, \pi_2] = 0$ it follows that $[g, \pi_D] = 0$ and thus that $g = \lambda \in \mathbb{R}$. We are left to show that this constant is the inverse of the elliptic residue. Instead of proving this directly, we will show in Lemma 5.2.31 that π_2 is inverse to an elliptic symplectic form ω_2 from which it will follow that λ is minus the inverse of the elliptic residue. □

Corollary 5.2.26. *Let $\pi \in \mathfrak{X}^2(N)$ be elliptic symplectic with non-zero elliptic residue. Then there exists a tubular neighbourhood of the degeneracy locus on which π is Poisson diffeomorphic to π_2 from* (5.2.4)*, via a Poisson diffeomorphism which is the identity on D.*

Zero residue We now inspect Proposition 5.2.21 for zero residue elliptic symplectic structures. As discussed in Chapter 1 these structures are in one-to-one correspondence with gauge equivalence classes of stable generalized complex structures. The normal form results of these are discussed in [16].

Proposition 5.2.27. *Let $E \to D^{2n-2}$ be a complex vector bundle. Let $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$ be a quadratic elliptic symplectic structure with zero elliptic residue. Then there exists a connection ∇ on E such that $\varphi^\nabla(\pi_2) = (0, X \otimes \mathrm{id} + Y \otimes J, \pi_D)$, where φ^∇ is as in* (5.1.13). *The triple $(X, Y, \pi_D) \in \mathfrak{X}^1(D) \times \mathfrak{X}^1(D) \times \mathfrak{X}^2(D)$ satisfies*

- $[X, Y] = \pi_D^\sharp(\iota_Y K_1) + \pi_D^\sharp(\iota_X K_2)$
- $[\pi_D, X] = \pi_D^\sharp(K_1)$
- $[\pi_D, Y] = -\pi_D^\sharp(K_2)$
- $[\pi_D, \pi_D] = 0$
- $X \wedge Y \wedge \pi_D^{n-2} \neq 0.$

Given any such triple (X, Y, π_D) and connection, the associated elliptic symplectic structure is given by:

$$\pi_2 = \mathrm{hor}^{\nabla^*}(X) \wedge r\partial_r + \mathrm{hor}^{\nabla^*}(Y) \wedge \partial_\theta + \mathrm{hor}^{\nabla^*}(\pi_D). \tag{5.2.5}$$

Proof. We again apply Proposition 5.2.21. Because π_2 has zero elliptic residue, we have by Proposition 5.2.2 that π_D has rank $2n-4$. Therefore, by Proposition 5.2.21 we have that for all connections ∇

$$\wedge^{2n}\pi_2 = r\partial_r \wedge \partial_\theta \wedge \mathrm{hor}^{\nabla^*}(X) \wedge \mathrm{hor}^{\nabla^*}(Y) \wedge \mathrm{hor}^{\nabla^*}(\pi_D)^{n-2},$$

and for π_2 to be non-degenerate we must thus have that $X \wedge Y \wedge \pi_D^{n-2}$ is nowhere vanishing.

Let $Q \in \mathscr{V}^1(E)_{1-\mathrm{pol}}$ denote the core of π_2. We will show that there exists a connection ∇ such that $Q = S^\nabla(\sigma_Q)$, where $S^\nabla : \mathscr{V}^1_1(E)_{1-\mathrm{pol}} \to \mathscr{V}^1(E)_{1-\mathrm{pol}}$ denotes the splitting of the symbol sequence from Lemma 4.2.42. Recall that $S^\nabla(\sigma_Q) = h_\nabla(\sigma_Q) - \frac{1}{2}h^2_\nabla(\sigma_Q^2)$, wich h_∇ as in (4.2.18). If we prove this we will have shown that $S_0 = Q - S^\nabla(\sigma_Q) = 0$.

Let ∇, ∇' be any two Hermitian connections compatible with a metric. We compute the difference $S^\nabla(\sigma_Q) - S^{\nabla'}(\sigma_Q)$. Because there exists $\beta \in \Omega^1(D)$ such that

$$\nabla - \nabla' = \beta \otimes J,$$

we have

$$(h_\nabla(\sigma_Q) - h_{\nabla'}(\sigma_Q))(s_0, s_1) = \langle \sigma_Q(s_0), \beta \rangle Js_1 - \langle \sigma_Q(s_1), \beta \rangle Js_0.$$

Moreover,

$$h^2_\nabla(\sigma^2_Q)(s_0, s_1) = \left\langle \sigma^2_Q, \nabla(s_0) \wedge \nabla(s_1) \right\rangle$$

and thus

$$\begin{aligned} h^2_\nabla(\sigma^2_Q)(s_0,s_1) - h^2_{\nabla'}(\sigma^2_Q)(s_0,s_1) &= \left\langle \sigma^2_Q, \nabla(s_0)\wedge(\beta\otimes Js_1)\right\rangle \\ &\quad - \left\langle \sigma^2_Q, (\beta\otimes Js_0)\wedge\nabla'(s_1)\right\rangle \\ &= \nabla_{\pi^\sharp(\beta)}(s_0)\cdot Js_1 - \nabla'_{\pi^\sharp(\beta)}(s_1)\cdot Js_0. \end{aligned}$$

Now we denote $\nabla_{\pi^\sharp(-)} - \nabla^\pi = X\otimes\mathrm{id} + Y\otimes J$, and $\nabla'_{\pi^\sharp(-)} - \nabla^\pi = X'\otimes\mathrm{id} + Y'\otimes J$. If we combine the two difference terms we end up with

$$\begin{aligned} h^2_\nabla(\sigma^2_Q)(s_0,s_1) - h^2_{\nabla'}(\sigma^2_Q)(s_0,s_1) &= X(\beta)s_0\cdot Js_1 - X'(\beta)s_1\cdot Js_0 \\ &\quad + (Y(\beta) - Y'(\beta))Js_0\cdot Js_1 \end{aligned}$$

Note that $X - X' = 0$ and $Y - Y' = \pi^\sharp(\beta)$, consequently $(Y(\beta) - Y'(\beta)) = 0$. We conclude that

$$(S^\nabla(\sigma_Q) - S^{\nabla'})(\sigma_Q)(s_0,s_1) = +X(\beta)(Js_1\cdot s_0 - Js_0\cdot s_1).$$

By Claim 5.2.22 we have that $(Q - S^\nabla(\sigma_Q)(s_0,s_1) = -gs_0\cdot Js_1 + gJs_0\cdot s_1$. Because the vector field X is nowhere vanishing we can take β such that $\beta(X) = g$. Doing this ensures that $(Q - S^{\nabla'}(\sigma_Q)) = 0$.

Using Lemma 4.2.54 to compute $[\pi_2,\pi_2] = 0$ ensures that X, Y and π_D satisfy the given conditions. □

Corollary 5.2.28 ([16])**.** *Let $\pi \in \mathfrak{X}^2(N)$ be elliptic symplectic with zero elliptic residue and co-orientable degeneracy locus. Then there exists a tubular neighbourhood of the degeneracy locus on which π is Poisson diffeomorphic to π_2 from* (5.2.5)*, via a Poisson diffeomorphism which is the identity on D. If the connection used is compatible with a metric, than X is the restriction of a modular vector field to D.*

In contrary to the non-zero elliptic residue case there is no canonical (flat) connection on E; the triple (X, Y, π_D) is therefore also dependent on the auxiliary connection. In the statement of Proposition 5.2.27 the term $Q \in \mathscr{V}^1(E)_{2-\mathrm{pol}}$ seems to drop out of the discussion, as there is a connection such that $Q = S^\nabla(\nabla^\pi)$. We now ask ourselves whether it is enough to remember ∇^π, and what the conditions are for such a connection ∇ to exist:

Lemma 5.2.29. *Let $E \to D$ be a complex line bundle. There is a one-to-one correspondence between:*

- *Zero-residue quadratic elliptic symplectic structures $\pi_2 \in \mathfrak{X}^2(E^*)_{2-\mathrm{pol}}$.*
- *Poisson structures $\pi_D \in \mathfrak{X}^2(D)$ together with flat contravariant connections ∇^{π_D} satisfying:*
  - *π_D has corank 2.*
  - *The transverse Higgs field of ∇^{π_D} (as in Definition 4.1.1) defines a frame for the normal bundle to the symplectic foliation.*

– *The symplectic foliation can be described as the intersection of the kernels of two closed one-forms.*
– $2\pi q(c_1(E)) = \mathcal{C}(\mathcal{F}, \omega_{\mathcal{F}})$ *(as in Section 4.1.2).*[7]

This can be seen as the purely Poisson geometric version of the same result for stable generalized complex structures in [16]. Proving such a result is easier by passing to the elliptic symplectic form ω_2 dual to π_2. So we will first give the description of this form and after that prove the above lemma.

Elliptic symplectic form To describe the elliptic symplectic forms associated to the Poisson structures in Proposition 5.2.27 and Proposition 5.2.25 we need the following construction, which is completely analogous to Lemma 5.2.16:

Lemma 5.2.30. *Let $\mathcal{N} \to D$ be a complex line bundle, with complex linear connection ∇ with curvature $C_1 \otimes \mathrm{id} + C_2 \otimes J \in \Omega^2(M; \mathrm{End}(\mathcal{N}))$. The connection-one form on the associated $\mathbb{C}^*$-bundle $\mathcal{N}^\times$ extends to a complex log-form $\zeta \in \Omega^1(\mathcal{A}_D)$.*

Recall from Section 1.3.3 that taking the imaginary part of a complex log form gives an elliptic form, and thus we can define

$$\rho := \mathrm{Re}(\zeta), \quad \Theta := \mathrm{Im}(\zeta) \in \Omega^2(\mathcal{A}_{|D|}).$$

And these satisfy $d\rho = C_1$ and $d\Theta = C_2$.

If the connection is moreover compatible with a metric with corresponding distance function r^2, then the corresponding ρ coincides with $d\log r$, and Θ is the connection-one form of the principal S^1-connection.

Non-zero elliptic residue The elliptic symplectic form dual to a non-zero residue elliptic symplectic form can be described as follows:

Lemma 5.2.31. *Let $\tau = \lambda^{-1}$ and $\omega_D = \pi_D^{-1}$, where λ and π_D are as in Theorem 5.2.25. When the same flat connection as in Theorem 5.2.25 is used to construct ρ and Θ then the elliptic symplectic form dual to π_2 is given by*

$$\omega_2 = \tau\rho \wedge \Theta + p^*\omega_D \in \Omega^2(E^*, \mathcal{A}_{|D|}^{E^*}), \tag{5.2.6}$$

where $\mathcal{A}_{|D|}^{E^*}$ is the elliptic tangent bundle induced by the complex structure on E.

Proof. The proof is completely similar as the one in Lemma 5.2.17. We will again compute $\pi_2^\sharp(\omega_2^\flat(W))$, for W either $r\partial_r, \partial_\theta$ or $\mathrm{hor}^{\nabla^*}(V)$, for $V \in \mathfrak{X}^1(D)$:

$$\pi_2^\sharp(\omega_2^\flat(r\partial_r)) = \pi_2^\sharp(\tau\Theta) = \lambda\tau r\partial_r$$
$$\pi_2^\sharp(\omega_2^\flat(\partial_\theta)) = \pi_2^\sharp(-\tau\rho) = \lambda\tau\partial_\theta$$
$$\pi_2^\sharp(\omega_2^\flat(\mathrm{hor}^{\nabla^*}(V)) = \pi_2^\sharp(p^*\omega_D^\flat(V)) = \mathrm{hor}^{\nabla^*}(V).$$ ☐

[7] A priori, $2\pi q(c_1(E)) \in H^2(D; \mathbb{C})$. But if we identify the coefficient in $\mathbb{C}$ with a coefficient in ν^* via $a + bi \mapsto a\theta_2 + b\theta_1$, then we may view $c_1(E) \in H^2(D; \nu^*)$ and apply q to it.

Zero elliptic residue The elliptic symplectic form dual to a zero residue elliptic symplectic form can be described as follows:

Lemma 5.2.32 ([16]). *Let $(X, Y, \pi_D) \in \mathfrak{X}^1(D) \times \mathfrak{X}^1(D) \times \mathfrak{X}^2(D)$ be the almost 2-Poisson structure as in Proposition 5.2.27, and let $(\alpha, \beta, \omega) \in \Omega^1(D) \times \Omega^1(D) \times \Omega^2(D)$ be the dual almost 2-cosymplectic structure (see Definition 4.1.11). Then the elliptic symplectic form dual to π_2 is given by*

$$\omega_2 = \rho \wedge p^*\theta_1 + \Theta \wedge p^*\theta_2 + p^*\beta \in \Omega^2(E^*, \mathcal{A}^{E^*}_{|D|}). \tag{5.2.7}$$

Proof. The proof is similar to the one in Lemma 5.2.31. We will again compute $\pi_2^\sharp(\omega_2^\flat(W))$ for W either $r\partial_r, \partial_\theta, \mathrm{hor}^{\nabla^*}(X), \mathrm{hor}^{\nabla^*}(Y)$ or $\mathrm{hor}^{\nabla^*}(V)$, with V tangent to the foliation on D:

$$\begin{aligned}
\pi_2^\sharp(\omega_2^\flat(\mathrm{hor}^{\nabla^*}(X))) &= \pi_2^\sharp(-\rho) = \mathrm{hor}^{\nabla^*}(X)\\
\pi_2^\sharp(\omega_2^\flat(\mathrm{hor}^{\nabla^*}(Y))) &= \pi_2^\sharp(-\Theta) = \mathrm{hor}^{\nabla^*}(Y)\\
\pi_2^\sharp(\omega_2^\flat(\mathrm{hor}^{\nabla^*}(V))) &= \pi_2^\sharp(p^*\beta^\flat(V)) = \mathrm{hor}^{\nabla^*}(V)\\
\pi_2^\sharp(\omega_2^\flat(r\partial_r)) &= \pi_2^\sharp(p^*\theta_1) = r\partial_r\\
\pi_2^\sharp(\omega_2^\flat(\partial_\theta)) &= \pi_2^\sharp(p^*\theta_2) = \partial_\theta.
\end{aligned}$$

□

Relating local data One can show directly that if (X, Y, π_D) is an almost 2-Poisson structure satisfying the conditions of Proposition 5.2.27, then the dual almost 2-cosymplectic structure $(\theta_1, \theta_2, \beta)$ satisfies the following conditions:

$$d\theta_1 = d\theta_2 = 0 \text{ and } d\beta = -K_1^* \wedge \theta_1 - K_2^* \wedge \theta_2 = -K_1 \wedge \theta_1 + K_2 \wedge \theta_2. \tag{5.2.8}$$

Proof. We assume we are given $(\theta_1, \theta_2, \beta)$ satisfying (5.2.8) and will show that the dual almost 2-Poisson structure (X, Y, π_D) satisfies the desired conditions. We have

$$\begin{aligned}
d\theta_1(X, Y) &= X(\theta_1(Y)) - Y(\theta_1(X)) - \theta_1([X, Y]) = -\theta_1([X, Y])\\
d\theta_2(X, Y) &= -\theta_2([X, Y]),
\end{aligned}$$

and therefore $[X, Y]$ is tangent to the foliation. Now for Z tangent to $\mathcal{F}$ we consider:

$$d\beta(X, Y, Z) = -\beta([X, Y], Z),$$

but at the same time

$$(-\theta_1 \wedge K_1 + \theta_2 \wedge K_2)(X, Y, Z) = -K_1(Y, Z) - K_2(X, Z).$$

Because this holds for all $Z \in T\mathcal{F}$ we conclude that

$$\beta^\flat([X, Y]) = \iota_Y K_1 + \iota_X K_2,$$

and as $[X, Y]$ is tangent to $\mathcal{F}$ by applying $\pi_D^\sharp$ to this expression we obtain

$$[X, Y] = \pi_D^\sharp(\iota_Y K_1) + \pi_D^\sharp(\iota_X K_2).$$

Let V, W be in $\mathcal{F}$ and $\alpha, \gamma \in T^*\mathcal{F}$ such that $\pi^\sharp(\alpha) = V$ and $\pi^\sharp(\gamma) = W$ then

$$\begin{aligned}[\pi_D, X](\alpha, \gamma) &= -X(\pi_D(\alpha, \gamma)) = -X(\beta(V, W)) \\ &= -d\beta(X, V, W) = K_1(V, W) = \pi^\sharp(K_1)(\alpha, \gamma).\end{aligned}$$

The proof for $[\pi_D, Y]$ is similar, and $[\pi_D, \pi_D] = 0$ already follows from the fact that $(\theta_1, \theta_2, \beta)$ defines a symplectic foliation. □

Remark 5.2.33. The one-forms θ_1, θ_2 as above arise canonically from a given elliptic symplectic form $\omega \in \Omega^2(\mathcal{A}_{|D|})$. Indeed they are the residues of ω: $\theta_1 = \mathrm{Res}_r(\omega)$ and $\theta_2 = \mathrm{Res}_\theta(\omega)$. ◇

We deduce the following intrinsic properties form the almost 2-cosymplectic structure $(\theta_1, \theta_2, \beta)$:

Lemma 5.2.34. *Let $\omega_2 \in \Omega^2(\mathcal{A}_{|D|})$ be an elliptic symplectic form. Then:*

- *The symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$ on D is defined by two closed one-forms $\theta_1, \theta_2 \in \Omega^2(D)$.*
- *The complex line bundle $E \to D$ satisfies:*

$$2\pi q(c_1(E)) = \mathcal{C}(\mathcal{F}, \omega_{\mathcal{F}}).$$

Proof. We have that $d\beta = -\theta_1 \wedge K_1^* - \theta_2 \wedge K_2^*$, consequently $C(\mathcal{F}, \omega_{\mathcal{F}}) = [-\theta_1 \wedge K_1 + \theta_2 \wedge K_2] \in H^3_{\mathcal{F}}(D)$. Moreover, $c_1(E) = \frac{1}{2\pi i}[K_1 + iK_2]$, which we view as element of $H^2(D; \nu^*)$ via $\frac{1}{2\pi}(-K_1 \wedge \theta_1 + K_2 \wedge \theta_2)$. Under the isomorphism of Lemma 4.1.31 we conclude that $q(c_1(E)) = \frac{1}{2\pi}(-K_1 \wedge \theta_2 + K_2 \wedge \theta_1)$, and thus $C(\mathcal{F}, \omega_{\mathcal{F}}) = 2\pi q(c_1(E))$, which finishes the proof. □

Here we view $c_1(E) \in H^2(D, \nu^*)$ using the isomorphism $\nu^* \simeq \mathbb{R}^2$ induced by θ_1 and θ_2.

Note that, by the discussing in Section 4.1.2 this in particular implies that $c_1(E)|_{\mathcal{F}} = \mathrm{var}_{\omega_{\mathcal{F}}}$. However the condition given above is slightly more.

Although the symplectic foliation together with the one forms is completely canonical, it does not remember enough to reconstruct the associated quadratic elliptic symplectic form; we also need the contravariant connection. However having only the symplectic foliation does ensure that an extension of the foliated form, β, satisfying (5.2.8) can be found:

Lemma 5.2.35. *A manifold admits a symplectic foliation as in Lemma 5.2.34 if and only it admits an almost 2-cosymplectic structure $(\theta_1, \theta_2, \beta)$ satisfying* (5.2.8).

Proof. Pick any extension $\beta \in \Omega^2(D)$ of $\omega_{\mathcal{F}}$ and let ∇ be any connection. Because $2\pi q(c_1(E)) = \mathcal{C}(\mathcal{F}, \omega_{\mathcal{F}})$, we thus have that

$$d\beta = d\eta - K_1 \wedge \theta_1 + K_2 \wedge \theta_2,$$

with $\eta \in \Omega^2_{\mathcal{F}}(D)$. Let $\eta_1, \eta_2 \in \Omega^1(D)$ be such that $\eta = \eta_1 \wedge \theta_1 + \eta_2 \wedge \theta_2$, then

$$d\beta = -(K_1 - d\eta_1) \wedge \theta_1 - (K_2 + d\eta_2) \wedge \theta_2.$$

We conclude that $\nabla - \eta_1 \otimes \mathrm{id} + \eta_2 \otimes J$ is the desired connection. □

The discussion in this section allows us to describe the conditions required on the contravariant connection to induce an elliptic symplectic form, and thus prove Lemma 5.2.29.

Proof of Lemma 5.2.29. We let ∇ be any connection on E, and use it to define the triple (X, Y, π_D) as in Proposition 5.2.21. For this triple we have the desired expressions for $[\pi_D, \pi_D]$, $[\pi_D, X]$ and $[\pi_D, Y]$ already follow from the flatness of the contravariant connection. We however have no a priori control over $[X, Y]$. Invert (X, Y, π_D) to an almost-2-cosymplectic structure $(\theta_1, \theta_2, \beta)$. By assumption the one-forms θ_1, θ_2 are closed. As β is an extension of the foliated symplectic form, and $2\pi q(c_1(E)) = -\mathcal{C}(\mathcal{F}, \omega_{\mathcal{F}})$ we can argue exactly as in the proof of Lemma 5.2.35 that there exists a connection $\widetilde{\nabla}$ for which then $d\omega = -\widetilde{K}_1 \wedge \theta_1 - \widetilde{K}_2 \wedge \theta_2$. For this connection it thus follows that (X, Y, π_D) satisfy the conditions in Proposition 5.2.27 and consequently they define a Poisson structure via Equation (5.2.5). □

Remark 5.2.36. One might wish to construct examples of quadratic elliptic symplectic structures using the description in Lemma 5.2.29. However, finding such a setting in practice might prove to be difficult. Lemma 5.2.34 tells us that we only need to find a manifold D endowed with a symplectic foliation defined by two closed one forms and a vector bundle over D with the correct curvature. ◇

Remark 5.2.37 (Terminology)**.** We comment on the use of the adverbs constant, linear and quadratic for the above normal forms. From our point of view, the normal forms for log and elliptic symplectic structures should be called fibrewise quadratic, because the Poisson bracket of two linear functions is a quadratic function.

In [45, 16] these normal forms are usually referred to as a linearisation. This also sensible, because the coefficient functions in the normal forms (5.2.2) and (5.2.5) are fibrewise affine. The exception, (5.2.4), which has quadratic coefficient functions and should thus really be called quadratic.

However, in calling these normal forms a linearisation one could remark that the normal form does not use all data of the first jet of the original Poisson structure. Indeed, if we see the decomposition of a jet of a Poisson structure as in Lemma 5.1.51 we see that we only used the contravariant connection, and not the extra bivector with values in the normal bundle. However, as the local model had no need of this piece it is irrelevant for our discussion.

Finally, from the Lie algebroid point of view one could add a third competing terminology, namely constant. Indeed, viewed as Lie algebroid forms, the formulae (5.2.2), (5.2.6) and (5.2.7) have fibrewise constant coefficients. ◇

5.2.3 Existence of elliptic symplectic forms with non-zero elliptic residue

In this section we will show how the explicit normal forms for elliptic and log symplectic structures can be used to construct examples of such structures.

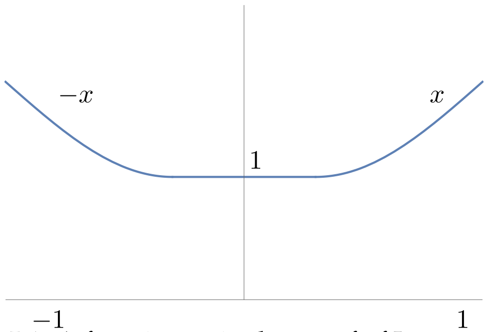


Figure 5.1: A function as in the proof of Lemma 5.2.38.

Lemma 5.2.38. *Let (M^{2n},ω) be a symplectic manifold and let $S^{2n-2}\subset M$ be a compact symplectic submanifold with flat normal bundle. Then there exists an elliptic symplectic structure $\widetilde{\omega}$ with non-zero elliptic residue on M with degeneracy locus precisely S.*

Proof. Because $S\subset M$ is an orientable submanifold of an orientable manifold, its normal bundle ν_S gets an induced orientation. Pick a metric g on ν_S and consider the induced complex structure. Let ∇ be a connection compatible with this metric, and let ρ,Θ denote the corresponding elliptic forms as in Lemma 5.2.30. Let $p:\mathcal{U}\to S$ be a tubular neighbourhood of S, and let $r^2:\mathcal{U}\to\mathbb{R}$ denote the distance function induced by g. On $\mathcal{U}$ we consider the following symplectic form:

$$\omega_2=r^2\rho\wedge\Theta+p^*\omega_S.$$

Note that this is indeed a well-defined symplectic form, because with respect to local fibre coordinates (x,y), in which $r^2=x^2+y^2$, we have $\rho\wedge\Theta=r^{-2}dx\wedge dy$. Because ω is cohomologous to ω_2 on $\mathcal{U}$, and $\iota_S^*\omega_2=\iota_S^*\omega$ we have that they are symplectomorphic on a, possibly smaller, open neighbourhood $\mathcal{U}'$. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function which is 1 on a neighbourhood of 0, and $|x|$ near -1 and 1. Then

$$\widetilde{\omega}:=f(|\cdot|^2)\rho\wedge\Theta+p^*\omega_D,$$

defines an elliptic symplectic form on $\mathcal{U}'$. Because $\widetilde{\omega}$ coincides with ω_2 near the boundary of $\mathcal{U}'$ we see that $\widetilde{\omega}$ extends to a globally defined form. □

The assumptions in this statement are (nearly) identical to a similar statement of constructing log symplectic structures, as proven in [14]. Moreover, it is also very similar in nature to another construction of log symplectic structures, which is folklore in the community:

Lemma 5.2.39. *Let (M^{2n},ω) be an orientable symplectic manifold and let $Z\subset M$ be a compact cosymplectic submanifold. Then there exists a log symplectic structure $\widetilde{\omega}$ on M with degeneracy locus two copies of Z.*

Proof. By the cosymplectic neighbourhood theorem we have that there exists a tubular neighbourhood $\mathcal{U}\simeq Z\times\mathbb{R}$ of Z on which

$$\omega=dx\wedge p^*\theta+p^*\beta,$$

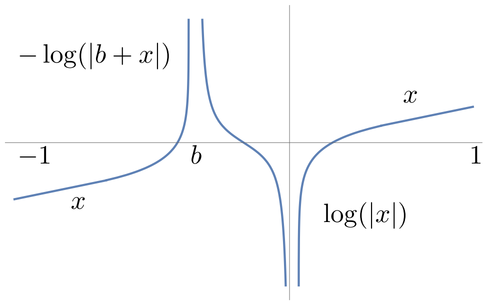


Figure 5.2: A function as in the proof of Lemma 5.2.39.

where (θ, β) is the cosymplectic structure on Z. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying:

- f has nowhere vanishing derivative at all points,
- f coincides with $\log |x|$ for 0,
- There are a, c with $-1 < a < b < c < 0$, such that f coincides with $-\log(|-b-x|)$ on $[a, c]$,
- f is equal to x near -1 and 1.

Then

$$\widetilde{\omega} = df \wedge p^*\theta + p^*\beta$$

is a log symplectic structure on $\mathcal{U}$. Because $\widetilde{\omega}$ coincides with ω near the boundary of $\mathcal{U}'$ it can be extended to a global log symplectic form on M. □

To close the circle, we compare with the construction in [14]:

Lemma 5.2.40. *Let ω be an elliptic symplectic structure on M with non-zero elliptic residue and degeneracy locus D. Then there exists a log symplectic structure on M with degeneracy locus $S^1 \times D$.*

Proof. Pick a metric g on ν_D and consider the induced complex structure. Let ∇ be a connection compatible with this metric, and let ρ, Θ denote the corresponding elliptic forms as in Lemma 5.2.30. By Lemma 5.2.31 we have that there exists a tubular neighbourhood $\mathcal{U}$ of D on which

$$\omega = \tau\rho \wedge \Theta + p^*\omega_D.$$

Let $0 < \varepsilon < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a function which coincides with $x^2/x^2 - \varepsilon$ near the origin, vanishes only at the origin and is equal to 1 near -1 and 1.

Then

$$\widetilde{\omega} = f(|\cdot|)\tau\rho \wedge \Theta + p^*\omega_D,$$

defines a log symplectic structure on $\mathcal{U}$ with degeneracy locus $\{|\cdot|^2 = \varepsilon\} \simeq D \times S^1$. Because $\widetilde{\omega}$ and ω coincide near the boundary of $\mathcal{U}$ we have that $\widetilde{\omega}$ can be extended to a globally defined logarithmic form. □

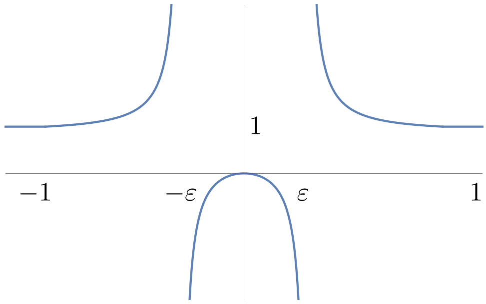


Figure 5.3: A function as in the proof of Lemma 5.2.40.

The proofs of Lemma 5.2.38 and Lemma 5.2.39 are similar, but there is one clear difference. For the log case we needed to introduce a disconnected degeneracy locus, whereas we could suffice with a connected degeneracy locus in the elliptic case. This gives a hint that reversing the process (i.e. turning a log symplectic structure into a symplectic one) might not be so easy. And indeed, as proven in [14, 58] there are examples of manifolds which admit log symplectic structures but not symplectic structures.

The problem of the disconnected degeneracy locus did not appear in the elliptic case, and in fact we can reverse Lemma 5.2.38 in general:

Lemma 5.2.41. *Let ω be an elliptic symplectic structure on M with non-zero elliptic residue and degeneracy locus D. Then there exists a symplectic structure on M which has D as symplectic submanifold.*

Proof. Pick a metric g on ν_D and consider the induced complex structure. Let ∇ be a connection compatible with this metric, and let ρ, Θ denote the corresponding elliptic forms as in Lemma 5.2.30. By Lemma 5.2.31 we have that there exists a tubular neighbourhood $\mathcal{U}$ of D on which

$$\omega = \tau\rho \wedge \Theta + p^*\omega_D.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function which is positive away from the origin, x^2 near the origin, and 1 near -1 and 1. Then

$$\widetilde{\omega} = f(|\cdot|)\tau\rho \wedge \Theta + p^*\omega_D$$

defines a symplectic structure on $\mathcal{U}$. Because ω and $\widetilde{\omega}$ coincide near the boundary of $\mathcal{U}$ we have that $\widetilde{\omega}$ can be extended to a globally defined symplectic form. □

Because elliptic symplectic structures with zero elliptic residue corresponds to generalized complex structures, we have studied them in Chapter 1 and 2 of this thesis. We have not paid much attention to the ones with non-zero elliptic residue. The above lemma states that, if one's goal is to construct geometric structures on manifolds which do not admit symplectic structures (as we have done in Chapter 1), one can disregard elliptic symplectic structures with non-zero elliptic residue.

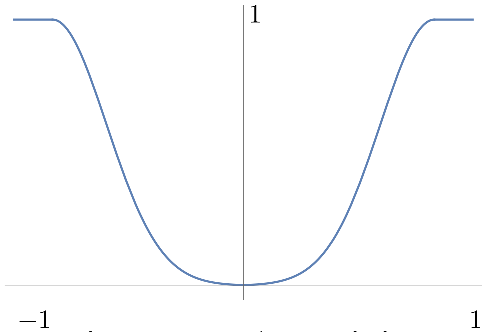


Figure 5.4: A function as in the proof of Lemma 5.2.41.

Remark 5.2.42. The results discussed in this section can also be used to understand the geometry of symplectic manifolds themselves. The results in [14, 58] provide obstructions on the existence of log-symplectic structures. If these results imply that a symplectic manifold (M, ω) can not admit a log-symplectic structure then we conclude that M cannot have a cosymplectic submanifold, or a codimension-two symplectic submanifold with trivial normal bundle. For if this was the case, then Lemma 5.2.39 or Lemma 5.2.40 would imply that M admits a log-symplectic structure. In particular, this implies that $\mathbb{C}P^2$ cannot have such submanifolds. ♢

Samenvatting

In dit proefschrift worden twee hoofdonderwerpen bestudeert: *gegeneraliseerde complexe meetkunde* en *Poisson meetkunde*. In deze samenvatting zullen we deze concepten aan niet-ingewijden uitleggen.

Differentiaalmeetkunde

De *differentiaalmeetkunde* is het vakgebied dat ruimtes genaamd *gladde variëteiten* bestudeerd. Een gladde variëteit van dimensie n is een ruimte met de eigenschap dat hij er lokaal uitziet als de Euclidische ruimte $\mathbb{R}^n$.

Bijvoorbeeld het oppervalk van de aarde, een sfeer S^2 genaamd, is een gladde variëteit van dimensie twee aangezien deze lokaal met behulp van een twee dimensionale kaart beschreven kan worden. Maar ook de rand van een donut, een torus T^2 genaamd, of de rand van een krakeling zijn voorbeelden van twee-dimensionale gladde variëteiten. Hoger dimensionale gladde variëteiten zijn vaak lastig om je voor te stellen, maar dat weerhoudt ons er niet van om ze te bestuderen.

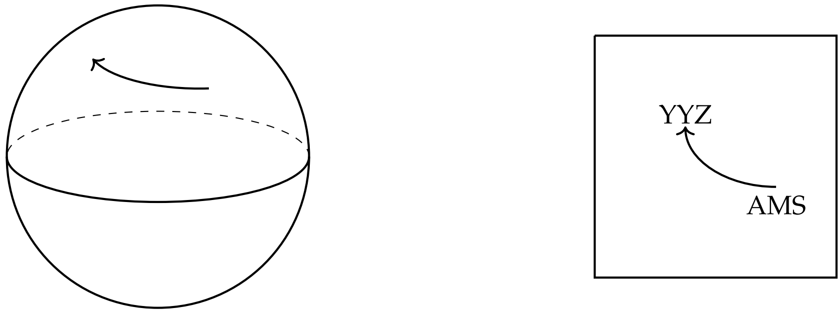
Gladde variëteiten bieden de omgeving om bepaalde fysische systemen te modelleren. Maar om dit te doen is het vaak handig om extra data, genaamd *meetkundige structuren*, te gebruiken. Een goed voorbeeld is een begrip van afstand gegeven door een *Riemanniaanse metriek*. In de Euclidische ruimte is de standaardmetriek gedefinieerd door het feit dat de kortste paden gegeven worden door rechte lijnen. Dit is echter niet het enige voorbeeld.

Als we een stuk wereldkaart nemen en de snelste route voor een vliegtuig tussen Amsterdam en New York willen aangeven dan is dit een gekromde lijn in plaats van een rechte; dit correspondeert met een andere metriek op $\mathbb{R}^2$, gegeven door de kromming van de aarde.

Er zijn heel veel verschillende soorten meetkundige structuren, in dit proefschrift bestuderen we er twee: *Poisson structuren* en *gegeneraliseerde complexe structuren.*

Poissonmeetkunde

Poissonmeetkunde geeft de wiskundige basis voor Hamiltoniaanse mechanica, een formalisme binnen de natuurkunde dat gebruikt wordt om bewegingen te



Figuur 5.5: Lokaal ziet de aarde er uit als een plat vlak. De kortste route voor een vliegtuig tussen Amsterdam en Toronto ziet er op een kaart uit als een cirkelboog.

modelleren. Een *Poissonhaak* op een gladde varieteit M is een afbeelding

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

die twee functies f, g op M pakt en een derde functie, $\{f, g\}$, teruggeeft en tevens aan de volgende eigenschappen voldoet:

- $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$,
- $\{f, g\} = -\{g, f\}$,
- $\{fg, h\} = f\{g, h\} + \{f, h\}g$,
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,

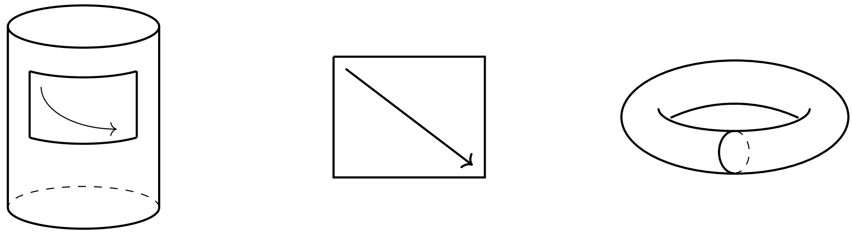
voor alle functies $f, g, h \in C^\infty(M)$ en getallen $a, b \in \mathbb{R}$. De Poissonhaak beschrijft de dynamica op de volgende manier: Als de functie f een fysische grootheid is, en de functie H de energie van het systeem dan is $\{f, H\}$ de verandering van de grootheid f door de tijd heen.

De Poissonhaak werd reeds door wiskundigen en natuurkundigen in het begin van de negentiende eeuw gebruikt om de beweging van hemellichamen te beschrijven. Sinds de jaren 70 heeft de Poissonmeetkunde zich als levendig vakgebied met vele interacties met andere takken van de wiskunde en natuurkunde ontwikkeld.

Normaalvormen

Gladde variëteiten zien er lokaal uit als de Euclidische ruimte $\mathbb{R}^n$. Wat $\mathbb{R}^n$ speciaal maakt is dat het een *vectorruimte is*: Je kan punten in $\mathbb{R}^n$ bij elkaar optellen en vermenigvuldigen met getallen $\lambda \in \mathbb{R}$.

Laat M een variëteit en $x \in M$ een punt in M zijn. Als we M zien als deel van $\mathbb{R}^k$, dan kunnen we alle vectoren rakend aan x beschouwen (zie figuur 3?). Deze vormen een vectorruimte T_xM genaamd de *raakruimte* aan x. De definitie



Figuur 5.6: De cylinder met de kortste afstand tussen twee punten weergegeven. Lokaal ziet de cylinder er uit als een plat vlak, en wordt de kortste afstand door een rechte lijn gegeven. Een omgeving van een cirkel in de torus ziet er uit als een cylinder.

van variëteit verteld ons dus dat voor elk punt $x \in M$ er een omgeving U van x bestaat zodat U er uit ziet als T_xM.

Een van de grote vragen in de studie van meetkundige structuren is of dit ook geldt voor gladde variëteiten tezamen met meetkundige structuren. Dat wil zeggen, gegeven een meetkundige structuur op M en een punt $x \in M$ bestaat er een modelstructuur op T_xM en een omgeving U zodat U tezamen met de meetkundige structuur lijkt op T_xM tezamen met het model.

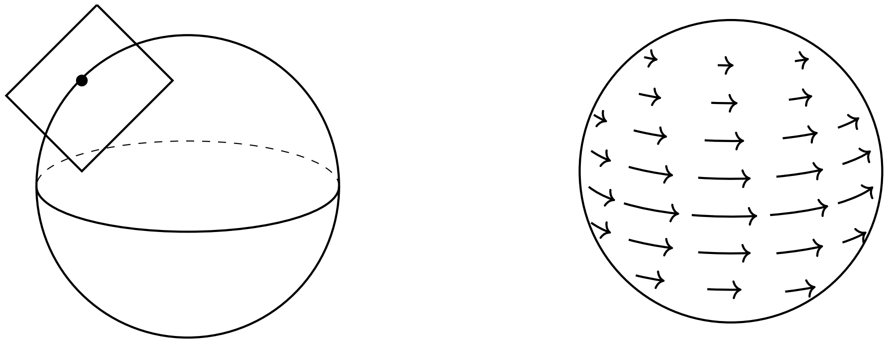
Als voorbeeld beschouwen we wederom Riemanniaanse metrieken, en in het bijzonder de cylinder $C = S^1 \times \mathbb{R}$. Deze heeft een metriek waarbij het kortste pad tussen twee punten wordt gegeven door tegelijkertijd langs de cirkel te bewegen en omhoog te lopen.

Als we een stukje van de cylinder identificeren met het vlak, komt dit overeen met het pad gegeven door een rechte lijn. Dus geldt inderdaad dat voor ieder punt er een omgeving is welke lijkt op $\mathbb{R}^2$ met de standaardmetriek. Daarom zeggen we dat $\mathbb{R}^2$ een lokaal model geeft voor C. Daarentegen zijn er ook ruimtes met metrieken te bedenken die niet lokaal op de standaard metriek lijken, een voorbeeld is S^2 met de metriek die we eerder hebben gezien.

Voor metrieken is het precies bekend welke lokaal op de standaardmetriek in $\mathbb{R}^n$ lijken (deze worden *plat* genoemd). Maar voor andere meetkundige structuren is dit nog niet het geval, zoals bijvoorbeeld voor Poisson structuren.

Naast kijken rond punten, is het ook erg interessant om rond grotere stukken van een variëteiten te kijken. Een *deelvariëteit* $N \subset M$ is een deelverzameling van M die zelf ook weer een variëteit is. Als $x \in N$ een punt in de deelvarieteit is, dan kunnen we alle vectoren in de richting van N beschouwen, T_xN, deze vormen een deelvectorruimte van alle vectoren T_xM. De *normaalrichting* van N op x is de vectorruimte die bestaat uit alle vectoren die "loodrecht" op N staan. In andere worden is dit de vectorruimte $\mathcal{N}_x$ met de eigenschap dat elke vector $v \in T_xM$ te schrijven valt als een som $v_1 + v_2$ waarbij $v_1 \in T_xN$ in de richting van N is en $v_2 \in \mathcal{N}_x$ in de normaalrichting staat.

Als je de vectoruimtes $\mathcal{N}_x$ voor alle $x \in N$ aan elkaar plakt krijg je de *normaalbundel* van N in M. Dit is een specifiek geval van een *vectorbundel*: Een variëteit



Figuur 5.7: De raakruimte aan een sfeer. Het vectorveld weergegeven op de sfeer stelt de rotatiesnelheid aan het aardoppervlak voor.

E, met een afbeelding $\pi : E \to N$ naar een andere variëteit N met de eigenschap dat de ruimtes E_x (*vezels* genaamd) bestaande uit alle punten in E die naar een vast punt $x \in N$ worden gestuurd een vectorruimte zijn.

Neem bijvoorbeeld de torus T^2, als we inzoomen rond een boogcirkel dan ziet het eruit als een cylinder $S^1 \times \mathbb{R}$. Dit is een speciaal geval van een algemeen resultaat genaamd de *tubulaire omgeving stelling*: Elke deelvariteit $N \subset M$ heeft een omgeving die lijkt op de normaalbundel $\mathcal{N}$ van N in M.

Idem als hiervoor, is een grote vraag in de studie van meetkundige structuren of er gegeven een deelvariëteit $N \subset M$ een modelstructuur op de normaalbundel $\mathcal{N}$ bestaat waarop de originele structuur lijkt in een omgeving van de deelvariëteit.

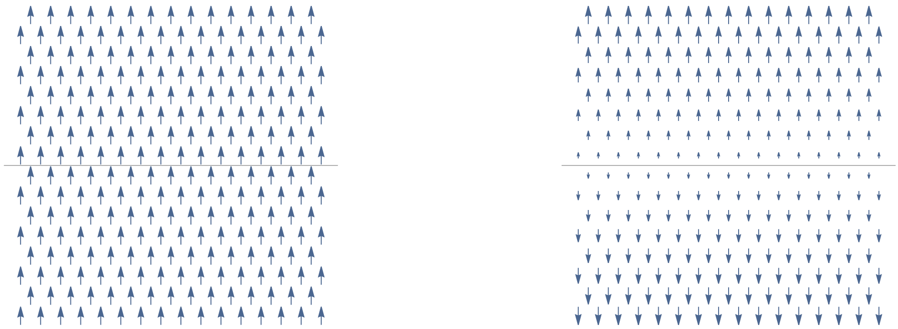
Homogene vectorvelden

Als we alle raakruimtes van een variëteit M samenbinden krijgen we een vectorbundel genaamd de *raakbundel* TM. Een *vectorveld* op M, is een afbeelding $X : M \to TM$ die aan elk punt $p \in M$ een vector X_p in T_pM associeert.

Een *multi-vectorveld* op M, is (ongeveer) een object wat aan elk punt $p \in M$ een collectie $(Q_{1,p}, \ldots, Q_{n,p})$ van vectorvelden associeert.

Als onze variëteit zelf een vectorbundel, E, is dan kunnen we vectorvelden bestuderen die goed samenspelen met de vectorruimtestructuur op de vezels van E. We zeggen dat een vectorveld X op E *homogeen van graad* d is als voor alle punten $v \in E$, de vector $X_{\lambda \cdot v}$ direct gerelateerd is aan $\lambda^d X_v$. We illustreren dit hieronder aan de hand van een voorbeeld (Figuur 4).

Homogene (multi-)vectorvelden zijn vaak de bouwstenen om normaalvormen, zoals in de vorige sectie beschreven, te bestuderen. Daarom is het heel interessant om hun eigenschappen te bestuderen. Dat is wat we doen in Hoofdstukken 4 en 5 van dit proefschrift. In Hoofdstuk 4 bestuderen we de ruimte van homogene multi-vectorvelden, en geven een concrete beschrijving daarvan. In Hoofdstuk 5 gebruiken we de opgedane kennis om normaalvormen in Poisson



Figuur 5.8: We beschouwen hier de vectorbundel gegeven door een interval aan rechte lijnen. Het linker vectorveld is op elke hoogte even lang, en dus homogeen van graad $d = 0$. Het rechter vectorveld neemt linear in lengte toe en is dus homogeen van graad $d = 1$.

meetkunde te bestuderen.

Gegeneraliseerde complexe structuren

Er zijn ontzettend veel verschillende meetkundige structuren, elk met hun eigen theorie. Wat men vaak in de wiskunde doet is het zoeken naar overeenkomsten en kaders om verschillende structuren tegelijk te kunnen bestuderen. Ook van dit soort kaders zijn er weer veel variaties. In Hoofdstuk 1, 2 en 3 van dit proefschrift bestuderen we *gegeneraliseerde complexe meetkunde*, dat een kader is om twee verschillende meetkundige structuren, *complexe en symplectische structuren* te verenigen.

Symplectische structuren zijn een heel speciaal soort Poisson structuren. Ze beschrijven fysische systemen waarbij de beweging door de ruimte alleen beperkt wordt door de beginenergie. Complexe structuren zijn meetkundige structuren op variëteiten M, die elk van de raakruimtes T_pM een complexe structuur geven. Dat wil zeggen, de vectorruimtes T_pM krijgen een afbeelding $J : T_pM \to T_pM$ ("vermenigvuldiging met i") die aan $J(J(v)) = -v$ voldoet.

Gegeneraliseerde complexe structuren bouwen hier op voort. Een gegeneraliseerde complexe structuur induceert een decompositie $T_xM = V_1 \oplus V_2$ in twee vectorruimtes, waarbij V_1 een symplectische structuur en V_2 complexe structuur krijgt. Het aantal complexe richtingen (oftewel de dimensie van V_2) noemen we het *type* van x. Echter hoeft het type voor verschillende punten in de variëteit niet hetzelfde te zijn.

Naast het bieden van een overkoepelend raamwerk voor symplectische en complexe meetkunde, heeft gegeneraliseerde complexe meetkunde ook andere interessante implicaties. Er zijn toepassingen binnen de theoretische fysica (in het bijzonder in de snaartheorie en spiegelsymmetrie), maar ook binnen andere takken van de wiskunde. Een gegeneraliseerde complexe structuur induceert bijvoorbeeld een Poisson structuur.

Een interessante vraag die veel bestudeerd wordt in de differentiaalmeetkunde is: Welke variëteiten hebben een bepaald type meetkundige structuur? Het beantwoorden van zo'n vraag levert vaak niet alleen inzichten op over de specifieke meetkundige structuur, maar ook over de vorm van de variëteiten in het algemeen.

Voor symplectische en complexe structuren is deze vraag goed bestudeerd (wat niet wil zeggen dat het helemaal is opgelost). Een natuurlijke vraag die oprijst na het geven van de definitie van gegeneraliseerde complexe structuren is of er variëteiten zijn die geen symplectische en complexe structuren hebben, maar wél gegeneraliseerde complexe structuren. Dit blijkt het geval te zijn, maar om zulke voorbeelden te construeren moet men wel wat werk verzetten.

In hoofdstuk 1 en 2 zetten we een theorie op om dit te doen. De voorbeelden die we construeren zijn onderdeel van een klasse van gegeneraliseerde complexe structuren, genaamd *stabiel*. Dit zijn gegeneraliseerde complexe structuren met de eigenschap dat het type bijna overal 0 is, en op een deelverzameling van dimensie twee lager type 2 heeft. De eerste constructies van stabiele gegeneraliseerde complexe structuren waren vrij ingewikkeld. De theorie die we in hoofdstuk 1 en 2 beschrijven is een stuk inzichtelijker.

Acknowledgements

Marius, when I first met you (ages ago) as my lecturer in topology I both respected you and thought you were a pretty strange fellow. After doing the PhD with you for four years that image has only been strengthened. Thank you for putting you faith in me (especially at times where I lacked it myself), for creating and nurturing the amazing atmosphere in the group, the endless discussions about non-math topics, the bike rides and the BBQ's. We can quite heavily disagree on certain things (like "appointments"), but I always found myself able to speak with you about such matters. Thanks for learning me many non-math things as well, like brewing elderberry juice.

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I would like to thank the people who read this thesis, Camille Laurent-Gengoux, Brent Pym, Silvia Sabatini, András Stipsicz and Marco Zambon, for their time and valuable comments. My apologies for the way the university handled the communication with you, this was really unacceptable.

The (extended) academic family I am part of is a truly wonderful bunch of people. Thank you all for making me feel at home at so many different conferences all over the world, and I hope to see you all again soon.

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Mum, I know you will not read this thesis but I'm happy you are proud of me nonetheless. Also, thank you for providing the picture on the cover which brightens up the thesis.

Curriculum vitae

Geraldo Arend Witte was born on August 13, 1994 in Haarlem, the Netherlands. After obtaining his high-school degree in 2012, he enrolled in the double bachelor's programme Mathematics and Physics at Utrecht University, which he completed in 2015. Consequently, he enrolled in the Mathematical Sciences master's programme at Utrecht University, which he completed in 2017. Thereafter, he started his PhD at the Utrecht Geometry Centre under supervision of Gil Cavalcanti and Marius Crainic.

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