Universiteit Utrecht

The Gauss-Bonnet Theorem for Surfaces

Bachelor thesis for Mathematics
Utrecht University

Lay de Jong

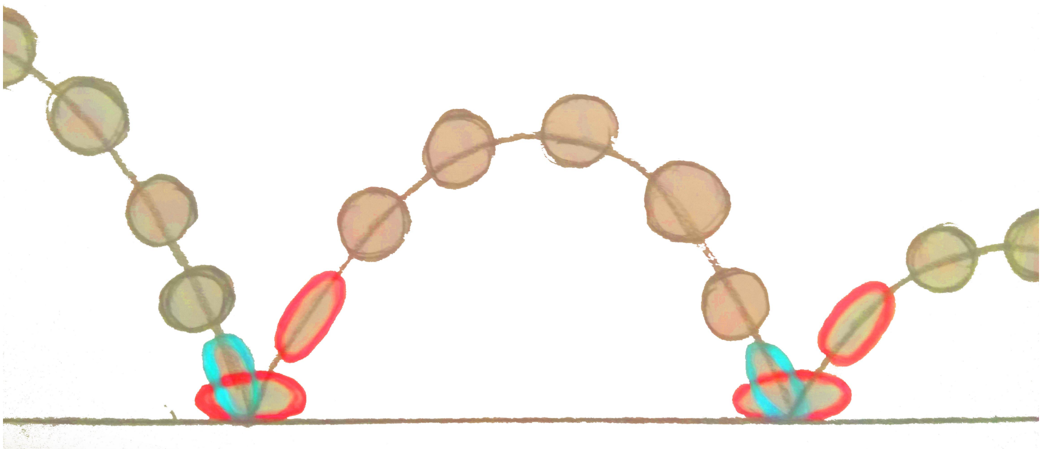
Supervisor: Dr. G.R. Cavalcanti

July 2018

Contents

1 Introduction

Imagine skipping on a skippyball. When you look at it whilst it is not being used, you will probably observe a normal sphere, if not: you might want to pump up the ball a bit further. If you proceed to sit on the ball, or bounce with it, you will notice it deforming. When mid-bounce, it will press down on the ground, flattening on both the top and bottom side, whilst spreading out a bit more on the side. And post-bounce it will again resume it's original sphere shape. All the while, hopefully, not bursting. If you bounce it harder and harder, it will flatten more and more upon impact. But in order to keep all the air in, it will have to stay closed, and so the edges of the ball become sharper and sharper curved. If you stop bouncing, and want to exact some kind of vengeance on the ball, you can try puncturing it. When doing this you will also notice that it becomes more and more concave at the point you are pressing on it, whilst extruding around this crater you are forming.



Seeing as how you were already intently observing the deformation of a skippyball, you probably are a mathematician, and you begin to wonder. 'How could the ball get so flat, but still close in on itself?', 'Is the flattened disc shape it assumes just as "round" as the sphere it was?', 'Are all inflatable objects equally "round"?'. Or maybe you did not, but now you do.

Luckily, you are not the first mathematician, and the "roundness" of shapes has been studied extensively. The deformation of the skippyball is closely linked to the subject of this thesis; the Gauss-Bonnet theorem. Which determines that indeed the total curvature of a surface, without boundary, is constant, even under deformations of it. The Gauss-Bonnet theorem states;

$$\int_{\Sigma} K(x)\sigma = 2\pi\chi(\Sigma) \tag{1}$$

Here Σ is our surface, $K(x)$ is our curvature at a point x, σ is a volume 2-form, and $\chi(\Sigma)$ is the Euler characteristic of our surface.

Later on we will go more in depth about what these exactly mean. But intuitively it states that the integral of the curvature over the entire surface equals a constant, which is determined by the topology of the surface.

The goal of this thesis is to give an understanding of the mathematics used to describe such things as surfaces and their curvature. And to give a proof of the Gauss-Bonnet theorem for surfaces.

We will start by introducing the general concept of vector bundles and connections on a manifold. From these definitions we will design the Euler class of a manifold. Then we look at the specific manifestation of vector bundles and connections needed for the Gauss-Bonnet theorem: the tangent bundle and the Levi-Civita connection. Finally we will apply this specific tangent bundle and connection to a 2-dimensional manifold; a surface. This will conclude in the proof of the Gauss-Bonnet theorem.

The reader should have a basic understanding of - differential - topology.

2 Vector Bundles and Connections

A key concept in defining the curvature on a manifold, is being able to work with and manipulate vector bundles and connections. In this section we will define what a vector bundle on a manifold is, and several operations we can apply to these mathematical objects. From there on we will define sections, which are maps which map to these introduced vector bundles. After we have defined sections on vector-bundles, we will explain connections; an operator on sections similar to the exterior derivative as known from calculus. We will explain how to expand these connections to the more generalized concept of covariant exterior derivatives. These three concepts together, vector bundles, sections and connections, will form our basic toolbox for the rest of the work done.

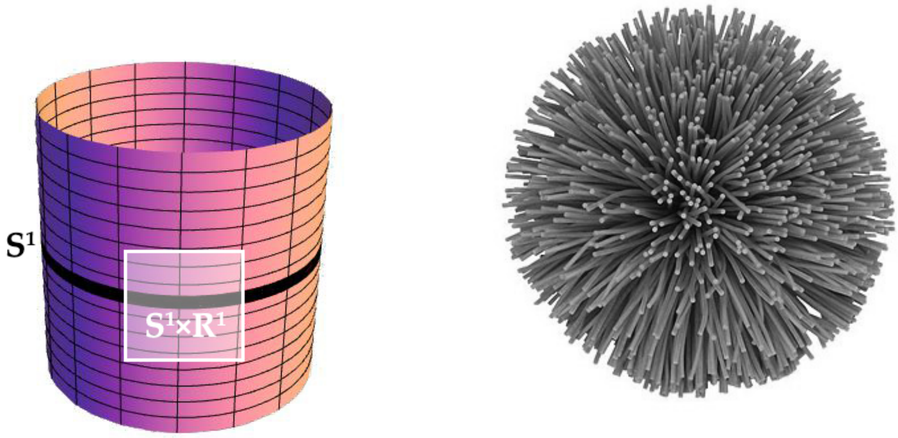
2.1 Vector Bundles

If we imagine a manifold M, then we can try and use this manifold M as a base for a different, seemingly larger manifold. Such as expanding the unit circle into a cylinder, or constructing a koosh-ball from a sphere. Vector bundles formalize this notion; they give a mathematical framework of how we can define a collection of vector spaces, which are smoothly indexed by our base space. More formally;

Definition 2.1. *a* ***vector bundle*** *of rank r over a manifold M is the triple (E, π, M) where E and M are manifolds, and $\pi : E \to M$ a continuous surjective map, for which the following holds;*

- *For each $x \in M$, $\pi^{-1}(x) := E_x$, the fibre of x in E, is a r-dimensional real valued vector space.*
- *For each $x \in M$ there is an open neighbourhood $U_x \subset M$, such that $\pi^{-1}(U_x)$ is diffeomorphic to $U_x \times \mathbb{R}^r$. And each fibre E_x is isomorphic to $\{x\} \times \mathbb{R}^r$.*

Strictly speaking the vector bundle is the entire triplet (E, π, M), for brevity we will speak of a vector bundle E, and will only denote the entire triplet if we want to emphasize the base manifold. The space E is the collection of fibres E_x, which if only defined individually at each point x, are disjoint spaces. However, we use the smooth structure of our manifold M, to form this disjoint collection of vector spaces itself, into a smooth manifold E. Without the information of the base space M on how to smoothly combine these vector spaces, it would not make sense to speak of a new manifold E.



(a) Cylinder $S^1 \times I$ (b) Koosh Ball

Figure 1: Examples of Vector Bundles

To visualize this definition one can observe figure 1. In figure 1a one can see a very trivial example of a vectorbundle; the cylinder. The cylinder as a manifold E can be seen as the vector bundle of rank 1 over the manifold $M = S^1$. In fact, one can always form the so-called trivial vector bundle of rank r by simply setting $E = M \times \mathbb{R}^r$. In figure 1b our base manifold M would be given by the sphere S^2. And to each point x one assigns a hair $\mathbb{R}$, it's fibre E_x. The map π would be given by the projection of each fibre onto the sphere. One can imagine this as constructing the Koosh ball as in 1b. But then with infinite many fibres; becoming the solid sphere from which the core has been removed. This would be the trivial vector bundle of rank 1 over S^2. A non-trivial vector bundle of rank 1 over S^1 is given by the Möbius strip. This vector bundle assigns to each x the vector space $\mathbb{R}$. But taken over the whole, there is a "twist" in the bundle, so globally it is not the same as the trivial bundle $S^1 \times \mathbb{R}$.

2.2 Sections

We are now able to define new manifolds E called vector bundles. Being manifolds over a base space M, we could be interested about maps sending a point $x \in M$ to a point in it's fibre E_x in a smooth way. A section is such a function; spanning the different fibres of the vector bundle smoothly like a graph. We will see that sections allow us to define a uniform bases for each and every individual vector space E_x. We start with the definition;

Definition 2.2. *a smooth* ***section*** $s : M \to E$ *of a vector bundle* E *is a smooth map such that* $(\pi \circ s) : Id_M$*.*
The space of all smooth sections of E *is denoted by* $\Gamma(E)$*.*

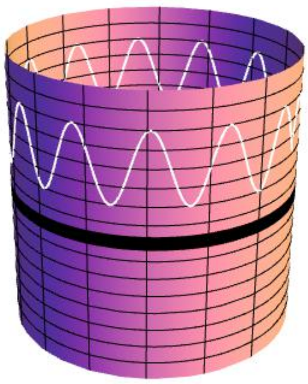


Figure 2: Section on the Cylinder

An example of a section would be to draw a sine-like function on the cylinder as before; the white line of figure 2 is a section of the vector bundle. The zero section, the section $s_0 : M \to E_x$; $x \mapsto 0_x$, maps a point $x \in M$ to the 0_x of it's fibre E_x, where we have notated 0_x to emphasise the fact that it is the zero-vector of E_x specifically. We can visualise it as the inclusion of the base space M into it's vector bundle, and would be the black line in figure 2.

We define section addition and function multiplication as being pointwise;

$$(s_1 + s_2)(x) := s_1(x) + s_2(x), \text{ for } s_1, s_2 \in \Gamma(E),$$
$$(fs)(x) := f(x)s(x), \text{ for } f \in C^\infty(M).$$

These operations make $\Gamma(E)$ into a left $C^\infty(M)$-module.

An equivalent, but insightful, interpretation for $\Gamma(E)$ arises when we compare it with k-forms on our manifold M. From the definition it follows that a k-form $\omega \in \Omega^k(M)$ is skew-symmetric; Taking k vector fields and producing a function. Thus we can see ω as an element of the wedge, as it skew-symmetric, of the dual, as it returns functions, of the tangent space, as it takes vector fields; so it follows $\omega \in \bigwedge^k T^*M$. The the entire space of k-forms can actually be seen as a section of this space. So $\Omega^k(M) \cong \Gamma(\bigwedge^k T^*M)$. Now if we pair ω, which returns a smooth function $f : M \to (R)$, with $e \in E$, we obtain a smooth section $f \otimes e : M \to E$, or alternatively; an E-valued k-form. The space of all E-valued k-forms we will denote with $\Omega^k(M; E)$. And so it follows that $\Omega^k(M; E) \cong \Gamma(\bigwedge^k T^*M \otimes E)$. And specifically that $\Gamma(E) \cong \Omega^0(M; E)$.

Let us return to the vector bundle E, in particular the vector spaces E_x assigned to each point $x \in M$. Being a vector space of rank r, we naturally find a collection of r vectors as a basis for this space E_x. We could just choose an arbitrary basis for each individual E_x. But seeing how E is a smooth manifold, we would also like to be able to translate the bases smoothly between fibres. This is where the fact that $\Gamma(E)$ is a $C^\infty(M)$-module comes into play;

Definition 2.3. *A **frame** S of a vector bundle, of rank r, E is a collection $S := (s_1, s_2, \cdots s_r)$, such that for each x, $S(x) := (s_1(x), s_2(x), \cdots s_r(x))$ forms a basis of vector space E_x.*

We see that a frame S gives rise to a way of choosing a basis for each of our fibres E_x. Frames can be global, in which case they are defined for all $x \in M$, or can be local, in which case they only apply at an open $U \subset M$. Note that frames need not be unique. For example, the section drawn in figure 2 defines a global frame of the cylinder. However, so would any line which is non-vanishing does not cross the black border, i.e.: $s(x) \neq 0_x$ for all $x \in M$.

One does not often have the luxury of a global frame S defined by a single collection of global non-vanishing sections. As such it is often necessary to stitch together different, locally defined, frames, to form a global one. This is possible as any atlas of our base space M would be smoothly compatible on all its opens. Let's say one has two locally defined frames $S_U := (s_1, s_2, \cdots s_r)$ and $S_{U'} := (s'_1, s'_2, \cdots s'_r)$, over opens U, $U' \subset M$. These opens overlap, such that $U \cap U' \neq \emptyset$. Because S_U, $S_{U'}$ are frames over U and U' respectively, they are also each frames over $U \cap U'$. Because $\Gamma(E)$ is a $C^\infty(M)$-module, we can transform one section into another by using functions;

$$s'_i(x) = \sum_{j=1}^{r} f_{i,j}(x) s_j(x).$$

Because s'_i and s_i are smooth sections, we know that the functions $f_{i,j}$ must be smooth too. A way to see the functions $f_{i,j}$ is as the transformation of basess at the overlapping fibres of U and U'; as S_U and $S_{U'}$ are by definition bases for the vector spaces E_x for x in the overlap of U and U'.

2.3 Operations on Vector Bundles

Being vector spaces, we have the usual set of operations to combine different types of vector spaces.

The way of constructing this vector bundle starts by defining the fibres fibre-wise at each point. We do this by applying the usual rules of the operation. As given two fibres of different vector bundles, combining them in the usual linear algebraic fashions, will give us a new vector space. Doing this for all the fibres of the vector bundles will produce a new collection of fibres. But as we have not indexed them smoothly, they are simply disjoint. We then need to observe however that this new vector bundle can also be indexed in a smooth way by our base manifold. Luckily the procedure for this is fairly straightforward once one has seen how to do it. We will highlight a few of these operations; being the dual bundle E^*, tensor product $E \otimes E'$, the endomorphism bundle $\mathrm{End}(E)$, the direct sum $E \oplus E'$, and finally the pull-back bundle f^*E

2.3.1 Dual Bundle

As stated above, we start by forming the dual bundle by defining the individual vector spaces fibre-wise. We set $E_x^* := (E_x)^*$, thus we define a fibre of the dual space, as simply being the dual of the original fibre. Now we take a collection of open sets $\{U_i, i \in I\}$ which cover our base space M. We want that it follows that E^* too, is indexed smoothly by M. For this we set a local frames $S_i = (s_1, s_2, \cdots, s_r)$ for each U_i over E. Then this induces a local frame $S_i^* = (s_1^*, s_2^*, \cdots, s_r^*)$ for each U_i over E^*. We know from the fact that E is a vector bundle, that each local frame is smoothly compatible. That is, given two opens U_i and U_j where the overlap is non-empty. Then over $U_i \cap U_j$ we could write $s_i(x) = \sum_{j=1}^r f_{i,j}(x)s_j(x)$. Now we find a similar construction for the dual bundle; the transition functions $f_{i,j} : S_i \to S_j$ for the dual bundle are given by $(f_{i,j}^T)^{-1} : S_i^* \to S_j^*$. As the duals of the smooth functions are also smooth, we have now obtained a way to smoothly combine the disjoint E_x^* into the dual vector bundle E^*.

When constructing the tensor product bundle $E \otimes E'$, the direct sum bundle $E \oplus E'$, and the endomorphism bundle $\mathrm{End}(E)$, the way of constructing follows a similar structure. The disjoint fibres of the vector bundle after the operation follows by definition of the operation applied fibre-wise to the corresponding fibres of each space. Then one can find a way to make them into a smooth manifold by looking at how the operation affects local frames. Therefore we will only give a short explanation of how to form these spaces, and some of their properties.

For the pull-back bundle $E_{f(x)}$ the procedure is slightly different, so we will highlight this one more extensively.

2.3.2 Tensor Product and Endomorphism Bundle

For the tensor product between two vector bundles E' and E, the fibre-wise construction is $(E \otimes E')_x := E_x \otimes E'_x$. And given a local frames S_i and S'_i, the new frames are $\{S_i \otimes S'_j\}_{(i,j)}$. The rank of this new vector bundle is given by multiplication $r \cdot r'$. Remark that the tensor product is isomorphic to the space of homomorphisms as following; $E \otimes E' \cong \mathrm{Hom}(E^*, E')$. So an element $e \otimes e'$ is a map working on an element $\epsilon \in E^*$ as follows; $(e \otimes e')(\epsilon) = \epsilon(e)e'$.

Remark for the endomorphism bundle End(E) that it is isomorphic to the space End(E) $\cong E^* \otimes E$. So we form the bundle, and it's smoothness, by combining what we know of the dual bundle and the tensor product bundle. Note that it now follows that the rank of this vector bundle is r^2.

2.3.3 Direct Sum Bundle

The direct sum $E \oplus E'$ is formed by the ordered pairing; $(E \oplus E')_x := E_x \oplus E'_x$. Where now the local frames S_i and S'_j form the bases $\{(S_i \oplus S'_j)\}_{i,j}$. The rank of this new vector bundle is $r + r'$.

2.3.4 Pull-back Bundle

The pull-back bundle is constructed slightly different from the bundles discussed before; if we are given a vector bundle (E, π_M, M), a manifold N, and a smooth function $f : N \to N$, then we can construct a pull-back bundle f^*E. Because this is not as much a vector space operation as the above examples, we will highlight its construction.

We start very similarly by defining the pull-back bundle fibre-wise. Namely; $(f^*E)_x := E_{f(x)}$. And we choose a collection of open sets $\{U_i\}_{i \in I}$, such that these opens cover M, and each open is small enough so that the vector bundle E is trivial over each open. We want that from this it follows that f^*E, too, is trivial over small enough opens, thus making it into a vector bundle by definition. It is reasonable to think that if we take our opens to be $\{f^{-1}(U_i)\}_{i \in I}$, then a trivialization follows. Indeed, if we let $S^\iota = (s^\iota_1, s^\iota_2, \cdots s^\iota_r)$ be a local frame for $E|_{U_\iota}$, then let us define the function $f^*s^\iota_k(x) := s^\iota_k(f(x))$, which maps from N to $E_{f(x)}$. Then we now define a local frame for $(f^*E)|_{f^{-1}(U_\iota)}$ as; $f^*S^\iota = (f^*s_1^\iota, f^*s_2^\iota, \cdots, f^*s_r^\iota)$. Notice now that because of how we defined $f^*s_k^\iota$, and the fact that we set $(f^*E)_x := E_{f(x)}$, it follows that f^*S^ι is a basis for $(f^*E)_x$ for all $x \in f^{-1}(U_\iota)$. And because E was trivial over U_ι, it follows that these now form a trivialization of $(f^*E)|_{f^{-1}(U_\iota)}$.

Now that we know that for each of the opens in $\{U_i\}_{i\in I}$ it follows that for each of the opens in $\{f^{-1}(U_i)\}_{i\in I}$ we have f^*E is trivial. This makes We only need to check that each $(f^*E)|_{f^{-1}(U_i)}$ and $(f^*E)|_{f^{-1}(U_j)}$ are smoothly compatible if $f^{-1}(U_i)$ and $f^{-1}(U_j)$ overlap. Recall that for the vector bundle E over M we had the smooth compatibility as follows; $s_i'(x) = \sum_{j=1}^r g_{i,j}(x)s_j(x)$. We form this into a smooth compatiblity for the vector bundle f^*E over N by simply taking the composition with our function f as follows; $(f^*s_i')(x) = \sum_{j=1}^r g_{i,j}(f(x))(f^*s_j)(x)$. Thus the transition functions become $g_{i,j} \circ f$, which is a composition of smooth functions, thus smooth itself.

These two properties; the smoothness of the overlaps, and the trivialization on small opens, makes f^*E into a vector bundle.

The vector space operations such as the tensor product and direct sum have their use in properly obtaining a new vector bundle from the old ones. On its own account, the pull-back bundle allows us to take pull-backs of every section; if s is a section of (E, π_M, M), then f^*s is a smooth section of (f^*E, π_N, N). These are in fact precisely the elements of $\Gamma(f^*E)$. The rank of the pull-back bundle is still r.

2.4 Connections

So far sections on vector bundles have been shown as smooth functions spanning over the different fibres E_x. Seeing how sections are indeed functions, the question, as could be expected in differential geometry, arises whether we could take the derivative of these functions. Normally when taking the, possibly directional, derivative of a function, we compare the tangent spaces of the points we are interested in. In “flat” Euclidean space this is a very natural thing to do; the tangent space at each and every point is the same space, and thus comparing them creates no issues whatsoever. This makes it possible to have a canonical derivative. However, with arbitrary vector bundles, we will find that points which are in different fibres, may have different tangent spaces. This makes the old concept of differentiating with d, ill-defined.

To solve this problem we introduce the concept of connections; the equivalent of exterior derivatives d for arbitrary vector bundles. We will see that this concept of connection can be expanded into the even more generalized covariant exterior derivative, also known as the exterior connection. We will then explain how we can obtain new connections from given vector bundles, and the operations explored in the previous section.

Definition 2.4. *A **connection** $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a linear map such that the Leibniz identity holds;*

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Recall that $\Gamma(\bigwedge^k T^*M \otimes E) = \Omega^k(M;E)$, and so; $\Gamma(E) = \Omega^0(M;E)$ and $\Gamma(T^*M \otimes E) = \Omega^1(M;E)$. Thus a connection ∇ can be seen as a linear differential operator acting on E-valued 0-forms;

$$\Omega^0(M;E) \xrightarrow{\nabla} \Omega^1(M;E).$$

Remark that this is already getting similar to the regular exterior derivative d, which was also a linear differential operator $d : \Omega^\bullet \to \Omega^{\bullet+1}$. So we would like to expand connections from linear differential operators on just E-valued 0-forms, to any E-valued k-form.

For this purpose we have to look at the connection together with a vector field. Take $X \in \mathfrak{X}(M)$ a smooth vector field on M, we remark that X is actually a section of the tangent bundle of M, so $X \in \Gamma(TM)$. Keeping in mind that $\nabla(s) \in \Omega^1(M;E)$ an E-valued 1-form, and thus can be naturally seen as a map $\nabla(s) : \Gamma(TM) \to \Gamma(E)$. Then we now define as follows;

Definition 2.5. *The **covariant derivative** along X; $\nabla_X : \Gamma(E) \to \Gamma(E)$, is defined by the tensor contraction;*

$$\nabla_X(s) \coloneqq i_x\nabla(s) = \nabla(s)(X).$$

We can now define an operator which on vector bundles E and connections ∇, will serve the same function as the exterior derivative.

Definition 2.6. *Given a vector bundle E and a connection ∇, then the **covariant exterior derivative** d_∇ is a linear differential operator of order two;*

$$\Omega^p(M;E) \xrightarrow{d_\nabla} \Omega^{p+1}(M;E),$$

defined on an E-valued p-form $\omega \in \Omega^p(M;E)$ as follows;

$$\begin{aligned} d_\nabla(\omega)(X_1, \cdots, X_{p+1}) \coloneqq & \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{p+1}) \\ & + \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i}(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1})) \in \Omega^{p+1}(M;E), \end{aligned}$$

It satisfies the following Leibniz identity for $\omega \in \Omega^p(M;E)$, and $\eta \in \Omega^r(M;E)$;

$$d_\nabla(\omega \odot \eta) = d_\nabla(\omega) \odot \eta + (-1)^p \omega \odot d_\nabla(\eta),$$

where $\odot$ signifies any bilinear tensor operation.

For example on a 1-form $\phi \in \Omega^1(M;E)$ we have;

$$d_\nabla(\phi)(X_0, X_1) = -\phi([X_0, X_1] + \nabla_{X_0}\phi(X_1)) - \nabla_{X_1}\phi(X_1).$$

From the definition and the linearity of the connection ∇, we can derive the following properties for d_∇;

1. For $p = 0$ it reduces to $d_\nabla = \nabla$;
2. Given $\omega, \eta \in \Gamma(E)$, we have $d_\nabla(\omega + \eta) = d_\nabla(\omega) + d_\nabla(\eta)$;
3. Given $\omega \in \Omega^p(M)$ and $s \in \Gamma(E)$, $d_\nabla(\omega s) = d\omega \otimes s + (-1)^p \omega \cdot d_\nabla(s)$;

The covariant exterior derivative is actually completely determined by the vector bundle E and the connection ∇, thus how it acts upon $\Omega^0(M;E) = \Gamma(E)$.

2.5 Constructed Connections

As d_∇ is so determined by the vector bundle E and the original connection ∇, we would like to know how to construct different covariant exterior derivatives, given from operations on vector bundles E and E', with connections ∇ and ∇' respectively. We will look at the same operations we have handled before; the dual E^*, tensor product $E \otimes E'$, direct sum $E \oplus E'$, endomorphism $\mathrm{End}(E)$, and the pull-back bundle f^*E. The tensor product and direct sum connections follow fairly straightforward by defining maps which work on the individual elements in a proper way regarding the vector space operations. For the dual-, endomorphism-, and pull-back- connection, the way of deriving the induced connections is slightly different.

2.5.1 Tensor Product Connection

Given vector bundles E and E', we formed the tensor product bundle by taking the fibres $(E_x \otimes E'_x)$. Now form the section $(s \otimes s') \in \Gamma(E \otimes E')$. As we would like the Leibniz identity for $\nabla^\otimes$ to hold, we define;

$$\nabla^\otimes(s \otimes s') := \nabla(s) \otimes s' + s \otimes \nabla(s').$$

We need to check that this indeed is a well-defined linear differential operator of order 1 on the vector bundle $E \otimes E'$. Remark that indeed $s \otimes s' \in \Omega^0(M; E \otimes E')$. Furthermore, $\nabla(s) \in \Gamma(T^*M \otimes E)$, which means that it maps to $T^*M \otimes E$, and $s' \in \Gamma(E')$, means it maps to E'. So the tensor product of these maps to $T^*M \otimes E \otimes E'$ which implies that $\nabla(s) \otimes s' \in \Gamma(T^*M \otimes E \otimes E') \cong \Omega^1(E \otimes E')$. So the induced connection does indeed map from and towards where we want it to.

The Leibniz identity is adhered to if we write it out, obtaining;

$$\begin{aligned}
\nabla^{\otimes}(f(s \otimes s')) &= \nabla^{\otimes}(fs \otimes s'), \\
&= \nabla(fs) \otimes s' + fs \otimes \nabla(s'), \\
&= df \otimes s \otimes s' + f\nabla(s) \otimes s' + fs \otimes \nabla(s'), \\
&= f(\nabla^{\otimes}(s \otimes s')) + df \otimes (s \otimes s').
\end{aligned}$$

Apart from it acting on the right spaces, and adhering to the Leibniz identity, we want the connection to be linear. So given $\eta, \theta \in \Gamma(E \otimes E')$ we want the following to hold;

$$\begin{aligned}
\nabla^{\otimes}(\eta + \theta) &= \nabla^{\otimes}((\eta_1 + \theta_1) \otimes (\theta_2 + \eta_2)), \\
&= \nabla(\eta_1 + \theta_1) \otimes (\theta_2 + \eta_2) + (\theta_1 + \eta_2) \otimes \nabla'(\theta_2 + \eta_2), \\
&\sim \nabla(\eta_1) \otimes \theta_2 + \eta_1 \otimes \nabla'(\eta_2) + \nabla(\theta_1) \otimes \theta_2 + \theta_1 \otimes \nabla(\theta_2), \\
&= \nabla^{\otimes}(\eta) + \nabla^{\otimes}(\theta).
\end{aligned}$$

This linearity combined with the property that under composition;

$$\begin{aligned}
\nabla^{\otimes}(\eta \circ \theta) &= \nabla^{\otimes}(\eta_1(\theta_1) \otimes \eta_2(\theta_2)), \\
&= \nabla(\eta_1 \circ \theta_1) \otimes \eta_2(\theta_2) + \eta_1(\theta_1) \otimes \nabla(\eta_2(\theta_2)), \\
&= [\nabla(\eta_1) \circ \theta_1 + \eta_1 \circ \nabla(\theta_1)] \otimes \eta_2(\theta_2) + \eta_1(\theta_1) \otimes [\nabla(\eta_2) \circ \theta_2 + \eta_2 \circ \nabla(\nabla_2)], \\
&= (\nabla(\eta_1) \otimes \eta_2) \circ \theta + (\nabla(\eta_2) \otimes \eta_1) \circ \theta + (\nabla(\theta_1) \otimes \theta_2) \circ \eta + (\nabla(\theta_2) \otimes \theta_1) \circ \eta, \\
&= \nabla^{\otimes}\eta \circ \theta + \nabla^{\otimes}\theta \circ \eta.
\end{aligned}$$

Now we remark that we have not actually checked whether this newly defined map, is actually tensorial. Simply that it is a linear map satisfying the Leibniz identity. Note that because $(\nabla^{\otimes}\eta) \in \Gamma(T^*M \otimes E \otimes E')$. Therefore we should be able to apply it to a section $s \in \Gamma(E^*)$, or $s' \in \Gamma(E'^*)$ or $t \in \Gamma(TM)$. In particular, we defined sections to be C^∞-linear, thus when applied to a section (fs), this linearity should hold.

$$\begin{aligned}
(\nabla^{\otimes}\eta)(fs) &= (\nabla\eta_1 \otimes \eta_2)(fs) + (\eta_1 \otimes \nabla\eta_2)(fs), \\
&= f(\nabla\eta_1 \otimes \eta_2 + \eta_1 \otimes \nabla\eta_2)s, \\
&= f(\nabla^{\otimes}\eta)s
\end{aligned}$$

All of the above makes $\nabla^{\otimes}$ into a linear differential operator.

2.5.2 Direct Sum Connection

Given vector bundles E and E', we recall that the direct sum $E \oplus E'$ vector bundle consists of the paired individual fibres (E_x, E'_x). Now take sections $s \in \Gamma(E)$ and $s' \in \Gamma(E')$ to form the section $(s, s') \in \Gamma(E \oplus E')$. Then we define the induced connection $\nabla^{\oplus}$ as being the matrix;

$$\nabla^{\oplus} := \begin{pmatrix} \nabla & 0 \\ 0 & \nabla' \end{pmatrix}$$

This works on a section $(s, s') \in \Gamma(E \oplus E')$;

$$\nabla^{\oplus}(s, s') = (\nabla s, \nabla' s').$$

Observing that this is indeed a map to $\Omega^1(M, E \oplus E')$, we readily see that all properties of a connection are satisfied as ∇ and ∇' individually already are connections.

2.5.3 Endomorphism Connection

For the endomorphism bundle we remark that an element $L \in \Gamma(\mathrm{End}(E))$ is a function $L : M \to \mathrm{End}(E)$. As such, given a section $s \in \Gamma(E)$, we see that Ls is also section of E. Applying ∇ to this, we would like the Leibniz identity to hold; $\nabla(Ls) = (\nabla^{\mathrm{End}} L)s + L(\nabla s)$. This prompts us to define;

$$(\nabla^{\mathrm{End}} L)s := \nabla(Ls) - L(\nabla s).$$

Remark that everything above on the right-hand side has already been defined. We want to check to be certain that $\nabla^{\mathrm{End}(E)}$ is indeed a map $\nabla^{\mathrm{End}(E)} : \Omega^0(M; \mathrm{End}(E)) \to \Omega^1(M; \mathrm{End}(E))$. Note that if we interpret $\nabla(L)(s) := \nabla(Ls)$ and $L(\nabla)(s) := L(\nabla s)$, then both $\nabla(L)$ and $L(\nabla)$ can be seen as a connections on E. This makes their difference $\nabla(L) - L\nabla$ an endomorphism. So $\nabla(L) - L\nabla \in \Omega^1(M; \mathrm{End}(E))$.

So it is indeed a well-defined map from and to the spaces we would like it to go to. We also need to check that it is indeed a linear differential operator by checking the Leibniz identity, the linearity, it's application to a section fs and it's compatibility under compositions.

Regarding the Leibniz identity we see;

$$\begin{aligned}(\nabla^{\mathrm{End}} fL)s &= \nabla(fLs) - fL(\nabla s), \\ &= df Ls + f\nabla(Ls) - fL(\nabla s), \\ &= df Ls + f(\nabla^{\mathrm{End}} L)s,\end{aligned}$$

removing the s we see that;

$$(\nabla^{\text{End}} fL) = df L + f(\nabla^{\text{End}} L).$$

So it does indeed satisfy the Leibniz identity.

We also need to check whether or not ∇^{End} is actually tensorial. That is; C^∞-linearity when applied to a section s. We check;

$$\begin{aligned}(\nabla^{\text{End}} L)fs &= \nabla(Lfs) - L\nabla(fs),\\ &= df Ls + f\nabla(Ls) - L(dfs) + Lf(\nabla s),\\ &= f\nabla(Ls) - fL(\nabla s),\\ &= f(\nabla^{\text{End}} L)s.\end{aligned}$$

Thus indeed making it tensorial. For normal linearity on $L \in \Gamma(\text{End}(E))$ we check;

$$\begin{aligned}\nabla^{\text{End}}(L_1 + L_2) &= \nabla(L_1 + L_2) - (L_1 + L_2)\nabla,\\ &= \nabla(L_1) - L_1\nabla + \nabla(L_2) - L_2\nabla,\\ &= \nabla^{\text{End}} L_1 + \nabla^{\text{End}} L_2.\end{aligned}$$

And so it adheres to the regular linearity we expect connections to adhere, furthermore we check whether is is compatible under compositions;

$$\begin{aligned}(\nabla^{\text{End}} L_1 \circ L_2)s &= \nabla(L_1 \circ L_2 s) - L_1 \circ L_2(\nabla s),\\ &= \nabla(L_1(L_2 s)) - L_1 \circ L_2(\nabla s),\end{aligned}$$

applying the Leibniz identity,

$$\begin{aligned}&= (\nabla^{\text{End}} L_1) \circ L_2 s + L_1 \circ \nabla(L_2 s) - L_1 \circ L_2(\nabla s),\\ &= (\nabla^{\text{End}} L_1) \circ L_2 s + L_1 \circ (\nabla^{\text{End}} L_2)s,\end{aligned}$$

and removing the s, to obtain;

$$= (\nabla^{\text{End}} L_1) \circ L_2 + L_1 \circ (\nabla^{\text{End}} L_2).$$

So we see that ∇^{End} is indeed a well-defined induced connection on the endomorphism bundle.

2.5.4 Dual Connection

In a very similar way one constructs the dual connection ∇^* on the dual bundle E^*. Starting with an element $\sigma \in \Gamma(E^*)$ and a $s \in \Gamma(E)$, we remark that $\sigma : M \to E^*$, and so pairing it with s, we obtain a regular function $\sigma(s) : M \to \mathbb{R}$. Thus $\nabla(\sigma(s)) = d(\sigma(s))$. Applying the Leibniz identity as before, we declare;

$$(\nabla^*\sigma)(s) := d(\sigma(s)) - \sigma(\nabla s).$$

One must check the same properties as before. For brevity we will just quickly show the Leibniz identity;

$$\begin{aligned}(\nabla^* f\sigma)s &= d(f\sigma(s)) - f\sigma(\nabla s),\\ &= df \otimes \sigma(s) + fd(\sigma(s)) - \sigma(\nabla s)),\end{aligned}$$

removing the s, and applying the induced connection we obtain;

$$(\nabla^* f\sigma) = df \otimes \sigma + f(\nabla^*\sigma).$$

Thus this makes it into a connection on E^*

2.5.5 Pull-back Connection

The pull-back connection too is retrieved slightly different from the above connections. Recall that the pull-back bundle f^*E over a manifold N was obtained if we had a smooth map $f : N \to M$, and already an existing vector bundle (E, π_M, M).

The pull-back connection $f^*\nabla$ is now defined by wanting the following diagram to commute;

$$\begin{array}{ccc}\Omega^1(N; f^*E) & \xleftarrow[f^*]{} & \Omega^1(M; E)\\ {\scriptstyle f^*\nabla}\uparrow & & {\scriptstyle \nabla}\uparrow\\ \Omega^0(N; f^*E) & \xleftarrow[f^*]{} & \Omega^0(M; E).\end{array}$$

Hence we define the pullback connection as follows;

$$(f^*\nabla)(f^*s) := f^*(\nabla s),$$

where we recall that $f^*s \in \Gamma(f^*E)$ is the pull-back of a section $s \in \Gamma(E)$.

Showing that this is indeed a linear differential operator follows the same path as shown above for the other induced connection. We will again only quickly show the Leibniz identity;

$$\begin{aligned}(f^*\nabla)(gf^*s) &= f^*(\nabla gs),\\ &= f^*(dg \otimes s + g\nabla s),\\ &= [dg \otimes s + g(\nabla s)]\,(f(x)),\\ &= dg \otimes f^*s + g(f^*\nabla s),\\ &= dg \otimes f^*s + g((f^*\nabla)(f^*s)).\end{aligned}$$

2.6 Curvature of a Connection

We now have an understanding of connections and how to extend them to the covariant exterior derivative d_∇. If we look at these operators as linear differential operators of order two, we will see that;

$$\Omega^0(M;E) \xrightarrow{\nabla} \Omega^1(M;E) \xrightarrow{d_\nabla} \Omega^2(M;E) \xrightarrow{d_\nabla} \Omega^3(M;E) \xrightarrow{d_\nabla} \cdots,$$

induces a co-chain.

Unfortunately, this co-chain is not a co-chain-complex, as generally speaking $d_\nabla \circ d_\nabla \neq 0$. As we will find out, d_∇^2 is defined to be the curvature-form of a connection ∇. In the following section we will define this curvature form and how it acts upon its spaces. We will explain why this 2-form d_∇^2 is called the curvature form of a connection. From there on we will introduce the Bianchi identity, and show how this identity is used to form a co-chain-complex from the above chain.

Definition 2.7. *The curvature form of a connection ∇ is the operator*

$$\kappa_\nabla := d_\nabla^2 = d_\nabla \circ d_\nabla.$$

It is a map

$$\Omega^p(M;E) \xrightarrow{k_\nabla} \Omega^{p+2}(M;E).$$

Notice how we refer more to the curvature form rather than the curvature operator. This is because we can show that $\kappa_\nabla(fs) = f\kappa_\nabla(s)$, thus it is C^∞-linear, instead of adhering to the Leibniz identity, making it strictly tensorial. And so it is actually a 2-form. We will show this as follows; we start by observing the d_∇ operator locally. Recall that if we are locally enough, our vector bundle is trivial. Our covariant exterior derivative d_∇ can then be written as the sum of the regular exterior derivative and a connection matrix. So locally we have;

$$d_\nabla^{local} = d + A,$$

here A is the $r \times r$ connection matrix consisting of 1-forms on M;

$$A := \begin{pmatrix} \omega_1^1 & \cdots & \omega_r^1 \\ \vdots & \ddots & \vdots \\ \omega_1^r & \cdots & \omega_r^r \end{pmatrix}.$$

As A is an $r \times r$ matrix of 1-forms, it is itself a 1-form with values in $\mathrm{End}(E)$. And as such $A \in \Omega^1(M; \mathrm{End}(E)) \cong \Gamma(T^*M \otimes \mathrm{End}(E))$. This leads to a local expression of the curvature κ_∇ too. The curvature form can now locally be written as;

$$\begin{aligned}\kappa_\nabla^{local} &= (d+A)(d+A),\\ &= dA + A\wedge A\end{aligned}$$

Note that because we saw that $A \in \Omega^1(M; \mathrm{End}(E))$, it follows that $A \wedge A \in \Omega^2(M; \mathrm{End}(E))$. So locally we already see that $\kappa_\nabla \in \Omega^2(M; \mathrm{End}(E))$.

Now we are going to use this fact to show that κ_∇ is completely determined by ∇, and thus does not depend on the k of the particular k-form of the space $\Omega^k(M; E)$ it is working on. If we take a local frame for $E|_U$. Then any element $\omega \in \Omega^k(M; E)$ can be rewritten in terms of this frame;

$$\omega = \sum_i \omega_i \otimes s_i,$$

with each $\omega_i \in \Omega^k(M)$, and each $s_i \in \Omega^0(M; E)$. Then we obtain via the Leibniz identity that;

$$d_\nabla \omega = \sum_i^r d\omega_i \otimes s_i + (-1)^k \omega_i \nabla s_i$$

Now applying d_∇ again, we obtain the curvature form κ_∇^k, where we emphasize with k that it works on the space $\Omega^k(M; \mathrm{End}(E))$. And rewriting gives us;

$$\begin{aligned}\kappa_\nabla^k \omega = d_\nabla^2 \omega &= \sum_i d_\nabla(d\omega_i \otimes s_i) + d_\nabla((-1)^k \omega_i \otimes \nabla s_i),\\ &= \sum_i d^2\omega_i \otimes s_i + (-1)^{k+1} d\omega_i \otimes \nabla s_i + (-1)^k d\omega_i \otimes \nabla s_i + (-1)^{2k} \omega_i \otimes \nabla^2 s_i,\\ &= \sum_i \omega_i \otimes \kappa_\nabla^0 s_i.\end{aligned}$$

Thus we see that κ_∇^k is independent of k. Now we apply κ_∇ to a section $f\omega$, we will see that κ_∇ is C^∞-linear, thus strictly tensorial on each space. Let f a smooth function and $\omega \in \Omega^p(M; E)$, then;

$$\kappa_\nabla(f\omega) = d_\nabla(d_\nabla(f\omega)),$$

rewriting in a similar matter as above, yields us;

$$\kappa_\nabla(f\omega) = f\kappa_\nabla(\omega),$$

proving the C^∞-linearity of our curvature form κ_∇, thus proving that $\kappa_\nabla \in \Omega^2(M; \mathrm{End}(E))$ acts tensorially on every k-form and not just 0-forms. The above result can be generalized if one observes that for any $\eta \in \Omega^n(M)$ it follows that;

$$\kappa_\nabla(\eta \otimes \omega) = \eta \otimes \kappa_\nabla(\omega).$$

Thus the C^∞-linearity is just the special case that $n = 0$.

So why is this 2-form κ_∇ actually called the curvature form of a connection ∇? For this we observe how κ_∇ acts on the trivial vector bundle of a manifold M being $E = M \times R^r$. Note that a smooth function $f \in C^\infty$ is a map from M to $\mathbb{R}$. So in the case that E is trivial, these smooth functions are actually sections of the vector bundle; $f \in \Gamma(M \times \mathbb{R})$, by taking $(x, f(x)) \in E_x$. Now note that the regular exterior derivative d becomes a connection; indeed it takes a 0-form, $(\cdot, f) \in \Omega^0(M; M \times \mathbb{R})$ to a 1-form $(\cdot, df) \in \Omega^1(M; M \times \mathbb{R})$. Now we already know d satisfies all the properties which makes it into a linear differential operator. And also very importantly, we know that d makes the co-chain into a co-chain complex, i.e.: $d^2 = \kappa_\nabla^{flat} = 0$. Now remark that because the vector bundle is trivial, the local representation of the curvature form, actually becomes global representation. Which implies $dA + A \wedge A = 0$, this is only the case when A is the zero-matrix. This gives us the reason to call κ_∇ the curvature form; because when our vector bundle is trivial, κ_∇ will reduce to 0, and we can observe $\kappa_\nabla \neq 0$ as related to the failure of the vector bundle to be flat. As such we will define as follows;

Definition 2.8. *A connection for which we have;*

$$\kappa_\nabla = 0,$$

is called a flat connection.

As said before, $\kappa_\nabla \neq 0$ can be as related to the failure of the vector bundle to be flat. And most spaces are not flat. Luckily so; otherwise differential topology would bore rather quickly. However, the fact that our co-chain does not become a co-chain-complex in that case, is regrettable. Fortunately there is a way we can construct a co-chain complex; by using the so-called Bianchi identity.

Lemma 2.1. *The second Bianchi identity is the property that;*

$$(d_\nabla \kappa_\nabla)\omega = 0. \tag{2}$$

Proof. The proof involves manipulating the expression algebraically using the Leibniz identity. Let $\omega \in \Omega^p(M;E)$, then it follows that;

$$d_\nabla(\kappa_\nabla \omega) = (d_\nabla \kappa_\nabla)\omega + (-1)^2 \kappa_\nabla(d_\nabla \omega),$$

re-arranging gives us;

$$\begin{aligned}(d_\nabla \kappa_\nabla)\omega &= d_\nabla(\kappa_\nabla \omega) - \kappa_\nabla(d_\nabla \omega), \\ &= d_\nabla(d_\nabla(d_\nabla \omega)) - d_\nabla(d_\nabla(d_\nabla \omega)), \\ &= 0\end{aligned}$$

□

The Bianchi identity induces an alternate co-chain complex, from which we can deduce so-called characteristic classes. The one we are interested in, for the Gauss-Bonnet theorem, is the so called Euler class of a vector bundle.

3 The Euler Class

We are now interested in designing a characteristic class; the Euler class. For this class we need to have a connection which is compatible with a metric. In this section we will explain how we can define a metric on a vector bundle. We will then introduce the concept of metric compatibility of a connection. Once we have defined these two key concepts, we will show how this leads to a co-homology class, via the Bianchi identity, which we call a characteristic class. The Euler class we will define is one of such classes. We will explore it's properties and it's usefulness.

3.1 Connections and Metrics

Let us start by specifying what a metric on a vector bundle is. Recall that a vector bundle is actually a union of individual vector spaces, which we indexed together in a smooth way determined by our base space M. As such, a metric on each individual bundle is already given by the property of it being a vector space. This will be the basis of our vector bundle wide metric as follows;

Definition 3.1. *A metric $g(s_1, s_2)$ on E is the family of indiviual metrics of each E_x;*

$$g(s_1, s_2)(x) := \{g|_{E_x}(s_1(x), s_2(x))\}_{x \in M}.$$

We see that the metric is defined fibre-wise, and we cannot compare the distance between $s_1(x)$ and $s_2(x')$ if x and x' lie in different fibres. From now on we will denote $g|_{E_x}$ simply with g, and let it be clear from the context that it is the specific metric on the fibre E_x.

Note that, by definition of a metric, g is a symmetric function. And g takes the inner product of two sections at a point x to return a scalar in $\mathbb{R}$, which we interpret as the distance. So we can think of $g(s1, s2)$ as a section of the trivial bundle of M; $g(s1, s2) \in \Gamma(\mathbb{R})$. And g as a section of the trivial bundle of $\mathrm{Sym}^2(E^*)$, the smooth manifold of functions. We can now see g as a bundle map $\mathrm{Sym}^2(E) \to \mathbb{R}$, which is equivalent to $g \in \mathrm{Sym}^2(E^*)$.

We now know what a metric on a vector bundle is. So let us establish a relationship between metrics and connections.

Definition 3.2. *A connection ∇ is called a metric connection of g, or metric compatible with g if;*

$$(\nabla g) = 0.$$

This is equivalent to the following identity;

$$\begin{aligned} d(g(s_1, s_2)) &= (\nabla g)(s_1, s_2) + g(\nabla s_1, s_2) + g(s_1, \nabla s_2), \\ &= g(\nabla s_1, s_2) + g(s_1, \nabla s_2). \end{aligned}$$

If a connection is metric compatible, then it follows that d_∇ is also metric compatible.

As seen above $\nabla(g)$ becomes the regular exterior derivative as the metric is just a regular smooth function. This together with the fact that $d^2 = 0$, implies the following identity for the curvature 2-form;

Lemma 3.1. *κ_∇ is skew-symmetric with respect to g, i.e.:*

$$g(\kappa_\nabla s_1, s_2) = -g(s_1, \kappa_\nabla s_2)$$

Proof. Rewriting the identity, using the fact that d_∇ too is metric compatible, we find;

$$\begin{aligned} \kappa_\nabla(g(s_1, s_2) &= d^2(g(s_1, s_2)) = 0, \\ &= d_\nabla(g(\nabla s_1, s_2) + g(s_1, \nabla s_2)), \\ &= g(\kappa_\nabla s_1, s_2) - g(\nabla s_1, \nabla s_2) + g(\nabla s_1, \nabla s_2) + g(s_1, \kappa_\nabla s_2), \\ &= g(\kappa_\nabla s_1, s_2) + g(s_1, \kappa_\nabla s_2), \end{aligned}$$

which, because it equals 0, after rearranging implies;

$$g(\kappa_\nabla s_1, s_2) = -g(s_1, \kappa_\nabla s_2).$$

□

One can see that a metric compatible connection has some nice properties which we would like to use. However, it seems like a rather strong demand to make of a connection. And such nice properties would be useless if a connection is only very rarely metric compatible. Fortunately, it is possible to construct a metric compatible connection for every vector bundle E with a metric g.

Lemma 3.2. *Every vector bundle E with a metric g, admits a metric compatible global connection.*

Proof. Let us start by trivializing the vector bundle, which is to say to cover M with opens $\{U_i\}_{i \in I}$, such that $E|_{U_i}$ is trivial. Now let us observe a local orthonormal frame $(s_{i_1}, \cdots s_{i_r})$. Because $E|_{U_i}$ is trivial, it inherits the flat local connection. Thus on the local orthonormal frame, which just consists of the regular basis vectors of $\mathbb{R}^r$, we get that $\nabla_i^{local} s_i = ds_i + A_i s_i = ds_i = 0$. Because it is an orthonormal frame, we have that $g(s_{i_r}, s_{i_{r'}}) = \delta^r_{r'}$, and so $d(g(s_{i_r}, s_{j_{r'}})) = 0$. Now we see that;

$$d(g(s_{i_r}, s_{i_{r'}})) = 0 = g(0, s_{i_{r'}}) + g(s_{i_r}, 0) = g(\nabla_i^{local} s_{i_r}, s_{i_{r'}}) + g(s_{i_r}, \nabla_i^{local} s_{i_{r'}}),$$

which makes the local connection metric compatible. Now note that, by definition of a frame, any arbitrary section $\sigma|_{U_i}$ can be written as $\sigma_k|_{U_i} = \sum_r f_{r,k} s_{i_r}$. And thus for two arbitrary sections $\sigma_k|_{U_i}$ and $\sigma_m|_{U_i}$ we have that;

$$d(g(\sigma_k, \sigma_m)) = d(g(\sum_r f_{r,k} s_{i_r}, \sum_r f_{r,m} s_{i_r})).$$

For the left hand side of this equation, note that a metric is a C^∞-linear function, so we obtain;

$$\begin{aligned} g(\sum_r f_{r,k} s_{i_r}, \sum_r f_{r,m} s_{i_r}) &= \sum_{r,r'} f_{r,k} f_{r,m} g(s_{i_r}, s_{i_{r'}}), \\ &= \sum_r f_{r,k} f_{r,m}. \end{aligned}$$

Applying the exterior derivative is a matter of calculus and leads to;

$$d(g(\sigma_k, \sigma_m)) = \sum_r df_{r,k} \cdot f_{r,m} + f_{r,k} \cdot df_{r,m}.$$

Now let us look at what happens if we apply ∇_i^{local} inside the metric;

$$\begin{aligned} g(\nabla_i^{local} \sigma_k, \sigma_m) &= g(d \sum_r f_{r,k} s_{i_r}, \sum_r f_{r,m} s_{i_m}), \\ &= g(\sum_r df_{r,k} s_{i_r} + f_{r,k} \nabla_i^{local} s_{i_r}, \sum_r f_{r,m} s_{i_m}), \\ &= \sum_{r,r'} df_{r,k} \cdot f_{r',m} g(s_{i_r}, s_{i_{r'}}), \\ &= \sum_r df_{r,k} \cdot f_{r,m}. \end{aligned}$$

A similar result follows if we apply ∇_i^{local} to σ_m. Thus we see that because metric compatibility holds for a local orthonormal frame, it holds for arbitrary sections;

$$d(g(\sigma_k, \sigma_m)) = g(\nabla_i^{local} \sigma_k, \sigma_m) + g(\sigma_k, \nabla_i^{local} \sigma_m).$$

Now we use these local connections ∇_i^{local} to form a global connection ∇. Consider a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$. As said before, on each of these opens one can find a local orthonormal frame $s_i = (s_{i_1}, \cdots s_{i_r})$. Now let us set the global connection ∇ as;

$$(\nabla s)(x) := \sum_i \rho_i(x)\nabla_i^{local}(s|_{U_i}(x))$$

One should check that this new connection does indeed adhere to all the properties a linear differential operator should adhere to, but we will for brevity only show it adheres to the Leibniz identity;

$$\begin{aligned}
(\nabla fs)(x) &= \sum_i \rho_i(x)\nabla_i^{local}(fs|_{U_i}(x)), \\
&= \sum_i \rho_i(x)\left[f(x)\nabla_i^{local}(s|_{U_i}(x)) + df \otimes s|_{U_i}(x)\right], \\
&= df \otimes s|_{U_i}(x) + f(x)\sum_i \rho_i(x)\nabla_i^{local}(s|_{U_i}(x)), \\
&= df \otimes s|_{U_i}(x) + f(x)\nabla(fs(x))
\end{aligned}$$

This connection is metric compatible as;

$$\begin{aligned}
\nabla(g(\sigma_k, \sigma_m)) &= \sum_i \rho_i \nabla_i^{local} g(\sigma_k|U_i, \sigma_m|U_i), \\
&= \sum_i g(\rho_i\nabla_i^{local}\sigma_k|_{U_i}, \sigma_m|_{U_i}) + g(\sigma_k|_{U_i}, \rho_i\nabla_i^{local}\sigma_m|_{U_i}), \\
&= g(\nabla\sigma_k, \sigma_m) + g(\sigma_k, \nabla\sigma_m).
\end{aligned}$$

So we see that given a metric g on a vector bundle E, we can construct a metric compatible connection ∇. □

3.2 Euler Class of a Vector Bundle

Now we understand what a metric on a vector bundle and when a connection is metric compatible, we can start designing our Euler class.

The Euler class is a specific manifestation of the so-called characteristic classes. These are formed by using the Bianchi identity $(d_\nabla \kappa_\nabla) = 0$ of lemma 2.6 to construct co-homology classes on M. In this section we will explain how the Bianchi identity forms co-homology classes, and how we specifically manipulate the Bianchi identity and the curvature form to obtain our specific characteristic class; the Euler class.

We remind ourselves that two k-forms ω and ω' are called co-homologous in the de-Rham co-homology, the co-homology using the regular exterior derivative d, if they are closed and their difference is an exact form, i.e. ; $\omega - \omega' = d\eta$. This eventually induces the class $H^k(M) \coloneqq \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)}$. So to design our Euler co-homology class, we would like a similar construction relating to our covariant exterior derivative d_∇. We see a semblance of this happening because of the skew-symmetric property of κ_∇. To find Ker(d) we start by defining the folllowing form;

Lemma 3.3. *Let* $\omega(s_1, s_2) \coloneqq g(\kappa_\nabla s_1, s_2)$*. Then*

$$(\nabla\omega) = 0$$

Proof. We start by observing what kind of form ω actually is. Note that κ_∇ was skew-symmetric, and that g is a symmetric function. Manipulating the arguments we see that;

$$\omega(s_1, s_2) = g(\kappa_\nabla s_1, s_2) = -g(s_1, \kappa_\nabla s_2) = -g(\kappa_\nabla s_2, s_1) = -\omega(s_2, s_1).$$

So we see that ω is skew-symmetric in it's arguments, and maps two sections to the metric g, which maps to a scalar. Thus $\omega \in \Omega^2(M; \bigwedge^2 E^*)$. Now it comes in handy that we derived in which function spaces ω, κ_∇ and g exsisted. Note that $\omega \in \Omega^2(M; \bigwedge^2 E^*) \subset \Omega^2(M; \bigotimes^2 E^*)$. For g we showed that $g \in \Gamma(E^* \otimes E)$. And finally for κ_∇ we derived that $\kappa_\nabla \in \Omega^2(M; \mathrm{End}(E))$, which implies it's pullback is a map $\kappa_\nabla^* : E^* \times E \to \mathrm{End(E)} \cong E^* \otimes E$. Now we observe the following commutation diagram;

$$\begin{array}{ccc} E^* \times E & \xrightarrow{\kappa_\nabla^*} & E^* \otimes E \\ & \overset{\omega}{\searrow} & \downarrow g \\ & & \mathbb{R} \end{array}$$

we see that $\omega \coloneqq g \circ \kappa_\nabla$, by the universal property, is unique. Thus we have;

$$\begin{aligned} \nabla\omega &= \nabla(g \circ \kappa_\nabla^*), \\ &= dg \circ \kappa_\nabla^* + g \circ \nabla^* \kappa_\nabla^*, \\ &= 0. \end{aligned}$$

Where the left hand side goed to 0 because of metric compatibility, and the right hand side goes to 0 because of the Bianchi identity. $\square$

We have found ourselves a 2-form ω for which we have $\nabla\omega = 0$. This means that $\omega \in \mathrm{Ker}(\nabla)$. We now would like to extend this to the entire chain, to form a co-chain complex. Observe the wedge-product between to forms as generating a higher form, that is; $\wedge : \Omega^i \times \Omega^j \to \Omega^{i+j}$. We use this, together with ω, to obtain $\omega^k \coloneqq \bigwedge^k \omega \in \Omega^{2k}(M; \bigwedge^{2k} E^*)$. And we will prove;

Lemma 3.4. *For a $2k$-form ω^k we have that $\omega^k \in Ker(\nabla)$, i.e.;*

$$\nabla \omega^k = 0$$

Proof. The proof follows by induction. We know that for $k = 1$, we have that $\nabla\omega = 0$. Now assume that for $k-1$ we have $\nabla\omega^{k-1} = 0$, then it follows;

$$\begin{aligned}\nabla(\omega^k) &= \nabla(\omega \wedge \omega^{k-1}),\\ &= \nabla\omega \wedge \omega^{k-1} + \omega \wedge \nabla\omega^{k-1},\\ &= 0.\end{aligned}$$

So it holds for every k. □

So now we have that the property for the 2-form we have $\omega \in \mathrm{Ker}(\nabla)$ 2-form, extends to the property of the $2k$-form $\omega^k \in \mathrm{Ker}(\nabla)$. Now we want to be able to apply the regular exterior derivative d to what we have just found. The problem in this lies in the fact that d only works on $\Omega^\bullet(M)$, whilst our forms live in the space $\Omega^\bullet(M; \bigwedge^{2k} E^*)$. We solve this problem by recalling that there exists a natural pairing between linear functionals of E^* and elements of E such that $E^* \otimes E \to \mathbb{R}$, becomes a regular C^∞ function. Upon this function we can just apply the regular exterior derivative d. We obtain this by taking the tensor contraction $\frac{1}{k!}\langle\omega^k, \sigma'\rangle = \frac{1}{k!}\omega^k(\sigma')$, where $\sigma' \in \Gamma(E)$. We then observe that;

$$d\frac{1}{k!}\langle\omega^k, \sigma\rangle = \frac{1}{k!}(\nabla\omega^k)\sigma' + \omega^k(\nabla\sigma') = \frac{1}{k!}\omega^k(\nabla\sigma') = \frac{1}{k!}\langle\omega^k, \nabla\sigma'\rangle$$

For this to be a closed form, i.e. an element of $\mathrm{Ker}(d)$, we need this to be 0. In order to obtain that $\frac{1}{k!}\langle\omega^k, \sigma\rangle = 0$, we will again turn to orthonormal frames of E.

Lemma 3.5. *Let $\sigma = \{s_1, s_2, \cdots s_{2k}\}$ be a local orthonormal frame for E. Then $\nabla^{local}(\sigma) = 0$*

Proof. By the definition of orthonormality it follows that $\langle s_j, s_j\rangle = 1$. We then see;

$$\begin{aligned}\nabla^{local}\langle s_j, s_j\rangle &= \nabla^{local}(1) = 0,\\ &= \langle\nabla^{local} s_j, s_j\rangle + \langle s_j, \nabla^{local} s_j\rangle,\\ &= 2\langle\nabla^{local} s_j, s_j\rangle.\end{aligned}$$

So we see that $\langle\nabla^{local} s_j, s_j\rangle = 0$. Recall that locally $\nabla^{local} = d + A$, and thus;

$$\langle\nabla^{local} s_j, s_j\rangle = \langle d(\sum_i f_i^j s_i) + A(\sum_i f_i^j s_i), s_j\rangle,$$

remark that for an orthonormal frame $f_i^j = 0$ if $i \neq j$, and $f_j^j = 1$ so $df_j^j = 0$,

$$\begin{aligned}
&= \langle \sum_{i'} A_j^{i'} s_{i'}, s_j \rangle, \\
&= \sum_{i'} A_j^{i'} \langle s_{i'}, s_j \rangle, \\
&= A_j^j = 0
\end{aligned}$$

We now have the following identity;

$$\nabla^{local} s_j = \sum_{i' \neq j} A_j^{i'} s_i.$$

And so we can rewrite $\nabla^{local}\sigma$ as follows;

$$\begin{aligned}
\nabla^{local}\sigma &= \nabla(s_1 \wedge s_2 \wedge \cdots \wedge s_{2k}), \\
&= \nabla s_1 \wedge s_2 \wedge \cdots \wedge s_{2k}) + s_1 \wedge \nabla(s_2 \wedge \cdots \wedge s_{2k}, \\
&= \sum_{i' \neq j} A_j^{i'} s_i \wedge s_2 \wedge \cdots \wedge s_{2k} + s_1 \wedge \nabla s_2 \wedge \cdots \wedge s_{2k} + s_1 \wedge s_2 \wedge \nabla(\cdots \wedge s_{2k}),
\end{aligned}$$

using that $s_i \wedge s_i = 0$, we find that by induction we end up with;

$$= 0.$$

□

Now remark that if we are given two local orthonormal frames σ and σ' which determine the same orientation, then $\sigma = \sigma'$ where they overlap. This is because the transformation of basis can be given by $\det(A)\sigma = \sigma'$, and if the orientation is the same $\det(A) = 1$ for orthonormal frames.

We know that we can stitch together local frames by using partitions of unity as in Lemma 3.2. Combining this with Lemma 3.5, we finally retrieve $\nabla\sigma = 0$.

Lemma 3.6. *If E is an orient vector bundle of rank 2k which admits a local orthonormal frame, then we can construct a global nowhere vanishing section σ such that $\nabla\sigma = 0$*

This is what we need for $\langle \omega^k, \sigma \rangle \in \mathrm{Ker}(d)$, and thus we have a cohomology class. We define $e := \langle \frac{1}{k!}\omega^k, \sigma \rangle$ where ω as before, and σ an orthonormal frame.

Definition 3.3. *$\mathcal{E}(E) = [e] \in H^{2k}$ is the Euler class of a vector bundle E.*

3.3 Properties of the Euler Class

The Euler Class $\mathcal{E}$ is just one of many characteristic classes which can be formed on vector bundles. So why are we so interested in this specific one? For this to be clear we have to analyse some important properties of the Euler Class.

Lemma 3.7. *We define $\mathcal{E}(E) = 0$ if the vector bundle E has odd rank.*

The Euler class follows from the pairing of our $2k$-form ω^k with σ. We can think of ω^k as being the top-form of our vector bundle. Therefore our vector bundle too should have rank $2k$. If this is not the case, then ω^k cannot be the top form of our vector bundle, and thus the expression makes no sense.

In defining the Euler class we started with a metric compatible connection, which eventually defined our local frame σ.

Lemma 3.8. *The Euler Class $\mathcal{E}$ is independent of the chosen connection.*

Proof. Let ∇_0 and ∇_1 be two different connections on (E, π, M). Let us define a map p by projection on the first factor;

$$p : M \times I \to M,$$
$$p : (m, t) \mapsto (m).$$

This map is a smooth map to M, which already had a vector bundle E. So this allows us to construct the pullback bundle p^*E over $M \times I$. We now also define a family of connections given by a linear combination of ∇_0 and ∇_1;

$$\nabla_t \coloneqq (1 - t)\nabla_0 + t\nabla_1.$$

This is a linear combination of connections, which implies ∇_t is a connection on E too. We are going to use this to define a connection $\tilde{\nabla}$ on p^*E.

We start by using a trivialization of E. So let $\{U_\alpha\}_(\alpha \in A)$ a collection of opens which covers M, and for each U_α we have that $E|_{U_\alpha}$ is trivial. Then on each of these opens, the connections ∇_1 and ∇_2 can locally be seen as $d + A_1$ and $d + A_2$. And for the family of connections we have $d + [(1 - t)A_0 + tA_1]$. Now remark that these same opens induce a similar trivialization of the vector bundle p^*E over $M \times I$, by setting $U'_\alpha = U_\alpha \times I$. We now define our connection locally, and pointwise as;

$$\tilde{\nabla}(s)(x, t) \coloneqq ((d + (1 - t)A_0 + tA_1)(s))(x)$$

Where we see that;

$$\tilde{\nabla}(s)(x, 0) = \nabla_0(s),$$
$$\tilde{\nabla}(s)(x, 1) = \nabla_1(s),$$
$$\tilde{\nabla}(s)(x, t) = \nabla_t(s)$$

Now let us define an inclusion map $i_t : M \hookrightarrow M \times I$ which, for a fixed t, is defined as $i_t(x) = (x, t)$. This would also induce a pull-back bundle $i_t^* p^* E$, however we see that for fixed t, $i_t^* p^* E \cong E$. And so we obtain the induced connection $i_t^* \tilde{\nabla} = \nabla_t$. The following commutative diagram helps visualize these maps. Where some notational abuse has been taken; one should note that the Ω have been omitted, and the ∇ must be seen as connections on the vector bundle they originate from, rather than maps in the direction the arrows are pointing;

$$\begin{array}{ccccc} E & & p^*E & \longrightarrow & E \\ \downarrow{\scriptstyle \nabla_t} & & \downarrow{\scriptstyle \tilde{\nabla}} & & \downarrow{\scriptstyle \nabla_0, \nabla_t, \nabla_1} \\ M & \xhookrightarrow{i_t} & M \times I & \xrightarrow{p} & M \end{array}$$

Now let $e_{\tilde{\nabla}_t} \in H^{\cdot}(M \times I)$, be the Euler form for the connection $\tilde{\nabla}_t$. And since we have the two specific cases;

$$\nabla_0 = i_0^* \tilde{\nabla},$$
$$\nabla_1 = i_1^* \tilde{\nabla}.$$

We obtain;

$$e_{\nabla_0} = i_0^* e_{\tilde{\nabla}},$$
$$e_{\nabla_1} = i_1^* e_{\tilde{\nabla}}.$$

Now because i_0 and i_1 are homotopic they induce equal maps in homology, thus i_0^* and i_1^* are both the same map $H^{\cdot}(M \times I) \to H^{\cdot}(M)$. And thus it follows that $[e_{\nabla_0}] = i_0^*[e_{\tilde{\nabla}}] = i_1^*[e_{\tilde{\nabla}}] = [e_{\nabla_1}]$.

And thus we see that the Euler classes $\mathcal{E}_{\nabla_0} = \mathcal{E}_{\nabla_1}$ do not depend on the connection chosen.

□

The fact that the Euler class is independent of the connection chosen, together with the fact that one can always find a connection given a vector bundle and a metric, means that the Euler class is an intrinsic property of the manifold M. This is important as it means we can freely choose any representation of our manifold M to calculate the Euler class. And nothing withholds us of choosing the easiest representation to work with every time.

Furthermore the Euler class is well-behaved under direct sums of vector bundles, i.e.;

Lemma 3.9. *Given vector bundles E and E', then it follows that;*

$$\mathcal{E}(E \oplus E') = \mathcal{E}(E) \smile \mathcal{E}(E').$$

Proof. Let us observe the ranks over the vector bundles E and E'. We remark that if E is even and E' is odd, then $\dim(E \oplus E')$ is also odd, thus we see;

$$\mathcal{E}(E \oplus E') = 0 = \mathcal{E}(E) \smile 0 = \mathcal{E}(E) \smile \mathcal{E}(E').$$

Now let us go to when $\text{rank}(E) = k$ and $\text{rank}(E') = l$, where k,l either both even or both odd. Remark that if $\omega_E^i \in \Omega^{2i}(M; \bigwedge^{2i} E^*)$ and $\omega_{E'}^j \in \Omega^{2j}(M; \bigwedge^{2j} E'^*)$, then we have that $\omega_E^i \wedge \omega_{E'}^j \in \Omega^{2(i+j)}(M \bigwedge^{2i} E^* \wedge \bigwedge^{2j} E'^*)$ Now observe that given the base sections ω_E and $\omega_{E'}$, we can set up a connection matrix;

$$\omega_{E\oplus E'} \begin{pmatrix} s \\ s' \end{pmatrix} = \begin{pmatrix} \omega_E & 0 \\ 0 & \omega_{E'} \end{pmatrix} \begin{pmatrix} s \\ s' \end{pmatrix} = \omega_E(s) + \omega_{E'}(s')$$

From this we deduce the following identity;

$$\omega_{E\oplus E'}^{k+l} = (\omega_E + \omega_{E'})^{k+l}$$

Some algebraic manipulation will show us;

$$\omega_{E\oplus E'}^{k+l} = \sum_{i=0}^{k+l} \binom{k+l}{i} \omega_E^{k+l-i} \wedge \omega_{E'}^i,$$

because $\Omega^x(M; E) = 0$ if $x > \text{rank}(E)$, the sum reduces drastically,

$$\begin{aligned} &= \sum_{l \le i \le l} \binom{k+l}{i} \omega_E^{k+l-i} \wedge \omega_{E'}^i, \\ &= \binom{k+l}{l} \omega_E^k \wedge \omega_{E'}^l, \\ &= \frac{(k+l)!}{k!l!} \omega_E^k \wedge \omega_{E'}^l. \end{aligned}$$

Now we recall that the definition of the Euler class was $[\langle \frac{\omega^n}{n!}, \sigma \rangle]$, and that $[\mathcal{E}(E)] \smile [\mathcal{E}(E')] = [\mathcal{E}(E) \wedge \mathcal{E}(E')]$. Then we can rewrite;

$$\begin{aligned} \left[\langle \frac{\omega_{E\oplus E'}^{k+l}}{(k+l)!}, \sigma \rangle \right] &= \left[\langle \frac{\omega_E^k}{k!} \wedge \frac{\omega_{E'}^l}{l!}, \sigma \rangle \right], \\ &= [e \wedge e'], \\ &= [e] \smile [e'] \end{aligned}$$

$\square$

We see that the above lemma's are useful properties to have when explicitly computing the Euler class of a manifold. However, they do not really describe the Euler class. One of the most advantageous properties of the Euler class is to be able to classify surfaces which admit a non-vanishing section. More precisely;

Lemma 3.10. *If there exists a global non-vanishing section* $\omega \in \Gamma(E)$*. Then* $\mathcal{E}(E) = 0$

Proof. Remark that we can use ω to split up the vector bundle E in a part defined by ω as $L \subset E$ and a part orthogonal to it $L^{\perp} = L^c$, by setting;

$$L := \{\lambda \cdot \omega | \lambda \in \mathbb{R}\},$$
$$L^{\perp} := \{\sigma \in E | \langle \sigma, \omega \rangle = 0\}$$

We then see that $E = L \oplus L^{\perp}$. Now by remarking that $L \cong$ the trivial bundle, of odd rank, and applying both Lemmas 3.9 and 3.7, we see that in this case $\mathcal{E}(E) = 0$. □

The above lemma allows us to interpret the Euler class of a space as a measure of how "twisted" this space is. To give a visual example, let us re-observe the cylinder of figure 2. The white line we see is indeed a global non-vanishing section of the vector bundle, being the cylinder. Thus we can conclude that $\mathcal{E}(\text{Cylinder}) = 0$. This makes sense intuitively, because the cylinder is a very normally stretched space; it is the trivial bundle $S^1 \times \mathbb{R}^{`}$. We quickly see that this applies to any trivial bundle; given a space $M$, we can always construct a global non-vanishing section of $M \times \mathbb{R}^{`}$ by simply taking the translation of M into $M \times \mathbb{R}$. Or more formally; we can define a global non-vanishing section $s : M \to M \times \mathbb{R}$ by taking $s : x \mapsto (x, \lambda)$, where $\lambda \in \mathbb{R}^{`}$ a constant vector. Thus trivial vector bundles allow us to have global non-vanishing sections, and thus always have Euler class equal to 0.

But let us give an example proof of a space which is twisted. The most straightforward being the space obtained from twisting a strip onto itself; the Möbius strip. An intrinsic property of the Möbius strip is the fact that if you draw a straight line on it, you will have gone around twice, and end up at the same spot where you started. We can obtain the Möbius strip by gluing as in figure 3. This is the same as starting with a square $[0,1] \times [-1,1]$ and applying the equivalence relation; $(0, y) \sim (1, -y)$. Giving us a vector bundle $(S^1, \pi, \text{Möbius})$. Now let s be a global section. Then remark that $E_0 = E_1$, but that in this fibre, s assumes the values $s(0)$ and $s(1)$, and for s to be smooth, we have to have $s(0) = -s(1)$. This means that either $s(0) = s(1) = 0$, in which case it vanishes in this fibre, or $s(0) = c$ and

Figure 3: Obtain the Mobius strip by a glueing

$s(1) = -c$. In this case, as s is a smooth function, the intermediate value theorem states it should be 0 somewhere else on the vector bundle, thus it vanished in another fibre. As such the Möbius strip does not admit a global non-vanishing section. Thus a priori we cannot conclude it's Euler class to be zero.

4 The Levi-Civita Connection

In the above theory we have explored the abstract notion of vector bundles and connections. Which led us to designing the Euler class. Important results were that the Euler class did not depend on the connection chosen, and that for every metric and vector bundle, we could construct a connection. This allows us a large degree of freedom in choosing the specific vector bundle we want to work on, which we have not yet made concrete in the above discussion.

For the Gauss-Bonnet theorem, we want to work on the tangent bundle TM of a manifold M. We will explain how TM is a vector bundle. From there on we will introduce a property for connections on the tangent bundle; the so-called torsion of a connection. We will show that we can construct a connection for which the aforementioned torsion vanishes. This connection is appears to be unique, and is called the Levi-Civita connection ∇^{LC}. We will show the properties of this specific connection. This specific manifestation of the above theory; the tangent bundle TM as our vector space, together with a Riemannian metric g, and the Levi-Civita connection ∇^{LC}, will be what we use for the proof of the Gauss-Bonnet theorem for surfaces.

4.1 Tangent Bundle Connections

For a manifold of dimension M we can construct the tangent space T_xM at each point $x \in M$. Note that by definition, if M is a m-dimensional manifold, then each tangent space T_xM is a m-dimensional vector space. Also remark that because M is a m-dimensional manifold, every point x has a neighbourhood U_x which is homeomorphic to R^m, this is done via charts on M. The tangent bundle of $\mathbb{R}^m$ is again $\mathbb{R}^m$, so $\pi^{-1}(U_x)$ is diffeomorphic to $\mathbb{R}^m \times \mathbb{R}^m$, and each fibre T_xM is isomorphic to $x \times \mathbb{R}^m$. Thus we see that TM is a vector bundle of rank m, the same as the manifold;

Lemma 4.1. *The rank of the vector bundle (TM, π, M) is the same as the dimension of the manifold M.*

Now we remark something interesting about connections on the vector bundle TM. Recall that sections $s \in \Gamma(E)$ are maps $s : M \to E$, and that connections are maps $\Gamma(E) \to \Gamma(T^*M \otimes E)$. If we now replace E with TM. We find that s is now a map assigning a tangent vector $s(x) \in TM$ to each point x. Thus s is by definition now a vector field over M, and we denote the space of all smooth vector fields on M by $\mathfrak{X}(M)$. For which we now have; $\Gamma(TM) \cong \mathfrak{X}(M)$. Now observe that ∇ becomes a map from this vector field to $\Gamma(T^*M \otimes TM) \cong \Gamma(\mathrm{End}(TM))$. Recall that we defined a covariant derivative along a vector field X as $\nabla_X(s') \coloneqq \nabla(s')(X)$ which was a map $\nabla_X : \Gamma(E) \to \Gamma(E)$. So this can now become a map $\nabla_s(s') \coloneqq \nabla(s')(s)$ which maps as $\nabla_s : \mathfrak{X}(M) \to \mathfrak{X}(M)$. This implies that ∇ can work on two sections, and thus smooth vector fields, s and s', and return another vector field. So if we take the vector bundle to be the tangent bundle, then we can observe connections as maps;

$$\nabla(s, s') \coloneqq \nabla_s(s') = \nabla(s')(s), \qquad \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

From now on we will interchangeably use s, s' which lie in $\Gamma(TM) \cong \mathfrak{X}(M)$, with vector spaces $X, Y \in \mathfrak{X}(M) \cong \Gamma(TM)$, depending on where we want the emphasis to lay. Note that given an atlas on M, we have charts working on opens $\{U_\alpha\}_{\alpha \in A}$ which cover M. On each of these charts TM is trivializable. The basis of the $TM|_{U_\alpha}$ is given by $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_m}\right)$. Note that each $\frac{\partial}{\partial x_i}$ is in itself a vector field, and as such a smooth section. Thus the canonical basis for $TM|_{U_\alpha}$ can be seen as local frame for the vector bundle.

Recall that the curvature of a connection was directly related to the connection itself as $\kappa_\nabla \coloneqq \nabla^2$ on sections. As we could see ∇ as a function working on two vector fields, we now can also observe κ_∇ as working on two vector fields. Let us observe how it behaves as such;

$$\begin{aligned}
\kappa_\nabla(X, Y)(Z) &= \nabla_X(\nabla_Y(X, Y))(Z), \\
&= \nabla_X \nabla_Y(X, Y)(Z) - \nabla_Y \nabla_X(X, Y)(Z), \\
&= (\nabla_X \nabla_Y - \nabla_Y \nabla_X)(X, Y)(Z), \\
&= \nabla_{[X,Y]}(X, Y)(Z).
\end{aligned}$$

So if we use the tangent bundle as our vector bundle, then the curvature of a connection can be written as;

$$\kappa_\nabla(X, Y) = \nabla_{[X,Y]}$$

We see that the curvature now becomes skew symmetric because of the skew symmetry in $[X, Y]$ and the C^∞-linearity of ∇_X in X, i.e.; $\nabla_{fX} = f\nabla_X$.

Lemma 4.2. *On the tangent bundle, the curvature of a connection is skew-symmetric;*

$$\kappa_\nabla(X, Y) = -\kappa_\nabla(Y, X).$$

4.2 Torsion

Earlier we defined the curvature of a connection κ_∇, and we saw that it said something about our connection ∇; We could see the curvature as an intrinsic property of the connection itself. For connections on the tangent bundle we still have the curvature κ_∇. But we also have another property, called the torsion T_∇ of the connection.

Definition 4.1. *The Torsion T_∇ of a connection ∇ is;*

$$T_\nabla(X,Y) \coloneqq \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$

We will explore some properties of T_∇. Note that T_∇ is skew-symmetric in it's arguments;

$$\begin{aligned} T_\nabla(Y,X) &= \nabla_Y(X) - \nabla_X(Y) - [Y,X], \\ &= -\nabla_X(Y) + \nabla_Y(X) + [X,Y], \\ &= -T_\nabla(X,Y). \end{aligned}$$

Furthermore T_∇ is tensorial, i.e. it is C^∞-linear in each of it's arguments;

$$\begin{aligned} T_\nabla(fX,Y) &= \nabla_{fX}(Y) - \nabla_Y(fX) - [fX,Y], \\ &= f\nabla(X)(Y) - ((df \otimes X)(Y) + f\nabla_Y(X)) - f[X,Y] + Y(f)X, \\ &= Y(f)X - L_Y f(X) + f(\nabla_X(Y) - \nabla_Y(X) - [X,Y]), \\ &= f(T_\nabla(X,Y)). \end{aligned}$$

For the other argument we have that $T_\nabla(X,fY) = -T_\nabla(fY,X) = -f(T_\nabla(Y,X)) = fT_\nabla(X,Y)$. Because we now have that T_∇ is skew-symmetric, tensorial and is a map $T^*M \times T^*M \to TM$, we see that $T_\nabla \in \Gamma(\bigwedge^2 T^*M \otimes TM) = \Omega^2(M;TM)$.

So we see parallels between the curvature of a connection $\kappa_\nabla \in \Omega^2(M;\mathrm{End}(TM))$, and the torsion of a connection $T_\nabla \in \Omega^2(M;TM)$. Both are intrinsic properties of the connection, and just like we called a connection for which $\kappa_\nabla = 0$ a flat connection. So too do we classify connections for which $T_\nabla = 0$.

Definition 4.2. *A connection is called torsion-free if;*

$$T_\nabla = 0.$$

Which is equivalent to saying;

$$\nabla_X(Y) - \nabla_Y(X) = [X,Y].$$

To imagine what torsion-free means concretely, is slightly less intuitive than imagining flatness. To give an intuitive, but not mathematically rigorous, way to think about torsion, observe again the Möbius strip. Imagine a basis of a tangent space at a point a on the Möbius strip. Let's say one of the basis vectors points forwards along the Möbius strip, and the other points towards the left. These vectors are coordinate vector fields, thus also orthonormal, so $[X, Y] = 0$. If we now displace this basis along the Möbius strip, we will see we end up again out our point a. Whilst the forward point basis vector still points in the same direction, we find that our basis vector which started pointing to the left, now points to the right. If we now think of $\nabla_X(Y)$ as being the change of the initially left pointing along the path we walked, and of $\nabla_Y(X)$ as the change of the forward facing if we step sideways. Then we can imagine $\nabla_Y(X) = 0$, as forwards stays forwards even if we side-step. But $\nabla_X(Y) \neq 0$, as evidently the vector flipped from left to right somewhere along the path. So of this "connection" we walked, $T_\nabla \neq 0$, because our basis flipped on the Möbius trip.

Whereas having a flat connection was too strong a demand to place on a connection, as that only naturally arises when the vector bundle is trivial, having a torsion-free connection is something we can construct.

4.3 Levi-Civita Connection

We can construct a torsion-free connection on specific manifolds, with a specific metric. Even though these may sound as very specific demands, there is a whole family of manifolds upon which this applies; Riemannian manifolds. Riemannian manifolds are manifolds M combined with a Riemannian metric $\langle \cdot, \cdot \rangle$, which is to say a defined inner product for vector field on each x. However, if we use the tangent bundle as our vector bundle. Then the metric g on the tangent bundle, is a defined inner product for the vector field on each x, because of the equivalency of vector fields and sections of the tangent bundle. Thus this metric makes the base manifold into a Riemannian manifold. Thus we are able to construct a metric compatible, torsion-free connection, which we call the Levi-Civita connection ∇^{LC}.

Let us start with a metric compatible connection ∇, which is torsion-free. And let X, Y, Z be three vector fields. Then we observe the following the following summation, also generally known as the Koszul-formula;

$$L_X g(Y, Z) + L_Y g(X, Z) - L_Z g(X, Y).$$

Writing it out we obtain;

$$= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y),$$

note that the Lie-derivatives here are the covariant exterior derivatives along a vector field, applied to a regular function, which the metric is. Because the metric is a linear function, we can group some terms together;

$$= g(\nabla_Y Z - \nabla_Z Y, X) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_X Y + \nabla_Y (X), Z),$$

now we use the property that ∇ is torsion-free to obtain;

$$= g([Y, Z], X) + g([X, Z], Y) + g(-[X, Y], Z) + g(2\nabla_X Y, Z).$$

We now define according to the following equality;

$$\begin{aligned} g(\nabla_X Y, Z) &:= \frac{1}{2}[L_X g(Y, Z) + L_Y g(X, Z) - L_Z g(X, Y) \\ &\quad + g([Z, Y], X) + g([Z, X], Y) + g([X, Y], Z)]. \end{aligned}$$

Now if we take X and Y fixed, then $g(\nabla_X(Y), Z)$, which we remind ourselves is the inner product, then only Z ultimately determines the terms. Note that we have not yet proven much; we have only shown that starting from a metric compatible and torsion-free connection, we can obtain a unique linear map, by fixing X and Y, which maps as $g(\nabla_X Y, \cdot) : \Gamma(TM) \to \Gamma(T^*M)$. First we will show that it is tensorial in Z, making it into a section of $T^* \otimes E$, and thus an element $\Omega^1(M; TM)$. We see from the fact that g is C^∞-linear that

$$g(\nabla_X Y, fZ) = f(\nabla_X, Z),$$

so indeed it is an element of $\Omega^1(M; TM)$. We notice that because ∇ satisfies the Leibniz rule, then;

$$g(\nabla_X fY, Z) = g(L_X(f)Y, Z) + g(f\nabla_X Y, Z),$$

satisfies the Leibniz rule in Y. And similarly we can check that it is tensorial in X too. This makes $g(\nabla_X Y, \cdot)$ into a connection. Checking whether this is torsion-free involves evaluating the identity;

$$g(\nabla_X Y - \nabla_Y X, Z) = g(\nabla_X Y, Z) - g(\nabla_X Y, Z).$$

From how we defined this expression, it now follows that this is;

$$\begin{aligned} &= \frac{1}{2}[L_X g(Y, Z) + L_Y g(X, Z) - L_Z g(X, Y) - L_Y g(X, Z) - L_X g(Y, Z) + L_Z g(Y, X)] \\ &+ \frac{1}{2}[g([Z, Y], X) + g([Z, X], Y) + g([X, Y], Z) - g([Z, X], Y) - g([Z, Y], X) - g([Y, X], Z)], \\ &= \frac{1}{2}(2g([X, Y], Z)), \\ &= g([X, Y], Z). \end{aligned}$$

So we see that this new connection is indeed torsion-free. For metric compatibility we want that;

$$L_x(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y).$$

And indeed, for the right hand side we find that;

$$\begin{aligned} = g(\nabla_X Y, Z) &+ \frac{1}{2}[L_X g(Y,Z) + L_Z g(X,Y) - L_Y g(X,Z)] \\ &- \frac{1}{2}[g([Z,Y],X) + g([X,Y],Z) + g([Z,X],Y)], \\ &= \frac{1}{2}(2L_x g(Y,Z)), \\ &= L_x g(Y,Z). \end{aligned}$$

We have now proven that the above identity is unique, a connection, is torsion-free, and is metric compatible. So now let us finally define the Levi-Civita connection;

Definition 4.3. *The Levi-Civita connection ∇^{LC} on the tangent bundle of a Riemannian manifold is the unique connection which is metric compatible and torsion-free, ∇_X^{LC} defined by;*

$$g(\nabla_X^{LC} Y, Z) \tag{3}$$

and fixing X and Y.

And from the above derivation of how to construct the Levi-Civita connection we also see;

Lemma 4.3. *Every Riemannian manifold has a Levi-Civita connection.*

We almost never use the explicit definition of the Levi-Civita connection, but rather its nice properties of torsion-freeness, metric compatibility and it's uniqueness.

Even though $T_\nabla^{LC} = 0$, there is no reason for it's curvature to vanish too. So being a connection, we naturally also have κ_∇^{LC}, the curvature of the Levi-Civita connection. Earlier we introduced the second Bianchi identity, pertaining to the curvature of the connection. For the Levi-Civita connection we also have the so-called first Bianchi identity;

Lemma 4.4. *For the Levi-Civita connection we have the following identity;*

$$\kappa_\nabla^{LC}(X,Y)Z + \kappa_\nabla^{LC}(Z,X)Y + \kappa_\nabla^{LC}(Y,Z)X = 0.$$

Which is called the first Bianchi idenity

Proof. The identity follows by writing out the terms and applying the torsion-freeness;

$$\begin{aligned}
&= (\nabla_X \nabla Y - \nabla_Y \nabla X - \nabla[X,Y])Z \\
&+ (\nabla_Z \nabla X - \nabla_X \nabla Z - \nabla[Z,X])Y \\
&+ (\nabla_Y \nabla Z - \nabla_Z \nabla Y - \nabla[Y,Z])X, \\
&= (\nabla_X \nabla_Z Y - \nabla_Y \nabla_Z X - \nabla_Z[X,Y]) \\
&+ (\nabla_Z \nabla_Y X - \nabla_X \nabla_Y Z - \nabla_Y[Z,X]) \\
&+ (\nabla_Y \nabla_X Z - \nabla_Z \nabla_X Y - \nabla_X[Y,Z]),
\end{aligned}$$

now we can group terms together, and using torsion-freeness; i.e. $\nabla_i j - \nabla_j i = [i,j]$, we simplify to;

$$\begin{aligned}
&= \nabla_X([Z,Y] - [Y,Z]) + \nabla_Y([Z,X] - [Z,X]) + \nabla_Z([Y,X] - [X,Y]), \\
&= -2(\nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y]), \\
&= -2([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]),
\end{aligned}$$

this is the Jacobi identity, which vanishes, so

$$= 0$$

□

4.4 Curvature Tensor

So far all the properties of curvature and torsion pertained to our connections. And we have only spoken of curvature on a manifold or vector bundle itself when this was quickly followed by the fact that the space was trivializable and thus the curvature vanished anyway. We will now finally start taking steps towards actually speaking of curvature on our manifold. And to do this we will define the curvature tensor $\mathcal{R}$. After we have defined the curvature tensor, we will explore some algebraic properties of it.

Definition 4.4. *The curvature tensor is a map $\mathcal{R} : \bigotimes^4 TM \to \mathbb{R}$ defined as;*

$$\mathcal{R}(X,Y,Z,W) = g(\kappa_\nabla^{LC}(X,Y)Z, W).$$

Note that κ_∇ and g are both C^∞-linear. And as such $\mathcal{R}$ is C^∞-linear in each of its arguments, thus tensorial, so it is indeed justified to call it a tensor.

The curvature tensor has some convenient algebraic properties which eventually show that $\mathcal{R} \in \mathrm{Sym}^2(\bigwedge^2 T^*M)$ the space of functions which are symmetric in its two arguments $\alpha, \beta \in T^*M$, which in their turn are skew-symmetric functions from $TM \times TM$, i.e $\alpha(a,b) = -\alpha(b,a)$.

The first property is skew-symmetry in it's first two arguments, i.e.: $\mathcal{R}(X,Y,Z,W) = -\mathcal{R}(Y,X,Z,W)$. Remark that flipping these arguments only flips the part on which κ_∇^{LC} works on. We can apply lemma 4.2, to see that κ_∇^{LC} is skew-symmetric, and thus this implies $\mathcal{R}(X,Y,Z,W)$ is skew-symmetric in its first two arguments.

For skew-symmetry on the second pair of arguments, i.e.: $\mathcal{R}(X,Y,Z,W) = -\mathcal{R}(X,Y,W,Z)$. We use the fact that the connection is metric compatible;

$$L_X(g(Z,W)) = g(\nabla_X Z, W) + g(Z, \nabla_X W),$$

together with the fact that the Lie-derivative is exact; $L_X(L_Y(g)) = 0$. Applying both these fact to $g(Z,W)$, we obtain;

$$\begin{aligned} L_X(L_Y g(Z,W)) - L_Y(L_X g(Z,W)) = 0, \\ L_X(L_Y g(Z,W)) = L_Y(L_X g(Z,W)), \\ \nabla_X(g(\nabla_Y Z, W) + g(Z, \nabla_Y W)) = \nabla_Y(g(\nabla_X Z, W) + g(Z, \nabla_X W)), \end{aligned}$$

applying metric compatibility again, we find that the terms where ∇ is not doubly applied, cancel each other out. Grouping g's together when one of the arguments is equal, we obtain;

$$\begin{aligned} g((\nabla_X\nabla_Y - \nabla_Y\nabla_X)Z, W) &= g(Z, (\nabla_Y\nabla_X - \nabla_X\nabla_Y)W), \\ g(\nabla_{[X,Y]}Z, W) &= -g(Z, \nabla_{[X,Y]}W), \\ g(\kappa_\nabla(X,Y)Z, W) &= -g(\kappa_\nabla(X,Y)W, Z), \\ \mathcal{R}(X,Y,Z,W) &= -\mathcal{R}(X,Y,W,Z), \end{aligned}$$

which proves it is skew-symmetric in the second pair of arguments too. The two skew-symmetric properties combine to also give us $\mathcal{R}(X,Y,Z,W) = \mathcal{R}(Y,X,W,Z)$

To prove pairwise symmetry, we use the first Bianchi identity proven in lemma 4.4; we can think of this identity applied to $\mathcal{R}$ in the sense that an even permutation of the first three arguments is the negative sum of the other two even permutations. A combination of flipping and permuting will give us the desired result. To save space we denote $\mathcal{R}(X,Y,Z,W)$ with $\mathcal{R}_{xyzw}$.

Then a way to obtain the desired identity is as follows;

$$
\begin{aligned}
\mathcal{R}_{xyzw} &= -\mathcal{R}_{zxyw} - \mathcal{R}_{yzxw}, \\
&= \mathcal{R}_{zxwy} - \mathcal{R}_{yzwx}, \\
&= -(\mathcal{R}_{wzxy} + \mathcal{R}_{xwzy}) - (\mathcal{R}_{wyzx} + \mathcal{R}_{zwyx}), \\
&= \mathcal{R}_{zwxy} + \mathcal{R}_{xwyz} + \mathcal{R}_{wyxz} + \mathcal{R}_{zwxy}, \\
&= 2\mathcal{R}_{zwxy} - \mathcal{R}_{yxwz}, \\
&= 2\mathcal{R}_{zwxy} - \mathcal{R}_{xyzw}, \\
2\mathcal{R}_{xyzw} &= 2\mathcal{R}_{zwxy}, \\
\mathcal{R}_{xyzw} &= \mathcal{R}_{zwxy}.
\end{aligned}
$$

We now have proven the three algebraic properties of the curvature tensor we wanted to have;

1. $\mathcal{R}_{xyzw} = -\mathcal{R}_{yxzw}$,
2. $\mathcal{R}_{xyzw} = -\mathcal{R}_{xywz}$,
3. $\mathcal{R}_{xyzw} = \mathcal{R}_{zwxy}$.

and thus we see that $\mathcal{R}$ is pairwise symmetric in its arguments; which would imply $\mathcal{R} \in \mathrm{Sym}^2(\bigotimes^2 T^*M)$, but is antisymmetric within these pairs, thus this implies $\mathcal{R} \in \mathrm{Sym}^2(\bigwedge^2 T^*M)$.

5 The Scalar Curvature

We are now ready to define the quintessential quantity used in the Gauss-Bonnet theorem; the scalar curvature of a point in our manifold. Though we disclaim that we will be proving the theorem for 2-dimensional compact orientable Riemannian manifolds. The above theory is generalized enough to prove it for Riemannian manifolds with boundary, which would add an extra term to account for the boundary. And can also be expanded to prove the generalized Gauss-Bonnet theorem for higher dimensional manifolds. We remind ourselves that the questions arisen were related to our skippyball; so compact orientable 2-dimensional Riemannian manifolds will do just fine. We will denote these surfaces with Σ.

We will start by defining a scalar curvature function $K(x)$ on our orientable surface. Then we will show some actual examples of what the Levi-Civita connection is on the flat plane, torus, and sphere. And show how we can calculate the scalar curvature of these points. We will then show what the relation is between the earlier designed Euler class and this new-found scalar curvature function.

5.1 Scalar Curvature Function

Armed with the curvature tensor, we are now finally ready to define a curvature function $K : M \to \mathbb{R}$, more aptly called the scalar curvature of a manifold M. This is the function we will eventually use for the Gauss-Bonnet theorem.

The theory we have discussed so far is applicable to any orientable Riemann surface Σ, as this is a smooth manifold of dimension 2. As such there exists an atlas such that each x is in a neighbourhood U_α which is homeomorphic to a subset $A_{U_x} \subset \mathbb{R}^2$. Now if a surface is orientable, there are only two possible orientations of the surface, these correspond to the normal vector pointing inwards, to be imagined as being inside on the surface of the object, or outwards, to be imagined as being outside on the surface of the object. Recall that according to lemma 4.1, the tangent bundle of a Riemann surface, is a vector bundle of rank 2. We now define on each of these U_α of the atlas, a local orthonormal frame of the tangent bundle: $\{s_{\alpha,1}s_{\alpha,2})\}_\alpha$, and its dual $\{(s^*_{\alpha,1}, s^*_{\alpha,2})\}_\alpha$, which are all oriented the same way, let's say positively. Then we have according to lemma 3.6 a global nowhere vanishing section $\sigma = s_1 \wedge s_2$ on $\bigwedge^2 T\Sigma$, and let σ^* the dual global nowhere vanishing section for $\bigwedge^2 T^*\Sigma$. Recall that because $s_i \in \Gamma(TM)$, it is by definition a map $M \to TM$, and so we have elements $s_i(x) \in TM$. This implies that $s_1(x) \wedge s_2(x) \in TM \wedge TM$. And as such their duals

$s_1^*(x) \wedge s_2^*(x) \in T^*M \wedge T^*M$. Now remark that elements in $T^*M \wedge T^*M$ are skew-symmetric linear functionals with two vector fields as arguments. So multiplying two of these with the regular dot product makes sense as we are just multiplying functions, the regular dot product is symmetric. So we can construct a function $(\sigma^* \cdot \sigma^*)(X, Y, Z, W) := \sigma^*(X, Y) \cdot \sigma^*(Z, W)$, and this function is an element of the function space $\mathrm{Sym}^2(\bigwedge^2 T^*M)$.

Now let us return to $\mathcal{R}$, and let us a define a slightly different curvature tensor by setting $\mathfrak{R} := -\mathcal{R}$. We still have the fact that $\mathfrak{R} \in \mathrm{Sym}^2(\bigwedge^2 T^*M)$. Now because function spaces are vector spaces, there should be a third vector connecting $\mathfrak{R}$ and $\sigma^* \cdot \sigma^*$. And so there must exist a function such that we have the following;

Definition 5.1. *For a 2-dimensional orientable Riemann surface Σ we have that there exists a function $K(x)$ such that;*

$$\mathfrak{R}(X, Y, Z, W)(x) = K(x) \cdot \sigma^*(X, Y) \cdot \sigma^*(Z, W).$$

We define this function as the scalar curvature function.

So $K(x)$ is a function $K : \Sigma \to \mathbb{R}$ which returns for every point on the function a scalar, which we interpret as the curvature of the surface at that point. We might find it strange that it is just a scalar, one would expect that for a 2-dimensional manifold, one would expect some kind of vector instead of a scalar to denote the curvature. We admit that the scalar curvature is a specific definition of the curvature; there are other ways to define a curvature, which have other properties. The main reason we defined the scalar curvature is because it is the specific curvature used in the Gauss-Bonnet theorem. But we will show that it seems to agree with our a priori intuition of curvature on a surface with the following examples.

5.1.1 The Plane

Let us first observe the flat plane $\mathbb{R}^2$. It is an astute observation that this is indeed not a compact manifold, but remark that the definition of the scalar curvature does not yet depend on compactness. Our intuition tells us that the curvature of a point $\vec{x}$ should be 0, and moreover that it should be constant.

Because this surface is trivial, we see that a global frame orthonormal frame is given by (x, y), with the corresponding orthonormal dual frame of the tangent bundle (∂_x, ∂_y). The metric given on the flat plane is naturally just the flat metric, that is to say;

$$g_{\mathbb{R}^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Not let us pick $X = Z = \partial_x$ and $Y = W = \partial_y$. Then we see that;

$$\begin{aligned}\mathfrak{R}(\partial_x, \partial_y, \partial_x, \partial_y)(\vec{x}) &= -g(\nabla_{[x,y]}x, y), \\ &= -g(0, y), \\ &= 0.\end{aligned}$$

So the metric curvature tensor is constant over the entire plane, and even vanishes, this leads us to;

$$\begin{aligned}\mathfrak{R}(\partial_x, \partial_y, \partial_x, \partial_y)(\vec{x}) = 0 &= K_{\mathbb{R}^2}(\vec{x})(\partial_x \wedge \partial_y) \otimes (\partial_x \wedge \partial_y)(\vec{x}), \\ &= K_{\mathbb{R}^2}(x)(\partial_x \wedge \partial_y)x \cdot (\partial_x \wedge \partial_y)y, \\ &= K_{\mathbb{R}^2}(\vec{x}).\end{aligned}$$

We remark that if we can find an orthogonal frame for our manifold, then $\mathfrak{R}$ will always just be equal to the scalar curvature function. So we see that the scalar curvature $K_{\mathbb{R}^2}(\vec{x})$ of the Euclidean plane $\mathbb{R}^2$ is constant, as we would have intuitively expected for non varying shape, and is 0 everywhere, which is on its turn in accordance with the intuitive thought that the plane is flat.

5.1.2 The Sphere

Now let us observe a less trivial Riemann surface; the unit sphere S^2. This actually is a compact Riemann surface without boundary, and the results found here for the sphere will actually satisfy the Gauss-Bonnet theorem. We will later on use the results found here for the sphere in the proof for the Gauss-Bonnet theorem.

The surface is globally non-trivial, so we cannot, as we did for the plane, use a single global section as a frame. Locally however we can. Let us start by looking at a parametrisation of $S^2 = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi))$. Using the spherical coordinates (θ, ϕ). When given a parametrisation for a surface $f : \mathbb{R}^2 \to \mathbb{R}^3$, we can find a metric g for this space via the Jacobian matrices J_f via; $g = J_f^T J_f$. The Jacobian for this specific parametrisation is given by;

$$J_{f_{S^2}} = \begin{pmatrix} -\sin(\theta)\cos(\phi) & -\cos(\theta)\cos(\phi) \\ \cos(\theta)\cos(\phi) & -\sin(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{pmatrix}$$

And so a metric on S^2 is given by;

$$g_{S^2} = J_{f_{S^2}}^T J_{f_{S^2}} = \begin{pmatrix} \cos^2(\phi) & 0 \\ 0 & 1 \end{pmatrix}$$

Remark that this is a diagonal matrix, so we have $g(\theta,\phi) = g(\phi,\theta) = 0$, as we would have expected for a orthogonal frame. And $g(\theta,\theta) = \cos^2(\phi)$ $g(\phi,\phi) = 1$, which shows that it is indeed not an orthonormal frame, as then the manifold would be trivializable, and we would just end up with the exact same curvature as the plane. Now recall the definition of the Levi-Civita connection with a given metric g. If we choose coordinate vector fields as a basis, then the Lie-brackets vanish, and so we have for coordinate vector fields X, Y, Z;

$$g(\nabla_X Y, Z) := \frac{1}{2}[L_X g(Y,Z) + L_Y g(X,Z) - L_Z g(X,Y)]$$

We can use this expression to find identities the Levi-Civita connection identites of $\nabla^{LC}_\theta \theta$, $\nabla^{LC}_\theta \phi$, $\nabla^{LC}_\phi \theta$, and $\nabla^{LC}_\phi \phi$. Recall that a covariant exterior derivative along a vector field was a map sending sections to section; $\nabla_i : \Gamma(TM) \to \Gamma(TM)$, as such, by property of lying in the same vector spaces, we can express the above identities as;

$$\begin{aligned}
\nabla^{LC}_\theta \theta &= f_{1,\theta,\theta}\theta + f_{2,\theta,\theta}\phi, \\
\nabla^{LC}_\theta \phi &= f_{1,\theta,\phi}\theta + f_{2,\theta,\phi}\phi, \\
\nabla^{LC}_\phi \theta &= f_{1,\phi,\theta}\theta + f_{2,\phi,\theta}\phi, \\
\nabla^{LC}_\phi \phi &= f_{1,\phi,\theta}\theta + f_{2,\phi,\theta}\phi.
\end{aligned}$$

Now finding these f is a matter of evaluating the Lev-Civita connection explicitly for each of the vector fields. Luckily the non-diagonal terms of the metric g vanish as we have an orthogonal frame. We will show how to compute $\nabla^{LC}_\theta \theta$, where in the expressions to come it may be cleear that when we speak of ∇, we speak specifically of ∇^{LC}.

$$\begin{aligned}
g(\nabla_\theta \theta, \theta) &= g(f_{1,\theta,\theta}\theta, \theta) + g(f_{2,\theta,\theta}\phi, \theta), \\
&= f_{1,\theta,\theta}\, g(\theta,\theta),
\end{aligned}$$

so re-arranging the terms gives us,

$$\begin{aligned}
f_{1,\theta,\theta} &= \frac{g(\nabla_\theta \theta, \theta)}{g(\theta,\theta)}, \\
&= \frac{L_\theta(\cos^2\phi)}{2\cos^2(\phi)}, \\
f_{1,\theta,\theta} &= 0
\end{aligned}$$

To find $f_{2,\theta,\theta}$ we find in a similar fashion;

$$\begin{aligned} f_{2,\theta,\theta} &= \frac{g(\nabla_\theta \theta, \phi)}{g(\phi,\phi)}, \\ &= L_\theta g(\theta,\phi) - \tfrac{1}{2} L_\phi g(\theta,\theta), \\ &= -\frac{1}{2} L_\phi \cos^2(\phi), \\ &= \sin(\phi)\cos(\phi). \end{aligned}$$

So we retrieve the expression;

$$\nabla^{LC}_\theta \theta = (\sin(\phi)\cos(\phi))\phi,$$

doing the same calculation for the other expressions yields;

$$\begin{aligned} \nabla^{LC}_\theta \phi &= -\frac{\sin(\phi)}{\cos(\phi)}\theta, \\ \nabla^{LC}_\phi \theta &= -\frac{\sin(\phi)}{\cos(\phi)}\theta, \\ \nabla^{LC}_\phi \phi &= 0 \end{aligned}$$

Now we are ready to evaluate $\mathfrak{R}(X,Y,Z,W)$. As before, we pick vector fields $X = Z = \frac{\partial_\theta}{\cos(\phi)}$ and $Y = W = \partial_\phi$. Then we find;

$$\begin{aligned} \mathfrak{R}(\tfrac{\partial_\theta}{\cos(\phi)}, \partial_\phi, \tfrac{\partial_\theta}{\cos(\phi)}, \partial_\phi)(\vec{x}), &= -\frac{1}{\cos^2(\phi)}\mathcal{R}(\partial_\theta, \partial_\phi, \partial_\theta, \partial_\phi)(\vec{x}), \\ &= -\frac{1}{\cos^2(\phi)} g(\nabla_{[\theta,\phi]}\theta, \phi), \\ &= -\frac{1}{\cos^2(\phi)} g(-\cos^2(\phi)\phi, \phi), \\ &= 1. \end{aligned}$$

So we see that the curvature tensor $\mathfrak{R}$ is a constant, equal to 1. Thus we find that;

$$1 = K_{S^2}(\vec{x}),$$

the scalar curvature function is also constant and equal to 1. Remark that if in the above calculation we had taken into account a constant radius r, not necessarily equal to 1, of the sphere. Then we would have found that $g_{S^2_r} = r^2 g_{S^2}$ which would have manifested itself by setting $K_{S^2}(\vec{x}) = \frac{1}{r^2}$. So the curvature would still be constant, but would be inversely proportional to r^2.

Lemma 5.1. *The scalar curvature of a sphere with radius r is constant and*

$$K_{S^2}(\vec{x}) = \frac{1}{r^2}$$

Again we see that the scalar curvature agrees with our intuition; the sphere is equally curved all over the surface, so we would indeed expect the curvature to be a constant. The inverse proportionality is a hit or miss regarding intuition, as it implies that smaller spheres are in a sense more curved than larger spheres. We can however agree on the fact that a sphere of infinite radius, should be akin to the plane, so we would indeed want that if $r \to \infty$, then $K(\vec{x}) \to 0$. A better way to imagine scalar curvature might not be via the lexical idea of curvature, but to what extent the surface does not curve like a unit sphere.

5.1.3 The Torus

So far the notion of scalar curvature has been very intuitive, and it almost seems useless that we devised such a large theory to just confirm what we already seemed to know about curvature. A shape however where the rigorous definitions of scalar curvature definitely benefits us in the calculation, as well as giving us a slightly counter-intuitive result, is for the torus. The torus, like the sphere, is also a compact Riemannian surface, and so too will use the results here in the proof of the Gauss-Bonnet theorem.

As with the sphere, we start by parametrizing the torus, in order to obtain a metric on the torus. Even though the torus can be seen as embedded in $\mathbb{R}^3$, this parametrization of the torus is not actually the easiest to calculate with. Lemma 4.3 says that every Riemannian manifold has a Levi-Civita connection, so we see that we are free to choose any parametrization of the torus. We will find out that the flat torus, a parametrization which cannot be embedded in $\mathbb{R}^3$, is very easy to calculate with. The flat torus is parametrized by;

$$\mathbb{T} = (\cos(u), \sin(v), \cos(u), \sin(v)),$$

it is obtained by the well known glueing shown in figure 4. The parametrization gives rise to a Jacobi matrix.

$$J_{\mathbb{T}} = \begin{pmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & -\sin(v) \\ 0 & \cos(v) \end{pmatrix}$$

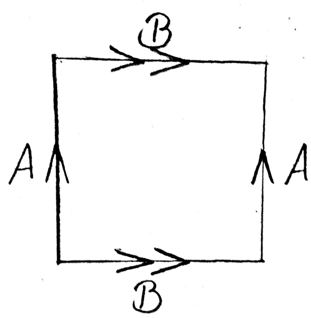


Figure 4: The Flat Torus by Glueing

Now we see something very satisfying happening, the metric given by $g = J^T J$ actually reduces to the flat metric, i.e.:

$$g_{\mathbb{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the metric on the flat torus, is actually the same as the metric for the plane. And so the calculation simplify as, like with the flat plane, all the $\nabla_i^{LC} j$ vanish. Thus giving us;

$$\mathfrak{R}(\partial_u, \partial_v, \partial_u, \partial_v)(\vec{x}) = 0 = K_{\mathbb{T}}(\vec{x}).$$

So this is where the scalar curvature function might not actually agree with our intuition. Even though we probably agree that, when looking just at the glueing, the flat torus has constant curvature, and is indeed flat; as subtly implied by the name. When looking at the embedding of the torus in $\mathbb{R}^3$, we would intuitively say that the curvature is not constant, we would actually expect the curvature to vary in some sense, depending on wether we are on the inside or outside radius of the torus. Now it is good to remark that this is, in fact, true. If we take that embedding of the torus, we will indeed find that the scalar curvature is not a constant, and as we would expect it is positive on the outside radius, and negative on the inside radius. However, this curvature is much harder to calculate; *because* it is not a constant. It is true that by using the flat torus, we in some sense have lost some information. But one could also argue that the embedding of the torus obfuscates a property of the torus, namely that it's curvature in a sense cancels out over the entire surface, which is actually vital for the Gauss-Bonnet theorem over surfaces.

Lemma 5.2. *The scalar curvature of the Torus is constant and;*

$$K_{\mathbb{T}}(\vec{x}) = 0.$$

5.2 Euler Class and Scalar Curvature

Now we have an actual mathematical definition of what the curvature at a point is. Let us try and relate this to the earlier defined Euler class of a vector bundle. Recall that the Euler class was defined for a vector bundle of rank $2k$, as the equivalence class of $e = [\frac{1}{k!}\langle \omega^k, \sigma^* \rangle]$, where we changed to σ^* to keep more in line with the notation used in the previous sections. Now for a 2-dimensional Riemannian surface, we tangent bundle is a vector bundle of rank 2. So it simplifies to the equivalence class of just $e[\langle \omega, \sigma \rangle]$. So what are ω and σ when interpreted in the context of a Riemann surface? Letting sections of the vector bundle be vector fields, the connection chosen the Levi-Civita connection, then we can now interpret $\omega(X,Y) = g(\kappa^{LC}_{\nabla} X, Y)$, and $\omega \in \Omega^2(\Sigma, \bigwedge^2 T^*M)$. Now observe that we could construct a global section which locally was expressed as $\sigma^* = s_1^* \wedge s_2^*$ where (s_1^*, s_2^*) is the dual frame for the orthonormal frame (s_1, s_2). So the pairing becomes;

$$\begin{aligned}\langle \omega, \sigma^* \rangle(\vec{x}) &= \omega(\sigma^*)(\vec{x}), \\ &= g(\kappa^{LC}_{\nabla} s_1^*(\vec{x}), s_2^*(\vec{x}))\end{aligned}$$

Now let us return again to the curvature tensor $\mathfrak{R}$, which evaluated at a point x became $\mathfrak{R}(X,Y,Z,W)\vec{x} = K(\vec{x})\sigma^*(X,Y)\sigma^*(Z,W)$. Now note that is we set $(X,Y) = \sigma$, then because of orthonormality it reduces to; $\mathfrak{R}(\sigma, Z, W)(\vec{x}) = K(\vec{x})\sigma^*(Z,W)$. We now see that pointwise $\langle \omega, \sigma^* \rangle(\vec{x}) = K(\vec{x})\sigma^*$. And hence for compact orientable Riemann surfaces we obtain;

Lemma 5.3. *Let Σ a compact orientalbe Riemann surface with vector bundel $T\Sigma$, then we have that;*

$$\int_{\Sigma} \mathcal{E}(T\Sigma) = \int_{\Sigma} K(\vec{x})\sigma^*.$$

We now finally have all the tools to prove the Gauss-Bonnet theorem, for compact orientable Riemann Surfaces.

6 The Gauss-Bonnet Theorem

The Gauss-Bonnet theorem for surfaces is already very similar to what we have shown in lemma 5.3. The exact statement is;

Theorem 6.1. *Given a compact 2-dimensional orientable Riemannian manifold without boundary* Σ*. Then;*

$$\int_{\Sigma} K(\vec{x})\sigma^* = 2\pi\chi(\Sigma). \tag{4}$$

Where $K(\vec{x})$ *is the scalar curvature at a point* $\vec{x}$ *and*$\chi(\Sigma)$ *is the Euler characteristic of the manifold* Σ*.*

Proof. The proof follows from induction, from the cases we have already done, namely the unit sphere, shown in lemma 5.1, to have constant scalar curvature $+1$, and the torus $\mathbb{T}$, shown in lemma 5.2 to have constant curvature 0. Because the curvature is constant, and we have orthonormal frames (u, v) and θ, ϕ. We get that σ^*, the volume form signifying the area dA, become $d\phi \wedge d\theta$ and $du \wedge dv$, the integral becomes just the surface area multiplied by a constant, this gives us;

$$\int_{S^2} 1 \; d\phi \wedge d\theta = 4\pi,$$
$$\int_{\mathbb{T}} 0 \; du \wedge dv = 0.$$

Now for the Euler characteristic χ for an orientable surface, we know that it is given by $\chi(\Sigma) = 2 - 2g$, where g is the genus of the surface. The lowest genus an orientable surface can have is 0, corresponding to the sphere, or any sphere-like manifold. Their is no upper limit to the genus an orientable manifold can have. For the sphere and torus we then see that $\chi(S^2) = 2$ and $\chi(\mathbb{T}) = 0$. Relating this to the above, we already see that the Gauss-Bonnet theorem holds for orientable Riemann surfaces of genus 0 and 1;

$$\int_{S^2} 1 \; d\phi \; d\theta = 2\pi\chi(S^2) = 4\pi,$$
$$\int_{\mathbb{T}} 0 \; du \; dv = 2\pi\chi(\mathbb{T}) = 0.$$

Now we start the induction step. Let us assume that the Gauss-Bonnet theorem holds for a genus g orientable manifold. Let us observe the connected sum of this genus g surface σ_g, and another torus. That is, let $D_1 : D^1 \hookrightarrow \sigma_g$

be a smooth embedding of the unit disc into Σ_g, and let $D_2 : D^1 \hookrightarrow \mathbb{T}$ be a smooth embedding of the unit disc into $\mathbb{T}$. Then with the connected smash sum $\Sigma_g \# \mathbb{T}$ we mean the new manifold $\Sigma_g \# \mathbb{T} = (\Sigma_g \sqcup \mathbb{T})/\sim$, where $\sim$ is the equivalence relation $\partial(D_1) \sim \partial(D_2)$. Now one has smoothly combined the genus g surface σ_g and the torus $\mathbb{T}$. We can visualize this operation as intersecting the torus and the genus g surface. Now because integration is linear over it's limits, we can see the integration of this entire new surface, as the sum of the surface integral of the $g+1$ surface summed with the integral of the unit sphere, that is;

$$\int_{\Sigma_g \# S^2} K(\vec{x})\sigma^* = \int_{\Sigma_{g+1}} K(\vec{x})\sigma^* + \int_{S^2} K(\vec{x})(\sigma^*).$$

A visualization of this process can be seen in figure 5.

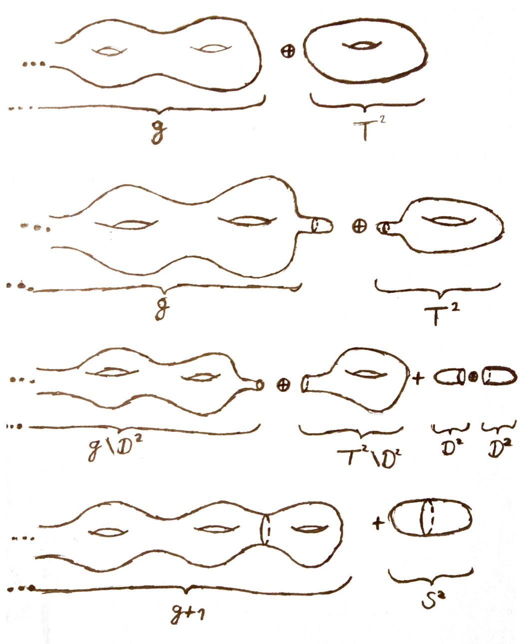


Figure 5: Induction on g

We now work out;

$$\begin{aligned}
\int_{\mathbb{T}} K(\vec{x})\sigma^* + \int_{\Sigma_g} K(\vec{x})\sigma^* &= \int_{\Sigma_g \# S^2} K(\vec{x})\sigma^*, \\
0 + 2\pi(2-2g) &= \int_{\Sigma_{g+1}} K(\vec{x})\sigma^* + \int_{S^2} K(\vec{x})\sigma^*, \\
2\pi(2-2g) &= \int_{\Sigma_{g+1}} K(\vec{x})\sigma^* + 4\pi, \\
2\pi(2-2g-2) &= \int_{\Sigma_{g+1}} K(\vec{x})\sigma^*, \\
2\pi(2-2(g+1)) &= \int_{\Sigma_{g+1}} K(\vec{x})\sigma^*, \\
2\pi\chi(\Sigma_{g+1}) &= \int_{\Sigma_{g+1}} K(\vec{x})\sigma^*.
\end{aligned}$$

And so by induction on $g = 1$, we see that it must be true for any orientable compact Riemann surface without boundary, and thus we find;

$$\int_{\Sigma} K(\vec{x})\sigma^* = 2\pi\chi(\Sigma). \tag{5}$$

Thus proving the Gauss-Bonnet theorem for surfaces. □

7 Conclusion

The beauty of the Gauss-Bonnet theorem lies in the fact that it relates a local property to a global invariant of the surface. One might forget that the scalar curvature is a local property of a point, even though we found and worked on surfaces which had a constant, and so in some sense global, curvature. Most surfaces however do not have constant curvature, but still satisfy the Gauss-Bonnet theorem. So a change locally of the curvature of a

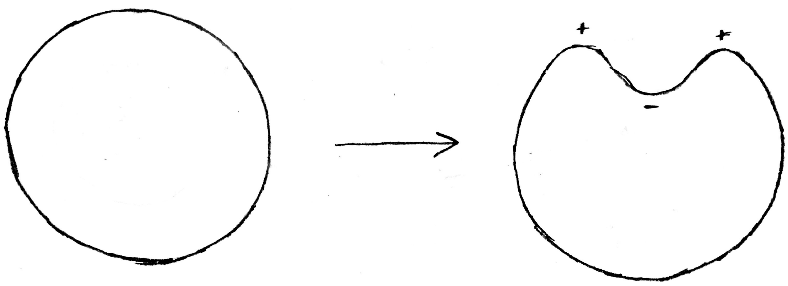


Figure 6: Local deformations are compensated

manifold, must be in some sense compensated by the rest of the surface, by changing the curvature at other points of the manifold, as such we can not change the curvature at a single point alone and keep the rest of the manifold unchanged.

Let us think back of the skippy-ball, the flattening of the top and bottom, thus decreasing the curvature there, was compensated by the increase of curvature at the sides. Trying to puncture the skippyball deforms the skippy ball as in the cross section of it in figure 6. It gives us a notion that surfaces, though in topology often deformed very freely, are still very much connected objects.

References

[1] Crainic, Marius. *Mastermath Course Differential Geometry*, Utrecht University, 2015.

[2] Hatcher, Allen. *Algebraic Topology*, Cambridge University Press, 2001.

[3] Lee, John M. *Riemannian Manifolds*, Springer-Verlag New York, 1997.

[4] Lee, John M.*Introduction to Smooth Manifolds.*, Springer Science+Business Media New York, 2013. 2nd Edition.

[5] Loring, T. Wu. *Differential Geometry*, Springer International Publishing, 2017.

[6] Madsen, Ib.; Tornehave Jørgen. *From Calculus to Cohomology*, Cambridge University Press, 1997.

[7] Milnor, John W.; Stasheff, James D. Characteristic Classes, Princeton University Press and Tokyo University Press, 1974.

[8] Petersen, Peter. *Riemannian Geometry*, Springer International Publishing, 2016. 3rd edition.