Complex Geometry – Exam 2

Questions

Exercise 1 ($D^4 \neq D^2 \times D^2$)**.** In this exercise we will prove that the unit open ball $B \subset \mathbb{C}^2$ is not biholomorphic to the product of two unit discs. For this exercise you can use without proof that any biholomorphism of the disc is of the form

$$\phi\colon D_1 \subset \mathbb{C} \to D_1, \qquad \phi(z) = e^{i\theta}\frac{\alpha - z}{1 - \bar{\alpha} z},$$

for some real constant θ and a complex number α with $|\alpha| < 1$.

1. Show that the restriction of any element of $U(2)$ to B gives a biholomorphism of B. Conclude that the group of biholomorphisms of B is not Abelian.
2. Show that for every $z \in D_{(1,1)}$ there is a biholomorphism $f\colon D_{(1,1)} \to D_{(1,1)}$ with $f(z) = 0$.
3. Show that a biholomorphism of $D_{(1,1)}$ which fixes the origin is made of two biholomorphisms of D_1 acting independently on each D_1 factor by rotations.
4. Conclude that the group of biholomorphisms of $D_{(1,1)}$ is different from the group of biholomorphisms of B.

Exercise 2.

1. Let V be a vector space and $\sigma \in \wedge^2 V$ be a 2-form. Show that if $\sigma^2 = 0$ then σ is decomposable, that is, there are $\alpha, \beta \in V$ such that $\sigma = \alpha \wedge \beta$.
2. Let M be a real four-dimensional manifold and let $\sigma \subset \Omega^2(M; \mathbb{C})$ be 2-form such that $\sigma \wedge \sigma = 0$ and $\sigma \wedge \bar{\sigma}$ is everywhere non-zero. Show that there exists a unique almost complex structure I on M such that $\sigma \in \Omega^{2,0}(M)$.
3. In the same situation of the previous exercise, show that if σ is closed, then the complex structure is integrable and σ is a holomorphic two-form on (M, I)

Exercise 3.

1. Show that for a polydisc $B \subset \mathbb{C}^n$ the sequence

   $$\Omega^{p-1,q-1}(B) \xrightarrow{\partial\overline{\partial}} \Omega^{p,q}(B) \xrightarrow{d} \Omega^{p+q+1}(B)$$

   is exact.
2. Let X be a complex manifold and consider the sequence of differential operators:

   $$\Omega^{p-1,q-1}(X) \overset{\partial\overline{\partial}}{\to} \Omega^{p,q}(X) \overset{\partial + \overline{\partial}}{\to} \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(X).$$

   Show that this is a complex of differential operators and that the corresponding symbol sequence is exact.

3. Let X be a complex manifold. Verify that the following definition of the Bott-Chern cohomology

$$H^{p,q}_{BC}(X) := \frac{\ker\{d : \Omega^{p,q}(X) \to \Omega^{p+q+1}(X)\}}{\mathrm{Im}\,(\partial\overline{\partial} : \Omega^{p-1,q-1}(X) \to \Omega^{p,q}(X))}$$

makes sense. Deduce that $H^{p,q}_{BC}(B) = 0$ for a polydisc $B \in \mathbb{C}^n$ and $p, q \geq 1$. Show that there are natural maps

$$H^{p,q}_{BC}(X) \longrightarrow H^{p,q}(X) \qquad H^{p,q}_{BC}(X) \longrightarrow H^{p+q}(X)$$

Exercise 4. In a complex manifold we define the Bott–Chern cohomology by

$$H^{p,q}_{BC} = \frac{\ker(d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M))}{\mathrm{Im}\,(\partial\overline{\partial} : \Omega^{p-1,q-1}(M) \to \Omega^{p,q}(M))}$$

1. Show that is a compact Kähler manifold the Bott–Chern cohomology $H^{p,q}_{BC}(M)$ is isomorphic to the Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(M)$
2. Let X be a compact Kähler manifold. Show that for two cohomologous Kähler forms ω and ω', i.e. $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$, there exists a real function f such that

   $$\omega = \omega' + i\partial\overline{\partial} f.$$

   You can use without proof the results from Exercise 3 above.

Exercise 5. Show that a complex submanifold of a Kähler manifold is Kähler. By considering linear subspaces of $\mathbb{C}^n$ or otherwise give an example of a symplectic submanifold of a Kähler manifold which is not a complex submanifold.

Exercise 6 (Calibrations)**.** A *k-calibration* in a Riemannian manifold (M, g) is a closed k-form ρ such that for all $p \in M$ and all k-dimensional oriented subspaces $V \subset T_pM$ we have $\rho \leq \sigma_V$, where σ_V denotes volume form induce by the metric g on V and the inequality makes sense because the orientation of V determines a positive direction on the real line $\wedge^k V$.

A k-dimensional oriented submanifold, N, of a manifold with calibration, (M, g, ρ), is *calibrated* if for every $p \in N$ $\rho|_{T_pN}$ agrees with the volume form determined by the restriction of the metric to N.

1. Show that if $N \hookrightarrow (M, g, \rho)$ is a calibrated submanifold, then the k-volume of any deformation of N is bigger than or equal to the volume of N.
2. Let (M, I, g) be a Kähler manifold and let $\omega = g(\cdot, I\cdot)$ be the associated symplectic form. Show that for all i ω^i is a calibration in (M, g).
3. Show that if $N \hookrightarrow (M, g, I)$ is a complex manifold of a Kähler manifold, then N is calibrated with respect to the appropriate power of ω.
4. Show that if $N \hookrightarrow (M, g, I)$ is calibrated with respect to ω^i, then N is a complex submanifold.

Exercise 7.

1. Let L be a holomorphic line bundle on a compact complex manifold X. Show that L is trivial if and only if L and its dual L^* admit non-trivial global sections. (Hint: Use the non-trivial sections to construct a non trivial section of $\mathcal{O}\ L \otimes L^*$.)
2. Let L_1 and L_2 be two holomorphic line bundles on a complex manifold X. Suppose that $Y \subset X$ is a submanifold of codimension at least two such that L_1 and L_2 are isomorphic on $X \backslash Y$. Prove that $L_1 \cong L_2$.

Exercise 8. Show that any holomorphic map from $\mathbb{C}P^1$ into a Riemann surface of genus $g > 0$ is constant. What about maps from $\mathbb{C}P^n$ into Σ_g?

Exercise 9 (Blow-up of Kähler manifolds)**.** Let (M, I, g, ω) be a Kähler manifold and $p \in M$. Following the steps below or otherwise, show that the blow-up of M at p admits a Kähler metric.

1. Let $r^2 = \sum_i z_i \bar{z}_i$ be the square distance to the origin in $\mathbb{C}^{n+1}$, let $\pi\colon \widetilde{\mathbb{C}^{n+1}} \to \mathbb{C}^{n+1}$ be the blow-down map and let $\sigma = -\frac{1}{2i}\partial\overline{\partial}\log(\pi^* r^2)$. Show that

a) for all $X \in T\widetilde{\mathbb{C}^{n+1}}$, $\sigma(X, IX) \geq 0$ and

b) for $X \neq 0$, X tangent to E, the exceptional divisor, $\sigma(X, IX) > 0$.

2. Show that there is a closed 2-form $\tilde{\sigma}$ which is equal to σ in an arbitrary neighbourhood U of E and has support in a slightly bigger neighbourhood $V \supset U$ of E.

3. On $\widetilde{M}$, the blow-up of M at p, with projection $\pi_M\colon \widetilde{M} \to M$, show that the form

$$\tilde{\omega} = \pi_M^* \omega + \epsilon\tilde{\sigma}$$

is symplectic and compatible with the complex structure for ε small. Conclude that $\widetilde{M}$ is Kähler

Exercise 10 (From sections of line bundles to maps)**.** Let $L \to M$ be a holomorphic line bundle over M, let $s_1, \cdots, s_{k+1}\colon M \to L$ be holomorphic sections and let B be the *base locus* of the sections $s_1, \cdots, s_{k+1}$, that is,

$$B = \{p \in M\colon s_i(p) = 0 \quad \forall i\}.$$

1. For $U_\alpha \subset M \backslash B$ an open set over which L is trivial, pick a trivialization and let s_i^α be the function corresponding to the section s_i. Show that the map

$$f\colon U_\alpha \to \mathbb{C}P^k, f(p) = [s_1^\alpha(z), \dots, s_{k+1}^\alpha]$$

is independent of the chosen trivialization. Conclude that the sections $s_1, \cdots, s_{k+1}$ give rise to a map

$$f\colon M\backslash B \to \mathbb{C}P^k$$

given in locally be the expression above.

2. Assume that the sections $s_1, \cdots, s_{k+1}$ have transverse zeros, that is, in a neighbourhood of each point $p \in B$ and for any trivializations of L in that neighbourhood, the functions $\{s_i^\alpha\}_{i=1,\dots,k+1}$ can be complemented to a coordinate chart:

$$(z_1 = s_1^\alpha, \dots, z_{k+1} = s_{k+1}^\alpha, z_{k+2}, \dots, z_n).$$

Show that B is a complex submanifold of M.

3. Letting $\widetilde{M}$ be the blow-up of M along B, show that f from the first step extends as a holomorphic map

$$\tilde{f}\colon \widetilde{M} \to \mathbb{C}P^k.$$

Side remark: this exercise is typically used in one of two ways. We can either take only two sections, blow up the base locus and obtain something that resembles a fibration $f\colon M \to \mathbb{C}P^1$ or pick so many independent sections that M embeds on $\mathbb{C}P^k$.

Exercise 11. Let $g > 0$ and let Σ_g be the Riemann surface of genus g.

1. Show that Σ_g is not biholomorphic to $\mathbb{C}P^1$ for $g > 0$.

2. For $p, q \in \Sigma_g$, $p \neq q$, using the previous exercise or otherwise, show that the holomorphic line bundle $L_p \otimes L_q^*$ over Σ_g is nontrivial (as a holomorphic vector bundle)

3. Show that $L_p \otimes L_q^*$ has zero first Chern class and hence is trivial as a complex vector bundle.

Exercise 12. Let Σ_g be a Riemann surface of genus g.

1. For $g > 0$, compute $h^0(L)$ and $h^1(L)$ for L a line bundle with negative degree (that is the integral of the first Chern class over Σ_g is negative).

2. For $g = 0$, Compute $h^0(L)$ and $h^1(L)$ for L a line bundle with negative degree.

3. Show that if the degree of L is big enough, then $h^0(L) \neq 0$, that is, L has holomorphic sections.

4. Compute $h^0(T\Sigma_g)$ and $h^1(T\Sigma_g)$. Note that you may need to consider separately the cases $g = 0$, $g = 1$ and $g > 1$.

Exercise 13. Let $: L \to M$ be a Hermitian line bundle over M and let $S_L \to M$ be the sphere bundle of L, that is,

$$S_L = \{v \in L \colon \|v\|^2 = 1\},$$

where the norm of v is computed using the fiberwise Hermitian metric. Pick a cover $\mathcal{U}$ for M and unitary trivializations of L over the elements of $\mathcal{U}$ (that is, the local frames are length one sections).

1. Let $\{g^\alpha_\beta \colon U_{\alpha\beta} \to \mathbb{C}\}_{\alpha,\beta\in A}$ be the transition functions for the chosen trivializations. Show that $\|g^\alpha_\beta\| = 1$ and conclude that if $U_{\alpha\beta}$ are simply connected there are functions $\theta^\alpha_\beta \colon U_{\alpha\beta} \to \mathbb{R}$ such that $g^\alpha_\beta = e^{i\theta^\alpha_\beta}$.

2. Let ∇ be a metric connection on L. In the local trivializations above, $\nabla = d + iA_\alpha$. Show that $A_\alpha \colon U_\alpha \to \mathbb{R}$ and $A_\alpha - A_\beta = d\theta^\alpha_\beta$.

3. Using the trivialization of L over U_α parametrize $SL|_{U_\alpha} = U_\alpha \times S^1$ and define a 1-form on $SL|_{U_\alpha}$ by $\phi_\alpha = d\theta + A_\alpha$, where θ is the 'coordinate' on the circle. Show that on the overlap $\phi_\alpha = \phi_\beta$ and hence there is a global 1-form ϕ on SL which equals ϕ_α on $SL|_{U_\alpha}$.

4. Show that the form ϕ above satisfies

$$\int_{S^1} \phi = 2\pi; \quad d\phi = F_\nabla.$$

5. For $i = 1, 2$, let $\pi_i \colon L_i \to M_i$ be Hermitian line bundles with connection over complex manifolds. Let ϕ_i be the form in L_i constructed in the previous part. Define an almost complex structure on $SL_1 \times SL_2$ by declaring that its canonical bundle is

$$K = (\phi_1 + i\phi_2) \wedge \pi_1^* K_{M_1} \wedge \pi_2^* K_{M_2}.$$

   Show that this indeed defines an almost complex structure. Show that this structure is integrable.

6. Conclude that $S^{2k+1} \times S^{2l+1}$ is a complex manifold for $k, l \in \mathbb{N}$.

7. For which values of k and l are these manifolds Kähler?

Exercise 14. Let M^{2n} be a compact complex manifold of complex dimension n and $E \to M$ be a holomorphic vector bundle over M. Show that there is a nondegenerate pairing

$$H^q(M; E \otimes \wedge^{p,0} T^*M) \times H^{n-q}(M; E^* \otimes \wedge^{n-p,0} T^*M) \to \mathbb{C}$$

Conclude that these spaces have the same dimension.

Exercise 15. Show that holomorphic forms, i.e., elements of $H^0(X, \Omega^p)$, on a compact Kähler manifold X are harmonic with respect to any Kähler metric.