Complex Geometry – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.0 pt)**.** Let $n > 1$ and let $f\colon \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Show that for every $w \in \mathrm{Im}(f)$ there is $z \in f^{-1}(w)$ with $\|z\| > 0$.

Exercise 2 (2.0 pt)**.** Let $f\colon M^n \to N^n$ be a map between compact connected oriented manifolds of the same dimension. The degree of f is defined by

$$\deg(f) = \frac{\int_M f^*\rho}{\int_N \rho},$$

where ρ is any top degree form on N with $\int_N \rho \neq 0$.

- Show that $\deg(f)$ does not depend on the form ρ (hint observe that the computation defining the degree takes place in cohomology and use that for compact, connected, orientable manifolds $H^{top}(M;\mathbb{R}) = \mathbb{R}$).
- Let $p \in N$ be a regular value of f, so that p has a neighbourhood U for which $f^{-1}(U) = \cup_{i=1}^k U_i$ and $f\colon U_i \to U$ is a diffeomorphism for all i. Pick $\rho \in \Omega^n(N)$ a top degree form supported in U and show that

$$\deg(f) = \sum_{i=1}^{k} \epsilon(p_i),$$

  where $\epsilon(p_i) = 1$ if $df_{p_i}\colon Tp_iM \to T_pN$ is orientation preserving and $\epsilon(p_i) = -1$ otherwise.
- Show that if M and N are complex manifolds and f is holomorphic, then $\deg(f)$ is non-negative and if f has invertible derivative at a point, then $\deg(f) > 0$ and equals the number of points in the pre-image of a regular value.

Exercise 3 (2.0 pt)**.** Let Σ be a compact Riemann surface and let $p \in \Sigma$. Consider the complex line bundle $\pi\colon L_p \to \Sigma$ which has two trivializations, one over $\Sigma\backslash\{p\}$ and one over a disc, D, centered at p,

$$\Phi_0\colon L_p|_{\Sigma\backslash\{p\}} \to \Sigma\backslash\{p\} \times \mathbb{C}, \qquad \Phi_1\colon L_p|_D \to D \times \mathbb{C},$$

with transition function

$$\Phi_1 \circ \Phi_0^{-1}\colon D\backslash\{0\} \times \mathbb{C} \to D\backslash\{0\} \times \mathbb{C}, \qquad \Phi_1 \circ \Phi_0^{-1}(z, v) = (z, zv),$$

where z denotes the coordinate on the disc.

- Show that L_p has a section which vanishes at p.
- Show that L_p is not isomorphic to the trivial bundle.
- Let $p_1 \neq p_2$. Show that $L_{p_1} \otimes L_{p_2}^{-1}$ has a section with a zero at p_1 and a simple pole at p_2.
- Let $p_1 \neq p_2$. Show that if $L_{p_1} \otimes L_{p_2}^{-1}$ is isomorphic to the trivial bundle then $\Sigma = \mathbb{C}P^1$.

Exercise 4 (2.0 pt)**.** Let (M^{2n}, I, g) be an almost complex manifold with compatible Riemannian metric.

- Show that the two form defined by $\omega(X, Y) = g(X, IY)$ is of type (1,1).
- Let $\Omega \in \Omega^{n,0}(M)$ be a nonvanishing local section of the canonical bundle of M. Show that if $d\omega \wedge \Omega \neq 0$ then I is not integrable.
- Show that the almost complex structure on S^6 determined by octonionic multiplication is compatible with the Euclidean metric of $\mathbb{R}^7$ restricted to the sphere.
- Show that the almost complex structure of S^6 is not integrable.

Exercise 5 (2.0 pt)**.** Let $\mathbb{Z}_2 = \langle a\colon a^2 = e\rangle$ act on $\mathbb{C}^2$ by $a \cdot (z_1, z_2) = -(z_1, z_2)$.

- Resolve the singularity of $\mathbb{C}^2/\mathbb{Z}_2$,
- Let $\Omega = dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^2)$. Show that Ω extends as a nonvanishing section of the canonical bundle of most natural resolution $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$.

Let $\mathbb{Z}^2$ act on $\mathbb{C}^2$ by $(m, n) \cdot (z_1, z_2) = (z_1 + m, z_2 + n)$.

- Show that $\Omega = dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^2)$ gives rise to a nonvanishing section of the canonical bundle of the torus $T^4 = \mathbb{C}^2/\mathbb{Z}^2$,
- Show that the $\mathbb{Z}_2$ action of the first part of this exercise gives rise to a $\mathbb{Z}_2$ action on T^4 and compute its fixed points.
- Show that the canonical bundle of the natural resolution of the singularities of $T^4/\mathbb{Z}_2$ has a nowhere vanishing holomorphic section.