A survey of Drinfeld modular forms

25.03.97

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The possibility of a theory of modular forms for function fields was perhaps implicit in Drinfeld's original work, but the notion was only really introduced by Goss and Gekeler in the beginning of the eighties. A lot of it developped along "classical" lines (by the word "classical" we will always refer to modular forms for subgroups of $SL(2, \mathbf{Z})$), but there are also a few very intriguing novel facts to the function field side, and on these we want to concentrate. People who want to compare with the theory of classical modular forms are warmly refered to the well known classical literature.

One can see this paper as a kind of (very personal) filter applied to the original works of Gekeler, Goss and Teitelbaum. It should serve as an introduction to (and motivation for) reading their original papers in the reference list, which inevitably contain much more information then can be presented here.

In this paper, most interest will go to the "analytic" theory of modular forms and curves, and thus I have neglected parts of the intriguing theory of "algebraic" modular forms, developped in particular by Goss in 1980. For compensation, one should at least glimpse at his papers.

Most novelties over function fields occur in connection with (a) the consideration of an arbitrary "ground ring" A, which already makes the moduli space without level structure look complicated; (b) the characteristic-p situation, which causes loss of information in taking derivatives of objects; (c) the analytic reduction of the upper half space to a combinatorial object, namely the Bruhat-Tits tree, and the interpretation of objects "down there"; (d) the fact that "modular forms" are only half of the story, the other side being complex valued Jacquet-Langlands like automorphic forms – and the interaction between these two notions.

The organisation of the paper will be as follows: in the first paragraph I will show how modular forms arise naturally in the search for a genus formula for the moduli space without level for $GL(2, A)$, and a few modular forms are introduced. The second paragraph contains a sketch of the explicit calculation of these genera. In the third part, function fields of modular curves are described by means of modular forms, and the fourth paragraph calculates the Hilbert function and presentation of a few rings of modular forms. In paragraph five, we present a few highlights in the theory of modular forms modulo primes and its connection with supersingular invariants. Paragraph six sketches the interpretation of modular forms on the tree via Teitelbaum's residue-map, and its application to modular forms of low weight. In paragraph seven, the absolute value of some modular forms is studied as a complex-valued object on the tree.

Needless to say, I do not claim originality in this survey, except for minor generalisations here and there, in particular in (1.5.3), (4.3) and (4.6.2).

List of notations:
$\mathbf{F}_q$ = finite field of characteristic $\mathrm{char}(\mathbf{F}_q) = p$
K = function field of a complete connected curve $\mathcal{C}$ over $\mathbf{F}_q$
$Z_K(S) = \zeta_K(s)$ = the zeta-function of K, ($S = q^{-s}$) and $P(S)$: the numerator of $Z_K(S)$; in particular, $P(1)$ = the class number of K.

[1] supported as "aspirant" by the Belgian National Fund for Scientific Research (NFWO)

∞ = a fixed place of K, of degree δ, with associated normalised absolute value $|\cdot|$
K_∞ = completion of K w.r.t. $|\cdot|$, and π = a uniformizing element for $|\cdot|$
$\mathcal{O}_\infty$ = the ring of integers in K_∞
C = completion of an algebraic closure of K_∞
$\Omega = C - K_\infty$ = Drinfeld's "half" plane
A = the ring of elements of A that are regular outside ∞, of class number $\delta \cdot P(1)$
$GL(2, A)$ = the invertible 2×2 matrices over A
$\mathcal{Z}$ = the center of $GL(2, A)$
$\mathfrak{n}, p, ...$ (Fraktur letters) = places of A, of degree $\deg(\mathfrak{n})$, where $\mathfrak{p}$ = prime place
$\Gamma(\mathfrak{n}) = \{\gamma \in GL(2, A) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n}\}$
$\Gamma_0(\mathfrak{n}) = \{\gamma \in GL(2, A) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{n}\}$
$\Gamma \subseteq GL(2, A)$ = an arithmetic group (*i.e.* it contains $\Gamma(\mathfrak{n})$ for some $\mathfrak{n}$)
$\gamma \cdot z = \frac{az+b}{cz+d}$ = the action of an element $\gamma \in GL(2, A)$ on $z \in \Omega$ via fractional linear transformation
$M_\Gamma = \Gamma \backslash \Omega$ the quotient space for this action
$\bar{M}_\Gamma = \Gamma \backslash \Omega \cup \mathbf{P}^1(K)$ = the compactification of M_Γ
$\bar{M}_\Gamma - M_\Gamma = \Gamma \backslash \mathbf{P}^1(K)$ the cusps of Γ
$\infty = (1 : 0) \in \mathbf{P}^1(K)$ the cusp at infinity
$\mathrm{Stab}_G(x)$ = the stabiliser subgroup of x in G
ϕ = an A-Drinfeld module of rank two over C, given by a map $\phi : A \to C\{\tau\}$, where $C\{\tau\}$ is the "twisted" polynomial ring with multiplication rule $\tau x = x^q \tau$.
${}_{\mathfrak{n}}\phi$ = the $\mathfrak{n}$-torsion of ϕ
ρ = the Carlitz module = the rank one Drinfeld module over $\mathbf{F}_q[T]$ goven by $\rho_T = T + \tau$, and $\bar{\pi}$ = it's period, i.e. ρ corresponds to the lattice $\bar{\pi}\mathbf{F}_q[T]$.
$g(X)$ = the geometric genus of a projective curve X
$\mathrm{Div}(X)$ = the group of divisors on a projective curve X, $\deg D$ = the degree of a divisor D
$\mathcal{D}$ = the canonical bundle on a fixed curve
$\mathcal{O}(D)$ = the line bundle associated with a divisor D on a curve
dim = the dimension of a $C-$vector space
$[\cdot]$ = the integer part, and $\langle\cdot\rangle$ = the fractional part of a real number

1. From modular curves to modular forms

(1.1) Since Riemann, understanding algebraic curves has become almost synonymous with the study of meromorphic functions and regular differentials (say, tangents) on them; for instance the number of independent differentials on a curve equals its fundamental invariant, the genus.
The same general philosophy should apply to the Drinfeld modular curves $\bar{M}_\Gamma$ associated with arithmetic groups $\Gamma \subseteq GL(2, A)$. The modular curves are (compactified) quotients of the Drinfeld "half" space Ω, so it seems only natural to look for functions and differentials "up" on Ω, let them descend on $\Gamma \backslash \Omega$ and extend them to the compactification.

(1.2) To make this descending procedure possible, we will first indicate how the analytic structures of Ω and $\bar{M}_\Gamma$ are related. There are two kinds of special points on $\bar{M}_\Gamma$ where the analytic structure ramifies — the local parameter at all other points is just $z \in \Omega$ itself.

(1.2.1) First of all, the *cusps* $\bar{M}_\Gamma - M_\Gamma$. Let $s \in \Gamma\backslash\mathbf{P}^1(K)$ be a cusp, and $\nu \in GL(2,K)$ such that $s = \nu \cdot \infty$. The stabilizer $\mathrm{Stab}_{\nu^{-1}\Gamma\nu}(\infty)$ contains a maximal subgroup of matrices $\begin{pmatrix} 1 & \mathrm{b} \\ 0 & 1 \end{pmatrix}$, where b runs through some fractional ideal $\mathfrak{b} \subseteq A$. Then we set $t_s(z) = \bar{\pi}^{-1} e_{\mathfrak{b}}^{-1}(z)$. (Note : The normalising factor $\bar{\pi}$ is introduced for rationality questions of expansions, and this is not done everywhere in the literature). The *genuine* analytic parameter on $\bar{M}_\Gamma$ can be a strict power of t_s: we set w_s = the order of the group of homotheties $z \mapsto az$ contained in $\mathrm{Stab}_{\nu^{-1}\Gamma\nu}(\infty)$, then $t_s^{w_s}$ is the local analytic parameter on $\bar{M}_\Gamma$. For instance, if $\Gamma = GL(2,A)$, then $w_s = q-1$ for all cusps s. For $\Gamma = GL(2,A)$, the number of such cusps is $\delta P(1)$, the class number of A.

(1.2.2) There is one more "special" kind of points, those $e \in \Omega$ for which the stabilizer $Stab_\Gamma(e)$ of $e \in \Omega$ in Γ (for the action of fractional transform) is strictly bigger then the trivial scalar matrices $\mathcal{Z}(\Gamma)$. Such points are called *elliptic*; the set of Γ-equivalence classes of elliptic points is denoted by $\mathcal{E}$.

The local parameter at an elliptic point $e \in \mathcal{E}$ on $\bar{M}_\Gamma$ is $t_e = (z-e)^{q+1}$. If $\gamma_e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a non-scalar matrix such that $\gamma_e.e = e$ for some elliptic point, then $ce^2 + (d-a)e - b = 0$, and this equation has the same discriminant as the characteristic equation of γ_e, which has two distinct roots of unity as eigenvalues (γ_e is of finite order, since it fixes the image of e under the "building map" $\Omega \to \mathcal{T}$. This order is prime to p, since p-order elements in $GL(2,A)$ are conjugate to elements of the Borel group $\mathcal{B}$, which don't fix any element). Hence e lies in the constant extension $K(e) = \mathbf{F}_{q^2} \otimes K$.
Also note that if δ is even, $K(e) \subset \mathbf{F}_{q^\delta}[[X]] \cong K_\infty$, but $e \in \Omega$ cannot be in K_∞, so there are no elliptic points in this case. On the other hand, for odd δ, the number of elliptic points can be calculated using some arithmetic in $K(e)$ ([5], V.4.5), and one finds that $|\mathcal{E}| = P(-1)$ for δ odd.

(1.3) We are now ready to describe the descending procedure for functions and differentials: A meromorphic function $f : \Omega \to C$ descends nicely if $f(\gamma \cdot z) = f(z)$ for all $z \in \Omega$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
As for differentials, $f(\gamma \cdot z) d(\gamma \cdot z) = f(\gamma \cdot z)(cz+d)^{-2} \det \gamma dz$, so $f(z)dz$ is invariant if

$$(*) \quad f(\gamma \cdot z) = \det \gamma^{-1} (cz+d)^2 f(z) \text{ for all } z \in \Omega, \gamma \in \Gamma.$$

In order to extend these functions and differentials nicely to the cusps $\bar{M}_\Gamma - M_\Gamma$, we will need an additional mero- or holomorphy condition at those points.
As will become clear in a moment, it is sensible to extend $(*)$ to include "multidifferentials" $f(z)dz^{\otimes k}$, since it will turn out that some of these are easier to construct and/or control. Hence we arrive at the following definition:

(1.4) Definition A meromorphic (in the sense of rigid analysis) function $f : \Omega \to C$ which is invariant under Γ, i.e. $f(\gamma \cdot z) = f(z)$, $\forall z \in \Omega, \gamma \in \Gamma$; and such that f is developable in a Laurent series in the parameter t_s at the cusps of Γ, is called a *modular function for* Γ. A (rigid) holomorphic function $f : \Omega \to C$ which satisfies

$$(1.4.1) \quad f(\gamma \cdot z) = \det \gamma^{-l} (cz+d)^k f(z), \ \forall z \in \Omega, \gamma \in \Gamma$$

and which is developable in a Taylor series in the parameter t_s at the cusps of Γ, is called a *modular form of weight k and type l for* Γ.

(1.4.2) Note that, although a modular form f is not invariant under Γ, it's order ord $_x f$ at a point $x \in \Omega$ is (which is the lowest non-trivial power of $z - x$ occuring in the Taylor series development of f around x — also, we define the order of f at a cusp s to be the lowest non-trivial power of the parameter t_s occuring in the Taylor series of f in t_s. Note: this definition is in accordance with [6], but not with [5], where the parameter $t_s^{w_s}$ is used. But if one considers modular forms f of type $l \neq 0$, only $t_s^{-l} f$ admits a series development in $t_s^{w_s}$.) We define the "divisor" of a modular form f for Γ to be the "rational" divisor $\mathrm{div} f \in \mathrm{Div}(\bar{M}_\Gamma) \otimes \mathbf{Q}$ on the modular curve $\bar{M}_\Gamma$ given by

$$\mathrm{div} f = \sum_{x \in \Gamma \backslash \Omega - \mathcal{E}} \mathrm{ord}\,_x f + \sum_{e \in \mathcal{E}} \frac{\mathrm{ord}\,_e f}{q+1} + \sum_{s \in \bar{M}_\Gamma - M_\Gamma} \frac{\mathrm{ord}\,_s f}{w_s}.$$

(1.4.3) The C-vector space of modular forms of weight k and type l for Γ is denoted by $M_{k,l}(\Gamma)$. If $l = 0$, we write $M_k(\Gamma)$ instead. We will also denote the graded rings of modular forms as follows :

$$M(\Gamma) := \bigoplus_{\substack{k \in \mathbf{Z}_{>0} \\ l \in \mathbf{Z}/(q-1)}} M_{k,l}(\Gamma), \quad M_0(\Gamma) := \bigoplus_{k \in \mathbf{Z}_{>0}} M_k(\Gamma).$$

A superscript $M^n_{*,*}$ will denote the space of n-fold cusp forms, i.e. forms that vanish of order $\geq n$ at all cusps.

(1.4.4) *Remark* If $\gamma = \mathrm{diag}(\epsilon, \epsilon) \in \Gamma$ for some $\epsilon \in \mathbf{F}_q^*$ of multiplicative order m, and if $k - 2l \neq 0 \bmod m$, then $M_{k,l} \equiv 0$. In particular, $M_k(GL(2, A)) = 0$ if $k \neq 0 \bmod q - 1$. This can be seen by substituting γ in (1.4.1).

(1.5) *Examples* We will give some examples of non-trivial modular forms.

(1.5.1) **Eisenstein series** (Goss, [10]) The function

$$E_k(z) := \sum_{(0,0) \neq (a,b) \in A^2} (\frac{1}{az+b})^k,$$

is called the Eisenstein series of weight k for $GL(2, A)$. The transformation equation (1.4.1) is easy to check, for holomorphy at the cusps, see *loc. cit.*

(1.5.2) **Coefficient forms** If ϕ^{Λ_z} is the rank two Drinfeld module associated with the lattice $\Lambda_z = zA \oplus A$ for some $z \in \Omega$, then for $a \in A$,

$$\phi_a^{\Lambda_z} = a + \sum_{i=1}^{2\deg(a)} l_i(z, a) \tau^i,$$

the "coefficient forms" $l_i(z, a)$ are modular forms of weight $q^i - 1$ for Γ. To check the transformation formula, use the fact that $\phi^{c.\Lambda} = c\phi^\Lambda c^{-1}$ for any rank two lattice Λ and scaling constant $c \in C^*$. For holomorphy, once can use the following "mysterious" connection between this and the first example:

$$aE_{q^k-1}(z) = \sum_{i+j=k} E_{q^i-1}(z) l_j^{q^i}(z, a),$$

where we set $E_0 \equiv -1$. (see [5], II.2.11). In particular, the leading term

$$\Delta_a := l_{2\deg(a)}(z,a)$$

is a modular forms of weight $q^{2\deg(a)}-1$ for $GL(2,A)$. Furthermore, by the definition of a rank two Drinfeld module, $\Delta_a(z) \neq 0$ if $z \in \Omega$. This means Δ_a has its divisor supported at the cusps.
As far as notation is concerned, if $A = \mathbf{F}_q[T]$, we let $g(z) = \bar{\pi}^{1-q} l_1(z,T)$ and $\Delta(z) = \bar{\pi}^{1-q^2} l_2(z,T)$. Let me also remark that while reading the paper [6] by Gekeler, one should beware of the change of notation for g, Δ on page 683. Here, we have normalised as in the second part of that paper.

(1.5.3) **Derivatives** The derivative f' of a modular form f with respect to z isn't always a modular form. If $f \in M_{k,l}(\Gamma)$ and $\gamma \in \Gamma$, one has

$$f'(\gamma.z) = \det\gamma^{-(l+1)}(cz+d)^{k+2} f'(z) + kc\det\gamma^{-(l+1)}(cz+d)^{k+1} f(z).$$

But on the other hand, if we choose a non-constant $a \in A$, and let $E_a(z) := \bar{\pi}^{-1}\Delta_a'/\Delta_a$, then

$$E_a(\gamma.z) = \det\gamma^{-1}(cz+d)^2 E_a(z) - c\bar{\pi}^{-1}\det\gamma^{-1}(cz+d).$$

If we define an operator (the "Serre-derivative") on $M_{k,l}(\Gamma)$ by

(1.5.4) $\quad \partial_a f(z) := \bar{\pi}^{-1} f'(z) - kE_a(z) f(z),$

then it follows from the above calculations that $\partial_a f$ transforms like a form of weight $k+2$ and type $l+1$. Also, it is holomorphic where f is. (Since $\Delta_a(z) \neq 0$ if $z \in \Omega$, the only possible problem would occur at the cusps. But one sees easily from the product development of Δ_a given in [5], VI (4.12), that its "logarithmic derivative" $E_a(z)$ is holomorphic. — Note that for the uniformisers t_s one has $t_s'/t_s = -t_s$ since the derivative of an "exponential" function $e_\Lambda(z)$ is trivial.)
The derivation operator ∂_a gives a way of constructing forms of non-trivial type from examples of type 0. In particular, for $A = \mathbf{F}_q[T]$, we set $\partial := \partial_T$, and $h := \partial g \in M_{q+1,1}(GL(2,\mathbf{F}_q[T])$. One then has (*cf.* Gekeler, [6]) : $h^{q-1} = -\Delta$, and

$$h(z) := \sum_{\substack{c,d\in A\\(c,d)=1}} \frac{t_\infty(\gamma_{c,d}.z)}{(cz+d)^{q+1}},$$

is a kind of Poincaré-series, where $\gamma_{c,d}$ is any chosen matrix of determinant one and bottom row (c,d).

(1.5.5) **Division values** Let $\mathfrak{n} \subset A$ be an ideal, and $u = (u_1,u_2) \in \mathfrak{n}^{-1}A^2 - A^2$. Define $e_u(z) := e_{\Lambda_z}(u_1 z + u_2)$. Then e_u^{-1} is a modular form of weight one for the principal congruence subgroup $\Gamma(\mathfrak{n}) = \{\gamma \in GL(2,A) : \gamma \equiv \mathbf{1} \bmod \mathfrak{n}\}$. As a matter of fact,

$$e_u^{-1}(z) = \sum_{\substack{0\neq a\in K^2\\ a\equiv u \bmod A}} \frac{1}{a_1 z + a_2}$$

is a kind of Eisenstein series of weight one for $\Gamma(\mathfrak{n})$, where the non-archimedean convergence of the right hand side is easy. Furthermore, the values of $e_u(z)$ for different u give the $\mathfrak{n}$-torsion of the Drinfeld module ϕ^{Λ_z} corresponding to the lattice $\Lambda_z = zA \oplus A$. (Note that

classically, x-coordinates of division points are modular forms of weight *two*, connected with Heckes "non-modular" Eisenstein series of weight two).
This example connects up with the previous ones via the fact that for a principal ideal $\mathfrak{n} = (n), \Delta_n$ and $\prod_u e_u^{-1}(z)$ differ only up to a constant, where u runs through a system of representatives of $n^{-1}A^2/A^2$. Similarly, E_{q-1} is a scalar multiple of $\sum e_u^{1-q}$. Also, for $A = \mathbf{F}_q[T]$, h is homothetic to

$$\sum_{u,v \in \mathbf{F}_q^2/\mathbf{F}_q^*} (u_1 v_2 - u_2 v_1) e_{T^{-1}u}^{-q} e_{T^{-1}v}^{-1},$$

which follows from [6], (9.3).
The expansion of e_u at the cusps of $\Gamma((n))$ can be calculated (*cfr.* Gekeler [5], VI); and from this a product expansion of the "discriminant function" Δ_n follows. For $A = \mathbf{F}_q[T]$, it takes on the following form:

$$\bar{\pi}^{1-q^2}\Delta = -t_\infty^{q-1} \prod_{a \in A \text{ monic}} (\rho_a(t_\infty^{-1}) t^{q^{\deg(a)}})^{(q-1)(q^2-1)}.$$

This is indeed analogous to Jacobi's formula for the classical discriminant function, if one regards $\rho_a(X)$ as similar to the cyclotomic $X^n - 1$. For general A, a similar formula exists, albeit somewhat involved ([5],VI). But one can calculate exactly the divisor of Δ_n, and hence its degree. Let $d = \deg(n)$. If we suppose that $f \in M_k(\Gamma)$ is a modular form of weight k, then $f^{1-q^{2d}} \Delta_n^k$ is a modular function, hence of degree zero. And since degree is additive, we find that

$$\text{(1.5.6)} \quad f \in M_k(GL(2,A)) \quad : \quad \deg(\text{div} f) = \frac{k(q^\delta - 1)}{(q^2-1)(q-1)} P(q),$$

using the formula for deg (div Δ_a).
Suppose $A = \mathbf{F}_q[T]$. The divisors of the modular functions $e_u e_v^{-1}$ are supported at the cusps of $\Gamma(\mathfrak{n})$, and one can show that they generate a group of finite index in the group of cuspidal divisors of degree zero. Hence the group generated by the cusps in the Jacobian of $\Gamma(\mathfrak{n})$ is finite (*loc. cit.*); this is the theorem of Manin-Drinfeld for function fields ([12], IV.2).

(1.6) There are Hecke-operators acting on modular forms, just like classically. A fact which is "badly understood" is that for $A = \mathbf{F}_q[T]$, the eigenvalues of the Hecke operator $T_\mathfrak{p}$ for a prime $\mathfrak{p}$ on g and Δ are *the same*. Also, something like $T_{\mathfrak{p}^2} = T_\mathfrak{p}^2$ is different from the classical situation.

2. The genus of modular curves

(2.1) The modular curve for $GL(2, \mathbf{F}_q[T])$ is easy to describe : if $\phi_T^{\Lambda_z} = T + g(z)\tau + \Delta(z)\tau^2$ is the rank two Drinfeld module of lattice Λ_z, then ϕ^{Λ_z} is isomorphic to $\phi^{\Lambda_{z'}}$ if and only if there exists a constant $c \in C^*$ such that $g(z) = c^{1-q} g(z')$ and $\Delta(z) = c^{1-q^2}\Delta(z')$, and hence

$$j : M_{GL(2,\mathbf{F}_q[T])} \to C : z \mapsto j(z) := \frac{g^{q+1}(z)}{\Delta(z)}$$

is an isomorphism. In particular, the modular curve $\bar{M}_{GL(2,\mathbf{F}_q[T])}$ is of genus zero. But already for a more general A, such an easy description of moduli of rank two Drinfeld modules is not available. As a matter of fact, already the construction of Drinfeld modules is non-trivial, *cfr.* Dummit and Hayes [3].

(2.2) To extract information about the modular curve for a general $\Gamma = GL(2, A)$, we will use the dictionary between modular curves and modular forms.

(2.2.1) As we saw, the local parameters at the cusps are t_s^{q-1}, and for differential forms this gives the formula

$$f(z)dz^{\otimes k} = f(z(t_s^{q-1}))(t_s^{q-1})^{\frac{-kq}{q-1}} d(t_s^{q-1})^{\otimes k}.$$

(2.2.2) At elliptic points $e \in \mathcal{E}$ we have a uniformizer $t_e = (z - e)^{q+1}$. Hence for differential forms we find locally at elliptic points :

$$f(z)dz^{\otimes k} = f(z(t_e))t_e^{\frac{-kq}{q+1}} dt_e^{\otimes k}.$$

(2.3) If f is a modular form of weight $2k$ and type k for $GL(2, A)$, we find that $f(z)dz$ is a k-fold differential form on $\bar{M}_\Gamma$ which (by the relation between the uniformizers on Ω and $\bar{M}_\Gamma$ which was just established)

- is holomorphic at $z \in \Gamma\backslash\Omega - \mathcal{E}$
- has a pole of order $\leq \frac{kq}{q+1}$ at $z \in \mathcal{E}$
- has a pole of order $\leq \frac{kq}{q-1}$ at cusps z.

If we view $\Delta_a(z)dz$ as a section of the divisor of meromorphic $\frac{q^{2d}-1}{2}$-fold differential forms satisfying the above three criteria, we find that

$$\deg(\mathrm{div}\Delta_a) = \frac{q^{2d} - 1}{2}((2g - 2) + |\mathcal{E}|\frac{q}{q + 1} + |\bar{M}_\Gamma - M_\Gamma|\frac{q}{q - 1}),$$

where g is the genus of $\bar{M}_\Gamma$. Hence using formula (1.5.6) for the left hand side, and the expression for the number of cusps and elliptic points (1.2) on the right hand side, we find that

(2.3) Theorem (Gekeler, [5]) *The genus of the Drinfeld modular curve for $GL(2, A)$ equals*

$$g(\bar{M}_{GL(2,A)}) = 1 + \frac{1}{q^2 - 1}(\frac{q^\delta - 1}{q - 1}P(q) - \frac{q(q + 1)}{2}\delta P(1) + \eta),$$

where $\eta = 0$ if δ is even, and $\eta = -\frac{q(q-1)}{2}P(-1)$ if δ is odd, where as before, P is the numerator polynomial of the zeta-function of K.

(2.4) *Remark* The form Δ_a was preferable, because its divisor is so well understood. On the other hand, the distribution of the zeroes of e.g. the Eisenstein series is still not very well known, but for weight $q^k - 1$ and $A = \mathbf{F}_q[T]$, see [1]. For instance, one has

(2.4.1) Proposition ([5], VII.3.3) *The order of E_{q-1} at all elliptic points is one.*

(2.5) *Remark* The above theorem (2.3) also calculates the first Betti number of the groups $GL(2, A)$, generalising Serre [13] ([5], V.A.11).

(2.6) For arithmetic subgroups $\Gamma \subseteq GL(2,A)$, one can now calculate the genus of the corresponding modular curve as is done classically, via the Hurwitz-formula.

(2.6.1) ([5],VII.5) For a principal congruence subgroup $\Gamma(\mathfrak{n})$ of level $\mathfrak{n}$, the group of the covering $\bar{M}_\Gamma \to \bar{M}_{GL(2,A)}$ is $G(\mathfrak{n}) := GL(2,A)/\Gamma(\mathfrak{n}) \cdot \mathcal{Z}$ (where $\mathcal{Z}$ is the center of $GL(2,A)$) of order

$$|G(\mathfrak{n})| = q^{3\deg(\mathfrak{n})} \prod_{\substack{\mathfrak{p}|\mathfrak{n} \\ \mathfrak{p} \text{ prime}}} (1 - \frac{1}{q^{2\deg(\mathfrak{p})}}),$$

and the ramification is tame of order $q+1$ at the elliptic points. There is wild ramification at the cusps, with only a non-trivial first ramification group of order $q^{\deg(\mathfrak{n})}$. Applying Hurwitz, we find

$$g(\bar{M}_{\Gamma(\mathfrak{n})}) = 1 + |G(\mathfrak{n})|(\frac{q^\delta - 1}{(q-1)(q^2-1)}P(q) - \frac{\delta P(1)}{q^{\deg(\mathfrak{n})}(q-1)}).$$

Note that apparently, there is a change of notation in [5], VII.5.8: from there on, h is the class number of A (whereas it used to be the class number of K).

(2.6.2) An even more tedious calculation using the covering $\bar{M}_{\Gamma(\mathfrak{p})} \to \bar{M}_{\Gamma_0(\mathfrak{p})}$ show that the genus of $\bar{M}_{\Gamma_0(\mathfrak{p})}$ for the Hecke group $\Gamma_0(\mathfrak{p}) := \{\gamma \in GL(2,A) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{p}\}$ of *prime* level $\mathfrak{p}$ equals

$$g(\bar{M}_{\Gamma_0(\mathfrak{p})}) = 1 + \frac{1}{q^2-1}(\frac{q^{(\deg(\mathfrak{p})}+1)(q^\delta - 1)}{q-1}P(q) - \delta q(q+1)P(1) + 2\eta),$$

where η is as in theorem (2.3).

$g(K)$	δ	$g(\bar{M}_{GL(2,A)})$	$\deg(\mathfrak{p})$, $\mathfrak{p}$ prime	$g(\bar{M}_{\Gamma(\mathfrak{p})})$	$g(\bar{M}_{\Gamma_0(\mathfrak{p})})$
0	1	0	1	0	0
			2	$q^4 - q^3 - q$	0
			3	—	q
	2	0	1	$q^2 - q - 1$	0
			2	—	q
	3	0	1	$q^3 + q^2 - 2q - 2$	q
	4	q	—	—	—
1	1	0	1	$(q^4 - q^2 - q)+$ $+(q^2 - q - 1)(P(1) - q - 1)$	q^2
	2	q^2	—	—	—
	3	$q(q^2 + P(1) - 1)$	—	—	—

A table of low genera

3. Function fields of some modular curves

(3.1) Modular forms also give a natural construction of modular functions: the quotient of two modular forms of the same weight is a meromorphic Γ-invariant function. We have already encountered "the" modular function j in (2.1) as an example of this. More generally, a Drinfeld module is determined by its coefficient forms (1.5.2), so it seems only natural to consider suitable quotients of them as functions on the modular curve $\bar{M}_{GL(2,A)}$. As a matter of fact,

(3.1.1) Proposition ([5], VII.1.3) *The function field of $\bar{M}_{GL(2,A)}$ is the field F consisting of quotients of homogeneous polynomials of the same degree in the coefficient forms (1.5.2) $l_i(a,z)$, (or E_{q^i-1}, which is the same) for $1 \leq i \leq 2\deg(a)$ and an arbitrary non-constant $a \in A$.*

Note that the term "same degree" in the above proposition is taken with respect to the natural grading by weight. (weight$(l_i) = q^i - 1$).
The proof consists essentially in checking that if all functions in F take on the same value at two different points of Ω, then these points are $GL(2,A)$-equivalent.

(3.1.2) *Remark* Application of the Riemann-Roch theorem to divisors of the form $k.\infty$ on K shows immediately that there exists a non-constant $a \in A$ of degree $= [\max\{\frac{g(K)}{\delta}, 2\frac{g(K)-1}{\delta}\}]$, where $[\cdot]$ indicates the integer part.

(3.2) Proposition (loc. cit.) *The function field of of $\bar{M}_{\Gamma(\mathfrak{n})}$ is generated over the function field of $\bar{M}_{GL(2,A)}$ by $E_{q-1}(z)e_u^{q-1}(z)$, where u runs through a set of representatives of $\mathfrak{n}^{-1}A^2/A^2$.*

Again, the proof consists in verifying that two points of Ω at which all the above functions agree are $\Gamma(\mathfrak{n})$-equivalent. Also note that, for $A = \mathbf{F}_q[T]$, the above two proposition are the analogues of what is very well known classically ([14], 6.1), which is why we leave out the details.

4. From modular curves to modular forms

(4.1) We have seen that modular forms seem to be very effective in studying the C-geometry of modular curves. Let's consider them for a moment as objects of study on their own; a natural question would be to ask about the structure of the rings $M(\Gamma)$ of modular forms for different Γ. (As a matter of fact, this remains linked with more subtle geometric aspects of the modular curves, namely some of their smooth embeddings in projective spaces).
A few answers along classical lines are known: first of all, the graded parts $M_{k,l}(\Gamma)$ are finite-dimensional C- vector spaces, and an application of the Riemann-Roch theorem on the modular curve will allow the calculation of their dimension in most cases (a few very interesting ones left aside for the moment). I will start by sketching the results for $\Gamma = GL(2,A)$, where A is general and the type $l = 0$; and then for $A = \mathbf{F}_q[T]$ and arbitrary type (although a detailed analysis of the behaviour of ∂E_{q-1} would certainly lead to a satisfactory formula in the case of arbitrary weight and type).

Because of the funny way in which parameters on Ω and $\bar{M}_\Gamma$ are linked, we found the divisor of a modular form on $\bar{M}_\Gamma$ having rational coefficients. But if we want to apply the Riemann-Roch theorem we will need "integral" divisors. Fortunately, if we define the integral part of a rational divisor $D = \sum_{x \in \bar{M}_\Gamma} a_x x \in \mathrm{Div}(\bar{M}_\Gamma) \otimes \mathbf{Q}$ to be $[D] = \sum_{x \in \bar{M}_\Gamma} [a_x]x$, where $[a_x]$ is the integral part of a_x, then it is easy to see that $H^0(\bar{M}_\Gamma, D) = H^0(\bar{M}_\Gamma, [D])$ (Shimura, [14], 2.21).
If f is a non-zero modular form for $GL(2,A)$, its weight k is divisible by $q-1$, as we noted in (1.4.4). Here is the basic trick: we send a modular form f of weight k and type zero to the mod-

ular function $f.E_{q-1}^{-\frac{k}{q-1}}$, which establishes an isomorphism of $M_k(\Gamma)$ with $H^0(\bar{M}_\Gamma, [k\mathrm{div}E_{q-1}])$. To apply Riemann-Roch, we want to know the degree of this divisor; from (1.5.6) we know the degree of the rational divisor $\deg(\mathrm{div}E_{q-1})$, but there is no definite general way in which $\deg D$ and $\deg[D]$ are linked for arbitrary D. Fortunately we also know that the order of E_{q-1} at cusps s is always divisible by $q-1$ (E_{q-1} has a Taylor series in t_s^{q-1} at all cusps, because it is of type zero), so that the "cuspidal" part of $[\mathrm{div}E_{q-1}]$ is integral. But even more is known, all zeroes at elliptic points are simple by (2.4.1). Hence

$$\deg[k\mathrm{div}E_{q-1}] = k\deg(\mathrm{div}E_{q-1}) - \langle\frac{k}{q+1}\rangle|\mathcal{E}|,$$

where $\langle x\rangle = x-[x]$ is the fractional part of x. We see that unless $k=1, q=2$, this degree is bigger than $2g(\bar{M}_\Gamma)-2$, so that the divisor is non-special, and the Riemann-Roch theorem gives:

(4.2) Theorem (Gekeler, [5], VII.4.6) *If $(q,k)\neq(2,1)$, the dimension of the vectorspace of modular forms of weight k for $GL(2,A)$ is zero if $q-1$ doesn't divide k, and otherwise*

$$\dim M_k(GL(2,A)) = 1 - g(\bar{M}_{GL(2,A)}) + \frac{k(q^\delta-1)}{(q-1)(q^2-1)}P(q) - \mu,$$

where $\mu=0$ when δ is even, and $\mu = \langle\frac{k}{q^2-1}\rangle P(-1)$ if δ is odd.
If we now let $A=\mathbf{F}_q[T]$, and f is a modular form of weight k and type l for $GL(2,\mathbf{F}_q[T])$, then $k-2l\equiv k-l(q+1)\equiv 0 \bmod q-1$, as we saw in (1.4.4). Therefore

$$M_{k,l}(GL(2,\mathbf{F}_q[T]) \to H^0(\bar{M}_{GL(2,\mathbf{F}_q[T])}, [\frac{k-l(q+1)}{q-1}\mathrm{div}g + l\mathrm{div}h]) \ : \ f\mapsto fh^{-l}g^{\frac{l(q+1)-k}{q-1}}$$

is an isomorphism. We note that (1.5.6) again allows the calculation of the degree of div h, and we see that $\mathrm{ord}_\infty h = 1$ from the series development. Furthermore, as $h=\partial g$, and the zero of g at the elliptic point is simple ($g'(e)\neq 0$), it follows that $h(e)\neq 0$. Similar to (4.2), one arrives at:

(4.3) Proposition *The dimension of the vectorspace of modular forms of weight k and type l for $GL(2,\mathbf{F}_q[T])$ is zero unless $k\equiv 2l \bmod q-1$, when*

$$\dim M_{k,l}(GL(2,\mathbf{F}_q[T]) = 1 + \frac{k}{q^2-1} - \langle\frac{k-l(q+1)}{q^2-1}\rangle - \langle\frac{l}{q-1}\rangle.$$

(4.4) The situation for a principal congruence subgroup is even more advantageous (compare [10], [5], VII.6): there are no elliptic points for $\Gamma(\mathfrak{n}), \mathfrak{n}\subset A$, since all elements of finite order in it are of order divisible by p (see the discussion of elliptic points). Also, since all elements in $\Gamma(\mathfrak{n})$ have determinant one, the type doesn't play a role.
The parameter at cusps s is just t_s, and we see that if $f\in M_2(\Gamma(\mathfrak{n}))$, then $f(z)dz = f(z(t_s))t_s^{-2}dt_s$ is a differential form on the modular curve which is everywhere holomorphic; except at the cusps, where it can have a pole of order 2.
If f is a modular form of weight k for $\Gamma(\mathfrak{n})$, then $fe_u^k\in H^0(\bar{M}_{\Gamma(\mathfrak{n})},\mathcal{M}^{\otimes k})$, where $\mathcal{M} := \mathrm{div}(e_u^{-1})$ is the linebundle of modular forms of weight one. By the above identification of modular forms

of weight 2 and differential forms, we see that if we let $\mathcal{D}$ denote the canonical bundle on $\bar{M}_{\Gamma(\mathfrak{n})}$, and Σ the "divisor of cusps" $\sum_{x \text{ cusp}} x$, we have

$$\mathcal{M}^{\otimes 2} \cong \mathcal{D} \otimes \mathcal{O}(2\Sigma).$$

One can immediately calculate the degree of $\mathcal{M}$ from this to find

$$\deg \mathcal{M} = \frac{q^\delta - 1}{(q-1)(q^2-1)} P(q)|G(\mathfrak{n})|,$$

and it is easy to see that this degree is $> 2g(\bar{M}_{\Gamma(\mathfrak{n})}) - 2$ if $k > 1$, hence an application of Riemann-Roch and duality can be made.
If $k = 1$, there is no obvious (degree) reason why the "speciality index" $h^1(\bar{M}_{\Gamma(\mathfrak{n})}, \mathcal{M})$ should vanish. Note however that we can interpret it in terms of modular forms: via duality, $H^1(\bar{M}_{\Gamma(\mathfrak{n})}, \mathcal{M}) = H^0(\bar{M}_{\Gamma(\mathfrak{n})}, \mathcal{M}^{\otimes -1} \otimes \mathcal{D}) = H^0(\bar{M}_{\Gamma(\mathfrak{n})}, \mathcal{M} \otimes \mathcal{O}(-2\Sigma)) = M^2_{1,0}(\Gamma(\mathfrak{n}))$, which is the space of so-called *double cusp forms of weight one.*

(4.4.1) Proposition ([5], VII.6) *The dimension of the space of modular forms of weight $k > 1$ for a principal congruence subgroup $\Gamma(\mathfrak{n})$ is*

$$\dim M_k(\Gamma(\mathfrak{n})) = 1 - g(\bar{M}_{\Gamma(\mathfrak{n})}) + \frac{q^\delta - 1}{(q-1)(q^2-1)} P(q)|G(\mathfrak{n})|,$$

and the dimension of $M_1(\Gamma(\mathfrak{n}))$ is the number of cusps of $\Gamma(\mathfrak{n})$ plus the number of double cusp forms of weight one.

(4.5) A very similar argument works for the case of a Hecke group $\Gamma_0(\mathfrak{p})$, say of prime level $\mathfrak{p}$.

(4.5.1) Proposition ([5], VII.6) *If $(q, k) \neq (2, 1)$, the dimension of the space of modular forms of weight k for a Hecke subgroup $\Gamma_0(\mathfrak{p})$ of prime level $\mathfrak{p}$ is zero unless $q - 1$ divides k, when it is*

$$\dim M_k(\Gamma_0(\mathfrak{p})) = 1 - g(\bar{M}_{\Gamma_0(\mathfrak{p})}) + k\frac{(q^\delta - 1)(q^{\deg(\mathfrak{p})} - 1)}{(q-1)(q^2-1)} P(q) - 2\mu,$$

where μ is as in (4.2).

$g(K)$	δ	$\dim M_{k(q-1)}(GL(2, A))$	$\deg(\mathfrak{p})$, $\mathfrak{p}$ prime	$\dim M_k(\Gamma(\mathfrak{p}))$ $k > 1$	$\dim M_{k(q-1)}(\Gamma_0(\mathfrak{p}))$
0	1	$1 + [\frac{k}{q+1}]$	1	$kq + 1$	$k + 1$
			2	$k(q^4 + q^2 + 1)+$ $-q^4 + q^3 + q$	$1 + k\frac{q^2+1}{q+1} - 2\langle\frac{k}{q+1}\rangle$
			3	—	$1 - q + k(q^2 - q + 1)$
	2	$1 + k$	1	$k(q^2 + 1) - q^2 + q$	$1 + k(q + 1)$
			2	—	$1 - q + k(q^2 + 1)$
	3	$1 + k\frac{q^2+q+1}{q+1} - \langle\frac{k}{q+1}\rangle$	1	$k(q^3 + q^2 - q - 1)+$ $-q^3 - q^2 + 2q + 2$	$1 - q + k(q^2 + q + 1)$
	4	$1 - q + k(q^2 + 1)$	—	—	—
1	1	$1 + k\frac{P(q)}{q+1} - \langle\frac{k}{q+1}\rangle P(-1)$	1	$k(q^4 - q^3 + (P(1) - 1)q^2+$ $+q) - q^4 + q^3 + q^2+$ $-q - P(1)(q^2 - q - 1)$	$1 - q^2 + kP(q)$
	2	$1 - q^2 + kP(q)$	—	—	—

(4.6) The full structure of the algebra of modular forms is only known in the most “trivial” cases. One has for instance, along classical lines:

(4.6.1) Proposition (Goss [9] and Gekeler [6] (5.12), (5.13)) *The algebra of modular forms for* $GL(2, \mathbf{F}_q[T])$ *is isomorphic to polynomial ring* $C[X, Y]$*, doubly graded by weight* $\in \mathbf{Z}_{>0}$ *and type* $\in \mathbf{Z}/(q-1)$*, where weight*$(X) = q-1$*, weight*$(Y) = q+1$*, type*$(X) = 0$ *and type*$(Y) = 1$*; the isomorphism is given by sending* X *to* g *and* Y *to* h*.*
The algebra of modular forms of type zero for $GL(2, \mathbf{F}_q[T])$ *is isomorphic to the graded polynomial ring* $C[X, Y]$*, where weight*$(X) = q-1$ *and weight*$(Y) = q^2-1$*; the isomorphism is given by sending* X *to* g *and* Y *to* Δ*.*

The proof of these two statements is analogous and classical, so we will stick to a sketch of the first one. It suffices to show that g and h are algebraically independent. Because then, one can calculate the dimension of the (k,l)-graded part $\dim C[g,h]_{k,l}$ of the subring $C[g,h]$ of $M(GL(2, \mathbf{F}_q[T]))$, and it is easy to see that it coincides with $\dim M_{k,l}(GL(2, \mathbf{F}_q[T]))$, calculated in (4.3).
Suppose there exists an algebraic relation between g, h. We might as well suppose it is homogeneous with respect to the natural grading by weight, and of minimal weight d. If $g^{\frac{d}{q-1}}$ occurs, then we evaluate the expression at ∞, and since $h(\infty) = 0$, this would imply that $g(\infty) = 0$, a contradiction. On the other hand, if $h^{\frac{d}{q+1}}$ occurs, we evaluate at an elliptic point instead, and using $g(e) = 0$, we would find $h(e) = 0$, but $h^{q-1}(e) = -\Delta(e) \neq 0$. Hence the algebraic relation is divisible by gh, and performing this division, we would find a relation of smaller degree, contradicting the minimality of d. This proves the theorem. □

(4.6.2) Proposition *The ring of modular forms of type zero for the Hecke group* $\Gamma_0((T))$ *of level* T *over* $\mathbf{F}_q[T]$ *is isomorphic to the graded ring* $C[X, Y]$*, where* X, Y *are variables of weights* $q-1$*; the isomorphism is given by sending* X *to* $g(z)$ *and* Y *to* $g(Tz)$*.*

The fact that $g(Tz)$ is a modular form for $\Gamma_0((T))$ follows as in [11],III prop. 17 *et seq.* The same technique of the previous proof applies, where we now first evaluate a possible algebraic relation between $g(z)$ and $g(Tz)$ at an elliptic point e, and then at $T^{-1}e$. All we have to note is that g has its unique zero at e, and that $T^{-1}e$ cannot be an elliptic point: since e satisfies a quadratic equation of discriminant $D(e) \in \mathbf{F}_q$ (*see* (1.2.2)), one has $D(T^{-1}e) = T^2 D(e) \notin \mathbf{F}_q$, hence $T^{-1}e$ is not elliptic. □

5. Congruences for modular forms and supersingular invariants

(5.1) Untill now, we have not shown very much interest in the coefficients of modular forms, by which we mean the coefficients of the Taylor series at different cusps. Nevertheless, it is known classically that after suitable normalisation, these have a significant arithmetical relevance. Although some of the interesting coefficients in the function field case will occur for automorphic, complex valued forms (think of number of representations of polynomials by quadratic forms), it is still possible to normalise say Eisenstein series, and extract information about supersingular invariants, along the same lines as the classical study of Swinnerton-Dyer in the Antwerp volumes [15]. Throughout this paragraph, we will assume $A = \mathbf{F}_q[T]$.

(5.2) If $\mathfrak{p}$ is a prime (irreducible polynomial) of $\mathbf{F}_q[T]$, then we let $M_{\mathfrak{p}}$ be the ring of modular forms f for $GL(2, \mathbf{F}_q[T])$ for which the Taylor series of f at the cusp ∞ lies in the localisation

$\mathbf{F}_q(T)_\mathfrak{p}[[t_\infty]]$, i.e. such that all coefficients lie in K, and have denominator prime to $\mathfrak{p}$. It then makes sense to consider the ring

$$\bar{M}_\mathfrak{p} = \{f \in \mathbf{F}_q[T]/(\mathfrak{p})[[t_\infty]] : f \equiv f' \bmod \mathfrak{p} \text{ for some } f' \in M_\mathfrak{p}\}$$

of modular forms modulo $\mathfrak{p}$ (congruence is meant termwise in t_∞).

(5.3) Here is the prototype: we will let

$$g_k := (-1)^{k+1}\bar{\pi}^{1-q^k}(T^{q^k} - T)(T^{q^{k-1}} - T)...(T^q - T)E_{q^k-1},$$

where E_{q^k-1} is the Eisenstein series for $A = \mathbf{F}_q[T]$ of weight $q^k - 1$. It turns out that the series development of g_k at the cusp ∞ has A-integral coefficients, *i.e.* belongs to $A[[t_\infty]]$. This is true in particular for g and hence for h.

(5.3.1) Theorem (Gekeler, [6]) *Let $\mathfrak{p} \in \mathbf{F}_q[T]$ be a prime of degree k. Then $g_k \equiv 1 \bmod \mathfrak{p}$, and if we let $B_k \in A[X,Y]$ denote the polynomial such that $g_k = B_k(g,h)$ (which exists by (4.6.1)), then*

$$\bar{M}_\mathfrak{p} \cong \mathbf{F}_q[T]/(\mathfrak{p})[X,Y]/(b_k(X,Y) - 1),$$

where b_k is obtained from B_k by reducing coefficients modulo $\mathfrak{p}$. Also, $\bar{M}_\mathfrak{p}$ is a normal ring.

Although the result is entirely similar to the one by Swinnerton-Dyer, the proof is slightly different; whereas the classical proof uses a double application of the Serre-derivative to the Eisenstein series, it is known that for function fields (*loc. cit.*) $\partial^2 g = \partial h = 0$, so $\partial^2 \equiv 0$ on the whole of $M(GL(2,A))$.

(5.4) We will consider the formal "Tate"-Drinfeld module $\phi_T^{Ta} = TX + g(t_\infty)X^q + \Delta(t_\infty)X^{q^2}$ over $\mathbf{F}_q[T]((t_\infty))$, where we subsitute $g(t_\infty), \Delta(t_\infty)$ for their expansions in t_∞. One can reduce this formal module modulo $\mathfrak{p}$ to find a Drinfeld module over $\mathbf{F}_q[T]/(\mathfrak{p})((t_\infty))$ (Note that the series expansions start of as $g = 1 + ..., \Delta = t_\infty^{q-1} + ...$). Then $\phi_\mathfrak{p}^{Ta} \bmod \mathfrak{p}$ will have lowest degree term $H_\mathfrak{p}(t_\infty)X^{q^{\deg \mathfrak{p}}}$ for some power series $H_\mathfrak{p} \in \mathbf{F}_q[T]/(\mathfrak{p})((t_\infty))$, and it turns out that $H_\mathfrak{p} \in \bar{M}$, namely $H_\mathfrak{p} \equiv g_k \bmod \mathfrak{p}$ (*loc. cit.*). $H_\mathfrak{p}$ is called the *Hasse-invariant*. The name arises as follows: if one gives t_∞ a fixed value such that the series for g, Δ converge, then the reduction $\phi_T^{Ta} \bmod \mathfrak{p}$ is a Drinfeld module ϕ' of finite characteristic $\mathfrak{p}$, which will be *supersingular* $\iff$ $H_\mathfrak{p}(t_\infty) = 0$ $\iff$ the $\mathfrak{p}$-torsion ${}_\mathfrak{p}\phi' = 0$ is trivial $\iff$ the endomorphism ring of ϕ' is non-commutative. This observation opens up the possibility of studying supersingular Drinfeld modules via reductions of Eisenstein series.

6. From modular forms to C-valued harmonic cochains

(6.1) The upper half space Ω has a cannonical reduction, which is the (intersection dual of) the Bruhat-Tits tree $\mathcal{T}$. One might ask what remains of modular forms on Ω when they are reduced to $\mathcal{T}$. It turns out that the result is analogous to the classical theory of periods of modular forms ([12], VI).

(6.2) Suppose f is a weight two modular form for some group Γ. Then as we saw, $f(z)dz$ is a holomorphic differential form on Ω. It was shown by Drinfeld how to "descend" with such a differential form to the tree: Let e be an edge of $\mathcal{T}$ corresponding to the interior of an analytic annulus $D_{x,n}^0 = \{z \in \Omega : |\bar{\pi}_\infty|^{n+1} < |z - x| < |\bar{\pi}_\infty|^n\}$, and assume that the inner circle $|z - x| = |\bar{\pi}_\infty|^{n+1}$ reduces to the *origin* of e. The restriction of $f(z)dz$ to D_i^0 admits a series development $f(z)dz = \sum a_k t^k dt$, where $t = \bar{\pi}^{-n}(z-x)$ is the boundary parameter. We define the residue of $f(z)dz$ at e to be $\text{res}(f(z)dz, e) := a_{-1}$. The residue theorem applied to the arc $D^0_{x,n}$ implies that $\sum \text{res}(f(z)dz, e) = 0$, where the sum is over all edges starting at an arbitrary vertex, i.e. the function res is harmonic; and it is Γ-invariant for the natural action of Γ on $\mathcal{T}$. To be able to generalize this construction, we recall

(6.3) Definition For an abelian group B (written additively), we let $\underline{\text{H}}(\mathcal{T}, B)$ denote the set of *harmonic cochains* on $\mathcal{T}$, i.e. functions from the set of edges of $\mathcal{T}$ to B, such that if the orientation of an edge is reversed, so is the value of the cochain; and the sum of the values of the cochain over all edges leaving a fixed vertex is zero. If Γ is an arithmetic group, we let $\underline{\text{H}}(\mathcal{T}, B)^\Gamma$ denote Γ-invariant cochains, and we will let $\underline{\text{H}}_!(\mathcal{T}, B)^\Gamma$ denote the set of *cuspidal harmonic cochains for* Γ, i.e. cochains with compact support modulo Γ.

(6.4) Let V be a two-dimensional vectorspace over C with the standard action of Γ. Set

$$W_{k,l} = [(\det)^{l-1} \otimes \text{Sym}^{k-2}(V^*)]^*,$$

for k, l are positive integers (one should understand "det" as C with Γ-action by multiplication with the determinant). This definition is very formal, but can be understood practically as follows : suppose $\varphi \in H(\mathcal{T}, W_{k,l})$, and choose a basis X, Y of V^*. For any edge e, $\varphi(e) \in W_{k,l}$ can be seen as a function on the space of $\text{Sym}^{k-2}(V^*)$ spanned by monomials of degree $k-2$ in the variables X, Y. So a harmonic cochain like this is really just a map, associating to any edge of $\mathcal{T}$ a "vector" of $k-1$ values in C, namely the values $\varphi(e)(X^jY^{k-2-j})$ for $j = 0, ..., k-2$ (but with the right Γ-action).

(6.5) Suppose $f \in M_{k,l}(\Gamma)$ is a modular form of weight $k > 1$ and type l. Define a map

$$\text{Res} : M_{k,l}(\Gamma) \to \underline{\text{H}}_!(\mathcal{T}, W_{k,l})^\Gamma$$

by the following :

$$\text{Res}(f, e)(X^jY^{k-2-j}) = \text{res}(z^j f(z)dz, e).$$

(The expression $z^j f(z)dz$ should be seen as a differential form on Ω, it makes no sense on the modular curve.) This is again a harmonic cocycle by the rigid residue theorem, and one computes immediately that $\text{Res}(f, \gamma \cdot e) = \gamma \cdot \text{Res}(f, e)$, which implies Γ-equivariance. Also, the resulting cochain is cuspidal, since Teitelbaum ([16],3) has shown the "strange" positive characteristic phenomenon that every Γ-equivariant cochain taking values in a group killed by $p = \text{char}(\mathbf{F}_q)$ is cuspidal.

(6.6) The prototype of non-cusp forms, the Eisenstein series have residue = 0 since the associated differential forms are holomorphic on Ω. On the other hand, Teitelbaum has constructed an explicit inverse to the map Res for cusp forms, using a "non-archimedean" integral, and shown this inverse to be injective. He can then use the dimension formulae for $M_{k,l}(\Gamma)$ and an explicit construction of sufficiently many harmonic cochains to make Res bijective:

(6.7) Theorem (Teitelbaum, [16]) *The map*

$$Res : M^1_{k,l}(\Gamma) \to \underline{\text{H}}_!(\mathcal{T}, W_{k,l})^\Gamma$$

is an isomorphism for $k > 1$.

(6.8) *Remark* In *loc. cit.*, Teitelbaum uses the interpretation of cocycles through modular forms to obtain results about the relative homology of Γ.

(6.9) The most striking application of theorem 6.7 however, is already a few years old, but remains unpublished untill now. Here it is:

(6.9.1) Theorem (Gekeler — unpublished) *Let $\Gamma \subseteq GL(2, A)$ be an arithmetic subgroup. Then there are no cusp forms of weight one for Γ whatsoever, i.e. $M^1_{1,l}(\Gamma) = 0, \forall l$.*

Proof. Suppose $f \in M^1_{1,l}(\Gamma)$, then it's q-th power $f^q \in M^1_{q,l}(\Gamma)$. Choose an edge $e \in \mathcal{T}$, corresponding to a annulus $D^0_{x,n}$ with parameter t on Ω as in (6.2). If $f(z)dz = \sum_{k\in\mathbf{Z}} a_k t^k dt$ on $D^0_{x,n}$, then $z^i f^q(z)dz = \sum_{k\in\mathbf{Z}} a^q_k t^{kq+i} dt$. In particular, $\mathrm{res}(z^i f^q(z)dz, e) = 0$ for $0 \le i \le q-2$. Hence $\mathrm{Res}(f^q) \equiv 0$, and by Teitelbaum's theorem, $f^q \equiv 0$, so $f \equiv 0$. □

(6.9.2) Corollary *One can complement the results in paragraph 4 as follows:*
$\dim M_1(\Gamma(\mathfrak{n}))$ *is exactly equal to the number of cusps of* $\Gamma(\mathfrak{n})$. *Furthermore, (4.2) and (4.5.1) remain valid if* $(q, k) = (2, 1)$.

(6.9.3) *Remark* The theorem might seem disturbing at first; classical modular forms in $SL_2(\mathbf{Z})$ of weight one are intimately connected with Artin's conjecture for representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ in $GL(2, \mathbf{C})$, and counting, constructing and interpreting $M_1(\Gamma)$ over the integers is very deep. Notice however that the classical "equivalence" between modular forms and automorphic forms/Galois representations doesn't remain valid in a direct way for function fields: whereas modular forms and modular curves are the natural characteristic-p defined moduli objects, defined starting from Drinfeld's "fine" space Ω, the automorphic side is about *complex* valued harmonic forms on the tree $\mathcal{T}$.

7. From modular forms to complex valued harmonic cochains

(7.1) The "automorphic" side of function fields arithmetic involves complex harmonic analysis on the tree $\mathcal{T}$. And much of the succes of the "modular" theory over function fields lies in the intimate relation between these distinct concepts of modular and automorphic form, *see* Gekeler & Reversat [8]. We will only touch upon a few recent applications of it to questions about modular forms and modular curves.

(7.2) For this, we need the r-map, introduced by van der Put in his study of invertible maps on rigid analytic spaces. It should be considered as a "good" substitute for the ordinary logarithmic derivative in the case of positive characteristic (recall *e.g.* that ∂^2 is bad, since it kills all of $M(GL(2, \mathbf{F}_q[T]))$!). The map r associates to every *invertible* holomorphic function on Ω a $\mathbf{Z}$-valued harmonic cochain. If e is an edge of $\mathcal{T}$, then $q^{r(f)(e)}$ is (by definition) the quotient of the maximal absolute value of f attained on the circle corresponding to the endpoint of e, by the maximal absolute value of f attained on the circle corresponding to the origin of e. It is connected with the "characteristic-p" cochains of the previous paragraph via

$$\mathrm{res}(f'(z)/f(z)dz, e) = r(f)(e) \bmod p,$$

which explains its relation with the logarithmic derivative. Also, if two points $z_1, z_2 \in \Omega$ reduce to vertices v_1, v_2 on $\mathcal{T}$, then the value

(*) $\log_q |f(z_2)/f(z_1)| =$ the sum of the values of $r(f)$ along the path from v_1 to v_2 in $\mathcal{T}$.

For proofs of all of this, see [4],I.8.9, [8], (1.7).

(7.3) A modular form f takes on values in C. But if we take its absolute value, then $|f| : \Omega \to \mathbf{R}$ takes on real values. And via (*), we can study this absolute value as an $r(f)$-weighted distance function on the tree, at least if f is invertible. *e.g.* the discriminant function:

(7.4) Theorem (Gekeler [7], (2.8)) *Let $z \in \Omega$ reduce to a vertex $[\pi^{-k}\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$ of $\mathcal{T}$. Then*

$$\log_q |\Delta(z)| = \begin{cases} q^2 + q - q^{1-k} & \text{if } k \leq 1 \\ q^2 + q + k(q^2-1) - q^{k+1} & \text{if } k \geq 1 \end{cases}$$

I have only indicated the formula for points reducing to special vertices, but actually, this allows the calculation of the value everywhere. It would also be interesting to know $|e_u|$ for division values e_u.

(7.5) There are immediate applications of the above result to the question of extracting "roots" out of functions like Δ and $\Delta(z)/\Delta(nz)$ in the ring of invertible functions on Ω, and in the function field of $\bar{M}_{\Gamma_0((n))}$, and from this, one can proceed to study the group of divisor classes supported at the cusps of $\bar{M}_{\Gamma_0((n))}$, *e.g.*

(7.6) Theorem (Gekeler [7], 3.23) *The cuspidal divisor class group of $\bar{M}_{\Gamma_0(\mathfrak{p})}$ for a prime ideal $\mathfrak{p}$ is cyclic of order $(q^{\deg \mathfrak{p}}-1)/(q^2-1)$ if $\deg \mathfrak{p}$ is even and $(q^{\deg \mathfrak{p}}-1)/(q-1)$ if $\deg \mathfrak{p}$ is odd.*

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