Analysis on Manifolds lecture notes for the 2009/2010 Master Class

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LECTURE 1
Differential operators

1.1. Differential operators I: trivial coefficients

In this section we discuss differential operators acting on spaces of functions on a manifold, while in the next section we will move to those acting on spaces of sections of vector bundles. We first discuss differential operators on an open subset $U \subset \mathbb{R}^n$.

We use the following notation for multi-indices $\alpha \in \mathbb{N}^n$:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j; \quad \alpha! = \prod_{j=1}^{n} \alpha_j!.$$

Moreover, if $\beta \in \mathbb{N}^n$ we write $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $1 \leq j \leq n$. If $\alpha \leq \beta$ we put

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} := \prod_{j=1}^{n} \begin{pmatrix} \beta_j \\ \alpha_j \end{pmatrix}.$$

Finally, we put $\partial_j = \partial/\partial x_j$ and

$$x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}. \tag{1.1}$$

Lemma 1.1.1. (Leibniz' rule) *Let $f, g \in C^\infty(U)$ and $\alpha \in \mathbb{N}^n$. Then*

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta f \, \partial^{\alpha-\beta} g.$$

Proof Exercise. □

Definition 1.1.2. A differential operator of order at most $k \in \mathbb{N}$ on U is an endomorphism $P \in \mathrm{End}(C^\infty(U))$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x) \, \partial^\alpha, \tag{1.2}$$

with $c_\alpha \in C^\infty(U)$ for all α.

The linear space of differential operators on U of order at most k is denoted by $\mathcal{D}_k(U)$. The union of these, for $k \in \mathbb{N}$, is denoted by $\mathcal{D}(U)$. Via Leibniz' rule one easily verifies that the composition of two differential operators from $\mathcal{D}_k(U)$ and $\mathcal{D}_l(U)$ is again a differential operator, in $\mathcal{D}_{k+l}(U)$. Accordingly, the set $\mathcal{D}(U)$ of differential operators is a (filtered) algebra with unit.

Next, we look at the effect of coordinate changes. More precisely, let h be a diffeomorphism from U onto a second open subset $U' \subset \mathbb{R}^n$. Then by pull-back, h induces the bijection $h^* : C^\infty(U') \to C^\infty(U)$. Thus, $h^*f(x) = f(h(x))$.

Accordingly, we have an induced map $h_* : \mathrm{End}(C^\infty(U)) \to \mathrm{End}(C^\infty(U'))$, given by

$$h_*(T) = h^{*-1} \circ T \circ h^*.$$

Lemma 1.1.3. *The map h_* maps $\mathcal{D}(U)$ bijectively onto $\mathcal{D}(U')$.*

Proof It follows by repeated application of the chain rule for differentiation, in combination with Leibniz' rule. □

We now move to arbitrary manifolds now.

Definition 1.1.4. Let M be a smooth manifold. A linear operator $P \in \mathrm{End}(C^\infty(M))$ is called *local* if

$$\mathrm{supp}(P(f)) \subset \mathrm{supp}(f) \quad \forall\, f \in C^\infty(M).$$

Since the complement of the support of a function is the largest open on which the function vanishes, the previous condition is equivalent to: for any open $U \subset M$, $f \in C^\infty(M)$, one has the implication:

$$f|_U = 0 \Longrightarrow P(f)|_U = 0.$$

Lemma 1.1.5. *There is a unique way to associate to any local operator $P \in \mathrm{End}(C^\infty(M))$ on a manifold M and any open $U \subset M$, a "restricted operator"*

$$P_U = P|_U \in \mathrm{End}(C^\infty(U))$$

such that, if $V \subset U$, then $(P|_U)|_V = P|_V$.

Proof For $f \in C^\infty(U)$, let's look at what the value of $P_U(f) \in C^\infty(U)$ at an arbitrary point $x \in U$ can be. We choose a function $f_x \in C^\infty(M)$ which coincides with f in an open neighborhood $V_x \subset U$ of x. From the condition in the statement, we must have

$$P_U(f)(x) = P(f_x)(x).$$

We are left with checking that this can be taken as definition of P_U. All we have to check is the independence of the choice of f_x. But if g_x is another one, then $f_x - g_x$ vanishes on a neighborhood of x; since P is local, we deduce that $P(f_x) - P(g_x) = P(f_x - g_x)$ vanishes on that neighborhood, hence also at x. □

Local operators can be represented in local charts: if (U, κ) is a coordinate chart, then $P|_U$ can be moved to $\kappa(U)$ using the pull-back map $k^* : C^\infty(\kappa(U)) \to C^\infty(U)$, to obtain an operator

$$P_\kappa : C^\infty(\kappa(U)) \to C^\infty(\kappa(U)),\ P_\kappa = \kappa_*(P|_U) = (\kappa^*)^{-1} \circ P|_U \circ \kappa^*.$$

Definition 1.1.6. Let M be a smooth manifold. A **differential operator** of order at most k on M is a local linear operator $P \in \mathrm{End}(C^\infty(M))$ with the property that, for any coordinate chart (U, κ), $P_\kappa \in \mathcal{D}_k(\kappa(U))$.

The space of operators on M of order at most k is denoted by $\mathcal{D}_k(M)$.

Note that, in the previous definition, it would have been enough to require the condition only for a family of coordinate charts whose domains cover M.

Note also that the condition on a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$ simply means that $P|_U$ is of type

$$P_U = \sum_{|\alpha| \leq k} c_\alpha(x)\, \partial_\kappa^\alpha,$$

with $c_\alpha \in C^\infty(U)$. Here ∂_κ^α act on $C^\infty(U)$ and are defined analogous to ∂^α but using the derivative along the vector fields $\partial/\partial x_j^\kappa$ induce by the chart.

Next, we discuss the symbols of differential operators.

Definition 1.1.7. Let $U \subset \mathbb{R}^n$ and let $P \in \mathcal{D}_k(U)$ be of the form (1.2). The **full symbol** of the operator P is the function $\sigma(P) : U \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leq k} c_\alpha(x)(i\xi)^\alpha.$$

The **principal symbol** of order k of P is the function $\sigma_k(P) : U \times \mathbb{R}^n \to \mathbb{C}$,

$$\sigma_k(P)(x, \xi) = \sum_{|\alpha| = k} c_\alpha(x)(i\xi)^\alpha.$$

A nice property of the principal symbol has, which fails for the total one is its multiplicativity property (see Exercise 1.5.2).

It is not difficult to check the following formulas for the symbols:

$$\sigma(P)(x, \xi) = e^{-i\xi} P(e^{i\xi})(x), \sigma_k(P)(x, \xi) = \lim_{t \to \infty} t^{-k} e^{-it\xi} P(e^{it\xi})(x).$$

Here we have identified ξ with the linear functional $x \mapsto \sum \xi_j x_j$. Accordingly, $e^{i\xi}$ stands for the function $x \mapsto e^{i\xi x}$. See also below.

Although the total symbol may look more natural then the principal one, the situation is the other way around: it is the principal symbol that can be globalized to manifolds (hence expressed coordinate free). Continuing the discussion above, it is most natural to view ξ as a variable in the dual of $\mathbb{R}^n$. Accordingly, $U \times \mathbb{R}^n$ should be viewed as the cotangent bundle T^*U. In other words, the principal symbol should be viewed as the function

$$\sigma_k(P) : T^*U \to \mathbb{C}, \quad \xi_1(dx_1)_x + \ldots + \xi_n(dx_n)_x \mapsto \sum_{|\alpha| = k} c_\alpha(x)(i\xi)^\alpha.$$

The following lemma supports this interpretation: it shows that the symbol can be characterized intrinsically (without reference to the coordinates).

Lemma 1.1.8. *Let $U \subset \mathbb{R}^n$, $P \in \mathcal{D}_k(U)$.*

For $\xi_x = (x, \xi) \in T_x^*\mathbb{R}^n$ ($x \in U$), choose $\varphi \in C^\infty(U)$ such that $(d\varphi)_x = \xi_x$. Then

$$\sigma_k(P)(x, \xi) = \lim_{t \to \infty} t^{-k} e^{-it\varphi(x)} P(e^{it\varphi})(x).$$

Proof Left to the reader. The proof follows by application of Leibniz' rule. $\square$

Although this should be already clear from the previous lemma, let us check explicitly that the symbol behaves well under coordinate changes. More precisely, let h be a diffeomorphism from U onto a second open subset $U' \subset \mathbb{R}^n$. It induces the map $T^*h : T^*U \to T^*U'$ given by $T^*h(x, \xi) = (h(x), \xi \circ T_x h^{-1})$.

Accordingly we have the map $h_* : C^\infty(T^*U) \to C^\infty(T^*U')$ given by $h_*\sigma = \sigma \circ (T^*h)^{-1}$. Thus,

$$h_*\sigma(x,\xi) = \sigma(h^{-1}(x), \xi \circ T_x h).$$

Lemma 1.1.9. *For all $P \in \mathcal{D}_k(U)$,*

$$\sigma_k(h_*(P)) = h_*(\sigma_k(P)).$$

Proof Fix $(x,\xi) \in U \times \mathbb{R}^n \simeq T^*U$. Put $y = h(x)$ and $\eta = \xi \circ T_x h^{-1}$. Select select $\varphi \in C^\infty(U')$ with $d\varphi(y) = \eta$. Then

$$\begin{aligned} \sigma_k(h_*(P))(y,\eta) &= \lim_{t\to\infty} t^{-k} e^{-it\varphi(y)} (h_*P)(e^{it\varphi})(y) \\ &= \lim_{t\to\infty} t^{-k} e^{-ith^*\varphi(x)} P(e^{ith^*\varphi})(x) \\ &= \sigma_k(P)(x,\xi). \end{aligned}$$

This establishes the desired formula. □

Corollary 1.1.10. *Let M be a manifold and $P \in \mathcal{D}_k(M)$. Then there is a well-defined smooth function*

$$\sigma_k(P) : T^*M \to \mathbb{C}$$

such that, for any coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$,

$$T_x^*M \ni \xi_1(dx_1^\kappa)_x + \ldots + \xi_n(dx_n^\kappa)_x \overset{\sigma_k(P)}{\longrightarrow} \sigma_k(P_\kappa)(x, \xi_1, \ldots, \xi_n) \in \mathbb{C}.$$

Definition 1.1.11. Let $P \in \mathcal{D}_k(M)$. The function $\sigma_k(P) : T^*M \to \mathbb{C}$ from the previous lemma is called **the principal symbol** of order k of the operator P.

You should now try to solve Exercises 1.5.3, 1.5.5 and 1.5.6.

1.2. Differential operators II: arbitrary coefficients

We shall now introduce the notion of a differential operator between smooth vector bundles E and F on a smooth manifold M, acting at the level of sections

$$P : \Gamma(E) \to \Gamma(F).$$

It is useful to have in mind that degree zero differential operators correspond to sections $C \in \Gamma(\underline{\mathrm{Hom}}\,(E,F))$ i.e. smooth maps

$$M \ni x \mapsto C_x \in \mathrm{Hom}(E_x, F_x).$$

More precisely, any such C defines an operator $C : \Gamma(E) \to \Gamma(F)$ acting on sections by

$$C(s)(x) = C_x(s(x)).$$

This construction identifies sections of $\underline{\mathrm{Hom}}\,(E,F)$ with $C^\infty(M)$-linear maps $\Gamma(E)$ to $\Gamma(F)$ (see Exercise 1.5.7).

First, we place ourselves in the following situation:

1. $M = U$ is the domain of a coordinate chart $(U, \kappa = (x_1^\kappa, \ldots, x_n^\kappa))$.
2. E is trivializable and we have a fixed trivialization $E \cong U \times \mathbb{C}^r$ with associated frame $\{s_1, \ldots\}$.

Note that, in this case, we have "higher order derivatives operators"

$$\partial_\kappa^\alpha : \Gamma(E) \to \Gamma(E), \quad f^1 s_1 + \ldots \mapsto \partial_\kappa^\alpha(f^1)s_1 + \ldots.$$

A differential operator of order at most k from E to F is a linear map $P : \Gamma(U, E) \to \Gamma(U, F)$ of the form

$$P = \sum_{|\alpha| \leq k} C_\alpha \circ \partial^\alpha,$$

with $C_\alpha \in \Gamma(U, \underline{\mathrm{Hom}}\,(E, F))$. The space of such differential operators is denoted by $\mathcal{D}_k(U, E, F)$.

Note that if also F is trivialized, with trivializing frame $\{s_1', \ldots\}$, then each C_α is uniquely determined by a matrix of smooth functions on U, $c(\alpha)_j^i \in C^\infty(U)$ (characterized by $C_\alpha(s_i) = \sum_j c(\alpha)_i^j s_j'$. With respect to the identification $\Gamma(U, E) \cong C^\infty(U, \mathbb{C})^r$, $(f^1 s_1 + \ldots) \mapsto (f^1, \ldots)$, and similarly for F, P becomes

$$P(f_1, \ldots) = (\sum_{\alpha, j} \partial_\kappa^\alpha(f^j)c(\alpha)_j^1, \ldots).$$

Next, we explain that $\mathcal{D}_k(U, E, F)$ does not depend on the trivialization of E. Another trivialization (over U) is associated with a vector bundle isomorphism $\varphi_E : E \to E$. Then $\varphi_E(x, v) = (x, \Phi_E(x)v)$, with Φ_E a smooth map $U \to \mathrm{GL}\,(\mathbb{C}^r)$. The map φ_E induces a linear isomorphism φ_{E*} of $\Gamma(U, E)$ onto itself, given by $\varphi_{E*}s = \varphi_E \circ s$. Accordingly, we have an induced linear isomorphism φ_* from $\mathrm{Hom}(\Gamma(U, E), \Gamma(U, F))$ onto itself given by $\varphi_*(T) = T \circ \varphi_{E*}^{-1}$. By an easy repeated application of Leibniz' formula, we see that the map φ_* maps $\mathcal{D}_k(U, E, F)$ bijectively onto $\mathcal{D}_k(U, E, F)$. Hence the space $\mathcal{D}_k(U, E, F)$ defined above only depends on the coordinate patch (U, κ) and on the fact that E is trivializable (in turn, one can already proceed as in the previous section and show that it does not depend on the choice of the coordinates on U either).

Proceeding as in the previous section, we say that a linear operator between spaces of sections of two vector bundles E and$F$ over a manifold M,

$$P : \Gamma(E) \to \Gamma(F)$$

is local if, for any $s \in \Gamma(E)$,

$$\mathrm{supp}(P(s)) \subset \mathrm{supp}(s).$$

By the same arguments as before, any such local operator can be restricted to arbitrary opens $U \subset M$, obtaining new operators $P|_U$ from $E|_U$ to $F|_U$. Also, for a coordinate chart (U, κ), we obtain an operator P_κ acting on the resulting bundles over $\kappa(U)$: from $E_\kappa := \kappa_*(E|U) = (\kappa^{-1})^*(E|U)$ to F_κ defined similarly.

Definition 1.2.1. Let E, F be smooth vector bundles over a smooth manifold M. A **differential operator of order at most k from E to F** is a linear local operator $P : \Gamma(E) \to \Gamma(F)$ with the property that, for any coordinate chart (U, κ) with the property that $E|_U$ is trivializable, $P_\kappa \in \mathcal{D}_k(E_\kappa, F_\kappa)$.

The space of differential operators of order at most k from E to F is denoted by $\mathcal{D}_k(E, F)$.

We extend the definition of principal symbol as follows. We denote by $\pi : T^*M \to M$ the canonical projection. For a vector bundle E over M, let π^*E be the pull-back of E to T^*M (whose fiber above $\xi_x \in T^*_xM$ is E_x). For two vector bundles E and F over M, we consider the vector bundle $\underline{\mathrm{Hom}}\,(E,F)$ over M (whose fiber above $x \in M$ is $\mathrm{Hom}(E_x, F_x)$) and its pull-back to T^*M,

$$\pi^*\underline{\mathrm{Hom}}\,(E,F) \cong \underline{\mathrm{Hom}}\,(\pi^*E, \pi^*F)$$

whose fiber above $\xi_x \in T^*_xM$ is $\mathrm{Hom}(E_x, F_x)$.

Lemma 1.2.2. *Let E, F be smooth vector bundles on M and let $P \in \mathcal{D}_k(E,F)$. There exists a unique section $\sigma_k(P)$ of $\pi^*\underline{\mathrm{Hom}}\,(E,F)$ (called again* **the principal symbol** *of P), i.e. a smooth function*

$$T^*_x \ni \xi_x \mapsto \sigma_k(P)(\xi_x) \in \mathrm{Hom}(E_x, F_x),$$

with the following property: for each $x_0 \in M$ and all $s \in \Gamma(E)$ and $\varphi \in C^\infty(M)$,

$$\sigma_k(P)((d\varphi)_{x_0})(s(x_0)) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x_0)} P(e^{it\varphi}s)(x_0). \tag{1.3}$$

Moreover, for each $x \in M$ the function $\xi \mapsto \sigma_k(P)(x,\xi)$ is a degree k homogeneous polynomial function $T^*_xM \to \mathrm{Hom}(E_x, F_x)$.

Proof Uniqueness follows from the fact that for every $(x,\xi) \in T^*M$ and $v \in E_x$ there exists a $s \in \Gamma^\infty(E)$ such that $s(x) = v$ and a function $\varphi \in C^\infty(M)$ such that $d\varphi(x) = \xi$. Let $x_0 \in M$ be given and select a coordinate patch $U \ni x_0$ over which E and F admit trivializations $\tau_E : E|_U \to U \times E_0$ and $\tau_F : F|_U \to U \times F_0$. trivialize. Let $(x_1, \dots, x_n)$ be a system of local coordinates on U. Then

$$\tau_*(P) = \sum_{|\alpha|\le k} C_\alpha \partial^\alpha,$$

with $C_\alpha \in C^\infty(U, \underline{\mathrm{Hom}}\,(E_0, F_0))$. It follows by application of Lemma 1.1.8 that the limit on the right-hand side of (1.3) is given by

$$\sum_{|\alpha|\le k} [\tau_F^{-1}] \circ C_\alpha(x_0)\tau_E[s(x_0)]\eta^\alpha.$$

Here we have used the multi-index notation with $\eta_j = \partial_j\varphi(x_0)$, $\partial_1, \dots, \partial_n$ being the derivations induced by the choice of coordinates. It follows that the limit on the right-hand side of (1.3) depends on s and ϕ through the values $s(x_0)$ and $d\phi(x_0)$. This implies the existence of a section $\sigma_k(P)$ of $\underline{\mathrm{Hom}}\,(\pi^*E, \pi^*F)$ with the property (1.3). The local computation just given also implies the final assertions about smoothness and homogeneity. $\square$

Example 1.2.3. We consider the complexified version of the DeRham complex. I.e., we define $\Omega^k(M)_\mathbb{C} = \Omega^k(M) \otimes_\mathbb{R} \mathbb{C}$, which should be interpreted as the space of sections of the complex vector bundle $\Lambda_\mathbb{C}T^*M$ whose fiber at $x \in M$ consists of antisymmetric, k-multilinear maps from T_xM to $\mathbb{C}$. The exterior differentiation clearly extends to a $\mathbb{C}$-linear map $d = d_k : \Omega^k(M)_\mathbb{C} \to \Omega^{k+1}(M)_\mathbb{C}$. Let U be a coordinate patch of M with local coordinates $x_1, \dots, x_n$. Then for each $a \in U$, the one forms $dx_1(a), \ldots, dx_n(a)$ span the cotangent space T_a^*M. Thus, $\wedge^k T_a^* M$ has the basis

$$dx_{j_1}(a) \wedge \cdots \wedge dx_{j_k}(a), \quad \text{with} j_1 < \cdots < j_k.$$

With respect to this basis, the restriction of a section $s \in \Omega^k(M)$ to U may be expressed as

$$s|_U = \sum_{j_1 < \cdots < j_k} s_{j_1,\ldots,j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Exterior differentiation is given by

$$ds|_U = \sum_{j_1 \cdots < j_k} d(s_{j_1,\ldots,j_k}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where $ds_{j_1,\ldots,j_k} = \sum_i \partial_i s_{j_1,\ldots,j_k}$. From this we see that d is a differential operator of order one from $\wedge^k T^*M$ to $\wedge^{k+1} T^*M$. For its principal symbol, see Exercise 1.5.9.

For E_1, E_2 smooth vector bundles on M and $P \in \mathcal{D}_k(E_1, E_2)$, the principal symbol $\sigma_k(P)$ is a section of the bundle $\underline{\mathrm{Hom}}\,(\pi^*E_1, \pi^*E_2)$. Equivalently, the symbol may be viewed as a homomorphism from the bundle π^*E_1 to π^*E_2. Thus, if E_3 is a third vector bundle and $Q \in \mathcal{D}_l(E_2, E_3)$ then the composition $\sigma_l(Q) \circ \sigma_k(P)$ is a vector bundle homomorphism from E_1 to E_3.

Lemma 1.2.4. *Let E_1, E_2, E_3 be smooth vector bundles on M. Let $P \in \mathcal{D}_k(E_1, E_2)$ and $Q \in \mathcal{D}_l(E_2, E_3)$. Then the composition $Q \circ P$ belongs to $\mathcal{D}_{k+l}(E_1, E_3)$ and*

$$\sigma_{k+l}(Q \circ P) = \sigma_l(Q) \circ \sigma_k(P).$$

Finally, we discuss the notion of formal adjoint. Assume now that E and F are equipped with hermitian inner products $\langle -,- \rangle^E$[1] and $\langle -,- \rangle^F$. We also choose a strictly positive density on M, call it $d\mu$. One has an induced inner-product on the space $\Gamma_c(E)$ of compactly supported sections of E given by

$$\langle s, s' \rangle^E := \int_M \langle s(x), s' \rangle_x^E \, d\mu,$$

and similarly an inner product on $\Gamma_c(F)$. Given $P \in \mathcal{D}_k(E, F)$, a **formal adjoint** of P (with respect to the hermitian metrics and the density) is an operator $P^* \in \mathcal{D}_k(F, E)$ with the property that

$$\langle P(s_1), s_2 \rangle^F = \langle s_1, P^*(s_2) \rangle^E, \quad \forall\ s_1 \in \Gamma_c(E), s_2 \in \Gamma_c(F).$$

Proposition 1.2.5. *For any $P \in \mathcal{D}_k(E, F)$, the formal adjoint $P^* \in \mathcal{D}_k(F, E)$ exists and is unique. Moreover, the principal symbol of P^* is $\sigma_k(P^*) = \sigma_k(P)^*$, where $\sigma_k(P)^*(\xi_x)$ is the adjoint of the linear map*

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

(with respect to the inner products $\langle -,- \rangle_x^E$ and $\langle -,- \rangle_x^F$).[2]

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[1]hence $\langle -,- \rangle^E$ is a family $\{\langle -,- \rangle_x^E : x \in M\}$ of inner products on the vector spaces E_x, which "varies smoothly with respect to x". The last part means, e.g., that for any $s, s' \in \Gamma(E)$, the function $\langle s, s' \rangle^E$ on M, sending x to $\langle s(x), s'(x) \rangle_x^E$ is smooth; equivalently, it has the obvious meaning in local trivializations.

[2]note that P^* depends both on the hermitian metrics on E and F as well as on the density, while it principal symbol does not depend on the density.

Proof Due to the local property of differential operators (or, more precisely to its sheaf property, cf. Exercise 1.5.8), it suffices to prove the statement (both the existence as well as the uniqueness) locally. So assume that $M = U \subset \mathbb{R}^n$, where we can write $P = \sum_{|\alpha|\leq k} C_\alpha \circ \partial^\alpha$. We have $d\mu = \rho|dx|$ for some smooth function ρ on U. Writing out $\langle P(s_1), s_2\rangle^F$ and integrating by parts $|\alpha|$ times (to move ∂^α from s to s'), we find the operator P^* which does the job:

$$P^*(s') = \sum_{|\alpha|\leq k} \frac{1}{\rho}\partial^\alpha(\rho C_\alpha^* s').$$

Clearly, this is a differential operator of order at most k. For the principal symbol, we see that the only terms in this sum which matter are:

$$\sum_{|\alpha|=k} (-1)^{|\alpha|}\frac{1}{\rho}\rho C_\alpha^*\partial^\alpha(s') = \sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*\partial^\alpha(s'),$$

i.e. the symbol is given by

$$\sum_{|\alpha|=k} (-1)^{|\alpha|} C_\alpha^*(i\xi)^\alpha = (\sum_{|\alpha|=k} C_\alpha^*(i\xi)^\alpha)^*.$$

The uniqueness follows from the non-degeneracy property of the integral: if $\int_U fg = 0$ for all compactly supported smooth functions, then $f = 0$. $\square$

1.3. Ellipticity and a preliminary version of the Atiyah-Singer index theorem

Definition 1.3.1. Let $P \in \mathcal{D}_k(E, F)$ be a differential operator between two vector bundles E and F over a manifold M. We say that P is an **elliptic operator** of order k if, for any $\xi_x \in T_x^*M$ non-zero,

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

is an isomorphism.

The aim of these lectures is to explain and complete the following theorem (a preliminary version of the Atiyah-Singer index theorem).

Theorem 1.3.2. *Let M be a compact manifold and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator. Then $Ker(P)$ and $Coker(P)$ are finite dimensional,*

$$Index(P) := dim(ker(P)) - dim(Coker(P))$$

depends only on the principal symbol $\sigma_k(P)$, and $Index(P)$ can be expressed in terms of (precise) topological data associated to $\sigma_k(P)$.

Due to the the way that elliptic operators arise in geometry (via "elliptic complexes"), it is worth giving a slightly different dress to this theorem.

Definition 1.3.3. A **differential complex** over a manifold M,

$$\mathcal{E} : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \dots$$

consists of:

1. For each $k \geq 0$, a vector bundle E_k over M, with $E_k = 0$ for k large enough.
2. For each $k \geq 0$, a differential operator P_k from E_k to E_{k+1}, of some order d independent of k

such that, for all k, $P_{k+1} \circ P_k = 0$.

Example 1.3.4. Let $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ be exterior differentiation. Then $d_{k+1} \circ d_k = 0$ for all k. Therefore, the sequence of differential operators $d_k \in \mathcal{D}_1(\wedge^k T^*M, \wedge^{k+1} T^*M)$ forms a complex; it is called the de Rham complex.

Note that, from Lemma 1.2.4 it follows that for a complex of differential operators as above, the associated sequence $\sigma_{d_k}(P_k)$ of principal symbols is a complex of homomorphisms of the vector bundles $\pi^* E_k$ on M, i.e., for any $\xi_x \in T_x^* M$, the sequence

$$E_x^0 \overset{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E_x^1 \overset{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E_x^2 \overset{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is a complex of vector space. In turn, this means that the composition of any two consecutive maps in this sequence is zero. Equivalently,

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) \subset \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

Definition 1.3.5. A differential complex $\mathcal{E}$ is called an **elliptic complex** if, for any $\xi_x \in T_x^* M$ non-zero, the sequence

$$E_x^0 \overset{\sigma_d(P_0)(\xi_x)}{\longrightarrow} E_x^1 \overset{\sigma_d(P_1)(\xi_x)}{\longrightarrow} E_x^2 \overset{\sigma_d(P_2)(\xi_x)}{\longrightarrow} \dots$$

is exact, i.e.

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) = \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

For a general differential complex $\mathcal{E}$, one can define

$$Z^k(\mathcal{E}) = \mathrm{Ker}(P_k), \quad B^k(\mathcal{E}) = \mathrm{Im}(P_{k_1}),$$

and the k-th cohomology groups

$$H^k(\mathcal{E}) = Z^k(\mathcal{E})/B^k(\mathcal{E}).$$

The space $H^k(M, P_*)$ defined as above, is called the k-th cohomology group of the elliptic complex.

Theorem 1.3.6. *If $\mathcal{E}$ is an elliptic complex over a compact manifold M, then all the cohomology groups $H^k(\mathcal{E})$ are finite dimensional and the resulting Euler characteristic*

$$\chi(\mathcal{E}) := \sum_k (-1)^k dim(H^k(\mathcal{E}))$$

can be expressed in terms of topological invariants of the principal symbols associated to $\mathcal{E}$.

Example 1.3.7. The De Rham complex of a manifold M is elliptic (see Exercise 1.5.10). Recall that the resulting cohomology in a degree k, i called the k-th de Rham cohomology of M, denoted $H^k_{\mathrm{dR}}(M)$. is defined to be the k-th cohomology of the de Rham complex. According to the above result, the de Rham cohomology of a compact manifold is finite dimensional. For a simpler proof of this result, involving Meyer-Vietoris sequences, we refer the reader to the book by Thornehaeve-Madsen or the book by Bott and Tu.

Example 1.3.8. Any elliptic operator $P \in \mathcal{D}_k(E, F)$ can be seen as an elliptic complex with $E^0 = E$, $E^1 = F$ and $E^k = 0$ for other k's, $P_0 = P$. Moreover, its Euler characteristic is just the index of P. Hence the last theorem seems to be a generalization of Theorem 1.3.1. However, there is a simple trick to go the other way around. This is explained in the last three exercises of this lecture.

1.4. General tool: Fredholm operators

As we have already mentioned, the aim of these lectures is to understand Theorem 1.3.2. The first few lectures will be devoted to proving that the index of any elliptic operator (over compact manifolds) is finite; after that we will spend some lectures to explain the precise meaning of "topological data associated to the symbol" (and the last lectures will be devoted to some examples). The nature of these three parts is Analysis- Topology- Geometry.

For the first part- on the finiteness of the index, we will rely on the fact that indices of operators are well behaved in the framework of Banach spaces. This is some very general theory that belongs to Functional Analysis, which we recall in this section. In the next few lectures we will show how this theory applies to our problem (on short, we have to pass from spaces of sections of vector bundles to appropriate "Banach spaces of sections" and show that our operators have the desired compactness properties).

So, for this section[3] we fix two Banach spaces $\mathbb{E}$ and $\mathbb{F}$ and we discuss Fredholm operators between them- i.e. operators which have a well-defined index. More formally, we denote by $\mathcal{L}(\mathbb{E}, \mathbb{F})$ the space of bounded (i.e. continuous) linear operators from $\mathbb{E}$ to $\mathbb{F}$ and we take the following:

Definition 1.4.1. A bounded operator $T : \mathbb{E} \to \mathbb{F}$ is called Fredholm if $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A)$ are finite dimensional. We denote by $\mathcal{F}(\mathbb{E}, \mathbb{F})$ the space of all Fredholm operators from $\mathbb{E}$ to $\mathbb{F}$.

The index of a Fredholm operator A is defined by

$$\mathrm{Index}(A) := \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Coker}(A)).$$

Note that a consequence of the Fredholmness is the fact that $R(A) = \mathrm{Im}(A)$ is closed. Here are the first properties of Fredholm operators.

Theorem 1.4.2. *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces.*

(i) If $B : \mathbb{E} \to \mathbb{F}$ and $A : \mathbb{F} \to \mathbb{G}$ are bounded, and two out of the three operators A, B and AB are Fredholm, then so is the third, and

$$\mathrm{Index}(A \circ B) = \mathrm{Index}(A) + \mathrm{Index}(B).$$

[3] all the theorem stated in this section are proved in the auxiliary set of notes handed out to you

(ii) If $A : \mathbb{E} \to \mathbb{F}$ is Fredholm, then so is $A^* : \mathbb{F}^* \to \mathbb{E}^*$ and [4]

$$\mathrm{Index}(A^*) = -\mathrm{Index}(A).$$

(iii) $\mathcal{F}(\mathbb{E}, \mathbb{F})$ is an open subset of $\mathcal{L}(\mathbb{E}, \mathbb{F})$, and

$$\mathrm{Index} : \mathcal{F}(\mathbb{E}, \mathbb{F}) \to \mathbb{Z}$$

is locally constant.

What will be important for us is an equivalent description of Fredholm operators, in terms of compact operators. First we recall the following:

Definition 1.4.3. A linear map $T : \mathbb{E} \to \mathbb{F}$ is said to be compact if for any bounded sequence $\{x_n\}$ in $\mathbb{E}$, $\{T(x_n)\}$ has a convergent subsequence.

Equivalently, compact operators are those linear maps $T : \mathbb{E} \to \mathbb{F}$ with the property that $T(B_{\mathbb{E}}) \subset \mathbb{F}$ is relatively compact, where $B_{\mathbb{E}}$ is the unit ball of $\mathbb{E}$. Here are the first properties of compact operators.

We point out the following improvement/consequence of the Fredholm alternative for compact operators (discussed in the appendix- see Theorem **??** there).

Theorem 1.4.4. *Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators.*

More precisely:

(i) If $K \in \mathcal{K}(\mathbb{E}, \mathbb{F})$ and $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $A + K \in \mathcal{F}(\mathbb{E}, \mathbb{F})$ and $\mathrm{Index}(A + K) = \mathrm{Index}(A)$.

(ii) If $A \in \mathcal{F}(\mathbb{E}, \mathbb{F})$, then $\mathrm{Index}(A) = 0$ *if and only if $A = A_0 + K$ for some invertible operator A_0 and some compact operator K.*

Finally, there is yet another relation between Fredholm and compact operators, know as the Atkinson characterization of Fredholm operators:

Theorem 1.4.5. *Fredholmness= invertible modulo compact operators.*

More precisely, given a bounded operator $A : \mathbb{E} \to \mathbb{F}$, the following are equivalent:

(i) A is Fredholm.

(ii) A is invertible modulo compact operators, i.e. there exist and operator $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ and compact operators K_1 and K_2 such that

$$BA = 1 + K_1, AB = 1 + K_2.$$

[4] here, $A^*(\xi)(e) = \xi(A(e)))$

1.5. Some exercises

We advise you to do the following exercises (in this order): Exercise 1.5.2, 1.5.3, 1.5.9, 1.5.7 , 1.5.12, 1.5.4, 1.5.8 (you should do at least three of them!). The take home exercise is Exercise 1.5.1.

Exercise 1.5.1. This exercise provides another possible (inductive) definition of the spaces $\mathcal{D}_k(M)$. For each $f \in C^\infty(M)$, let $m_f \in \mathrm{End}(C^\infty(M))$ be the "multiplication by f" operator. The commutator of two operators P and Q is the new operator $[P, Q] = P \circ Q - Q \circ P$.

Starting with $\mathcal{D}_{-1}(M) = 0$, show that $\mathcal{D}_k(M)$ is the space of linear operators P with the property that

$$[P, m_f] \in \mathcal{D}_{k-1}(M) \ \ \forall f \in C^\infty(M).$$

Exercise 1.5.2. Let $P \in \mathcal{D}_k(U)$ and $Q \in \mathcal{D}_l(U)$. Then the composition QP belongs to $\mathcal{D}_{k+l}$ and

$$\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P).$$

Exercise 1.5.3. Let V be a vector field on M. Show that $\partial_V : f \mapsto Vf := df{\cdot}V$ is a first order differential operator on M. Show that its principal symbol is given by $\sigma_1(P)(x, \xi) = \xi(v_x)$.

Exercise 1.5.4. Show that any differential operator $P \in \mathcal{D}_1(M)$ can be written as

$$P(\phi) = f\phi + \partial_V(\phi)$$

for some unique function $f \in C^\infty(M)$ and vector field V on M.

Exercise 1.5.5. Let $P \in \mathcal{D}_k(M)$ and $Q \in \mathcal{D}_l(M)$. Show that $QP \in \mathcal{D}_{k+l}(M)$. Moreover, $\sigma_{k+l}(QP) = \sigma_l(Q)\sigma_k(P)$. Hint: use reduction to charts.

Exercise 1.5.6. Lemma 1.1.8 gives a different description of the principal symbol without reference to charts. Show that, actually, for any $P \in \mathcal{D}_k(M)$ and any $f \in C^\infty(M)$ and all $\varphi \in C^\infty(M)$

$$f(x)\sigma_k(P)((d\varphi)_x) = \lim_{t\to\infty} t^{-k} e^{-it\varphi(x)} P(e^{it\varphi} f)(x). \tag{1.4}$$ [5]

Exercise 1.5.7. Recall that at the beginning of section 1.2, we associated to a section $C \in \Gamma(\underline{\mathrm{Hom}}\,(E, F))$ an operator (denoted by the same letter) $C : \Gamma(E) \to \Gamma(F)$. Show that this construction defines a 1-1 correspondence between sections of $\underline{\mathrm{Hom}}\,(E, F)$ and maps from $\Gamma(E)$ to $\Gamma(F)$ which are $C^\infty(M)$-linear.

Exercise 1.5.8. Show that, for any two vector bundles E and F over a manifold M, the assignment

$$U \mapsto \mathcal{D}_k(E|_U, F|_U)$$

($U \subset M$ open) is a sheaf on M.

Exercise 1.5.9. Show that the principal symbol of exterior differentiation $d : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1} T^*M)$ is given by

$$\sigma_1(d)(x, \xi) : \wedge^k T_x^*M \to \wedge^{k+1} T_x^*M, \ \ \omega \mapsto i\xi \wedge \omega.$$

[5] recall that $(d\varphi)_x \in T_x^*M$ sends $X_x \in T_xM$ to $T_x\varphi(X_x)$

Exercise 1.5.10. Let V be a finite dimensional complex vector space. Let $v \in V\setminus\{0\}$. Show that the complex of linear maps $T_k : \wedge^k V \to \wedge^{k+1}V$, $x \mapsto v\wedge x$, is exact.

Deduce that the DeRham complex a manifold is an elliptic complex.

In the following three exercises we explain how to relate ellipticity for elliptic complexes, and their Euler characteristic, to ellipticity of differential operators, and their index. We start with a differential complex $\mathcal{E}$ as before and we fix hermitian inner products on each E^k and a density on M. This allows us to talk about the adjoints of P_k and to form the Laplacians

$$\Delta_k := P_k^* P_k + P_{k-1}P_{k-1}^* : \Gamma(E^k) \to \Gamma(E^k).$$

It will be useful to put everything together and consider

$$E = \oplus_k E^k, P = \oplus_k P_k \in \mathcal{D}_d(E), \Delta = P^*P + PP^* = \oplus_k \Delta_k \in \mathcal{D}_{2d}(E).$$

Exercise 1.5.11.

1. Show that Δ is self-adjoint (i.e. it coincides with its formal adjoint), and then deduce that

$$\mathrm{Ker}(\Delta) \oplus \mathrm{Im}(\Delta) \subset \Gamma(E). \tag{1.5}$$

   (the only thing you have to show here is that the sum inside $\Gamma(E)$ is direct).
2. Then show that

$$\mathrm{Ker}(\Delta) = \mathrm{Ker}(P) \cap \mathrm{Ker}(P^*), \mathrm{Im}(\Delta) \subset \mathrm{Im}(P) \oplus \mathrm{Im}(P^*).$$

   and that the map

$$\mathrm{Ker}(\Delta_k) \ni s \mapsto s \bmod B^k(\mathcal{E}) \in H^k(\mathcal{E})$$

   is an injection.
3. Finally, show that if (1.5) becomes equality, then also the last inclusion becomes equality, and the last map becomes an isomorphism.

Exercise 1.5.12. Show that $\mathcal{E}$ is an elliptic complex if and only if Δ is an elliptic operator.

From the general properties of elliptic operators (to be developed later in the course), we will deduce that the inclusion (1.5) is actually an equality for any self-adjoint elliptic operator Δ. Hence, in this case, $\chi((E))$ will be equal to

$$\sum_k (-1)^k \mathrm{dim}(\mathrm{Ker}(\Delta_k)).$$

To relate this to the index of an operator, we introduce

$$E^+ = \oplus_{k-\mathrm{even}} E^k, \quad E^- = \oplus_{k-\mathrm{odd}} E^k, \quad \Delta_+ = \oplus_{k-\mathrm{even}} \Delta_k \in \mathcal{D}_{2d}(E^+, E^-).$$

Exercise 1.5.13. Show that, if the inclusion (1.5) becomes equality, then

$$\chi(\mathcal{E}) = \mathrm{Index}(\Delta_+).$$

LECTURE 2
Distributions on manifolds

As explained in the previous lecture, to show that an elliptic operator between sections of two vector bundles E and F,

$$P : \Gamma(M, E) \to \Gamma(M, F)$$

has finite index, we plan to use the general theory of Fredholm operators between Banach spaces. In doing so, we first have to interpret our P's as operators between certain "Banach spaces of sections". The problem is that the usual spaces of smooth sections $\Gamma(M, E)$ have no satisfactory Banach space structure. Given a vector bundle E over M, by a "Banach space of sections of E", $\mathbb{B}(M, E)$, one should understand (some) Banach space which contains the space $\Gamma(M, E)$ of all the smooth sections of E as a (dense) subspace. One way to introduce such Banach spaces is to consider the completion of $\Gamma(M, E)$ with respect to various norms of interest. This can be carried out in detail, but the price to pay is the fact that the resulting "Banach spaces of sections" have a rather abstract meaning (being defined as completions). We will follow a different path, which is based on the following remark: there is a very general (and natural!!) notion of "generalized sections of a vector bundle E over M", hence a space $\Gamma_{\text{gen}}(M; E)$ of such generalized sections (namely the space $\mathcal{D}'(M, E)$ of distributions, discussed in this lecture), so general that all the other "Banach spaces of sections" are subspaces of $\Gamma_{\text{gen}}(M; E)$. The space $\Gamma_{\text{gen}}(M; E)$ itself will not be a Banach space, but all the Banach spaces of sections which will be of interest for us can be described as subspaces of $\Gamma_{\text{gen}}(M; E)$ satisfying certain conditions (and that is how we will define them).

Implicit in our discussion is the fact that all the spaces we will be looking at will be vector spaces endowed with a topology (t.v.s.'s= topological vector spaces). Although our final aim is to deal with Banach spaces, the general t.v.s.'s will be needed along the way (however, all the spaces we will be looking at will be l.c.v.s.'s= locally convex vector spaces, i.e., similarly to Banach spaces, they can be defined using certain seminorms).

In this lecture, after recalling the notion of t.v.s. (topological vector space) and the special case of l.c.v.s. (locally convex vector space), we will discuss the space of generalized functions (distributions) on opens in $\mathbb{R}^n$ and then their generalizations to functions on manifolds or, more generally, to sections of vector bundles over manifolds. Since t.v.s.'s, l.c.v.s.' and the local theory of generalized functions (distributions) on opens in $\mathbb{R}^n$ have already been discussed in the intensive reminder, our job will be to pass from local (functions on opens in $\mathbb{R}^n$) to global (sections of vector bundles over arbitrary manifolds). However, these lecture notes also contain some of the local theory that has been discussed in the "intensive reminder".

2.1. Locally convex vector spaces

We start by recalling some of the standard notions from functional analysis (which have been discussed in the intensive reminder).

Topological vector spaces

First of all, a **t.v.s.** (**topological vector space**) is a vector space V (over $\mathbb{C}$) together with a topology $\mathcal{T}$, such that the two structures are compatible, i.e. the vector space operations

$$V \times V, (v, w) \mapsto v + w, \ \mathbb{C} \times V \to V, (\lambda, v) \mapsto \lambda v$$

are continuous. Recall that associated to the topology $\mathcal{T}$ and to the origin $0 \in V$, one has the family of all open neighborhoods of 0:

$$\mathcal{T}(0) = \{D \in \mathcal{T} : 0 \in D\}.$$

Since the translations $\tau_x : V \to V$, $y \mapsto y + x$ are continuous, the topology $\mathcal{T}$ is uniquely determined by by $\mathcal{T}(0)$: for $D \subset V$, we have

$$D \in \mathcal{T} \iff \forall \, x \in D \quad \exists \, B \in \mathcal{T}(0) \text{ such that } \ x + B \subset D. \tag{2.1}$$

In this characterization of the opens inside V, one can replace $\mathcal{T}(0)$ by any basis of neighborhoods of 0, i.e. by any family $\mathcal{B}(0) \subset \mathcal{T}(0)$ with the property that

$$D \in \mathcal{T}(0) \Longrightarrow \ \exists \, B \in \mathcal{B} \text{ such that } B \subset D.$$

In other words, if we knows a basis of neighborhoods $\mathcal{B}(0)$ of $0 \in V$, then we know the topology $\mathcal{T}$.

Exercise 2.1.1. Given a family $\mathcal{B}(0)$ of subsets of V containing the origin, what axioms should it satisfy to ensure that the resulting topology (defined by (2.1)) is indeed a topology which makes $(V, \mathcal{T})$ into a t.v.s.?

Note that, in a t.v.s. $(V, \mathcal{T})$, also the convergence can be spelled out in terms of a (any) basis if neighborhoods $\mathcal{B}(0)$ of 0: a sequence $(v_n)_{n \geq 1}$ of elements of V **converges to** $v \in V$, written $v_n \to v$, if and only if:

$$\forall \, B \in \mathcal{B}(0), \quad \exists \, n_B \in \mathbb{Z}_+ \ \text{ such that } \ v_n - v \in B \ \ \forall \, n \geq n_B.$$

Of course, this criterion can be used for $\mathcal{B}(0) = \mathcal{T}(0)$, but often there are smaller bases of neighborhoods $\mathcal{B}(0)$ at hand (after all, "b" is just the first letter of the word "ball"). For instance, if $(V, || - ||)$ is a normed space, then the resulting t.v.s. has as basis of neighborhoods

$$\mathcal{B}(0) = \{B(0, r) : r \geq 0\},$$

where

$$B(0, r) = \{v \in V : ||v|| < r\}.$$

In a t.v.s. $(V, \mathcal{T})$, one can also talk about the notion of Cauchy sequence: a sequence $(v_n)_{n\geq 1}$ in V is called a **Cauchy sequence** if:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ n_D \in \mathbb{Z}_+\ \text{ such that }\ v_n - v_m \in D\ \ \forall\ n, m \geq n_D.$$

Again, if we have a basis of neighborhoods $\mathcal{B}(0)$ at our disposal, it suffices to require this condition for $D = B \in \mathcal{B}(0)$.

In particular, one can talk about completeness of a t.v.s: one says that $(V, \mathcal{T})$ is (sequentially) **complete** if any Cauchy sequence in V converges to some $v \in V$.

Locally convex vector spaces

Recall also that a **l.c.v.s.** (**locally convex vector space**) is a t.v.s. $(V, \mathcal{T})$ with the property that "there are enough convex neighborhoods of the origin". That means that

$$\mathcal{T}_{\text{convex}}(0) := \{C \in \mathcal{T}(0) : C \text{ is convex}\}$$

is a basis of neighborhoods of $0 \in V$ or, equivalently:

$$\forall\ D \in \mathcal{T}(0)\ \ \exists\ \ C \in \mathcal{T}(0) \text{ convex, such that }\ C \subset D.$$

In general, l.c.v.s.'s are associated to families of seminorms (and sometimes this is taken as "working definition" for locally convex vector spaces). First recall that a **seminorm** on a vector space V is a map $p : V \to [0, \infty)$ satisfying

$$p(v + w) \leq p(v) + p(w), \quad p(\lambda v) = |\lambda| p(v),$$

for all $v, w \in V$, $\lambda \in \mathbb{C}$ (and it is called a norm if $p(v) = 0$ happens only for $v = 0$).

Associated to any family

$$P = \{p_i\}_{i\in I}$$

of seminorms (on a vector space V), one has a notion of balls:

$$B^r_{i_1,\ldots,i_n} := \{v \in V : p_{i_k}(v) < r,\ \forall 1 \leq k \leq n\},$$

defined for all $r > 0$, $i_1, \ldots, i_n \in I$. The collection of all such balls form a family $\mathcal{B}(0)$, which will induce a locally convex topology $\mathcal{T}_P$ on V (convex because each ball is convex). Note that, the convergence in the resulting topology is the expected one:

$$v_n \to v\ \text{ in } (V, \mathcal{T}_P) \Longleftrightarrow\ p_i(v_n - v) \to 0\ \ \forall\ i \in I.$$

(and there is a similar characterization for Cauchy sequences). The fact that, when it comes to l.c.v.s.'s it suffices to work with families of seminorms, follows from the following:

Theorem 2.1.2. *A t.v.s. $(V, \mathcal{T})$ is a l.c.v.s. if and only if there exists a family of seminorms P such that $\mathcal{T} = \mathcal{T}_P$.*

Proof Idea of the proof: to produce seminorms, one associates to any $C \subset V$ convex the functional

$$p_C(v) = \inf\{r > 0 : x \in rC\}.$$

Choosingn C "nice enough", this will be a seminorm. One then shows that one can find a basis of neighborhoods of the origin consisting of "nice enough" convex neighborhoods. $\square$

By abuse of terminology, we also say that (V, P) is a l.c.v.s. (but one should keep in mind that all that matters is not the family of seminorms P but just the induced topology $\mathcal{T}_P$).

Remark 2.1.3. In most of the examples of l.c.v.s.'s, the seminorms come first (quite naturally), and the topology is the associated one. However, there are some examples in which the topology comes first and one may not even care of what the seminorms are (see the general construction of inductive limit topologies at the end of this section).

On the other hand, one should be aware that different sets of seminorms may induce the same l.c.v.s. (i.e. the same topology). For instance, if $P_0 \subset P$ is a smaller family of seminorms, but which has the property that, for any $p \in P$, there exists $p_0 \in P_0$ such that $p_0 \leq p$ (i.e. $p_0(v) \leq p(v)$ for all $v \in V$), then P and P_0 define the same topology. This trick will be repeatedly used in the examples.

Exercise 2.1.4. Prove the last statement.

Next, it will be useful to have a criteria for continuity of linear maps between l.c.v.s.'s in terms of the seminorms. The following is a very good exercise.

Proposition 2.1.5. *Let (V, P) and (W, Q) be two l.c.v.s.'s and let*

$$A : V \to W$$

be a linear map. Then T is continuous if and only if, for any $q \in Q$, there exist $p_1, \ldots, p_n \in P$ and a constant $C > 0$ such that

$$q(A(v)) \leq C \cdot max\{p_1(v), \ldots, p_n(v)\} \ \ \forall\, v \in V.$$

Note that we will deal only with l.c.v.s.'s which are separated (Hausdorff).

Exercise 2.1.6. Let (V, P) be a l.c.v.s., where $P = \{p_i\}_{i \in I}$ is a family of seminorms on V. Show that it is Hausdorff if and only if, for $v \in V$, one has the imlication:

$$p_i(v) = 0 \ \ \forall\, i \in I \Longrightarrow \ v = o.$$

.

Finally, recall that a **Frechet space** is a t.v.s. V with the following properties:

1. it is complete.
2. its topology is induced by a countable family of semi-norms $\{p_1, p_2, \ldots\}$.

In this case, it follows that V is metrizable, i.e. the topology of V can also be induced by a (complete) metric:

$$d(v, w) := \sum_{n \geq 1} \frac{1}{2^n} \frac{p_n(v - w)}{1 + p_n(v - w)}.$$

Example 2.1.7. Of course, any Hilbert or Banach space is a l.c.v.s. This applies in particular to all the familiar Banach spaces such as the L^p-spaces on an open $\Omega \subset \mathbb{R}^n$

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is measurable }, \int_\Omega |f|^p < \infty\},$$

with the norm

$$||f||_{L^p} = (\int_\Omega |f|^p)^{1/p}$$

Recall that, for $p = 2$, this is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2} = \int_\Omega f\overline{g}.$$

Example 2.1.8. Another class of examples come from functions of a certain order, eventually with restrictions on their support. For instance, for an open $\Omega \subset \mathbb{R}^n$, $r \in \mathbb{Z}_+$ and $K \subset \Omega$ compact, we consider the space

$$\mathcal{C}^r_K(\Omega) = \{\phi : \Omega \to \mathbb{C} : \phi \text{ is of class } C^r \text{ and } \mathrm{supp}(\phi) \subset K\}.$$

The norm which is naturally associated to this space is $|| \cdot ||_{K,r}$ defined by

$$||\phi||_{r,K} = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \leq r\}.$$

With this norm, $\mathcal{C}^r_K(\Omega)$ becomes a Banach space. Note that convergence in this space is uniform convergence on K of all derivatives up to order r.

However, if we consider $r = \infty$, then $C^\infty_K(\Omega)$ should be considered with the family of seminorms $\{|| \cdot ||_{K,r} : r \in \mathbb{Z}_+\}$. The result is a Frechet space. Note that convergence in this space is uniform convergence on K of all derivatives.

Yet another natural space is the space of all smooth functions $C^\infty(\Omega)$. A nice topology on this space is the one induced by the family of seminorms

$$\{|| \cdot ||_{K,r} : K \subset \Omega \text{ compact}, r \in \mathbb{Z}_+\}.$$

Using an exhaustion of Ω by compacts, i.e. a sequence $(K_n)_{n\geq 0}$ of compacts with

$$\Omega = \cup_n K_n, \quad K_n \subset \mathrm{Int}(K_{n+1}),$$

we see that the original family of seminorms can be replaced by a countable one:

$$\{|| \cdot ||_{K_n,r} : n, r \in \mathbb{Z}_+\}$$

(using Remark 2.1.3, check that the resulting topology is the same!). Hence $C^\infty(\Omega)$ with this topology has the chance of being Frechet- which is actually the case.

Note that convergence in this space is uniform convergence on compacts of all derivatives.

Example 2.1.9. As a very general construction: for any t.v.s. (locally convex or not), there are (at least) two important l.c. topologies on the continuous dual:

$$V^* := \{u : V \to \mathbb{R} : u \text{ is linear and continuous}\}.$$

The first topology, denoted $\mathcal{T}_s$, is the one induced by the family of seminorms $\{p_v\}_{v\in V}$, where

$$p_v : V^* \to \mathbb{R},\ p_v(u) = |u(v)|.$$

This topology is called the weak* topology on V^*, or the topology of simple convergence. Note that $u_n \to u$ in this topology if and only if $u_n(v) \to u(v)$ for all $v \in V$.

The second topology, denoted $\mathcal{T}_b$, called the strong topology (or of uniform convergence on bounded sets) is defined as follows. First of all, recall that a subset $B \subset V$ is called bounded if, for any neighborhood of the origin, there exists $\lambda > 0$ such that $B \subset \lambda V$. If the topology of V is generated by a family of seminorms P, this means that for any $p \in P$ there exists $\lambda_p > 0$ such that

$$B \subset B_p(r_p) = \{v \in V : p(v) < r_p\}.$$

This implies (see also Proposition 2.1.5) that for any continuous linear functional $u \in V^*$,

$$p_B(u) := \sup\{|u(v)| : v \in B\} < \infty.$$

In this way we obtain a family $\{p_B\}_B$ of seminorms (indexed by all the bounded sets B), and $\mathcal{T}_b$ is defined as the induced topology.

A related topology on V^* is the topology $\mathcal{T}_c$ of uniform convergence on compacts, induced by the family of seminorms $\{p_C : C \subset V^* \text{ compact}\}$.

Some explanations (for your curiosity): In this course, when dealing with a particular l.c.v.s. V, what will be of interest to us is to understand the convergence in V, understand continuity of linear maps defined on V or the continuity of maps with values in V (i.e., in practical terms, one may forget the l.c. topology and just keep in mind convergence and continuity). From this point of view, in almost all the cases in which we consider the dual V^* of a l.c.v.s. V (e.g. the space of distributions), in this course we will be in the fortunate situation that it does not make a difference if we use $\mathcal{T}_s$ or $\mathcal{T}_b$ on V^* (note: this does not mean that the two topologies coincide- it just means that the specific topological aspects we are interested in are the same for the two).

What happens is that the spaces we will be dealing with in this course have some very special properties. Axiomatising these properties, one ends up with particular classes of l.c.v.s.'s which can be understood as part of the general theory of l.c.v.s.'s. Here we give a few more details of what is really going on (the references below send you to the book "Topological vector spaces, distributions and kernels" by F. Treves).

First of all, as a very general fact: for any t.v.s. V, $\mathcal{T}_s$ and $\mathcal{T}_c$ induce the same topology on any equicontinuous subset $H \subset V^*$ (Prop. 32.5, pp 340). Recall that H is called equicontinuous if, for every $\epsilon > 0$, there exists a neighborhood B of the origin such that

$$|u(v)| \leq \epsilon, \quad \forall\ v \in B, \quad \forall\ u \in H.$$

An important class of t.v.s.'s is the one of barreled space, which we now recall. A barrel in a t.v.s. V is a non-empty closed subset $A \subset V$ with the following properties:

1. A is absolutely convex: $|\alpha|A + |\beta|A \subset A$ for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$.
2. A is absorbing: $\forall\ \ v \in V, \ \ \exists\ \ r > 0$ such that $v \in rA$.

A t.v.s. V is said to be barreled if any barrel in V is a neighborhood of zero. For instance, all Frechet spaces are barreled.

For a barreled space V, given $H \subset V^*$, the following are equivalent (Theorem 33.2, pp.349):

1. H is weakly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_s)$).
2. H is strongly bounded (i.e. bounded in the l.c.v.s. $(V^*, \mathcal{T}_b)$).

3. H is relatively compact in the weak topology (i.e. the closure of H in $(V^*, \mathcal{T}_s)$ is compact there).
4. H is equicontinuous.

Hence, for such spaces, the notion of "bounded" is the same in $(V^*, \mathcal{T}_s)$ and $(V^*, \mathcal{T}_b)$, and we talk simply about "bounded subsets of V^*. However, the notion of convergence of sequences may still be different; of course, strong convergence implies weak convergence, but all we can say about a weakly convergent sequence is that it is bounded in the strong topology. More can be said for a more special class of t.v.s.'s.

A t.v.s. is called a Montel space if V is barreled and every closed bounded subset of E is compact. Note that this notion is much more restrictive then that of barreled space. For instance, while all Banach spaces are barreled, the only Banach spaces which are Montel are the finite dimensional ones (because the unit ball is compact only in the finite dimensinal Banach spaces). On the other hand, while all Frechet spaces are barreled, there are Frechet spaces which are Montel, but also others which are not Montel. The main examples of Montel spaces which are of interest for us are: the space of smooth functions, and the space of test functions (discussed below).

For a Montel space V, it follows that the topologies $\mathcal{T}_c$ and $\mathcal{T}_b$ are the same (Prop. 34.5, pp. 357). From the general property of equicontinuous subsets H mentioned above, we deduce that on such H's,

$$\mathcal{T}_s|_H = \mathcal{T}_b|_H.$$

(also, by the last result we mentioned, H being equicontinuous is equivalent to being bounded). Taking for H to be the elements of a weakly convergent sequence and its weak limit (clearly weakly bounded!), it follows that the sequence is also strongly convergent; hence convergence w.r.t. $\mathcal{T}_s$ and w.r.t. $\mathcal{T}_b$ is the same. Note that this not implies that the two topologies are the same: we know from point-set topology that the notion of convergence w.r.t. a topology $\mathcal{T}$ does not determine the topology uniquely unless the topology satisfies the first axiom of countability (e.g. if it is metrizable).

As a summary, for Montel spaces V,

1. the notion of boundedness in $(V^*, \mathcal{T}_s)$ and in $(V^*, \mathcal{T}_b)$ is the same (and coincides with equicontinuity).
2. $\mathcal{T}_s$ and $\mathcal{T}_b$ induce the same topology on any bounded $H \subset V^*$.
3. a sequence in V^* is weakly convergent if and only it is strongly convergent.

Inductive limits

As we saw in all examples (and we will see in almost all the other examples), l.c.v.s.'s usually come with naturally associated seminorms and the topology is just the induced one. However, there is an important example in which the topology comes first (and one usually doesn't even bother to find seminorms inducing it): the space of test functions (see next section). This example fits into a general construction of l.c. topologies, known as "the inductive limit". The general framework is the following. Start with

$$X = \text{vector space }, X_\alpha \subset X \text{ vector subspaces such that } \; X = \cup_\alpha X_\alpha,$$

where α runs in an indexing set I. We also assume that, for each α, we have given:

$$\mathcal{T}_\alpha - \text{locally convex topology on } X_\alpha.$$

One wants to associate to this data a topology $\mathcal{T}$ on X, so that

1. $(X, \mathcal{T})$ is a l.c.v.s.

2. all inclusions $i_\alpha : X_\alpha \to X$ become continuous.

There are many such topologies (usually the "very small" ones, e.g. the one containing just $\emptyset$ and X itself) and, in general, if $\mathcal{T}$ works, then any $\mathcal{T}' \subset \mathcal{T}$ works as well. The question is: is there "the best one" (i.e. the smallest one)? The answer is yes, and that is what the inductive limit topology on X (associated to the initial data) is. On short, this is induced by the following basis of neighborhoods:

$$\mathcal{B}(0) := \{B \subset X : B - \text{convex such that } B \cap X_\alpha \in \mathcal{T}_\alpha(0) \text{ for all } \ \alpha \in I\},$$

(show that one gets a l.c. topology and it is the largest one!). One should keep in mind that what is important about $(X, \mathcal{T})$ is to recognize when a function on X is continuous, and when a sequence in X converges. The first part is a rather easy exercise with the following conclusion:

Proposition 2.1.10. *Let X be endowed with the inductive limit topology $\mathcal{T}$, let Y be another l.c.v.s. and let*

$$A : X \to Y$$

be a linear map. Then A is continuous if and only if each

$$A_\alpha := A|_{X_\alpha} : X_\alpha \to Y$$

is.

The recognition of convergent subsequences is a bit more subtle and, in order to have a more elegant statement, we place ourselves in the following situation: the indexing set I is the set $\mathbb{N}$ of positive integers,

$$X_1 \subset X_2 \subset X_3 \subset \ldots, \quad , X_n - \text{closed in } X_{n+1}, \quad \mathcal{T}_n = \mathcal{T}_{n+1}|_{X_n}$$

(i.e. each $(X_n, \mathcal{T}_n)$ is embedded in $(X_{n+1}, \mathcal{T}_{n+1})$ as a closed subspace). We assume that all the inclusions are strict. The following is a quite difficult exercise.

Theorem 2.1.11. *In the case above, a sequence$(x_n)_{n\geq 1}$ of elements in X converges to $x \in X$ (in the inductive limit topology) if and only if the following two conditions hold:*

1. $\exists$ n_0 such that $x, x_m \in X_{n_0}$ for all m.
2. $x_m \to x$ in X_{n_0}.

(note: one can also show that $(X, \mathcal{T})$ cannot be metrizable).

2.2. Distributions: the local theory

In this section we recall the main functional spaces on $\mathbb{R}^n$ or, more generally, on any open $\Omega \subset \mathbb{R}^n$. Recall that, for $K \subset \mathbb{R}^n$ and $r \in \mathbb{N}$, one has the seminorm $|| \cdot ||_{K,r}$ on $C^\infty(\Omega)$ given by:

$$||f||_{r,K} = \sup\{|\partial^\alpha f(x)| : x \in K, |\alpha| \leq r\}.$$

$\mathcal{E}(\Omega)$: smooth functions:

One defines
$$\mathcal{E}(\Omega) := C^{\infty}(\Omega),$$
endowed with the locally convex topology induced by the family of seminorms $\{||\cdot||_{K,r}\}_{K\subset\Omega \text{ compact}, r\in\mathbb{Z}_+}$. This was already mentioned in Example 2.1.8. Hence, in this space, convergence means: $f_n \to f$ if and only if for each multi-index α and each compact $K \subset \Omega$, $\partial^{\alpha} f_n \to \partial^{\alpha} f$ uniformly on K.

As a l.c.v.s, it is a Frechet space (and is also a Montel space).

Algebraically, $\mathcal{E}(\Omega)$ is also a ring (or even an algebra over $\mathbb{C}$), with respect to the usual multiplication of functions. Note that this algebraic operation is continuous.

$\mathcal{D}(\Omega)$: compactly supported smooth functions (test functions):

One defines
$$\mathcal{D}(\Omega) := C_c^{\infty}(\Omega),$$
the space of smooth functions with compact support, with the following topology. First of all, for each $K \subset \Omega$, we consider
$$\mathcal{E}_K(\Omega) := C_K^{\infty}(\Omega),$$
the space of smooth functions with support inside K, endowed with the topology induced from the topology of $\mathcal{E}(\Omega)$ (which is the same as the topology discussed in Example 2.1.8., i.e. induced by the family of seminorms $\{||\cdot||_{K,r}\}_{r\in\mathbb{Z}_+}$. While, set theoretically (or as vector spaces),
$$\mathcal{D}(\Omega) = \cup_K \mathcal{E}_K(\Omega)$$
(union over all compacts $K \subset \Omega$), we consider the inductive limit topology on $\mathcal{D}(\Omega)$ (see the end of the previous section).

Convergent sequences are easy to recognize here: $f_n \to f$ in $\mathcal{D}(\Omega)$ if and only if there exist a compact K such that $f_n \in \mathcal{E}_K$ for all n, and $f_n \to f$ in $\mathcal{E}_K$ (indeed, using an exhaustion of Ω by compacts (see again Example 2.1.8), we see that we can place ourselves under the conditions which allow us to apply Theorem 2.1.11).

As a l.c.v.s., $\mathcal{D}(\Omega)$ is complete but it is not Frechet (see the end of Theorem 2.1.11). (However, it is a Montel space).

Algebraically, $\mathcal{D}(\Omega)$ is also an algebra over $\mathbb{C}$ (with respect to pointwise multiplication), which is actually an ideal in $\mathcal{E}(\Omega)$ (the product between a compactly supported smooth function and an arbitrary smooth function is again compactly supported).

$\mathcal{D}'(\Omega)$: distributions:

The space of distributions on $\mathbb{R}^n$ is defined as the (topological) dual of the space of test functions:
$$\mathcal{D}'(\Omega) := (\mathcal{D}(\Omega))^*$$
(see also Example 2.1.9). An element of this space is called a distribution on Ω. Unraveling the inductive limit topology on $\mathcal{D}(\Omega)$, one gets a more explicit

description of these space. More precisely, using Proposition 2.1.10 to recognize the continuous linear maps by restricting to compacts, and using Proposition 2.1.5 to rewrite the resulting continuity conditions in terms of seminorms, one finds the following:

Corollary 2.2.1. *A distribution on Ω is a linear map*

$$u : C_c^{\infty}(\Omega) \to \mathbb{C}$$

with the following property: for any compact $K \subset \Omega$, there exists $C = C_K > 0$, $r = r_K \in \mathbb{N}$ such that

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall \ \phi \in C_K^{\infty}(\Omega).$$

As a l.c.v.s., $\mathcal{D}'(U)$ will be endowed with the strong topology (the topology of uniform convergence on bounded subsets- see Example 2.1.9)). Note however, when it comes to convergence of sequences (u_n) of distributions, the strong convergence is equivalent to simple (pointwise) convergence. [1]

In general, any smooth function f induces a distribution u_f

$$\phi \mapsto \int_{\mathbb{R}^n} f\phi,$$

and this correspondence defines a continuous inclusion of

$$i : \mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

For this reason, distributions are often called "generalized functions", and one often identifies f with the induced distribution u_f.

Algebraically, the multiplication on $\mathcal{E}(\Omega)$ extends to a $\mathcal{E}(\Omega)$-module structure on $\mathcal{D}'(\Omega)$

$$\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), \ (f, u) \mapsto fu,$$

where

$$(fu)(\phi) = u(f\phi).$$

---

[1] Explanation (for your curiosity): When it comes to the following notions:

1. bounded subsets of $\mathcal{D}'(\Omega)$,
2. convergence of sequences in $\mathcal{D}'(\Omega)$,
3. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a Frechet space V (e.g. $V = \mathcal{E}(\Omega')$).
4. continuity of a linear map $A : V \to \mathcal{D}'(\Omega)$ defined on a l.c.v.s. V which is the inductive limit of Frechet spaces (e.g. $V = \mathcal{D}(\Omega')$).

(notions which depend on what topology we use on $\mathcal{D}'(\Omega)$), it does not matter whether we use the strong topology $\mathcal{T}_b$ or the weak topology $\mathcal{T}_s$ on $\mathcal{D}'(\Omega)$: the priory different resulting notions will actually coincide.

For boundedness and convergence this follows from the fact that $\mathcal{D}(\Omega)$ is a Montel space (Theorem 34.4, pp. 357 in the book by Treves). For continuity of linear maps defined on a Frechet space, one just uses that, because V is metrizable, continuity is equivalent to sequential continuity (i.e. the property of sending convergent sequences to convergent sequences) and the previous part. If V is an inductive limit of Frechet spaces one uses the characterization of continuity of linear maps defined on inductive limits (Proposition 2.1.10).

$\mathcal{E}'(\Omega)$: compactly supported distributions:

The space of compactly supported distributions on Ω is defined as the (topological) dual of the space of all smooth functions

$$\mathcal{E}'(\Omega) := (\mathcal{E}(\Omega))^*.$$

Using Proposition 2.1.5 to rewrite the continuity condition, we find:

Corollary 2.2.2. *A compactly supported distribution on Ω is a linear map*

$$u : C^\infty(\Omega) \to \mathbb{C}$$

with the following property: there exists a compact $K \subset \Omega$, $C > 0$ and $r \in \mathbb{N}$ such that

$$|u(\phi)| \leq C||\phi||_{K,r} \quad \forall\, \phi \in C^\infty(\Omega).$$

Again, as in the case of $\mathcal{D}'(\Omega)$, we endow $\mathcal{E}'(\Omega)$ with the strong topology.[2]

Note that the dual of the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}$ induces a continuous inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Explicitly, any linear functional on $C^\infty(\Omega)$ can be restricted to a linear functional on $C_c^\infty(\Omega)$, and the estimates for the compactly supported distributions imply the ones for distributions.

Hence the four distributional spaces fit into a diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}' \arrow[r] & \mathcal{D}' \end{tikzcd},$$

in which all the arrows are (algebraic) inclusions which are continuous, and the spaces on the left are (topologically) the compactly supported version of the spaces on the right.

Supports of distributions

Next, we recall why $\mathcal{E}'(\Omega)$ is called the space of *compactly supported* distributions. The main remark is that the assignment

$$\Omega \mapsto \mathcal{D}'(\Omega)$$

defines a sheaf and, as for any sheaf, one can talk about sections with compact support. What happens is that the elements in $\mathcal{D}'(\Omega)$ which have compact support in this sense, are precisely the ones in the image of the inclusion $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.

Here are some details. First of all, for any two opens $\Omega \subset \Omega'$, one has an inclusion ("extension by zero")

$$\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}(\Omega'),\ f \mapsto \tilde{f},$$

[2]Explanation (for your curiosity): The same discussion as in the case of $\mathcal{D}'(\Omega)$ applies also to $\mathcal{E}'(\Omega)$. This is due to the fact that also $\mathcal{E}(\Omega)$ is a Montel space (with the same reference as for $\mathcal{D}(\Omega)$). Hence, when it comes to bounded subsets, convergent sequences, continuity of linear maps from a (inductive limit of) Frechet space(s) to $\mathcal{E}'(\Omega)$, it does not matter whether we use the strong topology $\mathcal{T}_b$ or the simple topology $\mathcal{T}_s$ on $\mathcal{E}'(\Omega)$.

where $\tilde{f}$ is f on Ω and zero outside. Dualizing, we get a "restriction map",

$$\mathcal{D}'(\Omega') \to \mathcal{D}'(\Omega), \ u \mapsto u|_{\Omega}.$$

The sheaf property of the distributions is the following property which follows immediately from a partition of unity argument:

Lemma 2.2.3. *Assume that $\Omega = \cup_i \Omega_i$, with $\Omega_i \subset \mathbb{R}^n$ opens, and that u_i are distributions on Ω_i such that, for all i and j,*

$$u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j}.$$

Then there exists a unique distribution u on Ω such that

$$u|_{\Omega_i} = u_i$$

for all i.

Proof See your notes from the last lecture in the intensive reminder. □

From this it follows that, for any $u \in \mathcal{D}'(\Omega)$, there is a largest open $\Omega_u \subset \Omega$ on which u vanishes (i.e. $u|_{\Omega_u} = 0$).

Definition 2.2.4. For $u \in \mathcal{D}'(\Omega)$, define its support

$$\mathrm{supp}(u) = \Omega - \Omega_u = \{x \in \Omega : u|_{V_x} = 0 \text{ for any neighborhood } V_x \subset \Omega \text{ of } x\}.$$

We say that u is compactly supported if $\mathrm{supp}(f)$ is compact.

Example 2.2.5. For any $x \in \Omega$, one has the distribution δ_x defined by

$$\delta_x(\phi) = \phi(x).$$

It is not difficult to check that its support is precisely $\{x\}$.

Exercise 2.2.6. Show that $u \in \mathcal{D}'(\Omega)$ has compact support if and only if it is in the image of the inclusion

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Derivatives of distributions and Sobolev spaces

Finally, we discuss one last property of distributions which is of capital importance: one can talk about the partial derivatives of any distribution! The key (motivating) remark is the following, which follows easily from integration by parts.

Lemma 2.2.7. *Let $f \in C^{\infty}(\Omega)$ and let u_f be the associated distribution.*

Let $\partial^{\alpha} f \in C^{\infty}(\Omega)$ be the higher derivative of f associated to a multi-index α, and let $u_{\partial^{\alpha} f}$ be the associated distribution.

Then $u_{\partial^{\alpha} f}$ can be expressed in terms of u_f by:

$$u_{\partial^{\alpha} f}(\phi) = (-1)^{|\alpha|} u_f(\partial^{\alpha} f).$$

This shows that the action of the operator ∂^{α} on smooth functions can be extended to distributions.

Definition 2.2.8. Fora distribution u on Ω and a multi-index α, one defines the new distribution $\partial^{\alpha} u$ on Ω, by

$$(\partial^{\alpha} u)(\phi) = (-1)^{\alpha} u(\partial^{\alpha} \phi), \ \ \forall \ \phi \in C^{\infty}_c(\Omega).$$

Example 2.2.9. The distribution u_f makes sense not only for smooth functions on Ω, but also for functions $f : \Omega \to \mathbb{C}$ with the property that $\phi f \in L^1(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$ (so that the integral defining u_f) is absolutely convergent. In particular it makes sense for any $f \in L^2(\Omega)$ and, as before, this defines an inclusion

$$L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

We now see one of the advantages of the distributions: any $f \in L^2$, although it may even not be continuous, has derivatives $\partial^\alpha f$ of any order! Of course, they may fail to be functions, but they are distributions. In particular, it is interesting to consider the following spaces.

Definition 2.2.10. For any $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, we define the Sobolev space on Ω of order r as:

$$H_r(\Omega) := \{u \in \mathcal{D}'(\Omega) : \partial^\alpha(u) \in L^2(\Omega) \text{ whenever } |\alpha| \leq r\},$$

endowed with the inner product

$$\langle u, u' \rangle_{H_r} = \sum_{|\alpha| \leq r} \langle \partial^\alpha u, \partial^\alpha u' \rangle_{L^2}.$$

In this way, $H_r(\Omega)$ becomes a Hilbert space.

2.3. Distributions: the global theory

The l.c.v.s.'s $\mathcal{E}(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ can be extended from opens $\Omega \subset \mathbb{R}^n$ to arbitrary manifold M (allowing us to talk about distributions on M, or generalized functions on M) and, more generally, to arbitrary vector bundles E over a manifold M (allowing us to talk about distributional sections of E, or generalized sections of E). To explain this extension, we fix M to be an n-dimensional manifold, and let E be a complex vector bundle over M of rank p.

$\underline{\mathcal{E}(M; E)}$ **(smooth sections):**

One defines

$$\mathcal{E}(M; E) := \Gamma(E),$$

the space of all smooth sections of E endowed with the following local convex topology. To define it, we choose a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M by opens which are domains of "total trivializations" of E, i.e. both of charts (U_i, κ_i) for M as well as of trivializations $\tau_i : E|_{U_i} \to U_i \times \mathbb{C}^p$ for E. This data clearly induce an isomorphism of vector spaces

$$\phi_i : \Gamma(E|_{U_i}) \to C^\infty(\kappa_i(U_i))^p$$

(see also subsection 2.3 below). Altogether, and after restricting sections of E to the various $U_i's$, these define an injection

$$\phi : \Gamma(E) \to \Pi_i C^\infty(\kappa_i(U_i))^p = \Pi_i \mathcal{E}(\kappa_i(U_i))^p.$$

Endowing the right hand side with the product topology, the topology on $\Gamma(E)$ is the induced topology (via this inclusion). Equivalently, considering as indices $\gamma = (i, l, K, r)$ consisting of $i \in I$ (to index the open U_i), $1 \leq l \leq p$ (to index the

l-th component of $\phi(s|_{U_i})$), $K \subset \kappa(U_i)$ compact and r- non-negative integer, one has seminorms $||\cdot||_\gamma$ on $\Gamma(E)$ as follows: for $s \in \Gamma(E)$, restrict it to U_i, move it to $\mathcal{E}(\kappa_i(U_i))^p$ via ϕ_i, take its l-th component, and apply the seminorm $||\cdot||_{K,r}$ of $\mathcal{E}(\kappa_i(U_i))$:

$$||s||_\gamma = ||\phi(s|_{U_i})^l||_{r,K}.$$

Putting together all these seminorms will define the desired l.c. topology on $\Gamma(E)$.

Exercise 2.3.1. Show that this topology does not depend on the choices involved.

Note that, since the cover $\mathcal{U}$ can be chosen to be countable (our manifolds are always assumed to satisfy the second countability axiom!), it follows that our topology can be defined by a countable family of seminorms. Using the similar local result, you can now do the following:

Exercise 2.3.2. Show that $\mathcal{E}(M; E)$ is a Frechet space.

Finally, note that a sequence $(s_m)_{m\geq 1}$ converges to s in this topology if and only if, for any open U which is the domain of a local chart κ for M and of a local frame $\{s_1, \ldots, s_p\}$ for E, and for any compact $K \subset U$, writing $s_m = (f_m^1, \ldots, f_m^p)$, $s = (f^1, \ldots, f^p)$ with respect to the frame, all the derivatives $\partial_\kappa^\alpha(f_m^i)$ converge uniformly on K to $\partial_\kappa^\alpha(f^i$ (when $m \to \infty$).

When $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{E}(M)$. As in the local theory, this is an algebra (with continuous multiplication). Also, the multiplication of sections by functions make $\mathcal{E}(M, E)$ into a module over $\mathcal{E}(M)$.

$\underline{\mathcal{D}(M, E)}$ (compactly supported smooth sections)

One defines

$$\mathcal{D}(M, E) := \Gamma_c(E),$$

the space of all compactly supported smooth sections endowed with the following l.c. topology defined exactly as in the local case: one writes

$$\mathcal{D}(M, E) = \cup_K \mathcal{E}_K(M; E),$$

where the union is over all compacts $K \subset M$, and $\mathcal{E}_K(M; E) \subset \mathcal{E}(M; E)$ is the space of smooth sections supported in K, endowed with the topology induced from $\mathcal{E}(M; E)$; on $\mathcal{D}(M, E)$ we consider the inductive limit topology.

Exercise 2.3.3. Describe more explicitly the convergence in $\mathcal{D}(M, E)$.

Again, when $E = \mathbb{C}_M$ is the trivial line bundle over M, we simplify the notation to $\mathcal{D}(M)$.

$\underline{\mathcal{D}'(M; E)}$ (generalized sections):

This is the space of distributional sections of E, or the space of generalized sections of E. To define it, we do not just take the dual of $\mathcal{D}(M, E)$ as in the local case, but we first:

1. Consider the complex density line bundle $D = D_M$ on M.[3] All we need to know about D is that its compactly supported sections can be integrated

[3]This is the complex version of the density bundle from the intensive reminder. Here is the summary (the complexified version of what we have already discussed). Given an n-dimensional vector space V, one defines $D_r(V)$, the space of r-densities (for any real number $r > 0$), as the set of all maps $\omega : \Lambda^n V \to \mathbb{C}$ satisfying

$$\omega(\lambda\xi) = |\lambda|^r \omega(\xi), \ \forall\ \xi \in \Lambda^n V.$$

Equivalently (and maybe more intuitively), one can use the set $\mathrm{Fr}(V)$ of all frames of V (i.e. ordered sets $(e_1, \ldots, e_n)$ of vectors of V which form a basis of V). Then $D_r(V)$ can also be described as the set of all functions

$$\omega : Fr(V) \to \mathbb{C}$$

with the property that, for any invertible n by n matrix A, and any frame e, for the new frame $A(e)$ one has

$$\omega(A(e)) = |\det(A)|^r \omega(e).$$

Intuitively, one may think of an r-density on V as some rule of computing volumes of the hypercubes (each frame determines such a hypercube). For each r, $D_r(V)$ is one dimensional (hence isomorphic to $\mathbb{C}$), but in a non-canonical way. Choosing a frame e of V, one has an induced r-density denoted

$$\omega_e = |e^1 \wedge \ldots \wedge e^n|_r$$

uniquely determined by the condition that $\omega_e(e) = 1$ (the e^i's in the notation stand for the dual basis of V^*).

For a manifold M, we apply this construction to all the tangent spaces to obtain a line bundle $D_r(M)$ over M, whose fiber at $x \in M$ is $D_r(T_xM)$. For $r = 1$, $D_1(M)$ is simply denoted D, or D_M whenever it is necessary to remove ambiguities. The sections of D are called densities on M.

Any local chart $(U, \kappa = (x^1_\kappa, \ldots, x^n_\kappa))$ induces a frame $(\partial/\partial x^i_\kappa)_x$ for T_xM with the dual frame $(dx^i_\kappa)_x$ for T^*_xM, for all $x \in U$. Hence we obtain an induced trivialization of $D_r(M)$ over U, with trivializing section

$$|dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

(and, as usual, the smooth structure on D is so that these sections induced by the local charts are smooth).

An r-density on M is a section ω of $D_r(M)$. Hence, locally, with respect to a coordinate chart as before, such a density can be written as

$$\omega = f_\kappa \circ \kappa \cdot |(dx^1_\kappa \wedge \ldots \wedge dx^n_\kappa|_r$$

for some smooth function defined on $\kappa(U)$. If we consider another coordinate chart κ' on the same U then, after a short (but instructive) computation, wee see that f_κ changes according to the rule:

$$f_\kappa = |\mathrm{Jac}(h)|^r f_{\kappa'} \circ h,$$

where $h = \kappa' \circ \kappa^{-1}$ is the change of coordinates, and $\mathrm{Jac}(h)$ is the Jacobian of h. The case $r = 1$ reminds us of the usual integration and the change of variable formula: the usual integration of compactly supported functions on an open $\Omega \subset \mathbb{R}^n$ defines a map

$$\int_\Omega : C^\infty_c(\Omega) \to \mathbb{C}$$

and, if we move via a diffeomorphism $h : \Omega \to \Omega'$, one has the change of variables formula

$$\int_\Omega f = \int_\Omega |\mathrm{Jac}(h)| \cdot f \circ h.$$

Hence, for 1-densities on the domain U of a coordinate chart, one has an induced integration map

$$\int_U : \Gamma_c(U, D|_U) \to \mathbb{C}$$

over M without any further choice, i.e. there is an integral

$$\int_M : \Gamma_c(D) \to \mathbb{C}.$$

If you are more familiar with integration of (top-degree) forms, you may assume that M has an orientation, $D = \Lambda^n T^*M \otimes \mathbb{C}$- the space of $\mathbb{C}$-valued n-forms (an identification induced by the orientation), and $\int_M$ is the integral that you already know. Or, if you are more familiar with integration of functions on Riemannian manifolds, you may assume that M is endowed with a metric, D is the trivial line bundle (an identification induced by the metric) and that $\int_M$ is the integral that you already know.

2. Consider the "functional dual" of E:

$$E^\vee := E^* \otimes D = \mathrm{Hom}(E, D),$$

the bundle whose fiber at $x \in M$ is the complex vector space consisting of all ($\mathbb{C}$-)linear maps $E_x \to D_x$.

The main point about $E^\vee$ is that it comes with a "pairing" (pointwise the evaluation map)

$$< -, - >: \Gamma(E^\vee) \times \Gamma(E) \to \Gamma(D)$$

(and its versions with supports) and then, using the integration of sections of D, we get *canonical* pairings

$$[-,-] : \Gamma_c(E^\vee) \times \Gamma(E) \to \mathbb{C}, \quad (s_1, s_2) \mapsto \int_M < s_1, s_2 > .$$

We now define

$$\mathcal{D}'(M; E) := (\mathcal{D}(M, E^\vee))^*$$

(endowed with the strong topology). Note that, it is precisely because of the way that $E^\vee$ was constructed, that we have canonical (i.e. independent of any choices, and completely functorial) inclusions

$$\mathcal{E}(M; E) \hookrightarrow \mathcal{D}'(M; E),$$

sending a section s to the functional $u_s :=< \cdot, s >$. And, as before, we identify s with the induced distribution u_s.

When $E = \mathbb{C}_M$, we simplify the notation to $\mathcal{D}'(M)$.

As for the algebraic structure, as in the local case, $\mathcal{D}'(M; E)$ is a module over $\mathcal{E}(M)$, with continuous multiplication

$$\mathcal{E}(M) \times \mathcal{D}'(M; E) \to \mathcal{D}'(M; E)$$

(by sending ω to $\int_{\kappa(U)} f_\kappa$) which does not depend on the choice of the coordinates. For the global integration map

$$\int_M : \Gamma_c(M, D) \to \mathbb{C},$$

one decomposes an arbitrary compactly supported density Ω on M as a finite sum $\sum_i \omega_i$, where each ω_i is supported in the domain of a coordinate chart U_i (e.g. use partitions of unity) and put

$$\int_M \omega = \sum_i \int_{U_i} \omega_i.$$

Of course, one has to prove that this does not depend on the way we decompose ω as such a sum, but this basically follows from the additivity of the usual integral.

defined by

$$(fu)(s) = u(fs).$$

$\underline{\mathcal{E}'(M;E)}$ (compactly supported generalized sections):

This is the space of compactly supported distributional sections of E, or the space compactly supported generalized sections of E. It is defined as in the local case (but making again use of $E^\vee$), as

$$\mathcal{E}'(M;E) := (\mathcal{E}(M, E^\vee))^*.$$

Note that, by the same pairing as before, one obtains an inclusion

$$\mathcal{D}(M,E) \hookrightarrow \mathcal{E}'(M,E).$$

Hence, as in the local case, we obtain a diagram of inclusions

$$\begin{tikzcd} \mathcal{D}(M,E) \arrow[r] \arrow[d] & \mathcal{E}(M,E) \arrow[d] \\ \mathcal{E}'(M,E) \arrow[r] & \mathcal{D}'(M,E) \end{tikzcd} .$$

Example 2.3.4.

1. when $E = \mathbb{C}_M$ is the trivial line bundle over M, we have shorten the notations to $\mathcal{D}(M)$, $\mathcal{E}(M)$ etc. Hence, as vector spaces,
$$\mathcal{D}(M) = \Gamma_c(M),\ \mathcal{E}(M) = C^\infty(M),$$
while the elements of $\mathcal{D}'(M)$ will be called distributions on M.
2. staying with the trivial line bundle, but assuming now that $M = \Omega$ is an open subset of $\mathbb{R}^n$, we recover the spaces discussed in the previous section. Note that, in the case of distributions, we are using the identification of the density bundle with the trivial bundle induced by the section $|dx^1 \dots dx^n|$.
3. when $E = \mathbb{C}^p_M$ is the trivial bundle over M of rank p, then clearly
$$\mathcal{D}(M, \mathbb{C}^p_M) = \mathcal{D}(M)^p,\ \mathcal{E}(M, M\times\mathbb{C}^p_M) = \mathcal{E}(M)^p.$$
On the other hand, using the canonical identification between E^* and E, we also obtain
$$\mathcal{D}'(M, \mathbb{C}^p_M) = \mathcal{D}'(M)^p,\ \mathcal{E}'(M, \mathbb{C}^p_M) = \mathcal{E}'(M)^p.$$

Note that, as in the local theory, distributions $u \in \mathcal{D}'(M,E)$ can be restricted to arbitrary opens $U \subset M$, to give distributions $u|_U \in \mathcal{D}'(U, E|_U)$. More precisely, the restriction map

$$\mathcal{D}'(M,E) \to \mathcal{D}'(U, E|_U)$$

is defined as the dual of the map

$$\mathcal{D}'(U, E^\vee|_U) \to \mathcal{D}(M, E^\vee)$$

which takes a compactly supported section defined on U and extends it by zero outside U.

Exercise 2.3.5. For a vector bundle E over M,

1. Show that $U \mapsto \mathcal{D}(U, E|_U)$ forms a sheaf over M.
2. Define the support of any $u \in \mathcal{D}(M, E)$.
3. Show that the injection
$$\mathcal{E}'(M, E) \hookrightarrow \mathcal{D}'(M, E)$$
identifies $\mathcal{E}'(M, E)$ with the space of compactly supported distributional sections (as a vector space only!).

Exercise 2.3.6. Show that $\mathcal{D}(M, E)$ is dense in $\mathcal{E}(M, E)$, $\mathcal{D}'(M, E)$ and $\mathcal{E}'(M, E)$ (you are allowed to use the fact that this is known for trivial line bundles over opens in $\mathbb{R}^n$).

Invariance under isomorphisms

Given two vector bundles, E over M and F over a manifold N, an isomorphism h between E and F is a pair (h, h_0), where $h_0 : M \to N$ is a diffeomorphism and $h : E \to F$ is a map which covers h_0 (i.e. sends the fiber E_x to $F_{h_0(x)}$ or, equivalently, the diagram below is commutative) and such that, for each $x \in M$, it restricts to a linear isomorphism between E_x and $F_{h_0(x)}$.

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d] & F \arrow[d] \\ M \arrow[r, "h_0"] & N \end{tikzcd} .$$

We now explain how such an isomorphism h induces isomorphisms between the four functional spaces of E and those of F (really an isomorphism between the diagrams they fit in). The four isomorphism from the functional spaces of E to those of F will be denoted by the same letter h_*, while its inverse by h^*. At the level of smooth sections, this is simply

$$h_* : \mathcal{E}(M, E) \to \mathcal{E}(N, F), \ h_*(s)(y) = h(s(h_0^{-1}(y))),$$

which also restricts to the spaces $\mathcal{D}$. At the level of generalized sections, it is more natural to describe the map

$$h^* : \mathcal{D}'(N, F) \to \mathcal{D}'(M, E),$$

(and h_* will be its inverse). This will be dual of a map

$$h^\vee : \mathcal{D}(F^\vee) \to \mathcal{D}(E^\vee)$$

defined by

$$h^\vee(u)(e_x) = h_0^*(u(h(e_x))), \quad (e_x \in E_x),$$

where we have used the pull-back of densities, $h_0^* : D_{N,h_0(x)} \to D_{M,x}$.

The same formula defines h^* on the spaces $\mathcal{E}'$.

Example 2.3.7. Given a rank p vector bundle E over M, one often has to choose opens $U \subset M$ which are domains of both a coordinate chart (U, κ) for M as well as the domains of a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ for E. We say that (U, κ, τ) is a total trivialization for E over U. Note that such a data defines an isomorphism h between the vector bundle $E|_U$ over U and the trivial bundle $\kappa(U) \times \mathbb{C}^p$:

$$h_0 = \kappa,\ h(e_x) = (\kappa(x), \tau(e_x)).$$

Hence any total trivialization (U, κ, τ) induces isomorphisms

$$h_{\kappa,\tau} : \mathcal{D}(U, E|_U) \to \mathcal{D}(\kappa(U))^p,\ h_{\kappa,\tau} : \mathcal{E}(U, E|_U) \to \mathcal{E}(\kappa(U))^p,\ \text{etc.}$$

(see also Example 2.3.4).

2.4. General operators and kernels

Given two vector bundles, E over a manifold M and F over a manifold N, an operator from E to F is, roughly speaking, a linear map which associates to a "section of E" a "section of F". The quotes refer to the fact that there are several different choices for the meaning of sections: ranging from smooth sections to generalized sections, or versions with compact supports (or other types of sections). The most general type of operators are is following.

Definition 2.4.1. If E is a vector bundle over M and F is a vector bundle over F, a general operator from E to F is a linear continuous map

$$P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Remark 2.4.2. Note that general operators are often described with different domains and codomains. For instance, if $\mathcal{F}_1$ and $\mathcal{F}_2$ is any of the symbols $\mathcal{E}$, $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{D}'$ (or any of the other functional spaces that will be discussed in the next lecture), one can look at continuous linear operators

$$P : \mathcal{F}_1(M, E) \to \mathcal{F}_2(M, F). \tag{2.2}$$

But since in all cases $\mathcal{D} \subset \mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathcal{D}'$ (with continuous inclusions), P does induce a general operator

$$P_{\text{gen}} : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Conversely, since $\mathcal{D}(M, E)$ is dense in all the other functional spaces that we have discussed (Exercise 2.3.6), P_{gen} determines P uniquely. Hence, saying that we have an operator (2.2) is the same as saying that we have a general operator P_{gen} with the property that it extends to $\mathcal{F}_1(M, E)$, giving rise to a continuous operator taking values in $\mathcal{F}_2(M, F)$.

On the other extreme, one has the so called smoothing operators, i.e. operators which transforms generalized sections into smooth sections.

Definition 2.4.3. If E is a vector bundle over M and F is a vector bundle over F, a smoothing operator from E to F is a linear continuous map

$$P : \mathcal{E}'(M, E) \to \mathcal{E}(N, F).$$

We denote by $\Psi^{-\infty}(E, F)$ the space of all such smoothing operators. When E and F are the trivial line bundles, we will simplify the notation to $\Psi^{-\infty}(M)$.

In other words, a smoothing operator is a general operator $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$ which

1. P takes values in $\mathcal{E}(N, F)$.

2. P extends to a continuous linear map from $\mathcal{E}'(M, E)$ to $\mathcal{E}(N, F)$.

A very useful way of interpreting operators is in terms of their so called "kernels". The idea of kernel is quite simple- and to avoid (just some) notational complications, let us first briefly describe what happens when $M = U \subset \mathbb{R}^m$ and $N = V \subset \mathbb{R}^n$ are two open, and the bundles involved are the trivial line bundles. Then the idea is the following: any $K \in C^\infty(V \times U)$ induces an operator

$$P_K : \mathcal{D}(U) \to \mathcal{E}(V), \quad K(\phi)(y) = \int_U K(y, x)\phi(x)dx.$$

Even more: composing with the inclusion $\mathcal{E}(V) \hookrightarrow \mathcal{D}'(V)$, i.e. viewing P_K as an application

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V),$$

this map does not depend on K as a smooth functions, but just on K as a distribution (i.e. on $u_K \in \mathcal{D}'(V \times U)$). Indeed, for $\phi \in \mathcal{D}(U)$, $P_K(\phi)$, as a distribution on V, is

$$u_{P_K(\phi)} : \psi \mapsto \int_V P_K(\phi)\psi = \int_{V\times U} K(y, x)\psi(y)\phi(x)dydx = u_K(\psi \otimes \phi),$$

where $\psi \otimes \phi \in C^\infty(V \times U)$ is the map $(y, x) \mapsto \psi(y)\phi(x))$. In other words, any

$$K \in \mathcal{D}'(V \times U)$$

induces a linear operator

$$P_K : \mathcal{D}(U) \to \mathcal{D}'(V), \; P_K(\phi)(\psi) = K(\psi \otimes \phi)$$

which can be shown to be continuous. Moreover, this construction defines a bijection between $\mathcal{D}'(V \times U)$ and the set of all general operators (even more, when equipped with the appropriate topologies, this becomes an isomorphism of l.c.v.s.'s).

The passing from the local picture to vector bundles over manifolds works as usual, with some care to make the construction independent of any choices. Here are the details. Given the vector bundles E over M and F over N, we consider the vector bundle over $N \times M$:

$$F \boxtimes E^\vee := \mathrm{pr}_1^*(F) \otimes \mathrm{pr}_2^*(E^\vee),$$

where pr_j is the projection on the j-th component. Hence, the fiber over $(y, x) \in N \times M$ is

$$(F \boxtimes E^\vee)_{(y,x)} = F_y \otimes E_x^* \otimes D_{M,x}.$$

Note that the functional dual of this bundle is canonically identified with:

$$(F \boxtimes E^\vee)^\vee \cong F^\vee \boxtimes E.$$

Exercise 2.4.4. Work out this isomorphism. (Hints: the density bundle of $N \times M$ is canonically identified with $D_N \otimes D_M$; $D_M^* \otimes D_M \cong \underline{\mathrm{Hom}}(D_M, D_M)$ is canonically isomorphic to the trivial line bundle.)

As before, one may decide to use a fixed positive density on M and one on N, and then replace $F \boxtimes E^\vee$ by $F \boxtimes E^*$ and $F^\vee \boxtimes E$ by $F^* \boxtimes E$.

Fix now a distribution

$$K \in \mathcal{D}'(N \times M, F \boxtimes E^\vee).$$

We will associate to K a general operator

$$P_K : \mathcal{D}(M, E) \to \mathcal{D}'(N, F).$$

Due to the definition of the space of distributions, and to the identification mentioned above, K will be a continuous functions

$$K : \mathcal{D}(N \times M, F^\vee \boxtimes E) \to \mathbb{C}.$$

For $\psi \in \mathcal{D}(M, F^\vee)$ and $\phi \in \mathcal{D}(M, E)$ we denote by

$$\psi \otimes \phi \in \mathcal{D}(N \times M, F^\vee \boxtimes E)$$

the induced section $(y, x) \mapsto \psi(y) \otimes \phi(x)$. To describe P_K, let $\phi \in \mathcal{D}(M, E)$ and we have to specify $P_K(\phi) \in \mathcal{D}'(N, F)$, i.e. the continuous functional

$$P_K(\phi) : \mathcal{D}(N, F^\vee) \to \mathbb{C}.$$

We define

$$P_K(\phi)(\psi) := K(\psi \otimes \phi).$$

The general operator P_K is called the general operator associated to the kernel K. Highly non-trivial is the fact that any general operator arises in this way (and then K will be called the kernel of P_K).

Theorem 2.4.5. *The correspondence $K \mapsto P_K$ defines a 1-1 correspondence between*

1. *distributions $K \in \mathcal{D}'(N \times M; F \boxtimes E^\vee)$.*
2. *general operators $P : \mathcal{D}(M, E) \to \mathcal{D}'(N, F)$.*

Moreover, in this correspondence, one has

$$K \in \mathcal{E}(N \times M; F \boxtimes E^\vee) \Longleftrightarrow P \text{ is smoothing.}$$

Note (for your curiosity): the 1-1 correspondence actually defines an isomorphism of l.c.v.s.'s between

1. $\mathcal{D}'(N \times M; F \boxtimes E^\vee)$ with the strong topology.
2. the space $\mathcal{L}(\mathcal{D}(M, E), \mathcal{D}'(N, F))$ of all linear continuous maps, endowed with the strong topology.

Exercise 2.4.6. Let

$$P = \frac{d}{dx} : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}).$$

Compute its kernel, and show that this is not a smoothing operator.

LECTURE 3
Functional spaces on manifolds

The aim of this section is to introduce Sobolev spaces on manifolds (or on vector bundles over manifolds). These will be the Banach spaces of sections we were after (see the previous lectures). To define them, we will take advantage of the fact that we have already introduced the very general spaces of sections (the generalized sections, or distributions), and our Banach spaces of sections will be defined as subspaces of the distributional spaces.

It turns out that the Sobolev-type spaces associated to vector bundles can be built up from smaller pieces and all we need to know are the Sobolev spaces H_r of the Euclidean space $\mathbb{R}^n$ and their basic properties. Of course, it is not so important that we work with the Sobolev spaces themselves, but only that they satisfy certain axioms (e.g. invariance under changes of coordinates). Here we will follow an axiomatic approach and explain that, starting with a subspace of $\mathcal{D}'(\mathbb{R}^n)$ satisfying certain axioms, we can extend it to all vector bundles over manifolds. Back to Sobolev spaces, there is a subtle point: to have good behaved spaces, we will first have to replace the standard Sobolev spaces H_r, by their "local versions", denoted $H_{r,\mathrm{loc}}$. Hence, strictly speaking, it will be these local versions that will be extended to manifolds. The result will deserve the name "Sobolev space" (without the adjective "local") only on manifolds which are compact.

3.1. General functional spaces

When working on $\mathbb{R}^n$, we shorten our notations to

$$\mathcal{E} = \mathcal{E}(\mathbb{R}^n), \mathcal{D} = \mathcal{D}(\mathbb{R}^n), \mathcal{D}' = \dots$$

and, similarly for the Sobolev space of order r:

$$H_r = H_r(\mathbb{R}^n) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^2 : \quad \forall \, |\alpha| \leq r\}.$$

Definition 3.1.1. A functional space on $\mathbb{R}^n$ is a l.c.v.s. space $\mathcal{F}$ satisfying:

1. $\mathcal{D} \subset \mathcal{F} \subset \mathcal{D}'$ and the inclusions are continuous linear maps.
2. for all $\phi \in \mathcal{D}$, multiplication by ϕ defines a continuous map $m_\phi : \mathcal{F} \to \mathcal{F}$.

Similarly, given a vector bundle E over a manifold M, one talks about functional spaces on M with coefficients in E (or just functional spaces on (M, E)).

As in the case of smooth functions, one can talk about versions of $\mathcal{F}$ with supports. Given a functional space $\mathcal{F}$ on $\mathbb{R}^n$, we define for any compact $K \subset \mathbb{R}^n$,

$$\mathcal{F}_K = \{u \in \mathcal{F},\ \mathrm{supp}(u) \subset K\},$$

endowed with the topology induced from $\mathcal{F}$, and we also define

$$\mathcal{F}_{\mathrm{comp}} := \cup_K \mathcal{F}_K$$

(union over all compacts in $\mathbb{R}^n$), endowed with the inductive limit topology. In terms of convergence, that means that a sequence (u_n) in $\mathcal{F}_{\mathrm{comp}}$ converges to $u \in \mathcal{F}_{\mathrm{comp}}$ if and only if there exists a compact K such that

$$\mathrm{supp}(u_n) \subset K \ \ \forall n, \quad u_n \to u \ \text{ in } \mathcal{F}.$$

Note that $\mathcal{F}_{\mathrm{comp}}$ is itself a functional space.

Finally, for any functional space $\mathcal{F}$, one has another functional space (dual in some sense to $\mathcal{F}_{\mathrm{com}}$), defined by:

$$\mathcal{F}_{\mathrm{loc}} = \{u \in \mathcal{D}'(\mathbb{R}^n) : \phi u \in \mathcal{F} \ \ \forall\, \phi \in C_c^{\infty}(\mathbb{R}^n)\}.$$

This has a natural l.c. topology so that all the multiplication operators

$$m_\phi : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F},\ u \mapsto \phi u \quad (u \in \mathcal{D})$$

are continuous- namely the smallest topology with this property. To define it, we use a family P of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^{\infty}(\mathbb{R}^n)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}_{\mathrm{loc}}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

The l.c. topology that we use on $\mathcal{F}_{\mathrm{loc}}$ is the one induced by the family $\{q_{p,\phi} : p \in P, \phi \in \mathcal{D}$. Hence, $u_n \to u$ in this topology means $\phi u_n \to \phi u$ in $\mathcal{F}$, for all ϕ's.

Exercise 3.1.2. Show that, for any l.c.v.s. V, a linear map

$$A : V \to \mathcal{F}_{\mathrm{loc}}$$

is continuous if and only if , for any test function $\phi \in \mathcal{D}$, the composition with the multiplication m_ϕ by ϕ is a continuous map $m_\phi \circ A : V \to \mathcal{F}$.

Example 3.1.3. The four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}'$ are functional spaces and

$$\mathcal{D}_{\mathrm{comp}} = \mathcal{E}_{\mathrm{comp}} = \mathcal{D},\ (\mathcal{D}^{'})_{\mathrm{comp}} = (\mathcal{E}^{'})_{\mathrm{comp}} = \mathcal{E}',$$

$$\mathcal{D}_{\mathrm{loc}} = \mathcal{E}_{\mathrm{loc}} = \mathcal{E},\ (\mathcal{D}^{'})_{\mathrm{loc}} = (\mathcal{E}^{'})_{\mathrm{loc}} = \mathcal{D}'.$$

The same holds in the general setting of vector bundles over manifolds.

Regarding the Sobolev spaces, they are functional spaces as well, but the inclusions

$$H_{r,\mathrm{com}} \hookrightarrow H_r \hookrightarrow H_{r,\mathrm{loc}}$$

are strict (and the same holds on any open $\Omega \subset \mathbb{R}^n$).

The local nature of the spaces $\mathcal{F}_K$ is indicated by the following partition of unity argument which will be very useful later on.

Lemma 3.1.4. *Assume that $K \subset \mathbb{R}^n$ is compact, and let $\{\eta_j\}_{j\in J}$ a finite partition of unity over K, i.e. a family of compactly supported smooth functions on $\mathbb{R}^n$ such that $\sum_j \eta_j = 1$ on K. Let $K_j = K \cap supp(\eta_j)$. Then the linear map*

$$I : \mathcal{F}_K \to \Pi_{j\in J}\mathcal{F}_{K_j},\ u \mapsto (\eta_j u)_{j\in J}$$

is a continuous embedding (i.e. it is an isomorphism between the l.c.v.s. $\mathcal{F}_K$ and the image of I, endowed with the subspace topology) and the image of I is closed.

Proof The fact that I is continuous follows from the fact that each component is multiplication by a compactly supported smooth function. The main observation is that there is a continuous map R going backwards, namely the one which sends $(u_j)_{j\in J}$ to $\sum_j u_j$, such that $R \circ I = \mathrm{Id}$. The rest is a general fact about t.v.s.'s: if $I : X \to Y$, $R : Y \to X$ are continuous linear maps between two t.v.s.'s such that $R \circ I = \mathrm{Id}$, then I is an embedding and $D(X)$ is closed in Y. Let's check this. First, I is open from X to $D(X)$: if $B \subset X$ is open then, remarking that

$$I(B) = I(X) \cap R^{-1}(B)$$

and using the continuity of R, we see that $I(B)$ is open in $I(X)$. Secondly, to see that $I(X)$ is closed in Y, one remarks that

$$I(X) = \mathrm{Ker}(\mathrm{Id} - I \circ R).$$

□

Similar to Lemma 3.1.4, we have the following.

Lemma 3.1.5. *Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$ and let $\{\eta_i\}_{i\in I}$ be a partition of unity, with $\eta_i \in \mathcal{D}$. Let K_i be the support of η_i. Then*

$$I : \mathcal{F}_{loc} \to \Pi_{i\in I}\mathcal{F}_{K_i},\quad u \mapsto\ (\mu_i u)_{i\in I}$$

is a continuous embedding with closed image.

Proof This is similar to Lemma 3.1.4 and the argument is identical. Denoting by X and Y the domain and codomain of I, we have $I : X \to Y$. On the other hand, we can consider $R : Y \to X$ sending $(u_i)_{i\in I}$ to $\sum_i u_i$ (which clearly satisfies $R \circ I = \mathrm{Id}$). What we have to make sure is that, if $\{K_i\}_{i\in I}$ is a locally finite family of compact subsets of $\mathbb{R}^n$, then one has a well-defined continuous map

$$R : \Pi_i\mathcal{F}_{K_i} \to \mathcal{F}_{\mathrm{loc}},\quad (u_i)_{i\in I} \mapsto \sum_i u_i.$$

First of all, $u = \sum_i u_i$ makes sense as a distribution: as a linear functional on test functions,

$$u(\phi) := \sum_i u_i(\phi)$$

(this is a finite sum whenever $\phi \in \mathcal{D}$). Even more, when restricted to $\mathcal{D}_K$, one finds I_K finite such that the previous sum is a sum overall $i \in I_K$ for all $\phi \in \mathcal{D}_K$. This shows that $u \in \mathcal{D}'$. To check that it is in $\mathcal{F}_{\mathrm{loc}}$, we look at ϕu for $\phi \in \mathcal{D}$ (and want to check that it is in $\mathcal{F}$). But, again, we will get a finite sum of ϕu_i's, hence an element in $\mathcal{F}$. Finally, to see that the map is continuous, we have to check (see Exercise 3.1.2) that $m_\phi \circ A$ is continuous as a map to $\mathcal{F}$, for all ϕ. But, again, this is just a finite sum of the projections composed with m_ϕ. □

3.2. The Banach axioms

Regarding the Sobolev spaces H_r on $\mathbb{R}^n$, one of the properties that make them suitable for various problems (and also for the index theorem) is that they are Hilbert spaces. On the other hand, as we already mentioned, we will have to use variations of these spaces for which this property is lost when we deal with manifolds which are not compact. So, it is important to realize what remains of this property.

Definition 3.2.1. (**Banach axiom**) Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is Banach if the topology of $\mathcal{F}$ is a Banach topology.
2. $\mathcal{F}$ is locally Banach if, for each compact $K \subset \mathbb{R}^n$, the topology of $\mathcal{F}_K$ is a Banach topology.

Similarly, we talk about "Frechet", "locally Frechet", "Hilbert" and "locally Hilbert" functional spaces on $\mathbb{R}^n$, or, more generally, on a vector bundle E over a manifold M.

Of course, if $\mathcal{F}$ is Banach then it is also locally Banach (and similarly for Frechet and Hilbert). However, the converse is not true.

Example 3.2.2. $\mathcal{E}$ is Frechet (but not Banach- not even locally Banach). $\mathcal{D}$ is not Frechet, but it is locally Frechet. The Sobolev spaces H_r are Hilbert. Their local versions $H_{r,\mathrm{loc}}$ are just Frechet and locally Hilbert. The same applies for the same functional spaces on opens $\Omega \subset \mathbb{R}^n$.

Proposition 3.2.3. *A functional space $\mathcal{F}$ is locally Banach if and only if each $x \in \mathbb{R}^n$ admits a compact neighborhood K_x such that $\mathcal{F}_{K_x}$ has a Banach topology (similarly for Frechet and Hilbert).*

Proof From the hypothesis it follows that we can find an open cover $\{U_i : i \in I\}$ of $\mathbb{R}^n$ such that each $\overline{U}_i$ is compact and $\mathcal{F}_{\overline{U}_i}$ is Banach. It follows that, for each compact K inside one of these opens, $\mathcal{F}_K$ is Banach. We choose a partition of unity $\{\eta_i\}_{i \in I}$ subordinated to this cover. Hence each $\mathrm{supp}(\eta)_i$ is compact inside U_i, $\{\mathrm{supp}(\eta)_i\}_{i \in I}$ is locally finite and $\sum_i \eta_i = 1$.

Now, for an arbitrary compact K, $J := \{j \in I : \eta_j|_K \neq 0\}$ will be finite and then $\{\eta_j\}_{j \in J}$ will be a finite partition of unity over K hence we can apply Lemma 3.1.4. There K_j will be inside U_j, hence the spaces $\mathcal{F}_{K_j}$ have Banach topologies. The assertion follows from the fact that a closed subspace of a Banach space (with the induced topology) is Banach. □

Remark 3.2.4. If $\mathcal{F}$ is of locally Banach (or just locally Frechet), then $\mathcal{F}_{\mathrm{comp}}$ (with its l.c. topology) is a complete l.c.v.s. which is not Frechet (hint: Theorem 2.1.11).

3.3. Invariance axiom

In general, a diffeomorphism $\chi : \mathbb{R}^n \to \mathbb{R}^n$ (i.e. a change of coordinates) induces a topological isomorphism χ_* on $\mathcal{D}'$:

$$\chi_*(u)(\phi) := u(\phi \circ \chi^{-1}).$$

To be able to pass to manifolds, we need invariance of $\mathcal{F}$ under changes of coordinates. In order to have a notion of local nature, we also consider a local version of invariance.

Definition 3.3.1. Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is invariant if for any diffeomorphism χ of $\mathbb{R}^n$, χ_* restricts to a topological isomorphism
$$\chi_* : \mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$
2. $\mathcal{F}$ is locally invariant if for any diffeomorphism χ of $\mathbb{R}^n$ and any compact $K \subset \mathbb{R}^n$, χ_* restricts to a topological isomorphism
$$\chi_* : \mathcal{F}_K \xrightarrow{\sim} \mathcal{F}_{\chi(K)}.$$

Similarly, we talk about invariance and local invariance of functional spaces on vector bundles over manifolds.

Clearly, invariant implies locally invariant (but not the other way around).

Example 3.3.2. Of course, the standard spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$, $\mathcal{E}'$ are all invariant (in general for vector bundles over manifolds). However, H_r is not invariant but, fortunately, it is locally invariant (this is a non-trivial result which will be proved later on using pseudo-differential operators). As a consequence (see also below), the spaces $H_{r,\mathrm{loc}}$ are invariant.

Proposition 3.3.3. *A functional space $\mathcal{F}$ is locally invariant if and only if for any diffeomorphism χ of $\mathbb{R}^n$, any $x \in \mathbb{R}^n$ admits a compact neighborhood K_x such that χ_* restricts to a topological isomorphism*

$$\chi_* : \mathcal{F}_{K_x} \xrightarrow{\sim} \mathcal{F}_{\chi(K_x)}.$$

Proof We will use Lemma 3.1.4, in a way similar to the proof of Proposition 3.2.3. Let K be an arbitrary compact and χ diffeomorphism. We will check the condition for K and χ. As in the proof of Proposition 3.2.3, we find a finite partition of unity $\{\eta_j\}_{j\in J}$ over K such that each $K_j = K \cap \mathrm{supp}(\eta_j)$ has the property that

$$\chi_* : \mathcal{F}_{K_j} \xrightarrow{\sim} \mathcal{F}_{\chi(K_j)}.$$

We apply Lemma 3.1.4 to K and the partition $\{\eta_j\}$ (with map denoted by I) and also to $\chi(K)$ and the partition $\{\chi_*(\eta_i) = \eta_j \circ \chi^{-1}\}$ (with the map denoted I_χ). Once we show that $\chi_*(\mathcal{F}_K) = \mathcal{F}_{\chi(K)}$ set-theoretically, the lemma clearly implies that this is also a topological equality. So, let $u \in \mathcal{F}_K$. Then $\eta_j u \in \mathcal{F}_{K_j}$ hence

$$\chi_*(\eta_i) \cdot \chi_*(u) = \chi_*(\eta_j \cdot u) \in \mathcal{F}_{\chi(K_j)} \subset \mathcal{F}_K,$$

hence also

$$\chi_*(u) = \sum_j \chi_*(\eta_i) \cdot \chi_*(u) \in \mathcal{F}_K.$$

□

3.4. Density axioms

We briefly mention also the following density axioms.

Definition 3.4.1. Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

1. $\mathcal{F}$ is normal if $\mathcal{D}$ is dense in $\mathcal{F}$.
2. $\mathcal{F}$ is locally normal if, for any compact $K \subset \mathbb{R}^n$, $\mathcal{F}_K$ is contained in the closure of $\mathcal{D}$ in $\mathcal{F}$.

Similarly, we talk about normal and locally normal functional spaces on vector bundles over manifolds.

Again, normal implies locally normal and one can prove a characterization of local normality analogous to Proposition 3.2.3 and Proposition 3.3.3.

Example 3.4.2. All the four basic functional spaces $\mathcal{D}$, $\mathcal{E}$, $\mathcal{D}'$ and $\mathcal{E}'$ are normal (also with coefficients in vector bundles). Also the space H_r is normal. However, for arbitrary opens $\Omega \subset \mathbb{R}^n$, the functional spaces $H_r(\Omega)$ (on Ω) are in general not normal (but they are locally normal). The local spaces $H_{r,\mathrm{loc}}$ are always normal (see also the next section).

The normality axiom is important especially when we want to consider duals of functional spaces. Indeed, in this case a continuous linear functional $\xi : \mathcal{F} \to \mathbb{C}$ is zero if and only if its restriction to $\mathcal{D}$ is zero. It follows that the canonical inclusions dualize to continuous injections

$$\mathcal{D} \hookrightarrow \mathcal{F}^* \hookrightarrow \mathcal{D}'.$$

The duality between $\mathcal{F}_{\mathrm{loc}}$ and $\mathcal{F}_{\mathrm{comp}}$ can then be made more precise- one has:

$$(\mathcal{F}_{\mathrm{loc}})^* = (\mathcal{F}^*)_{\mathrm{comp}}, (\mathcal{F}_{\mathrm{comp}})^* = (\mathcal{F}^*)_{\mathrm{loc}}$$

(note: all these are viewed as vector subspaces of $\mathcal{D}'$, each one endowed with its own topology, and the equality is an equality of l.c.v.s.'s).

3.5. Locality axiom

In general, the invariance axiom is not enough for passing to manifolds. One also needs a locality axiom which allows us to pass to opens $\Omega \subset \mathbb{R}^n$ without loosing the properties of the functional space (e.g. invariance).

Definition 3.5.1. (**Locality axiom**) We say that a functional space $\mathcal{F}$ is local if, as l.c.v.s.'s,

$$\mathcal{F} = \mathcal{F}_{\mathrm{loc}}.$$

Similarly we talk about local functional spaces on vector bundles over manifolds.

Note that this condition implies that $\mathcal{F}$ is a module not only over $\mathcal{D}$ but also over $\mathcal{E}$.

Example 3.5.2. From the four basic examples, $\mathcal{E}$ and $\mathcal{D}'$ are local, while $\mathcal{D}$ and $\mathcal{E}'$ are not. Unfortunately, H_r is not local- and we will soon replace it with $H_{r,\mathrm{loc}}$ (in general, for any functional space $\mathcal{F}$, $\mathcal{F}_{\mathrm{loc}}$ is local).

With the last example in mind, we also note that, in general, when passing to the localized space, the property of being of Banach (or Hilbert, or Frechet) type does not change.

Exercise 3.5.3. Show that, for any functional space $\mathcal{F}$ and for any compact $K \subset \mathbb{R}^n$

$$(\mathcal{F}_{\text{loc}})_K = \mathcal{F}_K,$$

as l.c.v.s.'s. In particular, $\mathcal{F}$ is locally Banach (or locally Frechet, or locally Hilbert), or locally invariant, or locally normal if and only if $\mathcal{F}_{\text{loc}}$ is.

With the previous exercise in mind, when it comes to local spaces we have the following:

Theorem 3.5.4. *Let $\mathcal{F}$ be a local functional space on $\mathbb{R}^n$. Then one has the following equivalences:*

1. *$\mathcal{F}$ is locally Frechet if and only if it is Frechet.*
2. *$\mathcal{F}$ is locally invariant if and only if it is invariant.*
3. *$\mathcal{F}$ is locally normal if and only if it is normal.*

Note that in the previous theorem there is no statement about locally Banach. As we have seen, this implies Frechet. However, local spaces cannot be Banach.

Proof (of Theorem 3.5.4) In each part, we still have to prove the direct implications. For the first part, if $\mathcal{F}$ is locally Frechet, choosing a countable partition of unity and applying the previous lemma, we find that $\mathcal{F}_{\text{loc}}$ is Frechet since it is isomorphic to a closed subspace of a Frechet space (a countable product of Frechet spaces is Frechet!). For the second part, the argument is exactly as the one for the proof of Proposition 3.3.3, but using Lemma 3.1.5 instead of Lemma 3.1.4. For the last part, let us assume that $\mathcal{F}$ is locally normal. It suffices to show that $\mathcal{E}$ is dense in $\mathcal{F}$: then, for any open $U \subset \mathcal{F}$, $U \cap \mathcal{E} \neq \emptyset$; but $U \cap \mathcal{E}$ is an open in $\mathcal{E}$ (because $\mathcal{E} \hookrightarrow \mathcal{F}$ is continuous) hence, since $\mathcal{D}$ is dense in $\mathcal{E}$ (with its canonical topology), we find $U \cap \mathcal{D} \neq \emptyset$.

To show that $\mathcal{E}$ is dense in $\mathcal{F}$, we will need the following variation of Lemma 3.1.5. We choose a partition of unity η_i as there, but with $\eta_i = \mu_i^2$, $\mu_i \in \mathcal{D}$. Let

$$A = \mathcal{F}, \quad X = \Pi_{i \in I} \mathcal{F},$$

(X with the product topology). We define

$$i : A \to X, \ u \mapsto (\mu_i u)_i, \quad p : X \to A, \ (u_i)_i \mapsto \sum \mu_i u_i.$$

As in the lemma, these make A into a closed subspace of X. Hence we can place ourselves into the setting that we have a subspace $A \subset X$ of a l.c.v.s. X, which has a projection into A, $p : X \to A$ (not that we will omit writing i from now on). Consider the *subset*

$$Y = \Pi_{i \in I} \mathcal{D} \subset X.$$

Modulo the inclusion $A \hookrightarrow X$, $B = A \cap Y$ becomes $\mathcal{E}$ and $p(Y) = B$. Also, since A (or its image by i) is inside the closed subspace of X which is $\Pi_i \mathcal{F}_{K_i}$, we see that the hypothesis of local normality implies that $A \subset \overline{Y}$ (all closures are w.r.t. the topology of X). We have to prove that A is in the closure of B. Let

$a \in A$, V an open neighborhood of a in X. We have to show that $V \cap B \neq \emptyset$. From V we make $V' = p^{-1}(A \cap V)$- another open neighborhood of a in X. Since $A \subset \overline{Y}$, we have $V' \cap Y \neq \emptyset$. It now suffices to remark that $p(V') \subset V \cap B$. □

Finally, let us point out the following corollary which shows that, in the case of locally Frechet spaces, locality can be checked directly, using test functions and without any reference to $\mathcal{F}_{\text{loc}}$.

Corollary 3.5.5. *If $\mathcal{F}$ is a functional space which is locally Frechet, then $\mathcal{F}$ is local if and only if the following two (test-)conditions are satisfied:*

1. *$u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{F}$ if and only if $\phi u \in \mathcal{F}$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*
2. *$u_n \to u$ in $\mathcal{F}$ if and only if $\phi u_n \to \phi$ in $\mathcal{F}$, for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*

Proof The direct implication is clear. For the converse, assume that $\mathcal{F}$ is a functional space which satisfies these conditions. The first one implies that $\mathcal{F} = \mathcal{F}_{\text{loc}}$ as sets and we still have to show that the two topologies coincide. Let $\mathcal{T}$ be the original topology on $\mathcal{F}$ and let $\mathcal{T}_{\text{loc}}$ be the topology coming from $\mathcal{F}_{\text{loc}}$. Since $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{loc}}$ is always continuous, in our situation, this tells us that $\mathcal{T}_{\text{loc}} \subset \mathcal{T}$. On the other hand, $\text{Id} : (\mathcal{F}, \mathcal{T}_{\text{loc}}) \to (\mathcal{F}, \mathcal{T})$ is clearly sequentially continuous hence, since $\mathcal{F}_{\text{loc}}$ is metrizable, the identity is also continuous, hence $\mathcal{T} \subset \mathcal{T}_{\text{loc}}$. This concludes the proof. □

3.6. Restrictions to opens

The main consequence of the localization axiom is the fact that one can restrict to opens $\Omega \subset \mathbb{R}^n$. The starting remark is that, for any such open Ω, there is a canonical inclusion

$$\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

which should be thought of as "extension by zero outside Ω", which comes from the inclusion $\mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$ (obtained by dualizing the restriction map $C^\infty(\mathbb{R}^n) \to C^\infty(\Omega)$). In other words, any compactly supported distribution on Ω can be viewed as a (compactly supported) distribution on $\mathbb{R}^n$. On the other hand,

$$\phi u \in \mathcal{E}'(\Omega), \quad \forall\ \phi \in C_c^\infty(\Omega),\ u \in \mathcal{D}'(\Omega).$$

Hence the following makes sense:

Definition 3.6.1. Given a <u>local</u> functional space $\mathcal{F}$, for any open $\Omega \subset \mathbb{R}^n$, we define

$$\mathcal{F}(\Omega) := \{u \in \mathcal{D}'(\Omega) : \phi u \in \mathcal{F}\ \ \forall\ \phi\ \in C_c^\infty(\Omega)\},$$

endowed with the following topology. Let P be a family of seminorms defining the l.c. topology on $\mathcal{F}$ and, for every $p \in P$ and $\phi \in C_c^\infty(\Omega)$ we consider the seminorm $q_{p,\phi}$ on $\mathcal{F}$ given by

$$q_{p,\phi}(u) = p(\phi u).$$

We endow $\mathcal{F}(\Omega)$ with the topology associated to the family $\{q_{p,\phi} : p \in P, \phi \in C_c^\infty(\Omega)$.

Theorem 3.6.2. *For any local functional space $\mathcal{F}$ and any open $\Omega \subset \mathbb{R}^n$, $\mathcal{F}(\Omega)$ is a local functional space on Ω and, as such,*

1. *$\mathcal{F}(\Omega)$ is locally Banach (or Hilbert, or Frechet) if $\mathcal{F}$ is.*
2. *$\mathcal{F}(\Omega)$ is invariant if $\mathcal{F}$ is.*
3. *$\mathcal{F}(\Omega)$ is normal if $\mathcal{F}$ is.*

Proof For the first part, one remarks that $\mathcal{F}(\Omega)_K = \mathcal{F}_K$. For the second part, applying Theorem 3.5.4 to the local functional space $\mathcal{F}(\Omega)$ on Ω, it suffices to show local invariance. I.e., it suffices to show that for any $\chi : \Omega \to \Omega$ diffeomorphism and $x \in \Omega$, we find a compact neighborhood $K = K_x$ such that χ_* is an isomorphism between $\mathcal{F}(\Omega)_K (= \mathcal{F}_K)$ and $\mathcal{F}(\Omega)_{\chi(K)} (= \mathcal{F}_{\chi(K)})$. The difficulty comes from the fact that χ is not defined on the entire $\mathbb{R}^n$. Fix χ and x. Then we can find a neighborhood Ω_x of $x \in \Omega$ and a diffeomorphism $\tilde{\chi}$ on $\mathbb{R}^n$ such that

$$\tilde{\chi}|_{\Omega_x} = \chi|_{\Omega_x}$$

(this is not completely trivial, but it can be done using flows of vector fields, on any manifold). Fix any compact neighborhood $K \subset \Omega_x$. Using the invariance of $\mathcal{F}$, it suffices to show that $\chi_*(u) = \tilde{\chi}_*(u)$ for all $u \in \mathcal{F}_K$. But

$$\chi_*(u),\ \tilde{\chi}_*(u) \in \mathcal{F} \subset \mathcal{D}'$$

are two distributions whose restriction to $\chi(\Omega_x)$ is the same and whose restrictions to $\mathbb{R}^n - \chi(K)$ are both zero. Hence they must coincide. For the last part, since we deal with local spaces, it suffices to show that $\mathcal{F}(\Omega)$ is locally normal, i.e that for any compact $K \subset \Omega$, $\mathcal{F}_K(\Omega) = \mathcal{F}_K \subset \mathcal{F}(\Omega)$ is contained in the closure of $\mathcal{D}(\Omega)$. I.e., for any $u \in \mathcal{F}_K$ and any open $U \subset \mathcal{F}(\Omega)$ containing u, $U \cap \mathcal{D}(\Omega) =\neq \emptyset$. But since $\mathcal{F}$ is locally normal and the restriction map $r : \mathcal{F} \to \mathcal{F}(\Omega)$ is continuous, we have $r^{-1}(U) \cap \mathcal{D} \neq \emptyset$ and the claim follows. $\square$

Exercise 3.6.3. By a sheaf of distributions $\hat{\mathcal{F}}$ on $\mathbb{R}^n$ we mean an assignment

$$\Omega \mapsto \hat{\mathcal{F}}(\Omega)$$

which associates to an open $\Omega \subset \mathbb{R}^n$ a functional space $\hat{\mathcal{F}}(\Omega)$ on Ω (local or not) such that:

1. if $\Omega_2 \subset \Omega_1$ and $u \in \hat{\mathcal{F}}(\Omega_1)$, then $u|_{\Omega_2}$ is in $\hat{\mathcal{F}}(\Omega_2)$. Moreover, the map
$$\hat{\mathcal{F}}(\Omega_1) \to \hat{\mathcal{F}}(\Omega_2),\ u \mapsto u|_{\Omega_1}.$$
is continuous.
2. If $\Omega = \cup_{i\in I}\Omega_i$ with $\Omega_i \subset \mathbb{R}^n$ opens (I some index set), then the map
$$\hat{\mathcal{F}}(\Omega) \to \Pi_{i\in I}\hat{\mathcal{F}}(\Omega_i),\ u \mapsto (u|_{\Omega_i})_{i\in I}$$
is a topological embedding which identifies the l.c.v.s. on the left with the closed subspace of the product space consisting of elements $(u_i)_{i\in I}$ with the property that $u_i|_{\Omega_i\cap\Omega_j} = u_j|_{\Omega_i\cap\Omega_j}$ for all i and j.

Show that

1. If $\mathcal{F}$ is a local functional space on $\mathbb{R}^n$ then $\Omega \mapsto \mathcal{F}(\Omega)$ is a sheaf of distributions.
2. Conversely, if $\hat{\mathcal{F}}$ is a sheaf of distributions on $\mathbb{R}^n$ then
$$\mathcal{F} := \hat{\mathcal{F}}(\mathbb{R}^n)$$
is a local functional space on $\mathbb{R}^n$ and $\hat{\mathcal{F}}(\Omega) = \mathcal{F}(\Omega)$ for all Ω's.

Below, for diffeomorphisms $\chi : \Omega_1 \to \Omega_2$ between two opens, we consider the induced $\chi_* : \mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$.

Corollary 3.6.4. *Let $\mathcal{F}$ be a local functional space. If $\mathcal{F}$ is invariant then, for any diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two opens in $\mathbb{R}^n$, χ_* induces a topological isomorphism*

$$\chi_* : \mathcal{F}(\Omega_1) \to \mathcal{F}(\Omega_2).$$

Proof The proof of 2. of Theorem 3.6.2, when showing invariance under diffeomorphisms $\chi : \Omega \to \Omega$ clearly applies to general diffeomorphism between any two opens. □

3.7. Passing to manifolds

Throughout this section we fix

$$\mathcal{F} = \underline{\text{local, invariant functional space on } \mathbb{R}^n}.$$

and we explain how to induce functional spaces $\mathcal{F}(M, E)$ (of "generalized sections of E of type $\mathcal{F}$") for any vector bundle E over an n-dimensional manifold M.

To define them, we will use local total trivializations of E, i.e. triples (U, κ, τ) consisting of a local chart (U, κ) for M and a trivialization $\tau : E|_U \to U \times \mathbb{C}^p$ of E over U. Recall (see Example 2.3.7) that any such total trivialization induces an isomorphism

$$h_{\kappa,\tau} : \mathcal{D}'(U, E|_U) \to \mathcal{D}'(\Omega_\kappa)^p \quad (\text{where } \Omega_\kappa = \kappa(U) \subset \mathbb{R}^n)^1.$$

Definition 3.7.1. We define $\mathcal{F}(M, E)$ as the space of all $u \in \mathcal{D}'(M, E)$ with the property that for any domain U of a total trivialization of E, $h_{\kappa,\tau}(u|_U) \in \mathcal{F}(\Omega_\kappa)^p$.

We still have to define the topology on $\mathcal{F}(M, E)$, but we make a few remarks first. Since the previous definition applies to all n-dimensional manifolds:

---

[1] Let us make this more explicit. The total trivialization induces

1. a local frame $s_1, \ldots, s_p$ for E over U. Then, for any $s \in \Gamma(E)$ we find (local) coefficients $f_s^i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$s(x) = \sum_i f_s^i(\kappa(x)) s_i(x) \ \text{ for } x \in U.$$
2. the local dual frame $s^1, \ldots, s^p$ of E^* and a local frame (i.e. non-zero section on U) of the density bundle of M, $|dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|$. Then, for any $\xi \in \Gamma(E^\vee)$ we find (local) coefficients $\xi_i \in C^\infty(\Omega_\kappa)$, i.e. satisfying
$$\xi(x) = \sum_i \xi_i(\kappa(x)) s^i(x) |dx_\kappa^1 \wedge \ldots \wedge x_\kappa^n|_x \quad (x \in U).$$
3. any $u \in \mathcal{D}'(M, E)$ has coefficients $u^i \in \mathcal{D}'(\Omega_\kappa)$, i.e. satisfying
$$u(\xi) = \sum_i u_i(\xi_i) \quad (\xi \in \Gamma_c(U, E^\vee)).$$

The map $h_{\kappa,\tau}$ sends u to $(u_1, \ldots, u_n)$.

1. when applied to an open $\Omega \subset \mathbb{R}^n$ and to the trivial line bundle $\mathbb{C}_\Omega$ over Ω, one recovers $\mathcal{F}(\Omega)$- and here we are using the invariance of $\mathcal{F}$.

2. it also applies to all opens $U \subset M$, hence we can talk about the spaces $\mathcal{F}(U, E|_U)$. From the same invariance of $\mathcal{F}$, when U is the domain of a total trivialization chart (U, κ, τ), to check that $u \in \mathcal{D}'(U, E|_U)$ is in $\mathcal{F}(U, E|_U)$, it suffices to check that $h_{\kappa,\tau}(u) \in \mathcal{F}(\Omega_\kappa)^p$- i.e. we do not need to check the condition in the definition for <u>all</u> total trivialization charts.

3. If $\{U_i\}_{i\in I}$ is one open cover of M and $u \in \mathcal{D}'(M, E)$, then
$$u \in \mathcal{F}(M, E) \Longleftrightarrow u|_{U_i} \in \mathcal{F}(U_i, E|_{U_i}) \ \ \forall\, i \in I.$$
This follows from the similar property of $\mathcal{F}$ on opens in $\mathbb{R}^n$.

Exercise 3.7.2. Given a vector bundle E over M and $U \subset M$, $\mathcal{F}$ induces two subspaces of $\mathcal{D}'(U, E|_U)$:
1. $\mathcal{F}(U, E|_U)$ just defined.
2. thinking of $\mathcal{F}(M, E)$ as a functional space on M (yes, we know, we still have to define the topology, but that is irrelevant for this exercise), we have an induced space:
$$\{u \in \mathcal{D}'(U, E|_U) : \phi u \in \mathcal{F}(M, E), \ \ \forall\, \phi \in \mathcal{D}(U)\}.$$

Show that the two coincide.

Next, we take an open cover $\{U_i\}_{i\in I}$ by domains of total trivialization charts (U_i, κ_i, τ_i). It follows that we have an inclusion
$$h : \mathcal{F}(M, E) \to \Pi_{i\in I}\mathcal{F}(\Omega_{\kappa_i})^p.$$
We endow $\mathcal{F}(M, E)$ with the induced topology.

Exercise 3.7.3. Show that the topology on $\mathcal{F}(M, E)$ does not depend on the choice of the cover and of the total trivialization charts.

Theorem 3.7.4. *For any vector bundle E over an n dimensional manifold M, $\mathcal{F}(M, E)$ is a local functional space on (M, E).*

Moreover, if $\mathcal{F}$ is locally Banach, or locally Hilbert, or Frechet (= locally Frechet since $\mathcal{F}$ is local), or normal (= locally normal), then so is $\mathcal{F}(M, E)$.

Finally, if F is a vector bundle over another n-dimensional manifold N and (h, h_0) is an isomorphism between the vector bundles E and F, then $h_* : \mathcal{D}'(M, E) \to \mathcal{D}'(N, F)$ restricts to an isomorphism of l.c.v.s.'s
$$h_* : \mathcal{F}(M, E) \to \mathcal{F}(N, F).$$

Proof We just have to put together the various pieces that we already know (of course, here we make use of the fact that all proofs that we have given so far work for vector bundles over manifolds). To see that $\mathcal{F}(M, E)$ is a functional space we have to check that we have continuous inclusions
$$\mathcal{D}(M, E) \hookrightarrow \mathcal{F}(M, E) \hookrightarrow \mathcal{D}'(M, E).$$
We just have to remark that the map h used to define the topology of $\mathcal{F}(M, E)$ also describes the topology for $\mathcal{D}$ and $\mathcal{D}'$. To show that $\mathcal{F}(M, E)$ is local, one

uses the sheaf property of $\mathcal{F}(M,E)_{\mathrm{loc}}$ (see Exercise 3.6.3) where the U_i's there are chosen as in the construction of h above. This reduces the problem to a local one, i.e. to locality of $\mathcal{F}$.

All the other properties follow from their local nature (i.e. Proposition 3.2.3 and the similar result for normality, applied to manifolds) and the fact that, for $K \subset M$ compact inside a domain U of a total trivialization chart (U, κ, τ), $\mathcal{F}_K(M,E)$ is isomorphic to $\mathcal{F}_K(\Omega_\kappa)^p$. □

Corollary 3.7.5. *If $\mathcal{F}$ is locally Banach (or Hilbert) and normal then, for any vector bundle E over a compact n dimensional manifold M, $\mathcal{F}(M,E)$ is a Banach (or Hilbert) space which contains $\mathcal{D}(M,E)$ as a dense subspace.*

Definition 3.7.6. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local, invariant functional spaces on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and assume that $\mathcal{F}_1$ is normal. Let M be an m-dimensional manifold and N an n-dimensional one, and let E and F be vector bundles over M and N, respectively. We say that a general operator

$$P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$$

is of type $(\mathcal{F}_1, \mathcal{F}_2)$ if it takes values in $\mathcal{F}_2$ and extends to a continuous linear operator

$$P_{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1(M,E) \to \mathcal{F}_2(N,E).$$

Note that, due to the normality axiom, the extension $P_{\mathcal{F}_1,\mathcal{F}_2}$ will be unique, hence the notation is un-ambiguous.

3.8. Back to Sobolev spaces

We apply the previous constructions to the Sobolev spaces H_r on $\mathbb{R}^n$. Let us first recall some of the standard properties of these spaces:

1. they are Hilbert spaces.
2. $\mathcal{D}$ is dense in H_r.
3. if $s > n/2 + k$ then $H_s \subset C^k(\mathbb{R}^n)$ (with continuous injection) (Sobolev's lemma).
4. for all $r > s$ and all $K \subset \mathbb{R}^n$ compact, the inclusion $H_{s,K} \hookrightarrow H_{s,K}$ is compact (Reillich's lemma).

Also, as we shall prove later, H_r are locally invariant (using pseudo-differential operators and Proposition 3.3.3). Assuming all these, we now consider the associated local spaces

$$H_{r,\mathrm{loc}} = \{u \in \mathcal{D}' : \phi u \in H_r, \ \ \forall\, \phi \in \mathcal{D}\}$$

and the theory we have developed imply that:

1. $H_{r,\mathrm{loc}}$ is a functional space which is locally Hilbert, invariant and normal.
2. $\cap_r H_{r,\mathrm{loc}} = \mathcal{E}$. Even better, for $r > n/2 + k$, any $s \in H_{r,\mathrm{loc}}$ is of class C^k.

Hence these spaces extend to manifolds.

Definition 3.8.1. For a vector bundle E over an n-dimensional manifold M,

1. the resulting functional spaces $H_{r,\mathrm{loc}}(M,E)$ are called the local r-Sobolev spaces of E.

2. for $K \subset M$ compact, the resulting K-supported spaces are denoted $H_{r,K}(M,E)$.
3. the resulting compactly supported spaces are denoted $H_{r,\mathrm{comp}}(M,E)$ (hence they are $\cup_K H_{r,K}(M,E)$ with the inductive limit topology).

If M is compact, we define the r-Sobolev space of E as

$$H_r(M,E) := H_{r,\mathrm{loc}}(M,E)(= H_{r,\mathrm{comp}}(M,E)).$$

Corollary 3.8.2. *For any vector bundle E over a manifold M,*

1. *$H_{r,loc}(M,E)$ are Frechet spaces.*
2. *$\mathcal{D}(M,E)$ is dense in $H_{r,loc}(M,E)$.*
3. *if a distribution $u \in \mathcal{D}'(M,E)$ belongs to all the spaces $H_{r,loc}(M,E)$, then it is smooth.*
4. *for $K \subset M$ compact, $H_{r,K}(M,E)$ has a Hilbert topology and, for $r > s$, the inclusion*
$$H_{r,K}(M,E) \hookrightarrow H_{s,K}(M,E)$$
is compact.

Proof The only thing that may still need some explanation is the compactness of the inclusion. But this follows from the Reillich's lemma and the partition of unity argument, i.e. Lemma 3.1.4 . □

Corollary 3.8.3. *For any vector bundle E over a <u>compact</u> manifold M, $H_r(M,E)$ has a Hilbert topology, contains $\mathcal{D}(M,E)$ as a dense subspace,*

$$\cap_r H_{r,loc}(M,E) = \Gamma(E)$$

and, for $r > s$, the inclusion

$$H_r(M,E) \hookrightarrow H_s(M,E)$$

is compact.

Finally, we point out the following immediate properties of operators. The first one says that differential operators of order k are also operators of type (H_r, H_{r-k}).

Proposition 3.8.4. *For $r \geq k \geq 0$, given a differential operator $P \in \mathcal{D}_k(E,F)$ between two vector bundles over M, the operator*

$$P : \mathcal{D}(M,E) \to \mathcal{D}(M,F)$$

admits a unique extension to a continuous linear operator

$$P_r : H_{r,loc}(M,E) \to H_{r-k,loc}(M,F).$$

The second one says that smoothing operators on compact manifolds, viewed as operators of type (H_r, H_s), are compact.

Proposition 3.8.5. *Let E and F be two vector bundles over a compact manifold M and consider a smoothing operator $P \in \Psi^{-\infty}(E,F)$. Then for any r and s, P viewed as an operator*

$$P : H_r(M,E) \to H_s(M,F)$$

is compact.

Remark 3.8.6. Back to our strategy of proving that the index of an elliptic differential operator $P \in \mathcal{D}_k(E, F)$ (over a compact manifold) is well-defined, our plan was to use the theory of Fredholm operators between Banach spaces. We have finally produced our Banach spaces of sections on which our operator will act:

$$P_r : H_r(M, E) \to H_{r-k}(M, F).$$

To prove that P_r is Fredholm, using Theorem 1.4.5 on the characterization of Fredholm operators and the fact that all smoothing operators are compact, we would need some kind of "inverse of P_r modulo smoothing operators", i.e. some kind of operator "of order $-k$" going backwards, such that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing operators. Such operators "of degree $-k$" cannot, of course, be differential. What we can do however is to understand what make differential operators behave well w.r.t. (e.g.) Sobolev spaces- and the outcome is: it is not important that their total symbols are polynomials (of some degree k) in ξ, but only their symbols have a certain k-polynomial-like behaviour (in terms of estimates). And this is a property which makes sense even for k-negative (and that is where we have to look for our Q). This brings us to pseudo-differential operators ...

Exercise 3.8.7. Show that if such an operator $Q : H_{r-k}(M, F) \to H_r(M, E)$ is found (i.e. with the property that $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing), then the kernel of the operator $P : \Gamma(E) \to \Gamma(F)$ is finite dimensional. What about the cokernel?

LECTURE 4
Fourier transform

4.1. Schwartz functions

Recall that $L^1(\mathbb{R}^n)$ denotes the Banach space of functions $f : \mathbb{R}^n \to \mathbb{C}$ that are absolutely integrable, i.e., $|f|$ is Lebesgue integrable over $\mathbb{R}^n$. The norm on this space is given by

$$\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Given $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we put

$$\xi x := \xi_1 x_1 + \cdots + \xi_n x_n.$$

For each $\xi \in \mathbb{R}^n$, the exponential function

$$e^{i\xi} :\ x \mapsto e^{i\xi x},\ \mathbb{R}^n \to \mathbb{C},$$

has absolute value 1 everywhere. Thus, if $f \in L^1(\mathbb{R}^n)$ then $e^{-i\xi} f \in L^1(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$.

Definition 4.1.1. For a function $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\hat{f} = \mathcal{F}f : \mathbb{R}^n \to \mathbb{C}$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx. \tag{4.1}$$

We will use the notation $C_b(\mathbb{R}^n)$ for the Banach space of bounded continuous functions $\mathbb{R}^n \to \mathbb{C}$ equipped with the sup-norm.

Lemma 4.1.2. *The Fourier transform maps $L^1(\mathbb{R})$ continuous linearly to the Banach space $C_b(\mathbb{R}^n)$.*

Proof Let f be any function in $L^1(\mathbb{R}^n)$. The functions $fe^{-i\xi}$ are all dominated by f in the sense that $|fe^{-i\xi}| \leq |f|$ (almost) everywhere. Let $\xi_0 \in \mathbb{R}^n$; then it follows by Lebesgue's dominated convergence theorem that $\mathcal{F}f(\xi) \to \mathcal{F}f(\xi_0)$ if $\xi \to \xi_0$. This implies that $\mathcal{F}f$ is continuous. It follows that $\mathcal{F}$ defines a linear map from $L^1(\mathbb{R})$ to $C(\mathbb{R}^n)$. It remains to be shown that $\mathcal{F}$ maps $L^1(\mathbb{R})$ continuously into $C_b(\mathbb{R})$.

For this we note that for $f \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$|\mathcal{F}f(\xi)| = |\int_{\mathbb{R}^n} f(x)\, e^{-i\xi x}\, dx\,| \leq \int_{\mathbb{R}^n} |f(x)\, e^{-i\xi x}|\, dx = \|f\|_1.$$

Thus, $\sup |\mathcal{F}f| \leq \|f\|_1$. It follows that $\mathcal{F}$ is a linear map $L^1(\mathbb{R}) \to C_b(\mathbb{R}^n)$ which is bounded for the Banach topologies, hence continuous. $\square$

Remark 4.1.3. We denote by $C_0(\mathbb{R}^n)$ the subspace of $C_b(\mathbb{R}^n)$ consisting of functions f that vanish at infinity. By this we mean that for any $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f| < \epsilon$ on the complement $\mathbb{R}^n \setminus K$. It is well known that $C_0(\mathbb{R}^n)$ is a closed subspace of $C_b(\mathbb{R}^n)$, thus a Banach space of its own right.

The well known *Riemann-Lebesgue lemma* asserts that, actually, $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.

The above amounts to the traditional way of introducing the Fourier transform. Unfortunately, the source space $L^1(\mathbb{R}^n)$ is very different from the target space $C_b(\mathbb{R}^n)$. We shall now introduce a subspace of $L^1(\mathbb{R}^n)$ which has the advantage that it is preserved under the Fourier transform: the so-called Schwartz space.

Definition 4.1.4. A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is called rapidly decreasing, or Schwartz, if for all $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty. \tag{4.2}$$

The linear space of these functions is denoted by $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.5. Show that the function

$$f(x) = e^{-\|x\|^2}$$

belongs to $\mathcal{S}(x)$.

Condition (4.2) for all α, β is readily seen to be equivalent to the following condition, for all $N \in \mathbb{N}, k \in \mathbb{N}:$

$$\nu_{N,k}(f) := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\partial^\alpha f(x)| < \infty.$$

We leave it to the reader to check that $\nu = \nu_{N,k}$ defines a norm, hence in particular a seminorm, on $\mathcal{S}(\mathbb{R}^n)$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the locally convex topology generated by the set of norms $\nu_{N,k}$, for $N, k \in \mathbb{N}$.

The Schwartz space behaves well with respect to the operators (multiplication by) x^α and ∂^β.

Exercise 4.1.6. Let α, β be multi-indices. Show that

$$x^\alpha : f \mapsto x^\alpha f \quad \text{and} \quad \partial^\beta : f \mapsto \partial^\beta f$$

define continuous linear endomorphisms of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.7.

(a) Show that $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, with continuous inclusion map.

(b) Show that
$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n),$$
with continuous inclusion maps.

Lemma 4.1.8. *The space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space.*

Proof As the given collection of seminorms is countable it suffices to show completeness, i.e., every Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$ should be convergent. Let (f_n) be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$. Then by continuity of the second inclusion in Exercise 4.1.7 (b), the sequence is Cauchy in $C^\infty(\mathbb{R}^n)$. By completeness of the latter space, the sequence f_n converges to f, locally uniformly, in all derivatives. We will show that $f \in \mathcal{S}(\mathbb{R}^n)$ and $f_n \to f$ in $\mathcal{S}(\mathbb{R}^n)$. First, since (f_n) is Cauchy, it is bounded in $\mathcal{S}(\mathbb{R}^n)$. Let $N, k \in \mathbb{N}$; then there exists a constant $C_{N,k} > 0$ such that $\nu_{N,k}(f_n) \leq C_{N,k}$, for all $n \in \mathbb{N}$. Let $x \in \mathbb{R}^n$, then from $\partial^\alpha f_n(x) \to \partial^\alpha f(x)$ it follows that
$$(1 + \|x\|)^N \partial^\alpha f_n(x) \to (1 + \|x\|)^N \partial^\alpha f(x), \qquad \text{as} \quad n \to \infty.$$
In view of the estimates $\nu_{N,k}(f_n) \leq C_{N,k}$, it follows that $|(1 + \|x\|)^N \partial^\alpha f(x)| \leq C_{N,k}$, for all α with $|\alpha| \leq k$. This being true for arbitrary x, we conclude that $\nu_{N,k}(f) \leq C_{N,k}$. Hence f belongs to the Schwartz space.

Finally, we turn to the convergence of the sequence f_n in $\mathcal{S}(\mathbb{R}^n)$. Let $N, k \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists a constant M such that
$$n, m > M \Rightarrow \nu_{N,k}(f_n - f_m) \leq \epsilon/2.$$
Let $|\alpha| \leq k$ and fix $x \in \mathbb{R}^n$. Then it follows that
$$(1 + \|x\|)^N |\partial^\alpha f_n(x) - \partial^\alpha f_m(x)| \leq \frac{\epsilon}{2}$$
As $\partial^\alpha f_n \to \partial^\alpha f$ locally uniformly, hence in particular pointwise, we may pass to the limit for $m \to \infty$ and obtain the above estimate with f_m replaced by f, for all $x \in \mathbb{R}^n$. It follows that $\nu_{N,k}(f_n - f) < \epsilon$ for all $n \geq M$. $\square$

Another important property of the Schwartz space is the following.

Lemma 4.1.9. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof Fix a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on the closed unit ball in $\mathbb{R}^n$. For $j \in \mathbb{Z}_+$ define the function $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ by
$$\varphi_j(x) = \varphi(x/j).$$
Let now $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi_j f \in C_c^\infty(\mathbb{R}^n)$ for all $j \in \mathbb{Z}_+$. We will complete the proof by showing that $\varphi_j f \to f$ in $\mathcal{S}(\mathbb{R}^n)$ as $j \to \infty$.

Fix $N, k \in \mathbb{N}$. Our goal is to find an estimate for $\nu_{N,k}(\varphi_j f - f)$, independent of f. To this end, we first note that for every multi-index β we have $\partial^\beta \varphi_j(x) = (1/j)^{|\beta|} \partial^\beta \varphi(x/j)$. It follows that
$$\sup_{\mathbb{R}^n} |\partial^\beta \varphi_j| \leq \frac{1}{j}, \qquad (j \in \mathbb{Z}_+, \ 0 < |\beta| \leq k).$$
Let $|\alpha| \leq k$. Then by application of Leibniz' rule we obtain, for all $x \in \mathbb{R}^n$, that
$$|\partial^\alpha(\varphi_j f - f)(x)| \leq |(\varphi_j(x) - 1)\, \partial^\alpha f(x)| + \frac{1}{j} \sum_{0 < \beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\partial^{\alpha - \beta} f(x)|.$$

The first term on the right-hand side is zero for $\|x\| \leq j$. For $\|x\| \geq j$ it can be estimated as follows:

$$\begin{aligned} |(\varphi_j(x) - 1)\partial^\alpha f(x)| &\leq (1 + \sup|\varphi|)(1+j)^{-1}(1 + \|x\|)|\partial^\alpha f(x)| \\ &\leq 2j^{-1}(1 + \|x\|)|\partial^\alpha f(x)|. \end{aligned}$$

We derive that there exists a constant $C_k > 0$, only depending on k, such that for every $N \in \mathbb{N}$,

$$\nu_{N,k}(\varphi_j f - f) \leq \frac{C_k}{j}\nu_{N+1,k}(f).$$

It follows that $\varphi_j f \to f$ in $\mathcal{S}(\mathbb{R}^n)$. $\square$

The following lemma is a first confirmation of our claim that the Schwartz space provides a suitable domain for the Fourier transform.

Lemma 4.1.10. *The Fourier transform is a continuous linear map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Moreover, for each $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\alpha \in \mathbb{N}^n$, the following hold.*

(a) $\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha \mathcal{F}f$;
(b) $\mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}f$.

Proof Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $1 \leq j \leq n$. Then it follows by differentiation under the integral sign that

$$\frac{\partial}{\partial \xi_j}\int_{\mathbb{R}^n} f(x)\, e^{-i\xi x}\, dx = \int_{\mathbb{R}^n} f(x)(-ix_j)e^{-i\xi x}\, dx.$$

The interchange of integration and differentiation is justified by the observation that the integrand on right-hand side is continuous and dominated by the integrable function $(1 + \|x\|)^{-n-1}\nu_{n+1,0}(f)$ (check this). It follows that $\mathcal{F}(-x_j f) = \partial_j \mathcal{F}f$. By repeated application of this formula, we see that $\mathcal{F}f$ is a smooth function and that (b) holds. Since the inclusion map $\mathcal{S}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and the Fourier transform $L^1(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ are continuous, it follows that $\mathcal{F}$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $C_b(\mathbb{R}^n)$. As multiplication by x^α is a continuous endomorphism of the Schwartz space, it follows by application of (b) that $\mathcal{F}$ is a continuous linear map $\mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

Let $f \in C_c^\infty(\mathbb{R}^n)$ and $1 \leq j \leq n$. Then by partial integration it follows that

$$\int_{\mathbb{R}^n} \partial_j f(x) e^{-i\xi x}\, dx = (i\xi_j)\int_{\mathbb{R}^n} f(x)e^{-i\xi x}\, dx$$

so that $\mathcal{F}(\partial_j f) = (i\xi_j)\mathcal{F}(f)(\xi)$. By repeated application of this formula, it follows that (a) holds for all $f \in C_c^\infty(\mathbb{R}^n)$. By density of $C_c^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ combined with continuity of the endomorphism $\partial^\alpha \in \mathrm{End}(\mathcal{S})$ and continuity of $\mathcal{F}$ as a map $\mathcal{S}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ it now follows that (a) holds for all $f \in \mathcal{S}(\mathbb{R}^n)$.

It remains to establish the continuity of $\mathcal{F}$ as an endomorphism of $\mathcal{S}(\mathbb{R}^n)$. For this it suffices to show that $\xi^\alpha \partial^\beta \mathcal{F}$ is continuous linear as a map $\mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$. This follows from $\xi^\alpha\partial^\beta\mathcal{F} = \mathcal{F} \circ (-i\partial)^\alpha(-ix)^\beta$ (by (a), (b)) and the fact that $(-i\partial)^\alpha \circ (-ix)^\beta$ is a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$. $\square$

Later on, we will see that it is convenient to write

$$D^\alpha = (-i\partial)^\alpha,$$

so that formula (a) of the above lemma becomes

$$\mathcal{F}(D^\alpha f) = \xi^\alpha \mathcal{F} f.$$

Given $a \in \mathbb{R}^n$ we write T_a for the translation $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto x + a$ and T_a^* for map $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ induced by pull-back. Thus, $T_a^* f(x) = f(x+a)$.

Lemma 4.1.11. *The map T_a^* restricts to a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$. Moreover, for all $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathcal{F}(T_a^* f) = e^{i\xi a}\mathcal{F}(f); \quad \mathcal{F}(e^{-iax} f) = T_a^* \mathcal{F} f.$$

Exercise 4.1.12. Prove the lemma.

We write S for the point reflection $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto -x$ and S^* for the induced linear endomorphism of $C^\infty(\mathbb{R}^n)$. It is readily seen that S^* defines a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.13. The map S^* defines a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$ which commutes with $\mathcal{F}$.

We can now give the full justification for the introduction of the Schwartz space.

Theorem 4.1.14. (Fourier inversion)

(a) *$\mathcal{F}$ is a topological linear isomorphism $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*
(b) *The endomorphism $S^*\mathcal{F}\mathcal{F}$ of $\mathcal{S}(\mathbb{R}^n)$ equals $(2\pi)^n$ times the identity operator. Equivalently, for every $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F} f(\xi)\, e^{i\xi x}\, d\xi, \qquad (x \in \mathbb{R}^n).$$

Proof We consider the continuous linear operator $\mathcal{T} := S^*\mathcal{F}\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to itself. By Lemma 4.1.10 it follows that

$$\mathcal{T} \circ x^\alpha = S^*\mathcal{F} \circ \partial^\alpha \circ \mathcal{F} = S^* \circ x^\alpha \circ \mathcal{F}\mathcal{F} = x^\alpha \circ \mathcal{T}.$$

In other words, $\mathcal{T}$ commutes with multiplication by x^α, for every multi-index α. In a similar fashion it is shown that $\mathcal{T}$ commutes with T_a^*, for every $a \in \mathbb{R}^n$.

We will now show that any continuous linear endomorphism $\mathcal{T}$ of $\mathcal{S}(\mathbb{R}^n)$ with these properties must be equal to a constant times the identity. For this we use the Gaussian function $G(x) = \exp(-\|x\|/2)$. Let $f \in C_c^\infty(\mathbb{R}^n)$ and put $\varphi = G^{-1} f$. Then φ is smooth compactly supported as well. Moreover, in view of the formula

$$\begin{aligned} \varphi(x) &= \varphi(0) + \int_0^1 \frac{\partial}{\partial t}\varphi(tx)\, dt \\ &= \varphi(0) + [\int_0^1 D\varphi(tx)\, dt]x, \end{aligned}$$

we see that there exists a smooth map $L : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{C})$ such that $\varphi(x) = \varphi(0) + L(x)x$ for all $x \in \mathbb{R}^n$. It is easily seen that each component $L_j(x)$ is smooth with partial derivatives that are all bounded on $\mathbb{R}^n$. Hence, $L_jG \in \mathcal{S}(\mathbb{R}^n)$. It now follows that

$$\begin{aligned}\mathcal{T}(f) &= \mathcal{T}(\varphi G))\\ &= \mathcal{T}(\varphi(0)G) + \mathcal{T}(\sum_j x_j L_j G)\\ &= \varphi(0)\mathcal{T}(G) + \sum_j x_j \mathcal{T}(L_j G).\end{aligned}$$

Evaluating at $x = 0$ we find that $\mathcal{T}(f)(0) = cf(0)$, with c the constant $\mathcal{T}(G)(0)$.

We now use that $\mathcal{T}$ commutes with translation:

$$\mathcal{T}(f)(x) = [T_x^*\mathcal{T}(f)](0) = \mathcal{T}(T_x^* f)(0) = c\,T_x^* f(0) = cf(x).$$

This proves the claim that $\mathcal{T} = cI$. To complete the proof of (b) we must show that $c = (2\pi)^n$. This is the subject of the exercise below.

It follows from (b) and the fact that S^* commutes with $\mathcal{F}$ that $\mathcal{F}$ has $(2\pi)^{-n}S^*\mathcal{F}$ as a continuous linear two-sided inverse. Hence, $\mathcal{F}$ is a topological linear automorphism of $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.1.15. We consider the Gaussian function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = e^{-\frac{1}{2}x^2}$.

(a) Show that $\mathcal{F}g$ satisfies the differential equation $\frac{d}{dx}\mathcal{F}g = -x\mathcal{F}g$.
(b) Determine the Fourier transform $\mathcal{F}g$.
(c) Prove that for the Gaussian function $G : \mathbb{R}^n \to \mathbb{R}^n$ we have $\mathcal{T}(G) = (2\pi)^n G$.

In order to get rid of the constant $(2\pi)^n$ in formulas involving Fourier inversion, we change the normalization of the measures dx and $d\xi$ on $\mathbb{R}^n$, by requiring both of these measures to be equal to $(2\pi)^{-n/2}$ times Lebesgue measure. The definition of $\mathcal{F}$ is now changed by using formula (4.1) but with the new normalization of measures. Accordingly, the Fourier inversion formula becomes, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, e^{i\xi x}\, d\xi. \tag{4.3}$$

4.2. Convolution

The Schwartz space is also very natural with respect to convolution. In the following we shall make frequent use of the following easy estimates, for $x, y \in \mathbb{R}^n$

$$(1 + \|x\|)(1 + \|y\|)^{-1} \le (1 + \|x + y\|) \le (1 + \|x\|)(1 + \|y\|). \tag{4.4}$$

The inequality on the right is an easy consequence of the triangle inequality. The inequality on the left follows from the one on the right if we first substitute $-y$ for y and then, in the resulting inequality, $x + y$ for x.

Assume that $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$ are continuous functions with

$$\nu_N(f_j) := \sup(1 + \|x\|)^N f_j(x) < \infty$$

for all $N \in \mathbb{N}$ (Schwartz functions are of this type). Then it follows that $|f_j(x)| \leq (1 + \|x\|)^{-N} \nu_N(f_j)$ for all $x \in \mathbb{R}^n$. Therefore,

$$\begin{aligned} f_1(y) f_2(x-y) &\leq (1+\|y\|)^{-M}(1+\|x-y\|)^{-N} \nu_M(f_1) \nu_N(f_2) \\ &\leq (1+\|y\|)^{N-M}(1+\|x\|)^{-N} \nu_M(f_1) \nu_N(f_2). \end{aligned}$$

Choosing $N = 0$ and $M > n$ we see that the function $y \mapsto f_1(y) f_2(x-y)$ is integrable for every $x \in \mathbb{R}^n$.

Definition 4.2.1. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we define the convolution product $f * g : \mathbb{R}^n \to \mathbb{C}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy.$$

Lemma 4.2.2.

(a) *The convolution product defines a continuous bilinear map*

$$(f, g) \mapsto f * g, \quad \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

(b) *For all* $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}(f * g) = \mathcal{F}f \mathcal{F}g \qquad \textit{and} \qquad \mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g.$$

Proof Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let α be a multi-index of order at most k. Let $K \in \mathbb{N}$. Then it follows from the above estimates with $f_1 = f$ and $f_2 = \partial^\alpha g$ that

$$(1+\|x\|)^K |f(y) \partial^\alpha g(x-y)| \leq (1+\|y\|)^{N-M}(1+\|x\|)^{K-N} \nu_{M,0}(f) \nu_{N,k}(g).$$

We now choose $N = K$ and $M > N + n$. Then the function on the right-hand side is integrable with respect to y. It now follows by differentiation under the integral sign that the function $f * g$ is smooth and that for all α we have $\partial^\alpha(f * g) = f * \partial^\alpha g$. Moreover, it follows from the estimate that

$$\nu_{K,k}(f * g) \leq \nu_{M,0}(f) \nu_{N,k}(g) \int_{\mathbb{R}^n} (1+\|y\|)^{N-M}\, dy.$$

We thus see that the map $(f, g) \mapsto f * g$ is continuous bilinear from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Moreover, the above estimates justify the following application of Fubini's theorem:

$$\begin{aligned} \mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y) e^{-i\xi x}\, dy\, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y) e^{-i\xi x}\, dx\, dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(z) e^{-i\xi(z+y)}\, dz\, dy \\ &= \mathcal{F}f(\xi) \mathcal{F}g(\xi). \end{aligned}$$

To obtain the second equality of (b), we use that $S^*\mathcal{F} = \mathcal{F}S^*$ is the inverse to $\mathcal{F}$ (by our new normalization of measures). Put $\varphi = \mathcal{F}S^* f$ and $\psi = \mathcal{F}S^* g$. Then $fg = \mathcal{F}\varphi \mathcal{F}\psi = \mathcal{F}(\varphi * \psi)$. By application of $\mathcal{F}$ we now readily verify that

$$\mathcal{F}(fg) = S^*(\varphi * \psi) = S^*(\varphi) * S^*(\psi) = \mathcal{F}f * \mathcal{F}g.$$

□

Corollary 4.2.3. *The convolution product $*$ on $\mathcal{S}(\mathbb{R}^n)$ is continuous bilinear, associative and commutative, turning $\mathcal{S}(\mathbb{R}^n)$ into a commutative continuous algebra.*

Proof This follows from the above lemma combined with the fact that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological linear isomorphism. □

Exercise 4.2.4. By using Fourier transform, show that the algebra $(\mathcal{S}(\mathbb{R}^n), +, *)$ has no unit element.

On $\mathcal{S}(\mathbb{R}^n)$ we define the L^2-inner product $\langle \cdot\,,\,\cdot \rangle_{L^2}$ by

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\;dx.$$

Accordingly, the space $L^2(\mathbb{R}^n)$ may be identified with the Hilbert completion of $\mathcal{S}(\mathbb{R}^n)$.

Proposition 4.2.5. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2} = \langle f, g \rangle_{L^2}$. The Fourier transform has a unique extension to a surjective isometry $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Proof We define the function $\check{g} : \mathbb{R}^n \to \mathbb{C}$ by

$$\check{g}(x) = \overline{g(-x)}.$$

Then g belongs to the Schwartz space, and $\mathcal{F}(\check{g}) = \overline{\mathcal{F}g}$. Moreover,

$$\langle f, g \rangle_{L^2} = f * \check{g}(0).$$

By the Fourier inversion formula it follows that the latter expression equals

$$\int_{\mathbb{R}^n} \mathcal{F}(f * \check{g})(\xi)\;d\xi = \int_{\mathbb{R}^n} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}\;d\xi = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}.$$

Thus, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometry for $\langle \cdot\,,\,\cdot \rangle_{L^2}$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, so is $\mathcal{S}(\mathbb{R}^n)$ and it follows that $\mathcal{F}$ has a unique continuous linear extension to an endomorphism of the Hilbert space $L^2(\mathbb{R}^n)$; moreover, the extension is an isometry. Likewise, S^* is isometric hence extends to an isometric endomorphism of $L^2(\mathbb{R}^n)$. By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ the composition of extended maps $S^*\mathcal{F}$ is a two-sided inverse to the extended map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Therefore, $\mathcal{F}$ is surjective. □

4.3. Tempered distributions and Sobolev spaces

By means of the Fourier transform we shall give a different characterization of Sobolev spaces, which will turn out to be very useful in the context of pseudo-differential operators. We start by introducing the notion of tempered distributions.

Definition 4.3.1. The elements of $\mathcal{S}'(\mathbb{R}^n)$, the continuous linear dual of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$ are called *tempered distributions*.

Here we note that a linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if and only if there exist constants $N, k \in \mathbb{N}$ and $C > 0$ such that

$$|u(f)| \leq C\, \nu_{N,k}(f) \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The name distributions is justified by the following observation. By transposition the continuous inclusions

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

give rise to continuous linear transposed maps between the continuous linear duals of these spaces. Here we assume to have the duals equipped with the strong dual topologies (of uniform convergence on bounded sets). Moreover, as $C_c^\infty(\mathbb{R}^n)$ is dense in both $\mathcal{S}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$, it follows that the transposed maps are injective:

$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

We note that the transposed maps are given by restriction. Thus, $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is given by $u \mapsto u|_{\mathcal{S}(\mathbb{R}^n)}$. Moreover, the map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is given by $v \mapsto v|_{C_c^\infty(\mathbb{R}^n)}$. In this sense tempered distributions may be viewed as distributions.

We recall that the operators $x^\alpha\cdot$ and ∂^α on $\mathcal{D}'(\mathbb{R}^n)$ were defined through transposition:

$$x^\alpha u = u \circ (x^\alpha \cdot), \quad \text{and} \quad \partial^\alpha u = u \circ (-\partial)^\alpha,$$

for $u \in \mathcal{D}'(\mathbb{R}^n)$.

Exercise 4.3.2. Show that $\mathcal{S}'(\mathbb{R}^n)$ is stable under the operators ∂^α and x^α for all multi-indices α.

We recall that there is a natural continuous linear injection $L^2_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$. If $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ then the associated distribution is given by

$$f \mapsto \langle \varphi, f \rangle := \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx, \;\; C_c^\infty(\mathbb{R}^n) \to \mathbb{C}.$$

Lemma 4.3.3. *The continuous linear injection $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ maps $L^2(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$.*

Proof Denote the injection by j. Let $\varphi \in L^2(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Fix $N > n/2$. Then

$$\begin{aligned} \langle \varphi, f \rangle &= \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx \\ &\leq \int_{\mathbb{R}^n} \varphi(x)(1 + \|x\|)^{-N} \nu_{N,0}(f)\, dx \\ &\leq C\, \|\varphi\|_2\, \nu_{N,0}(f) \end{aligned}$$

where C is the L^2-norm of $(1+\|\xi\|)^{-N}$. It follows that the pairing $(\varphi, f) \mapsto \langle \varphi, f \rangle$ is continuous bilinear $L^2(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. This implies that j maps $L^2(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$. $\square$

The inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous. Accordingly, the natural injection $\mathcal{S}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ maps $\mathcal{S}(\mathbb{R}^n)$ continuous linearly into $\mathcal{S}'(\mathbb{R}^n)$.

Exercise 4.3.4. Let $s \in \mathbb{R}$. We denote by $L^2_s(\mathbb{R}^n)$ the space of $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ with $(1 + \|x\|)^s f \in L^2(\mathbb{R}^n)$. Equipped with the inner product

$$\langle f, g \rangle_{L^2,s} := \int_{\mathbb{R}^n} f(x)\overline{g(x)}(1 + \|x\|)^{2s}\, dx$$

this space is a Hilbert space.

Show that the continuous linear injection $L^2_s(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ maps $L^2_s(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$.

The following result will be very useful for our understanding of Sobolev spaces.

Proposition 4.3.5. *The Fourier transform has a continuous linear extension to a continuous linear map $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. For all $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\langle \mathcal{F}u, f \rangle = \langle u, \mathcal{F}f \rangle.$$

The extension to $\mathcal{S}'(\mathbb{R}^n)$ is compatible with the previously defined extension to $L^2(\mathbb{R}^n)$.

Remark 4.3.6. It can be shown that $C_0^\infty(\mathbb{R}^n)$, hence also $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, the continuous linear extension is uniquely determined. However, we shall not need this.

Proof The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous linear. Therefore its tranposed $\mathcal{F}^t : u \mapsto u \circ \mathcal{F}$ is a continuous linear map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

We claim that $\mathcal{F}^t$ restricts to $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$. Indeed, let us view $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as a tempered distribution. Then by a straightforward application of Fubini's theorem, it follows that, for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \mathcal{F}^t\varphi, f \rangle &= \langle \varphi, \mathcal{F}f \rangle \\
&= \int_{\mathbb{R}^n} \varphi(\xi) \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx\, d\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(\xi) e^{-i\xi x}\, d\xi\, f(x)\, dx \\
&= \langle \mathcal{F}\varphi, f \rangle.
\end{aligned}$$

This establishes the claim. We have thus shown that $\mathcal{F}$ has $\mathcal{F}^t$ as a continuous linear extension to $\mathcal{S}'(\mathbb{R}^n)$.

It remains to prove the asserted compatibility. Let $u \in L^2(\mathbb{R}^n)$. There exists a sequence of Schwartz functions $u_n \in \mathcal{S}(\mathbb{R}^n)$ such that $u_n \to u$ in $L^2(\mathbb{R}^n)$ for $n \to \infty$. It follows that $\mathcal{F}u_n \to \mathcal{F}u$ in $L^2(\mathbb{R}^n)$, hence also in $\mathcal{S}'(\mathbb{R}^n)$, by Lemma 4.3.3. On the other hand, we also have $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ by the same lemma. Hence $\mathcal{F}^t u_n \to \mathcal{F}^t u$ by what we proved above. Since $\mathcal{F}^t = \mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ it follows that $\mathcal{F}u_n = \mathcal{F}^t u_n$ for all n. Thus, $\mathcal{F}u = \mathcal{F}^t u$. □

From now on, we shall denote the extension of $\mathcal{F}$ to $\mathcal{S}'(\mathbb{R}^n)$ by the same symbol $\mathcal{F}$. The following lemma is proved in the same spirit as the lemma above. We leave the easy proof to the reader.

Lemma 4.3.7. *The operators ∂^α, $x^\alpha\cdot$, T_a^* and $e^{ia}\cdot$ have (unique) continuous linear extensions to endomorphisms of $\mathcal{S}'(\mathbb{R}^n)$. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$\partial^\alpha u = u \circ (-\partial)^\alpha, \quad x^\alpha u = u \circ x^\alpha, \quad T_a^* u = u \circ T_{-a}^*, \quad e^{ia} u = u \circ e^{ia}.$$

The formulas (a),(b) of Lemma 1.1.10 and the formulas of Lemma 1.1.11 are valid for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 4.3.8. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Then $\mathcal{F}u$ is a smooth function. Moreover, for every $\xi \in \mathbb{R}^n$,*

$$\mathcal{F}u(\xi) = \langle u, e^{-i\xi}\rangle.$$

Proof We sketch the proof. Not all details can be worked out because of time constraints. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then the function $\varphi : \xi \mapsto f(\xi)e^{-i\xi}$ with values in the Fréchet space $C^\infty(\mathbb{R}^n)$ is smooth and compactly supported. This implies that $\xi \mapsto u(\varphi(\xi))$ is smooth and compactly supported. Now

$$u(\varphi(\xi)) = f(\xi)u(e^{-i\xi})$$

and since f was arbitrary, we see that $\hat{u} : \xi \mapsto u(e^{-i\xi})$ is a smooth function.

Furthermore, the integral for $\mathcal{F}f$ may be viewed as an integral of the $C^\infty(\mathbb{R}^n)$-valued function φ. This means that in $C^\infty(\mathbb{R}^n)$ it can be approximated by $C^\infty(\mathbb{R}^n)$-valued Riemann sums. This in turn implies that

$$\begin{aligned} \langle \mathcal{F}u, f\rangle &= \langle u, \mathcal{F}f\rangle \\ &= u(\int_{\mathbb{R}^n} \varphi(\xi)\, d\xi) \\ &= \int_{\mathbb{R}^n} u(\varphi(\xi))\, d\xi \\ &= \int_{\mathbb{R}^n} f(\xi)u(e^{-i\xi})\, d\xi \\ &= \langle \hat{u}, f\rangle. \end{aligned}$$

Since this is true for any $f \in C_c^\infty(\mathbb{R}^n)$, it follows that $\hat{u} = \mathcal{F}u$. □

We recall from Definition 2.2.10 that for $r \in \mathbb{N}$ the Sobolev space $H_r(\mathbb{R}^n)$ is defined as the space of distributions $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \le r$. In particular, taking $\alpha = 0$ we see that $H_r(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Hence also $H_r(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Lemma 4.3.9. *Let $r \in \mathbb{N}$. Then*

$$H_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid (1 + \|\xi\|)^r \mathcal{F}(u) \in L^2(\mathbb{R}^n)\}.$$

Proof Let $u \in H_r(\mathbb{R}^n)$ and let α be a multi-index of order at most r. Then $\partial^\alpha u \in L^2(\mathbb{R}^n)$. It follows that

$$(i\xi)^\alpha \mathcal{F}u = \mathcal{F}(\partial^\alpha u) \in L^2(\mathbb{R}^n).$$

In view of the lemma below this implies that $(1 + \|\xi\|)^r \mathcal{F}u \in L^2(\mathbb{R}^n)$.

Conversely, let $u \in \mathcal{S}'(\mathbb{R}^n)$ and assume that $(1 + \|\xi\|)^r \mathcal{F}u \in L^2(\mathbb{R}^n)$. Then $\mathcal{F}u$ is locally square integrable, and in view of the obvious estimate

$$|\xi^\alpha| \le (1 + \|\xi\|)^{|\alpha|}, \qquad (\xi \in \mathbb{R}^n)$$

it follows that $(i\xi)^\alpha \mathcal{F}u \in L^2(\mathbb{R}^n)$. We conclude that

$$\partial^\alpha u = S^* \mathcal{F}((i\xi)^\alpha \mathcal{F}u) \in L^2(\mathbb{R}^n).$$

$\square$

Lemma 4.3.10. *Let $r \in \mathbb{N}$. There exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^n$,*

$$(1 + \|\xi\|)^r \leq C \sum_{|\alpha| \leq r} |\xi^\alpha|;$$

here ξ^0 should be read as 1.

Proof It is readily seen that there exists a constant $C > 0$ such that

$$(1 + \sqrt{n}|t|)^r \leq C(1 + |t|^r), \qquad (t \in \mathbb{R}),$$

where $|t|^0 \equiv 1$. Let $\xi \in \mathbb{R}^n$ and assume that k is an index such that $|\xi_k|$ is maximal. Then $\|\xi\| \leq \sqrt{n}|\xi_k|$. Hence,

$$(1 + \|\xi\|)^r \leq (1 + \sqrt{n}|\xi_k|)^r \leq C(1 + |\xi_k|^r) \leq C \sum_{|\alpha| \leq r} |\xi^\alpha|.$$

$\square$

Exercise 4.3.11. Show that the Fourier transform maps $H_r(\mathbb{R}^n)$ bijectively onto $L^2_r(\mathbb{R}^n)$. Thus, by transfer of structure, $H_r(\mathbb{R}^n)$ may be given the structure of a Hilbert space. Show that this Hilbert structure is not the same as the one introduced in Definition 2.2.10, but that the associated norms are equivalent.

The characterization of $H_r(\mathbb{R}^n)$ given above allows generalization to arbitrary real r.

Definition 4.3.12. Let $s \in \mathbb{R}$. We define the Sobolev space $H_s(\mathbb{R}^n)$ of order s to be the space of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1 + \|\xi\|)^s \mathcal{F}f \in L^2(\mathbb{R}^n)$, equipped with the inner product

$$\langle f, g \rangle_s = \int_{\mathbb{R}^n} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}\, (1 + \|\xi\|)^{2s}\, d\xi.$$

Equipped with this inner product, the Sobolev space $H_s(\mathbb{R}^n)$ is a Hilbert space. The associated norm is denoted by $\|\cdot\|_s$.

Exercise 4.3.13. The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined as the characteristic function of the interval $[0, \infty)$. For $R > 0$ we define u_R to be the characteristic function of $[0, R]$.

(a) Show that $u_R, H \in \mathcal{S}'(\mathbb{R})$ and that $u_R \to H$ in $\mathcal{S}'(\mathbb{R})$ (pointwise) as $R \to \infty$.
(b) Determine $\mathcal{F}u_R$ for every $R > 0$.
(c) Show that $u_R \in H_s(\mathbb{R})$ for every $s < \frac{1}{2}$, but not for $s = \frac{1}{2}$.
(d) Determine $\mathcal{F}H$ and show that $H \notin H_s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Lemma 4.3.14. *Let $s \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$, with continuous inclusion map. Furthermore, $C_c^\infty(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$.*

Proof If $f \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, let $N \in \mathbb{N}$ be such that $N > s + n/2$. Then $N = s + n/2 + \epsilon$, with $\epsilon > 0$, hence

$$|\mathcal{F}f(\xi)|^2 (1 + \|\xi\|)^{2s} \;\leq\; \nu_{N,0}(\mathcal{F}f)^2 (1 + \|x\|)^{-n-2\epsilon}.$$

This implies that $f \in H_s(\mathbb{R}^n)$ and that

$$\|f\|_s \;\leq\; \nu_{N,0}(\mathcal{F}f)\; \|(1 + \|x\|)^{-n-2\epsilon}\|_{L^1}^{1/2}.$$

Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous, it follows from this estimate that the inclusion map $\mathcal{S} \to H_s$ is continuous.

For the assertion about density it suffices to show that the orthocomplement of $C_c^\infty(\mathbb{R}^n)$ in the Hilbert space $H_s(\mathbb{R}^n)$ is trivial. Let $u \in H_s(\mathbb{R}^n)$, and assume that $\langle u, f\rangle_s = 0$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. This means that

$$\int_{\mathbb{R}^n} \mathcal{F}u(\xi)\, \mathcal{F}f(\xi)\; (1 + \|\xi\|)^{2s}\, d\xi = 0, \qquad (f \in \mathcal{S}(\mathbb{R}^n)).$$

Therefore, the tempered distribution $\mathcal{F}u(\xi)\,(1 + \|\xi\|)^{2s}$ vanishes on the space $\mathcal{F}(C_c^\infty(\mathbb{R}^n))$. The latter space is dense in $\mathcal{F}(\mathbb{R}^n)$, since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}$ is a topological linear automorphism of $\mathcal{S}(\mathbb{R}^n)$. We conclude that $\mathcal{F}u = 0$, hence $u = 0$. □

We conclude this section with two results that will allow us to define the local versions of the Sobolev spaces.

Lemma 4.3.15. *Let $s \in \mathbb{R}^n$. Then convolution $(f, g) \mapsto f * g$, $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) * L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$.*

Proof Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then for all $x, y \in \mathbb{R}^n$ we have

$$(1 + \|x\|)^s |f(y)g(x - y)| \leq (1 + \|y\|)^s |f(y)| (1 + \|x - y\|)^s |g(x - y)|.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then multiplying the above expression by $|\varphi(y)|$, followed by integration against $dxdy$, application of Fubini's theorem and of the Cauchy-Schwartz inequality for the L^2-inner product, we find

$$|\langle (1 + \|x\|)^s f * g, \varphi\rangle| \leq \int_{\mathbb{R}^n} (1 + \|y\|)^s |f(y)|\; dy\; \|g\|_{L^2,s}\; \|\varphi\|_{L^2}.$$

Since this holds for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^n)$, we obtain

$$\|(1 + \|x\|)^s (f * g)\|_{L^2} \leq \|(1 + \|y\|)^s f\|_{L^1}\; \|g\|_{L^2,s}.$$

The expression on the left-hand side equals $\|f * g\|_{L^2,s}$. Fix $N \in \mathbb{N}$ such that $s - N < -n$. Then the L^1-norm on the right-hand side is dominated by $C\nu_{N,0}(f)$, with C equal to the L^1-norm of the function $(1 + \|y\|)^{s-N}$. It follows that

$$\|f * g\|_{L^2,s} \leq C\nu_{N,0}(f)\; \|g\|_{L^2,s}.$$

As $C_c^\infty(\mathbb{R}^n)$ is dense in both $\mathcal{S}(\mathbb{R}^n)$ and $L^2_s(\mathbb{R}^n)$, the result follows. □

Lemma 4.3.16. *Let $s \in \mathbb{R}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in H_s(\mathbb{R}^n)$. Then $\varphi u \in H_s(\mathbb{R}^n)$. Moreover, the associated multiplication map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is continuous bilinear.*

Proof We recall that by definition the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometry for the norms $\| \cdot \|_s$ (from $H_s(\mathbb{R}^n)$) and $\| \cdot \|_{L^2,s}$. From the above lemma it now follows that the multiplication map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$. We need to check that this extension coincides with the restriction of the multiplication map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then we must show that $\langle fg, \varphi \rangle = \langle g, f\varphi \rangle$ for all $g \in H_s(\mathbb{R}^n)$. By continuity of the expressions on both sides in g (verify this!), it suffices to check this on the dense subspace $C_c^\infty(\mathbb{R}^n)$, where it is obvious. $\square$

In particular, it follows that $C_c^\infty(\mathbb{R}^n) H_s(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$. Therefore, we may define local Sobolev spaces.

Let $U \subset \mathbb{R}^n$ be open, and let $s \in \mathbb{R}$. We define the local Sobolev space $H_{s,\mathrm{loc}}$ in the usual way, as the space of distributions $u \in \mathcal{D}'(U)$ such that $\chi u \in H_s(\mathbb{R}^n)$ for every $\chi \in C_c^\infty(\mathbb{R}^n)$. At a later stage we will prove invariance of the local Sobolev spaces under diffeomorphisms, so that the notion of $H_{s,\mathrm{loc}}$ can be lifted to sections of a vector bundle on a smooth manifold.

Exercise 4.3.17. This exercise is a continuation of Exercise 4.3.13. Show that the Heaviside function $H = 1_{[0,\infty)}$ belongs to $H_{s,\mathrm{loc}}(\mathbb{R}^n)$ for every $s < \frac{1}{2}$ but not for $s = \frac{1}{2}$.

4.4. Some useful results for Sobolev spaces

We note that for $s < t$ the estimate $\|f\|_s \leq \|f\|_t$ holds for all $f \in H_t(\mathbb{R}^n)$. Accordingly, we see that

$$H_t(\mathbb{R}^n) \subset H_s(\mathbb{R}^n), \quad \text{for} \quad s < t,$$

with continuous inclusion map. We also note that, by the Plancherel theorem for the Fourier transform, $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Accordingly,

$$H_s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H_{-s}(\mathbb{R}^n) \qquad (s \geq 0). \tag{4.5}$$

Lemma 4.4.1. *Let $\alpha \in \mathbb{N}^n$. Then $\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ restricts to a continuous linear map $H_s(\mathbb{R}^n) \to H_{s-|\alpha|}(\mathbb{R}^n)$, for every $s \in \mathbb{R}$.*

Proof This is an immediate consequence of the definitions. $\square$

Given $k \in \mathbb{N}$ we define $C_b^k(\mathbb{R}^n)$ to be the space of C^k-functions $f : \mathbb{R}^n \to \mathbb{C}$ with

$$s_k(f) := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D_x^\alpha f(x)| < \infty.$$

Equipped with the norm s_k, this space is a Banach space.

Lemma 4.4.2. (Sobolev lemma) *Let $k \in \mathbb{N}$ and let $s > k + n/2$. Then*

$$H_s(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n)$$

with continuous inclusion map.

Proof In view of the previous lemma, it suffices to prove this for $k = 0$. We then have $s = n/2 + \epsilon$, with $\epsilon > 0$. Let $u \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} \mathcal{F}u(\xi) e^{i\xi x}\, dx \\ &= \int_{\mathbb{R}^n} e^{i\xi x} \mathcal{F}u(\xi)(1+\|\xi\|)^s (1+\|\xi\|)^{-n/2-\epsilon}\, d\xi \end{aligned}$$

From this we read off that u is bounded continuous, and

$$\sup|u| \leq \|u\|_s\, \|(1+\|\xi\|)^{-n/2-\epsilon}\|_{L^2}.$$

It follows that the inclusion $C_c^\infty \subset C_b$ is continuous with respect to the H_s topology on the first space. By density the inclusion has a unique extension to a continuous linear map $H_s \to C_b$. By testing with functions from $\mathcal{S}$ we see that the latter map coincides with the inclusion of these spaces viewed as subspaces of $\mathcal{S}'$. $\square$

In accordance with the above embedding, we shall view $H_s(\mathbb{R}^n)$, for $s > k + n/2$, as a subspace of $C_b^k(\mathbb{R}^n)$. We observe that as an important consequence we have the following result. Put

$$H_\infty(\mathbb{R}^n) = \bigcap_{s\in\mathbb{R}} H_s(\mathbb{R}^n).$$

Corollary 4.4.3.

(a) $H_\infty(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n)$.
(b) $H_\infty(\mathbb{R}^n)$ *equals the space of smooth functions* $f \in C^\infty(\mathbb{R}^n)$ *with* $\partial^\alpha f \in L^2(\mathbb{R}^n)$, *for all* $\alpha \in \mathbb{R}^n$.

Proof Assertion (a) is an immediate consequence of the previous lemma. For (b) we note that $H_r \subset H_s$ for $s < r$. We see that $H_\infty(\mathbb{R}^n)$ is the intersection of the spaces $H_r(\mathbb{R}^n)$, for $r \in \mathbb{N}$. Now use the original definition of $H_r(\mathbb{R}^n)$, Definition 2.2.10. $\square$

Let V, W be topological linear spaces. Then a pairing of V and W is a continuous bilinear map $\beta : V \times W \to \mathbb{C}$. The pairing induces a continuous map $\beta_1 : V \to W^*$ by $\beta_1(v) : w \mapsto \beta(v, w)$ and similarly a map $\beta_2 : W \to V^*$; the stars indicate the continuous linear duals of the spaces involved. The pairing is called non-degenerate if both the maps β_1 and β_2 are injective. It is called perfect if it is non-degenerate, and if β_1 is an isomorphism $V \to W^*$, and β_2 an isomorphism $W \to V^*$.

If V is a complex linear space, we denote by $\bar{V}$ the conjugate space. This is the complex space which equals V as a real linear space, whereas the complex scalar multiplication is given by $(z, v) \mapsto \bar{z}v$.

If V is a Banach space, the continuous linear dual V^* is equipped with the dual norm $\|\cdot\|^*$, given by

$$\|u\|^* = \sup\{|u(x)| \mid x \in V, \|x\| \leq 1\}.$$

This dual norm also defines a norm on the conjugate space $\bar{V}^*$.

If H is a Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, then the associated norm $\|\cdot\|$ may be characterized by

$$\|v\| = \sup_{\|w\|\leq 1} |\langle v, w\rangle|$$

It follows that $v \mapsto \langle v,\,\cdot\,\rangle$ induces a linear isomorphism $\varphi : H \to \bar{H}^*$ which is an isometry for the norm on H and the associated dual norm on H^*. The isometry φ may be used to transfer the Hilbert structure on H to a Hilbert structure on $\bar{H}^*$, called the dual Hilbert structure. It is readily seen that the norm associated with this dual Hilbert structure equals the dual norm $\|\cdot\|^*$ defined above.

Lemma 4.4.4. *Let $s \in \mathbb{R}$. Then the L^2-inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $C_c^\infty(\mathbb{R}^n)$ extends uniquely to a continuous bilinear pairing $H_s(\mathbb{R}^n) \times \bar{H}_{-s}(\mathbb{R}^n) \to \mathbb{C}$. The pairing is perfect and induces isometric isomorphisms $H_s(\mathbb{R}^n) \simeq \bar{H}_{-s}(\mathbb{R}^n)^*$ and $\bar{H}_{-s}(\mathbb{R}^n) \simeq H_s(\mathbb{R}^n)^*$.*

Proof Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}\langle f, g\rangle_{L^2} &= \int_{R^n} \mathcal{F}f(\xi)\mathcal{F}g(\xi)\, d\xi \\ &= \int_{R^n} \mathcal{F}f(\xi)(1+\|\xi\|)^s \mathcal{F}g(\xi)(1+\|\xi\|)^{-s}\, d\xi.\end{aligned}$$

By the Cauchy-Schwartz inequality, it follows that the absolute value of the latter expression is at most $\|f\|_s\|g\|_{-s}$. By density of $C_c^\infty(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$, this implies the assertion about the extension of the pairing. The above formulas also imply that

$$\sup_{g\in C_c^\infty(\mathbb{R}^n), \|g\|_{-s}=1} \langle f, g\rangle = \|f\|_s.$$

Thus, by density of $C_c^\infty(\mathbb{R}^n)$, the induced map $\beta_1 : H_s(\mathbb{R}^n) \to \bar{H}_{-s}(\mathbb{R}^n)^*$ is an isometry. Likewise, $\beta_2 : H_{-s}(\mathbb{R}^n) \to \bar{H}_s(\mathbb{R}^n)^*$ is an isometry. From the injectivity of β_1 it follows that β_2 has dense image. Being an isometry, β_2 must then be surjective. Likewise, β_1 is surjective. $\square$

4.5. Rellich's lemma for Sobolev spaces

In this section we will give a proof of the Rellich lemma for Sobolev spaces, which will play a crucial role in the proof of the Fredholm property for elliptic pseudo-differential operators on compact manifolds.

Given $s \in \mathbb{R}$ and a compact subset $K \subset \mathbb{R}^n$, we define

$$H_{s,K}(\mathbb{R}^n) = \{u \in H_s(R^n) \mid \operatorname{supp} u \subset K\}.$$

Lemma 4.5.1. *$H_{s,K}(\mathbb{R}^n)$ is a closed subspace of $H_s(\mathbb{R}^n)$.*

Proof Let $f \in C_c^\infty(\mathbb{R}^n)$. Then the space

$$f^\perp := \{u \in H_s(\mathbb{R}^n) \mid \langle u, f\rangle = 0\}$$

has Fourier transform equal to the space of $\varphi \in L_s^2(\mathbb{R}^n)$ with $\langle \varphi, \mathcal{F}f\rangle = 0$, which is the orthocomplement of $(1+\|\xi\|)^{-2s}\mathcal{F}f$ in $L_s^2(\mathbb{R}^n)$. As this orthocomplement is closed in $L_s^2(\mathbb{R}^n)$, it follows that $f^\perp$ is closed in $H_s(\mathbb{R}^n)$.

We now observe that $H_{s,K}(\mathbb{R}^n)$ is the intersection of the spaces $f^\perp$ for $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \cap K = \emptyset$. $\square$

Lemma 4.5.2. (Rellich) *Let $t < s$. Then the inclusion map $H_{s,K}(\mathbb{R}^n) \to H_t(\mathbb{R}^n)$ is compact.*

To prepare for the proof, we first prove the following result, which is based on an application of the Ascoli-Arzéla theorem.

Lemma 4.5.3. *Let B be a bounded subset of the Fréchet space $C^1(\mathbb{R}^n)$. Then B is relatively compact (i.e., has compact closure) as a subset of the Fréchet space $C(\mathbb{R}^n)$.*

Proof Boundedness of B means that every continuous semi-norm of $C^1(\mathbb{R}^n)$ is bounded on B. Let $K \subset \mathbb{R}^n$ be a compact ball. Then there exists a constant $C > 0$ such that $\sup_K \|df\| \leq C$ for all $f \in B$ and each $1 \leq j \leq n$. Since

$$f(x) - f(y) = \int_0^1 df(y + t(x-y))(x-y)\, dt$$

for all $y \in x$, we see that

$$|f(y) - f(x)| \leq C\|x - y\|, \quad \text{for all} \quad (x, y \in K).$$

It follows that the set of functions $B|_K = \{f|_K \mid f \in B\}$ is equicontinuous and bounded in $C(K)$. By application of the Ascoli-Arzèla theorem, the set $B|_K$ is relatively compact in $C(K)$. In particular, if (f_k) is a sequence in B, then there is a subsequence (f_{k_j}) which converges uniformly on K.

Let (f_k) be a sequence in B. We shall now apply the usual diagonal procedure to obtain a subsequence that converges in $C(\mathbb{R}^n)$.

For $r \in \mathbb{N}$ let K_r denote the ball of center 0 and radius r in $\mathbb{R}^n$. Then by repeated application of the above there exists a sequence of subsequences $(f_{k_{1,j}}) \succeq (f_{k_{2,j}}) \succeq \cdots$ ... such that $(f_{k_{r,j}})$ converges uniformly on K_r, for every $r \in \mathbb{N}$.

The sequence $(f_{k_{j,j}})_{j\in\mathbb{N}}$ is a subsequence of all the above sequences. Hence, it converges uniformly on each ball K_r. Therefore, it converges in $C(\mathbb{R}^n)$. $\square$

Remark 4.5.4. By a slight modification of the proof above, one obtains a proof of the compactness of each inclusion map $C^{k+1}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$. This implies that the identity operator of $C^\infty(\mathbb{R}^n)$ is compact. Equivalently, each bounded subset of $C^\infty(\mathbb{R}^n)$ is relatively compact. A locally convex topological vector space with this property is called Montel.

If B is a subset of $L^2_s(\mathbb{R}^n)$ (see Lecture 4) and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we write

$$\varphi * B := \{\varphi * f \mid f \in B\}.$$

Then $\varphi * B$ is a subset of $L^2_s(\mathbb{R}^n)$.

Lemma 4.5.5. *Let $s \in \mathbb{R}$ and let $B \subset L^2_s(\mathbb{R}^n)$ be bounded. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the set $\varphi * B$ is a relatively compact subset of $C(\mathbb{R}^n)$.*

Proof In view of the previous lemma, it suffices to prove that $f * B$ is bounded in $C^1(\mathbb{R}^n)$. For this we note that for each $1 \leq j \leq n$,

$$\begin{aligned}
&|\frac{\partial}{\partial x_j}[\varphi(x-y)f(y)]| \\
&\quad = \quad |\partial_j\varphi(x-y)|(1+\|y\|)^{-s}(1+\|y\|)^s|f(y)| \\
&\quad \leq \quad (1+\|x\|)^{-s}|(1+\|x-y\|)^{-s}\|\partial_j\varphi(x-y)\|(1+\|y\|)^s|f(y)|.
\end{aligned}$$

The right-hand side can be dominated by an integrable function of y, locally uniformly in x. It now follows by differentiation under the integral sign that $\varphi * f \in C^1(\mathbb{R}^n)$, that $\partial_j(\varphi * f) = \partial_j\varphi * f$ and that

$$\|\partial_j(\varphi * f)(x)\| \leq (1+\|x\|)^{-s}\,\|\varphi\|_{L^2,-s}\,\|f\|_{L^2,s}.$$

This implies that the set $\varphi * B$ is bounded in $C^1(B)$, hence relatively compact in $C(\mathbb{R}^n)$. □

Proposition 4.5.6. *Let $s > t$ and let B be a bounded subset of $L^2_s(\mathbb{R}^n)$ which at the same time is a relatively compact subset of $C(\mathbb{R}^n)$. Then B is relatively compact in $L^2_t(\mathbb{R}^n)$.*

Proof For $R > 0$ we denote by 1_R the characteristic function of the closed ball $B(R) := \bar{B}(0;R)$. Then for each $r \in \mathbb{R}$, the map $f \mapsto 1_R f$ gives the orthogonal projection from $L^2_s(\mathbb{R}^n)$ onto the closed subspace $L^2_{s,B(R)}$ of functions with support in $B(R)$. We now observe that the following estimate holds for every $f \in L^2_s(\mathbb{R}^n)$:

$$\begin{aligned}
\|(1-1_R)f\|^2_{L^2,t} &= \int_{\|x\|\geq R} (1+\|x\|)^{2t-2s}(1+\|x\|)^{2s}\|f(t)\|^2\,dt \\
&\leq (1+R)^{2(t-s)}\|f\|^2_{L^2,s}.
\end{aligned}$$

Fix $M > 0$ such that $\|f\|_{L^2,s} \leq M$ for all $f \in B$. Then we see that

$$\|(1-1_R)f\|_{L^2,t} \leq M\,(1+R)^{t-s}, \qquad (f \in B).$$

Let now (f_k) be a sequence in B. Then (f_k) has a subsequence (f_{k_j}) which converges in $C(\mathbb{R}^n)$, i.e., there exists a function $f \in C(\mathbb{R}^n)$ such that $f_{k_j} \to f$ uniformly on each compact set $K \subset \mathbb{R}^n$. It easily follows from this that $1_R f_{k_j}$ is a Cauchy-sequence in $L^2_t(\mathbb{R})$, for each $R > 0$. We will show that f_{k_j} is actually a Cauchy sequence in $L^2_t(\mathbb{R}^n)$. By completeness of the latter space, this will complete the proof.

Let $\epsilon > 0$. We fix $R > 0$ such that

$$M(1+R)^{t-s} < \frac{1}{3}\epsilon.$$

There exists a constant $N > 0$ such that

$$i, j \geq N \Rightarrow \|1_R f_{k_i} - 1_R f_{k_j}\|_{L^2,t} < \frac{1}{3}.$$

It follows that for all $i, j > N$,

$$\begin{aligned} &\|f_{k_i} - f_{k_j}\|_{L^2,t} \\ &\quad \leq \|1_R(f_{k_i} - f_{k_j})\|_{L^2,t} + \|(1 - 1_R)f_{k_i}\|_{L^2,t} + \|(1 - 1_R)f_{k_j}\|_{L^2,t} \\ &\quad < \epsilon. \end{aligned}$$

□

Proof of Lemma 4.5.2 Let $K \subset \mathbb{R}^n$ be compact and let B be a bounded subset of $H_{s,K}(\mathbb{R}^n)$. Fix a smooth compactly supported function $\chi \in C_c^\infty(\mathbb{R}^n)$ that is 1 on a neighborhood of K. Then $\chi f = f$ for all $f \in B$. It follows that

$$\mathcal{F}(B) = \varphi * \mathcal{F}(B),$$

with $\varphi = \mathcal{F}(\chi) \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 4.5.5 it now follows that $\mathcal{F}(B)$ is both bounded in $L^2_s(\mathbb{R}^n)$ and a relatively compact subset of $C(\mathbb{R}^n)$. By the previous proposition, this implies that $\mathcal{F}(B)$ is relatively compact in $L^2_t(\mathbb{R}^n)$. As $\mathcal{F}$ is an isometry from $H_t(\mathbb{R}^n)$ to $L^2_t(\mathbb{R}^n)$, it follows that B is relatively compact in $H_t(\mathbb{R}^n)$. □

LECTURE 5
Pseudo-differential operators, local theory

5.1. The space of symbols

We consider a differential operator P on $\mathbb{R}^n$ of the form

$$P = p(x, D_x) = \sum_{|\alpha| \le d} c_\alpha(x) D_x^\alpha; \tag{5.1}$$

here we recall that $D_x^\alpha = (-i\partial_x)^\alpha$. The coefficients c_α are assumed to be smooth functions on $\mathbb{R}^n$. The (full) symbol of P is the function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ given by

$$p(x, \xi) = \sum_{|\alpha| \le d} c_\alpha(x) \xi^\alpha.$$

If $f \in C_c^\infty(\mathbb{R}^n)$, then $\mathcal{F}(D^\alpha f) = \xi^\alpha \mathcal{F} f$, so that by the Fourier inversion formula we have

$$D^\alpha f(x) = \int_{\mathbb{R}^n} \xi^\alpha \widehat{f}(\xi)\, e^{i\xi x}\, d\xi,$$

where we have written $\widehat{f} = \mathcal{F} f$. It follows that the action of P on $C_c^\infty(\mathbb{R}^n)$ can be described by

$$Pf(x) = \int_{\mathbb{R}^n} p(x, \xi) \widehat{f}(\xi)\, e^{i\xi x}\, d\xi.$$

Pseudo-differential operators are going to be defined by the same formula, but with p from a larger class of spaces of functions, the so-called symbol spaces. The idea is to make the class large enough to allow a kind of division. This in turn will allow us to construct inverses to elliptic operators modulo smoothing operators, the so-called parametrices.

We return to the full symbol p of the differential operator P of degree at most d considered above. By the polynomial nature of the symbol p in the ξ-variable, there exists, for every compact subset $K \subset \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{N}^n$, a constant $C = C_{K,\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \le C(1 + \|\xi\|)^{d - |\beta|}, \qquad ((x, \xi) \in K \times \mathbb{R}^n).$$

Exercise 5.1.1. Prove this.

These observations motivate the following definition of the space of symbols of order d, for d a real number.

Definition 5.1.2. Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. The space of symbols on U of order at most d is defined to be the space of functions $q \in C^\infty(U \times \mathbb{R}^n)$ such that for each compact subset $K \subset U$ and all multi-indices α, β, there exists a constant $C = C_{K,\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leq C(1 + \|\xi\|)^{d-|\beta|}, \qquad ((x,\xi) \in K \times \mathbb{R}^n).$$

This space is denoted by $S^d(U)$.

We note that $S^d(U)$ can be equipped with the locally convex topology induced by the seminorms

$$\mu_{K,k}^d(q) := \max_{|\alpha|,|\beta| \leq k} \sup_{K \times \mathbb{R}^n} (1 + \|\xi\|)^{|\beta|-d} \, |\partial_x^\alpha \partial_\xi^\beta q(x,\xi)|,$$

for $K \subset U$ compact and $k \in \mathbb{N}$. Moreover, $S^m(U)$ is a Fréchet space for this topology.

Exercise 5.1.3. Show that $d_1 \leq d_2$ implies $S^{d_1}(U) \subset S^{d_2}(U)$ with continuous inclusion map.

We agree to write

$$S^\infty(U) = \cup_{d \in \mathbb{R}} S^d(U), \qquad S^{-\infty}(U) = \cap_{d \in \mathbb{R}} S^d(U).$$

Then $S^{-\infty}(U)$ equals the space of smooth functions $f : U \times \mathbb{R}^n \to \mathbb{C}$ such that for all $K \subset U$ compact, $N \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\nu_{K,k,N}(f) := \max_{|\alpha|,|\beta| \leq k} \sup_{K \times \mathbb{R}^n} (1 + \|\xi\|)^N |\partial_x \partial_\xi^\beta f(x,\xi)| < \infty.$$

Moreover, the norms $\nu_{K,k,N}$ induce a locally convex topology on $S^{-\infty}(U)$, which turn this space into a Fréchet space. Here we note that a function φ in the usual Schwartz space $\mathcal{S}(\mathbb{R}^n)$ can be viewed as the function $(x,\xi) \mapsto \varphi(\xi)$ in $S^{-\infty}(U)$. The corresponding natural linear map $\mathcal{S}(\mathbb{R}^n) \to S^{-\infty}(U)$ is a topological linear isomorphism onto the closed subspace of functions in $S^{-\infty}(U)$ that are constant in the x-variable. More generally, if $f \in C^\infty(U)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then the function

$$f \otimes \varphi : (x,\xi) \mapsto f(x)\varphi(\xi)$$

belongs to $S^{-\infty}(U)$. It can be shown that $S^{-\infty}(U)$ is the closure of the subspace $C^\infty(U) \otimes \mathcal{S}(\mathbb{R}^n)$ generated by these elements. Accordingly, we may view $S^{-\infty}(U)$ as a topological tensor product; this is expressed by the notation

$$S^{-\infty}(U) = C^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n).$$

Exercise 5.1.4. Show that the following functions are symbols on $U = \mathbb{R}^n$. What can be said about their orders?

(a) $p(x,\xi) = \|x\|^2(1 + \|\xi\|^2)^s$, for $s \in \mathbb{R}$.
(b) $p(x,\xi) = (1 + \|x\|^2 + \|\xi\|^2)^s$, for $s \in \mathbb{R}$.

Exercise 5.1.5. Let $U \subset \mathbb{R}^n$ be an open subset.

(a) Show that for each $\alpha \in \mathbb{N}^n$ the operator ∂_x^α gives a continuous linear map $S^d(U) \to S^d(U)$.
(b) Show that for each multi-index α as above the operator ∂_ξ^α restricts to a continuous linear map $S^d(U) \to S^{d-|\alpha|}(U)$, for every $d \in \mathbb{R}$.
(c) Show that the product map $(p, q) \mapsto pq$ restricts to a continuous bilinear map $S^d(U) \times S^e(U) \to S^{d+e}(U)$, for all $d, e \in \mathbb{N}$. Discuss what happens if $d = -\infty$ or $e = -\infty$.

Exercise 5.1.6. Let $U \subset \mathbb{R}^n$ be an open subset and let P be an elliptic differential operator of order d on U. This means that its principal symbol $\sigma^d(P)$ does not vanish on $U \times (\mathbb{R}^n \setminus \{0\})$. Let p the full symbol of P. The purpose of this exercise is to show that there exists a $q \in S^{-d}(U)$ such that $pq - 1 \in S^{-\infty}(U)$. We first address the local question. Let $V \subset U$ be an open subset with compact closure in U. We write p_V for the restriction of p to $V \times \mathbb{R}^n$.

(a) Show that there exists a constant $R = R_V > 0$ such that $p(x, \xi) \neq 0$ for $x \in V$ and $\xi \in \mathbb{R}^n \setminus B(0; R)$.
(b) Show that there exists a smooth function $\chi_V \in C_c^\infty(\mathbb{R}^n)$ such that the function $q : V \times \mathbb{R}^n \to \mathbb{C}$ defined by
$$q(x, \xi) := (1 - \chi(\xi)) p(x, \xi)^{-1}$$
if $p(x, \xi) \neq 0$ and by zero otherwise, is smooth.
(c) With χ and q as above, show that $q \in S^{-d}(V)$.
(d) Show that $p_V q - 1 \in \Psi^{-\infty}(V)$.
(e) Show that there exists a symbol $q \in S^{-d}(U)$ such that $pq - 1 \in \Psi^{-\infty}(U)$.

The following invariance result will allow us to extend the definition of the symbol space to an arbitrary smooth manifold. Let $\varphi : U \to V$ be diffeomorphism between open subsets of $\mathbb{R}^n$. We define the map $\varphi_* : C^\infty(U \times \mathbb{R}^{n*}) \to C^\infty(V \times \mathbb{R}^{n*})$ by
$$\varphi_* f(y, \eta) = f(\varphi^{-1}(y), \eta \circ d\varphi(\varphi^{-1}(y))).$$
Identifying $\mathbb{R}^n$ with its dual $\mathbb{R}^{n*}$ by using the standard inner product, we may view $S^d(U)$ as a subspace of $C^\infty(U \times \mathbb{R}^{n*})$.

Lemma 5.1.7. *For every $d \in \mathbb{R}$, the map φ_* restricts to a topological linear isomorphism $S^d(U) \to S^d(V)$.*

Proof Put $\psi = \varphi^{-1}$. Then, for $f \in S^d(M)$, the function $\varphi_* f$ is given by $\varphi_* f(y, \eta) = f(\psi(y), \eta \circ d\varphi(\psi(y)))$. The continuity of φ_* follows from checking that $\partial_y^\alpha \partial_\eta^\beta (\varphi_* f)$ satisfies the required estimates by a straightfoward but tedious application of the chain rule combined with the Leibniz rule. Similarly, ψ_* is seen to be continuous linear. $\square$

We define the symbol spaces on a manifold as follows.

Definition 5.1.8. Let M be a smooth manifold and let $d \in \mathbb{R}$. A symbol of order d is defined to be smooth function $\sigma : T^*M \to \mathbb{R}^n$ such that for each $x_0 \in M$ there exists a coordinate patch U_κ containing x_0 such that the natural map $\kappa_* : C^\infty(T^*U_\kappa) \to C^\infty(\kappa(U_\kappa) \times \mathbb{R}^{n*})$ maps $\sigma|_{T^*U}$ to an element $\kappa_*\sigma$ of $S^d(\kappa(U_\kappa))$. The space of these symbols is denoted by $S^d(M)$.

Remark 5.1.9. Let $\varphi : M \to N$ be a diffeomorphism of smooth manfolds. Then it follows by application of Lemma 5.1.7 that the natural map $\varphi_* : C^\infty(T^*M) \to C^\infty(T^*N)$, given by

$$\varphi_* f(\eta_{\varphi(x)}) = \eta_{\varphi(x)} \circ T_x\varphi, \qquad (x \in M,\, \eta_{\varphi(x)} \in T^*_{\varphi(x)}N)$$

restricts to a linear isomorphism

$$\varphi_* : S^d(M) \xrightarrow{\simeq} S^d(N).$$

In this lecture we will concentrate on the local theory of symbols and the associated pseudo-differential operators. The extension to manifolds will be rather straightforward, by using invariance and partitions of unity. In particular, one needs to localize on the x-variable in the symbol space. Accordingly, given $A \subset U$ compact and $d \in \widehat{\mathbb{R}}$ we define

$$S^d_A(U) = \{p \in S^d(U) \mid \mathrm{pr}_1(\operatorname{supp} p) \subset A\},$$

where $\mathrm{pr}_1 : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the natural projection map. The union of these spaces, for $A \subset U$ compact, is denoted by $S^d_c(U)$. Here we note that $S^d_c(U) \subset S^d_c(\mathbb{R}^n)$ naturally, by extension by zero outside $U \times \mathbb{R}^n$.

5.2. Pseudo-differential operators

In this section we will give the definition of the space $\Psi^\infty(U)$ of pseudo-differential operators on an open subset $U \subset \mathbb{R}^n$. For this, we first need to introduce the space $\Psi^{-\infty}(U)$ of smoothing operators on U. Given a smooth function $K \in C^\infty(U \times U)$, we define the integral operator T_K with integral kernel K to be the continuous linear operator $C^\infty_c(U) \to C^\infty(U)$ given by

$$T_K f(x) = \int_U K(x, y)\, f(y)\, dy.$$

We define $\Psi^{-\infty}$ to be the subspace of $\mathrm{Hom}(C^\infty_c(U), C^\infty(U))$ consisting of all operators of the form T_K, with $K \in C^\infty(U \times U)$. It is readily seen that $K \to T_K$ is injective, and thus provides a linear isomorphism from $C^\infty(U \times U)$ onto $\Psi^{-\infty}(U)$.

The name smoothing operator is derived from the following observation. We may extend the definition of T_K to the space $\mathcal{E}'(U)$ of compactly supported distributions on U by the formula

$$T_K u(x) = u(K(x, \cdot\,)).$$

Since $x \mapsto K(x, \cdot\,)$ is a smooth function on U with values in the Fréchet space $C^\infty(U)$, and since $u : C^\infty(U) \to \mathbb{C}$ is continuous linear, it follows that $T_K(u)$ is smooth. Moreover, the map

$$T_K : \mathcal{E}'(U) \to C^\infty(U)$$

is continuous linear. Conversely, by the Schwartz kernel theorem it follows that any continuous linear map $T : \mathcal{E}'(U) \to C^\infty(U)$ is of the form T_K, with K a uniquely determined smooth function on $U \times U$. We define $\Psi^{-\infty}(U)$ to be the space of all operators of the form T_K, for $K \in C^\infty(U \times U)$. Accordingly,

$$\Psi^{-\infty}(U) \simeq \mathrm{Hom}(\mathcal{E}'(U), C^\infty(U)).$$

We proceed to the definition of pseudo-differential operators on U. For $\varphi \in S^{-\infty}(U) = C^\infty(U)) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ the integral

$$W(\varphi)(x) := \int_{\mathbb{R}^n} e^{i\xi x} \varphi(x,\xi)\, dx$$

is absolutely convergent for every $x \in U$ and is readily seen to define a function $W(\varphi) \in C^\infty(U)$. More precisely, the following result holds.

Lemma 5.2.1. *For all $\varphi \in S^{-\infty}(U)$ and $\alpha \in \mathbb{N}^n$,*

(a) $\partial^\alpha W(\varphi) = W((\partial_x + i\xi)^\alpha \varphi)$;

(b) $(ix)^\alpha W(\varphi) = W(\partial_\xi^\alpha \varphi)$.

The map W is continuous linear from $S^{-\infty}(U)$ to $C^\infty(U)$.

Proof The proof is an obvious adaptation of the proof that Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ continuously to $C^\infty(\mathbb{R}^n)$. □

If $p \in S^d(U)$ and $F \in \mathcal{S}(\mathbb{R}^n)$ then it follows by a straightforward application of the Leibniz rule that the function $pF : (x,\xi) \mapsto p(x,\xi)F(\xi)$ belongs to $S^{-\infty}(U)$. Moreover, the map $(p,F) \mapsto pF$ is continuous and bilinear. These observations justify the following definition.

Definition 5.2.2. Let $p \in S^d(U)$. Then we define the operator $\Psi_p : C_c^\infty(U) \to C^\infty(U)$ by

$$\Psi_p f(x) := W(p\widehat{f})(x) = \int_{\mathbb{R}^n} e^{i\xi x} p(x,\xi) \widehat{f}(\xi)\, d\xi. \tag{5.2}$$

We note that $(p,f) \mapsto \Psi_p f$ is continuous and bilinear $S^d(U) \times C_c^\infty(U) \to C^\infty(U)$.

Lemma 5.2.3. *The (linear) map $p \mapsto \Psi_p$, $S^d(\mathbb{R}^n) \to \mathrm{Hom}(C_c^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ is injective.*

Proof Assume that $p \in S^d(\mathbb{R}^n)$ and $\Psi_p = 0$. Then for each $x \in \mathbb{R}^n$ the smooth function $e^{ix}p_x$ given by $\xi \mapsto e^{i\xi x}p(x,\xi)$ is perpendicular to all functions from $\mathcal{F}(C_c^\infty(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$. By density of $C_c^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ and the fact that $\mathcal{F}$ is a continuous linear automorphism of $\mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{F}(C_c^\infty(\mathbb{R}^n))$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Hence, $\xi \mapsto e^{i\xi x}p(x,\xi)$ is perpendicular to all functions from $\mathcal{S}(\mathbb{R}^n)$. In particular, p_x is perpendicular to $C_c^\infty(\mathbb{R}^n)$ and it follows that $p_x = 0$. □

If P is a differential operator with smooth coefficients of order d on U then its full symbol p belongs to $S^d(U)$ and

$$P = \Psi_p.$$

We now generalize the notion of differential operator as follows.

Definition 5.2.4. Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. A *pseudo-differential operator* of order d on U is a continuous linear operator $P : C_c^\infty(U) \to C^\infty(U)$ of the form

$$P = \Psi_p + T,$$

with $p \in S^d(U)$ and $T \in \Psi^{-\infty}(U)$. The space of these operators is denoted by $\Psi^d(U)$.

Lemma 5.2.5. *Let $A \subset U$ be compact.*

(a) *Let $p \in S^{-\infty}(U)$. Then there exists a unique $K \in C_c^\infty(U \times U)$ such that $\Psi_p = T_K$. In particular, $\Psi_p \in \Psi^{-\infty}(U)$. If Ψ_p vanishes on $C_c^\infty(U \setminus A)$ then* $\mathrm{pr}_2 \operatorname{supp} K \subset A$.

(b) *Let $K \in C^\infty(U \times U)$ be such that* $\mathrm{pr}_2(\operatorname{supp} K) \subset A$. *Then there exists a $p \in S^{-\infty}(U)$ such that the integral operator $T_K : C_c^\infty(U) \to C^\infty(U)$ equals Ψ_p.*

Proof Let $\mathcal{F}_2 F$ denote the Fourier transform of a function $F \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$ with respect to the second component. By straightforward estimation it follows that $\mathcal{F}_2$ is a continuous linear endomorphism of $C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. By application of Theorem 4.1.14 with respect to the second variable it follows that in fact $\mathcal{F}_2$ is a topogical linear automorphism.

We can now prove (a). Let p be as asserted and define

$$\widetilde{K}(x, y) = \mathcal{F}_2 p(x, y - x).$$

Then $\widetilde{K} \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. Moreover, by the Fourier inversion theorem we see that $p(x, \xi) = e^{-i\xi x} \mathcal{F}_2(\tilde{K})(x, -\xi)$. We put $K = \tilde{K}|_{U \times U}$. Then for all $f \in C_c^\infty(U)$ and $x \in U$ we have

$$T_K f(x) = T_{\tilde{K}} f(x) = \int_{\mathbb{R}^n} \mathcal{F}_2 K(x, -\xi) \widehat{f}(\xi)\, d\xi = \Psi_p f(x).$$

Uniqueness of K is obvious.

Now assume that Ψ_p vanishes on $C_c^\infty(U \setminus A)$. Then T_K vanishes on $C_c^\infty(U \setminus A)$. This implies that K is zero when tested with functions from $C_c^\infty(U \times (U \setminus A))$. Hence, $\operatorname{supp} K \subset U \times A$.

We turn to (b). Let $K \in C^\infty(U \times U)$ and assume $\operatorname{supp} K \subset U \times A$. Then $K \in C^\infty(U) \widehat{\otimes} \mathcal{S}(\mathbb{R}^n)$. Applying Proposition 4.2.5 with respect to the second variable, we see that

$$T_K = \Psi_p, \qquad p(x, \xi) = e^{-i\xi x} \mathcal{F}_2(K)(x, -\xi).$$

It is clear that $p \in S^{-\infty}(U) = S^{-\infty}(U)$.

□

Exercise 5.2.6. Let $U \subset \mathbb{R}^n$ be an open subset, and let $K \in C^\infty(U \times U)$. Show that T_K is a local operator if and only if $K = 0$.

In particular, it follows that pseudo-differential operators are not local in general, in contrast to differential operators. In fact, in view of the following result, differential operators are precisely those pseudo-differential operators that are local.

Theorem 5.2.7. (Peetre's theorem) *Let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear map such that* $\operatorname{supp} Pf \subset \operatorname{supp} f$ *for all $f \in C_c^\infty(\mathbb{R}^n)$. Then P is a differential operator (with bounded degree on every compact subset of U).*

It is remarkable that this result is true without any assumption of continuity for P. The analogous result is much easier to prove if P is required to be continuous. From this the above characterization of differential operators among the pseudo-differential operators already follows. A proof is suggested in the following exercise.

Exercise 5.2.8. Let $P : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator such that for all $f \in C_c^\infty(U)$ we have $\operatorname{supp} Pf \subset \operatorname{supp} f$.

(a) Show that for every $a \in U$ the map $u_a : f \mapsto Pf(a)$ is a distribution supported by $\{a\}$. Hint: use a suitable cut off function.

(b) Let $V \subset U$ be an open subset whose closure is compact in U. Show that there exists a constant $d = d_V \in \mathbb{N}$ such that for every $a \in V$ the distribution u_a has order at most d. This means that for every compact neighborhood K of a there exists a constant $C_K > 0$ such that

$$|u_a(f)| \leq C_K \max_{|\alpha| \leq d} \sup_K |\partial^\alpha f|.$$

(c) Show that there exist uniquely determined constants $c_\alpha(a) \in \mathbb{C}$, for $|\alpha| \leq d$ such that

$$u_a = \sum_{|\alpha| \leq d} c_\alpha(a)(-\partial)^\alpha \delta_a,$$

where δ_a denotes the Dirac measure at a (see the exercise below).

(d) Show that the functions c_α, for $|\alpha| \leq k$, are smooth on V.

(e) Show that the restriction of P to V is a differential operator of order at most d_V.

Exercise 5.2.9. The purpose of this exercise is to show the following. Let $a \in \mathbb{R}^n$ and let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with $\operatorname{supp} u \subset \{a\}$. Then there exists a constant $k \in \mathbb{N}$ and constants $c_\alpha \in \mathbb{C}$, for $\alpha \in \mathbb{N}^n$, such that

$$u = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta_\alpha.$$

(a) Show that without loss of generality we may assume that $a = 0$.

(b) Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be identically 1 in a neighborhood of 0. Show that

$$u(\chi f) = u(f)$$

for all $f \in C^\infty(\mathbb{R}^n)$.

(c) Let $f \in C^\infty(\mathbb{R}^n)$. Let p be the k-th order (multivariable) Taylor polynomial of f at 0 and let R be the remainder term, so that $f = p + R$. Show that for all α with $|\alpha| \leq k$, we have

$$\lim_{x \to 0} \|x\|^{|\alpha| - k - 1/2} |\partial^\alpha R(x)| = 0.$$

(d) For $\epsilon > 0$ define $\chi_\epsilon(x) = \chi(x/\epsilon)$. Show that

$$u(\chi_\epsilon R) \to 0 \ (\epsilon \to 0).$$

(e) Show that $u(f) = u(p)$.

(f) Conclude the proof.

5.3. Localization of pseudo-differential operators

We turn to the problem of localizing a pseudo-differential operator $P \in \Psi^d(U)$, for $U \subset \mathbb{R}^n$ open. More precisely, if $\chi \in C_c^\infty(U)$ we denote by M_χ or simply χ, the operator in $\mathrm{End}(C_c^\infty(U))$ given by multiplication by χ, i.e., $M_\chi(f) = \chi f$. The problem is whether $M_\chi \circ P \circ M_\psi$ is a pseudo-differential operator again, for $\chi, \psi \in C_c^\infty(U)$.

It is immediate from the definitions that

$$M_\chi \circ \Psi_p = \Psi_{(\chi\otimes 1)p}, \quad M_\chi \circ T_K = T_{(\chi\otimes 1)K},$$

for $p \in S^d(U)$ and $K \in C^\infty(U \times U)$. Moreover, it is also clear that $T_K \circ M_\psi = T_{K(1\otimes\psi)}$. The answer to the question whether $\Psi_p \circ M_\psi$ is pseudo-differential is provided by the following result.

Proposition 5.3.1. *Let $p \in S^d(U)$ and let $\psi \in C_c^\infty(U)$. Then there exists a $q \in C^d(U)$ such that*

$$\Psi_p \circ M_\psi = \Psi_q.$$

Proof Let $f \in C_c^\infty(U)$. Then $\mathcal{F}(\psi f) = \mathcal{F}(\psi) * \mathcal{F}(f)$ so that

$$\begin{aligned}
\Psi_p \circ M_\psi f(x) &= \int e^{i\xi x} p(x,\xi)(\widehat{\psi} * \widehat{f})(\xi)\, d\xi \\
&= \int\int e^{i\xi x} p(x,\xi)\widehat{\psi}(\xi-\zeta)\widehat{f}(\zeta)\, d\zeta\, d\xi \\
&= \int\int e^{i\xi x} p(x,\xi)\widehat{\psi}(\xi-\zeta)\widehat{f}(\zeta)\, d\xi\, d\zeta \\
&= \int\int e^{i\zeta x} e^{i\xi x}\, p(x,\xi+\zeta)\,\widehat{\psi}(\xi)\widehat{f}(\zeta)\, d\xi\, d\zeta \\
&= \int e^{i\zeta x} q(x,\zeta)\widehat{f}(\zeta)\, d\zeta,
\end{aligned}$$

where

$$q(x,\zeta) = \int_{\mathbb{R}^n} e^{i\xi x}\, p(x,\xi+\zeta)\,\widehat{\psi}(\xi)\, d\xi. \tag{5.3}$$

Note that all of the above integrals are absolutely convergent, because $\widehat{\psi}$ and $\widehat{f}$ are Schwartz functions. We will finish the proof by showing that q belongs to $S^d(U)$. More precisely, we will show that the map $(p, \psi) \mapsto q$ defined by (5.3) is continuous bilinear $S^d(U) \times \mathcal{S}(\mathbb{R}^n) \to S^d(U)$.

Let $K \subset U$ be compact, and $k \in \mathbb{N}$. Then for multi-indices α, β of order at most k, for $x \in K$ and $\xi, \zeta \in \mathbb{R}^n$ we have

$$\begin{aligned}
|\partial_x^\alpha \partial_\zeta^\beta [p(x,\xi+\zeta)\widehat{\psi}(\xi)]| &= |(\partial_x^\alpha \partial_\xi^\beta p)(x,\xi+\zeta)||\widehat{\psi}(\xi)| \\
&\leq (1+\|\xi+\zeta\|)^{d-|\beta|}(1+\|\xi\|)^{-N}\mu^d_{K,k}(p)\nu_{N,0}(\widehat{\psi}) \\
&\leq (1+\|\zeta\|)^{d-|\beta|}(1+\|\xi\|)^{k+|d|-N}\mu^d_{K,k}(p)\nu_{N,0}(\widehat{\psi}).
\end{aligned}$$

By application of the Leibniz rule, we now see that there exists a constant $C > 0$, only depending on K, k, N, such that

$$\begin{aligned}&|\partial_x^\alpha \partial_\zeta^\beta [e^{i\xi x}\, p(x, \xi + \zeta)\widehat{\psi}(\xi)]| \\ &\leq \; C\,(1 + \|\zeta\|)^{d-|\beta|}(1 + \|\xi\|)^{2k+|d|-N}\mu^d_{K,k}(p)\nu_{N,0}(\widehat{\psi}).\end{aligned}$$

Choosing N such that $2k + |d| - N < -n$, we see that in (5.3) differentiation under the integral sign is allowed, and leads to the estimate

$$\mu^d_{K,k}(q) \leq C'\mu^d_{K,k}(p)\nu_{N,0}(\widehat{\psi}), \qquad ((x,\zeta) \in K \times \mathbb{R}^n),$$

with C' a constant only depending on K, k and N. As $\psi \mapsto \widehat{\psi}$ is continuous in $\mathcal{S}(\mathbb{R}^n)$, the result follows. □

5.4. The full symbol

The formula (5.3) gives rise to an interesting characterization of q which we shall now describe. We recall that U is an open subset of $\mathbb{R}^n$.

If $\mathcal{N}$ is a countable set and $\nu \mapsto d_\nu$ a real-valued function on $\mathcal{N}$, then by $\lim_{\nu\to\infty} d_\nu = -\infty$ we mean that for every $m \in \mathbb{R}$ there exists a finite subset $F \subset \mathcal{N}$ such that $\nu \in \mathcal{N} \setminus F \Rightarrow d_\nu < m$.

Definition 5.4.1. Let $\mathcal{N}$ be a countable set, and $\nu \mapsto d_\nu$ a real-valued function on $\mathcal{N}$ with $d_\nu \to -\infty$ for $\nu \to \infty$. Let $p_\nu \in S^{d_\nu}(U)$, for each $\nu \in \mathcal{N}$, and let $p \in S^{d'}(U)$. Then

$$p \sim \sum_{\nu\in\mathcal{N}} p_\nu \tag{5.4}$$

means that for every $d \in \mathbb{R}$ there exists a finite subset $F_0 \subset \mathcal{N}$ such that for every finite subset $F \subset \mathcal{N}$ with $F \supset F_0$,

$$p - \sum_{\nu\in F} p_\nu \in S^d(U).$$

We observe that if a symbol $p' \in S^\infty(U)$ has the same expansion, then it follows that $p - p' \in S^d(U)$ for all d, hence $p - p' \in S^{-\infty}(U)$. We thus see that the asymptotic expansion (5.4) determines p modulo $S^{-\infty}(U)$.

The symbol q of (5.3) may now be characterized modulo $S^{-\infty}(U)$ as follows. We note that the operator ∂_ξ^α maps $S^d(U)$ continuous linearly to $S^{d-|\alpha|}(U)$.

Lemma 5.4.2. *Let $p \in S^d(U)$ let $\psi \in C_c^\infty(U)$ and let the symbol $q \in S^d(U)$ be defined as in (5.3). Then*

$$q \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, D_x^\alpha\psi\, \partial_\xi^\alpha p.$$

Proof By the multi-variable Taylor formula with remainder term, we have, for $M \in \mathbb{N}$,

$$p(x, \xi + \zeta) = \sum_{|\alpha|\leq M} \frac{\xi^\alpha}{\alpha!}\partial_\xi^\alpha p(x,\zeta) + R_M(x, \xi, \zeta),$$

with remainder term given by

$$R_M(x,\xi,\zeta) = -\frac{1}{M!}\int_0^1 (1-t)^M \partial_t^{M+1}[p(x,\zeta+t\xi)]\,dt. \tag{5.5}$$

This leads to

$$q(x,\zeta) = \sum_{|\alpha|\leq M} q_\alpha(x,\zeta) + q_M(x,\zeta),$$

where

$$q_M(x,\zeta) = \int_{\mathbb{R}^n} R_M(x,\xi,\zeta)\widehat{\psi}(\xi)\,d\xi,$$

and where

$$\begin{aligned} \alpha!\,q_\alpha(x,\zeta) &= \int e^{i\xi x}\,\xi^\alpha \partial_\xi^\alpha p(x,\zeta)\,\widehat{\psi}(\xi)\,d\xi \\ &= \partial_\xi^\alpha p(x,\zeta)\,\mathcal{F}^{-1}(\xi^\alpha \mathcal{F}(\psi))(x) \\ &= \partial_\xi^\alpha p(x,\zeta)\,D_x^\alpha \psi(x). \end{aligned}$$

Thus, to complete the proof, it suffices to show that

$$q_M \in S^{d-(M+1)}(U). \tag{5.6}$$

Let $K \subset U$ be compact. Then by differentiation under the integral sign in (5.5), application of the Leibniz rule, and a straightforward estimation of the resulting integrals we see that there exists a constant $C > 0$ only depending on K, M, k such that for all multi-indices α, β with $|\alpha|, |\beta| \leq k$ and all $x \in K <$ $\xi, \zeta \in \mathbb{R}^n$, we have

$$\begin{aligned} &|\partial_x^\alpha \partial_\zeta^\beta R_M(x,\xi,\zeta)| \\ &\quad\leq\; C(1+\|\xi\|)^{M+1} \sup_{0\leq t\leq 1}(1+\|\zeta+t\xi\|)^{d-(M+1)-|\beta|}\mu^d_{K,k+M+1}(p). \end{aligned}$$

We now observe that for $0 \leq t \leq 1$ we have

$$(1+\|\zeta+t\xi\|)^{d-(M+1)-|\beta|} \leq (1+\|\zeta\|)^{d-(M+1)-|\beta|}(1+\|\xi\|)^{|d|+M+1+|\beta|},$$

so that

$$\begin{aligned} &|\partial_x^\alpha \partial_\zeta^\beta R_M(x,\xi,\zeta)| \\ &\quad\leq\; C(1+\|\zeta\|)^{d-(M+1)-|\beta|}\,(1+\|\xi\|)^{|d|+2(M+1)+k}\mu_{K,k+M+1}(p). \end{aligned}$$

Now put

$$q_M(x,\zeta) = \int_{\mathbb{R}^n} R_M(x,\xi,\zeta)\widehat{\psi}(\xi)\,d\xi$$

Fix $N \in \mathbb{N}$ such that $|d| + 2(M+1) + k - N < -n$. Then by the usual method we infer that there exists a constant $C' > 0$, only depending on K, k, N such that

$$\mu^{d-(M+1)}_{K,k}(q_M) \leq C'\,\mu^d_{K,k+M+1}(p)\,\nu_{N,0}(\widehat{\psi})$$

The result follows.

Remark 5.4.3. From the estimate at the end of the proof we see that the map $(p,\psi) \mapsto q_M$ is continuous bilinear from $S^d(U) \times C_c^\infty(U)$ to $S^{d-(M+1)}(U)$. In this sense, the asymptotic expansion for q depends on (p,ψ) in a continuous bilinear fashion.

Using the asymptotic expansion above, we can now derive an important theorem. We begin by a useful sharpening of Proposition 5.3.1 Given an open subsets $V \subset U$ of $\mathbb{R}^n$, we define the restriction map $p \mapsto p_V$ from $S^d(U)$ to $S^d(V)$ by $p_V(x, \xi) = p(x, \xi)$, for $x \in V$ and $\xi \in \mathbb{R}^n$. Thus,

$$p_V := p|_{V \times \mathbb{R}^n}.$$

In the following we will use the notation $V \Subset U$ to indicate that V has compact closure in U.

Proposition 5.4.4. *Let $p \in S^d(U)$. Let $U' \Subset U$ be an open subset and let $\psi \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of $\mathrm{cl}\,(U')$. Then there exists a symbol $q \in S^d(U)$ such that*

(a) $\Psi_q = \Psi_p \circ M_\psi$;
(b) $q_{U'} - p_{U'} \in S^{-\infty}(U')$.

Proof Define q as in (5.3). Then (a) is valid, and we have the asymptotic expansion from Lemma 5.4.2. Since $D_x^\alpha \psi = 0$ on U', except when $\alpha = 0$, we see that $q_{U'} \sim p_{U'}$, or, equivalently, that (b) holds. □

Theorem 5.4.5. *The map $p \mapsto \Psi_p$ induces a linear isomorphism*

$$S^d(U)/S^{-\infty}(U) \xrightarrow{\simeq} \Psi^d(U)/\Psi^{-\infty}(U).$$

Proof From the definition of $\Psi^d(U)$ it follows that $p \mapsto \Psi_p$ induces a surjective linear map $S^d(U) \to \Psi^d(U)/\Psi^{-\infty}(U)$. We must show that its kernel equals $S^{-\infty}(U)$. Thus, let $p \in S^d(U)$ and assume that $\Psi_p \in \Psi^{-\infty}(U)$. Then $\Psi_p = T_K$ for a smooth function $K \in C^\infty(U \times U)$. We must show that this implies that $p \in S^{-\infty}(\mathbb{R}^n)$. By using a partition of unity we see that it suffices to show that $\chi p \in S^{-\infty}(\mathbb{R}^n)$ for all $\chi \in C_c^\infty(U)$. Now $\Psi_{\chi p} = M_\chi \circ \Psi_p = T_{(\chi \otimes 1)K}$. Thus, to prove the theorem, we may assume from the start that there exists a compact subset $A \subset U$ such that $\mathrm{supp}\, p \subset A \times U$ and $\mathrm{supp}\, K \subset A \times U$.

We now have that $p \in S_c^d(U) \subset S_c^d(R^n)$, so that p also defines a pseudo-differential operator $\widetilde{\Psi}_p : C_c^\infty(\mathbb{R}^n) \to C_A^\infty(\mathbb{R}^n)$. Of course, $\widetilde{\Psi}_p$ restricts to Ψ_p on $C_c^\infty(U)$.

Fix an open subset $U' \Subset U$. Then it suffices to show that $p_{U'} \in S^{-\infty}(U')$. To prove this, we select a cut off function $\psi \in C_c^\infty(U)$ which equals 1 on an open neighborhood of $\mathrm{cl}\,(U')$. Then

$$\widetilde{\Psi}_p \circ M_\psi = \Psi_p \circ M_\psi = T_{K(1 \otimes \psi)} \in \Psi^{-\infty}(\mathbb{R}^n). \tag{5.7}$$

As $K(1 \otimes \psi)$ is compactly supported with support in $U \times U$, it follows that

$$T_{K(1 \otimes \psi)} = \widetilde{\Psi}_r \tag{5.8}$$

for a symbol $r \in S^{-\infty}(\mathbb{R}^n)$.

On the other hand, by the above proposition applied with $\mathbb{R}^n$ in place of U, we see that

$$\widetilde{\Psi}_p \circ M_\psi = \widetilde{\Psi}_q, \tag{5.9}$$

with a symbol $q \in S^d(\mathbb{R}^n)$ that has the property that $q_{U'} - p_{U'} \in S^{-\infty}(U')$.

From (5.7), (5.7) and (5.9) we see that $\widetilde{\Psi}_r = \widetilde{\Psi}_q$, so that $r - q = 0$, by Lemma 5.2.5. This implies that

$$p_{U'} - r_{U'} = p_{U'} - q_{U'} \in S^{-\infty}(U').$$

Hence $p_{U'} \in S^{-\infty}(U')$ and the proof is complete. $\square$

Definition 5.4.6. The inverse of the linear isomorphism of Theorem 5.4.5, denoted

$$\sigma : \Psi^d(U)/\Psi^{-\infty}(U) \to S^d(U)/S^{-\infty}(U),$$

is called the (full) *symbol map.*

This symbol map is the appropriate generalization of the symbol map for differential operators. Just as the latter symbol map cannot be extended naturally to differential operators on manifolds, the present symbol map does not allow a coordinate invariant extension to manifolds either. Just as in the case of differential operators, there is an appropriate notion of principal symbol of order d, which can be extended to the setting of manifolds.

Definition 5.4.7. The principal symbol map σ^d of order d is defined to be the following map induced by the symbol map:

$$\sigma^d : \Psi^d(U)/\Psi^{d-1}(U) \longrightarrow S^d(U)/S^{d-1}(U). \tag{5.10}$$

Corollary 5.4.8. *The principal symbol map (5.10) is a linear isomorphism.*

5.5. Expansions in symbol space

The construction of parametrices for elliptic pseudo-differential operators will make use of a recurrence that is based on the following remarkable lemma.

Lemma 5.5.1. *Let $U \subset \mathbb{R}^n$ be open. Let $\{d_j\}_{j\geq 0}$ be a sequence of real numbers with $d_j \to -\infty$ as $j \to \infty$. Assume that for each j a symbol $p_j \in S^{d_j}(U)$ is given. Then there exists a symbol $p \in S^d(U)$, where $d = \max d_j$, such that $\mathrm{pr}_1(suppp)$ is contained in the closure of the union of the sets $\mathrm{pr}_1(\mathrm{supp}\, p_j)$, for $j \geq 0$, and such that*

$$p \sim \sum_{j=0}^{\infty} p_j \qquad \text{in} \quad S^d(U). \tag{5.11}$$

The symbol p is uniquely determined modulo $S^{-\infty}(U)$.

Proof The uniqueness assertion is an immediate consequence of the meaning of (5.11).

For the existence, we note that by using partitions of unity on U we can reduce to the local situation where a compact subset $A \subset U$ is given such that $p_j \in S^{d_j}_A(U)$. It then suffices to establish the existence of a symbol $p \in S^d_A(U)$ such that (5.11) is valid.

Taking suitable groups of terms we readily see that it suffices to consider the case that the sequence d_j is strictly decreasing. Then $d = d_0$.

Fix $\chi \in C_c^\infty(\mathbb{R}^n)$ with the property that $\chi(\xi) = 0$ for $\|\xi\| \leq 1$ and $\chi(\xi) = 1$ for $\|\xi\| \geq 2$. For $t > 0$ we define the function $\chi_t : (x, \xi) \mapsto \chi(t\xi)$ (constant in the x-variable). Note that $\operatorname{supp} \chi_t \cap (U \times B(0; t^{-1})) = \emptyset$.

We will select a sequence t_j of positive real numbers with $t_j \to 0$ and define

$$p := \sum_{j=0}^{\infty} \chi_{t_j} p_j.$$

The sum is locally finite with respect to the variable ξ, hence defines a smooth function $U \times \mathbb{R}^n \to \mathbb{C}$. Moreover, $\mathrm{pr}_1(\operatorname{supp} p)$ is contained in A. We claim that it is possible to select a sequence $\{t_j\}$ such that for every $l \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ the series

$$\sum_{j \geq l} (1 + \|\xi\|)^{|\beta| - d_l} \partial_x^\alpha \partial_\xi^\beta [\chi_{t_j} p_j]$$

converges uniformly on $U \times \mathbb{R}^n$. The proof of this claim is deferred to two lemmas below. Let $\{t_j\}$ be as claimed, then the proof can be finished as follows. From the above claim about convergence, it follows that the series

$$r_l := \sum_{j \geq l} \chi_{t_j} p_j$$

convergences absolutely in the symbol space $S_A^{d_l}(U)$, relative to its continuous seminorms. By completeness, this implies that $r_l \in S_A^{d_l}(U)$. In particular it follows that $p = r_0 \in S_A^d(U)$. Now

$$p - \sum_{j=0}^{l-1} p_j = \sum_{j=0}^{l-1} (\chi_{t_j} - 1) p_j + r_l. \tag{5.12}$$

The second sum defines a function with compact ξ-support, hence belongs to $S^{-\infty}(U)$. Since $r_l \in S^{d_l}(U)$, it follows that the difference on the left-hand side of (5.12) belongs to $S^{d_l}(U)$. This implies (5.11). □

To establish the claim of the above proof we need the following.

Lemma 5.5.2. *Let $j \geq 0$, $\alpha, \beta \in \mathbb{N}^n$. Then there exists a constant $C_{j,\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta [\chi_t p_j](x, \xi)| \leq C_{j,\alpha,\beta} (1 + \|\xi\|)^{d_j - \|\beta|}, \tag{5.13}$$

for all $(x, \xi) \in U' \times \mathbb{R}^n$ and all $0 < t \leq 1$.

Proof The estimate is trivially valid in the area $\|\xi\| \leq t^{-1}$, where $\chi_t = 0$. By definition of the symbol space, it is also valid in the area $\|\xi\| \geq 2t^{-1}$, where $\chi_t = 1$. Thus, it remains to establish an estimate of the above type in the area $t^{-1} \leq \|\xi\| \leq 2t^{-1}$, with a constant independent of t, x, ξ.

The estimate then follows by application of Leibniz' formula and bookkeeping, as follows. The expression on the left-hand side of (5.13) may be estimated by a sum of binomial coefficients times the expression $|\partial_x^\alpha(\partial_\xi^\gamma \chi_t \partial_\xi^{\beta - \gamma} p_j)(x, \xi)|$.

With the notation $(\partial_\xi^\gamma \chi)_t(\xi) = (\partial_\xi^\gamma \chi)(t\xi)$ the mentioned expression becomes, by application of the chain rule,

$$\begin{aligned} |\partial_x^\alpha(\,\partial_\xi^\gamma \chi_t \partial_\xi^{\beta-\gamma} p_j)(x,\xi)| &= t^{|\gamma|}\,|\,[\,(\partial_\xi^\gamma \chi)_t \partial_x^\alpha \partial_\xi^{\beta-\gamma} p_j](x,\xi)\,| \\ &\leq C'_{j,\alpha,\gamma}\, t^{|\gamma|}\,(1+\|\xi\|)^{d_j-|\beta|+|\gamma|}, \end{aligned} \tag{5.14}$$

with a constant independent of t, x, ξ. From $t \leq 1$ and $\|\xi\| \leq 2t^{-1}$ it follows that

$$t^{|\gamma|} = 3^{|\gamma|}(3t^{-1})^{-|\gamma|} \leq 3^{|\gamma|}(1+2t^{-1})^{-|\gamma|} \leq 3^{|\gamma|}(1+\|\xi\|)^{-|\gamma|}.$$

Substituting this in (5.14) we infer that the estimate (5.13) is valid in the area considered. □

Lemma 5.5.3. (Claim) *There exists a sequence $\{t_j\}$ of real numbers with $t_j \to 0$ such that for every $l \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ the series*

$$\sum_{j\geq l}(1+\|\xi\|)^{|\beta|-d_l}\partial_x^\alpha \partial_\xi^\beta[\chi_{t_j} p_j]$$

converges uniformly on $U \times \mathbb{R}^n$.

Proof Let $j \geq 0$. Then we may select $t_j > 0$ such that

$$C_{j,\alpha,\beta}(1+t_j^{-1})^{d_j-d_l} < 2^{-j}$$

for all α, β, l with $|\alpha|+|\beta|+l < j$. It follows that, for all such α, β, l and all $(x,\xi) \in U' \times \mathbb{R}^n$ with $\|\xi\| \geq t_j^{-1}$,

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta[\chi_t p_j](x,\xi)| &\leq C_{j,\alpha,\beta}(1+\|\xi\|)^{d_j-\|\beta|} \\ &\leq C_{j,\alpha,\beta}(1+\|\xi\|)^{d_j-d_l}(1+\|\xi\|)^{d_l-\|\beta|} \\ &\leq C_{j,\alpha,\beta}(1+t_j^{-1})^{d_j-d_l}(1+\|\xi\|)^{d_l-\|\beta|} \\ &\leq 2^{-j}(1+\|\xi\|)^{d_l-\|\beta|}. \end{aligned}$$

On the other hand, if $\|\xi\| \leq t_j^{-1}$, then $\chi_{t_j}(x,\xi) = \chi(t_j\xi) = 0$. We conclude that for all α, β, l with $|\alpha|+|\beta|+l < j$ and all $(x,\xi) \in U \times \mathbb{R}^n$ the following estimate is valid

$$|\partial_x^\alpha \partial_\xi^\beta[\chi_t p_j](x,\xi)| \leq 2^{-j}(1+\|\xi\|)^{d_l-\|\beta|}.$$

This implies the claim. □

LECTURE 6
Pseudo-differential operators, continued

6.1. The distribution kernel of a pseudo-differential operator

Let $U \subset \mathbb{R}^n$ be an open subset. If $p \in S^d(U)$, then the associated pseudo-differential operator $\Psi_p : C_c^\infty(U) \to C^\infty(U)$ may be viewed as a continuous linear operator $C_c^\infty(U) \to \mathcal{D}'(U)$. Hence, by the Schwartz kernel theorem, the operator Ψ_p has a (uniquely determined) distribution kernel $K_p \in \mathcal{D}'(U \times U)$. By this we mean that the pseudo-differential operator is given by

$$\langle \Psi_p f, g \rangle = \langle K_p, g \otimes f \rangle, \qquad (f, g \in C_c^\infty(U)).$$

It turns out that the kernel K_p can be quite easily determined, without reference to the Schwartz kernel theorem.

The idea is to extend the formula of Lemma 5.2.4 to the more general setting with $p \in S^m(U)$. Fourier transform with respect to the second variable defines a linear map

$$\mathcal{F}_2 : C_c^\infty(U \times \mathbb{R}^n) \to C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)$$

which is readily verified to be continuous. The transposed of this map is a continuous linear map $[C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)]' \to \mathcal{D}'(U \times \mathbb{R}^n)$ which we denote by $\mathcal{F}_2$ as well. Via the bilinear pairing

$$S^m(U) \times [C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)] \to \mathbb{C}$$

defined by integration, the space $S^m(U)$ is mapped continuously and injectively into $[C_c^\infty(U) \,\widehat{\otimes}\, \mathcal{S}(\mathbb{R}^n)]'$. Via composition with this injection, the Fourier transform $\mathcal{F}_2$ gives a a continuous linear map

$$\mathcal{F}_2 : S^m(U) \to \mathcal{D}'(U \times \mathbb{R}^n).$$

We note that on $C_c^\infty(U \times \mathbb{R})$ this map coincides with the usual Fourier transform in the second variable. Of course, this justifies the use of the notation $\mathcal{F}_2$ for it.

Proposition 6.1.1. *Let $p \in S^d(U)$. The distribution kernel $K_p \in \mathcal{D}'(U \times U)$ of Ψ_p is given by the formula*

$$K_p(x, y) = \mathcal{F}_2 p(x, y - x) \quad \text{(in distribution sense).} \tag{6.1}$$

Remark 6.1.2. The phrase 'in distribution sense' should be interpreted as follows. The formula is well defined for $p \in C_c^\infty(U \times \mathbb{R}^n)$ and can be rewritten in such a way that it extends continuously to the space of distributions. Indeed, since for a compactly supported smooth function p the transform $\mathcal{F}_2 p$ is a smooth function, $\mathcal{F}_2 p(x, y - x) = L^*(\mathcal{F}_2 p)(x, y)$, with L the map $(x, y) \mapsto (x, y - x)$ from $U \times \mathbb{R}^n$ to $U \times \mathbb{R}^n$. The map L is smooth with smooth inverse $L^{-1} : (x, y) \mapsto (x, y + x)$, hence is a diffeomorphism.

For all $f, g \in C^\infty(U \times \mathbb{R}^n)$ with $\operatorname{supp} g$ compact, we have

$$\langle L^* f, g \rangle = \langle f, L^{-1*} g \rangle,$$

by substitution of variables, so that $\langle L^* f, \cdot \rangle = L^{-1*t}(\langle f, \cdot \rangle)$. We thus see that the operator $L_*^{-1} := L^{-1*t} \in \operatorname{End}(\mathcal{D}'(U \times \mathbb{R}^n))$ is the unique continuous linear extension of the operator $L^* \in \operatorname{End}(C^\infty(U \times \mathbb{R}^n))$. Accordingly, the above formula (6.1) should be read as:

$$K_p = L_*^{-1}(\mathcal{F}_2(p))|_{U \times U}. \tag{6.2}$$

Proof Let $g \in C_c^\infty(U)$ and $f \in C_c^\infty(U)$. Then

$$\begin{aligned} \langle g, \Psi_p(f) \rangle &= \int_U g(x) \Psi_p f(x)\, dx \\ &= \int_U \int_{\mathbb{R}^n} g(x) e^{i\xi x} p(x, \xi) \widehat{f}(\xi)\, d\xi\, dx \\ &= \int_U \int_{\mathbb{R}^n} g(x) p(x, \xi) \mathcal{F}(T_x^* f)(\xi)\, d\xi\, dx \\ &= \int_U \int_{\mathbb{R}^n} g(x) \mathcal{F}_2 p(x, y) (T_x^* f)(y)\, d\xi\, dx \\ &= \langle \mathcal{F}_2 p, h \rangle, \end{aligned}$$

where $h(x, y) = g(x) f(y + x) = L^{-1*}(g \otimes f)(x, y)$. It follows that

$$\langle g, \Psi_p(f) \rangle = \langle \mathcal{F}_2(p), L^{-1*}(g \otimes f) \rangle = \langle L_*^{-1} \mathcal{F}_2(p), g \otimes f \rangle.$$

This implies that $\Psi_p = T_K$, with $K = L_*^{-1} \mathcal{F}_2(p)|_{U \times U}$. $\square$

Proposition 6.1.3. *Let $p \in S^m(U)$. Then the distribution kernel K_p of Ψ_p is smooth on the complement of the diagonal* $\operatorname{diag}(U)$ *in $U \times U$.*

Proof The diagonal $\operatorname{diag}(U)$ is the pre-image of $\mathbb{R}^n \times \{0\}$ under the map $L : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$. Therefore, it suffices to show that $\mathcal{F}_2 p$ is smooth outside $\mathbb{R}^n \times \{0\}$. For this it suffices to show that $(1 \otimes \psi)\mathcal{F}_2 p$ is smooth for all $\psi \in C_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp} \psi$. By the usual formulas for the Fourier transform of a differentiated function, we derive that

$$(1 \otimes \psi)\mathcal{F}_2 p = [1 \otimes \|y\|^{-2N} \psi]\, \mathcal{F}_2((-\Delta_\xi)^N p), \tag{6.3}$$

for every $N \in \mathbb{N}$, where Δ_ξ denotes the Laplace operator in the ξ-variable. Now Δ_ξ^N maps $S^d(U)$ to $S^{d-2N}(U)$. If $k \in \mathbb{N}$ and $d - 2N < -n - k$, it is readily seen that $\mathcal{F}_2$ maps $S^{d-2N}(U)$ continuously to $C^k(U \times \mathbb{R}^n)$. Thus, for such N the distribution on the right-hand side of (6.3) belongs to $C^k(U \times \mathbb{R}^n)$. As the left-hand side is independent of N, it follows that $(1 \otimes \psi)\mathcal{F}_2 p$ belongs to $C^k(U \times \mathbb{R}^n)$ for every $k \in \mathbb{N}$. The result follows. $\square$

Corollary 6.1.4. *Let $P \in \Psi^d(U)$. Then the distribution kernel of the operator P, denoted $K_P \in \mathcal{D}'(U \times U)$, is smooth outside the diagonal* $\mathrm{diag}(U)$.

Proof By definition, there exist $p \in S^d(U)$ and $K \in C^\infty(U \times U)$ such that $P = \Psi_p + T_K$. Now $K_P = K_p + K$. $\square$

Exercise 6.1.5. Let $k \in \mathbb{N}$ and assume that $d < -n-k$. Let $p \in S^d(U)$. Show that the distribution kernel K_p of Ψ_p belongs to $C^p(U \times U)$.

In the sequel we shall use the notation K_P for the distribution kernel of a pseudo-differential operator P on U. The operator P is said to be *properly supported* if the maps $\mathrm{pr}_j : \mathrm{supp}\, K_P \to U$ are proper, for $j = 1, 2$. Recall that a continuous map $\Phi : X \to Y$ between locally compact Hausdorff spaces is proper if and only if the preimage $\Phi^{-1}(A)$ is compact for each compact subset $A \subset Y$. Thus, the requirement that P is properly supported is equivalent to the requirement that the intersections $(A \times U) \cap \mathrm{supp}\,(K_P)$ and $(U \times A) \cap \mathrm{supp}\,(K_P)$ are compact, for each compact subset $A \subset U$.

The following lemma asserts that modulo a smoothing operator any pseudo-differential operator may be represented by a properly supported one, with kernel supported arbitrary close to the diagonal.

Lemma 6.1.6. *Let $P \in \Psi^d(U)$ and let Ω be an open neighborhood of the diagonal in $U \times U$. Then there exists a $q \in S^d(U)$ such that Ψ_q is properly supported with* $\mathrm{supp}\, K_q \subset \Omega$ *such that $P - P_q \in \Psi^{-\infty}(U)$.*

For the proof of this lemma, we need a generality about partitions of unity.

Lemma 6.1.7. *Let X be a paracompact locally compact Hausdorff space. Let Ω be an open neighborhood of the diagonal* $\mathrm{diag}(X)$ *in $X \times X$. Then for every open cover $\mathcal{U}$ of X there exists a locally finite refinement $\mathcal{V}$ such that for all $V, V' \in \mathcal{V}$*

$$V \cap V' \neq \emptyset \Rightarrow V \times V' \subset \Omega.$$

Proof Let $\mathcal{U}$ be an open cover of X. Then there exists an open cover $\mathcal{W}$ of X, finer than $\mathcal{U}$, such that $W \times W \subset \Omega$ for all $W \in \mathcal{W}$. By paracompactness of X, we may assume that $\mathcal{W}$ is locally finite. For each $x \in X$ there exists an open neighborhood V_x such that V_x is contained in every $W \in \mathcal{W}$ containing x, whereas it has empty intersection with every $W \in \mathcal{W}$ not containing x. Let $\mathcal{V}$ be a locally finite refinement of $\{V_x\}_{x \in X}$. Let $V, V' \in \mathcal{V}$ have a point x in common. Then $x \in W$ for some $W \in \mathcal{W}$. Now $V \subset W$ and $V' \subset W$ so that $V \times V' \subset W \times W \subset \Omega$. $\square$

Corollary 6.1.8. *Let X be a smooth manifold and let Ω be an open neighborhood of the diagonal* $\mathrm{diag}(X)$ *in $X \times X$. Then for every open covering $\mathcal{U}$ of X there exists a partition of unity $\{\psi_i\}_{i \in I}$, subordinate to $\mathcal{U}$, such that for all $i, j \in I$,*

$$\mathrm{supp}\, \psi_i \cap \mathrm{supp}\, \psi_j \neq \emptyset \Rightarrow \mathrm{supp}\, \psi_i \times \mathrm{supp}\, \psi_j \subset \Omega.$$

Proof of Lemma 6.1.6 Let $\{\psi_i\}_{i\in I}$ be a partition of unity on U as in the preceding corollary. Then the operators $P_{ij} := M_{\psi_i}\Psi_p M_{\psi_j}$ are all of the form $\Psi_{q_{ij}}$, with $q_{ij} \in S^d(U)$. The kernel of P_{ij} equals $K_{ij} := (\psi_i \otimes \psi_j)K_P$, hence has support inside $\operatorname{supp}\psi_i \times \operatorname{supp}\psi_j$. If $\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset$ then K_{ij} is smooth outside the diagonal and supported outside the diagonal, hence smooth everywhere. On the other hand, if $\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset$ then K_{ij} has support contained in Ω.

Let J be the set of $(i,j) \in I \times I$ with $\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset$. We note that $\mathrm{pr}_1\operatorname{supp} q_{ij} \subset \operatorname{supp}\psi_i$, so that the collection of sets $\mathrm{cl}\,\mathrm{pr}_1\operatorname{supp} q_{ij}$ for $(i,j) \in J$ is locally finite. It follows that

$$q := \sum_{(i,j)\in J} q_{ij}$$

is a symbol in $S^d(U)$. We note that

$$K := \sum_{(i,j)\in I\times I\setminus J} K_{ij}$$

is a locally finite sum of smooth functions on $U \times U$, hence a smooth function of its own right. It is now straightforward to verify that $\Psi_p = \Psi_q + T_K$. Moreover, the kernel K_q is given by the locally finite sum

$$K_q = \sum_{(i,j)\in J} K_{ij}.$$

It follows that $\operatorname{supp} K \subset \Omega$. We will finish the proof by showing that Ψ_q is properly supported. Let $A \subset U$ be compact. The set J_A of $(i,j) \in J$ with $\operatorname{supp}\psi_i \cap A \neq \emptyset$ is finite. As $(A \times U) \cap \operatorname{supp} K_q$ is contained in the union of the sets $\operatorname{supp}\psi_i \times \operatorname{supp}\psi_j$ for $(i,j) \in J_A$, it follows that $(A \times U) \cap \operatorname{supp} K_q$ is compact. The compactness of $(U \times A) \cap K$ is proved in a similar manner. $\square$

Exercise 6.1.9. Let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear operator such that for each $\chi, \psi \in C_c^\infty(U)$ the operator $M_\chi \circ P \circ M_\psi$ is a pseudo-differential operator in $\Psi^d(U)$. Show that $P \in \Psi^d(U)$.

Exercise 6.1.10. Let $P \in \Psi^d(U)$ be properly supported.

Show that for every compact subset $\mathcal{K} \subset U$ there exists a compact subset $\mathcal{K}' \subset U$ such that P maps $C^\infty_{\mathcal{K}}(U)$ to $C^\infty_{\mathcal{K}'}(U)$.

Exercise 6.1.11. Let $P \in \Psi^d(U)$ be properly supported.

Show that the operator P has a unique extension to a continuous linear operator $C^\infty(U) \to C^\infty(U)$.

6.2. The adjoint of a pseudo-differential operator

Let $U \subset \mathbb{R}^n$ be an open subset. It is a well known fact that the space $C_c^\infty(\mathbb{R}^n)$ is reflexive. By this we mean that the natural map $j : C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)'$ is a topological linear isomorphism. Here $\mathcal{D}'(R^n)$ is equipped with the strong dual topology, and so is the dual of $\mathcal{D}'(\mathbb{R}^n)$.

Accordingly, if $T : C_c^\infty(U) \to \mathcal{D}'(U)$ is a continuous linear map, then the transposed T^t may be viewed as a continuous linear map $C_c^\infty(U) \to \mathcal{D}'(U)$. According to this definition,

$$\langle Tf, g\rangle = \langle f, T^t g\rangle, \qquad (f, g \in C_c^\infty(U)).$$

Lemma 6.2.1. *The distribution kernel K_{T^t} (which exists by Schwartz' theorem) of the adjoint map can be expressed in terms of the kernel K_T by*

$$K_{T^t}(x, y) = K_T(y, x), \qquad ((x, y) \in U \times U).$$

Of course, this equality should be interpreted in the sense of distributions. Let $\tau : U \times U \to U \times U$ be defined by $\tau(x, y) = (y, x)$, then the above equality should be read as

$$K_{T^t} = \tau_*(K_T).$$

Proof Put $K = K_T$ and $K^1 = \tau_*(K_T)$. Then the integral operator T^1 with distribution kernel K^1 is given by

$$\langle T^1(f), g\rangle = \langle K^1, g \otimes f\rangle = \langle K, \tau^*(g \otimes f)\rangle = \langle K, f \otimes g\rangle = \langle f, Tg\rangle.$$

This shows that $T^1 = T^t$ and that $K^1 = K_{T^t}$. $\square$

It follows from the above proof that if K^t and K are distributions in $\mathcal{D}'(U \times U)$ related by $K^t = \tau_*(K)$, then the associated operator T_{K^t} is the adjoint of T_K. As each pseudo-differential operator $P \in \Psi(U)$ has a distribution kernel, we do not have to invoke the reflexivity of the space $C_c^\infty(U)$ to establish the existence of the adjoint operator P^t. However, the above reasoning puts the existence of an adjoint operator in a general framework.

In the following proposition an important role is played by the continuous linear map $e^{\langle D_x, \partial_\xi\rangle} \in \mathrm{End}(\mathcal{S}'(\mathbb{R}^{2n}))$, defined by

$$\mathcal{F} \circ e^{\langle D_x, \partial_\xi\rangle} = e^{i\hat{x}\hat{\xi}} \circ \mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform $\mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$. For details we refer to the appendix to this lecture. For a proper understanding of the proposition below, we mention that $e^{\langle D_x, \partial_\xi\rangle}$ maps $S_c^d(U)$ continuously into $S^d(U)$. Moreover, for each $p \in S_c^d(U)$,

$$e^{\langle D_x, \partial_\xi\rangle} p \sim \sum_{k \geq 0} \frac{1}{k!} \langle D_x, \partial_\xi\rangle^k = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha.$$

Proposition 6.2.2.

(a) *Let $p \in S_c^d(U)$. Then*

$$\Psi_p^t = \Psi_q, \qquad \text{with} \quad q = e^{\langle D_x, \partial_\xi\rangle} p^\vee, \quad p^\vee(x, \xi) = p(x, -\xi).$$

(b) *Let $P \in \Psi^d(U)$. Then $P^t \in \Psi(U)$. If $p \in S^d(U)$ represents the full symbol of P, then the full symbol of P^t is given by the expansion*

$$\sigma(P^t)(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \, \partial_\xi^\alpha p(x, -\xi).$$

Remark 6.2.3. Here we recall that the full symbol of P^t is defined modulo $S^{-\infty}(U)$ hence is completely determined by the given expansion.

Proof Since $\mathrm{pr}_1(p)$ has compact closure, p defines a tempered distribution on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. We denote the variables in $\mathbb{R}^{2n}$ by $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. The Fourier transform of $\mathbb{R}^{2n}$ defines a continuous linear isomorphism $\mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n})$. We agree to denote the (dual) variables on the Fourier transform side by $(\hat{x}, \hat{\xi})$. Let $\mathcal{F}_1$ denote the Fourier transform with respect to the first variable x (or $\hat{x}$) and $\mathcal{F}_2$ the Fourier transform with respect to the second variable ξ (or $\hat{\xi}$). Then both partial Fourier transforms $\mathcal{F}_1, \mathcal{F}_2$ are readily seen to be topological linear automorphisms of $\mathcal{S}(\mathbb{R}^{2n})$ and we have

$$\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2. \tag{6.4}$$

By transposition we see that all these remarks extend to tempered distributions. In particular, $\mathcal{F}_2 p \in \mathcal{S}'(\mathbb{R}^{2n})$ and we see that the kernel $K_p = L_*^{-1}\mathcal{F}_2 p$ of Ψ_p is a tempered distribution as well. Let

$$K_p^t = \tau_* K_p, \quad q = \mathcal{F}_2^{-1} L_*(K_p^t).$$

Then q is a tempered distribution on $\mathbb{R}^{2n}$. We will show that actually $q \in S^d(U)$. Then by (6.2) it follows that Ψ_q has kernel K_p^t hence is the transposed of Ψ_p.

We note that $\mathcal{F}_2 q = L_* \tau_* L_*^{-1} \mathcal{F}_2 p$. Applying $\mathcal{F}_1$ to both sides, and using (6.4) we see that

$$\mathcal{F}q = \mathcal{F}_1 L_* \tau_* L_*^{-1} \mathcal{F}_1^{-1} \mathcal{F}p.$$

To understand the composition of operators acting on $\mathcal{F}p$, we note that

$$\mathcal{F}_1 L_* \tau_* L_*^{-1} \mathcal{F}_1^{-1} = \mathcal{F}_1 (L \circ \tau \circ L^{-1})_* \mathcal{F}_1^{-1} = [\mathcal{F}_1^{-1}(L \circ \tau \circ L^{-1})^* \mathcal{F}_1]^t.$$

Now $L \circ \tau \circ L^{-1}(x, y) = (x + y, -y)$. It follows that for $f \in \mathcal{S}(\mathbb{R}^{2n})$ we have

$$\begin{aligned}
\mathcal{F}_1^{-1}(L \circ \tau \circ L^{-1})^* \mathcal{F}_1 f(\hat{x}, \hat{\xi}) &= \int_{\mathbb{R}^n} e^{i\hat{x}x} \mathcal{F}_1 f(x + \hat{\xi}, -\hat{\xi})\, d\hat{x} \\
&= \int_{\mathbb{R}^n} e^{i\hat{x}x - i\hat{\xi}\hat{x}} \mathcal{F}_1 f(x, -\hat{\xi})\, dx \\
&= e^{-i\hat{x}\hat{\xi}} f(\hat{x}, -\hat{\xi}) \\
&= e^{-i\hat{x}\hat{\xi}} S_2^* f(\hat{x}, \hat{\xi}),
\end{aligned}$$

where we have used the notation S_2 for the map $(x, \xi) \mapsto (x, -\xi)$. By transposition we now see that

$$[\mathcal{F}_1^{-1}(L \circ \tau \circ L^{-1})^* \mathcal{F}_1]^t = e^{i\hat{x}\hat{\xi}} S_{2*},$$

so that

$$\mathcal{F}q = e^{i\hat{x}\hat{\xi}} S_{2*} \mathcal{F}p = \mathcal{F}[e^{D_x \partial_\xi} p^\vee],$$

see appendix to this lecture. This proves the identity (a).

We turn to (b). Let $P \in \Psi^d(U)$ and let $p \in S^d(U)$ represent its symbol. There exists a symbol $p' \in S^d(U)$ such that $P' = \Psi_{p'}$ is properly supported and such that $P - P' = T \in \Psi^{-\infty}$. It follows that $p - p' \in S^{-\infty}(U)$, so that p' represents the symbol of P as well. The adjoint of T has a smooth kernel, hence belongs to $\Psi^{-\infty}$. Thus, it suffices to show that P' has an adjoint which is a pseudo-differential operator with the required symbol. We thus see that without loss of generality we may assume that P equals Ψ_p and is properly supported. Let $\{\psi_j\}$ be a partition of unity on U. Then each $P_j := M_{\psi_j} \circ \Psi_p$ is properly supported and of the form $P_j = \Psi_{p_j}$ with symbol $p_j(x, \xi) = \psi_j(x) p(x, \xi)$. Let K_j be the distribution kernel of P_j, then $K_j = (\psi_j \otimes 1) K_P$. From the fact that P is properly supported, it follows that the collection of subsets $\mathrm{pr}_2(\mathrm{supp}\, K_j)$ is locally finite in U. We also note that $p = \sum_j p_j$, with locally finite sum.

It follows from (a) that $P_j^t = \Psi_{q_j}$, where $q_j = e^{\langle D_x, \partial_\xi \rangle} p_j^\vee$. This implies that each q_j has an asymptotic expansion of the form

$$q_j = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha [p_j(x, -\xi)] = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p_j(x, -\xi)$$

We note that $\mathrm{pr}_1(\mathrm{supp}\, q_j) \subset \mathrm{pr}_2(K_j)$; hence the collection $\mathrm{pr}_1(\mathrm{supp}\,(q_j))$ is locally finite.

We define

$$q_\alpha(x, \xi) := \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p_j(x, -\xi)$$

This sum is locally finite in the x-variable, hence defines an element of $S^d(U)$. Let $q \in S^d(U)$ be a symbol with

$$q \sim \sum_{\alpha \in \mathbb{N}^n} q_\alpha.$$

Then $\Psi_q \in \Psi^d(U)$. We claim that $\Psi_q - \Psi_p^t \in \Psi^{-\infty}$. To see this, let $\varphi \in C_c^\infty(U)$. Then $M_\varphi \circ \Psi_q$ has symbol φq and

$$M_\varphi \circ \Psi_p^t = \sum_j M_\varphi \circ \Psi_p^t \circ M_{\psi_j} = \sum_j M_\varphi \Psi_{q_j} = \Psi_{\sum_j \varphi q_j},$$

which is a finite sum. Since $\varphi q \sim \sum_j \varphi q_j$ by construction, it follows that $M_\varphi \circ \Psi_q - M_\psi \circ \Psi_p^t \in \Psi^{-\infty}(U)$ for all $\varphi \in C_c^\infty(U)$. This implies that $\Psi_q - \Psi_p^t \in \Psi^{-\infty}(U)$ and the proof is complete. $\square$

Exercise 6.2.4. Let P be a differential operator of order d on U with full symbol p. Show by direct calculation that the full symbol q of the transposed operator P^t is given by the formula

$$q(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha \partial_\xi^\alpha p(x, -\xi).$$

Show that the terms in the above series are zero for $|\alpha| > d$.

Let $P \in \Psi^d(U)$. We define the conjugate $\bar{P}$ of P by the formula

$$\bar{P}(f) = \overline{P(\bar{f})}, \qquad (f \in C^\infty(U)).$$

It is readily seen that the kernel of $\bar{P}$ is given by $K_{\bar{P}} = \bar{K}_P$ (which should be interpreted in the sense of distributions). If $p \in S^d(U)$ then it is readily verified that

$$\overline{\Psi}_p = \Psi_{\bar{p}^\vee}.$$

Exercise 6.2.5. Verify this.

Thus, $\overline{\Psi}_p$ is a pseudo-differential operator. In general it follows that the conjugate of any pseudo-differential operator P is a pseudo-differential operator again. Moreover, its full symbol is given by

$$\sigma(\bar{P})(x,\xi) = \overline{\sigma(P)(x,-\xi)}.$$

We now define the adjoint P^* of P by $P^* = \bar{P}^t$. Then it is readily checked that

$$\langle Pf, g\rangle_{L^2} = \langle f, P^*g\rangle_{L^2}, \qquad (f,g \in C_c^\infty(U)).$$

where we have used the notation $\langle f,g\rangle_{L^2} = \langle f,\bar{g}\rangle$ for the usual sesquilinear pairing.

Corollary 6.2.6. *Let $P \in \Psi^d(U)$. Then the adjoint P^* belongs to $\Psi^d(U)$ and its full symbol is given by*

$$\sigma(P^*) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{\sigma(P)}$$

Proof This follows from the discussion preceding this corollary, combined with Proposition 6.2.2. □

Corollary 6.2.7. *Let $P \in \Psi^d(U)$. Then P extends uniquely to a continuous linear operator $\mathcal{E}'(U) \to \mathcal{D}'(U)$.*

Proof The transposed P^t is a pseudo-differential operator, hence defines a continuous linear operator $C_c^\infty(U) \to C^\infty(U)$. Its transposed $(P^t)^t$ is a continuous linear operator $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. Clearly, it restricts to P on $C_c^\infty(U)$. This establishes existence. Uniqueness follows by density of $C_c^\infty(U)$ in $\mathcal{E}'(U)$. □

6.3. Pseudo-locality

In the sequel our view-point will be that a pseudo-differential operator on a manifold M is an integral operator with distribution kernel K on $M \times M$, with K smooth away from the diagonal, whereas along the diagonal the operator is described as in the local setting.

If E is a smooth vector bundle on a smooth manifold M, then by $\Gamma^{-\infty}(E)$ we denote the space of generalized sections of E. Let $u \in \Gamma^{-\infty}(E)$ be a generalized section. Then u is said to be smooth on an open subset $U \subset M$ if $\chi u \in \Gamma^\infty(E)$ for all $\chi \in C_c^\infty(U)$. The generalized section u is said to be smooth at a point $x \in M$ if their exists an open neighborhood $U \ni x$ such that u is smooth on U. The singular support of u, denoted $\operatorname{singsupp}(u)$, is defined to be the subset of $x \in M$ such that u is not smooth at x. Clearly, $\operatorname{singsupp} u$ is a closed subset of M.

Definition 6.3.1. Let E, F be vector bundles on a smooth manifold M. A linear operator $T : \Gamma_c^{-\infty}(E) \to \Gamma^{-\infty}(F)$ is said to be *pseudo-local* if

$$\operatorname{singsupp} Tu \subset \operatorname{singsupp} u, \qquad \text{for all} \quad u \in \Gamma_c^{-\infty}(E).$$

Corollary 6.3.2. *Let $P \in \Psi^d(U)$. Then the operator $P : \mathcal{E}'(U) \to \mathcal{D}'(U)$ is pseudo-local.*

Proof The adjoint operator P has a distribution kernel $K \in \mathcal{D}'(U \times U)$ which is smooth outside the diagonal of U.

Let $u \in \mathcal{E}'(\mathbb{R}^n)$ have singular support equal to A. Then A is contained in $\operatorname{supp} u$, hence compact. Let $x \in \mathbb{R}^n \setminus A$. Then there exist open neighborhoods U_1 of x and U_2 of A in U such that $U_1 \cap U_2 = \emptyset$. Let $\varphi \in C_c^\infty(U_1)$. Then it suffices to show that $\varphi \Psi_r(u)$ is smooth. Fix $\psi \in C_c^\infty(U_2)$ such that $\psi = 1$ on an open neighborhood of A. Then $(1-\psi)u$ is in $C_c^\infty(\mathbb{R}^n)$ hence $P((1-\psi)u) \in C^\infty(U)$. Thus, it suffices to show that $\varphi P(\psi u)$ is smooth. The function $k : (x, y) \mapsto \varphi(x)\psi(y)K(x, y)$ is smooth on $U \times U$, since $\varphi \otimes \psi$ is supported in $U_1 \times U_2 \subset U \times U \setminus \operatorname{diag}(U)$.

Moreover, for all $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
\langle \varphi P(\psi u), f\rangle &= \langle P(\psi u), \varphi f\rangle \\
&= \langle \psi u, P^t(\varphi f\rangle \\
&= \langle u, \psi P^t(\varphi f)\rangle \\
&= \langle u, T_k f\rangle \\
&= \langle T_k^t(u), f\rangle,
\end{aligned}$$

where T_k denotes the integral operator with smooth kernel k. It follows that

$$\varphi P(\psi u) = T_k^t(u).$$

Since $T_k^t(u)$ is smooth, the result follows. $\square$

Appendix: A special map in symbol space

6.4. The exponential of a differential operator

In these notes we assume that A is a symmetric $n \times n$ matrix with complex entries and with $\operatorname{Re}\langle A\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard bilinear pairing $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$. The function

$$x \mapsto e^{-\langle A\xi, \xi\rangle} \tag{6.5}$$

is bounded on $\mathbb{R}^n$. Moreover, every derivative of (6.5) is polynomially bounded. Hence, multiplication by the function (6.5) defines a continuous linear endomorphism $M(A)$ of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As the operator $M(A)$ is symmetric with respect to the usual pairing $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ defined by integration, it follows that $M(A)$ has a unique extension to a continuous linear endomorphism $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Clearly, $M(A)$ leaves each subspace $L^2_s(\mathbb{R}^n)$, for $s \in \mathbb{R}$, invariant and restricts to a bounded linear endomorphism with operator norm at most 1 on it.

We define $E(A)$ to be the unique continuous linear endomorphism of $\mathcal{S}'(\mathbb{R}^n)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}'(\mathbb{R}^n) & \xrightarrow{M(A)} & \mathcal{S}'(\mathbb{R}^n) \\ {\scriptstyle \mathcal{F}}\uparrow & & \uparrow{\scriptstyle \mathcal{F}} \\ \mathcal{S}'(\mathbb{R}^n) & \xrightarrow{E(A)} & \mathcal{S}'(\mathbb{R}^n) \end{array}$$

As $\mathcal{F}$ restricts to a topological automorphism of $\mathcal{S}(\mathbb{R}^n)$ and to an isometric automorphism isomorphism from $H_s(\mathbb{R}^n)$ onto $L^2_s(\mathbb{R}^n)$, it follows that $E(A)$ restricts to a bounded endomorphism of $H_s(\mathbb{R}^n)$ of operator norm at most 1. Furthermore, $E(A)$ restricts to a continuous linear endomorphism of $\mathcal{S}(\mathbb{R}^n)$.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then clearly $\partial_t M(tA)\varphi + \langle A\xi, \xi\rangle M(tA)\varphi = 0$. By application of the inverse Fourier transform, we see that for a given function $f \in \mathcal{S}$ the function $f_t := E(tA)f$ satisfies:

$$\partial_t f_t = -\langle AD, D\rangle f_t, \quad \text{where} \quad -\langle AD, D\rangle = \sum_{ij} A_{ij}\partial_j\partial_i.$$

We note that $f_0 = f$, so that f_t may be viewed as a solution to the associated Cauchy problem with initial datum f.

For obvious reasons, we will write

$$E(tA) = E^{-t\langle AD, D\rangle}$$

from now on. The purpose of these notes is to derive estimates for E which are needed for symbol calculus.

Lemma 6.4.1. *The operator* $e^{\langle AD, D\rangle} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *commutes with the translations* T_a^* *translations and the partial differentiations* ∂_j*, for* $a \in \mathbb{R}^n$ *and* $1 \leq j \leq n$.

Proof This is obvious from the fact that translation and partial differentiation become multiplication with a function after Fourier transform; each such multiplication operator commutes with $M(A)$. $\square$

Lemma 6.4.2. *Assume that A is non-singular. Then the tempered function $x \mapsto e^{-\langle Ax,x\rangle/2}$ has Fourier transform*

$$\mathcal{F}(e^{-\langle Ax,x\rangle/2}) = c(A)e^{-\langle B\xi,\xi\rangle/2}$$

with $c(A)$ a non-zero constant.

Remark 6.4.3. It can be shown that $c(A) = (\det A)^{-1/2}$, where a suitable analytic branch of the square root must be chosen. However, we shall not need this here.

Proof For $v \in \mathbb{R}^n$ let ∂_v denote the directional derivative in the direction v. Thus, $\partial_v f(x) = df(x)v$. Then the tempered distribution f given by the function $x \mapsto \exp(-\langle Ax,x\rangle/2)$ satisfies the differential equations $\partial_v f = -\langle Av,x\rangle f$. It follows that the Fourier transform $\widehat{f}$ satisfies the differential equations $\langle v,\xi\rangle\widehat{f} = -\partial_{Av}\widehat{f}$ for all $v \in \mathbb{R}^n$, or, equivalently, $\partial_v f = -\langle Bv,\xi\rangle f$. This implies that the tempered distribution

$$\varphi = e^{\langle B\xi,\xi\rangle/2}\widehat{f}$$

has all partial derivatives equal to zero, hence is the tempered distribution coming from a constant function $c(A)$. □

Proposition 6.4.4. *For each $k \in \mathbb{N}$ there exists a positive constant $C_k > 0$ such that the following holds. Let A be a complex symmetric $n \times n$-matrix with* $\operatorname{Re} A \geq 0$. *Let $f \in \mathcal{S}(\mathbb{R})$ and let $x \in \mathbb{R}^n$ be a point such that the distance $d(x)$ from x to* $\operatorname{supp} u$ *is at least one. Then*

$$|e^{-\langle AD,D\rangle}f(x)| \leq C_k d(x)^{-k}\|A\|^{s+k} \max_{|\alpha|\leq 2s+k} \sup|D^\alpha f|. \tag{6.6}$$

Proof The function $e^{-\langle A\xi,\xi\rangle}\widehat{f}$ in $\mathcal{S}(\mathbb{R}^n)$ depends continuously on A and hence, so does $e^{\langle AD,D\rangle}f$. We may therefore assume that A is non-singular.

As $e^{-\langle AD,D\rangle}$ commutes with translation, we may as well assume that $x = 0$. We assume that f has support outside the unit ball B in $\mathbb{R}^n$.

For each j let Ω_j denote the set points y on the unit sphere $S = \partial B$ with $|\langle y,e_j\rangle| > 1/2\sqrt{n}$. Then the U_j form an open cover of S. Let $\{\psi_j\}$ be a partition of unity subordinate to this covering and define $\chi_j : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $\chi_j(y) = \psi_j(y/\|y\|)$. Then each of the functions $f_j = \chi_j f$ satisfies the same hypotheses as f and in addition, $|\langle y,e_j\rangle| \geq |y|/2\sqrt{n}$ for $y \in \operatorname{supp} f_j$. As $f = \sum_j f_j$, it suffices to prove the estimate for each of the f_j. Thus, without loss of generality, we may assume from the start that there exists a unit vector $v \in \mathbb{R}^n$ such that $|\langle y,v\rangle| \geq |y|/2\sqrt{n}$ for all $y \in \operatorname{supp} f$.

We now observe that

$$e^{-\langle AD,D\rangle}f(0) = \int e^{-\langle A\xi,\xi\rangle}\widehat{f}(\xi)\,d\xi = c\int e^{-\langle By,y\rangle/4}f(y)\,dy,$$

where $B = A^{-1}$. The idea is to apply partial differentiation with the directional derivative ∂_{Av} to this formula. For this we note that

$$e^{-\langle By,y\rangle/4} = -\frac{2}{\langle v,y\rangle}\partial_{Av}e^{-\langle By,y\rangle/4}$$

on $\operatorname{supp} f$, so that, for each $j \geq 0$,

$$\begin{aligned} e^{-\langle AD,D\rangle} f(0) &= c\, 2^j \int e^{-\langle By,y\rangle/4}\, [\langle v, y\rangle^{-1}\partial_{Av}]^j f(y)\, dy \\ &= [e^{-\langle AD,D\rangle}(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f](0). \end{aligned}$$

By using the Sobolev lemma, we find, for each natural number $s > n/2$, that

$$\begin{aligned} |e^{-\langle AD,D\rangle} f(0)| &\leq C' \max_{|\alpha|\leq s} \|D^\alpha e^{-\langle AD,D\rangle}(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2} \\ &= C' \max_{|\alpha|\leq s} \|e^{-\langle AD,D\rangle} D^\alpha(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2} \\ &\leq C' \max_{|\alpha|\leq s} \|D^\alpha(\langle v,\,\cdot\,\rangle^{-1}\partial_{Av})^j f\|_{L^2}. \end{aligned}$$

By application of the Leibniz rule and using that $|\langle v, y\rangle| \geq \|y\|/2\sqrt{n}$ and $\|y\| \geq d \geq 1$ for $y \in \operatorname{supp} f$, we see that, for $j > 2n$,

$$|e^{-\langle AD,D\rangle} f(0)| \leq C'_j \|A\|^j d^{n/2-j} \max_{|\alpha|\leq s+j} \sup |D^\alpha f|.$$

We now take $j = s + k$ to obtain the desired estimate. $\square$

Our next estimate is independent of supports.

Lemma 6.4.5. *Let $s > n/2$ be an integer. Then there exists a positive constant with the following property. Let $A \in \mathrm{M}_n(\mathbb{C})$ be symmetric with $\operatorname{Re} A \geq 0$. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,*

$$|e^{-\langle AD,D\rangle} f(x)| \leq C \max_{|\alpha|\leq s} \|D^\alpha f\|_{L^2}.$$

Proof By the Sobolev lemma we have

$$\begin{aligned} |e^{-\langle AD,D\rangle} f(x)| &\leq C \max_{|\alpha|\leq s} \|D^\alpha e^{-\langle AD,D\rangle} f\|_{L^2} \\ &= C \max_{|\alpha|\leq s} \|e^{-\langle AD,D\rangle} D^\alpha f\|_{L^2} \\ &\leq C \max_{|\alpha|\leq s} \|D^\alpha f\|_{L^2} \end{aligned}$$

$\square$

Corollary 6.4.6. *Let $s > n/2$ be an integer and let $C > 0$ be the constant of Lemma 6.4.5. Let $\mathcal{K} \subset \mathbb{R}^n$ a compact subset. Let $A \in \mathrm{M}_n(\mathbb{C})$ be symmetric and $\operatorname{Re} A \geq 0$. Then for every $f \in C^s_{\mathcal{K}}(\mathbb{R}^n)$, the distribution $e^{-\langle AD,D\rangle} f$ is a continuous function, and*

$$|e^{-\langle AD,D\rangle} f(x)| \leq C \sqrt{\operatorname{vol}(\mathcal{K})} \max_{|\alpha|\leq s} \sup |D^\alpha f|, \qquad (x \in \mathbb{R}^n).$$

Proof We first assume that $f \in C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ with $\mathcal{K}'$ compact. Then by straightforward estimation,

$$\|D^\alpha f\|_{L^2} \leq \operatorname{vol}(\mathcal{K}') \sup |D^\alpha f|$$

and the estimate follows with $\mathcal{K}'$ instead of $\mathcal{K}$. Let now $f \in C^s_{\mathcal{K}}(\mathbb{R}^n)$. Then by regularization there is a sequence $f_n \in C^\infty_{\mathcal{K}_n}(\mathbb{R}^n)$, with $\mathcal{K}_n \to \mathcal{K}$ and $f_n \to f$ in $C^s(\mathbb{R}^n)$. By the above estimate, the sequence $e^{-\langle AD,D\rangle} f_n$ is Cauchy in

$C(\mathbb{R}^n)$. By passing to a subsequence we may arrange that the sequence already converges to a limit φ in $C(\mathbb{R}^n)$. By continuity of $e^{-\langle AD,D\rangle}$ in $\mathcal{S}'(\mathbb{R}^n)$ it follows that $\varphi = e^{-\langle AD,D\rangle}f$. The required estimate for φ now follows from the similar estimates for $e^{-\langle AD,D\rangle}f_n$ by passing to the limit for $n \to \infty$. $\square$

In the sequel we shall frequently refer to a principle that is made explicit in the following lemma.

Lemma 6.4.7. *Let $L : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear endomorphism. Let V, W be linear subspaces of $\mathcal{S}'(\mathbb{R}^n)$ equipped with locally convex topologies for which the inclusion maps are continuous. Assume that $C_c^\infty(\mathbb{R}^n)$ is dense in V and that W is complete. If L maps $C_c^\infty(\mathbb{R}^n)$ into W, and the restricted map $L_0 : C_c^\infty(\mathbb{R}^n) \to W$ is continuous with respect to the V-topology on the first space, then $L(V) \subset W$.*

Proof The restricted map L_0 has a unique extension to a continuous linear map $L_1 : V \to W$. Thus, it suffices to show that $L_1 = L$ on V. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, the linear functional $\langle \cdot, \varphi\rangle$ is continuous on W. It follows that the linear functional μ on $C_c^\infty(\mathbb{R}^n)$ given by $\mu(f) = \langle L_1 f, \varphi\rangle$ is continuous linear for the V-topology.

From the assumption about the continuity of L is follows that the functional $\nu : f \mapsto \langle Lf, \varphi\rangle$ is continuous for the $\mathcal{S}'(\mathbb{R}^n)$ topology. In particular, this implies that ν is continuous for the V-topology.

As $\mu = \nu$ on $C_c^\infty(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ is dense in V it follows that $L_1 = L$ on V. $\square$

If $p \in \mathbb{N}$ we denote by $C_b^p(\mathbb{R}^n)$ the Banach space of p times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ with $\max_{|\alpha|\le p} \sup |D^\alpha f| < \infty$.

Proposition 6.4.8. *Let $s > n/2$ be an integer. Then there exists a constant $C > 0$ with the following property. For each symmetric $A \in \mathrm{M}_n(\mathbb{C})$ with $\mathrm{Re}\, A \geq 0$ and all $f \in C_b^{2s}(\mathbb{R}^n)$ the distribution $e^{-\langle AD,D\rangle}f$ is continuous and*

$$|e^{-\langle AD,D\rangle}f(x)| \le C\|A\|^s \max_{|\alpha|\le 2s} \sup |D^\alpha f|.$$

For x with $d(x) := d(x, \mathrm{supp}\, f) \geq 1$ the stronger estimate (6.6) is valid.

Proof As in the proof of the previous corollary, we first prove the estimate for $f \in C_c^\infty(\mathbb{R}^n)$. By translation invariance we may as well assume that $x = 0$.

We fix a function $\chi \in C_c^\infty(\mathbb{R}^n)$ which equals 1 on the unit ball and has support contained in $\mathcal{K} = B(0; 2)$ and such that $0 \le \chi \le 1$. Then the desired estimate follows from combining the estimate of Corollary 6.4.6 for χf with the estimate of Proposition 6.4.4 with $k = 0$ for $(1 - \chi)f$.

By density of $C_c^\infty(\mathbb{R}^n)$ in $C_c^s(\mathbb{R}^n)$ it follows that $e^{-\langle AD,D\rangle}$ maps $C_c^s(\mathbb{R}^n)$ continuously into $C_b(\mathbb{R}^n)$, with the desired estimate (apply Lemma 6.4.7). As $C_c^s(\mathbb{R}^n)$ is not dense in $C_b^s(\mathbb{R}^n)$ we need an additional argument to pass to the latter space.

Let χ be as above, and put $\chi_n(x) = \chi(x/n)$. Then it is readily seen that $\chi_n f \to f$ in $\mathcal{S}'(\mathbb{R}^n)$. Hence $e^{-\langle AD,D\rangle}f_n \to e^{-\langle AD,D\rangle}f$ in $\mathcal{S}'(\mathbb{R}^n)$. It follows by application of Proposition 6.4.4 that for each compact subset $\mathcal{K} \subset \mathbb{R}^n$ the sequence

$e^{-\langle AD,D\rangle} f_n|_{\mathcal{K}}$ is Cauchy in $C(\mathcal{K})$. This implies that $e^{-\langle AD,D\rangle} f_n$ converges to a limit φ in the Fréchet space $C(\mathbb{R}^n)$. In particular, φ is also the limit in $\mathcal{S}'(\mathbb{R}^n)$ so that $e^{-\langle AD,D\rangle} f = \varphi$ is a continuous function.

We now note that by application of the Leibniz rule,

$$\sup |D^\alpha f_n| \leq \sup |D^\alpha f| + \mathcal{O}(1/n).$$

Hence the desired estimate for f follows from the similar estimate for f_n by passing to the limit. $\square$

Theorem 6.4.9. *Let $s > n/2$ be an integer and let $k \in \mathbb{N}$. Then there exists a constant $C_k > 0$ with the following property. For each symmetric $A \in \mathrm{M}_n(\mathbb{C})$ with $\operatorname{Re} A \geq 0$ and all $f \in C_b^{2s+2k}(\mathbb{R}^n)$ the function $e^{-\langle AD,D\rangle} f$ is continuous, and*

$$|e^{-\langle AD,D\rangle} f(x) - \sum_{j<k} \frac{1}{j!}(-\langle AD,D\rangle)^j f(x)| \leq C_k \|A\|^s \max_{|\alpha|\leq 2s} \sup |D^\alpha \langle AD,D\rangle^k f|.$$

Proof Let $R_k(A)f(x)$ denote the expression between absolute value signs on the left-hand side of the above estimate. We first prove the estimate for a function $f \in C_c^\infty(\mathbb{R}^n)$. The function

$$f_t(x) := e^{-\langle tAD,D\rangle}(x)$$

is smooth in $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and satisfies the differential equation

$$\partial_t f_t(x) = -\langle AD,D\rangle f_t(x).$$

By application of Taylor's formula with remainder term with respect to the variable t at $t = 0$, we find that

$$f_1(x) = \sum_{j<k} \partial_t^j f_t(x) - \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, \partial_t^k f_t(x)\, dt.$$

This leads to

$$\begin{aligned} R_k(A)f(x) &= \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, (-\langle AD,D\rangle)^k\, f_t(x)\, dt \\ &= \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1}\, e^{-t\langle AD,D\rangle}(-\langle AD,D\rangle)^k\, f(x)\, dt. \end{aligned}$$

By estimation under the integral sign, making use of Proposition 6.4.8, we now obtain the desired estimate for $f \in C_c^\infty(\mathbb{R}^n)$. For the extension of the estimate to $C_c^{2s+2k}(\mathbb{R}^n)$ and finally to $C_b^{2s+2k}(\mathbb{R}^n)$ we proceed as in the proof of Proposition 6.4.8. $\square$

6.5. The exponential of a differential operator in symbol space

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$ and let $d \in \mathbb{R}$. Then the space of symbols $S^d_{\mathcal{K}}(\mathbb{R}^n)$ is a subspace of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{2n})$ with continuous inclusion map. Indeed, if $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$, then for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} \langle p, \varphi \rangle &= \int_{\mathbb{R}^{2n}} p(x,\xi)\, \varphi(x,\xi)\, dx\, d\xi \\ &\leq \int_{R^{2n}} (1+\|\xi\|)^{-d-n-1} |p(x,\xi)| (1+|(x,\xi)|)^{|d|+n+1} |\varphi(x,\xi)|\, dx\, d\xi \\ &\leq C\, \mu^d_{\mathcal{K},0}(p)\, \nu_{|d|+n+1,0}(\varphi), \end{aligned}$$

with $C > 0$ only depending on $n, \mathcal{K}$ and d.

We consider the second order differential operator

$$\langle D_x, \partial_\xi \rangle = i \sum_{j=1}^n \frac{\partial}{\partial x_j} \frac{\partial}{\partial \xi_j}.$$

Thus, with notation as in the previous section, $\langle D_x, \partial_\xi \rangle = -\langle AD, D \rangle$, where

$$A = i \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. The matrix A is complex, symmetric, and has real part equal to zero, hence fulfills all conditions of the previous section. Moreover, its operator norm $\|A\|$ equals 1.

In the rest of this section we will discuss the action of $e^{\langle D_x, \partial_\xi \rangle}$ on $S^d_{\mathcal{K}}(\mathbb{R}^n)$. The following lemma is obvious.

Lemma 6.5.1. *For each $k \in \mathbb{N}$,*

$$\frac{1}{k!} \langle D_x, \partial_\xi \rangle = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi.$$

In particular, $\langle D_x, \partial_\xi \rangle$ defines a continuous linear map $S^d(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$, preserving supports.

Theorem 6.5.2. *Let $k \in \mathbb{N}$. Then*

$$e^{\langle D_x, \partial_\xi \rangle} - \sum_{|\alpha|<k} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi, \tag{6.7}$$

originally defined as an endomorphism of $\mathcal{S}'(\mathbb{R}^n)$, maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ continuous linearly into $S^{d-k}(\mathbb{R}^n)$. In particular, $e^{\langle D_x, \partial_\xi \rangle}$ restricts to a continuous linear map $S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^d(\mathbb{R}^n)$.

Before turning to the proof of the theorem, we list a corollary that will be important for applications.

Corollary 6.5.3. *Let $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$. Then $e^{\langle D_x, \partial_\xi \rangle} p \in S^d(\mathbb{R}^n)$ and*

$$e^{\langle D_x, \partial_\xi \rangle} p \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi p.$$

We will prove Theorem 6.5.2 through a number of lemmas of a technical nature. The next lemma will be used frequently for extension purposes.

Lemma 6.5.4. *Let $\mathcal{K} \subset U$ be compact and let $d < d'$. Then the space $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ is dense in $S^d_{\mathcal{K}}(U)$ for the topology of $S^{d'}_{\mathcal{K}}(U)$.*

Proof Let $p \in S^d_{\mathcal{K}}(U)$. Select $\psi \in C^\infty_c(\mathbb{R}^n)$ such that $\psi = 1$ on a neighborhood of 0. Put $\psi_n(\xi) = \psi(\xi/n)$ and

$$p_n(x,\xi) = \psi_n(\xi)p(x,\xi).$$

Then by an application of the Leibniz rule in a similar fashion as in the proof of Lemma 4.1.9, it follows that $\nu^{d'}_{\mathcal{K},k}(p_n - r) \to 0$ as $n \to \infty$, for each $k \in \mathbb{N}$. $\square$

The expression (6.7) is abbreviated by $R_k(D)$. It will be convenient to use the notation

$$C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n}) := \{f \in C^\infty_c(\mathbb{R}^{2n}) \mid \operatorname{supp} f \subset \mathcal{K} \times \mathbb{R}^n\}.$$

Lemma 6.5.5. *Let $k \in \mathbb{N}$. Then for each $d < k$ the map $R_k(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ continuous linearly into $C_b(\mathbb{R}^{2n})$.*

Proof Let $s > n/2$ be an integer. Let $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n})$. Then by Theorem 6.4.9,

$$\begin{aligned}
|R_k(D)f(x,\xi)| &\leq C_k \max_{|\alpha|+|\beta|\leq 2s} \sup_{\mathcal{K}\times\mathbb{R}^n} |D^\alpha_x \partial^\beta_\xi \langle D_x, \partial_\xi\rangle^k f(x,\xi)| \\
&\leq C'_k \max_{|\alpha|+|\beta|\leq 2s, |\gamma|=k} \sup_{\mathcal{K}\times\mathbb{R}^n} |D^{\alpha+\gamma}_x \partial^{\beta+\gamma}_\xi f(x,\xi)| \\
&\leq C'_k \max_{|\alpha|+|\beta|\leq 2s, |\gamma|=k} \sup_{\mathcal{K}\times\mathbb{R}^n} (1+\|\xi\|)^{d-k}\, \nu^d_{\mathcal{K},2s+2k}(f) \\
&\leq C'_k\, \nu^d_{\mathcal{K},2s+2k}(f).
\end{aligned}$$

It follows that the map $R_k(D)$ is continuous $C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n}) \to C_b(\mathbb{R}^{2n})$, with respect to the $S^d_{\mathcal{K}}(\mathbb{R}^n)$-topology on the first space, for *each* $d < k$.

Let now $d < k$ and fix d' with $d < d' < k$. Then by density of $C^\infty_{\mathcal{K},c}(\mathbb{R}^{2n})$ in $S^d_{\mathcal{K}}(\mathbb{R}^{2n})$ for the $S^{d'}_{\mathcal{K}}(\mathbb{R}^{2n})$-topology, it follows by application of Lemma 6.4.7 that $R_k(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ to $C_b(\mathbb{R}^n)$ with continuity relative to the $S^{d'}_{\mathcal{K}}(\mathbb{R}^n)$-topology on the domain. As this topology is weaker than the original topology on $S^d_{\mathcal{K}}(\mathbb{R}^n)$, the result follows. $\square$

Lemma 6.5.6. *Let $d \in \mathbb{R}$ and assume that $k > |d|$. Let s be an integer $> n/2$. Then there exists a constant $C > 0$ such that for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x,\xi) \in \mathbb{R}^{2n}$ with $\|\xi\| \geq 4$ we have*

$$|R_k(D)f(x,\xi)| \leq C(1+\|\xi\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2k}(f). \tag{6.8}$$

Proof Let $\chi \in C^\infty_c(\mathbb{R}^n)$ be a smooth function which is identically 1 on the unit ball of $\mathbb{R}^n$, and has support inside the ball $B(0;2)$. For $t > 0$ we define the function $\chi_t \in C^\infty_c(\mathbb{R}^n)$ by $\chi_t(\xi) = \chi(t^{-1}\xi)$. Then $\chi_t(\xi)$ is identically 1 on $B(0;t)$ and has support inside the ball $B(0;2t)$. We agree to write $\psi = 1-\chi$ and $\psi_t(\xi) = \psi(t^{-1}\xi)$. In the following we will frequently use the obvious equalities

$$\sup|\partial^\alpha_\xi \chi_t| = t^{-|\alpha|}\sup|\partial^\alpha_\xi\chi|, \qquad \sup|\partial^\alpha_\xi\psi_t| = t^{-|\alpha|}\sup|\partial^\alpha_\xi\psi|.$$

Let $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. Then f is a Schwartz function, hence $e^{\langle D_x, \partial_\xi \rangle} f$ is a Schwartz function as well, and therefore, so is $R_k(D)f$. For $t > 0$ we agree to write $f_t(x,\xi) = \chi_t(\xi) f(x,t)$ and $g_t(x,\xi) = \psi_t(\xi)) f(x,\xi)$. Then $f = f_t + g_t$. From now on we assume that $(x,\xi) \in \mathbb{R}^{2n}$, that $\|\xi\| \geq 4$ and $t = \frac{1}{4}\|\xi\|$.

We will complete the proof by showing that both the values $|R_k(D) f_t(x,\xi)|$ and $|R_k(D) g_t(x,\xi)|$ can be estimated by $C' \nu^d_{\mathcal{K},2s+k}(f)$ with $C' > 0$ a constant independent of f, x, ξ. We start with the first of these functions. As f_t has support inside $B(0; 2t) = B(0; \|\xi\|/2)$, it follows that $d(\xi, \operatorname{supp} f_t) \geq \|\xi\|/2 \geq 2$. In view of Proposition 6.4.4 it follows that there exists a constant $C_k > 0$, only depending on k, such that

$$\begin{aligned}
|R_k(D) f(x,\xi)| &= |e^{\langle D_x, \partial_\xi \rangle} f(x,\xi)| \\
&\leq C_k (\|\xi\|/2)^{-k} \max_{|\alpha|+|\beta| \leq 2s+k} \sup |D_x^\alpha \partial_\xi^\beta (\chi_t f)| \\
&\leq C'_k (1 + \|\xi\|)^{-k} \max_{|\alpha|+|\beta_1+\beta_2| \leq 2s+k} \sup |\partial_\xi^{\beta_1} \chi_t \, D_x^\alpha \partial_\xi^{\beta_2} f|,
\end{aligned}$$

with $C'_k > 0$ independent of f, x and ξ. For $\eta \in \operatorname{supp} \chi_t$ we have $\|\eta\| \leq \|\xi\|/2$, so that

$$\begin{aligned}
|\partial_\xi^{\beta_1} \chi_t(\eta) \, D_x^\alpha \partial_\xi^{\beta_2} f(y,\eta)| &\leq C''_k t^{-|\beta_1|} (1 + \|\eta\|)^{d - |\beta_2|} \nu^d_{\mathcal{K},2s+k}(f) \\
&\leq C''_k (1 + \|\xi\|/2)^{|d|} \nu^d_{\mathcal{K},2s+k}(f) \\
&\leq C'''_k (1 + \|\xi\|)^{|d|} \nu^d_{\mathcal{K},2s+k}(f).
\end{aligned}$$

It follows that

$$|R_k(D) f_t(x,\xi)| \leq C' (1 + \|\xi\|)^{|d|-k} \nu^d_{\mathcal{K},2s+2k}(f).$$

We now turn to g_t. By application of Theorem 6.4.9 it follows that

$$\begin{aligned}
&|R_k(D)(g_t)(x,\xi)| \\
&\quad\leq D_k \max_{|\alpha|+|\beta| \leq 2s} \sup |D_x^\alpha \partial_\xi^\beta \langle D_x, \partial_\xi \rangle^k (\psi_t f)| \\
&\quad\leq D'_k \max_{|\alpha|+\|\beta\| \leq 2s, |\gamma| = k} \sup |\partial_\xi^{\gamma+\beta} (\psi_t D_x^{\alpha+\gamma} f)|
\end{aligned}$$

To estimate the latter expression, we concentrate on

$$|\partial_\xi^{\gamma+\beta} (\psi_t D_x^{\alpha+\gamma} f)(y,\eta)|, \tag{6.9}$$

for $y \in \mathcal{K}$ and $\eta \in \mathbb{R}^n$. Since $\psi_t(\eta)$ equals zero for $\|\eta\| \leq t = \|\xi\|/4$ and equals 1 for $\|\eta\| \geq 2t = \|\xi\|/2$, we distinguish two cases: (a) $\|\xi\|/4 \leq \|\eta\| \leq \|\xi\|/2$ and (b) $\|\eta\| \geq \|\xi\|/2$.

Case (a): the expression (6.9) can be estimated by a sum of derivatives of the form

$$|(\partial_\xi^{\gamma_1} \psi_t) D_x^{\alpha+\gamma} \partial_\xi^{\gamma_2} f(y,\eta)|, \qquad (\gamma_1 + \gamma_2 = \gamma + \beta),$$

with suitable binomial coefficients. Now

$$\begin{aligned}
|(\partial_\xi^{\gamma_1} \psi_t) D_x^{\alpha+\gamma} \partial_\xi^{\gamma_2} f(y,\eta)| &\leq D''_k t^{-|\gamma_1|} (1 + \|\eta\|)^{d - \|\gamma_2\|} \nu^d_{\mathcal{K},2s+2k}(f) \\
&\leq D'''_k (1 + \|\xi\|)^{-|\gamma_1|} (1 + \|\xi\|)^{d - |\gamma_2|} \nu^d_{\mathcal{K},2s+2k}(f) \\
&\leq D'''_k (1 + \|\xi\|)^{d-k} \nu^d_{\mathcal{K},2s+2k}(f).
\end{aligned}$$

Case (b): we now have that (6.9) equals $|D_x^{\alpha+\gamma}\partial_\xi^{\gamma+\beta} f(y,\eta)|$, and can be estimated by

$$\begin{aligned} |D_x^{\alpha+\gamma}\partial_\xi^{\gamma+\beta} f)(y,\eta)| &\leq (1+\|\eta\|)^{d-|\gamma+\beta|}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq (1+\|\eta\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq (1+\|\xi\|/2)^{d-k}\nu^d_{\mathcal{K},2s+2k}(f) \\ &\leq D(1+\|\xi\|)^{d-k}\nu^d_{\mathcal{K},2s+2k}(f). \end{aligned}$$

Collecting these estimates we see that

$$|R_k(D)g_t(x,\xi)| \leq D'(1+\|\xi\|)^{|d|-k}\,\nu^d_{2s+2k}(f),$$

with $D'>0$ a constant independent of f, x and ξ. □

Corollary 6.5.7. *Let d, k and s be as in the above lemma. With a suitable adaptation of the constant $C>0$, the estimate (6.8) holds for all $(x,\xi)\in\mathbb{R}^{2n}$.*

Proof It follows from Lemma 6.5.5 and its proof that there exists a constant $C_1>0$ such that $|R_k(D)f(x,\xi)| \leq C_1\nu^d_{\mathcal{K},2s+2k}(f)$. We now use that

$$(1+\|\xi\|)^{|d|-k} \geq 5^{|d|-k}$$

for all ξ with $|\xi\| \leq 4$. Hence, the estimate (6.8) holds with $C = 5^{k-|d|}C_1$ for $\|\xi\|\leq 4$. □

Corollary 6.5.8. *Let $d\in\mathbb{R}$ and $m\in\mathbb{N}$. Then there exist constants $C>0$ and $l\in\mathbb{N}$ such that for all $f\in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x,\xi)\in\mathbb{R}^{2n}$ we have*

$$|R_m(D)f(x,\xi)| \leq C(1+\|\xi\|)^{d-m}\nu^d_{\mathcal{K},l}(f). \tag{6.10}$$

Proof Let s be as in the previous corollary. Fix $k\in\mathbb{N}$ such that $|d|-k < d-m$. Let now $C'>0$ be constant as in the previous corollary. Then for all $f\in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ we have

$$|R_k(D)f(x,\xi)| \leq C'(1+\|\xi\|)^{|d|-k}\nu^d_{\mathcal{K},2s+2m}(f), \qquad (\,(x,\xi)\in\mathbb{R}^{2n}).$$

On the other hand,

$$R_m(D) - R_k(D) = \sum_{k\leq j\leq m}\langle D_x, \partial_\xi\rangle^j$$

is a continuous linear operator $S^d_{\mathcal{K}}(\mathbb{R}^n)\to S^{d-k}_{\mathcal{K}}(\mathbb{R}^n)$. In fact, there exists a constant $C''>0$ such that

$$|R_m(D)f(f(x,\xi) - R_k(D)f(x,\xi)| \leq C''(1+\|\xi\|)^{d-k}\nu^d_{\mathcal{K},2m-2}(f)$$

for all $f\in S^d_{\mathcal{K}}(\mathbb{R}^n)$ and $(x,\xi)\in\mathbb{R}^{2n}$. The result now follows with $C = C'+C''$ and with $l = \max(2s+2m, 2m-2)$. □

After these technicalities we can now finally complete the proof of the main theorem of this section.

Completion of the proof of Theorem 6.5.2 Let $k \in \mathbb{N}$, let $\alpha, \beta \in \mathbb{N}^n$ and put $m = k - |\beta|$. Then by the previous corollary, applied with $d - |\beta|$ in place of d there exist constants $C > 0$ and $l \in \mathbb{N}$ such that for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and all $(x, \xi) \in \mathbb{R}^{2n}$,

$$|R_k(D) f(x, \xi)| \leq (1 + \|\xi\|)^{d - |\beta|} \nu^{d-|\beta|}_{\mathcal{K},l}(f).$$

Moreover, by definition of the seminorms,

$$\nu^{d-|\beta|}_{\mathcal{K},l}(D^\alpha_x \partial^\beta_\xi f) \leq \nu^d_{\mathcal{K}, l+|\alpha|+|\beta|}(f)$$

for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. Combining these estimates and using that $R_k(D)$ commutes with $D^\alpha_x \partial^\beta_\xi$, we find that

$$\begin{array}{rcl} |D^\alpha_x \partial^\beta_\xi R_k(D) f(x,\xi)| & = & R_k(D)[D^\alpha_x \partial^\beta_\xi f](x, \xi) \\ & \leq & C \nu^d_{\mathcal{K}, l+|\alpha|+|\beta|}(f), \end{array}$$

for all $f \in C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ and $(x, \xi) \in \mathbb{R}^{2n}$.

It follows from the above that for each $d' \in \mathbb{R}$ the map

$$R_{k+1}(D) : C^\infty_{\mathcal{K},c}(\mathbb{R}^n) \to S^{d' - (k+1)}(\mathbb{R}^n)$$

is continuous with respect to the $S^{d'}_{\mathcal{K}}(\mathbb{R}^n)$-topology on $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$. In particular, this is valid for $d' = d + 1$. As $C^\infty_{\mathcal{K},c}(\mathbb{R}^n)$ is dense in $S^d_{\mathcal{K}}(\mathbb{R}^n)$ with respect to the topology of $S^{d+1}_{\mathcal{K}}(\mathbb{R}^n)$, it follows by application of Lemma 6.4.7 that $R_{k+1}(D)$ maps $S^d_{\mathcal{K}}(\mathbb{R}^n)$ into $S^{d-k}(\mathbb{R}^n)$ with continuity relative to the $S^{d+1}_{\mathcal{K}}(\mathbb{R}^n)$-topology on the first space. As this topology is weaker than the usual one, we conclude that $R_{k+1}(D) : S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$ is continuous. Now

$$R_{k+1}(D) - R_k(D) = \langle D_x, \partial_\xi \rangle^k$$

is continuous $S^d_{\mathcal{K}}(\mathbb{R}^n) \to S^{d-k}(\mathbb{R}^n)$ as well, and the result follows. $\square$

LECTURE 7
Pseudo-differential operators, continued

7.1. The symbol of the composition

In this section we will investigate the composition $P \circ Q$ of two pseudo-differential operators $P, Q \in \Psi(U)$; here $U \subset \mathbb{R}^n$ is an open subset. We first assume that $P = \Psi_p$ and $Q = \Psi_q$, with $p \in S^d(U)$ and $q \in S^e(U)$, where $d, e \in \mathbb{R}$. In general the operator Q maps $C_c^\infty(U)$ to $C^\infty(U)$, but not to $C_c^\infty(U)$. For the composition to exist we therefore require that the projection $\mathrm{pr}_1(\operatorname{supp} q)$ has compact closure A in U. Then Ψ_q maps $C_c^\infty(U)$ to $C_A^\infty(U)$ and $P \circ Q$ is a well-defined continuous linear operator $C_c^\infty(U) \to C^\infty(U)$.

Proposition 7.1.1. *Let $p \in S^d(U)$ and $q \in S_c^e(U)$. Then $\Psi_p \circ \Psi_q = \Psi_r$, with $r \in S^{d+e}(U)$ given by*

$$r(x,\xi) = e^{\langle D_y, \partial_\xi \rangle} p(x,\xi) q(y,\eta)|_{y=x,\eta=\xi}. \tag{7.1}$$

In particular, $\mathrm{pr}_1(\operatorname{supp} r) \subset \mathrm{pr}_1(\operatorname{supp} p)$ and

$$r \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi).$$

As a preparation we prove the following result.

Lemma 7.1.2. *The Fourier transform of the function $u : \mathbb{R}^{2n} \to \mathbb{C}$, $(x,\xi) \mapsto e^{i\xi x}$ (which defines a tempered distribution) is given by $\mathcal{F}u(\hat{x}, \hat{\xi}) = e^{-i\hat{\xi}\hat{x}}$.*

Proof Let $\Omega := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. For $z \in \Omega$ the function $v_z : x \mapsto e^{-zx^2/2}$ defines a function in $\mathcal{S}(\mathbb{R})$. By checking the Cauchy-Riemann equations relative to the variable z, using differentiation under the integral sign, we see that for $\hat{x} \in \mathbb{R}$, the Fourier transform

$$\mathcal{F}v_z(\hat{x}) = \int_{\mathbb{R}} e^{-zx^2/2}\, e^{-i\hat{x}x}\, dx$$

depends holomorphically on $z \in \Omega$. For $z \in \mathbb{R} \cap \Omega$ we find, by the substitution of variables $x \to z^{-1/2}x$, that $\mathcal{F}v_z(\hat{x}) = z^{-1/2}\mathcal{F}v_1(z^{-1/2}\hat{x})$, hence

$$\mathcal{F}v_z(\hat{x}) = z^{-1/2} e^{-\hat{x}^2/2z}. \tag{7.2}$$

By analytic continuation in z the latter formula is valid for all $z \in \Omega$, provided the branch of $z \mapsto z^{1/2}$ over Ω which is positive on $\mathbb{R} \cap \Omega$ is taken. If $f \in \mathcal{S}(\mathbb{R})$ then by continuity of $z \mapsto \langle \mathcal{F} v_z, f\rangle = \langle v_z, \mathcal{F} f\rangle$ we conclude that formula (7.2) remains valid for $z \in \overline{\Omega} \setminus \{0\}$, provided the continuous extension of the fixed branch of the square root is taken. In particular we find that

$$\mathcal{F} v_i(\hat{x}) = e^{-\pi i/4}\, e^{i\hat{x}^2/2}, \qquad \mathcal{F} v_{-i}(\hat{x}) = e^{\pi i/4}\, e^{-i\hat{x}^2/2}.$$

We now turn to the function u. Let $a : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the linear map defined by $a(x,\xi) = (x+\xi, x-\xi)/\sqrt{2}$. Then

$$[a^* u](x,\xi) = e^{i(x^2-\xi^2)/2} = \prod_{j=1}^n v_i(\xi_j) v_{-i}(x_j).$$

Therefore,

$$\mathcal{F}[a^* u](\hat{x},\hat{\xi}) = \prod_{j=1}^n e^{i(\hat{\xi}_j^2 - \hat{x}_j^2)/2} = [a^* u](-\hat{x},\hat{\xi}).$$

By orthogonality of a we have that $\mathcal{F} \circ a^* = a^* \circ \mathcal{F}$ on $\mathcal{S}(\mathbb{R})$, hence also on $\mathcal{S}'(\mathbb{R})$, and the result follows. $\square$

We also need an extension of the convolution product in order to be able to convolve tempered distributions with Schwartz functions.

Lemma 7.1.3. *The convolution product* $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *has a unique extension to a continuous bilinear map* $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. *For this extension, denoted by* $*$ *again,*

$$\mathcal{F}(f * u) = \mathcal{F} f\, \mathcal{F} u, \qquad (f \in \mathcal{S}(\mathbb{R}^n),\ u \in \mathcal{S}'(\mathbb{R}^n)).$$

Proof The multiplication map $(f,u) \to fu$, $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is readily seen to be continuous bilinear. Define $\widetilde{*} = \mathcal{F}^{-1} \circ * \circ (\mathcal{F} \times \mathcal{F})$. Then $\widetilde{*}$ is continuous bilinear and extends the convolution product on $\mathcal{S}(\mathbb{R}^n)$, by Lemma 4.2.2(b). Uniqueness of the extension follows by density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. $\square$

As a final preparation for the proof of Proposition 7.1.1 we need the following lemma.

Lemma 7.1.4. *Let* $u \in C^\infty(\mathbb{R}^{2n})$ *be defined by* $u(x,\xi) := e^{-i\xi x}$. *Then*

$$e^{\langle D_x, \partial_\xi\rangle} f = u * f \qquad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Proof In view of the definition of the operator $e^{\langle D_x,\partial_\xi\rangle} \in \mathrm{End}(\mathcal{S}(\mathbb{R}^{2n}))$ (see Section 6.4), this follows immediately by applicaton of Lemmas 7.1.3 and 7.1.2. $\square$

Proof of Proposition 7.1.1 We will first prove the result under the assumption that both p and q are smooth and compactly supported. Then

$p, q \in C_c^\infty(\mathbb{R}^{2n})$. Let $f \in C_c^\infty(U)$. Then

$$\begin{aligned} [\Psi_p \Psi_q f](x) &= \int\int e^{i\xi(x-y)} p(x,\xi)\, [\Psi_q f](y)\; dy\, d\xi \\ &= \int\int\int e^{i\xi(x-y)} e^{iy\eta}\, p(x,\xi)\, q(y,\eta) \widehat{f}(\eta)\; d\eta\, dy\, d\xi \\ &= \int e^{i\eta x} r(x,\eta) \widehat{f}(\eta)\; d\eta, \end{aligned}$$

where

$$r(x,\eta) = \int\int e^{i(\xi-\eta)(x-y)}\, p(x,\xi)\, q(y,\eta)\, d\xi dy. \tag{7.3}$$

Here we note that all integrands are compactly supported and continuous, so all integrals are convergent, and the order of the integrations is immaterial. For each $(x,\eta) \in \mathbb{R}^{2n}$ we define $R_{x,\eta} : \mathbb{R}^{2n} \to \mathbb{C}$ by

$$R_{x,\eta}(y,\xi) = p(x,\xi) q(y,\eta).$$

Then $R_{x,\eta}$ is smooth and compactly supported. Moreover, (7.3) can be rewritten as $r(x,\eta) = [u * R_{x,\eta}](x,\eta)$, with $u(y,\xi) = e^{-i\xi y}$. In view of Lemma 7.1.4 above, this implies that

$$r(x,\eta) = [e^{\langle D_y, \partial_\xi\rangle} R_{x,\eta}](x,\eta) = e^{\langle D_y, \partial_\xi\rangle} p(x,\xi) q(y,\eta)|_{y=x,\xi=\eta},$$

which in turn gives (7.1).

Fix a compact subset $\mathcal{K} \subset U$. Our next step is to extend the validity of (7.1) to $(p,q) \in S^d_{\mathcal{K}}(U) \times S^e_{\mathcal{K}}(U)$. We will do this by using a continuity argument. For $p, q \in S^\infty_{\mathcal{K}}(U)$ we observe that, for each $(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, the function $(y,\xi) \mapsto p(x,\xi) q(y,\eta)$ belongs to $S^\infty_{\mathcal{K}}(U)$ again, and we define the function $\rho(p,q) : U \times \mathbb{R}^n \to \mathbb{C}$ by

$$\rho(p,q)(x,\xi) = e^{\langle D_y, \partial_\xi\rangle} p(x,\xi) q(y,\eta)|_{y=x,\eta=\xi}. \tag{7.4}$$

In view of the lemma below, ρ has values in $S^\infty(U)$ and maps $S^d_{\mathcal{K}}(U) \times S^e_{\mathcal{K}}(U)$ continuous bi-linearly to $S^{d+e}(U)$. Fix $f \in C_c^\infty(U)$. Then in view of the remark below Definition 5.2.2 follows that

$$(p,q) \mapsto \Psi_{\rho(p,q)}(f) \tag{7.5}$$

maps $S^d_{\mathcal{K}}(U) \times S^e_{\mathcal{K}}(U)$ continuous bi-linearly to $C^\infty(U)$, for all $d, e \in \mathbb{N}$.

On the other hand, for all $d \in \mathbb{N}$ the map $(r, g) \mapsto \Psi_r(g)$ is continuous bilinear from $S^d_{\mathcal{K}}(U) \times C_c^\infty(U)$ to $C^\infty_{\mathcal{K}}(U)$ and by composition it follows that for all $d, e \in \mathbb{N}$ the map

$$(p,q) \mapsto \Psi_p \Psi_q(f) \tag{7.6}$$

is continuous bilinear from $S^d_{\mathcal{K}}(U) \times S^e_{\mathcal{K}}(U)$ to $C^\infty_{\mathcal{K}}(U)$. By the first part of the proof the maps (7.5) and (7.6) are equal on $C^\infty_{\mathcal{K},c}(U) \times C^\infty_{\mathcal{K},c}(U)$. By density of $C^\infty_{\mathcal{K},c}(U)$ in $S^d_{\mathcal{K}}(U)$ for the S^{d+1}-topology and in $S^e_{\mathcal{K}}(U)$ for the S^{e+1}-topology, it follows that the equality extends to $S^d_{\mathcal{K}}(U) \times S^e_{\mathcal{K}}(U)$.

It follows that $\Psi_p \circ \Psi_q = \Psi_{\rho(p,q)}$ on $C_c^\infty(U)$, for all $p \in S^d_{\mathcal{K}}(U)$ and $q \in S^e_{\mathcal{K}}(U)$.

Now assume more generally that $p \in S^d(U)$ and that $q \in S_c^e(U)$. Then by using a partition of unity we may write p as a locally finite sum $p = \sum_j p_j$, with $p_j \in S_c^d(U)$, and $\mathrm{pr}_1(\mathrm{supp}\, p_j)$ a locally finite collection of subsets of U. For each j we have $\Psi_{p_j} \circ \Psi_q = \Psi_{r_j}$ with r_j given in terms of p_j and q as in (7.3). In particular this implies that $\mathrm{pr}_1(\mathrm{supp}\, r_j) \subset \mathrm{pr}_1(\mathrm{supp}\, p_j)$. It follows that $\{\mathrm{pr}_1(\mathrm{supp}\, r_j)\}$ is a locally finite collection of subsets of U. Therefore, $r = \sum_j r_j$ defines a symbol in $S^{d+e}(U)$. Clearly, r satisfies (7.3), and $\Psi_p \circ \Psi_q = \Psi_r$. $\square$

Lemma 7.1.5. *Let $\mathcal{K} \subset U$ be a compact subset. Then the map ρ defined by (7.4) maps $S_{\mathcal{K}}^d(U) \times S_{\mathcal{K}}^e(U)$ continuous bilinearly to $S_{\mathcal{K}}^{d+e}(U)$.*

Proof By continuity of the map $e^{\langle D_y, \partial_\xi \rangle} : S_{\mathcal{K}}^d(U) \to S^d(\mathbb{R}^n)$ (see Theorem 6.5.2 there exist constants $C > 0, k \in \mathbb{N}$, such that

$$(1 + \|\xi\|)^{-d} |[e^{D_x \partial_\xi} f](x, \xi)| \leq C \nu_{\mathcal{K},k}^d(f),$$

for all $f \in S_{\mathcal{K}}^d(U)$, $x \in \mathcal{K}$ and $\xi \in \mathbb{R}^n$. Let $p \in S_{\mathcal{K}}^d(U)$ and $q \in S_{\mathcal{K}}^e(U)$. Then by application of the above estimate to $f = f_{x,\eta} : (y, \xi) \mapsto p(x, \xi) q(y, \eta)$, and observing that

$$(1 + \|\eta\|)^{-e} \nu_{\mathcal{K},k}^d(f_{x,\eta}) \leq \nu_{\mathcal{K},k}^d(p) \nu_{\mathcal{K},k}^e(q),$$

we find the estimate

$$(1 + \|\xi\|)^{-d-e} |\rho(p, q)(x, \xi)| \leq C \nu_{\mathcal{K},k}^d(p) \nu_{\mathcal{K},k}^e(q), \tag{7.7}$$

for all $(x, \xi) \in U \times \mathbb{R}^n$. (Note that the expression on the left-hand side vanishes for $x \in U \setminus \mathcal{K}$.) We now observe that for $\alpha, \beta \in \mathbb{N}^n$ we have

$$\partial_x^\alpha \partial_\xi^\beta \rho(p, q) = \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta}} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \beta \\ \beta_1 \end{pmatrix} \rho(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} p, \partial_x^{\alpha_2} \partial_\xi^{\beta_2} q).$$

Combining this with (7.7) we find that for every $l \in \mathbb{N}$ there exists a constant $C_l > 0$, only depending on l, such that

$$\nu_{\mathcal{K},l}^{d+e} \rho(p, q) \leq C\, C_l\, \nu_{\mathcal{K},k+l}^d(p)\, \nu_{\mathcal{K},k+l}^e(q).$$

The asserted continuity follows. $\square$

Exercise 7.1.6. Give a proof of Proposition 6.2.2 based on the idea of continuous extension used in the above proof.

We recall that a pseudo-differential operator $Q \in \Psi^e(U)$ has an operator kernel $K_Q \in \mathcal{D}'(U \times U)$ and is said to be properly supported if and only if the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{supp}\,(K_Q) \to \mathbb{R}^n$ are proper.

If $B \subset U$ is compact then so is $A := \mathrm{pr}_1(K_Q \cap \mathrm{pr}_2^{-1}(B))$ and it is easily verified that Q maps $C_B^\infty(U)$ into $C_A^\infty(U)$. Hence, Q is a continuous linear endomorphism of $C_c^\infty(U)$. Thus, for any $P \in \Psi^d(U)$ the composition $P \circ Q$ is a well defined continuous linear operator $C_c^\infty(U) \to C^\infty(U)$.

Theorem 7.1.7. *Let $P \in \Psi^d(U)$ and $Q \in \Psi^e(U)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(U)$. Moreover, the full symbol of $P \circ Q$ is given by*

$$\sigma(P \circ Q) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P)\, D_x^\alpha \sigma(Q).$$

Proof In view of Lemma 7.1.9 below, there exist symbols $p \in S^d(U)$ and $q \in S^e(U)$ such that $P = \Psi_p$ and $Q = \Psi_q$. Let $K_P \in \mathcal{D}'(U \times U)$ be the distribution kernel of P. Let $\{\psi_j\}$ be a partition of unity on U. Let $p_j = \psi_j p$; then $P = \sum_j P_j$, with $P_j = \Psi_{p_j}$. For each j the set $A_j := \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\mathrm{supp}\, \psi_j) \cap \mathrm{supp}\, K_P)$ is compact and contained in U. Hence, there exists a $\chi_j \in C_c^\infty(U)$ with $\chi_j = 1$ on A_j. It follows that $P_j \circ M_{\chi_j}$ has kernel $(\psi_j \otimes \chi_j) K_P = (\psi_j \otimes 1) K_P$ hence equals P_j. Therefore, $P_j \circ Q = P_j \circ Q_j$, where $Q = M_{\chi_j} \circ Q = \Psi_{q_j}$, with $q_j = \chi_j q$. It follows that $P_j \circ Q = \Psi_{r_j}$, with $r_j \in S^{d+e}(U)$ expressed in terms of p_j and q_j as in formula (7.3). In particular, $\mathrm{pr}_1(\mathrm{supp}\,(r_j)) \subset \mathrm{supp}\, \psi_j$, so that $r = \sum_j r_j$ is a locally finite sum defining an element of $S^{d+e}(U)$. We now have that

$$\Psi_r = \sum_j P_j \circ Q_j = \sum_j P_j \circ Q = P \circ Q.$$

on $C_c^\infty(U)$. From the construction, it follows that $q_j = q$ on an open neighborhood of $\mathrm{supp}\, \psi_j$, so that for all $\alpha \in \mathbb{N}^n$ we have

$$\partial_\xi^\alpha p_j\, D_x^\alpha q_j = \partial_\xi^\alpha p_j D_x^\alpha q.$$

This implies that

$$\begin{aligned} r \quad &\sim \quad \sum_{\alpha \in \mathbb{N}^n} \sum_j \frac{1}{\alpha!} \partial_\xi^\alpha p_j\, D_x^\alpha q_j = \sum_{\alpha \in \mathbb{N}^n} \sum_j \frac{1}{\alpha!} \partial_\xi^\alpha p_j\, D_x^\alpha q \\ &= \quad \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha p\, D_x^\alpha q. \end{aligned}$$

The result follows. □

Lemma 7.1.8. *Let $T \in \Psi^{-\infty}(U)$ be properly supported. Then there exists a $r \in S^{-\infty}(U)$ such that $T = \Psi_r$.*

Proof Let K be the integral kernel of T. Fix a partition of unity $\{\psi_j\}$ on U. Then $K_j(x, y) = \psi_j(x) K(x, y)$ defines a smooth function with compact support contained in $\mathrm{pr}_1^{-1}(\mathrm{supp}\, \psi_j) \cap \mathrm{supp}\, K$. By Lemma 5.2.5 (see also its proof) there exists a $p \in S^{-\infty}(U)$ with $\mathrm{pr}_1(\mathrm{supp}\, p) \subset \mathrm{pr}_1(\psi_j)$ such that $T_{K_j} = \Psi_{p_j}$. The locally finite sum $\sum_j p_j$ defines an element $p \in S^{-\infty}(U)$, and it is clear that $\Psi_p = \sum_j \Psi_{p_j} = \sum_j T_{K_j} = T_K$ on $C_c^\infty(U)$. □

Lemma 7.1.9. *Let $d \in \mathbb{R} \cup \{\infty\}$ and let $P \in \Psi^d(U)$ be properly supported. Then there exists a $p \in S^d(U)$ such that $P = \Psi_p$.*

Proof In view of the above lemma we may assume that $d > -\infty$. Then by Lemma 6.1.6 we may rewrite P as $P = \Psi_q + T$, with $q \in S^d(U)$ and $T \in \Psi^{-\infty}$ and with Ψ_q and hence also T properly supported. By the previous lemma there exists a $r \in S^{-\infty}(U)$ such that $T = \Psi_r$. The lemma now follows with $p = q + r$. □

We can now deduce the important result that the principal symbol behaves multiplicatively.

Corollary 7.1.10. *Let $P \in \Psi^d(U)$ and $Q \in \Psi^e(U)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(U)$. Moreover, the principal symbol of $P \circ Q$ of order $d+e$ is given by*

$$\sigma^{d+e}(P \circ Q) = \sigma^d(P)\,\sigma^e(Q).$$

7.2. Invariance of pseudo-differential operators

In order to be able to lift pseudo-differential operators to manifolds, we need to establish invariance under diffeomorphisms. For this it will turn out to be useful to have a different characterization of properly supported pseudo-differential operators.

We recall the definition of the (Fréchet) space $C^\infty(\mathbb{R}^n)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$ given in the text before Exercise 5.1.4. Note that $\mathcal{S}(\mathbb{R}^n)$ may be identified with the (closed) subspace of functions in $C^\infty(\mathbb{R}^n)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n)$ that are constant in the x-variable.

For $\varphi \in C^\infty(\mathbb{R}^n)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n)$ the integral

$$W(\varphi)(x) := \int_{\mathbb{R}^n} e^{i\xi x}\varphi(x,\xi)\,dx$$

is absolutely convergent for every $x \in X$ and defines a function $W(\varphi) \in C^\infty(\mathbb{R}^n)$. Moreover, by Lemma 5.2.1 the transform W is continuous linear from $C^\infty(\mathbb{R}^n)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We now define a more general symbol space as follows. We write (x,ξ,y) for points in $\mathbb{R}^{3n} \simeq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and denote by Σ^d the space of smooth functions $\mathbb{R}^{3n} \to \mathbb{C}$ such that for all compact $\mathcal{K} \subset \mathbb{R}^n$ and all $k \in \mathbb{N}$,

$$\nu^d_{\mathcal{K},k}(r) := \max_{|\alpha|,|\beta|,|\gamma| \le k}\ \sup_{\mathcal{K}\times\mathbb{R}^n\times\mathcal{K}} (1+|\xi|)^{|\beta|-d}|\partial_x^\alpha\partial_\xi^\beta\partial_y^\beta r(x,\xi,y)| < \infty. \tag{7.8}$$

Note that the symbol space $S^d(\mathbb{R}^n)$ may be viewed as the subspace of Σ^d consisting of functions that are constant in the y-variable.

Let now $r \in \Sigma^d$. Then for each $f \in \mathcal{S}(\mathbb{R}^n)$ the integral

$$r(x,\xi,f) := \int_{\mathbb{R}^n} e^{-i\xi y} r(x,\xi,y) f(y)\,dy$$

converges absolutely and defines a function in $C^\infty(\mathbb{R}^n)\,\widehat{\otimes}\,\mathcal{S}(\mathbb{R}^n)$. Moreover, the map $f \mapsto r(\,\cdot\,,f)$ is continuous for the obvious topologies.

The definition of pseudo-differential operator may now be extended to symbols in Σ^d by putting

$$\Psi_r(f)(x) := W(r(\,\cdot\,,\,\cdot\,,f)) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\xi(x-y)} r(x,\xi,y) f(y)\,dy\,d\xi. \tag{7.9}$$

Note that if r is independent of the variable y, then $r \in S^d(\mathbb{R}^n)$ and by carrying out the integration over y we see that (7.9) equals

$$\int_{\mathbb{R}^n} e^{i\xi x} r(x,\xi)\widehat{f}(\xi)\,d\xi,$$

which is compatible with the definition given before. In fact, we have not really extended our class of operators. For $\mathcal{K} \subset \mathbb{R}^n$ a compact subset, let $\Sigma^d_{\mathcal{K}}$ be the closed subspace of Σ^d consisting of all functions $r \in \Sigma^d$ with $\operatorname{supp} r \subset \mathcal{K}\times\mathbb{R}^n\times\mathcal{K}$. Moreover, let Σ^d_c be the union of the spaces $\Sigma^d_{\mathcal{K}}$, for $\mathcal{K} \subset \mathbb{R}^n$ compact.

Proposition 7.2.1. *Let $r \in \Sigma^d_{\mathcal{K}}$. Then*

$$p(x,\xi) = e^{\langle D_y, \partial_\xi \rangle} r(x,\xi,y)|_{y=x}$$

belongs to $S^d(\mathbb{R}^n)$, and $\Psi_r = \Psi_p$. In particular, Ψ_r belongs to $\Psi^d(\mathbb{R}^n)$ and its full symbol is given by

$$\sigma^d(\Psi_r) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} D^\alpha_y \partial^\alpha_\xi r(x,\xi,y)|_{y=x}.$$

To prepare for the proof we introduce the space

$$C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n}) := \{\varphi \in C^\infty_c(\mathbb{R}^{3n}) \mid \operatorname{supp}\varphi \subset \mathcal{K}\times\mathbb{R}^n\times\mathcal{K}\}.$$

The following lemma is proved in the same fashion as Lemma 4.1.9.

Lemma 7.2.2. *Let $d' > d$. Then $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ is dense in $\Sigma^d_{\mathcal{K}}$ for the $\Sigma^{d'}$-topology.*

Proof of Proposition 7.2.1 For $r \in \Sigma^d_{\mathcal{K}}$ and fixed $x \in \mathbb{R}^n$ the function $r_x : (y,\xi) \mapsto r(x,\xi,y)$ belongs to $S^d(\mathbb{R}^n)$. Moreover, $x \mapsto r_x$ is a smooth map $\mathbb{R}^n \to S^d_{\mathcal{K}}(\mathbb{R}^n)$, supported by $\mathcal{K}$ and it follows that

$$x \mapsto e^{\langle D_y,\partial_\xi\rangle}(r_x)$$

is a smooth map $\mathbb{R}^n \to S^d_{\mathcal{K}}(\mathbb{R}^n)$. The function $p = p(r)$ is given by the formula

$$p(r)(x,\xi) = e^{\langle D_y,\partial_\xi\rangle}(r_x)(x,\xi).$$

It is readily seen that $r \mapsto p(r)$ is a continuous linear map $\Sigma^d_{\mathcal{K}} \to S^d_{\mathcal{K}}(\mathbb{R}^n)$. Fix $f \in C^\infty_c(\mathbb{R}^n)$. Then both $r \mapsto \Psi_r(f)$ and $r \mapsto \Psi_{r(p)}(f)$ are continuous linear maps $\Sigma^d_{\mathcal{K}} \to C^\infty(\mathbb{R}^n)$, for every $d \in \mathbb{R}$. As $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ is dense in $\Sigma^d_{\mathcal{K}}$ for the Σ^{d+1}-topology, it suffices to prove the identity $\Psi_r(f) = \Psi_{p(r)}(f)$ for every $p \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$.

Thus, let $p \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ be fixed. Then

$$\begin{aligned}
\Psi_r(f) &= \int\!\!\int e^{i\xi(x-y)} r(x,\xi,y) f(y)\, dy\, d\xi \\
&= \int\!\!\int e^{i\xi(x-y)} r_x(y,\xi) \int e^{iy\eta} \widehat{f}(\eta)\, d\eta\, dy\, d\xi \\
&= \int e^{i\eta x}\, p(x,\eta) \widehat{f}(\eta)\, d\eta,
\end{aligned}$$

with

$$p(x,\eta) = \int\!\!\int e^{-i(\eta-\xi)(x-y)} r_x(y,\xi) dy d\xi.$$

Write $u(y,\xi) = e^{-i\xi y}$, then it follows that

$$\begin{aligned} p(x,\eta) &= (u * r_x)(x,\eta) \\ &= e^{\langle D_y, \partial_\xi \rangle} r_x(y,\eta)|_{y=x} \\ &= p(r)(x,\eta). \end{aligned}$$

All assertions now follow. □

Corollary 7.2.3. *Let $p \in S_c(\mathbb{R}^n)$. Then Ψ_p is properly supported if and only if there exists a $r \in \Sigma^d_{\mathcal{K}}$ such that*

$$\Psi_p = \Psi_r \qquad \text{on} \quad C_c^\infty(\mathbb{R}^n).$$

For any such r the d-th order principal symbol of p is represented by the symbol $(x,\xi) \mapsto r(x,\xi,x)$.

Proof The only if part as well as the statement about the principal symbol follows from Proposition 7.2.1 above. For the if part, assume that Ψ_p is properly supported. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$ such that $\operatorname{supp} p \subset \mathcal{K} \times \mathbb{R}^n$. Let K_p be the distribution kernel of Ψ_p. Then $\mathcal{K} \times \mathbb{R}^n \cap \operatorname{supp} K_p$ is compact, hence contained in a product of the form $K \times \mathcal{K}'$, with $\mathcal{K}' \subset \mathbb{R}^n$ compact. Let V be any open neighborhood of $\mathcal{K}'$ in $\mathbb{R}^n$. Then there exists a $\chi \in C_c^\infty(U)$ which is identically one on a neighborhood of $\mathcal{K}'$. It follows that the $\Psi_p = \Psi_p \circ M_\chi$ on $C_c^\infty(\mathbb{R}^n)$. This implies that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \Psi_p(f)(x) &= \Psi_p(\chi f(x)) \\ &= \int e^{i\xi x} p(x,\xi) \mathcal{F}(\chi f)(\xi)\, d\xi \\ &= \int e^{i\xi x} p(x,\xi) \int e^{-i\xi y} \chi(y) f(y)\, dy\, d\xi \\ &= \Psi_r(f)(x), \end{aligned}$$

with $r(x,\xi,y) = p(x,\xi)\chi(y)$. □

We now turn to the actual proof of the invariance. Given two points $x, y \in \mathbb{R}^n$, we denote the line segment from x to y by $[x,y]$. Thus,

$$[x,y] = \{x + t(y-x) \mid t \in [0,1]\}.$$

We agree to write $\mathrm{M}_n(\mathbb{R})$ for the space of $n \times n$-matrices with real entries, and $\mathrm{GL}\,(n,\mathbb{R})$ for the subset of invertible matrices.

Lemma 7.2.4. *Let U be an open subset of $\mathbb{R}^n$.*

(a) *There exists an open neighborhood Ω of the diagonal* $\operatorname{diag}(U)$ *in $U \times U$ such that $(x,y) \in \Omega \Rightarrow [x,y] \subset U$.*

(b) *If Ω is any such neighborhood, and if $f : U \to \mathbb{R}^n$ is a smooth map, then there exists a smooth map $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ such that*

$$f(y) - f(x) = T(x,y)(y-x), \qquad (x,y \in \Omega).$$

(c) *Any continuous map $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ with property (b) satisfies $T(x,x) = df(x)$ for all $x \in U$.*

Proof The map $a : [0,1] \times (U \times U) \to \mathbb{R}^n$ given by $a(t,x,y) = x + t(y-x)$ is continuous and maps $[0,1] \times \operatorname{diag}(U)$ to U. The preimage $a^{-1}(U)$ of U under a is open in $[0,1] \times U \times U$ and contains $[0,1] \times \operatorname{diag}(U)$. By compactness of $[0,1]$ it contains a subset of the form $[0,1] \times \Omega$ with Ω an open neighborhood of $\operatorname{diag}(U)$ in $U \times U$. For $(x,y) \in \Omega$ we have $[x,y] = a([0,1] \times \{(x,y)\}) \subset U$.

Let f be as in (b). Then

$$f(y) - f(x) = \int_0^1 \partial_t f(x + t(y-x))\, dt = T(x,y)(y-x),$$

with

$$T(x,y) = \int_0^1 df(x + t(y-x))\, dt.$$

Clearly, T is a smooth map $\Omega \to \mathrm{M}_n(\mathbb{R})$.

Let now $T : \Omega \to \mathrm{M}_n(\mathbb{R})$ be any continuous map satisfying property (b). Then for each $v \in \mathbb{R}^n$ we have

$$df(x)v = \lim_{t \to 0} t^{-1}[f(x+tv) - f(x)] = \lim_{t\to 0} T(x, x+tv)v = T(x,x)v.$$

$\square$

If $\varphi : M \to N$ is a diffeomorphism of smooth manifolds, and $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator, then the push-forward of P by φ, denoted $\varphi_*(P)$, is defined to be the continuous linear operator $\varphi_*(P) : C_c^\infty(N) \to C^\infty(N)$ given by

$$\varphi_*(P)(f) = P(f \circ \varphi) \circ \varphi^{-1}.$$

Proposition 7.2.5. *Let $\varphi : U \to V$ be a smooth diffeomorphism between open subsets of $\mathbb{R}^n$, with inverse ψ. Let Ω be an open neighborhood of* $\operatorname{diag}(V)$ *in $V \times V$ and assume that $T : \Omega \to \mathrm{GL}(n,\mathbb{R})$ is a smooth map such that $\psi(y) - \psi(x) = T(x,y)(y-x)$ for all $(x,y) \in \Omega$ (such a pair Ω, T exists). Let $\mathcal{K}$ be a compact subset of U such that $\varphi(\mathcal{K}) \times \varphi(\mathcal{K}) \subset \Omega$.*

(a) *For every $d \in \mathbb{N}$ and $r \in \Sigma^d_{\mathcal{K}}$, the function $\varphi_*(r) : \mathbb{R}^{3n} \to \mathbb{C}$ defined by*

$$(x,\eta,y) \mapsto r(\psi(x), T(x,y)^{-1t}\eta, \psi(y))\, |\det d\psi(y)|\, |\det T(x,y)|^{-1}$$

on $\mathcal{K} \times \mathbb{R}^n \times \mathcal{K}$, and by $\varphi_*(r) = 0$ elsewhere, belongs to $\Sigma^d_{\varphi(\mathcal{K})}$.

(b) *For every $d \in \mathbb{N}$, the map $r \mapsto \varphi_*(r)$ is continuous linear $\Sigma^d_{\mathcal{K}} \to \Sigma^d_{\varphi(\mathcal{K})}$.*

(c) *For every $d \in \mathbb{N}$ and all $r \in \Sigma^d_{\mathcal{K}}$,*

$$\varphi_*(\Psi_r) = \Psi_{\varphi_*(r)}.$$

(d) *The principal symbol of $\varphi_*(\Psi_r)$ is represented by the symbol*

$$(x,\xi) \mapsto r(\psi(x), d\varphi(\psi(x))^t\xi, \psi(x)).$$

Proof The proof of (a) and (b) is straightforward, be it somewhat tedious. Fix $f \in C_c^\infty(V)$. Then the equality $\varphi_*(\Psi_r)(f) = \Psi_{\varphi_*(r)}(f)$ is equivalent to

$$\Psi_r(f \circ \varphi) \circ \psi = \Psi_{\varphi_*(r)}(f). \tag{7.10}$$

The map $r \mapsto \Psi_r(f\circ\varphi)\circ\psi$ is continuous linear $\Sigma^d_{\mathcal{K}} \to C^\infty_{\mathcal{K}}(U)$ and so is the map $r \mapsto \varphi^*(\Psi_{\varphi^*(r)})$, for every $d \in \mathbb{N}$. Now $C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ is dense in $\Sigma^d_{\mathcal{K}}$ for the Σ^{d+1}-topology. Therefore, it suffices to prove the equality (7.10) for all $r \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$.

Fix $r \in C^\infty_{\mathcal{K},c}(\mathbb{R}^{3n})$ and let $x \in U$. Then

$$\begin{aligned}
\Psi(f\circ\varphi)(\psi(x)) &= \int\!\!\int e^{i\xi\psi(x)-i\xi z}\, r(\psi(x),\xi,z)\, f(\varphi(z))\,dz\,d\xi \\
&= \int\!\!\int e^{i\xi(\psi(x)-\psi(y))}\, r(\psi(x),\xi,\psi(y))\, f(y)\,|\det d\psi(y)|\,dy\,d\xi \\
&= \int\!\!\int e^{i[T(x,y)^t\xi](x-y)}\, r(\psi(x),\xi,\psi(y))\, f(y)\,|\det d\psi(y)|\,d\xi\,dy \\
&= \int\!\!\int e^{i\eta(x-y)}\varphi_*(r)(x,\eta,y)\, f(y)\,d\eta\,dy \\
&= \Psi_{\varphi_*(r)}(f)(x).
\end{aligned}$$

This establishes (c).

For (d) we note that $T(x,x) = d\psi(x)$ so that the principal symbol of $\Psi_{\varphi_*(r)}$ is given by

$$(x,\xi) \mapsto \varphi_*(r)(x,\xi,x) = \varphi(\psi(x), d\psi(x)^{-1t}\xi, \psi(x)).$$

Now use that $d\psi(x)^{-1} = dh(\psi(x))$. $\square$

Theorem 7.2.6. *Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. Then the following assertions are valid.*

(a) *For each $d \in \mathbb{R}\cup\{-\infty\}$ and all $P \in \Psi^d(U)$ the operator $\varphi_*(P)$ belongs to $\Psi^d(V)$.*

(b) *Let the principal symbol of P be represented by $p \in S^d(U)$. Then the principal symbol of $\varphi_*(P)$ is represented by*

$$\varphi_*(p) : (x,\xi) \mapsto p(\varphi^{-1}(x), d\varphi(\varphi^{-1}(x))^t\xi).$$

Proof Let $\psi : V \to U$ be the inverse to φ and let Ω, T be as in the statement of Proposition 7.2.5. First, assume that $d = -\infty$ and let $P \in \Psi^{-\infty}(U)$. Then P is an integral operator T_K with integral kernel $K \in C^\infty(U\times U)$. Let $f \in C^\infty_c(V)$ and $x \in V$, then by substitution of variables

$$\begin{aligned}
\varphi_*(P)(f)(x) &= \int_U K(\psi(x),z) f(\varphi(z))\,dz \\
&= \int_V K(\psi(x),\psi(y))|\det d\psi(y)|\,f(y)dy
\end{aligned}$$

from which we see that $\varphi_*(P)$ is the integral transformation with smooth integral kernel $\tilde{K} \in C^\infty(V\times V)$ given by

$$\widetilde{K}(x,y) = K(\psi(x),\psi(y))\,|\det d\psi(y)|.$$

We now assume that $d \in \mathbb{R}$ and that $P \in \Psi^d(U)$. Then by Lemma 6.1.6 we may write $P = \Psi_p + T$, with $T \in \Psi^{-\infty}(U)$ and $p \in S^d(U)$ such that Ψ_p is properly supported and has distribution kernel K_p supported inside $\Omega_U := (\varphi\times\varphi)^{-1}(\Omega)$. Since $\varphi_*(T)$ is a smoothing operator by the first part of the proof, it suffices to show that $\varphi_*(\Psi_p) \in \Psi^d(V)$. For this we proceed as follows.

Let $\{\chi_j\}$ be a partition of unity on U such that $\operatorname{supp}\chi_j \times \operatorname{supp}\chi_j \subset \Omega_U$ for every j. For each j we select an open neighborhood U_j of $\operatorname{supp}\chi_j$ with $U_j \times U_j \subset \Omega_U$ and a function $\chi_j' \in C^\infty_c(U_j)$ which is identically 1 on $\operatorname{supp}\chi_j$.

Then $P_j = P_j \circ M_{\chi_j'} + T_j$, with T_j a smoothing operator supported in $\operatorname{supp} \chi_j \times U_j$ and with $P_j \circ M_{\xi_j} = \Psi_{r_j}$, where $r_j(x, \xi, y) = \chi_j(x) p(x, \xi) \chi_j'(y)$.

We now observe that $\mathcal{K}_j = \operatorname{supp} \chi_j'$ is compact and that $\varphi(\mathcal{K}_j) \times \varphi(\mathcal{K}_j) \subset \Omega$. Morover, $r_j \in \Sigma^d_{\mathcal{K}_j}$. It follows that $\varphi_*(\Psi_{r_j}) = \Psi_{\varphi_*(r_j)}$. The supports of the kernels of the operators T_j form a locally finite set, so that $T = \sum_j T_j$ is a smoothing operator. Hence, so is $\varphi_*(T)$. The sum $\sum_j P_{r_j}$ is locally finite, and therefore so is $Q = \sum_j \varphi_*(P_{r_j})$. Hence $Q \in \Psi^d(V)$. We conclude that $\varphi_*(P) = Q + \varphi_*(T)$ is a pseudo-differential operator on V of order d. Its principal symbol equals the principal symbol of Q, which is represented by the symbol $q \in S^d(V)$ given by

$$\begin{aligned} q(x, \xi) &= \sum_j \varphi_*(r_j)(x, \xi, x) \\ &= \sum_j r_j(\psi(x), d\varphi(\psi(x))^t \xi, \psi(x)) \\ &= \sum_j p_j(\psi(x), d\varphi(\psi(x))^t \xi) \\ &= p(\psi(x), d\varphi(\psi(x))^t \xi) = \varphi_*(p)(x, \xi). \end{aligned}$$

□

7.3. Pseudo-differential operators on a manifold, scalar case

In view of the results of the previous section we can now extend the notion of a pseudo-differential operator to a smooth manifold M of dimension n.

Definition 7.3.1. Let $d \in \mathbb{R} \cup \{-\infty\}$. A pseudo-differential operator P of order d on M is a continuous linear operator $C_c^\infty(M) \to C^\infty(M)$ given by a distribution kernel $K_P \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ such that the following conditions are fulfilled.

(a) The kernel K_P is smooth outside the diagonal of $M \times M$.
(b) For each $a \in M$ there exists a chart (U_κ, κ) containing a such that the operator $P_\kappa : C_c^\infty(\kappa(U)) \to C^\infty(\kappa(U))$ given by

$$P_\kappa(f)(\kappa(x)) = P(f \circ \kappa)(x), \qquad (x \in U)$$

belongs to $\Psi^d(\kappa(U))$.

Remark 7.3.2. Of course, by the Schwartz kernel theorem, each continuous linear operator $P : C_c^\infty(M) \to C^\infty(M)$ is in particular continuous linear $C_c^\infty(M) \to \mathcal{D}'(M)$, hence given by a distribution kernel $K_P \in \mathcal{D}'(M \times M, D_M \otimes \mathbb{C}_M)$. In the above formulation, the existence of the kernel is demanded in order not to rely on the kernel theorem.

Condition (a) asserts that K_P has singular support contained in the diagonal of M, whereas condition (b) stipulates that the singularity along the diagonal is of the same type as that of the kernel of a pseudo-differential operator on $\mathbb{R}^n$.

If $\varphi : M \to N$ is a diffeomorphism then by the nature of the definition the map $\varphi_* : \Psi^d(M) \to \Psi^d(N)$ is readily seen to be a linear isomorphism. Before proceeding we will show that the definition of pseudo-differential operator coincides with the old one in case M is an open subset of $\mathbb{R}^n$.

Lemma 7.3.3. *Let M be an open subset of $\mathbb{R}^n$ and let $P : C_c^\infty(M) \to C^\infty(M)$ be a continuous linear operator with distribution kernel K_P. Then the following statements are equivalent.*

(1) *Conditions (a) and (b) of the above definition are fulfilled.*
(2) *P is a pseudo-differential operator in the sense of Definition 5.2.4.*

Proof Clearly (2) implies (1), since M can be taken as the coordinate patch. We assume (1) and will prove (2).

We first assume that $d = -\infty$. Then requirements (a) and (b) of Definition 7.3.1 guarantee that K_P is smooth on all of M, hence that P is an integral operator with smooth kernel $K_P \in \Gamma^\infty(M \times M, \mathbb{C}_M \boxtimes D_M)$. Thus, P is an operator in $\Psi^{-\infty}$ in the sense of Definition 7.3.1.

We now assume that $d \in \mathbb{R}$. By Lemma 6.1.6 each $a \in M$ has an open neighborhood U_a in M such that the operator $P_a : C_c^\infty(U_a) \to C^\infty(U_a)$ given by $f \mapsto (Pf)|_{U_a}$ may be written as $P_a = \Psi_{p_a} + T_a$, with p_a a symbol in $\Psi^d(U_a)$ and with $T_a \in \Psi^{-\infty}(U_a)$ a smoothing operator.

By paracompactness of M there exists a partition of unity $\{\chi_j\}$ on M such that for each j the support of χ_j is contained in some set U_{a_j} as above. We put $P_j = P_{a_j}$, $p_j = p_{a_j}$ and $T_j = T_{a_j}$. Then $P_j = \Psi_{p_j} + T_j$. For each j we choose a $\chi_j' \in C_c^\infty(U_j)$ such that $\chi_j' = 1$ on an open neighborhood of $\operatorname{supp}\chi_j$. Then $\chi_j(1 - \chi_j') = 0$ so $T_j' := M_{\chi_j} P M_{(1-\chi_j')}$ has kernel $[\chi_j \otimes (1 - \chi_j')]K_P$ which is smooth. The supports of these kernels form a locally finite collection, so that $T' = \sum_j T_j'$ is a smoothing operator.

Moreover, we may write

$$M_{\chi_j} \circ P \circ M_{\chi_j'} = \Psi_{q_j} + M_{\chi_j} \circ T_j \circ M_{\chi_j'},$$

with $q_j \in S^d(M)$ supported by $\operatorname{supp}\psi_j$. It follows that $q = \sum_j q_j$ is a locally finite sum and defines an element of $S^d(M)$. Moreover, the smooth kernels of the operators $M_{\chi_j} \circ T_j \circ M_{\chi_j'}$ are locally finitely supported in $M \times M$ so that the operators sum up to a smoothing operator T. For $f \in C_c^\infty(M)$ we now have that

$$Pf = \sum_j \chi_j P(\chi_j' f) + T'(f) = \Psi_q(f) + T(f) + T'(f)$$

and (2) follows. □

If M is a smooth manifold, $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator, and $U \subset M$ an open subset, we agree to write P_U for the operator $C_c^\infty(U) \to C^\infty(U)$ given by

$$P_U f = (Pf)|_U, \qquad (f \in C_c^\infty(U)).$$

The following results are now easy consequences of the definitions.

Exercise 7.3.4. If $K \in \mathcal{D}'(M \times M)$ is the distribution kernel of P, then the restriction $K|_{U\times U}$ is the distribution kernel of P_U.

Exercise 7.3.5. Let M be a smooth manifold, and $U \subset M$ an open subset. Then for each $P \in \Psi^d(M)$, the operator P_U belongs to $\Psi^d(U)$.

In the sequel we shall make frequent use of the following results.

Lemma 7.3.6. *Let $P \in \Psi^d(M)$ and let $\chi \in C^\infty(M)$.*

(a) *Let $\psi \in C^\infty(M)$ be such that* $\operatorname{supp}\psi \cap \operatorname{supp}\chi = \emptyset$. *Then*

$$M_\chi \circ P \circ M_\psi \in \Psi^{-\infty}(M).$$

(b) *Let $\chi' \in C_c^\infty(M)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp}\chi$. Then*

$$M_\chi \circ P - M_\chi \circ P \circ M_{\chi'} \in \Psi^{-\infty}(M).$$

Likewise, $P \circ M_\chi - M_{\chi'} \circ P \circ M_\chi \in \Psi^{-\infty}(M)$.

(c) *Let $\{P_j\}$ is a collection of operators from $\Psi^d(M)$ such that the supports* $\operatorname{supp} K_{P_j}$ *of the distribution kernels form a locally finite collection of subsets of $M \times M$. Then*

$$\sum_j P_j \in \Psi^d(M).$$

Proof Let K_P denote the distribution kernel of P. The distribution kernel K' of the operator $M_\chi \circ P \circ M_\psi$ equals $K' = (\chi \otimes \psi)K_P$. Since $\chi \otimes \psi = 0$ on an open neighborhood of the diagonal in $M \times M$, the kernel K' is smooth. Hence (a).

We turn to (b). There exists a function $\varphi \in C^\infty(M)$ such that $\varphi = 1$ on an open neighborhood of $\operatorname{supp}\chi$ and such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp}\varphi$. It follows that $(1-\chi)$ and φ have disjoint supports. Hence

$$M_\chi \circ P - M_\psi \circ M_\chi = M_\psi \circ (M_\varphi \circ P \circ M_{1-\chi}) \in \Psi^{-\infty}(M).$$

The second statement of (b) is proved in a similar way.

It remains to prove (c). Let $Q = \sum_j P_j$. Then Q is a well defined continuous linear operator $C_c^\infty(M) \to C^\infty(M)$ with distribution kernel $K_Q = \sum K_{P_j}$. On the complement of the diagonal in $M \times M$ the kernel K_Q is a locally finite sum of smooth functions, hence a smooth function. Let $a \in M$. There exists a coordinate patch $U \ni a$ whose closure in M is compact. The collection J of indices j for which $\operatorname{supp} K_{P_j} \cap (U \times U) \neq \emptyset$ is finite. It follows that the kernel of Q_U equals

$$K_Q|_{U\times U} = \sum_{j\in J} K_{P_j}|_{U\times U} = \sum_{j\in J} K_{P_{jU}}.$$

Hence, Q_U equals the finite sum $\sum_{j\in J} P_{jU}$ and belongs to $\Psi^d(U)$. It follows that $Q \in \Psi^d(M)$. □

Exercise 7.3.7. Let M be a smooth manifold, and $P : C_c^\infty(M) \to C^\infty(M)$ a continuous linear operator with a distribution kernel that it smooth outside the diagonal in $M \times M$. Let $\{U_j\}$ be an open covering of M. If $P_{U_j} \in \Psi^d(U_j)$ for each j, then $P \in \Psi^d(M)$.

The following result indicates that pseudo-differential operators modulo smoothing operators behave like sections of a sheaf.

Lemma 7.3.8. *Let $\{U_j\}$ be an open covering of the manifold M.*

(a) *Let $P, Q \in \Psi^d(M)$ be such that $P_{U_j} = Q_{U_j}$ for all j. Then $P - Q \in \Psi^{-\infty}(M)$.*
(b) *Assume that for each j a pseudo-differential operator $P_j \in \Psi^d(U_j)$ is given. Assume furthermore that $P_i = P_j$ on $C_c^\infty(U_i \cap U_j)$ for all indices i, j with $U_i \cap U_j \neq \emptyset$. Then there exist a $P \in \Psi^d(M)$ such that $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$ for all j. The operator P is uniquely determined modulo $\Psi^{-\infty}(M)$.*

Proof Let K_P and K_Q denote the distribution kernels of P and Q, respectively. Then $K_P|_{U_j\times U_j}$ and $K_Q|_{U_j\times U_j}$ are the distribution kernels of P_{U_j} and Q_{U_j}, respectively. It follows that $K_{P-Q} = K_P - K_Q$ is smooth on each of the sets $U_j \times U_j$, hence on the diagonal of $M \times M$. As K_{P-Q} is already smooth outside the diagonal, it follows that K_{P-Q} is smooth on $M \times M$. Hence, $P - Q \in \Psi^{-\infty}(M)$.

We turn to (b). Let $\Omega = \cup_j U_j \times U_j$. Then Ω is an open neighborhood of the diagonal in $M \times M$. Let $K_j \in \mathcal{D}'(U_j \times U_j)$ denote the distribution kernel of P_j. Let $U_{ij} := U_i \cap U_j \neq \emptyset$. Then from the assumption it follows that $K_i = K_j$ on $(U_i \times U_i) \cap (U_j \times U_j) = U_{ij} \times U_{ij}$. From the gluing property of the sheaf $\mathcal{D}'$ on Ω it follows that there exists a $K \in \mathcal{D}'(\Omega)$ such that $K = K_j$ on $U_j \times U_j$ for all j.

We will now use a cut off function to extend K to all of $M \times M$, leaving it unchanged on an open neighborhood of the diagonal. Let $\{\chi_\nu\}$ be a partition of unity of M, subordinate to the covering $\{U_j\}$. For each ν we select $j(\nu)$ such that $\operatorname{supp} \chi_\nu \subset U_{j(\nu)}$ and we fix a function $\chi'_\nu \in C_c^\infty(U_{j(\nu)})$ which is identically 1 on an open neighborhood of $\operatorname{supp} \chi_\nu$. The functions $\chi_\nu \otimes \chi'_\nu$ form a locally finitely supported family of functions in $C_c^\infty(\Omega)$. Put

$$\psi := \sum_\nu \chi_\nu \otimes \chi'_\nu.$$

Let $x \in M$ and let N_x be the finite collection of indices ν with $x \in \operatorname{supp} \psi_\nu$. Then the functions χ'_ν, for $\nu \in N_x$ are all 1 on a common open neighborhood V_x of x in M. Moreover, $\sum_{\nu\in N_x} \chi_\nu$ equals 1 on an open neighborhood U_x of x in M. It follows that $\psi = 1$ on $U_x \times V_x$. Hence, $\psi = 1$ on an open neighborhood of the diagonal in $M \times M$. Put $P = \sum_\nu M_{\chi_\nu} \circ P_{j(\nu)} \circ M_{\chi'_\nu}$. Then P is a pseudo-differential operator on M with kernel equal to

$$K_P = \sum_\nu (\chi_\nu \otimes \chi'_\nu) K_{P_j} = \psi K.$$

For each $j \in U$ we have that P_{U_j} has kernel

$$K_P|_{U_j\times U_j} = \psi K|_{U_j\times U_j} = \psi K_j.$$

It follows that $K_P - K_{P_j}$ is smooth on $U_j \times U_j$, hence $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$. The uniqueness statement follows from (a). □

In fact, with a bit more effort it can be shown that $\Psi^d/\Psi^{-\infty}$ defines a sheaf of vector spaces on M. More precisely, for two open subsets $U \subset V$ of M the map $P \mapsto P_U$, $\Psi^d(V) \to \Psi^d(U)$ induces a restriction map $\Psi^d(V)/\Psi^{-\infty}(V) \to \Psi^d(U)/\Psi^{-\infty}(U)$ which we claim to define a sheaf. The following exercise prepares for the proof of this fact.

Exercise 7.3.9. Let Ω be smooth manifold, and let $\{\Omega_j\}_{j\in J}$ be an open cover of Ω. Assume that for each pair of indices (i,j) with $\Omega_{ij} := \Omega_i \cap \Omega_j \neq \emptyset$ a smooth function $g_{ij} \in C^\infty(\Omega_{ij})$ is given such that

$$g_{ij} + g_{jk} + g_{kj} = 0 \qquad \text{on} \quad \Omega_{ijk} := \Omega_i \cap \Omega_j \cap \Omega_k$$

for all i, j, k with $\Omega_{ijk} \neq \emptyset$. Show that there exist functions $g_j \in C^\infty(U_j)$ such that $g_i - g_j = g_{ij}$ for all i, j. Hint: select a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ on Ω which is subordinate to the covering $\{\Omega_j\}$. Thus, a map $j : \mathcal{A} \to J$ is given such that $\operatorname{supp}\psi_\alpha \subset U_{j(\alpha)}$. Now consider $g_j := \sum_\alpha \psi_\alpha g_{jj(\alpha)}$.

Exercise 7.3.10.

(a) Show that with the restriction maps defined above the assignment $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ defines a presheaf on M.

(b) Show that $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ satisfies the restriction properety of a sheaf.

(c) Use the previous exercise combined with the arguments of the proof of Lemma 7.3.8 to show that $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ has the gluing property.

7.4. The principal symbol on a manifold

We will now extend the definition of principal symbol of a pseudo-differential operator to the setting of a manifold M. Let $\pi : T^*M \to M$ denote the cotangent bundle of M.

First we consider a coordinate patch U of M. Let $\kappa : U \to U'$ be a diffeomorphism onto an open subset U' of $\mathbb{R}^n$. We consider the induced diffeomorphism $T^*\kappa : T^*U \to U \times (\mathbb{R}^n)^* \simeq U' \times \mathbb{R}^n$ given by

$$T^*\kappa(\xi_x) = (\kappa(x)\,,\, \xi_x \circ T_x\kappa^{-1}), \qquad (x \in U,\ \xi_x \in T_x^*M).$$

Pull-back by the inverse of $T^*\kappa$ induces a linear isomorphism

$$\kappa_* : C^\infty(T^*U) \to C^\infty(U' \times \mathbb{R}^n).$$

For $d \in \mathbb{R} \cup \{-\infty\}$ we define

$$S^d(U) = \{p \in C^\infty(T^*U) \mid \kappa_*(p) \in S^d(U')\}.$$

It follows from Lemma 5.1.6 that this space is independent of the choice of κ.

Definition 7.4.1. We define $S^d(M)$ to be the space of smooth functions $p : T^*M \to \mathbb{C}$ with the property that for each $a \in M$ there exists a coordinate patch U containing a such that $p|_{T^*U} \in S^d(U)$.

If $\varphi : M \to N$ is a diffeomorphism of manifolds, then it follows from the above definition that the induced linear isomorphism $\varphi_* : C^\infty(T^*M) \to C^\infty(T^*N)$ maps $S^d(M)$ onto $S^d(N)$. Moreover, if $\Omega \subset M$ is an open subset then the restriction map $C^\infty(T^*M) \to C^\infty(T^*\Omega)$, $p \mapsto p|_{T^*\Omega}$ maps $S^d(M)$ to $S^d(\Omega)$.

If Ω is an open subset of M, and $\mathcal{K} \subset \Omega$ a compact subset, we define $S^d_{\mathcal{K}}(\Omega)$ to be the space of $p \in S^d(\Omega)$ with $\operatorname{supp} p \subset \pi^{-1}(\mathcal{K})$. Finally, we define $S^d_{\mathcal{K}}(\Omega)$ to be the union of the spaces $S^d_{\mathcal{K}}(\Omega)$, for $\mathcal{K} \subset \Omega$ compact. We note that elements of $S^d_c(\Omega)$ may be viewed as elements of $S^d_c(M)$ by defining them to be zero on $T^*M \setminus T^*\Omega$.

If (U, κ) is a chart of M, then the map $\kappa_* : C^\infty(T^*U) \to C^\infty(\kappa(U) \times \mathbb{R}^n)$ induces a linear bijection

$$\kappa_* : \; S^d(U)/S^{d-1}(U) \xrightarrow{\simeq} S^d(\kappa(U))/S^{d-1}(\kappa(U)).$$

The following definition is justified by Theorem 7.2.6 (b).

Definition 7.4.2. Let U be a coordinate patch of M. We define the map $\sigma^d_U : \Psi(U) \to S^d(U)/S^{d-1}(U)$ by

$$\kappa_* \sigma^d_U(P) = \sigma^d_{\kappa(U)}(\kappa_* P),$$

for κ a coordinate system on U. (Here $\sigma^d_{\kappa(U)}$ denotes the principal symbol map of $S^d(\kappa(U))$, defined in Definition 5.4.7.)

If $U \subset M$ is open, then the restriction map $p \mapsto p|_{T^*\Omega}$ induces a map $S^d(M)/S^{d-1}(M) \to S^d(\Omega)/S^{d-1}(\Omega)$, which we shall denote by $\sigma \mapsto \sigma_\Omega$. The support of an element $\sigma \in S^d(M)/S^{d-1}(M)$ is defined to be the complement of the largest open subset $\Omega \subset M$ such that $\sigma_\Omega = 0$.

Lemma 7.4.3. *Let $d \in \mathbb{R}$. There exists a unique linear map $\sigma^d : \Psi^d(M) \to S^d(M)/S^{d-1}(M)$ such that for every coordinate patch $U \subset M$ we have*

$$\sigma^d(P)_U = \sigma^d_U(P_U).$$

Proof Uniqueness of the map follows from the fact that an element $\sigma \in S^d(M)/S^{d-1}(M)$ is completely determined by its restrictions to the sets of an open covering of M. We will establish existence by using a partition of unity.

Let χ_j be a partition of unity on M such that for each j, the support $\mathcal{K}_j$ of χ_j is contained in a coordinate patch U_j. We define

$$\tilde{\sigma}^d : P \mapsto \sum_j \sigma^d_{U_j}((M_{\psi_j} \circ P)_{U_j})$$

Here we note that the term corresponding to j may be viewed as an element of $S^d(M)/S^{d-1}(M)$ with support contained in $\mathcal{K}_j$. In particular, the sum is locally finite and defines an element of $S^d(M)/S^{d-1}(M)$. Let $P \in \Psi^d(M)$ and let (U, κ) be a chart of M. We will show that $\tilde{\sigma}^d(P)_U = \sigma^d_U(P_U)$.

Let $\psi \in C^\infty_c(U)$. Then it suffices to show that $\psi\tilde{\sigma}^d(P)_U = \psi\sigma^d_U(P_U)$. From the definition given above it follows that $\psi\tilde{\sigma}^d(P)_U = \tilde{\sigma}^d(\psi P)_U$. Let J be the finite collection of indices for which $\operatorname{supp}\psi_j \cap \operatorname{supp}\psi \neq \emptyset$. For each $j \in J$ we fix a function $\chi_j \in C^\infty_c(U_j \cap U)$ which equals one on $\operatorname{supp}\psi\psi_j$. Then it follows that

$$\begin{aligned}\psi\tilde{\sigma}^d(P)_U &= \sum_{j\in J}\sigma^d_{U_j}(M_{\psi\psi_j}\circ P_{U_j})\\ &= \sum_{j\in J}\sigma^d_{U_j}(M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j})\\ &= \sigma^d_U(\sum_{j\in J}M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j}).\end{aligned}$$

Now modulo smoothing operators from $\Psi^{-\infty}(U)$ we have

$$\sum_{j\in J}M_{\psi\psi_j}\circ P_{U_j}\circ M_{\chi_j} = \sum_{j\in J}M_\psi M_{\psi_j}\circ P_U = M_\psi\circ P_U.$$

Hence,

$$\psi\tilde{\sigma}^d(P)_U = \sigma^d_U(M_\psi\circ P_U) = \psi\sigma^d_U(P_U).$$

The result follows. □

Theorem 7.4.4. *The principal symbol map $\sigma^d : \Psi^d(M)\to S^d(M)/S^{d-1}(M)$ induces a linear isomorphism*

$$\sigma^d:\ \Psi^d(M)/\Psi^{d-1}(M)\xrightarrow{\simeq}S^d(M)/S^{d-1}(M).$$

Proof We will first establish surjectivity. Let $p\in S^d(M)$. Let χ_j be a partition of unity on M such that for each j the support of χ_j is contained in a coordinate patch U_j. For each j there exists an operator $P_j\in\Psi^d(U_j)$ such that $\sigma^d_{U_j}(P_j) = p_{U_j} + S^{d-1}(U_j)$. For each j we select a function $\chi'_j\in C^\infty_c(U_j)$ such that $\chi'_j = 1$ on an open neighborhood of $\operatorname{supp}\chi_j$. Then

$$P = \sum_j M_{\chi_j}\circ P_j\circ M_{\chi'_j}$$

defines an element of $\Psi^d(M)$. We will show that $\sigma^d(P) = p + S^{d-1}(M)$. Let $\psi\in C^\infty_c(M)$ have support inside a coordinate patch U. Then it suffices to show that $\psi\sigma^d(P) = \psi p + S^{d-1}(M)$. Let $\psi'\in C^\infty_c(U)$ be identically one on a neighborhood of $\operatorname{supp}\psi$. Then

$$\begin{aligned}\psi\sigma^d(P) &= \psi\sigma^d_U(P_U) = \sigma^d_U(M_\psi\circ P_U\circ M_{\psi'})\\ &= \sum_j\sigma^d_U(M_{\psi\chi_j}\circ P_U\circ M_{\psi'})\\ &= \sum_j\sigma^d_U(M_{\psi\chi_j}\circ P_U\circ M_{\psi'\chi'_j})\\ &= \sum_j\sigma^d_{U_j}(M_{\psi\chi_j}\circ P_{U_j}\circ M_{\psi'\chi'_j})\\ &= \sum_j\psi\chi_j p + S^{d-1}(M) = \psi p + S^{d-1}(M).\end{aligned}$$

It remains to be established that $\sigma^d : \Psi^d(M)\to S^d(M)/S^{d-1}(M)$ has kernel equal to $\Psi^{d-1}(M)$. Clearly, the latter space is contained in the kernel. Conversely, let $P\in\Psi^d(M)$ and assume that $\sigma^d(P) = 0$. Then it follows that

$\sigma^d_U(P_U) = 0$ for each coordinate patch U. In view of Corollary 5.4.8 it follows that $P_U \in \Psi^{d-1}(U)$ for each coordinate patch U. This in turn implies that $P \in \Psi^{d-1}(M)$ by Definition 7.3.1. $\square$

7.5. Symbol calculus on a manifold

In this section we will discuss results concerning the principal symbols of adjoints and products of pseudo-differential operators on the manifold M. The proofs of these results will consist of reduction to the analogous local results.

Our first goal is to understand the behavior of the principal symbol under left and right composition with multiplication by smooth functions. Let $\chi \in C^\infty(M)$ and $p \in S^d(M)$, then the function $\pi^*(\chi)p : T^*_x M \ni \xi_x \mapsto \chi(x)p(\xi_x)$ belongs to $S^d(M)$ again. Indeed, this is an easy consequence of the analogous property in the local case. Accordingly, the space $S^d(M)$ becomes a $C^\infty(M)$-module and we write χp for $\pi^*(\chi)p$. As $S^{d-1}(M)$ is a submodule, it follows that the quotient $S^d(M)/S^{d-1}(M)$ is a $C^\infty(M)$-module in a natural way.

Lemma 7.5.1. *Let $P \in \Psi^d(M)$ and $\chi, \psi \in C^\infty(M)$. Then $M_\chi \circ P \circ M_\psi \in \Psi^d(M)$ and*

$$\sigma^d(M_\chi \circ P \circ M_\psi) = \chi\psi\sigma^d(P).$$

Proof The first assertion is a straightforward consequence of the definition and the fact that

$$(M_\chi \circ P \circ M_\psi)_U = M_{\chi|U} \circ P_U \circ M_{\psi|U}$$

for every open subset $U \subset M$.

Let now $U \subset M$ be a coordinate patch. Then

$$\begin{aligned} \sigma^d(M_\chi \circ P \circ M_\psi)_U &= \sigma^d_U((M_\chi \circ P \circ M_\psi)_U) \\ &= \sigma^d_U(M_{\chi|U} \circ P_U \circ M_{\psi|U}) \\ &= (\chi\psi)|_U \sigma^d_U(P_U) = [\chi\psi\sigma^d(P)]_U. \end{aligned}$$

The result follows. $\square$

Our next goal is to understand the symbol of the adjoint of a pseudo-differential operator relative to a smooth positive density dm on M. We assume such a density to be fixed for the rest of this section.

Given two smooth functions $f \in C^\infty_c(M)$ and $g \in C^\infty(M)$, we agree to write

$$(\, f\,,\, g\,) = \int_M f(x)g(x)\, dm(x).$$

The above pairing has a unique extension to a continuous bilinear pairing $C^\infty_c(M) \times \mathcal{D}'(M) \to \mathbb{C}$.

Lemma 7.5.2. *Let $P \in \Psi^d(M)$. Then there exists a unique continuous linear operator $R : \mathcal{E}'(M) \to \mathcal{D}'(M)$ such that*

$$(\, Pf\,,\, g\,) = (\, f\,,\, Rg\,), \qquad \text{for all} \quad f, g \in C^\infty_c(M).$$

The operator R belongs to $\Psi^d(M)$ and has principal symbol given by

$$\sigma^d(R)(\xi_x) = \sigma^d(P)(-\xi_x), \qquad (x \in M,\ \xi_x \in T^*_x M).$$

Proof The operator $\tilde{P} : f \mapsto (Pf)\, dm$ is continuous linear from $C_c^\infty(M)$ to $\Gamma^\infty(D_M)$, where D_M denotes the density bundle on M. It follows that the transposed $\tilde{P}^t : \mathcal{E}'(M) \to \mathcal{D}'(M, D_M)$ is continuous linear, hence of the form $u \mapsto R(u)\, dm$ with R a uniquely determined continuous linear operator $\mathcal{E}'(M) \to \mathcal{D}'(M)$. Clearly the operator R satisfies

$$(\, Pf \,,\, g\,) = \langle (Pf)\, dm, g\rangle = \langle f, \tilde{P}^t g\rangle = \langle f, (Rg) dm\rangle = (\, f \,,\, Rg\,).$$

Moreover, the operator R is uniquely determined by this property. It remains to be shown that $R \in \Psi^d(M)$. For this we first note that R has the distribution kernel $K_R \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ given by

$$K_R(x,y)(dm(x) \otimes 1) = K_P(y,x)(dm(y) \otimes 1),$$

with $K_P \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ the distribution kernel of P. Of course this equality should be interpreted in distribution sense. From the smoothness of K_P outside the diagonal, it follows that K_R is smooth outside the diagonal. Let now $U \subset M$ be a coordinate patch, with associated coordinate system $\kappa : U \to U' \subset \mathbb{R}^n$. Then there exists a unique strictly positive function $J \in C^\infty(U')$ such that $\kappa_*(dm) = J\, dx$. Let $f, g \in C_c^\infty(U)$, then it follows that

$$\begin{aligned}
\langle f dm, Rg\rangle &= (\, Pf \,,\, g\,)\\
&= \langle \kappa_*(Pf), \kappa_* g\, \kappa_*(dm)\rangle\\
&= \langle \kappa_*(P_U)\kappa_* f, J\kappa_* g\, dx\rangle\\
&= \langle \kappa_* f, \kappa_*(P_U)^t [J\kappa_* g]\, dx\rangle\\
&= \langle \kappa_*(f\, dm), J^{-1}\kappa_*(P_U)^t[J\kappa_*(g)]\rangle.
\end{aligned}$$

From this we conclude that

$$\kappa_*(R_U) = M_{J^{-1}} \circ \kappa_*(P_U)^t \circ M_J.$$

In view of Lemma 5.4.2 and Proposition 6.2.2 it now follows that $\kappa_*(R_U) \in \Psi^d(U')$ with principal symbol given by

$$\begin{aligned}
\sigma^d(\kappa_*(R_U))(x,\xi) &= \sigma^d(M_{J^{-1}} \circ \kappa_*(P_U)^t \circ M_J)(x,\xi)\\
&= J(x)^{-1} J(x) \sigma^d(\kappa_*(P_U))(x, -\xi) = \sigma^d(\kappa_*(P_U))(x,-\xi).
\end{aligned}$$

We conclude that $R_U \in \Psi^d(U)$ with principal symbol given by

$$\begin{aligned}
\sigma_U^d(R_U)([T_m\kappa]^t\xi) &= \sigma^d(\kappa_*(R_U))(\kappa(m),\xi)\\
&= \sigma^d(\kappa_*(P_U))(\kappa(m),-\xi) = \sigma_U^d(P_U)(-[T_m\kappa]^t\xi),
\end{aligned}$$

for all $m \in U$, $\xi \in \mathbb{R}^n$. □

We will end this section by discussing the principal symbol of the composition of two properly supported pseudo-differential operators.

From the similar local property of principal symbols, it follows that multiplication induces a bilinear map

$$S^d(M)/S^{d-1}(M) \times S^e(M)/S^{e-1}(M) \to S^{d+e}(M)/S^{d+e-1}(M)$$

for all $d, e \in \mathbb{R} \cup \{-\infty\}$.

Theorem 7.5.3. *Let $P \in \Psi^d(M)$ and $Q \in \Psi^e(M)$ be properly supported. Then $P \circ Q$ belongs to $\Psi^{d+e}(M)$ and has principal symbol given by*

$$\sigma^{d+e}(P \circ Q) = \sigma^d(P)\,\sigma^e(Q). \tag{7.11}$$

Proof We first consider the case that $e = -\infty$, i.e., Q is a smoothing operator. We will show that in this case $P \circ Q$ is smoothing. Let $K \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernel of $P \circ Q$. Let a be a point of M, U a coordinate patch containing a and $\psi \in C_c^\infty(U)$ a function with $\psi(a) \neq 0$. Then it suffices to show that $(1 \otimes \psi)K$ is smooth. The latter requirement is equivalent to the requirement that $P \circ Q \circ M_\psi$ be smoothing. Now this can be seen as follows. Since Q is proper, there exists a compact subset $\mathcal{K} \subset M$ such that the $\mathrm{supp}\,(1 \otimes \psi)K \subset \mathcal{K} \times \mathrm{supp}\,\psi$. Fix a non-vanishing smooth density dm on M. We may write $K_Q = \widetilde{K}_Q(1 \otimes dm)$, with $\widetilde{K}_Q \in \mathcal{D}'(M \times M)$. Then the map $k_Q : z \mapsto \widetilde{K}_Q(\,\cdot\,, z)\psi(z)$ is smooth from M to $C_{\mathcal{K}}^\infty(M)$. It follows that the map $k : z \mapsto P(k_Q(z))$ is smooth from M to $C^\infty(M)$. For each $f \in C_c^\infty(M)$ we have

$$\begin{aligned} P \circ Q(\psi f) &= P[\int_M \widetilde{K}_Q(\,\cdot\,, z)\psi(z)f(z)dz] \\ &= \int_M P[\widetilde{K}_Q(\,\cdot\,, z)\psi(z)]f(z)\,dz \\ &= \int_M k(z)f(z)\,dz \end{aligned}$$

which implies that $P \circ Q \circ M_\psi$ has the smooth integral kernel $(x, y) \mapsto k(z)(x)(1 \otimes dm)$. We conclude that $P \circ Q$ is smoothing whenever Q is.

Combining the above with Lemma 7.5.2 we see that $P \circ Q$ is also a smoothing operator whenever P is.

It remains to consider the case of arbitrary $d, e \in \mathbb{R}$. We will first show that $P \circ Q$ is a pseudo-differential operator of order $d + e$. Let $\mathcal{K} \subset M$ be compact. Then there exists a compact subset $\mathcal{K}' \subset M$ such that $\mathrm{supp}\,K_Q \cap (M \times \mathcal{K}) \subset \mathcal{K}' \times \mathcal{K}$ and a compact subset $\mathcal{K}'' \subset M$ such that $\mathrm{supp}\,K_P \cap (M \times \mathcal{K}') \subset \mathcal{K}'' \times \mathcal{K}'$. Then Q maps $C_{\mathcal{K}}^\infty(M)$ continuous linearly into $C_{\mathcal{K}'}^\infty(M)$ and P maps the latter space continuous linearly into $C_{\mathcal{K}''}^\infty(M)$. It follows that the composition $P \circ Q$ is continuous linear $C_{\mathcal{K}}^\infty(M) \to C_{\mathcal{K}''}^\infty(M)$. This implies that $R = P \circ Q$ is a continuous linear map $C_c^\infty(M) \to C_c^\infty(M)$.

Let ψ_j be a partition of unity on M such that each ψ_j is supported in a coordinate patch. Then by Lemma 7.3.6 (c) it suffices to show that each operator $R_j = R \circ M_{\psi_j}$ belongs to $\Psi^{d+e}(M)$. We fix j for the moment, put $\psi = \psi_j$ and let $U = U_j$ be a coordinate patch containing $\mathrm{supp}\,\psi$. There exists a compactly supported function $\chi \in C_c^\infty(U)$ such that $\chi = 1$ on an open neighborhood of $\mathrm{supp}\,\psi$. Now $M_{(1-\chi)} \circ Q \circ M_\psi$ is smoothing, so by the first part of the proof its left composition with P is smoothing as well. Put

$$Q_j = M_\chi \circ Q \circ M_\psi.$$

Then it suffices to show that $P \circ Q_j$ belongs to $\Psi^{d+e}(M)$. We select $\psi' \in C_c^\infty(U)$ such that $\psi' = 1$ on an open neighborhood of $\mathrm{supp}\,\chi$ and $\chi' \in C_c^\infty(U)$ such that $\chi' = 1$ on an open neighborhood of $\mathrm{supp}\,\psi'$. Then $M_{1-\chi'} \circ P \circ M_{\psi'}$ is smoothing, and by the first part of the proof, so is $[M_{1-\chi'} \circ P \circ M_{\psi'}]Q_j$. Put

$$P_j = M_{\chi'} \circ P \circ M_{\psi'}.$$

Then $P = P_j + M_{1-\chi'} \circ P \circ M_{\psi'}$ hence it suffices to show that $P_j \circ Q_j$ belongs to $\Psi^{d+e}(M)$. As the distribution kernels of the operators P_j, Q_j are contained in $\operatorname{supp} \chi' \times \operatorname{supp} \psi' \subset U \times U$, this result follows from the local result, Theorem 7.1.7. We conclude that $P \circ Q \in \Psi^{d+e}(M)$. By application of Corollary 7.1.10 we obtain that

$$\sigma^{d+e}(P_j \circ Q_j) = \sigma^d(P_j)\sigma^d(Q_j).$$

From the above it follows furthermore that $P \circ Q \circ M_{\psi_j} = P_j \circ Q_j$ modulo a smoothing operator, hence

$$\begin{aligned} \psi_j \sigma^{d+e}(P \circ Q) &= \sigma^{d+e}(P_j \circ Q_j) = \sigma^d(P_j)\sigma^e(Q_j) \\ &= \chi_j' \psi_j' \chi_j \psi_j \sigma^d(P)\sigma^d(Q) = \psi_j \sigma^d(P)\sigma^e(Q). \end{aligned}$$

This holds for every j. The identity (7.11) follows. $\square$

Exercise 7.5.4. Let $P \in \Psi^d(M)$.

(a) Show that P has a unique extension to a continuous linear map $\mathcal{E}'(M) \to \mathcal{D}'(M)$.
(b) Show that the extension is pseudo-local.

Exercise 7.5.5. Let dm be positive smooth density on M and let $P \in \Psi^d(M)$.

(a) Show that there exists a unique $P^* \in \Psi^d(M)$ such that

$$(\, Pf \,,\, \overline{g}\,) = (\, f \,,\, \overline{P^* g}\,) \qquad \text{for all} \quad f, g \in C_c^\infty(M).$$

(b) Show that the principal symbol of P^* is given by

$$\sigma^d(P^*) = \overline{\sigma^d(P)}.$$

LECTURE 8
Operators between vector bundles

8.1. Operators on manifolds

In this section we shall extend the definition of pseudo-differential operator to sections of vector bundles on a smooth manifold M. We start by recalling the notion of a smooth kernel or smoothing operator between vector bundles on manifolds.

Let M and N be smooth manifolds, and let $\pi_E : E \to M$ and $\pi_F : F \to N$ be smooth complex vector bundles over M and N respectively.

We fix a positive smooth density dy on M, i.e., a smooth section of the density bundle D_M on M that is positive at every point of M. For the definition of the density bundle and the integration of its sections, see Lecture 2.

The exterior tensor product $F \boxtimes E$ is the vector bundle on $N \times M$ defined by

$$F \boxtimes E := \mathrm{pr}_1^*(F) \otimes \mathrm{pr}_2^*(E);$$

here $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections of $N \times M$ to N and M, respectively. Let k be a smooth section of the bundle $F \boxtimes E^*$ on $N \times M$ and let $f \in \Gamma_c^\infty(M, E)$. For every $x \in N$ we may view $y \mapsto k(x, y) f(y)\, dy$ as a compactly supported density on M with values in the finite dimensional linear space F_x. By using a partition of unity it is readily seen that the integral of the mentioned density depends smoothly on x. Accordingly, we define the complex linear operator $T : \Gamma_c^\infty(M, E) \to \Gamma^\infty(N, F)$ by

$$Tf(x) = \int_M k(x, y)\, f(y)\, dy \qquad (y \in M).$$

Any T of the above form is called a smooth kernel operator or smoothing operator from $\Gamma_c^\infty(M, E)$ to $\Gamma^\infty(N, F)$. Obviously, for such an operator the Schwartz kernel $K_T \in \mathcal{D}'(N \times M, F \boxtimes E^\vee)$ is smooth and given by the formula

$$K_T(x, y) = k(x, y) \mathrm{pr}_2^*(dy).$$

In terms of local trivializations of the bundles, T is given by matrices of scalar smoothing operators. More precisely, this may be described as follows.

We recall that a trivialization of E over an open subset U of M is defined to be a vector bundle isomorphism $\tau : E_U \to U \times \mathbb{C}^k$, with k the rank of E.

A frame of E over an open subset U of M is an ordered set $s_1, \dots, s_k$ of sections in $\Gamma^\infty(U, E)$ such that $s_1(x), \dots, s_k(x)$ is an ordered basis of the fiber E_x, for every $x \in U$. Given a choice of frame $s_1, \dots, s_k$, we have a vector bundle isomorphism $\sigma : U \times \mathbb{C}^k \to E_U$ given by $(x, v) \mapsto a_1 s_1(x) + \cdots + a_k s_k(x)$. The inverse τ of σ is a vector bundle isomorphism $E_U \to U \times \mathbb{C}^k$, i.e., a trivialization of E over U.

Conversely, if $\tau : E_U \to U \times \mathbb{C}^k$ is a trivialization of the bundle, then there exists a unique frame $s_1, \dots, s_k$ of E over U, to which τ is associated in the above manner, i.e.,

$$\tau[\, a_1 s_1(x) + \cdots + a_k s_k(x)\,] = (x, a), \qquad ((x, a) \in U \times \mathbb{C}^k).$$

Assume that E trivializes over the open subset $U \subset M$ and that F trivializes over the open subset $V \subset N$. Then the operator $T_{V,U} : \Gamma_c^\infty(U, E) \to \Gamma^\infty(V, F)$ given by

$$T_{V,U}(f) = (Tf)|_V, \qquad (f \in \Gamma_c^\infty(U, E)),$$

has Schwartz kernel equal to $K\mathrm{pr}_2^*(dy)|_{V\times U}$. Let $s_1, \dots, s_k$ be a local frame for E over U and $t_1, \dots, t_l$ a local frame for F over V, then $T_{V,U}$ is given by

$$T_{U,V}\big(\sum_j f^j s_j\big) = T^i_j(f^j) t_i,$$

with T^i_j uniquely determined smoothing operators $C_c^\infty(U) \to C^\infty(V)$. Conversely, any such collection of smoothing operators defines a smoothing operator from $\Gamma_c^\infty(U, E)$ to $C^\infty(V, F)$. Let $t^1, \dots, t^l$ be the frame of F^* over U dual to $t_1, \dots, t_l$, i.e., $\langle t^i(y), t_j(y)\rangle = \delta^i_j$ for all $y \in V$. Then we note that the Schwartz kernel of T^i_j equals

$$\langle K(x, y), t^i(x) \otimes s_j(y)\rangle\, \mathrm{pr}_2(dy)_{(x,y)}.$$

The space of smooth kernel operators from E to F is denoted by $\Psi^{-\infty}(E, F)$. Since any positive smooth density on M is of the form $c(y)dy$, with c a strictly positive smooth function, the space $\Psi^{-\infty}(E, F)$ is independent of the particular choice of the density dy.

Let $\mathbb{C}_M$ denote the trivial line bundle $M \times \mathbb{C}$. Then the space $\Psi^{-\infty}(\mathbb{C}_M, \mathbb{C}_M)$ may be identified with the space of smoothing operators $C_c^\infty(M) \to C^\infty(M)$, which we previously denoted by $\Psi^{-\infty}(M)$.

We shall now give the definition of a pseudo-differential operator between smooth complex vector bundles $E, F \to M$. We first deal with the case that E and F are trivial bundles on an open subset $U \subset M$. Thus, $E = U \times \mathbb{C}^k$ and $F = U \times \mathbb{C}^l$. Then we have natural identifications $\Gamma_c^\infty(U, E) \simeq C_c^\infty(U, \mathbb{C}^k) \simeq C_c^\infty(U)^k$, and similar identifications for F. Accordingly, we define $\Psi^d(U, E, F) = \Psi^d(E_U, F_U) \subset \mathrm{Hom}(\Gamma_c^\infty(U, E), \Gamma^\infty(U, F))$ by

$$\Psi^d(E_U, F_U) := \mathrm{M}_{l,k}(\Psi^d(U)),$$

the linear space of $l \times k$ matrices with entries in $\Psi^d(U)$. With these identifications, the action of an element $P \in \Psi^d(E_U, F_U)$ on a section $f \in \Gamma_c^\infty(U, E)$ is given by

$$(Pf)_i = \sum_{1\le j\le k} P_{ij} f_j.$$

Assume that τ_E and τ_F are bundle automorphisms of the trivial bundles $E = U\times\mathbb{C}^k$ and $F = U\times\mathbb{C}^l$, respectively. Thus, τ_E is a map of the form $(x,v)\mapsto (x,\gamma_E(x)v)$ with $\gamma_E : U\to \mathrm{GL}\,(k,\mathbb{C})$ a smooth map, and τ_F is similarly given in terms of smooth map $\gamma_F : U\to \mathrm{GL}\,(l,\mathbb{C})$. Then we have an induced linear automorphism τ_{E*} of $\Gamma_c^\infty(U,E)\simeq C_c^\infty(U)^k$ given by

$$(\tau_{E*}f)(x) = \gamma_E(x)f(x), \qquad (x\in U).$$

Likewise, we have an induced linear automorphism τ_F^* of $\Gamma^\infty(U,F)$, and, accordingly, an induced linear automorphism τ_* of $\mathrm{Hom}(\Gamma_c^\infty(U,E),\Gamma_c^\infty(U,F))$. The latter is given by $\tau_*(Q) = \tau_{F*}\circ Q\circ\tau_{E*}^{-1}$, or

$$\tau_*(Q)(f) = \gamma_F Q(\gamma_E^{-1}f), \qquad (f\in\Gamma_c^\infty(U,E)\simeq C_c^\infty(U)^k).$$

It follows by component wise application of Lemma 7.5.1 that τ_* maps the linear space $\Psi^d(E_U,F_U)$ isomorphically onto itself.

The last observation paves the way for the definition of $\Psi^d(U,E,F) = \Psi^d(E_U,F_U)$ when E and F are smooth complex vector bundles on M that admit trivializations over an open subset $U\subset M$. Let $\tau_E : E\to E' = U\times\mathbb{C}^k$ and $\tau_F : F\to F' = U\times\mathbb{C}^l$ be trivializations; let $\tau_{E*} : \Gamma_c^\infty(U,E)\to\Gamma_c^\infty(U,E')$ be the induced map, and let $\tau_{F*} : \Gamma^\infty(U,F)\to\Gamma^\infty(U,F')$ be defined similarly. Then we define $\Psi^d(U,E,F)$ to be the space of linear maps $Q : \Gamma_c^\infty(U,E)\to\Gamma^\infty(U,F)$ such that

$$\tau_*(Q) := \tau_F^*\circ Q\circ\tau_{E*}^{-1} \in \Psi^d(U,E',F'). \tag{8.1}$$

This definition is independent of the particular choice of the trivializations for E_U and F_U, by the observation made above.

Definition 8.1.1. Let E,F be smooth vector bundles on a manifold M and let $d\in\mathbb{R}$. A pseudo-differential operator of order at most d from E to F is a continuous linear operator $P : \Gamma_c^\infty(E)\to\Gamma^\infty(F)$ with distribution kernel $K_P\in\mathcal{D}'(M\times M, F\boxtimes E^\vee)$ such that the following conditions are fulfilled.

(a) The kernel K_P is smooth outside the diagonal of $M\times M$.
(b) For each $a\in M$ there exists an open neighborhood $U\subset M$ on which E and F admit trivializations and such that the operator $P_U : \Gamma_c^\infty(U,E)\to \Gamma^\infty(U,F)$, $f\mapsto Pf|_U$ belongs to $\Psi^d(E_U,F_U)$.

The space of such pseudo-differential operators is denoted by $\Psi^d(E,F)$.

It follows from the above definition that the space $\Psi^d(E,F)$ transforms naturally under isomorphisms of vector bundles. More precisely, let $\varphi : M_1\to M_2$ be an isomorphism of smooth manifolds, and let $\varphi_E : E_1\to E_2$ be a compatible isomorphism of smooth complex vector bundles $\pi_{E_1} : E_1\to M_1$ and $\pi_{E_2} : E_2\to M_2$. Here the requirement of compatibility means that the pair (φ_E,φ) is an isomorphism of E_1 and E_2 in the sense of Lecture 2, §3. The isomorphism φ_E induces a linear isomorphism $\varphi_{E*} : \Gamma_c^\infty(M_1,E_1)\to\Gamma_c^\infty(M_2,E_2)$, given by $\varphi_{E*}f = \varphi_E\circ f\circ\varphi^{-1}$.

Likewise, let $\varphi_F : F_1 \to F_2$ be an isomorphism of vector bundles $F_j \to M_j$ which is compatible with φ. Then we have an induced linear isomorphism $\varphi_{F*} : \Gamma^\infty(M_1, F_1) \to \Gamma^\infty(M_2, F_2)$. Moreover, the map

$$\varphi_* : \mathrm{Hom}(\Gamma_c^\infty(M_1, E_1), \Gamma^\infty(M_1, F_1)) \to \mathrm{Hom}(\Gamma_c^\infty(M_2, E_2), \Gamma^\infty(M_2, F_2))$$

given by $\varphi_*(Q) = \varphi_{F*} \circ Q \circ \varphi_{E*}^{-1}$ restricts to a linear isomorphism

$$\varphi_* : \Psi^d(M_1, E_1, F_1) \xrightarrow{\simeq} \Psi^d(M_2, E_2, F_2).$$

We note that it also follows from the above definition that if E and F admit trivializations $\tau_E : E \to E' = U \times \mathbb{C}^k$ and $\tau_F : F \to F' = U \times \mathbb{C}^k$, respectively, then τ_* maps $\Psi^d(U, E, F)$ linearly isomorphically onto $\Psi^d(U, E', F') \simeq \mathrm{M}_{l,k}(\Psi^d(U))$. Indeed, by the previous remark and the remark below (8.1), it suffices to prove this for $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$. In this case

$$\mathrm{Hom}(\Gamma_c^\infty(E), C^\infty(F)) \simeq \mathrm{M}_{l,k}(\mathrm{Hom}(C_c^\infty(M), C^\infty(M))).$$

If $P \in \mathrm{Hom}(\Gamma_c^\infty(E), C^\infty(F))$, then $P \in \Psi^d(E, F)$ in the sense of Definition 8.1.1 if and only if all its components P_{ij} belong to $\Psi^d(M)$ in the sense of Definition 7.3.1.

Remark 8.1.2. The straightforward analogues of Exercise 7.3.4 and 7.3.5, Lemma 7.3.6, Exercise 7.3.7, Lemma 7.3.8, Exercise 7.3.10 are valid for operators from $\Psi^d(E, F)$, by reduction to trivial bundles and the scalar case, along the lines discussed above. We leave it to the reader to check the details.

It follows from Definition 8.1.1 and the corresponding fact for scalar operators that $\Psi^{-\infty}(E, F)$ equals the intersection of the spaces $\Psi^d(E, F)$, for $d \in \mathbb{R}$.

8.2. The principal symbol, vector bundle case

In this section we shall discuss the definition and basic properties of the principal symbol for a pseudo-differential operator between smooth complex vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$. We first concentrate on the definition of the appropriate symbol space.

Let $\pi_H : H \to M$ be a vector bundle. We agree to write

$$\Gamma^\infty(T^*M, H) = \{f \in C^\infty(T^*M, H) \mid \forall x \in M : \ f(T_x^*M) \subset H_x\}.$$

There is a natural identification of this space with the space of sections of the pull-back π^*H of the vector bundle H under the map $\pi : T^*M \to M$, so that $\Gamma^\infty(T^*M, H) \simeq \Gamma^\infty(T^*M, \pi^*H)$, but we shall not need this. If H is trivial of the form $H = M \times \mathbb{C}^N$, then the elements of $\Gamma^\infty(T^*M, H)$ are precisely the functions of the form $\xi_x \mapsto (x, f(\xi_x))$ with $f \in C^\infty(T^*M, \mathbb{C}^N)$. It follows that $\Gamma^\infty(T^*M, H) \simeq C^\infty(T^*M, \mathbb{C}^N)$. Accordingly, we define

$$S^d(M, H) := \{f \in \Gamma^\infty(T^*M, V) \mid \forall j : \ f_j \in S^d(M) \ \} \simeq S^d(M)^N.$$

Let $\tau : H_1 \to H_2$ be an isomorphism of two vector bundles on M. Then the map

$$\tau_* : \Gamma^\infty(T^*M, H_1) \to \Gamma^\infty(T^*M, H_2),$$

defined by

$$\tau_*(f)(\xi_x) = \tau_x(f(\xi_x)), \qquad (x \in M, \ \xi_x \in T_x^*M),$$

is a linear isomorphism. Assume now that τ is a bundle automorphism of the trivial bundle $H = M \times \mathbb{C}^N$. Then τ has the form $\tau(x, v) = (x, \tau_x(v))$, with $x \mapsto \tau_x$ a smooth map $M \to \mathrm{GL}\,(N, \mathbb{C})$. It follows that the linear automorphism τ_* of $\Gamma^\infty(T^*M, H) \simeq C^\infty(T^*M)^N$ is given by

$$\tau_*(f)(\xi_x) = \tau_x(f(\xi_x)).$$

It is readily checked that this map restricts to a linear automorphism of the symbol space $S^d(T^*M, H)$.

Now assume the bundle H is trivializable and let $\tau' : H \to H' = M \times \mathbb{C}^N$ be a trivialization. Then we define $S^d(M, H) := \tau_*^{-1} S^d(M, H')$. This definition is independent of the particular choice of the trivialization τ, in view of the preceding discussion.

Definition 8.2.1. Let $H \to M$ be a complex vector bundle. For $d \in \mathbb{R} \cup \{-\infty\}$, we define the symbol space $S^d(M, H)$ to be the space of sections $p \in \Gamma^\infty(T^*M, H)$ such that for every open neighborhood U on which H admits a trivialization, the restriction $p_U := p|_{T^*U}$ belongs to $S^d(U, H_U)$.

Clearly, for $p \in \Gamma^\infty(T^*M, H)$ to belong to $S^d(M, H)$ is suffices that for every $a \in M$ there exists an open trivializing neighborhood U such that $p_U \in S^d(U, H_U)$.

We also note that for $\varphi \in C^\infty(M)$ multiplication by $\pi^*\varphi \in C^\infty(T^*M)$ maps $S^d(T^*M, H)$ linear isomorphically to itself. Accordingly, $S^d(M, H)$ becomes a $C^\infty(M)$-module. It follows that the quotient

$$S^d/S^{d-1}(M, H) := S^d(M, H)/S^{d-1}(M, H)$$

is a $C^\infty(M)$-module as well.

Let $U \subset M$ be open and let $\mathcal{K} \subset U$ be compact. Then we write $S^d_{\mathcal{K}}(U, H_U)$ for the subspace of $S^d(U, H_U)$ consisting of p with support in $\mathcal{K}$ in the sense that $p_{U \setminus \mathcal{K}} = 0$. Equivalently, this means that the function $p : T^*U \to H_U$ vanishes on $\pi^*(U \setminus \mathcal{K})$. The extension of such a function to T^*M by the requirement $u(\xi_x) = 0_x \in H_x$ for every $\xi_x \in T^*M \setminus T^*U$ belongs to $S^d_{\mathcal{K}}(M, H)$. Accordingly, we have a linear injection

$$S^d_{\mathcal{K}}(U, H_U) \hookrightarrow S^d_{\mathcal{K}}(M, H).$$

Let $S^d_c(U, H_U)$ denote the union of the spaces $S^d_{\mathcal{K}}(U, H_U)$, for $\mathcal{K} \subset U$ compact. Then $S^d_c(U, H_U) \hookrightarrow S^d_c(M, H)$. Accordingly, we have an induced linear injection

$$S^d_c(U, H_U)/S^{d-1}_c(U, H_U) \hookrightarrow S^d_c(M, H)/S^{d-1}_c(M, H).$$

We will now see that the definition of the principal symbol map can be generalized to the context of bundles. Let E, F be two complex vector bundles on M. The principal symbol map associated with $\Psi^d(E, F)$ will be a map

$$\sigma^d : \Psi^d(M, E, F) \to S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)).$$

Here $\underline{\mathrm{Hom}}\,(E, F)$ is the vector bundle on M whose fiber at $x \in M$ is given by

$$\underline{\mathrm{Hom}}\,(E, F)_x = \mathrm{Hom}_{\mathbb{C}}(E_x, F_x).$$

If $U \subset M$ is an open subset on which both E and F admit trivializations $\tau_U : E_U \to U \times \mathbb{C}^k$ and $\tau_F : F_U \to U \times \mathbb{C}^l$, then the bundle $\underline{\mathrm{Hom}}\,(E,F)$ admits the trivialization

$$\tau : \underline{\mathrm{Hom}}\,(E,F)_U \to U \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^l)$$

given by

$$\tau_x(T) = (\tau_F)_x \circ T \circ (\tau_E)_x^{-1}.$$

Let $\varphi : E \to F$ be a vector bundle homomorphism. Then the map $\underline{\varphi} : x \mapsto \varphi_x \in \mathrm{Hom}(E_x, F_x)$ defines a smooth section of the bundle $\underline{\mathrm{Hom}}\,(E,F)$. Using trivializations we readily see that the map $\varphi \mapsto \underline{\varphi}$ defines a linear isomorphism

$$\mathrm{Hom}(E,F) \xrightarrow{\simeq} \Gamma^\infty(\underline{\mathrm{Hom}}\,(E,F)).$$

Initially we will give the definition of principal symbol for trivial bundles. Assume that $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$ so that $\underline{\mathrm{Hom}}\,(E,F) = M \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^l) \simeq \mathrm{M}_{l,k}(\mathbb{C})$. Then

$$S^d(M, \underline{\mathrm{Hom}}\,(E,F)) \simeq \mathrm{M}_{l,k}(S^d(M))$$

and, accordingly,

$$S^d(M, \underline{\mathrm{Hom}}\,(E,F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F)) \simeq \mathrm{M}_{l,k}(S^d(M)/S^{d-1}(M))$$

In this setting of trivial bundles, we define the principal symbol map $\sigma^d = \sigma^d_{E,F}$ component wise by

$$\sigma^d(P)_{ij} := \sigma^d(P_{ij}), \qquad (1 \le i \le l,\ 1 \le j \le k).$$

Assume now that τ_E and τ_F are automorphisms of the trivial bundles E and F, respectively and let τ be the induced automorphism of $\underline{\mathrm{Hom}}\,(E,F)$. We denote by τ_* the induced automorphisms of $\Psi^d(E,F)$ and of the quotient space $S^d(M, \underline{\mathrm{Hom}}\,(E,F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F))$.

Lemma 8.2.2. *For every $P \in \Psi^d(E,F)$,*

$$\sigma^d(\tau_*(P)) = \tau_*(\sigma^d(P)).$$

Proof We observe that for $f \in \Gamma^\infty(E) \simeq C^\infty(M)^k$ we have

$$(\tau_*(P)f)_i = \sum_{r,s,j} (\tau_{Fx})_{ir} P_{rs} (\tau_{Ex}^{-1})_{sj} f_j$$

so that by Lemma 7.5.1

$$\begin{aligned}
\sigma^d(\tau_*(P)_{ij}) &= \sum_{r,s} (\tau_F)_{ir} (\tau_E^{-1})_{sj} \sigma^d(P_{rs}) \\
&= \sum_{r,s} (\tau_F)_{ir} (\tau_E^{-1})_{sj} \sigma^d(P)_{rs} \\
&= (\tau_* \sigma^d(P))_{ij}.
\end{aligned}$$

This implies that $\sigma^d(\tau_*(P))_{ij} = \sigma^d(\tau_*(P)_{ij}) = (\tau_* \sigma^d(P))_{ij}$. $\square$

If E and F admit trivializations $\tau_E : E \to E' = M \times \mathbb{C}^k$ and $\tau_F : F \to F' = M \times \mathbb{C}^l$ we define the principal symbol map $\sigma^d_{E,F}$ on $\Psi^d(E,F)$ by requiring the following diagram to be commutative

$$\begin{array}{ccc} \Psi^d(E,F) & \xrightarrow{\tau_*} & \Psi^d(E',F') \\ \sigma^d_{E,F}\downarrow & & \downarrow \sigma^d_{E',F'} \\ S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F)) & \xrightarrow{\tau_*} & S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E',F')). \end{array} \tag{8.2}$$

Finally, we come to the case that $E \to M$ and $F \to M$ are arbitrary complex vector bundles of rank k and l respectively.

Lemma 8.2.3. *Let $P \in \Psi^d(E,F)$. Then there exists a unique*

$$\sigma^d(P) = \sigma^d_{E,F} \in S^d(M, \underline{\mathrm{Hom}}\,(E,F))/S^{d-1}(\underline{\mathrm{Hom}}\,(E,F))$$

such that for every open subset $U \subset M$ on which both E and F admit trivializations,

$$\sigma^d(P)_U = \sigma^d_{E_U,F_U}(P_U). \tag{8.3}$$

Proof Uniqueness is obvious. We will establish existence. Let $\{U_j\}$ be an open cover of M consisting of open subsets on which both E and F admit trivializations. We may assume that $\{U_j\}$ is locally finite and that $\{\psi_j\}$ is a partition of unity on M with $\operatorname{supp}\psi_j \subset U_j$ for all j. Given $P \in \Psi^d(E,F)$ we define $\sigma^d(P)$ by

$$\sigma^d(P) = \sum_j \psi_j \sigma^d_{E_{U_j},F_{U_j}}(P_{U_j}).$$

As this is a locally finite sum, it defines an element of $S^d(M,H)/S^{d-1}(M,H)$. It remains to verify (8.3) for an open subset U on which E and F admit trivializations τ_U. With $\sigma^d(P)$ as just defined we have

$$\begin{aligned} \sigma^d(P)_U &= \sum_j \psi_j|_U\, \sigma^d_{E_{U_j},F_{U_j}}(P_{U_j})_{U\cap U_j} \\ &= \sum_j \psi_j|_U\, \sigma^d_{E_U,F_U}(P_U)_{U\cap U_j} \\ &= \sum_j \psi_j|_U\, \sigma^d_{E_U,F_U}(P_U) \\ &= \sigma^d_{E_U,F_U}\big(\sum_j M_{\psi_j|U} \circ P_U\big) \\ &= \sigma^d_{E_U,F_U}(P_U). \end{aligned}$$

$\square$

Definition 8.2.4. Let $P \in \Psi^d(E,F)$. The d-th order principal symbol of P is defined to be the unique element $\sigma^d(P) \in S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E,F))$ satisfying the properties of Lemma 8.2.3.

Obviously, $P \mapsto \sigma^d(P)$ is a linear map. As should be expected, it follows from the above definition that the principal symbol map behaves well under bundle isomorphisms. Consider isomorphisms $\tau_E : E_1 \to E_2$ and $\tau_F : F_1 \to F_2$

of vector bundles on M. Then the definitions have been given in such a way that the following diagram commutes

$$
\begin{array}{ccc}
\Psi^d(M, E_1, F_1) & \xrightarrow{\sigma^d} & S^d(M, \underline{\mathrm{Hom}}\,(E_1, F_1))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E_1, F_1)) \\
\tau_* \downarrow & & \downarrow \tau_* \\
\Psi^d(M, E_2, F_2) & \xrightarrow{\sigma^d} & S^d(M, \underline{\mathrm{Hom}}\,(E_2, F_2))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E_2, F_2))
\end{array}
\tag{8.4}
$$

The local version of this result is true because of the local requirement (8.2). The global validity follows by the uniqueness part of the characterization of the symbol map in Lemma 8.2.3.

Lemma 8.2.5. *Let $\psi, \chi \in C^\infty(M)$. Then, for all $P \in \Psi^d(E, F)$,*

$$\sigma^d(M_\psi \circ P \circ M_\chi) = \psi\chi\sigma^d(P).$$

Proof For trivializable bundles E and F the result is a straightforward consequence of the analogous result in the scalar case. Let U be any open subset of M on which both E and F admit trivializations. Then

$$
\begin{aligned}
\sigma^d(M_\psi \circ P \circ M_\chi)_U &= \sigma^d_U(M_{\psi|U} \circ P_U \circ M_{\chi|U}) \\
&= \big(\psi|_U \chi|_U \sigma^d_U(P_U)\big) \\
&= \big(\psi\chi\sigma^d(P)\big)_U.
\end{aligned}
$$

The result follows. □

Theorem 8.2.6. *The principal symbol map σ^d induces a linear isomorphism*

$$\Psi^d(E, F)/\Psi^{d-1}(E, F) \xrightarrow{\simeq} S^d(M, \underline{\mathrm{Hom}}\,(E, F))/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)).$$

Proof If E, F are trivial, then the result is an immediate consequence of the analogous result in the scalar case. If E, F are trivializable, the result is still true in view of the commutativity of the diagram (8.4). Let now E, F be arbitrary complex vector bundles on M and put $H = \underline{\mathrm{Hom}}\,(E, F)$. We must show that the principal symbol map $\sigma^d : \Psi^d(E, F) \to S^d(M, H)/S^{d-1}(M, H)$ has kernel $\Psi^{d-1}(E, F)$ and is surjective. Let $P \in \Psi^d(E, F)$, then $\sigma^d(P) = 0$ if and only if for every open subset $U \subset M$ on which both E and F admit a trivialization, $\sigma^d(P)_U = 0$. The latter condition is equivalent to $\sigma^d_U(P_U) = 0$, hence by the first part of the proof to $P_U \in \Psi^{d-1}(E_U, F_U)$. It follows that $\ker \sigma^d = \Psi^{d-1}(E, F)$.

To establish the surjectivity, let $p \in S^d(M, H)$ and let $[p]$ denote its class in the quotient $S^d(M, H)/S^{d-1}(M, H)$. Let $\{U_j\}$ be an open cover of M such that both E and F admit trivializations over U_j, for all j. We may choose the covering such that there exists a partition of one, $\{\psi_j\}$, with $\mathrm{supp}\,\psi_j \subset U_j$ for all j. By the first part of the proof, there exists for each j a pseudo-differential operator $P_j \in \Psi^d(E_{U_j}, F_{U_j})$, such that

$$\sigma^d(P_j) = [p]_{U_j} \in S^d(U_j, H_{U_j})/S^{d-1}(U_j, H_{U_j}).$$

For each j we fix $\chi_j \in C^\infty_c(U_j)$ such that $\chi_j = 1$ on $\mathrm{supp}\,\psi_j$. Then $M_{\psi_j} \circ P_j \circ M_{\chi_j}$ is a pseudo-differential operator in $\Psi^d(E, F)$ with distribution kernel supported by $\operatorname{supp}\psi_j \times \operatorname{supp}\chi_j$. It follows that the distribution kernels are locally finitely supported. Hence

$$P := \sum_j M_{\psi_j} \circ P_j \circ M_{\chi_j}$$

is a well-defined pseudo-differential operator in $\psi^d(E, F)$. Let U be any relatively compact open subset of M on which both E and F admit trivializations, then

$$\sigma^d(P)_U = \sum_j \left(\psi_j \sigma^d(P_j)\right)_U = \sum_j \psi_j|_U [p]_U = [p]_U,$$

with only finitely many terms of the sums different from zero. It follows that $\sigma^d(P) = [p]$. We have established the surjectivity of the principal symbol map. □

8.3. Symbol of adjoint and composition

We now turn to the behavior of the principal symbol when passing to adjoints. Let $E \to M$ and $F \to M$ be complex vector bundles on M of rank k and l, respectively. Let E^* and F^* be the dual bundles of E and F respectively. We recall that $E^\vee := E^* \otimes D_M$ and $F^\vee = F^* \otimes D_M$, with D_M the density bundle on M.

Lemma 8.3.1. *Let V, W be finite dimensional complex linear spaces, and let L be a one-dimensional complex linear space. Then the map $T \mapsto T \otimes \mathrm{I}_L$ defines a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \simeq \mathrm{Hom}_{\mathbb{C}}(V \otimes L, W \otimes L).$$

Proof Straightforward. □

Corollary 8.3.2. *The map*

$$\underline{\mathrm{Hom}}\,(F^*, E^*)_x \ni T_x \mapsto T_x \otimes \mathrm{I}_{D_{Mx}} \in \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)_x$$

defines a natural isomorphism of vector bundles.

Given $p \in S^d(M, \underline{\mathrm{Hom}}\,(E, F))$, we define $p^\vee : T^*M \to \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)$ by

$$p^\vee(\xi_x) = p(-\xi_x)^* \otimes \mathrm{I}_{D_{Mx}},$$

for $x \in M$ and $\xi_x \in T^*_x M$. Then clearly, $p^\vee \in S^d(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee))$. The map $p \to p^\vee$ is readily seen to define a linear isomorphism

$$S^d(M, \underline{\mathrm{Hom}}\,(E, F)) \to S^d(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee),$$

for every $d \in \mathbb{R} \cup \{-\infty\}$. Moreover, $p^{\vee\vee} = p$ for all p. Accordingly, we have an induced linear isomorphism

$$S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F)) \xrightarrow{\simeq} S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(F^\vee, E^\vee)),$$

denoted by $\sigma \mapsto \sigma^\vee$.

Let $P \in \Psi^d(E, F)$. As P is a continuous linear operator $\Gamma^\infty_c(E) \to \Gamma^\infty(F)$, its adjoint P^t is a continuous linear operator from the topological linear dual

$\Gamma^\infty(F)' = \mathcal{E}'(F^\vee)$ to the topological linear dual $\Gamma_c^\infty(E)' = \mathcal{D}'(E^\vee)$. We recall that the natural continuous bilinear pairing $\Gamma^\infty(E^\vee) \times \Gamma_c^\infty(E) \to \mathbb{C}$ defined by

$$\langle f, g\rangle = \int_M (f, g)$$

induces a natural continuous linear embedding $\Gamma^\infty(E^\vee) \hookrightarrow \mathcal{D}'(E^\vee)$. Likewise, we have a natural continuous linear embedding $\Gamma_c^\infty(F^\vee) \hookrightarrow \mathcal{E}'(F^\vee)$.

Lemma 8.3.3. *Let $P \in \Psi^d(E, F)$. The adjoint P^t restricts to a continuous linear map $\Gamma_c^\infty(F^\vee) \to \Gamma^\infty(E^\vee)$. The restricted map is a pseudo-differential operator in $\Psi^d(F^\vee, E^\vee)$ with principal symbol given by*

$$\sigma^d(P^t) = \sigma^d(P)^\vee.$$

Proof Let $U \subset M$ be an open subset. Then it is readily seen from the definitions that $(P^t)_U = (P_U)^t$ and that $(\sigma^d(P)^\vee)_U = (\sigma^d(P)_U)^\vee = \sigma^d(P_U)^\vee$. Therefore the result is of a local nature, and we may as well assume that E and F admit trivializations on M. Let $e_1, \ldots, e_k$ be a frame for E and let $f_1, \ldots, f_l$ be a frame for F. Let $e^1, \ldots, e^k$ be the dual frame for the dual bundle E^*; i.e., $(e^i, e_j) = \delta_{ij}$, for all $1 \le i, j \le k$. Similarly, let $f^1, \ldots, f^l$ be the dual frame for the dual bundle F^*. Let dm be choice of smooth positive density on M, then $\{dm\}$ constitutes a frame for the density bundle D_M. It follows that $e^1 dm, \ldots, e^k dm$ is a frame for $E^\vee$ and $f^1 dm, \ldots, f^l dm$ a frame for $F^\vee$.

Let $P \in \Psi^d(E, F)$. The operator P has components P_{ij} relative to the frames $\{e_j\}$ and $\{f_i\}$. Given $\varphi, \psi \in C_c^\infty(M)$, we have

$$\psi P_{ij}\varphi = (\psi f^i, P(\varphi e_j)).$$

The components of the adjoint operator P^t are given by

$$\varphi (P^t)_{ji}(\psi) dm = (\varphi e^j, P^t(\psi f^i dm)).$$

Integrating the densities on both sides of the equality over M we find that

$$\langle \varphi dm, (P^t)_{ji}\psi\rangle = \langle P(\varphi e^j), \psi f^i dm\rangle = \langle P_{ij}(\varphi), \psi dm\rangle.$$

This implies that $(P^t)_{ji}$ equals the adjoint of P_{ij} in the sense of Lemma 7.5.2. Hence, $P^t \in \Psi^d(F^\vee, E^\vee)$ and

$$\sigma^d(P^t)_{ji}(\xi_x) = \sigma^d(P)_{ij}(-\xi_x) = \big(\sigma^d(P)^\vee(\xi_x)\big)_{ji}.$$

The result follows. $\square$

Let $P \in \Psi^d(E, F)$. Then it follows by application of the lemma above that P extends to a continuous linear map $\mathcal{E}'(E) \to \mathcal{D}'(F)$. Indeed, the extension equals the adjoint of the map $P^t : \Gamma_c^\infty(F^\vee) \to \Gamma^\infty(E^\vee)$. The extension is unique by density of $\Gamma_c^\infty(E)$ in $\mathcal{E}'(E)$.

It follows from the definitions that if the distributional kernel K_P has support $S \subset M \times M$, then the distributional kernel of the adjoint operator P^t has support $S^t = \{(y, x) \in M \times M \mid (x, y) \in S\}$. In analogy with the scalar case, the operator P is said to be *properly supported* if the restricted projection maps $\mathrm{pr}_j|_S : S \to M$ are proper, for $j = 1, 2$. Thus, if P is properly supported, then so is P^t. In this case P^t maps $\Gamma_c^\infty(F^\vee)$ continuous linearly into $\Gamma_c^\infty(E^\vee)$, and we see that P extends to a continuous linear operator $\mathcal{D}'(E) \to \mathcal{D}'(F)$.

Lemma 8.3.4. *Let $P \in \Psi^d(E, F)$ be properly supported. Then the continuous linear operator $P : \mathcal{D}'(E) \to \mathcal{D}'(F)$ is pseudo-local, i.e., for all $u \in \mathcal{D}'(E)$ we have*

$$\operatorname{singsupp}(Pu) \subset \operatorname{singsupp} u.$$

Proof Let $a \in M \setminus \operatorname{supp} u$. Then there exists an open neighborhood $\mathcal{O} \ni a$ with $\mathcal{O} \cap \operatorname{supp} u = \emptyset$. Let $\psi \in C_c^\infty(\mathcal{O})$ be equal to 1 on a neighborhood of a. By paracompactness of $\operatorname{supp} u$ there exists a smooth functions $\chi \in C^\infty(M)$ such that $\chi = 1$ on a neighborhood of $\operatorname{supp} u$ and such that $\operatorname{supp} \psi \cap \operatorname{supp} \chi = \emptyset$. It follows that $T := M_\psi \circ P \circ M_\chi$ is a properly supported smoothing operator. Hence $\psi P u = \psi P(\chi u) = Tu$ is smooth. It follows that Pu is smooth in a neighborhood of a. $\square$

As in the scalar case, modulo a smoothing operator each pseudo-differential operator can be represented by a properly supported one.

Lemma 8.3.5. *Let $\Omega \subset M \times M$ be an open neighborhood of the diagonal. Then for each $P \in \Psi^d(E, F)$ there exists a properly supported $P_0 \in \Psi^d(E, F)$ with $\operatorname{supp} K_{P_0} \subset \Omega$ such that $P - P_0 \in \Psi^{-\infty}(E, F)$.*

Proof The proof is an obvious adaptation of the proof of Lemma 6.1.6. By Lemma 6.1.7 there exists a locally finite open covering $\{U_j\}_{j \in J}$ of M such that for all $i, j \in J$, $U_i \cap U_j \neq \emptyset \Rightarrow U_i \times U_j \subset \Omega$. There exists a partition of unity $\{\psi_j\}$ with $\psi_j \in C_c^\infty(U_j)$. For each j we choose a $\chi_j \in C_c^\infty(U_j)$ which equals 1 on an open neighborhood of $\operatorname{supp} \psi_j$. We now define

$$P_0 = \sum_{j \in J} M_{\psi_j} \circ P \circ M_{\chi_j}.$$

The j-th term in the above sum is a pseudo-differential operator of order d with distribution kernel supported in $\operatorname{supp} \psi_j \times \operatorname{supp} \chi_j$. As this is a locally finite collection of sets, it follows that $P_0 \in \Psi^d(E, F)$. Moreover, the distribution kernel of P_0 has support contained in the union of the sets $U_j \times U_j$ which is contained in Ω.

Since $\operatorname{supp} \psi_j \cap \operatorname{supp}(1 - \chi_j) = \emptyset$, the operator

$$T_j := M_{\psi_j} \circ P \circ M_{1 - \chi_j}$$

is a smooth kernel operator, with smooth kernel supported inside the set $U_j \times M$. Since these sets form a locally finite collection in $M \times M$, the sum $T = \sum_j T_j$ is a well defined smoothing operator in $\Psi^{-\infty}(E, F)$. It is now readily checked that $P - P_0 = T$. $\square$

We end this section with a discussion of the composition of two properly supported pseudo-differential operators. To prepare for this, we will first study the product of bundle-valued symbols. Let E_1, E_2, E_3 be complex vector bundles on M. Given $p \in S^d(M, \underline{\mathrm{Hom}}(E_1, E_2)$ and $q \in S^e(M, \underline{\mathrm{Hom}}(E_2, E_3)$ we define $qp : T^*M \to \mathrm{Hom}(E_1, E_3)$ by

$$qp : T_x^*M \ni \xi_x \mapsto q(\xi_x) \circ_x p(\xi_x) \in \underline{\mathrm{Hom}}(E_1, E_3)_x,$$

where $\circ_x$ denotes the composition map from $\underline{\mathrm{Hom}}(E_1, E_2)_x \times \underline{\mathrm{Hom}}(E_2, E_3)_x$ to $\underline{\mathrm{Hom}}(E_1, E_3)_x$.

Lemma 8.3.6. *The assignment* $(p, q) \mapsto qp$ *defines a bilinear map*

$$S^d(M, \underline{\mathrm{Hom}}\,(E_1, E_2)) \times S^e(M, \underline{\mathrm{Hom}}\,(E_2, E_3)) \to S^{d+e}(M, \underline{\mathrm{Hom}}\,(E_1, E_3)).$$

Proof The bilinearity of the assignment as a map into $\Gamma^\infty(T^*M, \underline{\mathrm{Hom}}\,(E_1, E_3))$ is obvious. We will prove the remaining assertion that the assignment has image contained in S^{d+e}.

If U is an open subset of M, then for $p \in S^d(M, \underline{\mathrm{Hom}}\,(E_1, E_2))$ and $q \in S^e(M, \underline{\mathrm{Hom}}\,(E_2, E_3))$ we have $(qp)_U = q_U p_U$. Therefore, the result is of a local nature, and we may as well assume that each E_j admits a trivialization, for $j = 1, 2, 3$. For each such j, let $e_{j1}, \ldots, e_{jk_j}$ be a frame for E_j. Then the symbol p has components $p_{\beta\alpha} \in S^d(M)$ given by

$$p(\xi_x)(e_{1\alpha}(x)) = \sum_{\beta=1}^{k_2} p_{\beta\alpha}(\xi_x) e_{2\beta}(x)$$

for all $x \in M$ and $\xi_x \in T_x^*M$. Likewise, the symbol q has components $q_{\gamma\beta} \in S^e(M)$ given by

$$q(\xi_x)(e_{2\beta}(x)) = \sum_{\gamma=1}^{k_3} q_{\gamma\beta}(\xi_x) e_{3\gamma}(x).$$

It follows that the γ-component of $q(\xi_x)p(\xi_x)(e_{1\alpha}(x))$ relative to the basis $\{e_{3\gamma}(x)\}$ of E_{3x} is given by

$$(qp)_{\gamma\alpha} = \sum_{\beta=1}^{k_2} q_{\gamma\beta} p_{\beta\alpha}.$$

This shows that $qp \in S^{d+e}(M, \underline{\mathrm{Hom}}\,(E_1, E_3))$. $\square$

It follow from this lemma that the product map induces a bilinear map

$$\begin{aligned} S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E_1, E_2)) \times S^e/S^{e-1}(M, \underline{\mathrm{Hom}}\,(E_2, E_3)) \\ \longrightarrow \quad S^{d+e}/S^{d+e-1}(M, \underline{\mathrm{Hom}}\,(E_1, E_3)) \end{aligned}$$

denoted $(\sigma_1, \sigma_2) \mapsto \sigma_2\sigma_1$.

We now turn to the composition of pseudo-differential operators. If $P \in \Psi^d(E_1, E_2)$ is a properly supported pseudo-differential operator then P maps $\Gamma_c^\infty(E_1)$ continuous linearly to $\Gamma_c^\infty(E_2)$. Thus, if $Q \in \Psi^e(E_2, E_3)$ then the composition $Q \circ P$ is a well-defined continuous linear operator $\Gamma_c^\infty(E_1) \to \Gamma^\infty(E_3)$.

Theorem 8.3.7. *Let* $P \in \Psi^d(E_1, E_2)$ *and* $Q \in \Psi^e(E_2, E_3)$ *be properly supported. Then the composition* $Q \circ P$ *is a properly supported pseudo-differential operator in* $\Psi^{d+e}(E_1, E_3)$ *with principal symbol given by*

$$\sigma^{d+e}(Q \circ P) = \sigma^e(Q)\sigma^d(P). \tag{8.5}$$

Proof We first assume that $d = e = -\infty$ so that both P and Q are smoothing operators and will show that $Q \circ P$ is a smoothing operator. For this it suffices to be shown that the kernel of $Q \circ P$ is smooth at each point $(a, b) \in M \times M$. Let U and W be relatively compact open neighborhoods of a and b on which both E and F admit trivializations. Let $\chi \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of a and let $\chi' \in C_c^\infty(W)$ be equal to 1 on an open neighborhood of b. Then the kernel of $M_{\chi'} \circ Q \circ P \circ M_\chi$ equals the kernel of $Q \circ P$ on an open neighborhood of (b, a), so that it suffices to show that $M_{\chi'} \circ Q \circ P \circ M_\chi$ is a smoothing operator.

Let A be a compact subset of M such that $\operatorname{supp} K_P \subset A \times U$, and $\operatorname{supp} K_Q \subset W \times A$. Let $\{V_j\}$ be a finite cover of A by open subsets of M on which each of the bundles E, F admits a trivialization. Let $\psi_j \in C_c^\infty(V_j)$ be functions such that $\sum_j \psi_j = 1$ on an open neighborhood of A. For each j, let $\psi_j' \in C_c^\infty(V_j)$ be such that $\psi_j' = 1$ on an open neighborhood of $\operatorname{supp} \psi_j$. Then

$$M_{\chi'} \circ Q \circ P \circ M_\chi = \sum_j Q_j \circ P_j,$$

where $Q_j = M_{\chi'} \circ Q \circ M_{\psi_j'}$ and $P_j = M_{\psi_j} \circ P \circ M_\chi$. It suffices to show that each of the operators $Q_j \circ P_j$ is smoothing. Fix j. Let $e_{11}, \ldots, e_{1k_1}$ be a frame of E_1 on U, $e_{21}, \ldots, e_{2k_2}$ a frame of E_2 on V_j and $e_{31}, \ldots, e_{3k_3}$ a frame of E_3 on W. Let $P_{j\beta\alpha}$ be the components of $P_j : \Gamma_c^\infty(U, E_1) \to \Gamma_c^\infty(V_j, E_2)$ relative to the first two frames, and let $Q_{j\gamma\beta}$ be the components of $Q_j : \Gamma_c^\infty(V_j, E_2) \to \Gamma_c^\infty(W, E_3)$ relative to the second pair of frames. These components are scalar smoothing operators. The components of $Q_j \circ P_j : \Gamma_c^\infty(U) \to \Gamma_c^\infty(W)$ are given by

$$(Q_j \circ P_j)_{\gamma\alpha} = \sum_\beta Q_{j\gamma\beta} \circ P_{j\beta\alpha}.$$

As all operators in this sum are smoothing, it follows that $Q_j \circ P_j : \Gamma_c^\infty(U, E_1) \to \Gamma_c^\infty(W, E_3)$ is smoothing. As $Q_j \circ P_j$ vanishes on the complement of $\operatorname{supp} \chi$ and has image contained in $C_c^\infty(W)$, it follows that $Q_j \circ P_j$ is a smoothing operator.

We now assume that $d \in \mathbb{R}$ and $e = -\infty$ and will show that $Q \circ P$ is smoothing. Let $U \subset M$ be a relatively compact open subset on which each of the bundles E_j, for $j = 1, 2, 3$ admits a trivialization. Let $\chi \in C_c^\infty(U)$ then it suffices to show that $M_\chi \circ Q \circ P$ is smoothing. Let $\psi \in C_c^\infty(U)$ be such that $\psi = 1$ on an open neighborhood of $\operatorname{supp} \chi$. Then $M_\chi \circ Q$ differs from $M_\chi \circ Q \circ M_\psi$ by a smoothing operator, hence, by the first part of the proof it suffices to show that $M_\chi \circ Q \circ M_\psi \circ P$ is smoothing. Let $\chi' \in C_c^\infty(U)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp} \chi$. Then $M_\chi \circ Q \circ M_\psi \circ P = Q_0 \circ P_0$, where $Q_0 = M_\chi \circ Q \circ M_\psi$ and $P_0 = M_{\chi'} \circ P$. As P is properly supported, there exists a compact subset $B \subset M$ such that $\operatorname{supp} K_P \cap U \subset M \subset U \times B$. Let $\{V_j\}$ be a finite open cover of B such that the bundle E_1 admits a trivialization on each of the sets V_j. Let $\psi_j \in C_c^\infty(V_j)$ be such that $\sum_j \psi_j = 1$ on an open neighborhood of B. Then $P_0 = \sum_j P_j$, where $P_j = P_0 \circ M_{\psi_j}$. It suffices to show that each operator $Q_0 \circ P_j$ is smoothing. Fix j, let $e_{11}, \ldots, e_{1,k_1}$ be a frame of E_1 on V_j, let $e_{21}, \ldots e_{2k_2}$ be a frame of E_2 on U and $e_{31}, \ldots, e_{3k_3}$ a frame of E_3 on U. Then in terms of components of the operators $Q_0 : \Gamma_c^\infty(U, E_2) \to \Gamma_c^\infty(U, E_3)$ and $P_j : \Gamma_c^\infty(V_j, E_1) \to \Gamma_c^\infty(U, E_2)$ the operator $Q_0 \circ P_j : \Gamma_c^\infty(V_j, E_1) \to \Gamma_c^\infty(U, E_3)$ has components given by

$$(Q_0 \circ P_j)_{\gamma\alpha} = \sum_{\beta=1}^{k_2} (Q_0)_{\gamma\beta} \circ (P_j)_{\beta\alpha} \tag{8.6}$$

The operators $(P_j)_{\beta\alpha}$ are smoothing with kernels whose compact supports are contained in $U \times V_j$. Extending these kernels with value zero outside their supports, we obtain kernels on $M \times M$ such that the identities (8.6) still hold for the associated scalar operators. As $(Q_0)_{\gamma\beta} \in \Psi^e(M)$ for all γ, β and $(P_j)_{\beta\alpha} \in \Psi^{-\infty}(M)$ for all α, β, it follows from Theorem 7.5.3 that all components of $Q_0 \circ P_j$ are smoothing operators. Hence, $Q_0 \circ P$ is smoothing. We conclude that $Q \circ P$ is smoothing.

Likewise, if $e = -\infty$ and $d \in \mathbb{R}$, then $Q \circ P$ is a smoothing operator. To see this we may either imitate the argument in the previous part of the proof, or combine the result of that part with Lemma 8.3.3.

Finally, we discuss the case that e, d are arbitrary. We will show that $Q \circ P \in \Psi^{d+e}(E_1, E_3)$. Let U be an open subset of M on which each of the bundles E_j, for $j = 1, 2, 3$ trivializes. Let $\chi \in C_c^\infty(U)$; then it suffices to show that $M_\chi \circ Q \circ P \in \Psi^{d+e}$. Let $\chi' \in C_c^\infty(U)$ be such that $\chi' = 1$ on $\operatorname{supp} \chi$. Then $M_\chi \circ Q$ equals $Q_0 := M_\chi \circ Q \circ M_{\chi'}$ modulo a smoothing operator, hence by the first part of the proof it suffices to show that $Q_0 \circ P$ belongs to Ψ^{d+e}. The latter operator equals $Q_0 \circ M_\psi \circ P$ for $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on an open neighborhood of $\operatorname{supp} \chi'$. Let $\psi' \in C_c^\infty(U)$ be equal to 1 on an open neighborhood of $\operatorname{supp} \psi$, then $M_\psi \circ P$ equals $P_0 := M_\psi \circ P \circ M_{\psi'}$ modulo a smoothing operator, hence it suffices to show that $Q_0 \circ P_0 \in \Psi^{d+e}(E_1, E_3)$. As the supports of the kernels of Q_0 and P_0 are contained in $U \times U$, it suffices to show that $(Q_0)_U \circ (P_0)_U \in \Psi^{d+e}((E_1)_U, (E_3)_U)$.

Thus, to show that $Q \circ P$ belongs to $\Psi^{d+e}(E_1, E_3)$ we may as well assume that E_1, E_2, E_3 are trivial on M from the start. In this situation, $\Psi^d(E_1, E_2) \simeq M_{k_2,k_1}(\Psi^d(M))$ and $\Psi^e(E_2, E_3) \simeq M_{k_3k_2}(\Psi^e(M))$. Moreover, the composition $Q \circ P$ has components

$$(Q \circ P)_{\gamma\alpha} = \sum_\beta Q_{\gamma\beta} \circ P_{\beta\alpha}.$$

It follows by application of Theorem 7.5.3 that these components belong to $\Psi^{d+e}(M)$ and have principal symbols given by

$$\sigma^{d+e}((Q \circ P)_{\gamma\alpha}) = \sum_\beta \sigma^e(Q_{\gamma\beta}) \circ \sigma^d(P_{\beta\alpha}).$$

It follows that

$$\sigma^{d+e}(Q \circ P)_{\gamma\alpha} = \big(\sigma^e(Q)\sigma^d(P)\big)_{\gamma\alpha}$$

whence (8.5).

We now assume to be in the general situation again. It remains to be shown that $R := Q \circ P$ is properly supported. Let $A \subset M$ be compact. Then there exists a compact subset $B \subset M$ such that $\operatorname{supp} K_P \cap (M \times A) \subset B \times A$. There exists a compact subset $C \subset M$ such that $\operatorname{supp} K_Q \cap (M \times B) \subset C \times B$. It now easily follows that $\operatorname{supp} K_R \cap (M \times A) \subset C \times A$. Thus, $\mathrm{pr}_2|_{\operatorname{supp} K_R} : \operatorname{supp} K_R \to M$ is proper. The properness of $\mathrm{pr}_1|_{\operatorname{supp} K_R}$ is established in a similar way. $\square$

Exercise 8.3.8. Let $P \in \Psi^d(E, F)$. Show that the following two assertions are equivalent.

(a) The operator P is properly supported.
(b) The operator P maps $\Gamma_c^\infty(E)$ continuous linearly to $\Gamma_c^\infty(F)$ and its adjoint P^t maps $\Gamma_c^\infty(F^\vee)$ continuous linearly to $\Gamma_c^\infty(E^\vee)$.

8.4. Elliptic operators, parametrices

We assume that $E \to M$ and $F \to M$ are complex vector bundles on M of rank k and l, respectively.

The symbol space $S^0(M, \underline{\mathrm{End}}\,(E))$ has a distinguished element 1_E given by

$$1_E : T_x^*M \ni \xi_x \mapsto \mathrm{I}_{E_x} \in \underline{\mathrm{End}}\,(E)_x, \qquad (x \in M).$$

Likewise, $S^0(M, \underline{\mathrm{End}}\,(F))$ has a distinguished element 1_F.

A symbol $p \in S^d(M, \underline{\mathrm{Hom}}\,(E, F))$ is said to be *elliptic* (of order d) if there exists a symbol $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F, E)$ such that

$$pq - 1_F \in S^{-1}(M, \underline{\mathrm{End}}\,(F)), \quad \text{and} \quad qp - 1_E \in S^{-1}(M, \underline{\mathrm{End}}\,(E)).$$

For the classes $[p] \in S^d/S^{d-1}$ and $[q] \in S^{-d}/S^{-d-1}$ this means precisely that

$$[p][q] = [1_F], \quad \text{and} \quad [q][p] = [1_E].$$

Thus, the notion of ellipticity factors to the quotient space S^d/S^{d-1}.

Definition 8.4.1. A pseudo-differential operator $P \in \Psi^d(E, F)$ is said to be elliptic if its principal symbol $\sigma^d(P) \in S^d/S^{d-1}(M, \underline{\mathrm{Hom}}\,(E, F))$ is elliptic.

The notion of ellipticity of a pseudo-differential operator generalizes the similar notion for a differential operator.

Lemma 8.4.2. *Let $d \in \mathbb{N}$ and let P be a differential operator of order d from E to F. Then the following assertions are equivalent.*

(a) *P is elliptic as differential operator;*
(b) *P is elliptic as a pseudo-differential operator in $\Psi^d(E, F)$.*

Proof Let p be the principal symbol of P as a differential operator. Then $p \in \Gamma^\infty(T^*M, \underline{\mathrm{Hom}}\,(E, F))$ and for each $x \in M$, the map $\xi \mapsto p(x, \xi)$ is a $\mathrm{Hom}(E_x, F_x)$-valued polynomial function on T_xM^* that is homogeneous of degree d. By using local trivializations of E and F one sees that p is a symbol in $S^d(M, \underline{\mathrm{Hom}}\,(E, F))$, and that its class in S^d/S^{d-1} is the principal symbol of P in the sense of pseudo-differential operators.

Now assume (a). This means that $p(x, \xi_x)$ is an invertible homomorphism $E_x \to F_x$ for every $x \in M, \xi_x \in T_x^*M \setminus \{0\}$. Let $\chi : T^*M \to \mathbb{R}$ be a smooth function such that $\pi : \operatorname{supp} \chi \to M$ is proper and such that $\chi = 1$ on a neighborhood of M; as usual we identify M with the image of the zero section in T^*M. The existence of such a function can be established locally (relative to M) by using an atlas of M, and globally by using a partition of unity subordinate to this atlas.

We define the smooth function $q \in \Gamma^\infty(T^*M, \underline{\mathrm{Hom}}\,(F, E))$ by

$$q(\xi_x) := (1 - \chi(\xi_x))p(\xi_x)^{-1}, \qquad (x \in M,\, \xi_x \in T_x^*M).$$

By a local analysis we readily see that $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F,E))$. Moreover, from the definition of q it follows that

$$q(\xi_x)p(\xi_x) - I_{E_x} = \chi(\xi_x)I_{E_x}, \quad \text{and} \quad p(\xi_x)q(\xi_x) - I_{F_x} = \chi(\xi_x)I_{F_x}.$$

This implies that $qp - 1_E \in S^{-\infty}(E,E)$ and $pq - 1_F \in S^{-\infty}(F,F)$. Hence (b) follows.

Conversely, assume that (b) holds. Let p be the principal symbol of the differential operator P introduced above. Then $[p] = \sigma^d(P)$ in the sense of pseudo-differential operators, hence there exists a $q \in S^{-d}(M, \underline{\mathrm{Hom}}\,(F,E))$ such that $[q][p] = [1_E]$ and $[p][q] = [1_F]$. It follows that there exists a $r \in S^{-1}(M, \underline{\mathrm{End}}\,(E))$ such that

$$q(\xi_x)p(\xi_x) = \mathrm{I}_{E_x} + r(\xi_x), \qquad (x \in M,\ \xi_x \in T_x^*M).$$

Fix $x \in M$, and choose norms on the finite dimensional spaces T_x^*M and $\mathrm{End}(E_x)$. Then it follows that

$$q(\xi_x)p(\xi_x) - \mathrm{I}_{E_x} = \mathcal{O}(1 + \|\xi_x\|)^{-1} \qquad (\|\xi_x\| \to \infty).$$

This implies that $\det(q(\xi_x)p(\xi_x)) \to 1$ for $\|\xi_x\| \to \infty$. Hence, $p(\xi_x)$ is an invertible element of $\mathrm{Hom}(E_x, F_x)$ for $\|\xi_x\|$ sufficiently large. By homogeneity of $p|_{T_x^*M}$ this implies that $p(\xi_x)$ is invertible for all $\xi \in T_x^*M \setminus \{0\}$. As this holds for any $x \in M$, the operator P is elliptic as a differential operator. $\square$

Corollary 8.4.3. *Let $P \in \Psi^d(E,F)$ be properly supported and elliptic. Then there exists a properly supported $Q \in \Psi^{-d}(F,E)$ such that $QP - I \in \Psi^{-1}(E,E)$ and $PQ - I \in \Psi^{-1}(F,F)$.*

Proof By Definition 8.4.1 there exists a $q \in S^{-d}/S^{-d-1}(M, \underline{\mathrm{Hom}}\,(F,E))$ such that $\sigma^d(P)q = [1_E]$ and $\sigma^d(p)q = [1_F]$. By Theorem 8.2.6 there exists a $Q \in \Psi^{-d}(F,E)$ with $\sigma^{-d}(Q) = q$. By Lemma 8.3.5 there exists such a Q such that in addition Q is properly supported. It now follows from Theorem 8.3.7 that $\sigma^0(Q \circ P) = [1_E]$ and $\sigma^0(P \circ Q) = [1_F]$. As $[1_E]$ is the principal symbol of the identity operator $\mathrm{I}_E : \Gamma_c^\infty(E) \to \Gamma_c^\infty(E)$, it follows that $Q \circ P - \mathrm{I}_E \in \Psi^{-1}(E,E)$, by Theorem 8.2.6. Likewise, $P \circ Q - \mathrm{I}_F \in \Psi^{-\infty}(F,F)$. $\square$

The above corollary has the remarkable improvement that Q may be adapted in such a way that $QP - I$ and $PQ - I$ become smooth kernel operators. The proof of this fact is based on the following principle involving series of pseudo-differential operators. That principle in turn is the appropriate generalization of the similar principle for symbols, as formulated in Lemma 5.5.1.

We put

$$\Psi(E,F) = \bigcup_{d \in \mathbb{R}} \Psi^d(E,F).$$

Definition 8.4.4. Let $\{d_j\}$ be a sequence of real numbers with $\lim_{j\to\infty} d_j = -\infty$. Let $Q_j \in \Psi^{d_j}(E,F)$, for $j \in \mathbb{N}$. Let $Q \in \Psi(E,F)$. Then

$$Q \sim \sum_{j=0}^{\infty} Q_j$$

means that for each $d \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $k \geq N$

$$Q - \sum_{j=0}^{k} Q_j \in \Psi^d(E, F).$$

Theorem 8.4.5. *Let $\{d_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of real numbers with $\lim_{\nu \to \infty} d_\nu = -\infty$ and let for each $\nu \in \mathbb{N}$ a pseudo-differential operator $Q_\nu \in \Psi^{d_\nu}(E, F)$ be given. Then there exists a properly supported $Q \in \Psi(E, F)$ such that*

$$Q \sim \sum_{\nu=0}^{\infty} Q_\nu. \tag{8.1}$$

The operator Q is uniquely determined modulo $\Psi^{-\infty}(E, F)$.

Proof If Q' is a second pseudo-differential operator with this property, then it follows from the definition of $\sim$ that $Q - Q'$ belongs to $\Psi^d(E, F)$ for every d. Hence, $Q - Q' \in \Psi^{-\infty}(E, F)$ and uniqueness follows.

Let $d = \max_\nu d_\nu$. In view of Lemma 8.3.5 it suffices to establish the existence of an operator $Q \in \Psi^d(E, F)$ such that (8.1).

First we consider the case that M is an open subset of $\mathbb{R}^n$ and that E and F are trivial of the form $E = M \times \mathbb{C}^k$ and $F = M \times \mathbb{C}^l$. Then $\Psi(E, F) = M_{l,k}(\Psi(M))$. Let $1 \leq i \leq l$ and $1 \leq j \leq k$. For every $\nu \in \mathbb{N}$ there exists a symbol $(q_\nu)_{ij} \in S^{d_\nu}(M)$ such that

$$(Q_\nu)_{ij} = \Psi_{(q_\nu)_{ij}}.$$

By Lemma 5.5.1 there exists a symbol $q_{ij} \in S^d(M)$ such that

$$q_{ij} \sim \sum_{\nu \in \mathbb{N}} (q_\nu)_{ij}.$$

Let $Q \in \Psi^d(E, F)$ be the pseudo-differential operator with $Q_{ij} = \Psi_{q_{ij}}$ for all $1 \leq i \leq l$, $1 \leq j \leq k$. Then Q satisfies (8.1).

We now turn to the case that both E and F admit trivializations $\tau_E : E \to E' = M \times \mathbb{C}^k$ and $\tau_F : F \to F' = M \times \mathbb{C}^l$. Then by the first part of the proof there exists a $P \in \Psi(E', F')$ such that

$$P \sim \sum_{\nu \in \mathbb{N}} \tau_*(Q_\nu).$$

Put $Q = \tau_*^{-1}(P)$, then Q satisfies (8.1). It remains to establish the general case.

Let $\{U_j\}_{j \in J}$ be an open cover of M such that both bundles E and F admit trivializations on U_j, for every $j \in J$. We may select such a cover with the additional property that it is locally finite and that there exists a partition of unity $\{\psi_j\}_{j \in J}$ such that $\operatorname{supp} \psi_j \subset U_j$ for all $j \in J$. By the first part of the proof there exists for each j an operator $Q_j \in \Psi^d(E_{U_j}, F_{U_j})$ such that

$$Q_j \sim \sum_{\nu \in \mathbb{N}} (Q_\nu)_{U_j}.$$

Clearly, for all i, j such that $U_{ij} := U_i \cap U_j \neq \emptyset$, both operators $(Q_i)_{U_{ij}}$ and $(Q_j)_{U_{ij}}$ have the expansion $\sum_\nu (Q_\nu)_{U_{ij}}$. This implies that the difference of these operators belongs to $\Psi^{-\infty}(U_{ij})$. By the gluing property for $\Psi^d/\Psi^{-\infty}$, see Exercise 7.3.10 and Remark 8.1.2, it follows that there exists a $Q \in \Psi^d(E,F)$ such that $Q_{U_j} - Q_j \in \Psi^{-\infty}(E_{U_j}, F_{U_j})$ for all j. It follows that for all j we have

$$Q_{U_j} \sim \sum_{\nu \in \mathbb{N}} (Q_\nu)_{U_j}.$$

This implies (8.1). □

Theorem 8.4.6. *Let E, F be two complex vector bundles on a manifold M. Let $P \in \Psi^d(E,F)$ be a properly supported elliptic pseudo-differential operator. Then there exists a properly supported pseudo-differential operator $Q \in \Psi^{-d}(F,E)$ such that*

$$QP - I \in \Psi^{-\infty}(E,E). \tag{8.2}$$

The operator Q is uniquely determined modulo $\Psi^{-\infty}(F,E)$ and satisfies

$$PQ - I \in \Psi^{-\infty}(F,F). \tag{8.3}$$

Remark 8.4.7. An operator Q with the above properties is called a *parametrix* for P.

Proof It follows from Corollary 8.4.3 that there exists a properly supported operator $Q_0 \in \Psi^{-d}(F,E)$ such that $Q_0P - I \in \Psi^{-1}(E,E)$ and $PQ_0 - I \in \Psi^{-1}(E,E)$. Put $R = I - Q_0P$. Then R is properly supported. It follows that $R^k \in \Psi^{-k}(E,E)$. Hence, there exists a pseudo-differential operator $A \in \Psi^0(E,E)$ such that

$$A \sim \sum_{k=0}^{\infty} R^k.$$

It is now a straightforward matter to verify that $A(I-R) - I \in \Psi^{-n}$ for all $n \in \mathbb{N}$. It follows that $A(I-R) - I \in \Psi^{-\infty}$. Put $Q = AQ_0$. Then $Q \in \Psi^{-d}(F,E)$ is properly supported and

$$QP - I = AQ_0P - I = A(I-R) - I \in \Psi^{-\infty}(E,E).$$

This shows the existence of Q such that (8.2). We will show that Q also satisfies (8.3). Put $B = QP - I$. Then B is a properly supported smoothing operator in $\Psi^{-\infty}(E,E)$. By what we proved so far there exists a properly supported operator $P_1 \in \Psi^d(E,F)$ such that $P_1Q - I \in \Psi^{-\infty}(F,F)$. The operator $C := P_1Q - I$ is properly supported. We now observe that $P_1QP = P(I+B) = P + PB$, and that $P_1QP = (I+C)P_1 = P_1 + CP_1$. Hence $P - P_1 \in \Psi^{-\infty}(E,F)$ and we conclude that $D := PQ - I \in \Psi^{-\infty}(F,F)$. This establishes the existence of Q.

To establish uniqueness, let $Q' \in \Psi^{-d}(F,E)$ be a properly supported operator with the same property as Q. Then $E := Q'P - I$ is smoothing. It follows that

$$Q' - Q = Q'(PQ) - Q'D - Q = EQ - Q'D$$

is a smoothing operator. □

Corollary 8.4.8. *Let $P \in \Psi^d(M, E, F)$ be a properly supported elliptic pseudo-differential operator. Then for all $u \in \mathcal{D}'(E)$ we have*

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu.$$

In particular, if Pu is smooth, then u is smooth.

Proof Since P is pseudo-local, $\operatorname{singsupp} Pu \subset \operatorname{singsupp} u$. Let $Q \in \Psi^{-d}$ be a properly supported parametrix for P. Then $QP - I$ is a properly supported smoothing operator, hence $QPu - u$ is smooth. Since Q is pseudo-local, it follows that $\operatorname{singsupp} u \subset \operatorname{singsupp}(QPu) \subset \operatorname{singsupp} Pu$. $\square$

Remark 8.4.9. Let $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ be a differential operator of order d. Then by locality of P is follows that $\langle Pf, g\rangle = 0$ for all $f \in \Gamma_c^\infty(E)$ and $g \in \Gamma_c^\infty(F^\vee)$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. This implies that the distribution kernel of P is supported by the diagonal of $M \times M$. In particular, P is properly supported. It follows that the above corollary applies to elliptic differential operators.

LECTURE 9
The index of an elliptic operator

9.1. Pseudo-differential operators and Sobolev space

We recall the definition of the Sobolev space $H_s(\mathbb{R}^n)$, for every $s \in \mathbb{R}$, from Definition 4.3.12, The Sobolev space $H_s(\mathbb{R}^n)$ comes equipped with the inner product that makes $\mathcal{F}$ an isometry from $H_s(\mathbb{R}^n)$ to $L^2_s(\mathbb{R}^n)$. The associated norm on $H_s(\mathbb{R}^n)$ is denoted by $\| \cdot \|_s$. Since Fourier transform is an isometry from $L^2(\mathbb{R}^n)$ to itself, we see that

$$H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n) \tag{9.1}$$

as Hilbert spaces. We recall that for $s < t$ we have $H_t(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$ with continuous inclusion. The intersection of these spaces, denoted $H_\infty(\mathbb{R}^n)$, and equipped with all Sobolev norms $\| \cdot \|_s$, is a Fréchet space.

We note that for all $s \in \mathbb{R} \cup \{\infty\}$ we have $C_c^\infty(\mathbb{R}^n) \subset H_s(R^n) \subset \mathcal{D}'(\mathbb{R}^n)$ with continuous inclusion maps. Thus, the following lemma implies that $H_s(\mathbb{R}^n)$ is a local space in the sense of Lecture 3, Definition 3.1.1.

Lemma 9.1.1. *Let $s \in \mathbb{R} \cup \{\infty\}$. Then the multiplication map $(\varphi, f) \mapsto M_\varphi(f)$, $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, restricts to a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$.*

Remark 9.1.2. This should have been the statement of Lemma 4.3.19, at least for $s < \infty$, and a proof should have been inserted.

Proof It suffices to prove this for finite s. It follows from Lemma 7.1.3 that $\mathcal{F}(M_\varphi(f)) = \mathcal{F}(\varphi) * \mathcal{F}(f)$. Since $\mathcal{F}$ is a topological linear isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself, and from $H_s(\mathbb{R}^n)$ onto $L^2_s(\mathbb{R}^n)$, the result of the lemma is equivalent to the statement that convolution defines a continuous bilinear map $\mathcal{S}(\mathbb{R}^n) \times L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$. This is what we will prove.

Let $\varphi, f, g \in \mathcal{S}(\mathbb{R}^n)$. Then for all $\xi, \eta \in \mathbb{R}^n$ we have

$$(1 + \|\xi - \eta\|)^{-s}(1 + \|\xi\|)^s \leq (1 + \|\eta\|)^{|s|}, \tag{9.2}$$

hence

$$\begin{aligned}
&|\langle g, \varphi * f\rangle| \\
&\leq \iint |\, g(\xi)\varphi(\eta)f(\xi-\eta)\,|\; d\eta d\xi \\
&\leq \int |\varphi(\eta)|(1+\|\eta\|)^{|s|} \int |g(\xi)|(1+\|\xi\|)^{-s}|f(\xi-\eta)|(1+\|\xi-\eta\|)^{s}\; d\xi d\eta \\
&\leq \int |\varphi(\eta)|\,(1+\|\eta\|)^{|s|}\; d\eta\; \|f\|_{L^2_s}\|g\|_{L^2_{-s}},
\end{aligned}$$

by the Cauchy-Schwartz inequality for the L^2-inner product. Let $N \in \mathbb{N}$ be such that $|s| - N < -n$; then there exists a continuous seminorm ν_N on $\mathcal{S}(\mathbb{R}^n)$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|\varphi(\eta)|\,(1+\|\eta\|)^{|s|} \leq \nu_N(\varphi)(1+\|\eta\|)^{-N+|s|}, \qquad (\eta \in \mathbb{R}^n).$$

We conclude that

$$|\langle g, \varphi * f\rangle\| \leq C\nu_N(\varphi)\|f\|_{L^2_s}\|g\|_{L^2_{-s}},$$

with $C = \int (1+\|\eta\|)^{|s|-N}\, d\eta$ a positive real number. Now this is valid for all g in the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of the space $L^2_{-s}(\mathbb{R}^n)$, whose dual is isometrically isomorphic to $L^2_s(\mathbb{R}^n)$. Therefore,

$$\|\varphi * f\|_{L^2_s} \leq C\nu_N(\varphi)\|f\|_{L^2_s}, \qquad (\varphi, f \in \mathcal{S}(\mathbb{R}^n)).$$

By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2_s(\mathbb{R}^n)$ it now follows that the convolution product has a continuous bilinear extension to a map $\mathcal{S}(\mathbb{R}^n) \times L^2_s(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$. The latter space is included in $\mathcal{S}'(\mathbb{R}^n)$ with continuous inclusion map. Hence the present extension of the convolution product must be the restriction of the convolution product $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. □

As said, it follows from the above lemma that $H_s(\mathbb{R}^n)$ is a functional space. In view of the general discussion in Lecture 3 we may now define the local Sobolev space as follows.

Definition 9.1.3. Let $U \subset \mathbb{R}^n$ be open and let $s \in \mathbb{R} \cup \infty$. The *local Sobolev space* $H_{s,\,\mathrm{loc}}(U)$ is defined to be the space of $f \in \mathcal{D}'(U)$ with the property that $\psi f \in H_s(\mathbb{R}^n)$ for all $\psi \in C_c^\infty(\mathbb{R}^n)$. The space $H_s(\mathbb{R}^n)$ is equipped with the locally convex topology induced by the collection of seminorms $\nu_\psi : f \mapsto \|\psi f\|_s$, for $\psi \in C_c^\infty(U)$.

The space $L^2_{\mathrm{loc}}(U)$ is defined in a similar fashion. In view of (9.1),

$$L^2_{\mathrm{loc}}(U) = H_{0,\mathrm{loc}}(U),$$

including topologies. Since $H_s(\mathbb{R}^n)$ is a Hilbert space, if follows from the theory developed in Lecture 3 that $H_{s,\,\mathrm{loc}}(U)$ (with the specified topology) is a Fréchet space.

It follows from the Sobolev lemma, Lemma 4.3.16, that $C_c^\infty(\mathbb{R}^n) \subset H_\infty \subset C^\infty(\mathbb{R}^n)$ with continuous inclusion maps. This in turn implies that

$$H_{\infty,\mathrm{loc}}(U) = C^\infty(U),$$

including topologies.

The Sobolev spaces behave very naturally under the action of pseudo-differential operators.

Lemma 9.1.4. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact subset, and let $-d, s \in \mathbb{R} \cup \{\infty\}$. Then the map*

$$(p, f) \mapsto \Psi_p(f), \;\; S^d_{\mathcal{K}}(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \to C^\infty_{\mathcal{K}}(\mathbb{R}^n),$$

has a unique extension to a continuous bilinear map

$$S^d_{\mathcal{K}}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_{s-d,\mathcal{K}}(\mathbb{R}^n)$$

Proof It suffices to prove this for s, d finite, which we will now assume to be the case. Uniqueness follows from density of $C^\infty_c(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$. Thus, it suffices to establish existence. Given $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$, let $\mathcal{F}_1 p$ denote the Fourier transform of the function $(x, \xi) \mapsto p(x, \xi)$ with respect to the first variable. If $\alpha \in \mathbb{N}^n$, then

$$\begin{aligned}(1 + \|\xi\|)^{-d} |\eta^\alpha \mathcal{F}_1 p(\eta, \xi)| &= |\mathcal{F}_1((1 + \|\xi\|)^{-d} \partial^\alpha_x p)(\eta, \xi)| \\ &\leq \operatorname{vol}(\mathcal{K}) \, \sup \big[(1 + \|\xi\|)^{-d} |\partial^\alpha_x p(x, \xi)|\big].\end{aligned}$$

It follows that for every $N \in \mathbb{N}$ there exists a continuous seminorm μ_N on $S^d_{\mathcal{K}}(\mathbb{R}^n)$ such that

$$(1 + \|\eta\|)^N (1 + \|\xi\|)^{-d} |\mathcal{F}_1 p(\eta, \xi)| \leq \mu_N(p), \qquad ((\eta, \xi) \in \mathbb{R}^{2n}),$$

for all $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$.

Let now $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$ and $f, g \in C^\infty_c(\mathbb{R}^n)$. Then it follows that

$$\begin{aligned}\langle g, \Psi_p f\rangle &= \iint e^{i\xi x} g(x) p(x, \xi) \widehat{f}(\xi) \, d\xi \, dx \\ &= \iint e^{i\xi x} g(x) p(x, \xi) \widehat{f}(\xi) \, dx \, d\xi \\ &= \iint \widehat{g}(\xi - \eta) \mathcal{F}_1 p(\eta, \xi) \widehat{f}(\xi) \, d\eta \, d\xi.\end{aligned}$$

We obtain

$$\begin{aligned}&|\langle g, \Psi_p f\rangle| \\ &\quad\leq \iint F_p(\eta, \xi) \, (1 + \|\xi - \eta\|)^{d-s} |\widehat{g}(\xi - \eta)| \, (1 + \|\xi\|)^s |\widehat{f}(\xi)| \, d\xi \, d\eta \\ &\quad\leq \int \sup_{\xi \in \mathbb{R}^n} F_p(\eta, \xi) \, d\eta \;\; \|g\|_{d-s} \|f\|_s, \end{aligned} \tag{9.3}$$

where

$$\begin{aligned}F_p(\eta, \xi) &= |\mathcal{F}_1 p(\eta, \xi)| (1 + \|\xi - \eta\|)^{s-d} (1 + \|\xi\|^{-s} \\ &\leq |\mathcal{F}_1 p(\eta, \xi)| (1 + \|\xi\|)^{-d} (1 + \|\eta\|)^{|s-d|} \\ &\leq (1 + \|\eta\|)^{|s-d|-N} \, \mu_N(p).\end{aligned}$$

In the above estimation we have used (9.2) with $s - d$ in place of s. Fix N such that $|s - d| - N < -n$. Then combining the last estimate with (9.3) we see that there exists a constant $C > 0$ such that for all $f, g \in C^\infty_c(\mathbb{R}^n)$ and $p \in S^d_{\mathcal{K}}(\mathbb{R}^n)$,

$$|\langle g, \Psi_p f\rangle| \leq C \, \mu_N(p) \|g\|_{d-s} \|f\|_s.$$

The space $C_c^\infty(\mathbb{R}^n)$ is dense in the Hilbert space $H_{d-s}(\mathbb{R}^n)$ whose dual is isometrically isomorphic with $H_{s-d}(\mathbb{R}^n)$. This implies that

$$\|\Psi_p(f)\|_{s-d} \leq C\,\mu_N(p)\|f\|_s, \qquad (p \in S^d_{\mathcal{K}}(\mathbb{R}^n), f \in C_c^\infty(\mathbb{R}^n)).$$

it follows that the map $(p, f) \mapsto \Psi_p(f)$ has a continuous bilinear extension $\beta : S^d_{\mathcal{K}}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \to H_{s-d}(\mathbb{R}^n)$. Since β maps the dense subspace $S^d_{\mathcal{K}}(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ into the closed subspace $H_{s-d,\mathcal{K}}(\mathbb{R}^n)$ it follows that β maps continuous bilinearly into this closed subspace as well. $\square$

The *local* Sobolev spaces behave very naturally under the action of pseudo-differential operators as well.

Proposition 9.1.5. *Let $P \in \Psi^d(U)$ be properly supported, $d \in \mathbb{R} \cup \{-\infty\}$. Then for every $s \in \mathbb{R} \cup \{\infty\}$ the operator $P : \mathcal{D}'(U) \to \mathcal{D}'(U)$ restricts to a continuous linear operator $P_s : H_{s,\,\mathrm{loc}}(U) \to H_{s-d,\,\mathrm{loc}}(U)$.*

Proof By Lemma 7.1.9 there exists a $p \in S^d(U)$ such that $P = \Psi_p$. Let $\psi \in C_c^\infty(U)$ and put $B = \operatorname{supp}\psi$. Then it suffices to show that the operator $Q := M_\psi \circ P$ is continuous linear from $H_{s,\,\mathrm{loc}}(U)$ to $H_{s-d}(\mathbb{R}^n)$. We note that $Q = \Psi_q$, where $q \in S^d_c(U) \subset S^d_c(\mathbb{R}^n)$ is given by $q = \psi p$. Since P is properly supported, there exists a compact subset $\mathcal{K}$ of U such that the kernel of Q has support contained in $B \times \mathcal{K}$. Let $\chi \in C_c^\infty(U)$ be such that $\chi = 1$ on an open neighborhood of $\mathcal{K}$. Then $Q = Q \circ M_\chi$ on $C_c^\infty(U)$, hence on $\mathcal{D}'(U)$, hence on $H_{s,\,\mathrm{loc}}(U)$. Put $A = \operatorname{supp}\chi$. Then M_χ is continuous linear $H_{s,\,\mathrm{loc}}(\mathbb{R}^n) \to H_{s,A}(\mathbb{R}^n)$. Moreover, by Lemma 9.1.4 Ψ_q is continuous linear $H_{s,A}(\mathbb{R}^n) \to H_{s-d,B}(\mathbb{R}^n)$. Therefore, $Q = \Psi_q \circ M_\chi$ is continuous linear $H_{s,\,\mathrm{loc}}(\mathbb{R}^n) \to H_{s-d}(\mathbb{R}^n)$. $\square$

9.2. Sobolev spaces on manifolds

In the previous section we have seen that the local Sobolev spaces are functional in the sense of Lecture 3. In order to be able to extend these spaces to manifolds, we need to establish their invariance under diffeomorphims. We will do this through characterizing them by elliptic pseudo-differential operators, which are already known to behave well under diffeomorphisms. A first result in this direction is the following.

Proposition 9.2.1. *Let $s \in \mathbb{R}$, and let $P \in \Psi^s(U)$ be a properly supported elliptic pseudo-differential operator. Then*

$$H_{s,\,\mathrm{loc}}(U) = \{f \in \mathcal{D}'(U) \mid \; Pf \in L^2_{\mathrm{loc}}(U)\}.$$

Proof If $f \in H_{s,\mathrm{loc}}(U)$ then $f \in \mathcal{D}'(U)$ and $Pf \in H_{0,\mathrm{loc}}(U) = L^2_{\mathrm{loc}}(U)$. This proves one inclusion. To prove the converse inclusion, we use that by Theorem 8.4.6 applied with $M = U$ and $E = F = \mathbb{C}_U$, there exists a properly supported $Q \in \Psi^{-s}(U)$ such that $Q \circ P = I + T$, with $T \in \Psi^{-\infty}(U)$ a properly supported smoothing operator. If $f \in \mathcal{D}'(U)$ and $Pf \in L^2_{\mathrm{loc}}(U) = H_{0,\mathrm{loc}}(U)$, then $QPf \in H_{s,\,\mathrm{loc}}(U)$ and $Tf \in C^\infty(U) \subset H_{s,\,\mathrm{loc}}(U)$. Hence, $f = QPf - Tf \in H_{s,\,\mathrm{loc}}(U)$. $\square$

By combination of the above result with the following lemma, it can be shown that the local Sobolev spaces behave well under diffeomorphisms.

Lemma 9.2.2. *Let $U \subset \mathbb{R}^n$ be an open subset and let $d \in \mathbb{R}$. There exists a properly supported elliptic operator in $\Psi^d(U)$.*

Proof Let d be finite. Then the function $p = p_d : \mathbb{R}^{2n} \to \mathbb{C}$ defined by $p(x, \xi) = (1 + \|\xi\|^2)^{d/2}$ belongs to the symbol space $S^d(\mathbb{R}^n)$. Since $p_d p_{-d} = 1$, the operator $R = \Psi_{p_d} \in \Psi^d(\mathbb{R}^n)$ is elliptic. Its restriction $P = R_U$ to U belongs to $\Psi^d(U)$ and has principal symbol $[(p_d)_U]$ hence is elliptic as well. By Lemma 8.3.5 there exists a properly supported $P_0 \in \Psi^d(U)$ such that $P - P_0$ is a smoothing operator. Hence, P_0 has the same principal symbol as P and we see that P_0 is elliptic. □

Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. This diffeomorphism induces a topological linear isomorphism

$$\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V). \tag{9.4}$$

Theorem 9.2.3. *Let $\varphi : U \to V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$ and let $s \in \mathbb{R} \cup \{\infty\}$.*

(a) *The map (9.4) restricts to a linear isomorphism*

$$\varphi_{*s} : H_{s,\mathrm{loc}}(U) \to H_{s,\mathrm{loc}}(V).$$

(b) *The isomorphism φ_{*s} is topological.*

Proof It suffices to prove the result for finite s, the result for $s = \infty$ is then a consequence.

We will first obtain (a) as a consequence of Proposition 9.2.1 and Lemma 9.2.2. By the latter lemma there exists a properly supported elliptic operator $P \in \Psi^d(U)$. Let $P' = \varphi_*(P)$. Then $P' \in \Psi^d(V)$ and $P'\varphi_*(f) = \varphi_*(Pf)$ for all $f \in \mathcal{D}'(U)$. The principal symbol of P' is given by $\sigma^d(P') = \varphi_*(\sigma^d(P))$, hence elliptic. Therefore, P' is elliptic. The kernel of P' has support contained in $(\varphi \times \varphi)(\operatorname{supp} K_P)$, hence is properly supported.

By a straightforward application of the substitution of variables formula, it follows that φ_* restricts to a topological linear isomorphism $L^2_{\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(V)$. Let now $f \in \mathcal{D}'(U)$. Then

$$\begin{aligned} f \in H_{s,\mathrm{loc}}(U) \quad &\iff \quad Pf \in L^2_{\mathrm{loc}}(U) \iff \varphi_*(Pf) \in L^2_{\mathrm{loc}}(V) \\ &\iff \quad P'\varphi_*(f) \in L^2_{\mathrm{loc}}(V) \iff \varphi_*(f) \in H_{s,\mathrm{loc}}(V). \end{aligned}$$

This proves (a). In order to prove (b) we need characterizations of the topology on $H_{s,\mathrm{loc}}$ that behave well under diffeomorphisms. These will first be given in two lemmas below. The present proof will be completed right after those lemmas. □

Lemma 9.2.4. *Let $s \geq 0$ and let $P \in \Psi^s(U)$ be properly supported and elliptic. Then the topology on $H_{s,\mathrm{loc}}(U)$ is the weakest locally convex topology for which both the inclusion map $j : H_{s,\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(U)$ and the map $P : H_{s,\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(U)$ are continuous.*

Proof For the topology on $H_{s,\mathrm{loc}}(U)$ the mentioned maps j and P are continuous with values in $L^2_{\mathrm{loc}}(U)$. Let V be equal to $H_{s,\mathrm{loc}}(U)$ equipped with the weakest locally convex topology for which j and P are continuous. Then we must show that the identity map $V \to H_{s,\,\mathrm{loc}}(U)$ is continuous. Let $\varphi \in C_c^\infty(U)$; then it suffices to show that the map $M_\varphi : V \to H_s(\mathbb{R}^n)$ is continuous.

Let $Q \in \Psi^{-s}(U)$ be a properly supported parametrix for P. Then

$$\mathrm{I} = QP + T,$$

with $T \in \Psi^{-\infty}(U)$ a properly supported smoothing operator. Let $A = \operatorname{supp} \varphi$. There exists a compact subset $B \subset U$ such that the intersections of both $\operatorname{supp} K_Q$ and $\operatorname{supp} K_T$ with $A \times U$ are contained in $A \times B$. Let $\psi \in C_c^\infty(U)$ be such that $\psi = 1$ on an open neighborhood of B. Then $M_\varphi \circ Q = M_\varphi \circ Q \circ M_\psi$ and $M_\varphi \circ T = M_\varphi \circ T \circ M_\psi$. We now see that, for all $f \in H_{s,\,\mathrm{loc}}(U)$,

$$\varphi f = M_\varphi(QP + T)(f) = M_\varphi Q M_\psi P f + M_\varphi T M_\psi f.$$

As $M_\varphi \circ Q$ and $M_\varphi \circ T$ define continuous linear maps $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$, by Lemma 9.1.4, it follows that there exists a constant $C > 0$ such that

$$\|\varphi f\|_s \le C(\|\varphi P f\|_0 + \|\psi f\|_0),$$

for all $f \in H_{s,\,\mathrm{loc}}(U)$. The seminorms $f \mapsto \|\varphi P f\|_0$ and $f \mapsto \|\psi f\|_0$ are continuous on V. Hence, $M_\varphi : V \to H_s(\mathbb{R}^n)$ is continuous. $\square$

We will also need a characterization of the topology of $H_{s,\mathrm{loc}}$ by duality, which is invariant under diffeomorphisms for all negative s. Let $D = D_{\mathbb{R}^n}$ denote the density bundle on $\mathbb{R}^n$. Then we have the natural continuous bilinear pairing $\langle \cdot\,, \cdot \rangle : C_c^\infty(\mathbb{R}^n) \times \Gamma_c^\infty(\mathbb{R}^n, D) \to \mathbb{C}$ given by

$$\langle f, \gamma \rangle = \int_{\mathbb{R}^n} f\gamma.$$

This pairing induces a continuous injection of the space $C_c^\infty(\mathbb{R}^n)$ into the topological dual $\mathcal{D}'(\mathbb{R}^n)$ of $\Gamma_c^\infty(\mathbb{R}^n, D)$. This pairing also induces a continuous injection of $\Gamma_c^\infty(\mathbb{R}^n, D)$ into $\mathcal{D}'(\mathbb{R}^n, D) \simeq C_c^\infty(\mathbb{R}^n)'$. We note that the map $g \mapsto g dx$ defines a topological linear isomorphism $C_c^\infty(\mathbb{R}^n) \to \Gamma_c^\infty(\mathbb{R}^n, D)$ and extends to a continuous linear isomorphism $\mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n, D)$. For $s \in \mathbb{R}$, the image of the Sobolev space $H_s(\mathbb{R}^n)$ under this isomorphism, equipped with the transferred Hilbert structure, is denoted by $H_s(\mathbb{R}^n, D)$.

By transposition, the inclusion $\Gamma_c^\infty(\mathbb{R}^n, D) \to H_s(\mathbb{R}^n, D)$ induces a continuous linear map $H_s(\mathbb{R}^n, D)' \to \mathcal{D}'(\mathbb{R}^n)$ which is injective by density of $\Gamma_c^\infty(\mathbb{R}^n, D)$ in $H_s(\mathbb{R}^n, D)$. Of course the induced map is given by $u \mapsto u|_{\Gamma_c^\infty(\mathbb{R}^n, D)}$.

The perfectness of the pairing of Lemma 4.3.18 can now be expressed as follows.

Lemma 9.2.5. *Let $s \in \mathbb{R}$. The image of the injection $H_s(\mathbb{R}^n, D)' \to \mathcal{D}'(\mathbb{R}^n)$ equals $H_{-s}(\mathbb{R}^n)$. The associated bijection $H_s(\mathbb{R}^n, D)' \to H_{-s}(\mathbb{R}^n)$ is a topological linear isomorphism.*

More generally, if $U \subset \mathbb{R}^n$ is an open subset, we define $H_{s,\mathrm{comp}}(U, D)$ to be the image of $H_{s,\mathrm{comp}}(U)$ in $\mathcal{D}'(U, D)$ under the map $f \mapsto f dm$, equipped with the topology that makes this map a topological isomorphism. The natural inclusion $\Gamma_c^\infty(U, D) \to H_{s,\mathrm{comp}}(U, D)$ induces a continuous linear injection

$$H_{s,\mathrm{comp}}(U, D)' \hookrightarrow \mathcal{D}'(U).$$

Here, the space on the left is equipped with the strong dual topology.

Corollary 9.2.6. *Let $s \in \mathbb{R}$. Then the image of $H_{s,\mathrm{comp}}(U,D)'$ in $\mathcal{D}'(U)$ equals $H_{-s,\mathrm{loc}}(U)$. The associated bijection $H_{s,\mathrm{comp}}(U,D)' \to H_{-s,\mathrm{loc}}(U)$ is a topological linear isomorphism.*

Proof Let $j : H_{s,\mathrm{comp}}(U)' \to \mathcal{D}'(U)$ denote the natural linear injection. Let $\varphi \in C_c^\infty(M)$. Then for all $u \in H_{s,\mathrm{comp}}(U)'$ we have

$$M_\varphi \circ j(u) = j(u \circ M_\varphi).$$

The map $u \mapsto u \circ M_\varphi$ is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_s(\mathbb{R}^n)'$. It follows that $u \mapsto M_\varphi \circ j(u) = j(M_\varphi u)$ is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_{-s}(\mathbb{R}^n)$. Since this holds for all φ, it follows that j is continuous linear $H_{s,\mathrm{comp}}(U)' \to H_{-s,\mathrm{loc}}(U)$ as stated.

Conversely, let $\mathcal{K} \subset U$ be compact. Let $\mathcal{K}'$ be a compact neighborhood of $\mathcal{K}$ in U. Then $H_{s,\mathcal{K}}(\mathbb{R}^n)$ is contained in the closure of $C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$. Let $v \in H_{-s,\mathrm{loc}}(U)$ and let $\varphi \in C_c^\infty(U)$ be such that $\varphi = 1$ on a neighborhood of $\mathcal{K}'$. Then $M_\varphi v \in H_{-s}(\mathbb{R}^n)$, hence, $M_\varphi v = j(k_\varphi(v))$ for a unique $k_\varphi(v) \in H_s(\mathbb{R}^n)'$. Moreover, the map $k_\varphi : H_{-s,\mathrm{loc}}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)'$ is continuous linear by the above lemma. The restriction of $k_\varphi(v)$ to $C^\infty_{\mathcal{K}'}(\mathbb{R}^n)$ is independent of the choice of φ, and therefore so is the map $k_\mathcal{K} : v \mapsto k_\varphi(v)|_{H_{s,\mathcal{K}}}$. The map $k_\mathcal{K}$ is continuous linear. Moreover, if $\mathcal{K}_1 \subset \mathcal{K}_2$ are compact subsets of U then $k_{\mathcal{K}_1}(v) = k_{\mathcal{K}_2}(v)|\mathcal{K}_1$. It follows that there exists a unique linear map $k : H_{-s,\mathrm{loc}}(U) \to H_{s,\mathrm{comp}}(U)'$ such that $k_A(v) = k(v)|_A$ for all $v \in H_{-s,\mathrm{loc}}(U)$ and all $A \subset U$ compact. As all the k_A are continuous, it follows that k is continuous. Now $j \circ k = \mathrm{I}$ and it follows that j defines a continuous linear isomorphism from $H_{s,\mathrm{comp}}(U)'$ onto $H_{-s,\mathrm{loc}}(U)$. □

Completion of the proof of Theorem 9.2.3. First assume that $s \geq 0$. Let $P \in \Psi^s(U)$ be a properly supported elliptic operator, and let $P' = \varphi_*(P)$ be as in part (a) of the proof. Then by Lemma 9.2.4 the topology of $H_s(U)$ is the weakest locally convex topology for which both the inclusion $j_U : H_s(U) \to L^2_{\mathrm{loc}}(U)$ and the map $P : H_s(U) \to L^2_{\mathrm{loc}}(U)$ are continuous. Likewise, the topology on $H_s(V)$ is the weakest for which both the inclusion map $j_V : H_s(V) \to L^2_{\mathrm{loc}}(V)$ and $P' : H_s(V) \to L^2_{\mathrm{loc}}(V)$ are continuous. The map $\varphi_* : L^2_{\mathrm{loc}}(U) \to L^2_{\mathrm{loc}}(V)$ is a topological linear isomorphism and the following diagrams commute:

$$\begin{array}{ccc} H_s(U) & \xrightarrow{j_U} & L^2_{\mathrm{loc}}(U) \\ \varphi_{*s}\downarrow & & \downarrow \varphi_* \\ H_s(V) & \xrightarrow{j_V} & L^2_{\mathrm{loc}}(V) \end{array} \qquad \begin{array}{ccc} H_s(U) & \xrightarrow{P} & L^2_{\mathrm{loc}}(U) \\ \varphi_{*s}\downarrow & & \downarrow \varphi_* \\ H_s(V) & \xrightarrow{P'} & L^2_{\mathrm{loc}}(V) \end{array}$$

It follows that φ_{*s} is a topological linear isomorphism.

This proves (b) for $s \geq 0$. We will complete the proof by proving (b) for $s \leq 0$, using the duality expressed in Lemma 9.2.5. We put $t = -s$, so that $t \geq 0$. By the validity of (b) for $s \geq 0$ it follows that the map $\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V)$, restricts to a topological linear isomorphism $\varphi_{*t} : H_{t,\mathrm{comp}}(U) \to H_{t,\mathrm{comp}}(V)$. On the other hand, the map φ_* restricts to the topological linear isomorphism $\varphi_* : C_c^\infty(U) \to C_c^\infty(V)$ given by $f \mapsto f \circ \varphi^{-1}$. The map $f \mapsto f dx$ defines a topological linear isomorphism from $\mathcal{D}'(U)$ to $\mathcal{D}'(U,D)$. Likewise, $g \mapsto g dy$

defines a topological linear isomorphism from $\mathcal{D}'(V)$ to $\mathcal{D}'(V,D)$. Now

$$\varphi_*(fdx) = \varphi_*(f)\varphi_*(dx) = M_J\varphi_*(f)dy,$$

where $J : V \to (0,\infty)$ is the positive smooth function given by

$$J(y) = |\det D\varphi(\varphi^{-1})|^{-1}.$$

The map M_J defines a topological linear automorphism of $\mathcal{D}'(V)$ and restricts to a topological linear isomorphism of $H_t(V)$. We conclude that φ_* defines a topological linear isomorphism $\mathcal{D}'(U,D) \to \mathcal{D}'(V,D)$ and restricts to a topological linear isomorphism $\varphi_{*t} : H_{t,\mathrm{comp}}(U,D) \to H_{t,\mathrm{comp}}(V,D)$. Its restriction to $\Gamma_c^\infty(U,D)$ is given by $fdx \mapsto M_J(\varphi^{-1})^*(f)dy$, hence defines a topological linear isomorphism $\Gamma_c^\infty(U,D) \to \Gamma_c^\infty(V,D)$. By taking transposed maps in the commutative diagram

$$\begin{array}{ccc} H_{t,\mathrm{comp}}(U,D) & \longrightarrow & H_{t,\mathrm{comp}}(V,D) \\ \uparrow & & \uparrow \\ \Gamma_c^\infty(U,D) & \longrightarrow & \Gamma_c^\infty(V,D) \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{D}'(U) & \xleftarrow{\varphi_*^{-1}} & \mathcal{D}'(V) \\ \uparrow & & \uparrow \\ H_{t,\mathrm{comp}}(U,D)' & \xleftarrow{\simeq} & H_{t,\mathrm{comp}}(V,D)'. \end{array}$$

Here the bottom arrow is the transpose of a topological linear isomorphism, hence a topological linear isomorphism of its own right.

In view of Lemma 9.2.5, and using $s = -t$, we see that $\varphi_* : \mathcal{D}'(U) \to \mathcal{D}'(V)$ restricts to a topological linear isomorphism $\varphi_{*s} : H_{s,\mathrm{loc}}(U) \to H_{s,\mathrm{loc}}(V)$. $\square$

In particular, it follows from Theorem 9.2.3 that $H_{s,\mathrm{loc}}$ is an invariant local functional space in the terminology of Lecture 3. It follows that for E a complex vector bundle on a smooth manifold the spaces of sections $H_{s,\mathrm{comp}}(M,E)$ and $H_{s,\mathrm{loc}}(M,E)$ are well defined locally convex topological vector spaces. Moreover, the first of these spaces is contained in the second with continuous inclusion map, and the second space is a Fréchet space.

Lemma 9.2.7. *Let $s,t \in \mathbb{R}$ and let $s < t$. Then $H_{t,\mathrm{loc}}(M,E) \subset H_{s,\mathrm{loc}}(M,E)$ with continuous inclusion map. If M is compact, this inclusion map is compact.*

Proof Let $\{U_j\}_{j\in J}$ be a cover of M by relatively compact open coordinate patches on which the bundle E admits a trivialization. Passing to a locally finite refinement, we may assume that the index set J is countable. Let φ_j be a partition of unity subordinate to the cover. For each $j \in J$ we write $K_j = \operatorname{supp}\varphi_j$. Then the map $f \mapsto (\varphi_j f)_{j\in J}$ defines a continuous linear embedding

$$H_{s,\mathrm{loc}}(M,E) \longrightarrow \prod_{j\in J} H_{s,K_j}(U_j,E),$$

for every $s \in \mathbb{R}$. Via a trivialization of E over U_j we may identify $H_{s,K_j}(U_j,E) \simeq H_{s,K_j}(U_j)^k$. As $H_{t,K_j}(U_j)^k \subset H_{s,K_j}(U_j)^k$, for $s < t$, with continuous inclusion map, the first assertion of the lemma follows.

If M is compact, we may take the covering such that the index set J is finite. Then by the above reasoning and application of Rellich's lemma, Lemma 4.5.2, it follows that the following diagram commutes, and that the inclusion map represented by the vertical arrow on the right is the finite direct product of compact maps:

$$\begin{array}{ccc} H_s(M,E) & \longrightarrow & \prod_{j\in J} H_{s,K_j}(U_j,E) \\ \uparrow & & \uparrow \\ H_t(M,E) & \longrightarrow & \prod_{j\in J} H_{t,K_j}(U_j,E). \end{array}$$

As the maps represented by the horizontal arrows are embeddings, it follows that the inclusion $H_t(M,E) \to H_s(M,E)$, represented by the vertical arrow on the left, is compact as well. □

We define

$$H_{\infty,\mathrm{loc}}(M,E) = \cup_{s\in\mathbb{R}} H_{s,\mathrm{loc}}(M,E)$$

equipped with the weakest topology for which all inclusion maps $H_{\infty,\mathrm{loc}}(M,E) \to H_{s,\mathrm{loc}}(M,E)$ are continuous. Then by an argument similar to the one used in the proof of the above lemma, it follows from the corresponding local statement (see 9.1.5), that $H_{\infty,\mathrm{loc}}(M,E) = \Gamma^\infty(M,E)$, as topological linear (Fréchet) spaces.

Theorem 9.2.8. *Let E, F be vector bundles on the smooth manifold M. Let $d \in \mathbb{R}\cup\{-\infty\}$ and $s \in \mathbb{R}\cup\{\infty\}$. Finally, let $P \in \Psi^d(E,F)$ be properly supported. Then $P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,F)$ restricts to a continuous linear operator $P_s : H_{s,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$.*

Proof First assume that $d = -\infty$, so that P is a properly supported smoothing operator. Then P is continuous linear $\mathcal{D}'(M,E) \to \Gamma^\infty(M,F)$. Since the inclusion $H_{s,\mathrm{loc}}(M,E) \to \mathcal{D}'(M,E)$ is continuous linear, it follows that the restriction $P_s : H_{s,\mathrm{loc}}(M,E) \to \Gamma^\infty(M,F) = H_{\infty,\mathrm{loc}}(M,F)$ is continuous linear.

Now assume that $\{U_j\}_{j\in J}$ is a cover of M with relatively compact open coordinate patches. By paracompactness, we may assume that J is countable and that the cover is locally finite. Let $\{\psi_j\}$ be a partition of unity subordinate to this cover. For each $j \in J$ we select a function $\chi_j \in C_c^\infty(U_j)$ such that $\chi_j = 1$ on an open neighborhood of $\operatorname{supp}\psi_j$. Put $P_j := M_\psi \circ P \circ M_{\chi_j}$. Then it follows that

$$T_j = M_{\psi_j} \circ P - P_j$$

is a properly supported smoothing operator. The supports of the kernels of T_j form a locally finite set, hence $T = \sum_j T_j$ is a well-defined smoothing operator. Moreover, $P_* = \sum_j P_j$ is a well-defined operator in $\Psi^d(E,F)$, which is properly supported by local finiteness of the cover $\{U_j\}$. Moreover,

$$P = P_* + T,$$

so T is properly supported. By the first part of the proof, T maps $H_{s,\mathrm{loc}}(M,E)$ continuously into $C^\infty(M,F)$, hence also continuously into $H_{s-d}(M,F)$. Thus, it suffices to show that P_* is continuous linear $H_{s,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$. Let $\varphi \in C_c^\infty(M)$. Then it suffices to show that $M_\varphi \circ P_*$ is continuous linear $H_{s,\mathrm{loc}}(M,E) \to H_{s-d,B}(M,F)$, where $B = \operatorname{supp}\varphi$. Now $M_\varphi \circ P_* = \sum_j M_\varphi \circ P_j$, the sum extending over the finite set of j for which $\operatorname{supp}\psi_j \cap B \neq \emptyset$. Thus, it suffices to establish the continuity of $(P_j)_s : H_{s,\mathrm{loc}}(M,E) \to H_{s-d,\mathrm{loc}}(M,F)$.

This is equivalent to the continuity of $(P_j)_{U_j} : H_{s,\mathrm{loc}}(U_j, E) \to H_{s-d,\mathrm{loc}}(U_j, F)$ which by triviality of the bundles E_{U_j} and F_{U_j} follows from the local scalar result, Proposition 9.1.5. $\square$

Lemma 9.2.9. *Let M be a smooth manifold, and $E \to M$ be a vector bundle. For every $s \in \mathbb{R}$ the natural pairing $\Gamma_c^\infty(M,E) \times \Gamma^\infty(M, E^\vee) \to \mathbb{C}$ has a unique extension to a continuous bilinear pairing*

$$H_{s,\mathrm{comp}}(M,E) \times H_{-s,\mathrm{loc}}(M,E^\vee) \to \mathbb{C}.$$

Moreover, the pairing is perfect, i.e., the induced maps

$$H_{s,\mathrm{comp}}(M,E) \to H_{-s,\mathrm{loc}}(M,E^\vee)', \quad H_{-s,\mathrm{loc}}(M,E^\vee) \to H_{s,\mathrm{comp}}(M,E)'$$

are topological linear isomorphisms.

Proof The proof will be given in an appendix. $\square$

9.3. The index of an elliptic operator

We are now finally prepared to show that every elliptic operator between vector bundles E, F over a compact manifold M has a well defined index.

Theorem 9.3.1. *Let E, F be vector bundles over a compact manifold M. Let $d \in \mathbb{R}$ and let $P \in \Psi^d(E,F)$ be elliptic. Then the transpose $P^t \in \Psi^d(F^\vee, E^\vee)$ is elliptic as well.*

The operator $P_\infty : \Gamma^\infty(M,E) \to \Gamma^\infty(M,F)$ has a finite dimensional kernel, and closed image of finite codimension.

For all $s \in \mathbb{R}$ the operator $P_s : H_s(m,E) \to H_{s-d}(M,F)$ is Fredholm and has index

$$\operatorname{index} P_s = \operatorname{index}(P_\infty).$$

In particular, the index is independent of s.

Proof Let $p \in S^d(M, \underline{\mathrm{Hom}}\,(E,F))$ be a representative of the principal symbol $\sigma^d(P)$. Then the principal symbol of P^t is represented by

$$p^\vee : (x,\xi) \mapsto p(x,-\xi)^* \otimes \mathrm{I}_{D_x}.$$

Since p is elliptic, $p^\vee$ is elliptic as well, and we conclude that P^t is elliptic.

Since M is compact, the spaces $H_s(M,E)$ and $H_s(M,F)$ carry a Banach topology. Let $Q \in \Psi^{-d}(F,E)$ be a parametrix. Then $QP = \mathrm{I}_E + T$, with $T \in \Psi^{-\infty}(E,E)$ a smoothing operator. The operator T is continuous $H_s(M,E) \to C^\infty(M,E)$, hence continuous $H_s(M,E) \to H_{s+1}(M,E)$. As $H_{s+1}(M,E) \to H_s(M,E)$ with compact inclusion map, it follows that $T_s : H_s(E) \to H_s(E)$ is a compact operator. It follows that $Q_{s-d} \circ P_s = (\mathrm{I}_E)_s + T_s$, hence P_s has left inverse Q_{s-d} modulo a compact operator. Likewise, from $PQ - I_F = T' \in \Psi^{-\infty}(F,F)$ we see that the operator P_s has right inverse Q_{s-d} modulo a compact operator. This implies that P_s is Fredholm. In particular, $\ker P_s$ is finite dimensional.

Since $\ker P_\infty \subset \ker P_s$ and $\ker P_s \subset \Gamma^\infty(M,E)$ by the elliptic regularity theorem, Corollary 8.4.8, it follows that $\ker P_s = \ker P_\infty$, for all $s \in \mathbb{R}$. In particular, $\ker P_\infty$ is finite dimensional.

Likewise, $(P^t)_s : H_s(M, F\vee) \to H_{s-d}(M, E^\vee)$ is Fredholm, $(\ker P^t)_\infty$ is finite dimensional and $\ker(P^t)_s = \ker(P^t)_\infty$ for all s.

We consider the natural continuous bilinear pairing

$$(f, g) \mapsto \langle f, g\rangle, \quad \Gamma^\infty(M, E) \times \Gamma^\infty(M, E^\vee) \to \mathbb{C}. \tag{9.5}$$

The operator $P^t : \Gamma^\infty(M, F^\vee)) \to \Gamma^\infty(M, E^\vee)$ satisfies

$$\langle Pf, h\rangle = \langle f, P^t h\rangle \tag{9.6}$$

for all $f \in \Gamma^\infty(M, E)$ and $h \in \Gamma^\infty(M, F^\vee)$. The natural bilinear pairing (9.5) extends uniquely to a continuous bilinear pairing $H_s(M, E)\times H_{-s}(M, E^\vee) \to \mathbb{C}$, which is perfect by Lemma 9.2.9. The map P_s is the continuous linear extension of $P : \Gamma^\infty(M, E) \to \Gamma^\infty(M, F)$ to a map $H_s(M, E) \to H_{s-d}(M, F)$. Similarly, P^t extends to a continuous linear map $(P^t)_{d-s} : H_{d-s}(M, F^\vee) \to H_{-s}(M, E^\vee)$. By density and continuity, the identity (9.6) implies that more generally,

$$\langle (P^t)_{d-s} f, g\rangle = \langle f, P_s g\rangle$$

for all $f \in H_{d-s}(M, F^\vee)$ and $g \in H_s(M, E)$. In other words,

$$(P_s)^t = (P^t)_{d-s}.$$

This implies that $\ker(P_s)^t = \ker(P^t)_{d-s} = \ker(P^t)_\infty$.

The annihilator of $\operatorname{im} P_s$ in $H_{d-s}(M, F^\vee)$ relative to the natural pairing

$$H_{s-d}(M, F) \times H_{d-s}(M, F^\vee) \to \mathbb{C}$$

equals $\ker(P_s)^t$ hence $\ker(P^t)_\infty$. By perfectness of the pairing, it follows that

$$\begin{aligned}(\ker P^t)_\infty &\simeq \{u \in H_{d-s}(M, F)' \mid u = 0 \text{ on } \operatorname{im} P_s\}\\ &\simeq [H_{d-s}(M, F)/\overline{\operatorname{im}(P_s)}]'.\end{aligned}$$

Since P_s is Fredholm, its image is closed and of finite codimension. Hence $\ker(P^t)_\infty \simeq \operatorname{coker}(P_s)^*$ and it follows that

$$\operatorname{index}(P_s) = \dim\ker P_\infty - \dim\ker(P^t)_\infty.$$

We will complete the proof by showing that $\ker(P^t)_\infty \simeq (\operatorname{coker} P_\infty)^*$, naturally. For this we note that by the elliptic regularity theorem, Theorem 8.4.8,

$$\operatorname{im}(P_s) \cap \Gamma^\infty(M, F) = \operatorname{im} P_\infty.$$

Since $\Gamma^\infty(M, F) \subset H_{s-d}(M, F)$ with continuous inclusion map, and since $\operatorname{im}(P_s)$ is closed in $H_{s-d}(M, F)$, it follows that $\operatorname{im}(P_\infty)$ is closed in $\Gamma^\infty(M, F)$. The annihilator of $\operatorname{im} P$ in $\Gamma^\infty(M, F)' = \mathcal{D}'(M, F^\vee)$ equals the kernel of $(P^t)_{-\infty} : \mathcal{D}'(M, F^\vee) \to \mathcal{D}'(M, E^\vee)$, which in turn equals $\ker(P^t)_\infty$ by the elliptic regularity theorem. This implies that

$$\ker(P^t)_\infty \simeq [\Gamma^\infty(M, F)/\operatorname{im}(P_\infty)]'.$$

As the first of these spaces is finite dimensional, P_∞ has finite dimensional cokernel, and $\ker(P^t)_\infty \simeq [\Gamma^\infty(M, F)/\operatorname{im}(P_\infty)]^*$. □

For obvious reasons, the integer $\operatorname{index}(P_\infty)$ is called the index of P and will more briefly be denoted by $\operatorname{index}(P)$. The following result asserts that the index depends on P through its principal symbol.

Lemma 9.3.2. *Let M be compact, and E, F complex vector bundles on M. Let $P, P' \in \Psi^d(E, F)$ be elliptic operators. Then*

$$\sigma^d(P) = \sigma^d(P') \Rightarrow \text{index}\,(P) = \text{index}\,(P').$$

Proof From the equality of the principal symbols, it follows that $P - P' = Q \in \Psi^{d-1}(E, F)$. Let $s \in \mathbb{R}$. The operator Q maps $H_s(M, E)$ to $H_{s-d+1}(M, F)$. The latter space is contained in $H_{s-d}(M, F)$, with compact inclusion map. It follows that Q is compact as an operator $H_s(M, E) \to H_{s-d}(M, F)$. We conclude that $P_s - P'_s$ is compact, hence $\text{index}\,(P_s) = \text{index}\,(P'_s)$. $\square$

LECTURE 10
Characteristic classes

At this point we know that, for an elliptic differential operator of order d,

$$P : \Gamma(E) \to \Gamma(F)$$

its analytical index

$$\mathrm{Index}(P) = \dim(\mathrm{Ker}(P)) - \dim(\mathrm{Coker}(P))$$

is well-defined (finite) and only depends on the principal symbol

$$\sigma_d(P) : \pi^* E \to \pi^* F$$

(where $\pi : T^*M \to M$ is the projection). The Atiyah-Singer index theorem gives a precise formula for Index(P) in terms of topological data associated to $\sigma_d(P)$,

$$\mathrm{Index}(P) = (-1)^n \int_{TM} ch(\sigma_d(P)) Td(TM \otimes C).$$

The right hand side, usually called the topological index, will be explained in the next two lectures. On short, the two terms "ch" and "Td" are particular characteristic classes associated to vector bundles. So our aim is to give a short introduction into the theory of characteristic classes.

The idea is to associate to vector bundles E over a manifold M certain algebraic invariants which are cohomology classes in $H^*(M)$ which "measure how non-trivial E is", and which can distinguish non-isomorphic vector bundles. There are various approaches possible. Here we will present the geometric one, which is probably also the simplest, based on the notion of connection and curvature. The price to pay is that we need to stay in the context of smooth manifolds, but that is enough for our purposes.

Conventions: Although our main interest is on complex vector bundles, for the theory of characteristic classes it does not make a difference (for a large part of the theory) whether we work with complex or real vector bundles. So, we will fix a generic ground field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$) and, unless a clear specification is made, by vector bundle we will mean a vector bundle over the generic field $\mathbb{F}$.

Accordingly, when referring to $C^\infty(M)$, TM, $\Omega^p(M)$, $\mathcal{X}(M)$, $H^*(M)$ without further specifications, we mean in this lecture the versions which take into account $\mathbb{F}$; i.e., when $\mathbb{F} = \mathbb{C}$, then they denote the algebra of $\mathbb{C}$-valued smooth functions on M, the complexification of the real tangent bundle, complex-valued forms, complex vector fields (sections of the complexified real tangent bundle), DeRham cohomology with coefficients in $\mathbb{C}$. Also, when referring to linearity (of a map), we mean linearity over $\mathbb{F}$.

10.1. Connections

Throughout this section E is a vector bundle over a manifold M. Unlike the case of smooth functions on manifolds (which are sections of the trivial line bundle!), there is no canonical way of taking derivatives of sections of (an arbitrary) E along vector fields. That is where connections come in.

Definition 10.1.1. A connection on E is a bilinear map ∇

$$\mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s),$$

satisfying

$$\nabla_{fX}(s) = f\nabla_X(s), \ \nabla_X(fs) = f\nabla_X(s) + L_X(f)s,$$

for all $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$.

Remark 10.1.2. In the case when E is trivial, with trivialization frame

$$e = \{e_1, \ldots, e_r\},$$

giving a connection on E is the same thing as giving an r by r matrix whose entries are 1-forms on M:

$$\omega := (\omega_i^j)_{i,j} \in M_r(\Omega^1(U)).$$

Given ∇, ω is define by

$$\nabla_X^U(e_i) = \sum_{j=1}^r \omega_i^j(X)e_j.$$

Conversely, for any matrix ω, one has a unique connection ∇ on E for which the previous formula holds: this follows from the Leibniz identity.

Remark 10.1.3. Connections are local in the sense that, for a connection ∇ and $x \in M$,

$$\nabla_X(s)(x) = 0$$

for any $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$ such that $X = 0$ or $s = 0$ in a neighborhood U of x. This can be checked directly, or can be derived from the remark that ∇ is a differential operator of order one in X and of order zero in f.

Locality implies that, for $U \subset M$ open, ∇ induces a connection ∇^U on the vector bundle $E|_U$ over U, uniquely determined by the condition

$$\nabla_X(s)|_U = \nabla^U_{X|_U}(s_U).$$

Choosing U the domain of a trivialization of E, with corresponding local frame $e = \{e_1, \ldots, e_r\}$, the previous remark shows that, over U, ∇ is uniquely determined by a matrix

$$\theta := (\theta_i^j)_{i,j} \in M_r(\Omega^1(U)).$$

This matrix is called the connection matrix of ∇ over U, with respect to the local frame e (hence a more appropriate notation would be $\theta(\nabla, U, e)$).

Proposition 10.1.4. *Any vector bundle E admits a connection.*

Proof Start with a partition of unity η_i subordinated to an open cover $\{U_i\}$ such that $E|_{U_i}$ is trivializable. On each $E|_{U_i}$ we consider a connection ∇^i (e.g., in the previous remark consider the zero matrix). Define ∇ by

$$\nabla_X(s) := \sum_i \nabla_{X|_{U_i}})(\eta_i s).$$

□

Next, we point out s slightly different way of looking at connections, in terms of differential forms on M. Recall that the elements $\omega \in \Omega^p(M)$ (p-forms) can be written locally, with respect to coordinates $(x_1, \ldots, x_n)$ in M, as

$$\omega = \sum_{i_1,\ldots,i_p} f^{i_1,\ldots,i_p} dx_{i_1} \ldots dx_{i_p}, \tag{10.1}$$

with $f^{i_1,\ldots,i_p}$-smooth functions while globally, they are the same thing as $C^\infty(M)$-multilinear, antisymmetric maps

$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)}_{p \text{ times}} \to C^\infty(M),$$

where $\mathcal{X}(M)$ is the space of vector fields on M.

Similarly, for a vector bundle E over M, we define the space of E-valued p-differential forms on M

$$\Omega^p(M; E) = \Gamma(\Lambda^p T^* M \otimes E).$$

As before, its elements can be written locally, with respect to coordinates $(x_1, \ldots, x_n)$ in M,

$$\eta = \sum_{i_1,\ldots,i_p} dx_{i_1} \ldots dx_{i_p} \otimes e^{i_1,\ldots,i_p}. \tag{10.2}$$

with $e^{i_1,\ldots,i_p}$ local sections of E. Using also a local frame $e = \{e_1, \ldots, e_r\}$ for E, we obtain expressions of type

$$\sum_{i_1,\ldots,i_p,i} f_i^{i_1,\ldots,i_p} dx_{i_1} \ldots dx_{i_p} \otimes e_i.$$

Globally, such an η is a $C^\infty(M)$-multilinear antisymmetric maps

$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)}_{p \text{ times}} \to \Gamma(E).$$

Recall also that

$$\Omega(M) = \bigoplus_p \Omega^p(M)$$

is an algebra with respect to the wedge product: given $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$, their wedge product $\omega \wedge \eta \in \Omega^{p+q}(M)$, also denoted $\omega\eta$, is given by

$$(\omega\wedge\eta)(X_1, \ldots, X_{p+q}) = \sum_{\sigma} sign(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\cdot\eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}), \tag{10.3}$$

where the sum is over all (p, q)-shuffles σ, i.e. all permutations σ with $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$. Although this formula no longer makes sense when ω and η are both E-valued differential forms, it does make sense when one of them is E-valued and the other one is a usual form. The resulting operation makes

$$\Omega(M, E) = \bigoplus_{p} \Omega^p(M, E)$$

into a (left and right) module over $\Omega(M)$. Keeping in mind the fact that the spaces Ω are graded (i.e are direct sums indexed by integers) and the fact that the wedge products involved are compatible with the grading (i.e. $\Omega^p \wedge \Omega^q \subset \Omega^{p+q}$), we say that $\Omega(M)$ is a graded algebra and $\Omega(M, E)$ is a graded bimodule over $\Omega(M)$. As for the usual wedge product of forms, the left and right actions are related by[1]

$$\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega \quad \forall\ \omega \in \Omega^p(M), \eta \in \Omega^q(M, E).$$

In what follows we will be mainly using the left action.

Finally, recall that $\Omega(M)$ also comes with DeRham differential d, which increases the degree by one, satisfies the Leibniz identity

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d(\eta),$$

where $|\omega|$ is the degree of ω[2], and is a differential (i.e. $d \circ d = 0$). We say that $(\Omega(M)$ is a DGA (differential graded algebra). However, in the case of $\Omega(M, E)$ there is no analogue of the DeRham operator.

Proposition 10.1.5. *Given a vector bundle E over M, a connection ∇ on E induces a linear operator which increases the degree by one,*

$$d_{\nabla} : \Omega^{\bullet}(M, E) \to \Omega^{\bullet+1}(M, E)$$

which satisfies the Leibniz identity

$$d_{\nabla}(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d_{\nabla}(\eta)$$

for all $\omega \in \Omega(M)$, $\eta \in \Omega(M, E)$. The operator ∇ is uniquely determined by these conditions and

$$d_{\nabla}(s)(X) = \nabla_X(s)$$

for all $s \in \Omega^0(M, E) = \Gamma(E)$, $X \in \mathcal{X}(M)$.

Moreover, the correspondence $\nabla \leftrightarrow d_{\nabla}$ is a bijection between connections on E and operators d_{∇} as above.

[1] Important: this is the first manifestation of what is known as the "graded sign rule": in an formula that involves graded elements, if two elements a and b of degrees p and q are interchanged, then the sign $(-1)^{pq}$ is introduced

[2] Note: the sign in the formula agrees with the graded sign rule: we interchange d which has degree 1 and ω

Instead of giving a formal proof (which is completely analogous to the proof of the basic properties of the DeRham differential), let us point out the explicit formulas for d_∇, both global and local. The global one is completely similar to the global description of the DeRham differential- the so called Koszul formula: for $\omega \in \Omega^p(M)$, $d\omega \in \Omega^{p+1}(M)$ is given by

$$\begin{aligned} d(\eta)(X_1, \ldots, X_{p+1}) &= \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1})) \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} L_{X_i}(\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})). \end{aligned} \tag{10.4}$$

Replacing the Lie derivatives L_{X_i} by ∇_{X_i}, the same formula makes sense for $\eta \in \Omega(M, E)$; the outcome is precisely $d_\nabla(\eta) \in \Omega^{p+1}(M, E)$. For the local description we fix coordinates $(x_1, \ldots, x_n)$ in M and a local frame $e = \{e_1, \ldots, e_r\}$ for E. We have to look at elements of form (10.2). The Leibniz rule for d_∇ implies that

$$d_\nabla(\eta) = \sum_{i_1, \ldots, i_p} (-1)^p dx_{i_1} \ldots dx_{i_p} \otimes d_\nabla(e^{i_1, \ldots, i_p})$$

hence it suffices to describe d_∇ on sections of E. The same Leibniz formula implies that it suffices to describe d_∇ on the frame e. Unraveling the last equation in the proposition, we find

$$d_\nabla(e_i) = \sum_{j=1}^{r} \omega_i^j e_j, \tag{10.5}$$

where $\theta = (\theta_j^i)_{i,j}$ is the connection matrix of ∇ with respect to e.

Exercise 10.1.6. Let ∇ be a connection on E, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$, $x \in M$ and $\gamma : (-\epsilon, \epsilon) \to M$ a curve with $\gamma(0) = x$, $\gamma'(0) = X_x$. Show that if $X_x = 0$ or $s = 0$ along γ, then

$$\nabla_X(s)(x) = 0.$$

Deduce that, for any $X_x \in T_xM$ and any section s defined around x, it makes sense to talk about $\nabla_{X_x}(s)(x) \in E_x$.

10.2. Curvature

Recall that, for the standard Lie derivatives of functions along vector fields,

$$L_{[X,Y]} = L_X L_Y(f) - L_Y L_X(f).$$

Of course, this can be seen just as the definition of the Lie bracket $[X, Y]$ of vector fields but, even so, it still says something: the right hand side is a derivation on f (i.e., indeed, it comes from a vector field). The similar formula for connections fails dramatically (i.e. there are few vector bundles which admit a connection for which the analogue of this formula holds). The failure is measured by the curvature of the connection.

Proposition 10.2.1. *For any connection ∇, the expression*

$$k_{\nabla}(X,Y)s = \nabla_X\nabla_Y(s) - \nabla_Y\nabla_X(s) - \nabla_{[X,Y]}(s), \tag{10.6}$$

is $C^{\infty}(M)$-linear in the entries $X, Y \in \mathcal{X}(M)$, $s \in \Gamma(E)$. Hence it defines an element

$$k_{\nabla} \in \Gamma(\Lambda^2 T^*M \otimes End(E)) = \Omega^2(M; End(E)),$$

called the curvature of ∇.

Proof It follows from the properties of ∇. For instance, we have

$$\begin{aligned} \nabla_X\nabla_Y(fs) &= \nabla_X(f\nabla_Y(s) + L_Y(f)s) \\ &= f\nabla_X\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_XL_Y(f)s, \end{aligned}$$

and the similar formula for $\nabla_X\nabla_Y(fs)$, while

$$\nabla_{[X,Y]}(fs) = f\nabla_{[X,Y]}(s) + L_{[X,Y]}(f)s.$$

Hence, using $L_{[X,Y]} = L_XL_Y - L_YL_X$, we deduce that

$$k_{\nabla}(X,Y)(fs) = fk_{\nabla}(X,Y)(s),$$

and similarly the others. □

Remark 10.2.2. One can express the curvature locally, with respect to a local frame $e = \{e_1, \ldots, e_r\}$ of E over an open U, as

$$k_{\nabla}(X,Y)e_i = \sum_{j=1}^{r} k_i^j(X,Y)e_j,$$

where $k_i^j(X,Y) \in C^{\infty}(U)$ are smooth functions on U depending on $X, Y \in \mathcal{X}(M)$. The previous proposition implies that each k_i^j is a differential form (of degree two). Hence k_{∇} is locally determined by a matrix

$$k = (k_i^j)_{i,j} \in M_n(\Omega^2(U)),$$

called the curvature matrix of ∇ over U, with respect to the local frame e. Of course, we should be able to compute k in terms of the connection matrix θ. This will be done as bit later.

There is another interpretation of the curvature, in terms of forms with values in E. While ∇ defines the operator d_{∇} which is a generalization of the DeRham operator d, it is very rarely that it squares to zero (as d does). Again, k_{∇} measure this failure. To explain this, we first look more closely to elements

$$K \in \Omega^p(M, End(E)).$$

The wedge product formula (10.3) has a version when $\omega = K$ and $\eta \in \Omega^q(M,E)$:

$$(K \wedge \eta)(X_1, \ldots, X_{p+q}) = \sum_{\sigma} sign(\sigma)K(X_{\sigma(1)}, \ldots, X_{\sigma(p)})(\eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})),$$

Any such K induces a linear map

$$\hat{K} : \Omega^{\bullet}(M,E) \to \Omega^{\bullet+p}(M,E), \;\; \hat{K}(\eta) = K \wedge \eta.$$

For the later use not also that the same formula for the wedge product has an obvious version also when applied to elements $K \in \Omega^p(M, End(E))$ and $K' \in \Omega^q(M, End(E))$, giving rise to operations

$$\wedge : \Omega^p(M, End(E)) \times \Omega^q(M, End(E)) \to \Omega^{p+q}(M, End(E)) \tag{10.7}$$

which make $\Omega(M, End(E))$ into a (graded) algebra.

Exercise 10.2.3. Show that $\hat{K}$ is an endomorphism of the graded (left) $\Omega(M)$-module $\Omega(M, E)$ i.e., according to the graded sign rule (see the previous footnotes):

$$\hat{K}(\omega \wedge \eta) = (-1)^{pq} \omega \wedge K(\eta),$$

for all $\omega \in \Omega^q(M)$.

Moreover, the correspondence $K \mapsto \hat{K}$ defines a bijection

$$\Omega^p(M, End(E)) \cong End^p_{\Omega(M)}(\Omega(M, E))$$

between $\Omega^p(M, End(E))$ and the space of all endomorphisms of the graded (left) $\Omega(M)$-module $\Omega(M, E)$ which rise the degree by p.

Finally, via this bijection, the wedge operation (10.7) becomes the composition of operators, i.e.

$$\widehat{K \wedge K'} = \hat{K} \circ \hat{K}'$$

for all $K, K' \in \Omega(M, End(E))$.

Due to the previous exercise, we will tacitly identify the element K with the induced operator m_K. For curvature of connections we have

Proposition 10.2.4. *If ∇ is a connection on E, then*

$$d^2_\nabla = d_\nabla \circ d_\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+2}(M, E)$$

is given by

$$d^2_\nabla(\eta) = k_\nabla \wedge \eta$$

for all $\eta \in \Omega^*(M; E)$, and this determines k_∇ uniquely.

Proof Firs of all, d_∇ is $\Omega(M)$-linear: for $\omega \in \Omega^p(M)$ and $\eta \in \Omega^p(M, E)$,

$$\begin{aligned} d^2_\nabla(\omega \wedge \eta) &= d_\nabla(d(\omega) \wedge \eta + (-1)^p \omega \wedge d_\nabla(\eta)) \\ &= [d^2(\omega) \wedge \eta + (-1)^{p+1} d(\omega) \wedge d_\nabla(\eta)] + (-1)^p [d(\omega) \wedge d_\nabla(\eta) + (-1)^p \omega \wedge d^2_\nabla(\eta) \\ &= \omega \wedge d_\nabla(\eta). \end{aligned}$$

Hence, by the previous exercise, it comes from multiplication by an element $k \in \Omega^2(M)$. Using the explicit Koszul-formula for d_∇ to compute d^2_∇ on $\Gamma(E)$, we see that $d^2_\nabla(s) = k_\nabla \wedge s$ for all $s \in \Gamma(E)$. We deduce that $k = k_\nabla$. □

Exercise 10.2.5. Let θ and k be the connection and curvature matrices of ∇ with respect to a local frame e. Using the local formula (10.5) for d_∇ and the previous interpretation of the curvature, show that

$$k^j_i = d\theta^j_i - \sum_k \theta^k_i \wedge \theta^j_k,$$

or, in a more compact form,

$$\boxed{k = d\theta - \theta \wedge \theta}. \tag{10.8}$$

10.3. Characteristic classes

The local construction of characteristic classes is obtained by gluing together expressions built out of connection matrices associated to a connection. Hence it is important to understand how connection matrices change when the frame is changed.

Lemma 10.3.1. *Let ∇ be a connection on E. Let $e = \{e_1, \ldots, e_r\}$ be a local frame of E over an open U and let θ and k be the associated connection matrix and curvature matrix, respectively. Let $e' = \{e'_1, \ldots, e'_r\}$ be another local frame of E over some open U' and let θ' and k' be the associated connection and curvature matrix of ∇. Let*

$$g = (g_i^j) \in M_n(C^\infty(U \cap U'))$$

be the matrix of coordinate changes from e to e', i.e. defined by:

$$e_i^{'} = \sum_{j=1}^{r} g_i^j e_j$$

over $U \cap U'$. Then, on $U \cap U'$,

$$\boxed{\theta' = (dg)g^{-1} + g\theta g^{-1}.}$$

$$\boxed{k' = gkg^{-1}.}$$

Proof Using formula (10.5) for d_∇ we have:

$$\begin{aligned} d_\nabla(e'_i) &= d_\nabla(\sum_l g_i^l e_l) \\ &= \sum_l d(g_i^l)e_l + \sum_{l,m} g_i^l \theta_l^m e_m, \end{aligned}$$

where for the last equality we have used the Leibniz rule and the formulas defining θ. Using the inverse matrix $g^{-1} = (\overline{g}_j^i)_{i,j}$ we change back from the frame e to e' by $e_j = \sum_i \overline{g}_j^i \omega_i$ and we obtain

$$d_\nabla(e'_i) = \sum_{l,j} d(g_i^l)\overline{g}_l^j e'_j + \sum_{l,m,j} g_i^l \theta_l^m \overline{g}_m^j e'_j.$$

Hence

$$(\theta')_i^j = \sum_l d(g_i^l)\overline{g}_l^j + \sum_{l,m} g_i^l \theta_l^m \overline{g}_m^j,$$

i.e. the first formula in the statement. To prove the second equation, we will use the formula (refinvariance) which expresses k in terms of θ. We have

$$d\theta' = d(dg \cdot g^{-1} + g\theta g^{-1}) = -dgd(g^{-1}) + d(g)\theta g^{-1} + gd(\theta)g^{-1} - g\theta d(g^{-1}).$$

For $\theta' \wedge \theta'$ we find

$$dgg^{-1} \wedge d(g)g^{-1} + dgg^{-1} \wedge g\theta g^{-1} + g\theta g^{-1} \wedge dgg^{-1} + g\theta g^{-1} \wedge g\theta g^{-1}.$$

Since

$$g^{-1}dg = d(g^{-1}g) - d(g^{-1})g = -d(g^{-1})g,$$

the expression above equals to

$$-dgd(g^{-1}) + d(g)\theta g^{-1} - g\theta d(g^{-1}) + g\omega\omega g^{-1}.$$

Comparing with the expression for $d\theta'$, we find

$$k' = d\theta' - \theta' \wedge \theta' = g(d\theta - \theta \wedge \theta)g^{-1} = gkg^{-1}.$$

□

Since the curvature matrix stays the same "up to conjugation", it follows that any expression that is invariant under conjugation will produce a globally defined form on M. The simplest such expression is obtained by applying the trace:

$$Tr(k) = \sum_i k_i^i \in \Omega^2(U).$$

Indeed, it follows immediately that, if k' corresponds to another local frame e' over U', then $Tr(k) = Tr(k')$ on the overlap $U \cap U'$. Hence all these pieces glue to a global 2-form on M:

$$Tr(k_\nabla) \in \Omega^2(M).$$

As we will see later, this form is closed, and the induced cohomology class in $H^2(M)$ does not depend on the choice of the connection (and this will be, up to a constant, the first Chern class of E). More generally, one can use other "invariant polynomials" instead of the trace. We recall that we are working over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 10.3.2. We denote by $I_r(\mathbb{F})$ the space of all functions

$$P : M_r(\mathbb{F}) \to \mathbb{F}$$

which are polynomial (in the sense that $P(A)$ is a polynomial in the entries of A), and which are invariant under the conjugation, i.e.

$$P(gAg^{-1}) = P(A)$$

for all $A \in M_r(\mathbb{F})$, $g \in Gl_r(\mathbb{F})$.

Note that $I_r(\mathbb{F})$ is an algebra (the product of two invariant polynomials is invariant).

Example 10.3.3. For each $p \geq 0$,

$$\Sigma_p : M_r(\mathbb{F}) \to \mathbb{F},\ \Sigma_p(A) = Tr(A^p)$$

is invariant. One can actually show that the elements with $0 \leq p \leq r$ generate the entire algebra $I_r(\mathbb{F})$: any $P \in I_r(\mathbb{F})$ is a polynomial combination of the Σ_p's. Even more, one has an isomorphism of algebras

$$I_r(\mathbb{F}) = \mathbb{F}[\Sigma_0, \Sigma_1, \ldots, \Sigma_p].$$

Example 10.3.4. Another set of generators are obtained using the polynomial functions

$$\sigma_p : M_r(\mathbb{F}) \to \mathbb{F}$$

defined by the equation

$$det(I + tA) = \sum_{p=0}^{r} \sigma_p(A)t^p.$$

For instance, $\sigma_1 = \Sigma_1$ is just the trace while $\sigma_p(A) = det(A)$. One can also prove that

$$I_r(\mathbb{F}) = \mathbb{F}[\sigma_0, \sigma_1, \ldots, \sigma_p].$$

Remark 10.3.5. But probably the best way to think about the invariant polynomials is by interpreting them as symmetric polynomials, over the base field $\mathbb{F}$, in r variables $x_1, \ldots x_r$ which play the role of the eigenvalues of a generic matrix A. More precisely, one has an isomorphism of algebras

$$I_r(\mathbb{F}) \cong Sym_{\mathbb{F}}[x_1, \ldots, x_r]$$

which associates to a symmetric polynomial S the invariant function (still denoted by S) given by

$$S(A) = S(x_1(A), \ldots, x_r(A)),$$

where $x_i(A)$ are the eigenvalues of A. Conversely, any $P \in I_r(\mathbb{F})$ can be viewed as a symmetric polynomial by evaluating it on diagonal matrices:

$$P(x_1, \ldots, x_r) := P(diag(x_1, \ldots, x_r)).$$

For instance, via this bijection, the Σ_p's correspond to the polynomials

$$\Sigma_p(x_1, \ldots, x_r) = \sum_i (x_i)^p,$$

while the σ_p's correspond to

$$\sigma_p(x_1, \ldots, x_r) = \sum_{i_1 < \ldots < i_p} x_{i_1} \ldots x_{i_p},$$

With this it is now easier to express the Σ's in term of the σ's and the other way around (using "Newton's formulas": $\Sigma_1 = \sigma_1$, $\Sigma_2 = (\sigma_1)^2 - 2\sigma_2$, $\Sigma_3 = (\sigma_1)^3 - 3\sigma_1\sigma_2 + 3\sigma_3$, etc.).

From the previous lemma we deduce:

Corollary 10.3.6. *Let $P \in I_r(\mathbb{F})$ be an invariant polynomial of degree p. Then for any vector bundle E over M of rank r and any connection ∇ on E, there exists a unique differential form of degree $2p$,*

$$P(E, \nabla) \in \Omega^{2p}(M)$$

with the property that, for any local frame e of E over some open U,

$$P(E, \nabla)|_U = P(k) \in \Omega^{2p}(U),$$

where $k \in M_r(\Omega^2(U))$ is the connection matrix of ∇ with respect to e.

The following summarizes the construction of the characteristic classes.

Theorem 10.3.7. *Let $P \in I_r(\mathbb{F})$ be an invariant polynomial of degree p. Then for any vector bundle E over M of rank r and any connection ∇ on E, $P(E, \nabla)$ is a closed form and the resulting cohomology class*

$$P(E) := [P(E, \nabla)] \in H^{2p}(M)$$

does not depend on the choice of the connection ∇. It is called the P-characteristic class of E.

This theorem can be proven directly, using local connection matrices. Also, it suffices to prove the theorem for the polynomials $P = \Sigma_p$. This follows from the fact that these polynomials generate $I_r(\mathbb{F})$ and the fact that the construction

$$I_r(\mathbb{F}) \ni P \mapsto P(E, \nabla) \in \Omega(M)$$

is compatible with the products. In the next lecture we will give a detailed global proof for the Σ_p's; the price we will have to pay for having a coordinate-free proof is some heavier algebraic language. What we gain is a better understanding on one hand, but also a framework that allows us to generalize the construction of characteristic classes to "virtual vector bundles with compact support". Here we mention the main properties of the resulting cohomology classes (proven at the end of this lecture).

Theorem 10.3.8. *For any $P \in I_r(\mathbb{F})$, the construction $E \mapsto P(E)$ is natural, i.e.*

1. *If two vector bundles E and F over M, of rank r, are isomorphic, then $P(E) = P(F)$.*
2. *If $f : N \to M$ is a smooth map and*

$$f^* : H^\bullet(M) \to H^\bullet(N)$$

is the pull-back map induced in cohomology, then for the pull-back vector bundle f^*E,

$$P(f^*E) = f^*P(E).$$

10.4. Particular characteristic classes

Particular characteristic classes are obtained by applying the constructions of the previous section to specific polynomials. Of course, since the polynomials Σ_p (and similarly the σ_p's) generate $I(\mathbb{F})$, we do not loose any information if we restrict ourselves to these polynomials and the resulting classes. Why don't we do that? First, one would have to make a choice between the Σ_p's or σ_p's. But, most importantly, it is the properties that we want from the resulting characteristic classes that often dictate the choice of the invariant polynomials (e.g. their behaviour with respect to the direct sum of vector bundles- see below). Sometimes the Σ_p's are better, sometimes the σ_p's, and sometimes others. On top, there are situations when the relevant characteristic classes are not even a matter of choice: they are invariants that show up by themselves in a specific context (as is the case with the Todd class which really shows up naturally when comparing the "Thom isomorphism" in DeRham cohomology with the one in K-theory- but that goes beyond this course).

Here are some of the standard characteristic classes that one considers. We first specialize to the complex case $\mathbb{F} = \mathbb{C}$.

1. Chern classes: They correspond to the invariant polynomials

$$c_p = \left(\frac{1}{2\pi i}\right)^p \sigma_p,$$

with $0 \leq p \leq r$. Hence they associate to complex vector bundle E rank r a cohomology class, called the p-th Chern class of E:

$$c_p(E) \in H^{2p}(M) \quad (0 \leq p \leq r).$$

The total Chern class of E is defined as

$$c(E) = c_0(E) + c_1(E) + \ldots + c_r(E) \in H^{\text{even}}(M);$$

it corresponds to the inhomogeneous polynomial

$$c(A) = det(I + \frac{1}{2\pi i} A).$$

For the purpose of this lecture, the rather ugly constants in front of σ_p (and the similar constants below) are not so important. Their role will be to "normalize" some formulas so that the outcome (the components of the Chern character) are real, or even integral (they come from the cohomology with integral coefficients; alternatively, one may think that they produce integrals which are integers). If you solve the following exercise you will find out precisely such constants showing up.

Exercise 10.4.1. Let $M = \mathbb{CP}^1$ be the complex projective space, consisting of complex lines in $\mathbb{C}^2$ (i.e. 1-dimensional complex vector subspaces) in $\mathbb{C}^2$. Let $L \subset \mathbb{CP}^1 \times \mathbb{C}^2$ be the tautological line bundle over M (whose fiber above $l \in \mathbb{CP}^1$ is l viewed as a complex vector space). Show that

$$c_1(L) \in H^2(\mathbb{CP}^1)$$

is non-trivial. What is its integral? (the element $a := -c_1(L) \in H^2(\mathbb{CP}^1)$ will be called the canonical generator).

Here are the main properties of the Chern classes (the proofs will be given at the end of the section).

Proposition 10.4.2. *The Chern classes of a complex vector bundle, priory cohomology classes with coefficients in $\mathbb{C}$, are actually real (cohomology classes with coefficients in $\mathbb{R}$). Moreover,*

1. *The total Chern class has an exponential behaviour with respect to the direct sum of vector bundles i.e., for any two complex vector bundles E and F over M,*

$$c(E \oplus F) = c(E)c(F)$$

or, component-wise,

$$c_p(E \oplus F) = \sum_{i+j=p} c_i(E)c_j(F).$$

2. If $\overline{E}$ is the conjugated of the complex vector bundle E, then

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

Remark 10.4.3. One can show that the following properties of the Chern classes actually determines them uniquely:

C1: Naturality (see Theorem 10.3.8).

C2: The behaviour with respect to the direct sum (see the previous proposition).

C3: For the tautological line bundle L, $c_1(L) = -a \in H^2(\mathbb{CP}^1)$ (see the previous exercise).

2. Chern character: The Chern character classes correspond to the invariant polynomials

$$Ch_p = \frac{1}{p!}\left(-\frac{1}{2\pi i}\right)^p \Sigma_p.$$

Hence Ch_p associates to a complex vector bundle E a cohomology class, called the p-th component of the Chern character of E:

$$Ch_p(E) \in H^{2p}(M).$$

They assemble together into the full Chern character of E, defined as

$$Ch(E) = \sum_{p\geq 0} Ch_p(E) \in H^{\text{even}}(M);$$

it corresponds to the expression (which, strictly speaking is a powers series and not a polynomial, but which when evaluated on a curvature matrix produces a finite sum):

$$Ch(A) = Tr(e^{-\frac{1}{2\pi i}A}).$$

Note that $Ch(E)$ are always real cohomology classes; this follows e.g. from the similar property for the Chern classes, and the fact that the relationship between the Σ's and the σ's involve only real coefficients (even rational!). Here is the main property of the Chern character.

Proposition 10.4.4. *The Chern character is additive and multiplicative i.e., for any two complex vector bundles E and F over M,*

$$Ch(E \oplus F) = Ch(E) + Ch(F), \quad Ch(E \otimes F) = Ch(E)Ch(F).$$

3. Todd class: Another important characteristic class is the Todd class of a complex vector bundle. To define it, we first expand formally

$$\frac{t}{1-e^{-t}} = B_0 + B_1 t + B_2 t^2 + \ldots.$$

(the coefficients B_k are known as the Bernoulli numbers). For instance,

$$B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \text{etc},$$

(and they have the property that $B_k = 0$ for k-odd, $k \geq 3$ and they have various other interesting interpretations). For r-variables, we expand the resulting product

$$T := \Pi_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} = T_0 + T_1(x_1, \ldots, x_r) + T_2(x_1, \ldots, x_r) + \ldots$$

where each T_k is a symmetric polynomial of degree k. We proceed as before and define

$$Td_k(E) = \left(\frac{1}{2\pi i}\right)^k T_k(E) \in H^{2k}(M),$$

The Todd class of a vector bundle E is the resulting total characteristic class

$$Td(E) = \sum_k Td_k(E) \in H^{\text{even}}(M).$$

Again, $Td(E)$ are real cohomology classes and Td is multiplicative.

4. Pontryagin classes: We now pass to the case $\mathbb{F} = \mathbb{R}$. The Pontryagin classes are the analogues for real vector bundles of the Chern classes. They correspond to the polynomials

$$p_k = \left(\frac{1}{2\pi}\right)^{2k} \sigma_{2k},$$

for $2k \leq r$. The reason we restrict to the σ's of even degree $(2k)$ is simple: the odd dimensional degree produce zero forms (see below). Hence p_k associates to a real vector bundle E of rank r a cohomology class, called the k-th Pontryagin class of E:

$$p_k(E) \in H^{4k}(M) \quad (0 \leq k \leq r/2).$$

They assemble together into the full Pontryagin class of E, defined as

$$p(E) = p_0(E) + p_1(E) + \ldots + p_{[\frac{r}{2}]} \in H^{\bullet}(M).$$

The Pontryagin class has the same property as the Chern class: for any two real vector bundles,

$$p(E \oplus F) = p(E)p(F).$$

The relationship between the two is actually much stronger (which is expected due to their definitions). To make this precise, we associate to any real vector bundle E its complexification

$$E \otimes \mathbb{C} := E \otimes_{\mathbb{R}} \mathbb{C} = \{e_1 + ie_2 : e_1, e_2 \in E, \}.$$

Proposition 10.4.5. *For any real vector bundle E,*

$$c_l(E \otimes \mathbb{C}) = \begin{cases} (-1)^k p_k(E) & \text{if } l = 2k \\ 0 & \text{if } l = 2k+1 \end{cases}$$

Finally, one can go from a complex vector bundle E to a real one, denoted $E_{\mathbb{R}}$, which is just E vied as a real vector bundle. One has:

Proposition 10.4.6. *For any complex vector bundle E,*

$$p_0(E_{\mathbb{R}}) - p_1(E_{\mathbb{R}}) + p_2(E_{\mathbb{R}}) - \ldots = (c_0(E) + c_1(E) + \ldots)(c_0(E) - c_1(E) + \ldots).$$

5. The Euler classs: There are other characteristic classes which are, strictly speaking, not immediate application of the construction from the previous section, but are similar in spirit (but they fit into the general theory of characteristic classes with structural group smaller then GL_n). That is the case e.g. with the Euler class, which is defined for real, oriented vector bundles E of even rank $r = 2l$. The outcome is a cohomology class over the base manifold M:

$$e(E) \in H^{2l}(M).$$

To construct it, there are two key remarks:

1. For real vector bundles, connection matrices can also be achived to be antisymmetric. To see this, one choose a metric $\langle \cdot, \cdot \rangle$ on E (fiberwise an inner product) and, by a partition of unity, one can show that one can choose a connection ∇ which is compatible with the metric, in the sense that
$$L_x\langle s_1, s_2\rangle = \langle \nabla_X(s_1), s_2\rangle + \langle s_1, \nabla_X(s_2)\rangle$$
for all $s_1, s_2 \in \Gamma(E)$. Choosing orthonormal frames e, one can easily show that this compatibility implies that the connection matrix θ with respect to e is antisymmetric.
2. With the inner product and ∇ as above, concentrating on positively oriented frames, the change of frame matrix, denoted g in the previous section, has positive determinant.
3. In general, for for skew symmetric matrices A of even order $2l$, $det(A)$ is naturally a square of another expression, denoted $Pf(A)$ (polynomial of degree l):
$$det(A) = Pf(A)^2.$$
For instance,
$$det\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af + cd - be)^2.$$
Moreover, for g with positive determinant,
$$Pf(gAg^{-1}) = Pf(A).$$

It follows that, evaluating Pf on connection matrices associated to positive orthonormal frames, we obtain a globvally defined form

$$Pf(E, \nabla) \in \Omega^{2l}(M).$$

As before, this form is closed and the resulting cohomology class does not depend on the choice of the connection. The Euler class is

$$e(E) = \left[\left(\frac{1}{2\pi}\right)^l Pf(E, \nabla)\right] \in H^{2l}(M).$$

Note that, since *det* is used in the construction of $p_l(E)$, it follows that

$$p_l(E) = e(E)^2.$$

10.5. Proofs of the main properties

The proofs of the main properties of the characteristic classes (Theorem 10.3.8, Proposition 10.4.2, Proposition 10.4.4 and Proposition 10.4.5) are based on some basic constructions of connections: pull-back, direct sum, dual, tensor product.

Pullback of connections and the proof of Theorem 10.3.8: For Theorem 10.3.8 we need the construction of pull-back of connections: given a vector bundle E over M and a smooth map $f : N \to M$ then for any connection ∇ on E, there is an induced connection $f^*\nabla$ on f^*E. This can be described locally as follows: if e is a local frame e of E over U and θ is determined by the connection matrix (with respect to e) θ then, using the induced local frame f^*e of f^*E over $f^{-1}(U)$, the resulting connection matrix of $f^*\nabla$ is $f^*\theta$ (pull back all the one-forms which are the entries in the matrix θ). Of course, one has to check that these connection matrices glue together (which should be quite clear due to the naturality of the construction); alternatively, one can describe $f^*\nabla$ globally, by requiring

$$(f^*\nabla)_X(f^*s)(x) = \nabla_{(df)_x(X_x)}(s)(f(x)), \quad x \in N \tag{10.9}$$

where, for $s \in \Gamma(E)$, we denoted by $f^*s \in \Gamma(f^*E)$ the section $x \mapsto s(f(x))$; for the right hand side, see also Exercise 10.1.6.

Exercise 10.5.1. Show that there is a unique connection $f^*\nabla$ on f^*E which has the property (10.9). Then show that its connection matrices can be computed as indicated above.

With this construction, the second part of Theorem 10.3.8 is immediate: locally, the connection matrix of $f^*\nabla$ is just the pull-back of the one of ∇, hence we obtain $P(E, f^*\nabla) = f^*P(E, \nabla)$ as differential forms. The first part of the theorem is easier (exercise!).

Direct sum of connections and the proof of Proposition 10.4.2: For the proof of the behaviour of c with respect to direct sums (and similarly for the Pontryagin class) we need the construction of the direct sum of two connections: given connections ∇^0 and ∇^1 on E and F, respectively, we can form a new connection ∇ on $E \oplus F$:

$$\nabla_X(s^0, s^1) = (\nabla^0_X(s^0), \nabla^1_X(s^1)).$$

To compute its connection matrices, we will use a local frame of $E \otimes F$ which comes by putting together a local frame e for E and a local frame f for F (over the same open). It is then clear that the connection matrix θ for ∇ with respect to this frame, and similarly the curvature matrix, can be written in terms as the curvature matrices θ^i, $i \in \{0,1\}$ for E and F (with respect to e and f) as

$$\theta = \begin{pmatrix} \theta^0 & 0 \\ 0 & \theta^1 \end{pmatrix}, k = \begin{pmatrix} k^0 & 0 \\ 0 & k^1 \end{pmatrix}.$$

Combined with the remark that

$$det(I + t\begin{pmatrix} A^0 & 0 \\ 0 & A^1 \end{pmatrix}) = det(I + tA^0)\ det(I + tA^1)$$

for any two matrices A^0 and A^1, we find that

$$c(E \oplus F, \nabla) = c(E, \nabla^0)c(F, \nabla^1)$$

(as differential forms!) from which the statement follows.

Duals/conjugations of connections and end proof of Prop. 10.4.2: For the rest of the proposition we need the dual and the conjugate of a connection. First of all, a connection ∇ on E induces a connection ∇^* on E^* by:

$$\nabla_X(s^*)(s) := \text{Ł}_X(s^*(s)) - s^*(\nabla_X(s)), \quad \forall\ s \in \Gamma(E), s^* \in \Gamma(E^*).$$

(why this formula?).

Exercise 10.5.2. Show that this is, indeed, a connection on E^*.

Starting with a local frame e of E, it is not difficult to compute the connection matrix of ∇^* with respect to the induced dual frame θ^*, in terms of the connection matrix θ of ∇ with respect to e:

$$\theta^* = -\theta^t$$

(minus the transpose of θ). Hence the same holds for the curvature matrix. Since for any matrix A

$$det(I + t(-A^t)) = det(I - tA) = \sum (-1)^k t^k \sigma_k(A),$$

we have $\sigma_k(-A^t) = (-1)^k \sigma_k(A)$ from which we deduce

$$c_k(E^*) = (-1)^k c_k(E).$$

Similarly, any connection ∇ on E induces a conjugated connection $\overline{\nabla}$ on $\overline{E}$. While $\overline{E}$ is really just $\overline{E}$ but with the structure of complex multiplication changed to:

$$z \cdot v := \overline{z}v, \quad (z \in \mathbb{C}, v \in E)$$

$\overline{\nabla}$ is just ∇ but interpreted as a connection on $\overline{E}$. It is then easy to see that the resulting connection matrix of $\overline{\nabla}$ is precisely $\overline{\theta}$. Since for any matrix A,

$$det(I + \frac{1}{2\pi i}\overline{A}) = \overline{det(I - \frac{1}{2\pi i}A)},$$

we have $c_k(\overline{A}) = (-1)^k \overline{c_k(A)}$ and then

$$c_k(\overline{E}) = (-1)^k \overline{c_k(E)}.$$

Finally, note that for any complex vector bundle E, E^* and $\overline{E}$ are isomorphic (the isomorphism is not canonical- one uses a hermitian metric on E to produce

one). Hence $c_k(E^*) = c_k(\overline{E})$. Comparing with the previous two formulas, we obtain
$$c_k(\overline{E}) = (-1)^k \overline{c_k(E)} = (-1)^k c_k(E)$$
which shows both that $c_k(E)$ is real as well as the last formula in the proposition.

Tensor product of connections and the proof of Proposition 10.4.2: The additivity is proven, as above, using the direct sum of connections and the fact that for any two matrices A (r by r) and A' (r' by r'),
$$Tr \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}^k = Tr(A^k) + Tr(A'^k).$$
For the multiplicativity we need the construction of the tensor product of two connections: given ∇^0 on E and ∇^1 on F, one produces ∇ on $E \otimes F$ by requiring
$$\nabla_X(s^0 \otimes s^1) = \nabla^0_X(s^0) \otimes s^1 + s^0 \otimes \nabla^1_X(s^1)$$
for all $s^0 \in \Gamma(E)$, $s^1 \in \Gamma(F)$. Frames e and f for E and F induce a frame $e \otimes f = \{e_i \otimes f_p : 1 \leq i \leq r, 1 \leq p \leq r'\}$ for $E \otimes F$ (r is the rank of E and r' of F). After a straightforward computation, we obtain as resulting curvature matrix
$$k = k^0 \otimes I_{r'} + I_r \otimes k^1$$
where I_r is the identity matrix and, for two matrices A and B, one of size r and one of size r', $A \otimes B$ denotes the matrix of size rr' (whose columns and rows are indexed by pairs with (i, p) as before)
$$(A \otimes B)^{j,q}_{i,p} = A^j_i B^q_p.$$
Remarking that $Tr(A \otimes B) = Tr(A)Tr(B)$, we immediately find
$$Tr(e^{A \otimes I + I \otimes B}) = Tr(e^A)Tr(e^B)$$
from which the desired formula follows.

Complexifications of connections and the proof of Proposition 10.4.5: For this second part, i.e. when l is odd, it suffices to remark that, for $E \otimes \mathbb{C}$, it is isomorphic to its conjugation then just apply the last part of Proposition 10.4.2. For l even we have to go again to connections and to remark that a connection ∇ on a real vector bundle E can be complexified to give a connection $\nabla^{\mathbb{C}}$ on $E \otimes \mathbb{C}$: just extend ∇ by requiring $\mathbb{C}$-linearity. Comparing the connection matrices we immediately find that
$$c_{2k}(E \otimes \mathbb{C}) = \left(\frac{1}{i}\right)^{2k} p_k(E) = (-1)^k p_k(E).$$

Proof of Proposition 10.4.6: The main observation is that, for any complex vector bundle E, one has a canonical isomorphism
$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}$$
defined fiberwise by
$$v + \sqrt{-1}w \mapsto (v + iw, v - iw),$$

where $\sqrt{-1}$ is "the i used to complexify $E_{\mathbb{R}}$). Hence

$$c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E)c(\overline{E})$$

and, using the formulas from Proposition 10.4.5 and from end of Proposition 10.4.2, the desired formula follows.

10.6. Some exercises

Here are some more exercises to get used with these classes but also to see some of their use (some of which are rather difficult!).

Exercise 10.6.1. Show that, for any trivial complex vector bundle T (of arbitrary rank) and any other complex vector bundle E,

$$c(E \oplus T) = c(E).$$

Exercise 10.6.2. (the normal bundle trick). Assume that a manifold N is embedded in the manifold M, with normal bundle ν. Let τ_M be the tangent bundle of M and τ_N the one of N. Show that

$$c(\tau_M)|_N = c(\tau_N)c(\nu).$$

Exercise 10.6.3. For the tangent bundle τ of S^n show that

$$p(\tau) = 1.$$

Exercise 10.6.4. Show that the the tautological line bundle L over $\mathbb{CP}^1$ (see Exercise 10.4.1) is not isomorphic to the trivial line bundle. Actually, show that over $\mathbb{CP}^1$ one can find an infinite family of non-isomorphic line bundles.

Exercise 10.6.5. Let $L \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be the tautological line bundle over $\mathbb{CP}^n$ (generalizing the one from Exercise 10.4.1) and let

$$a := -c_1(L) \in H^2(\mathbb{CP}^n).$$

Show that

$$\int_{\mathbb{CP}^n} a^n = 1.$$

Deduce that all the cohomology classes $a, a^2, \ldots, a^n$ are non-zero.

Exercise 10.6.6. This is a continuation of the previous exercise. Let τ be the tangent bundle of $\mathbb{CP}^n$- a complex vector bundle (why?), and let $\tau_{\mathbb{R}}$ be the underlying real vector bundle. We want to compute $c(\tau)$ and $p(\tau_{\mathbb{R}})$.

Let $L^{\perp} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be the (complex) vector bundle over $\mathbb{CP}^n$ whose fiber at $l \in \mathbb{CP}^n$ is the orthogonal $l^{\perp} \subset \mathbb{C}^{n+1}$ of l (with respect to the standard hermitian metric).

1. Show that

$$Hom(L, L) \cong T^1, \ Hom(L, L^{\perp}) \oplus T^1 \cong \underbrace{L^* \oplus \ldots \oplus L^*}_{n+1},$$

where T^k stands for the trivial complex vector bundle of rank k.

2. Show that

$$\tau \cong Hom(L, L^{\perp}).$$

3. Deduce that
$$c(\tau) = (1+a)^{n+1},\ p(\tau_{\mathbb{R}}) = (1+a^2)^{n+1}.$$
4. Compute $Ch(\tau)$.

Exercise 10.6.7. Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.
(Hint: use the previous computation and the normal bundle trick).

Exercise 10.6.8. Show that $\mathbb{CP}^{2010}$ cannot be written as the boundary of a compact, oriented (real) manifold.
(Hint: first, using Stokes' formula, show that if a manifold M of dimension $4l$ can be written as the boundary of a compact oriented manifold then $\int_M p_l(\tau_M)) = 0$).

Exercise 10.6.9. Show that $\mathbb{CP}^{2010}$ can not be written as the product of two complex manifolds of non-zero dimension.
(Hint: for two complex manifolds M and N of complex dimensions m and n, respectively, what happens to $Ch_{m+n}(\tau_{M\times N}) \in H^{2m+2n}(M \times N)$?).

10.7. Chern classes: the global description and the proof of Theorem 10.3.7

In this section we look closer at the Chern character, which shows up naturally in the context of the Atiyah-Singer index theorem. We start by giving a slightly different dress (of a more global flavour) to the discussion of characteristic classes from the previous lecture and we prove Theorem 10.3.7. The algebraic formalism that we use is known as "Quillen's formalism". It has the advantage that it applies (with very litlle changes) to other settings (in particular, it can be addapted easily to non-commutative geometry).

Although we concentrate on the Chern character (which uses the invariant polynomials Σ_p), most of what we say in this section can be carried out for arbitrary invariant polynomials (see also the comments which follow Theorem 10.3.7).

This section can (and should) be viewed as a global presentation of the Chern classes already discussed (including proofs). There are three ingredients that we need. The first two have already been discussed

- the algebra $\Omega^*(M, End(E))$ of differential forms on M with coefficients in $End(E)$, and the wedge product (10.7) which makes it into a (graded) algebra.
- a connection ∇ on E and the associated curvature, interpreted globally as an element
$$k_{\nabla} \in \Omega^2(M, End(E)).$$

The last ingredient is a global version of the trace map. Due to the invariance of the usual trace map of matrices, it follows that it does not really depend on the choice of the basis; hence any finite dimensional (complex) vector space V comes with a trace map
$$Tr_V : End(V) \to \mathbb{C}$$

which can be computed via a (any) basis e_i of V; the any $T \in End(V)$ has a matrix representation $T = (t_i^j)$ and $Tr_V(T) = \sum_i t_i^i$. This map has "the trace property" (an infinitesimal version of invariance):

$$Tr_V([A, B]) = 0, \tag{10.10}$$

for all $A, B \in End(V)$, where $[A, B] = AB - BA$.

Since Tr_V is intrinsec to V, it can be applied also to vector bundles: for any vector bundle E, the trace map (applied fiberwise) is a vector bundle map from $End(E)$ to the trivial line bundle. In particular, one has an induced map, which is the final ingredient:

- The trace map:

$$Tr : \Omega^\bullet(M; End(E)) \to \Omega^\bullet(M). \tag{10.11}$$

Exercise 10.7.1. Show that Tr has the graded trace property:

$$Tr([A, B]) = 0,$$

for all $A \in \Omega^p(M; End(E))$, $B \in \Omega^q(M; End(E))$, where $[A, B]$ are the graded commutators:

$$[A, B] = A \wedge B - (-1)^{pq} B \wedge A.$$

and $\wedge$ is here the wedge operation on $\Omega(M, End(E))$ (see (10.7)).

Putting things together we find immediately:

Proposition 10.7.2. *For a connection ∇ on E,*

$$Ch_p(E, \nabla) = \frac{1}{p!} \left(-\frac{1}{2\pi i} \right)^p Tr(k_\nabla^p) \in \Omega^{2p}(M).$$

Hence, using the wedge product and the usual power seria development to define formally the exponential map

$$exp : \Omega(M, End(E)) \to \Omega(M, End(E)),$$

the Chern character can be written as

$$Ch(E, \nabla) = Tr(e^{-\frac{1}{2\pi\sqrt{-1}} k_\nabla}).$$

Next, we explain the relationship between these ingredients; putting everything together, we will obtain the proof of Theorem 10.3.7 (for the Chern classes). The plan is as follows:

- show that the connection ∇ on E induces a connection $\tilde{\nabla}$ on $End(E)$.
- show that $Tr : \Omega^\bullet(M; End(E)) \to \Omega^\bullet(M)$ is compatible with $d_{\tilde{\nabla}}$ and d.
- show that k_∇ is $d_{\tilde{\nabla}}$-closed.

Lemma 10.7.3. *Given a connection ∇ on E, there is an induced connection $\tilde{\nabla}$ on $End(E)$, given by*

$$\tilde{\nabla}_X(T)(s) = \nabla_X(T(s)) - T(\nabla_X(s)),$$

for all $X \in \mathcal{X}(M)$, $T \in \Gamma(End(E))$, $s \in \Gamma(E)$. Moreover, the induced operator

$$d_{\tilde{\nabla}} : \Omega^p(M; End(E)) \to \Omega^{p+1}(M; End(E))$$

is given by

$$d_{\tilde{\nabla}}(K) = [d_{\nabla}, K] \tag{10.12}$$

where, as in Exercise 10.2.3, we identify the elements in $K \in \Omega^*(M; End(E))$ *with the induced operators* $\hat{K}$ *and we use the graded commutators*

$$[d_{\nabla}, K] = d_{\nabla} \circ K - (-1)^p K \circ d_{\nabla}$$

In particular, $d_{\tilde{\nabla}}$ *also satisfies the Leibniz identity with respect to the wedge product (10.7) on* $\Omega(M, End(E))$.

Proof First one has to check that, for $X \in \mathcal{X}(M)$, $T \in \Gamma(End(E))$, $\tilde{\nabla}_X(T) \in \Gamma(End(E))$, i.e. the defininig formula is $C^{\infty}(M)$-linear on $s \in \Gamma(E)$:

$$\begin{aligned}
\tilde{\nabla}_X(T)(fs) &= \nabla_X(T(fs)) - T(\nabla_X(fs)) \\
&= [f\nabla_X(T(s)) + L_X(f)T(s)] - [fT(\nabla_X(s)) + L_X(f)T(s)] \\
&= f\tilde{\nabla}_X(T)(s).
\end{aligned}$$

The identities that $\tilde{\nabla}$ have to satisfy to be a connection are proven similarly. For the second part, we first have to show (similar to what we have just checked) that, for $K \in \Omega^p(M, End(E))$,

$$K' := [d_{\nabla}, K] \in \Omega^{p+1}(M, End(E)).$$

More porecisely, unraveling the identifications between the elements K and the operators $\hat{K}$, we have to show that

$$\hat{K}' := d_{\nabla} \circ \hat{K} - (-1)^p \hat{K} \circ d_{\nabla} : \Omega^{\bullet}(M, E) \to \Omega^{\bullet + p+1}(M, E)$$

is $\Omega(M)$-linear, i.e.

$$\hat{K}'(\omega \wedge \eta) = (-1)^{(p+1)|\omega|} \omega \wedge \hat{K}'(\eta)$$

for all $\omega \in \Omega^p(M)$, $\eta \in \Omega(M, E)$. writing out the left hand side we find

$$d_{\nabla}(\hat{K}(\omega \wedge \eta)) - (-1)^p k(d_{\nabla}(\omega \wedge \eta))$$

which is (using the $\Omega(M)$-linearity of $\hat{K}$ and the Leibniz identity for d_{∇}):

$$[(-1)^{p|\omega|} d(\omega) \wedge \hat{K}(\eta) + (-1)^{(p+1)|\omega|} \omega \wedge d_{\nabla}(\hat{K}(\eta))] -$$

$$-[(-1)^{p|\omega|} d(\omega) \wedge \hat{K}(\eta) + (-1)^{p+(p+1)|\omega|} \omega \wedge K(d_{\nabla}(\eta))]$$

which equals to

$$(-1)^{(p+1)|\omega|} \omega \wedge [d_{\nabla}(\hat{K}(\eta)) - (-1)^p \hat{K}(d_{\nabla}(\eta)),$$

i.e. the right hand side of the formula to be proven. Hence we obtain an operator

$$D : \Omega^p(M; End(E)) \to \Omega^{p+1}(M; End(E)),\ D(K) = [d_{\nabla}, K],$$

and we have to prove that it coincides with $d_{\tilde{\nabla}}$. By a computation similar to the one above, one shows that D satisfies the Leibniz identity. Hence it suffices to show that the two coincide on elements $T \in \Gamma(End(E))$. They are both elements of $\Omega^1(M, End(E))$ hence they act on $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$ and produce sections of E. For $d_{\tilde{\nabla}}(T)$ we obtain

$$d_{\tilde{\nabla}}(T)(X)s = \nabla_X(T(s)) - T(\nabla_X(s))$$

while for $D(T)$ we obtain

$$\begin{aligned}
\widehat{D(T)}(s)(X) &= (d_{\nabla} \circ \hat{T} - \hat{T} \circ d_{\nabla})(X)(s)] \\
&= d_{\nabla}(T(s))(X) - T(d_{\nabla}(s)(X)) \\
&= \nabla_X(T(s)) - T(\nabla_X(s)).
\end{aligned}$$

□

From the second part of the proposition we immediately deduce:

Corollary 10.7.4. *(Bianchi identity)* $d_{\tilde{\nabla}}(k_{\nabla}) = 0$.

We can now return to the issue of the relationship between the trace map and the DeRham differential.

Proposition 10.7.5. *For any connection* ∇*, the trace map* $Tr : \Omega^*(M; End(E)) \to \Omega^*(M)$ *satisfies*

$$Tr \circ d_{\tilde{\nabla}} = d \circ Tr.$$

Proof Consider

$$\Phi := Tr \circ d_{\tilde{\nabla}} - d \circ Tr : \Omega^{\bullet}(M, End(E)) \to \Omega^{\bullet+1}(M, End(E)).$$

By a computations similar to the previous ones (but simpler), one check the graded $\Omega(M)$-linearity of Φ:

$$\Phi(\omega \wedge K) = (-1)^{|\omega|}\omega \wedge \Phi(K).$$

Hence to show that Φ is zero, it suffices to show that it vanishes on elements $T \in \Gamma(End(E))$. I.e., we have to show that, for all such T's and all $X \in \mathcal{X}(M)$,

$$Tr(\tilde{\nabla}_X(T)) - L_X(Tr(T)) = 0.$$

This can be checked locall, using a local frame $e = \{e_i\}$ for E. Such a local frame induces a local frame $\{e^i_j\}$ for $End(E)$ given by $e^i_j(e_i) = e_j$ and zero on the other e_k's. If the matrix corresopnding to T is $\{t^j_i\}$ ($T(e_i) = \sum_j t^j_i e_j$), for $\tilde{T} := \tilde{\nabla}_X(T)$ we have:

$$\tilde{T}(e_i) = \sum_j \nabla_X(t^j_i e_j) - \sum_j T(\theta^j_i(X)e_j),$$

where $\{\theta^j_i\}$ is the connection matrix. Writing this out, we find the matrix corresponing to $\tilde{T}$:

$$\tilde{t}^k_i = \sum_j t^j_i \theta^k_j(X) - \sum_j \theta^j_i(X) t^k_j + L_X(t^k_i).$$

We deduce that

$$Tr(\tilde{\nabla}_X(T)) = L_X(\sum_i t^i_i) = L_X(Tr(T)).$$

□

Corollary 10.7.6. *For any connection ∇ on E,*

$$Tr(k_{\nabla}^{p}) \in \Omega^{2p}(M)$$

is closed.

Proof From the previous proposition,

$$dTr(k_{\nabla}^{p}) = Tr(d_{\tilde{\nabla}}(k_{\nabla}^{p})).$$

Due to the Leibniz rule for $d_{\tilde{\nabla}}$ and the Bianchi identity (the previous corollary) we find $d_{\tilde{\nabla}}(k_{\nabla}^{p}) = 0$. □

Finally, we prove the independence of the connection.

Proposition 10.7.7. *For any vector bundle E over M, the cohomology class*

$$[Tr(k_{\nabla}^{p})] \in H^{2p}(M)$$

does not depend on the choice of the connection ∇.

Proof Let ∇' be another connection. The expression

$$R(X)s = \nabla'_{X}(s) - \nabla_{X}(s)$$

is then $C^{\infty}(M)$-linear in X and s, hence defines an element

$$R \in \Omega^{1}(M, End(E)).$$

Related to this is the fact that the difference

$$\hat{R} = d_{\nabla'} - d_{\nabla} : \Omega^{\bullet}(M, E) \to \Omega^{\bullet+1}(M, E)$$

is $\Omega(M)$-linear (in the graded sense). This follows immediately from the Leibniz rule for d_{∇} and $d_{\nabla'}$. Our notation is not accidental: R and $\hat{R}$ correspond to each other by the bijection of Exercise 10.2.3 (why?). In what follows we will not distinguish between the two.

The idea is to join the two connections by a path (of connections): for each $t \in [0, 1]$ we consider

$$\nabla^{t} := \nabla + tR$$

and the induced operators

$$d_{\nabla^{t}} = d_{\nabla} + tR.$$

Computing the square of this operator, we find

$$k_{\nabla^{t}} = k_{\nabla} + t[d_{\nabla}, R] + t^{2}R^{2} = k_{\nabla} + td_{\tilde{\nabla}}(R) + t^{2}\frac{1}{2}[R, R]$$

from which we deduce

$$\frac{d}{dt}(k_{\nabla^{t}}) = d_{\tilde{\nabla}}(R) + t[R, R] = d_{\tilde{\nabla}^{t}}(R).$$

Using the Leibniz identity,

$$\frac{d}{dt}(k_{\nabla^{t}}^{p}) = \sum_{i} k_{\nabla^{t}}^{i-1} d_{\tilde{\nabla}^{t}}(R) k_{\nabla^{t}}^{p-i}$$

hence, using the fact that Tr is a graded trace,

$$\frac{d}{dt}(Tr(k_{\nabla^{t}}^{p})) = pTr(k_{\nabla^{t}}^{p-1} d_{\tilde{\nabla}^{t}}(R)).$$

Using again the Leibniz and Bianchi identity,

$$\frac{d}{dt}(Tr(k^p_{\nabla^t})) = pd[Tr(k^{p-1}_{\nabla^t}R)].$$

Integrating from 0 to 1 we find

$$Tr(k^p_{\nabla'}) - Tr(k^p_{\nabla}) = d[p\int_0^1 Tr(k^{p-1}_{\nabla^t}R)],$$

i.e. the two terms in the left hand side differ by an exact form, hence they represent the same element in the cohomology group. $\square$

LECTURE 11
K-theoretical formulation of the Atiyah-Singer index theorem

11.1. K-theory and the Chern character

For a manifold M we denote by $Vb(M)$ the set of isomorphism classes of vector bundles over M. Together with the direct sum of vector bundles, is an abelian semi-group, with the trvial 0-dimensional bundle as the zero-element.

In general, one can associate to any semi-group $(S, +)$ a group $G(S)$, called the Grothendick group of S, which is the "smallest group which can be made out of S". For instance, applied to the semigroup $(\mathbb{N}, +)$ of positive integers, one recovers $(\mathbb{Z}, +)$. Or, if S is already a group, then $G(S) = S$.

Given an arbitrary semi-group $(S, +)$, $G(S)$ consists of formal differences

$$[x] - [y]$$

with $x, y \in S$, where two such formal differences $[x]-[y]$ and $[x']-[y']$ are equal if and only if there exists $z \in S$ such that

$$x + y' + z = x' + y + z. \tag{11.1}$$

More formally, one defines

$$G(S) = S \times S / \sim,$$

where $\sim$ is the equivalence relation given by: $(x, y) \sim (x', y')$ if and only if there exists a $z \in S$ such that (11.1) holds. Denote by $[x, y]$ the equivalence class of (x, y). The componentwise addition in $S \times S$ descends to a group structure on $G(S)$: the zero element is $[0, 0]$, and the inverse of $[x, y]$ is $[y, x]$. For $x \in S$, we define

$$[x] := [x, 0] \in G(S),$$

and this defines a morphism of semigroups $S \mapsto G(S)$ (find the universal property of $G(S)$!). Also, since

$$[x, y] = [(x, 0)] - [(y, 0)] = [x] - [y],$$

we obtain the representation of $G(S)$ mentioned at the beginning. Note that there is a canonical map:

$$i : S \to G(S), x \mapsto [x],$$

which is a morphisms of semi-groups.

Exercise 11.1.1. Show that $G(S)$ has the following universal property: for any other group G and any morphism $i_G : S \to G$ of semi-groups, there is a unique morphism of groups, $\phi : G(S) \to G$ such that $i_G = \phi \circ i$.

Definition 11.1.2. Define the K-theory group of M as the group associated to the semi-group $Vb(M)$ of (isomorphism classes) of vector bundles over M.

Exercise 11.1.3. Is the map $i : Vb(M) \to K(M)$ injective? (hint: you can use that the tangent bundle of S^2 is not trivial).

One may think of $K(M)$ as "integers over M". Indeed, natural numbers are in bijection with isomorphism classes of finite dimensional vector spaces. Since vector bundles can be viewed as families of vector spaces indexed by the base manifold M, isomorphism classes of vector bundles play the role of "natural numbers over M", and $K(M)$ the one of "integers over M".

Note also that the tensor product of vector bundles induces a product on $K(M)$, and $K(M)$ becomes a ring.

Theorem 11.1.4. *The Chern character induces a ring homomorphism*

$$Ch : K(M) \to H^{even}(M),\ [E] - [F] \mapsto Ch(E) - Ch(F).$$

Proof The compatibility By the additive property of the Chern character,

$$Ch : Vb(M) \to H^{even}(M)$$

is a morphism of semi-groups hence by the universality of $K(M)$ (see Exercise 11.1.1), it induces a morphism of groups $K(M) \to H^{even}(M)$. This is precisely the map in the statement. The compatibility with the produtcs follows from the multiplicativity of the Chern character. $\square$

Remark 11.1.5. One can show that, after tensoring with $\mathbb{R}$,

$$Ch \otimes_{\mathbb{Z}} \mathbb{R} : K(M) \otimes_{\mathbb{Z}} \mathbb{R} \to H^{even}(M)$$

becomes an isomorphism.

11.2. K-theory and the Chern character with compact supports

Next, we discuss K-theory with compact supports. We consider triples

$$C = (E, \alpha, F) \tag{11.2}$$

where E and F are vector bundles over M and

$$\alpha : E \to F$$

is a vector bundle morphism with compact support, i.e. with the property that

$$\mathrm{supp}(C) := \{x \in M : \alpha_x : E_x \to F_x \text{ is not an isomorphism}\}$$

is a compact subset of M. We denote by $L_1(M)$ the set of equivalence classes of such triples. If $C' = (E', \alpha', F')$ is another one, we say that C and C are homotopic, and we write

$$C \equiv C'$$

if there exists a smooth family $C_t = (E_t, \alpha_t, F_t)$, indexed by $t \in [0, 1]$, of triples (still with compact supports), such that

$$C_0 = C, C_1 = C'.$$

More precisely, we require the existence of a triple $\tilde{C} = (\tilde{E}, \tilde{\alpha}, \tilde{F})$ over $M \times [0, 1]$ (with compact support), such that C is the restriction of $\tilde{C}$ via $i_0 : M \hookrightarrow M \times [0, 1]$, $i_0(x) = (x, 0)$ and C' is the restriction of $\tilde{C}$ via $i_1 : M \hookrightarrow M \times [0, 1]$, $i_1(x) = (x, 1)$.

Finally, we introduce a new equivalence relation $\sim$ on $L_1(M)$:

$$C \sim C' \iff C \oplus T \equiv C' \oplus T'$$

for some triples T and T' with empty support.

Definition 11.2.1. Define the K-theory of M with compact supports as the quotient

$$K_{\mathrm{cpct}}(M) = L_1(M)/\sim .$$

For $C \in L_1(M)$, we denote by $[C]$ the induced element in $K_{\mathrm{cpct}}(M)$.

With respect to the direct sum of vector bundles and morphisms of vector bundles, $K_{\mathrm{cpct}}(M)$ becomes a semi-group.

Exercise 11.2.2. Show that $K_{\mathrm{cpct}}(M)$ is a group. Also, when M is compact, show that

$$K_{\mathrm{cpct}}(M) \to K(M),\ [(E, \alpha, F)] \mapsto [E] - [F]$$

is an isomorphism of groups.

Example 11.2.3. Given an elliptic operator $P : \Gamma(E) \to \Gamma(F)$ of order d over a compact manifold M, its principal symbol

$$\sigma_d(P) : \pi^* E \to \pi^* F$$

(where $\pi : T^*M \to M$ is the projection) will induce an element

$$[(\pi^* E, \sigma_d(P), \pi^* F)] \in K_{\mathrm{cpct}}(T^*M).$$

Next, we discuss the Chern character on $K_{\mathrm{cpct}}(M)$. We start with an element

$$C = (E, \alpha, F) \in L_1(M).$$

Lemma 11.2.4. *There exists a connection ∇^E onb E and ∇^F on F and a compact $L \subset M$ such that α is an isomorpism outside L and*

$$\alpha^*(\nabla^F) = \nabla^E \quad \text{on } M - L.$$

Note that, here,

$$\alpha^*(\nabla^F)_X(s) = \alpha^{-1}\nabla^F_X(\alpha(s)).$$

Proof Let ∇^F be any connection on F. We will construct ∇^E. For that we first fix an arbitrary connection ∇ on E (to be changed). Let K be the support of α and L a compact in M with

$$K \subset \mathrm{Int}(L) \subset L.$$

Then $\alpha^*(\nabla^F)$ is well-defined as a connection on $U := M - K$. The difference

$$\nabla_X(s) - \alpha^*(\nabla^F)_X(s)$$

is $C^\infty(U)$-linear in X and s, hence defines a section

$$\theta \in \Gamma(U, T^*M \otimes End(E)).$$

From the general properties of sections of vector bundles, for any closed (in M) $A \subset U$, we find a smooth section

$$\tilde{\theta} \in \Gamma(M, T^*M \otimes End(E))$$

such that

$$\tilde{\theta}|_A = \theta|_A.$$

We apply this to $A = M - Int(L)$ and we define ∇^E by

$$\nabla^E_X(s) = \nabla_X(s) - \tilde{\theta}(X)s.$$

$\square$

Theorem 11.2.5. *For any pair of connections ∇^E and ∇^F as in Lemma 11.2.4,*

$$Ch(C, \nabla^E, \nabla^F) := Ch(E, \nabla^E) - Ch(F, \nabla^F) \in \Omega_{cpct}(M)$$

is a closed differential form on M with compact support and the induced cohomology class

$$Ch(C) := [Ch(C, \nabla^E, \nabla^F)] \in H^{even}_{cpct}(M)$$

does not depend on the choice of the connections. Moreover, Ch induces a group homomorphism

$$Ch : K_{cpct}(M) \to H^{even}_{cpct}(M),\ [C] \mapsto Ch(C).$$

Proof The naturality of the Chern character implies that

$$Ch(C, \nabla^E, \nabla^F)|_{M-L} = Ch(E|_{M-L}, \alpha^*(\nabla^F|_{M-L})) - Ch(F|_{M-L}, \nabla^F|_{M-L}) = 0.$$

Hence $Ch(C, \nabla^E, \nabla^F)$ is a closed from with compact support. To see that the cohomology class does not depend on the choice of the connections, we addapt the argument for the similar statement for the Chern character of vector bundles; see the proof of Proposition 10.7.7. With the same notations as there, another pair of connections as in Lemma 11.2.4 will be of type

$$\nabla^E + R^E, \nabla^F + R^F$$

with

$$R^E \in \Omega^1(M, End(E)), R^F \in \Omega^1(M, End(F))$$

and the two new connections correspond to each other via α outise some compact L' (which may be different from L). Let $K = L \cup L'$. We apply the construction in the proof of Proposition 10.7.7 and we obtain

$$Ch_p(E, \nabla^E) - Ch_p(E, \nabla^E + R^E) = d(\omega^E)$$

for some form ω^E which has compact support, and similarly for F. Looking at the explicit formulas for ω_E and ω_F, we see immediately that $\omega_E - \omega_F$ has compact support.

Finally, to see that Ch induces a map in the K-theory with compact support, we still have to show that if $(E, \alpha, F) \equiv (E', \alpha', F')$ then they have the sama Chern character (in cohomology). But this follows by an argument simialr to the last one, because the equivalence $\equiv$ means that the two triples can be joined by a smooth family of triple (E_t, α_t, F_t), $t \in [0, 1]$ (fill in the details!). □

11.3. The K-theoretical formulation of the Atiyah-Singer index theorem

Next, we give an outline of the K-theoretical formulation of the index theorem. First of all

Definition 11.3.1. Let M be a compact n-dimensional manifold. The topological index map is defined as

$$Ind_t : K_{\mathrm{cpt}}(T^*M) \to \mathbb{R},\ Index_t([C]) = (-1)^n \int_{T^*M} Ch(C)\pi^*(Td(T_{\mathbb{C}}M)),$$

where $\pi : T^*M \to M$ is the projection.

A few explanations are in order. First of all, $T_{\mathbb{C}}M$ is the complexification of the tangent bundle of M. Secondly,

$$Ch(C) \in H_{\mathrm{cpt}}(T^*M), \pi^*(Td(T^*_{\mathbb{C}}M)) \in H(T^*M)$$

hence their product is a cohomology class with compact support. Finally, T^*M has a canonical orientation: any coordinate chart $(U, \chi = (x_1, \ldots, x_n))$ on M induces a coordinate chart on $T * M$:

$$T^*U \ni y_1(dx_1)_x + \ldots + y_n(dx_n)_x \mapsto (x_1, \ldots, x, y_1, \ldots, y_n);$$

moreover, the resulting atlas for T^*M is oriented and defines the canonical orientation of T^*M. In particular, we have an integration map

$$\int_{T^*M} : H^*_{\mathrm{cpt}}(T^*M) \to \mathbb{R},$$

which kills all the cohomology classes except the ones in the top degree $2n$ ($dim(T^*M) = 2n$).

Next, we discuss the analytic index. We have already seen that the index of an elliptic pseudo-differential operator only depends on its principal symbol. However, what happens is the following:

1. For an elliptic pseudo-differential operator P of degree d, $Index(P)$ only depends on
$$[\sigma_d(P)] \in K_{\mathrm{cpt}}(T^*M).$$
2. Any element in $K_{\mathrm{cpt}}(T^*M)$ can be represented by the principal symbol of an elliptic operator.

Hence there is a unique map
$$Ind_a : K_{\mathrm{cpt}}(T^*M) \to \mathbb{Z}$$
with the property that, for any elliptic pseudo-differential operator of degree d,
$$Ind_a([\sigma_d(P)]) = Index(P).$$

Definition 11.3.2. The map Ind_a is called the analytic index map.

With these, we have the following formulation of the Atiyah-Singer index theorem.

Theorem 11.3.3. *On any compact manifold M, $Ind_t = Ind_a$.*

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