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Chapter 1
Vector bundles (& intermezzos: foliations and complex structures)

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1.1 Vector bundles

1.1.1 Definitions/terminology

Intuitively, a vector bundle over a manifold M is a collection

$$E = \{E_x\}_{x \in M}$$

of vector spaces "smoothly parametrized by $x \in M$". Such a family can be encoded in one single set, the disjoint union $E = \coprod_x E_x$ and a map $\pi : E \to M$- the one that sends E_x to x. Conversely, $E_x = \pi^{-1}(x)$. The example to have in mind is the tangent bundle of M, made of by putting together all the tangent spaces (see subsection 3.2.4 from the bachelor notes).

Definition 1.1. A (real) **vector bundle** of rank r over a manifold M consists of:

- a manifold E together with a surjective map $\pi : E \to M$,
- a structure of (real) vector space on each fiber

$$E_x := \pi^{-1}(x) \quad (x \in M)$$

satisfying the following local triviality condition: for each $x_0 \in M$, there exists an open neighborhood U of x_0 s.t., over U, E is isomorphic to the trivial vector bundle $U \times \mathbb{R}^r$, in the sense that there exists a diffeomorphism

$$h : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r$$

that takes the fiber E_x to $\{x\} \times \mathbb{R}^r$, linearly. Such an h is called a **trivialization of E over U**.

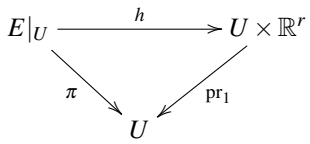
We either say that E is a rank r vector bundle over M and we use the generic notation π_E for the projection, or that $\pi : E \to M$ is a rank r vector bundle. Complex vector bundles are defined similarly, replacing $\mathbb{R}$ by $\mathbb{C}$.

Strictly speaking, a vector bundle is an entire triple (E, π, M); accordingly, one sometimes uses notations of type

$$\xi = (E, \pi, M) \tag{1.1.1}$$

one refers to the triple ξ as being a vector bundle, E is called the total space of ξ and denoted $E(\xi)$.

The trivialization h over U can be seen as part of a commutative diagram:



As hinted by the notation $E|_U$, vector bundles $\pi : E \to M$ can be restricted to opens $U \subset M$. More precisely,

$$E|_U := \pi^{-1}(U)$$

together with the restriction of π gives a vector bundle $\pi_U : E|_U \to U$ over U. The local triviality condition says that, locally (i.e. after restricting to small enough opens), the vector bundle is "isomorphic" to "the trivial vector bundle". The quoted concepts are properly defined below. The first one is based on the rather obvious notion of morphism $u : E \to F$ between two such-vector bundles, which gives a precise meaning to "a collection $\{u : E_x \to F_x\}_{x \in M}$ of linear maps smoothly parametrized by $x \in M$".

Definition 1.2. Given two vector bundles E and F over M, a **morphism** from E to F (of vector bundles over M) is a smooth map $u : E \to F$ with the property that, for each $x \in M$, u sends E_x to F_x and

$$u_x := u|_{E_x} : E_x \to F_x$$

is linear. We say that u is an isomorphism (of vector bundles over M) if each u_x is an isomorphism (or, equivalently, if u is also a diffeomorphism). From now on we will denote by

$$\mathrm{Vect}^r_{\mathbb{R}}(M)$$

the collection of isomorphism classes of real vector bundles of rank r over the manifold M

Example 1.3 (trivial vector bundles). The trivial vector bundle of rank r over M is the product $M \times \mathbb{R}^r$ together with the first projection

$$\mathrm{pr}_1 : M \times \mathbb{R}^r \to M$$

and the usual vector space structure on each fiber $\{x\} \times \mathbb{R}^r$. We sometimes use the notation $\mathbb{R}^r_M$. We say that a vector bundle E (of rank r) over M is **trivializable** if E is isomorphic to $M \times \mathbb{R}^r$. A **trivialization** of E is the choice of such an isomorphism.

We can now say it without quotes: the local triviality condition from the definition says that E is locally trivializable: around each point in M there is an open neighborhood U such that the restriction $E|_U$ is trivializable.

Remark 1.4 (not fixing the rank). It is possible to relax a bit the notion of vector bundle by now requiring a fixed r in advance and talk about **vector bundles of possibly nonconstant rank**; the only change in Definition 1.1 is that, when requiring the local triviality condition around an arbitrary x_0, since r is not fixed in advance, we demand also the existence of a number $r = r(x_0)$ such that that condition holds. This does imply that the dimension of each fiber E_x (also called **the rank of** E **at** x) is locally constant w.r.t. x. We deduce that:

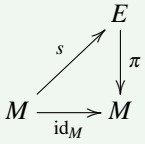
1. if M is connected then the rank is constant, hence E is a vector bundle of some rank r, as in Definition 1.1;
2. in general, writing M as the union of its connected components M_i, the only novelty is that $E \to M$ may have different ranks on different connected components.

1.1.2 Sections

Definition 1.5. Given a vector bundle $\pi : E \to M$, a **section** of E is a smooth map $s : M \to E$ satisfying

$$s(x) \in E_x \quad \forall\, x \in M. \tag{1.1.2}$$

or, more compactly, $\pi \circ s = \mathrm{Id}$, also described by a commutative diagram:



If we give up on the smoothness condition, we talk about **set-theoretical sections** of E. We denote by

$$\Gamma(M,E) = \Gamma(E)$$

the space of all smooth sections. For $U \subset M$ open, the space of **local sections** of E defined over U is

$$\Gamma(U,E) := \Gamma(E|_U).$$

Sections can be added pointwise:

$$(s+s')(x) := s(x)+s'(x)$$

and, similarly, can be multiplied by scalars $\lambda \in \mathbb{R}$:

$$(\lambda s)(x) := \lambda s(x).$$

With these, $\Gamma(E)$ becomes a vector space. Furthermore, any section $s \in \Gamma(E)$ can be multiplied pointwise by any real-valued smooth function $f \in C^\infty(M)$ giving rise to another section $fs \in \Gamma(E)$:

$$(fs)(x) := f(x)s(x).$$

The resulting operation

$$C^\infty(M) \times \Gamma(E) \to \Gamma(E), \quad (f,s) \mapsto fs$$

makes $\Gamma(E)$ into a module over the algebra $C^\infty(M)$. Actually the entire vector bundle E is fully encoded in the space of sections $\Gamma(E)$ together with this module structure- that is something that is made precise by the Serre-Swan theorem, to be discussed a bit later. For now, here is just a strong indication, itself a step in the proof of Serre-Swan, but very useful on its own.

Proposition 1.6. *Let E and F be vector bundles over M. Associating to a morphism $u : E \to F$ the map*

$$u_* : \Gamma(E) \to \Gamma(F), \quad u_*(s)(x) = u_x(s(x)),$$

gives rise to a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{morphisms } u : E \to F \\ \text{of vector bundles over } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{morphisms } u_* : \Gamma(E) \to \Gamma(F) \\ \text{of } C^\infty(M)\text{-modules} \end{array} \right\}$$

Proof. This proposition is proven by precisely the same arguments as Theorem 4.19 from the bachelor notes (that it actually generalises). Still referring to the bachelor notes: one needs, of course, the analogue of the surjectivity of the map (4.1.7) (which, locally, boils down to Taylor-like decompositions (3.1.5)). The two analogues are parts 1) and 2) of the Lemma below. 😄

Lemma 1.7. *For any vector bundle $\pi : E \to M$ and any $x \in M$:*

1). For any $v \in E_x$ there exists a section $s \in \Gamma(E)$ such that $s(x) = v$.
2). For any $s \in \Gamma(E)$ such that $s(x) = 0$, there exists a finite number of smooth functions $\{f_i\}_{i\in I}$ on M that vanish at x, and sections $\{s_i\}_{i\in I}$ of E, such that $s = \sum_{i\in I} f_i \cdot s_i$.

More algebraically: denoting by ev_x the evaluation of sections at x, by $I_x \subset C^\infty(M)$ the ideal of smooth functions that vanish at x and by $I_x \cdot \Gamma(E) \subset \Gamma(E)$ the submodule of $\Gamma(E)$ generated by products of sections with functions $f \in I_x$, one has a short exact sequence of vector spaces:

$$0 \longrightarrow I_x \cdot \Gamma(E) \xrightarrow{\text{inclusion}} \Gamma(E) \xrightarrow{\mathrm{ev}_x} E_x \longrightarrow 0\ .$$

Proof. Part 2) is based on Taylor-like decompositions (3.1.5) (bachelor notes) and works exactly was we did it in Lemma 4.20 for vector fields. Part 1) is completely similar to equation (4.1.7) from the bachelor notes. 😄

1.1.3 Frames

For the next discussion, recall that a frame in a vector space V is a an ordered collection

$$\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_r)$$

of vectors $\mathfrak{e}_i \in V$ that form a basis of V. While V is isomorphic to $\mathbb{R}^r$, choosing a frame of V is equivalent to choosing an isomorphism between the two. More precisely, any frame $\mathfrak{e}$ induces an isomorphism $\hat{\mathfrak{e}} : \mathbb{R}^r \to V$ obtained by sending the ith vector in the canonical basis of $\mathbb{R}^r$ to $\mathfrak{e}$ and, furthermore, this construction gives rise to a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{frames of } V \\ \mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_r) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{linear isomorphisms} \\ \hat{\mathfrak{e}} : \mathbb{R}^r \xrightarrow{\sim} V \end{array} \right\} \tag{1.1.3}$$

Definition 1.8. Let $\pi : E \to M$ be a vector bundle over M. A **frame** of E is a collection

$$\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$$

consisting of sections $\mathfrak{e}_i$ of E with the property that, for each $x \in M$,

$$\{\mathfrak{e}_1(x), \ldots, \mathfrak{e}_r(x)\}$$

is a frame of E_x (i.e. a basis of the vector space E_x). A **local frame of** E is a frame $\mathfrak{e}$ of $E|_U$ for some open $U \subset M$; we also say that $\mathfrak{e}$ is a local frame over U.

Local frames are useful in handling general sections through more explicit formulas. Indeed, fixing a local frame $e = (e_1, \ldots, e_r)$ of E over U, any section s can be written over U as

$$s(x) = f^1(x)\mathfrak{e}_1(x) + \ldots + f^r(x)\mathfrak{e}_r(x) \quad (x \in U),$$

hence it is encoded by r functions, $f^1, \ldots, f^r \in C^\infty(U)$- called the **coefficients of** s **w.r.t. the local frame** $\mathfrak{e}$. We also write $[s : \mathfrak{e}]$ for the column matrix formed by the coefficients:

$$[s : \mathfrak{e}] = \begin{pmatrix} f^1 \\ \ldots \\ f^r \end{pmatrix}.$$

Hence, thinking of $\mathfrak{e}$ as a row matrix, one can formally write:

$$s = \mathfrak{e} \cdot [s : \mathfrak{e}]. \tag{1.1.4}$$

Exercise 1.9. An instructive exercise is to show that, if s is only a set-theoretical section (i.e. no assumption on its smoothness), the smoothness of s is equivalent to the smoothness of its coefficients (w.r.t. all local frames).

The trivial vector bundle $M \times \mathbb{R}^k$ (of some rank k) comes with a canonical frame, made of the canonical basis of each copy $\{x\} \times \mathbb{R}^k$:

$$e_1^{\text{can}}, \ldots, e_n^{\text{can}} \in \Gamma(M \times \mathbb{R}^k).$$

Notice that a morphism $\mathfrak{e} : M \times \mathbb{R}^k \to E$ of vector bundles over M, to an arbitrary vector bundle E over M (of some rank r) is encoded on its values on the canonical frame, which is a collection of sections of E, $\mathfrak{e}_1, \ldots, \mathfrak{e}_k \in \Gamma(E)$. In turn, this is a frame iff $\mathfrak{e}$ is an isomorphism of vector bundles. Therefore, one obtains a 1-1 correspondence:

$$\left\{\begin{array}{c}\text{(local) trivializations}\\ h : E|_U \to U\times\mathbb{R}^r\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{local frames}\\ e_1,\dots,e_r\in\Gamma(E|_U)\end{array}\right\} \tag{1.1.5}$$

This will be precisely the way in which we will be using the local trivialization axiom.

Frames are especially well-suited when looking at geometric structures on vector bundles, when one can talk about frames that are "adapted to the given structure" and, ultimately, use them in order to make sense of the concept of "geometric structures on vector bundles" itself, in full generality.

Let us illustrate this in the case of inner products on vector bundles- a notion that also turns out to be a very valuable tool in proving some of the most basic properties of vector bundles. By a **(fiberwise) inner product** on a vector bundle $\pi : E \to M$ we mean a family $g = \{g_x\}$ of inner products on the fibers E_x which varies smoothly w.r.t. x in the sense that the inner product of any two sections $s_1, s_2 \in \Gamma(E)$,

$$g(s_1,s_2) : M \ni x \mapsto g_x(s_1(x),s_2(x))\mathbb{R},$$

is smooth. Given g, a local frame of E is said to be g**-orthonormal** if $g(\mathfrak{e}_i,\mathfrak{e}_j) = \delta_{ij}$ for all i,j.

Exercise 1.10. Show that, given a vector bundle E and an inner product g on E, a set-theoretical section s of E is smooth if and only of $g(s,s')$ is smooth for any smooth section s' of E.

Lemma 1.11. *Basic properties of inner products on vector bundles:*

(1) Any vector bundle E admits an inner product g.

(2) Given an inner product g on E then, around any $x \in M$, there exists a g-orthonormal local frame of E.

Proof. The idea for (1) is to use partitions of unity. Namely: start with an open cover $\{U^\alpha\}_{\alpha\in\Lambda}$ of M such that over each open U^α there exists a local frame $\mathfrak{e}^\alpha = \{\mathfrak{e}^\alpha_1,\dots,\mathfrak{e}^\alpha_r\}$. Then, over U^α one has a simple inner product g^α, uniquely characterized by the condition that $\mathfrak{e}^\alpha$ is orthonormal. The problem is that $g^\alpha(x)$ (an inner product on E_x) is defined only for $x \in U^\alpha$. Choose now a partition of unity $\{\eta_\alpha\}_{\alpha\in\Lambda}$ subordinated to our cover. Since each η_α is supported inside U_α, $\eta_\alpha \cdot g^\alpha$ becomes smooth on M if we set it to be zero outside U^α. Each such $\eta_\alpha \cdot g^\alpha$ is no longer an inner product (non-degeneracy fails), but it is still made of bilinear maps. However, when we sum up,

$$g := \sum_\alpha \eta_\alpha \cdot g^\alpha$$

will be again an inner product, and it is defined and smooth over the entire M.

Item (2) follows by applying the Gram-Schmidt procedure. 😀

1.1.4 Transition functions

While the local triviality axiom allows one to use local frames and decompose sections w.r.t. to such frames, when interested in global aspects, one of the first questions one faces is: how does one pass from a local frame to another one? That is basically a question about bases of vector spaces: how does one pass from one basis to another one- for which the answer is: via matrices of coordinate changes. Applied to (local) frames of a vector bundle, the resulting matrices (one for each point x in overlapping domains) are known under the name of "transition functions" and provide an alternative view-point on vector bundles. Let us make everything more explicit.

1. First of all, if $\mathfrak{e} = \{\mathfrak{e}_1,\dots,\mathfrak{e}_r\}$ is a local frame of E over an open U and $\mathfrak{e}' = \{\mathfrak{e}'_1,\dots,\mathfrak{e}'_r\}$ is another local frame of E over some open U' then, at points on the overlap, $x \in U \cap U'$, one can write

$$\mathfrak{e}'_i(x) = \sum_{j=1}^{r} \tau_i^j(x)\mathfrak{e}_j(x)$$

with $\tau_i^j(x) \in \mathbb{R}$ being part of an invertible matrix (of coordinate changes),

$$(\tau_i^j(x))_{i,j} \in GL_r(\mathbb{R}).$$

In other words, one obtains a (smooth) function

$$\tau = (\tau_i^j)_{1\leq i,j\leq r} = \begin{pmatrix} \tau_1^1 & \dots & \tau_r^1 \\ \dots & \dots & \dots \\ \tau_1^r & \dots & \tau_r^r \end{pmatrix} : U \cap U' \to GL_r(\mathbb{R}).$$

This matrix is called **the transition function from the local frame $\mathfrak{e}$ to the local frame $\mathfrak{e}'$** and will be denoted $[\mathfrak{e}' : \mathfrak{e}]$ (other possible notations would be $\mathfrak{e}' \cdot \mathfrak{e}^{-1}$ or $\frac{\mathfrak{e}'}{\mathfrak{e}}$, but those may be confusing). Thinking again of frames as row matrices we can formally write

$$\mathfrak{e}' = \mathfrak{e} \cdot [\mathfrak{e}' : \mathfrak{e}].$$

Exercise 1.12. Using (1.1.4) deduce that $[\mathfrak{e}' : \mathfrak{e}]$ also encodes the way the coefficients of sections w.r.t. local frames change- more precisely, prove that

$$[s : \mathfrak{e}] = [\mathfrak{e}' : \mathfrak{e}] \cdot [s : \mathfrak{e}']$$

2. To handle the entire (global) E and global sections of E, we start with an open cover

$$\mathscr{U} = \{U^\alpha\}_{\alpha\in\Lambda}, \tag{1.1.6}$$

for some index set Λ, together with local frames

$$\mathfrak{e}^\alpha = \{\mathfrak{e}_1^\alpha, \dots, \mathfrak{e}_r^\alpha\}$$

of E over U^α. Then, for $\alpha, \beta \in \Lambda$, the discussion above applied to $\mathfrak{e} = \mathfrak{e}^\alpha$ and $\mathfrak{e}' = \mathfrak{e}^\beta$ gives rise to the transition functions $[\mathfrak{e}^\beta : \mathfrak{e}^\alpha]$, denoted

$$\tau^{\alpha,\beta} : U^\alpha \cap U^\beta \to GL_r(\mathbb{R}). \tag{1.1.7}$$

These matrices satisfy the following identities (proving them is an instructive exercise)

1. $\tau^{\alpha,\alpha} = \mathrm{Id}$ (the constant function equal to the identity matrix).
2. for any $\alpha, \beta, \gamma \in \Lambda$, one the overlap $U^\alpha \cap U^\beta \cap U^\gamma$, $\tau^{\alpha,\beta} \cdot \tau^{\beta,\gamma} = \tau^{\alpha,\gamma}$.

The fact that one can recover E from this data may be surprising at first:

Proposition 1.13. *Given a manifold M, for any open cover (1.1.6) and any collection $\{\tau^{\alpha,\beta}\}_{\alpha,\beta\in\Lambda}$ of smooth functions (1.1.7) satisfying the conditions 1. and 2. above, there exists a vector bundle $E \to M$, unique up to isomorphism, admitting $\{\tau^{\alpha,\beta}\}_{\alpha,\beta\in\Lambda}$ as transition functions (w.r.t. to some local frames).*

Such collections $\{\tau^{\alpha,\beta}\}_{\alpha,\beta\in\Lambda}$ are called **transition data on M with values in GL_r**. Of course, changing the open cover $\mathscr{U}$, or just the local frames, produces a completely new different set of transition matrices. In particular, different sets of transition functions may give rise to the same vector bundle. It is not particularly nice, but it is not particularly difficult either to figure out when two sets of transition functions gives rise to isomorphic vector bundle- therefore providing an equivalence relation $\sim$ on the collection of all transition functions.

Conversely, starting with an abstract transition data $\{U^\alpha, \tau^{\alpha,\beta}\}$ on a manifold M, one can construct a vector bundle E compatible with the given data. The idea is to start by putting together all the trivial bundles $U^\alpha \times \mathbb{R}^r$, i.e. consider the disjoint union of these spaces, described as

$$\{(\alpha,x,v) : \alpha \in \Lambda, x \in U^\alpha, v \in \mathbb{R}^r\},$$

and define E as the quotient of this last space modulo the equivalence relation

$$(\alpha,x,v) \sim (\beta,y,w) \quad \Longleftrightarrow \quad y = x,\ w = \tau^{\beta,\alpha}(v).$$

The fact that this is an equivalence relation is an instructive exercise. Denoting by $[\alpha,x,v]$ the resulting equivalence classes, we can write

$$E = \{[\alpha,x,v] : \alpha \in \Lambda, x \in U^\alpha, v \in \mathbb{R}^r\}$$

and there is an obvious map π ("projection") from E to M: send $[\alpha,x,v]$ to x. Furthermore, for each $\alpha \in \Lambda$, there is also an obvious "local frame $\mathfrak{e}^\alpha$ over U^α", with components

$$\mathfrak{e}^\alpha_i : U^\alpha \to E, \quad x \mapsto [\alpha,x,e^{\text{can}}_i],$$

where $e^{\text{can}}_1,\dots,e^{\text{can}}_r$ is the usual basis of $\mathbb{R}^r$. Here in below we use quotes because the terms make sense only when we already know that E is a vector bundle (which is what we are trying to explain/prove here!). These are relevant for two reasons:

(i) first of all, with the mind at the fact that local frames encode local trivializations, we are led to consider the bijections

$$\pi^{-1}(U^\alpha) \to U^\alpha \times \mathbb{R}^r, \quad [(\alpha,x,v)] \mapsto (x,v_1,\dots,v_r).$$

These can now be used to put a smooth structure on E, and then to ensure the local triviality of E (this requires a bit of tedious work, let's defer that to the exercise sheet).

(ii) secondly, once we know that E is a vector bundle, since it comes with the local frames $\mathfrak{e}^\alpha$, by the general theory discussed above, one obtains a transition data on M. It is just a matter of checking the definition to see that the resulting transition data is precisely the one we started with!

Ultimately, one obtains a 1-1 correspondence:

$$\text{Vect}^r_{\mathbb{R}}(M) \equiv \left\{ \begin{array}{c} \text{vector bundles } E \\ \text{over } M, \text{ of rank } r \\ \text{(up to isomorphisms)} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{transition data } (\tau^{\alpha,\beta}) \\ \text{on } M, \text{ with values in } GL_r \\ \text{(up to } \sim \text{ equivalence)} \end{array} \right\} \tag{1.1.8}$$

3. Returning to the transition functions (data) associated to the open cover (1.1.6) and the local frames $\{\mathfrak{e}^\alpha : \alpha \in \Lambda\}$, an arbitrary global section $s \in \Gamma(E)$ is determined, over each U^α, by a function

$$f_\alpha = (f^1_\alpha,\dots,f^r_\alpha) : U^\alpha \to \mathbb{R}^r. \tag{1.1.9}$$

consisting of the coefficients of s w.r.t. $\mathfrak{e}^\alpha$, defined by the condition that

$$s|_{U^\alpha} = f^1_\alpha \cdot \mathfrak{e}^\alpha_1 + \dots + f^r_\alpha \cdot \mathfrak{e}^\alpha_r.$$

As before, it is sometimes useful to turn f_α into a column matrix f^T_α, denoted $[s : \mathfrak{e}^\alpha]$ earlier on:

$$f^T_\alpha = \begin{pmatrix} f^1_\alpha \\ \dots \\ f^r_\alpha \end{pmatrix} = [s : \mathfrak{e}^\alpha].$$

Moving to another open U^β, we see that the coefficients change, on $U^\alpha \cap U^\beta$, according to the formula

$$f^T_\alpha = \tau^{\alpha,\beta} \cdot f^T_\beta. \tag{1.1.10}$$

Or, in the older notations: $[s : \mathfrak{e}^\alpha] = [e^\beta : e^\alpha] \cdot [s : \mathfrak{e}^\beta]$- which is just an instance of the identity from Exercise 1.12. It should be clear that the process can be reversed and, therefore, one obtains a 1-1 correspondence

$$\Gamma(E) \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{collections of functions } \{f_\alpha : U^\alpha \to \mathbb{R}^r\}_{\alpha\in\Lambda} \\ \text{satisfying } f_\alpha^T = \tau^{\alpha,\beta} \cdot f_\beta^T \text{ on the overlaps } U_\alpha \cap U_\beta \end{array} \right\}$$

And here is a property of (smooth) vector bundles that is relevant to this discussion and extremely useful in establishing several fundamental results.

1.1.5 First examples of vector bundles

Here are some first examples of vector bundles, besides the trivial ones mentioned in 1.3.

Example 1.14 (the tangent bundle). For an n-dimensional manifold M, if you remember the tangent bundle from the basic course on manifolds you will recognize immediately that TM is a vector bundle over M of rank n. Nevertheless, it is still useful to have another look at that discussion and reinterpret it from the point of view of vector bundles. In that case the starting point was a collection of vector spaces, the tangent spaces themselves,

$$TM = \{T_xM\}_{x\in M},$$

put together in a larger space sitting above M:

$$TM = \{(x,v) : x \in M, v \in T_xM\}, \quad \text{with projection } \pi : TM \to M,\ (x,v) \mapsto x.$$

Then, for the smooth structure on TM the important ingredient was provided by charts: any chart (U,χ) was giving rise to a basis of T_xM at each $x \in U$, i.e., something like a local frame over U

$$U \ni x \longmapsto \left(\frac{\partial}{\partial x_j}\right)_x \in T_xM, \quad 1 \le j \le m.$$

Of course, the desire is (and was, implicitly) to turn these into smooth frames. That gives rise to the chart $(TU,\tilde{\chi})$ of TM; see equation (3.2.5) and Proposition 3.37 in the bachelor notes. Furthermore, the main property of these frames that ensures that any two charts of this type, $(TU,\tilde{\chi})$ and $(TU',\tilde{\chi}')$, are smoothly compatible, was the change of coordinates formula (3.2.6) (from the bachelor notes again) , formula that now can be understood as describing the transition functions associated to the change of frames. In other words, what was important was those transition functions were smooth.

Example 1.15 (a general construction). The last remarks we made on the tangent bundle shows that the entire discussion can be carried much more generally, whenever the natural starting point is just a collection $E = \{E_x\}_{x\in M}$ of vector spaces indexed by $x \in M$ and certain "local sections" that we "want to be smooth". The following makes this precise:

Lemma 1.16. *Let M be a manifold and let $E = \{E_x\}_{x\in M}$ be a collection of r-dimensional vector spaces parametrized by points $x \in M$. Assume that we are given an open cover $\mathscr{U}$ of M and, for each $U \in \mathscr{U}$, a discrete frame e_U of E over U - that is, a family parametrized by $x \in U$ consisting of frames*

$$\mathfrak{e}^U(x) = \left\{\mathfrak{e}_1^U(x),\dots,\mathfrak{e}_r^U(x)\right\} \ -\text{frame (basis) of } E_x \qquad (x \in U, U \in \mathscr{U})$$

and we assume the smooth compatibility condition that, for any $U,V \in \mathscr{U}$, writing

$$\mathfrak{e}_i^U(x) = \sum_{j=1}^{r} \tau_i^j(x)\mathfrak{e}_j^V(x) \qquad (\text{for } x \in U \cap V),$$

all the resulting functions $\tau_i^j : U \cap V \to \mathbb{R}$ are smooth. Then:

1. *E admits a unique smooth structure which makes it into a vector bundle over M with the property that all the sections $\mathfrak{e}_i^U$ become smooth local sections of E.*
2. *a discrete section s of E is smooth if and only if, for any $U \in \mathscr{U}$, writing*

$$s(x) = f^1(x)\mathfrak{e}_1^U(x) + \dots + f^r(x)\mathfrak{e}_r^U(x) \quad (x \in U),$$

all the functions $f^i : U \to \mathbb{R}$ are smooth.

Proof. As pointed out above, frames (over U) can be re-arranged into "local trivializations" ($U \times \mathbb{R}^r \cong E_U$). Combining those with the smooth charts of M one produces smooth charts, and therefore a smooth atlas, and therefore a smooths structure, on E. The details are left as an exercise. 😀

Example 1.17 (tautological bundles). Here is another interesting vector bundle- the so called tautological line bundle (vector bundle of rank 1) over the projective space $\mathbb{RP}^n$, which we will denote $L_{\mathbb{RP}^n}$.[1] To describe $L_{\mathbb{RP}^n}$ recall that a point $l \in \mathbb{RP}^n$ is a line $l \subset \mathbb{R}^{n+1}$ through the origin; the fiber of $L_{\mathbb{RP}^n}$ above such a point will be precisely the line l, interpreted as a 1-dimensional vector space. This can be realised concretely by defining

$$L_{\mathbb{RP}^n} = \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in l\},$$

with the projection $L_{\mathbb{RP}^n} \to \mathbb{RP}^n$ that sends (l, v) to l. This description also has the advantage that it exhibits $L_{\mathbb{RP}^n}$ as a subspace of the product manifold $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ and, as such, it is an embedded submanifold (exercise!). This provides $L_{\mathbb{RP}^n}$ with a smooth structure and we leave it as an exercise to check that, indeed, $L_{\mathbb{RP}^n}$ becomes a vector bundle of rank 1 over $\mathbb{RP}^n$.

Another way to describe $L_{\mathbb{RP}^n}$ would be by using again Lemma 1.16. The collection of 1-dimensional vector spaces over $\mathbb{RP}^n$ that we start with is the tautological one, $\{l\}_{l \in \mathbb{RP}^n}$. Then one uses the standard atlas for $\mathbb{RP}^n$, $(U_0, \chi^0), \ldots, (U_n, \chi_n)$. Using the notation $[x_0 : x_1 : \ldots : x_n]$ for the line through the origin and $(x_0, x_1, \ldots, x_n)$, recall that $U_i \subset \mathbb{R}^{n+1}$ is the open defined by the condition $x_i \neq 0$ and

$$\chi^i : U_i \to \mathbb{R}^m, \ \chi^i([x_0 : x_1 : \ldots : x_n]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

For each $l = [x_0 : x_1 : \ldots : x_n] \in U_i$, one has an obvious non-zero vector in l, namely

$$e_{U_i}(l) := \left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in l.$$

This defines a section over U_i, really the most obvious one, and which certainly deserves to be called smooth. And that is precisely what Lemma 1.16 is doing for us!

Let us also point out that such "tautological" vector bundles exist for arbitrary ranks: one replaces $\mathbb{RP}^n$ with the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ (whose points are the k-dimensional vector subspaces of $\mathbb{R}^{n+k}$- see also section 2.2.4 of the bachelor notes) and the resulting tautological bundle will be of rank k: the fiber above a $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the vector space V itself.

[1] in the notation (1.1.1), this is usually often denoted γ_1, hence $L_{\mathbb{RP}^n} = E(\gamma_1)$.

1.1.6 Sub-bundles

We now have a closer look to "sub-objects" in the realm of vector bundles. Since there are several viewpoints, each one interesting/useful on its own, and we start with a lemma showing that they are all equivalent.

Definition 1.18. A **vector sub-bundle** of a vector bundle $\pi : E \to M$ is an embedded submanifold $F \subset E$ with the property that

$$\pi_F := \pi|_F : F \to M$$

is a submersion ad each fiber $F_x = \pi_F^{-1}(x)$ is a vector subspace of E_x.

Equivalently, the condition can be restated as saying that F can be made into a vector bundle over M s.t. the inclusion $i : F \hookrightarrow E$ is a morphism of vector bundles. This and other equivalent reformulations are the subject of the following:

Lemma 1.19. *Let $\pi : E \to M$ be a vector bundle. Let $F \subset E$ be a subset and denote*

$$\pi_F := \pi|_F : F \to M, \quad F_x = F \cap E_x = \pi_F^{-1}(x).$$

The following are equivalent:

(1) F can be made into a vector bundle over M s.t. the inclusion $i : F \hookrightarrow E$ is a morphism of vector bundles;
(2) each F_x is a vector subspace of E_x, F is an embedded submanifold of E and $\pi_F : F \to M$ is a submersion.
(3) $\exists\, p \in \mathbb{N}$ such that around each $x \in M$ there exist a local frame $e_1, \dots, e_r$ of E over some $U \subset M$ with

$$F_y = \mathrm{Span}_{\mathbb{R}} \{e_1(y), \dots, e_p(y)\} \quad \forall\, y \in U.$$

(4) there exists $p \in \mathbb{N}$ such that all the fibers F_x are p-dimensional vector subspaces of E_x, and for each vector v in a fiber F_x, there exists a (local) section s of E such that $s(x) = v$ and s takes values in F.

Furthermore, when this is the case, the smooth structure on F appearing in (a) is unique and is the induced structure on F viewed as an embedded submanifold of E.

Proof. We show that $(1) \Longrightarrow (4) \Longrightarrow 3) \Longrightarrow (2) \Longrightarrow (1)$. The first implication is immediate. Assume now (4), we fix $x \in M$ and we look for a local frame as in (3). We start with a frame $v_1, \dots, v_r$ of the fiber E_x such that $v_1, \dots, v_p$ are a frame of F_x. Each v_i can be written as $v_i = e_i(x)$ for some local section e_i of E. By (4), we make assume that $e_1, \dots, e_p$ take values in F. Since $e_1, \dots, e_r$, when evaluated at x, gives a frame of E_x, by restricting to a possibly smaller neighborhood U of x, $e_1, \dots, e_r$ will become a local frame. Since for each $y \in U$, $e_1(y), \dots, e_p(y)$ are linearly independent vectors of the p-dimensional vector space F_y, the equality in (3) follows.

Next we assume (3) and prove (2). For that it is useful to use the interpretation of local frames as trivialisation- and then the frame from (3) is used to produce a diffeomorphism

$$\pi^{-1}(U) \cong U \times \mathbb{R}^r, \quad \lambda^1 e_1(y) + \ldots + \lambda^k e_k(y) \leftrightarrow (y, \lambda^1, \dots, \lambda^k).$$

This could be combined with a chart $U \cong \mathbb{R}^n$ of M (after possibly shrinking U) to produce a chart for E. On the other hand, by the condition in c), that this diffeomorphism restricts to one between $F \cap \pi^{-1}(U)$ and $U \times (\mathbb{R}^p \times \{0\})$. It follows right away that the submanifold condition is satisfied. Furthermore, via these diffeomorphisms the projection for F becomes the projection from the product $U \times (\mathbb{R}^p \times \{0\})$ to U. This implies that π_F is a submersion.

We now assume (2) and prove (1). The structure on F will be induced from the one on E: $\pi_F = \pi|F$, each F_x is a vector subspace of E_x, and the smooth structure on F comes from the fact that F is an embedded submanifold of E. The regular value theorem for $\pi_F : F \to M$ implies that all the fibers F_x have the same dimension $p := \mathrm{rank}F - \mathrm{rank}E$. The local form of submersions implies that π_F admits local sections: for any $v \in F$ there exists a

local section s of F such that $s(x) = v$ where $x = \pi_F(v)$. In particular (4) is satisfied and, therefore, also (3). Hence F admits local frames around any point and, therefore, $\pi_F : F \to M$ is locally trivial. This proves (1).

We still have to show that for the smooth structure on F that would make F into a vector bundle as in (1) is unique.The key point is the result from the bachelor course that says that if we have an embedded submanifold, than that submanifold admits only one smooth structure (the induced one) for which the inclusion becomes an immersion (Corollary 2.165 in section 2.4.4. of the bachelor notes). Hence it all boils down to showing that, for any injective morphism $i : F \to E$ of vector bundles, $i(F)$ is embedded in E and i is an immersion. The sequence of implications above proves that $i(F)$ is embedded. To prove that i is an immersion let $v \in F_x$ and we show that $(di)_v : T_vF \to T_vE$ is injective. Let $X_v \in T_vF$ be in the kernel of this map. Since $\pi_E \circ i = \pi_F$, the chain rule implies that X_v is killed also by $d\pi_F$. Using the regular value theorem we deduce that X_v is tangent to the fiber F_x. Denoting $i_x = i|_{F_x} : F_x \to E_x$, we deduce that X_v is sent by di_x to 0. But i_x is linear and injective and, therefore, $X_v = 0$. 😀

Example 1.20 (the tautological line bundle). The tautological bundles over $\mathbb{RP}^n$ (Example 1.17) sits inside the trivial bundle of rank $n+1$, as a vector sub-bundle:

$$E \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$

This follows since (3) and/or (4) of the previous lemma are clearly satisfied.

Example 1.21 (orthogonals). Given a vector bundle $\pi : E \to M$ endowed with a fiberwise inner product g and any vector sub-bundle $F \subset E$, the g-orthogonal of F (consisting of the g-orthogonals of all the fibers of F):

$$F^\perp \subset E$$

is a smooth vector sub-bundle. This follows from part (3) of Lemma 1.19. To check it, let $x \in M$ and we consider a local frame $\mathfrak{e}_1, \dots, \mathfrak{e}_r$ for E with the property that $\mathfrak{e}_1, \dots, \mathfrak{e}_k$ of F is a frame of F. The Gram-Schmidt algorithm produces then an orthonormal frame $\mathfrak{e}'_1, \dots, \mathfrak{e}'_r$ for E with the property that $\mathfrak{e}'_1, \dots, \mathfrak{e}'_k$ is still a frame of F, and it also follows that $\mathfrak{e}'_{k+1}, \dots, \mathfrak{e}'_r$ are sections of $F^\perp$ that provide a basis at each point $x \in U$.

Notice, in particular, the following interesting consequence:

Corollary 1.22. *Any vector sub-bundle $F \subset E$ of a vector bundle $\pi : E \to M$ admits a complement, i.e. a vector sub-bundle $F' \subset E$ with the property that*

$$E = F \oplus F',$$

in the sense that each E_x is the direct sum of the subspaces F_x and F'_x.

Proposition 1.23. *For any morphism of vector bundles $u : F \to E$ of constant rank, i.e., so that the rank of the maps $u_x : F_x \to E_x$ is constant w.r.t. x, the image and the kernel of u,*

$$Ker\ u \subset F, \quad Im\ u \subset E,$$

are vector sub-bundles of F and E, respectively.

Proof. First of all, it is clear that the image of u does satisfy the conditions from Lemma 1.19. For the image, we use a standard trick from linear algebra to relate kernels to images of linear maps: use inner products and the resulting adjoints. More precisely, we fix inner products g_E on E and g_F on F and we forms the resulting adjoint

$$u^* : F \to E,$$

defined on F_x as the adjoint of $u_x : E_x \to F_x$ w.r.t. the given inner products on the fibers. We claim that this is again a morphism of vector bundles. The only issue is smoothness, which is equivalent to the fact that $u^*(s')$ is a smooth

section of E whenever s' is a smooth section of F. But, by part (b) of Exercise 1.10, the smoothness of $u^*(s')$ is equivalent to the one of $g_F(u^*(s'),s)$, for any smooth section s of F. But

$$g_F(u^*(s'),s) = g_E(s',u(s)),$$

which is clearly smooth whenever s and s' are. Hence u^* is smooth, and standard linear algebra implies that $\operatorname{Ker} u = (\operatorname{Im} u)^\perp$ which, by the previous discussions, is smooth. 😄

1.1.7 Embeddings into trivial bundles

We can now mention another fundamental property of (smooth) vector bundles: the fact that they can always be embedded into a trivial one. Here, motivated by the previous proposition, by a **(vector bundle) embedding** of a vector bundle $E \to M$ into another vector bundle $\widetilde{E} \to M$ we mean a morphism $u : E \to \widetilde{E}$ that is injective. The proposition implies that E is isomorphic to a sub-bundle of $\widetilde{E}$, namely $u(E)$.

Theorem 1.24. *For any vector bundle $E \to M$ here exists an embedding of E into a trivial bundle $M \times \mathbb{R}^k$, for some $k \in \mathbb{Z}$.*

For the proof we need that following Proposition, which is actually of independent interest:

Proposition 1.25. *For any vector bundle E over a manifold M there exists a finite open cover $\{U^1, \dots, U^l\}$ for M such that the restriction of E to each U^i is trivializable.*

The proof below shows that one may actually choose $l = \dim M + 1$.

Proof (of Proposition 1.25). When M is compact this should be clear: start with any, possibly infinite, trivializing cover, and just extract a finite subcover. In general, the key point is the following topological property that m-dimensional manifolds have (related to the concept of covering dimension, see the last chapter in the lecture notes on topology): any open cover of M admits a refinement $\{U^\alpha\}_{i \in \Lambda}$ with the property that each $x \in M$ belongs to at most $m+1$ of the U^α's. Therefore there exists an open cover $\{U^\alpha\}_{\alpha \in \Lambda}$ which is trivializing, i.e. such that each $E|_{U^\alpha}$ is trivializable, and such that

$$U^{\alpha_0} \cap \ldots \cap U^{\alpha_{m+1}} = \emptyset \quad \forall\, \alpha_0, \dots, \alpha_{m+1} \in \Lambda \quad \textit{distinct}. \tag{1.1.11}$$

Choose also a partition of unity $\{\eta^\alpha\}_{\alpha \in \Lambda}$ subordinated to this cover. Set now V^0 to be the set of points $x \in M$ with the property that there exists $\alpha_0 \in \Lambda$ such that

$$\eta^\alpha(x) < \eta^{\alpha_0}(x) \quad \forall\, \alpha \in \Lambda \setminus \{\alpha_0\},$$

which is a disjoint union of opens (one for each α_0) over which E is trivializable (as the open corresponding to each α_0 is inside U^{α_0}). Hence E is trivializable over V^0. Similarly, we define V^1 to be the set of points $x \in M$ with the property that there exists $\alpha_0, \alpha_1 \in \Lambda$ such that

$$\eta^\alpha(x) < \eta^{\alpha_0}(x) \text{ and } \eta^\alpha(x) < \eta^{\alpha_1}(x) \quad \forall\, \alpha \in \Lambda \setminus \{\alpha_0, \alpha_1\}.$$

By the same reasons as for V^0, E is trivializable also over V^1. Proceeding similarly we define V^2, V^3, etc, obtaining an open trivializing cover. To see that it is a cover, look at arbitrary $x \in M$: then $\eta^\alpha(x) \neq 0$ happens only for a finite number of α's, hence the maximum $\{\eta^\alpha(x) : \alpha \in \Lambda\}$ is attained for a finite number of indices, call them $\alpha_0, \dots, \alpha_k$; then $x \in V_k$. Hence $\{V^n : n \in \mathbb{N}\}$ is indeed an open trivializing cover. Property (1.1.11) ensures that $V^\alpha = \emptyset$ for all $\alpha \geq m+1$. 😄

Proof (of Theorem 1.24). We use a trivializing cover $\{V^1, \dots, V^l\}$ as ensured by Proposition 1.25 and, for each $\alpha \in \{1, \dots, l\}$ we choose a trivialization $h^\alpha : E|_{V^\alpha} \to V^\alpha \times \mathbb{R}^r$. We also choose a partition of unity $\{\eta^1, \dots, \eta^l\}$ subordinated to the cover. Define u by sending $v \in E_x$ (hence $x = \pi(v)$) to

$$u(v) := (x, \underbrace{\eta^1(x) \cdot h^1(v)}_{\in \mathbb{R}^r}, \dots, \underbrace{\eta^l(x) \cdot h^l(v)}_{\in \mathbb{R}^r}) \in M \times \underbrace{\mathbb{R}^r \times \ldots \times \mathbb{R}^r}_{l \text{ times}},$$

which provides the desired embedding into the trivial vector bundle of rank $k := rl$. 😄

1.2 Constructions/operations with vector bundles

1.2.1 Pull-backs

One basic property of vector bundles is that they can be pulled back along smooth maps

$$f : M \to N;$$

in other words, any such smooth map gives rise to an operation

$$f^* : \mathrm{Vect}^r_{\mathbb{R}}(M) \to \mathrm{Vect}^r_{\mathbb{R}}(N)$$

of pulling back vector bundles E over N to vector bundles f^*E over M. Fiberwise, for all $x \in M$.

$$(f^*E)_x = E_{f(x)},$$

(as vector spaces). This forces the definition of the set f^*E, the projection and the fiberwise vector space structure:

$$f^*E = M \times_N E = \{(x,v) \in M \times E : f(x) = \pi_E(v)\}, \quad \pi_{f^*E}(x,v) = x, \quad (x,v)+(x,w) = (x, v+w).$$

To make f^*E into a smooth vector bundle one can proceed in several ways. One would be to notice that f^*E becomes an embedded submanifold of the product $M \times E$, and then check local triviality (exercise). A slightly different approach is based on the notion of pull-backs of sections,

$$f^* : \Gamma(E) \to \Gamma(f^*E), \quad (f^*s)(x) := s(f(x)), \tag{1.2.1}$$

and we postulate the condition that, for any smooth section s of E, f^*s is a smooth section of f^*E. This can be implemented right away using Lemma 1.16.

Exercise 1.26. Show that any section s of f^*E can be written as

$$s = f^1 \cdot f^*(s_1) + \ldots + f^k \cdot f^*(s_k) \quad \text{with } f^i \in C^\infty(M),\ s_i \in \Gamma(E)$$

In other words, as a $C^\infty(M)$-module, $\Gamma(f^*E)$ is generated by the sections that are pull-backs of sections of E.

If you like rings and modules, and you want to see a very natural appearance of tensor products: go on further and describe the $C^\infty(M)$-module $\Gamma(f^*E)$ more precisely, in terms of the $C^\infty(N)$-module $\Gamma(E)$ and the ring morphism $f^* : C^\infty(N) \to C^\infty(M)$. You will probably recognise (or discover) a very natural appearance of tensor products- in this case $C^\infty(M) \otimes_{C^\infty(N)} \Gamma(E)$.

(Hints: One idea is to embed your E into a trivial one (Theorem 1.24), check the claims for the trivial one and then show that, if the claims hold for a vector bundle $\widetilde{E}$, then it holds for any vector sub-bundle E (hint for the last point: Corollary 1.22)).

Yet another description of f^*E would be via transition functions: if (1.1.6) and (1.1.7) is a transition data for E, then f^*E is induced by the pull-back transition data:

$$f^*\mathscr{U} := \{f^{-1}(U^\alpha)\}_{\alpha\in\Lambda}, \quad f^*(g_{\alpha,\beta}) := g_{\alpha,\beta} \circ f.$$

Pull-backs are relevant also when looking at morphisms of vector bundles over possibly different manifolds.

Definition 1.27. Given two vector bundles $\pi_E : E \to M$ over a manifold M and $\pi_E : E \to N$ over a manifold N, and a smooth map $f : M \to N$, by a **vector bundle morphism covering** f, or f-morphisms, we mean any smooth map $\Phi : E \to F$ with the property that it is compatible with the projections:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[f]{} & N \end{array} \quad , \quad \pi_F \circ \Phi = f \circ \pi_E$$

and its restriction to each fiber is linear. We denote by $\mathrm{Hom}_f(E,F)$ the collection of such f-morphisms.

Notice that the compatibility with the projections ensures that its restriction to each fiber is a map

$$\Phi_x : E_x \to F_{f(x)},$$

so that an φ-vector bundle morphisms can be seen as a family $\{\Phi_x\}_{x\in M}$ of such linear maps which fit into a smooth map $\Phi : E \to F$. Since $F_{f(x)} = f^*(E)_x$, we immediately deduce that Φ can also be interpreted as a map $\Phi : E \to f^*F$ between vector bundles over M. Therefore, indeed,

$$\boxed{\mathrm{Hom}_f(E,F) = \mathrm{Hom}(E, f^*F).}$$

Of course, when $f = \mathrm{Id}$ is the identity map of M, one recovers the usual morphisms over M.

Exercise 1.28. Show that for any smooth map between manifolds, $f : M \to N$, its differential df makes sense as a morphism from TM to f^*TN.

Finally, here is another standard piece of terminology related to pull-backs: when f is an inclusion $i : M \hookrightarrow N$ of a submanifold M of N, i^*E will also be called **the restriction of** E **to** N, and it will also be denoted

$$\boxed{E|_M := i^*E \qquad (i : M \hookrightarrow N \text{ the inclusion}).}$$

1.2.2 Duals

Let us start with the operation that associates to a vector space V its dual V^*. Applied to the fibers of a vector bundle E over M, one obtains an operation

$$\boxed{(-)^* : \mathrm{Vect}^r_{\mathbb{R}}(M) \to \mathrm{Vect}^r_{\mathbb{R}}(M)}$$

associating to a vector bundle E its dual E^*, which is a new vector bundle over M with fibers

$$\boxed{(E^*)_x = (E_x)^*.}$$

This equality determines the set E^*, the projection $E^* \to M$ and the fiberwise vector space structure. For the smooth structure we use Lemma 1.16 combined with the fact that any smooth local frame $\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_r)$ of E induces a

local frame e^* for E^*; the lemma allows us to find the smooth structure on E^* with the property that all such dual frames e^* become smooth. Notice that this discussion can be used to produce yet another description of E^*, in terms of transition functions: any transition functions $\{\tau^{\alpha,\beta}\}$ for E gives rise to transitions functions $\tau^{\alpha,\beta}$ for E^* by

$$\tau^*_{\alpha,\beta}(x) = \left(\tau^{\alpha,\beta}(x)\right)^* - \text{the adjoint of } \tau^{\alpha,\beta}(x) \in \mathrm{GL}_r(\mathbb{R}). \tag{1.2.2}$$

Another basic observation on E^* is that its sections are nothing but morphisms of vector bundles $s^* : E \to \mathbb{R}_M$, from E to the trivial line bundle $\mathbb{R}_M = M \times \mathbb{R}$. This provides a pairing

$$\langle \cdot, \cdot \rangle : \Gamma(E^*) \times \Gamma(E) \to C^\infty(M), \quad \langle s^*, s \rangle(x) := s^*_x(s_x).$$

and, using Lemma 1.6, one obtains an algebraic interpretation of $\Gamma(E^*)$ as the dual of the $C^\infty(M)$-module $\Gamma(E)$:

$$\left\{ \begin{array}{c} \text{smooth sections} \\ s^* \in \Gamma(E^*) \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{morphisms of } C^\infty(M)\text{-modules} \\ s^* : \Gamma(E) \to C^\infty(M) \end{array} \right\}.$$

Exercise 1.29. Show that any inner product g on E induces an isomorphism of vector bundles $I_g : E \to E^*$ uniquely characterised by the condition that, for any $s_1, s_2 \in \Gamma(E)$,

$$\langle I_g(s_1), s_2 \rangle = g(s_1, s_2).$$

1.2.3 Direct sums

We now start with another basic operation: the direct sum $V \oplus W$ of vector spaces V and W. Applying it fiberwise we obtain an operation

$$- \oplus - : \mathrm{Vect}^r_{\mathbb{R}}(M) \times \mathrm{Vect}^s_{\mathbb{R}}(M) \to \mathrm{Vect}^{r+s}_{\mathbb{R}}(M)$$

which associates to two vector bundles over M, $\pi_E : E \to M$ and $\pi_F : F \to M$, another vector bundle over M, $E \oplus F$, with fibers

$$(E \oplus F)_x := E_x \oplus F_x. \tag{1.2.3}$$

As before, these equations force the definition of $E \oplus F$ as a set, of the projection into M, and of the fiberwise vector space structure.

$$E \times_M F := \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f)\}, \quad \pi_{E \oplus F}(e, f) = \pi_E(e) = \pi_F(f).$$

To exhibit the smooth structure of $E \oplus F$ one can remark that $E \times_M F$ is an embedded submanifold of $E \times F$. Alternatively, one can use again Lemma 1.16; indeed, for any (smooth) local frames $\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_p)$ of E and $f = (f_1, \ldots, f_q)$ of F, defined over the same open, one can form the direct sum local frame

$$\mathfrak{e} \oplus \mathfrak{f} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_p, \mathfrak{f}_1, \ldots, \mathfrak{f}_q)$$

and we consider the smooth structure on $E \oplus F$ which makes all such local frames of $E \oplus F$ smooth. Notice that

$$\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$$

(as $C^\infty(M)$-modules).

Exercise 1.30. Check that there is no contradiction with the use of direct sums in Corollary 1.22. More precisely, show that if E^1 and E^2 are two vector sub-bundles of a vector bundle E and such that, at each $x \in M$ one has $E_x = E^1_x + E^2_x$ and the sum is direct, then $E^1 \oplus E^2 \to E$, $(e^1, e^2) \to e^1 + e^2$, is an isomorphism of vector bundles.

Exercise 1.31. Show that if E admits transition functions $g_E^{\alpha,\beta} : U_\alpha \to \mathrm{GL}_r(\mathbb{R})$, and similarly $g_F^{\alpha,\beta} : U_\alpha \to \mathrm{GL}_s(\mathbb{R})$ for F, where $\mathscr{U} = \{U^\alpha\}_{\alpha\in\Lambda}$ is an open cover of M, then $E \oplus F$ admits as transition functions

$$g^{\alpha,\beta}_{E\oplus F} := \begin{pmatrix} g_E^{\alpha,\beta} & 0 \\ 0 & g_F^{\alpha,\beta} \end{pmatrix} : U^\alpha \cap U^\beta \to GL_{r+s}(\mathbb{R}).$$

Use this, together with Proposition 1.13, to introduce $E \oplus F$ through local (transition) data.

Exercise 1.32. Consider the tangent bundle TS^n of the n-dimensional sphere. Show that the direct sum of TS^n with the trivial bundle of rank one is isomorphic to the trivial vector bundle of rank $n+1$ (note: this does not imply that TS^n is isomorphic to a trivial bundle, and it is not!).

And here is another fundamental property of vector bundles.

Theorem 1.33. *For any vector bundle $E \to M$ there exists a vector bundle $F \to M$ such that $E \oplus F$ is trivializable.*

Proof. We use an embedding u as in Theorem 1.24 and then apply Corollary 1.22 to the sub-bundle $u(E)$ of the trivial bundle. 😀

This discussion is also closely related to short exact sequence. Recall here that a short exact sequence of vector spaces, described as

$$0 \longrightarrow W \xrightarrow{\ i\ } V' \xrightarrow{\ \pi\ } V \longrightarrow 0 \tag{1.2.4}$$

consists of three vector space W, V' and V and linear maps i and π between them, satisfying the conditions: i-injective (allowing us to interpret W as a subspace of V'), π-surjective, $\mathrm{Im}(i) = \mathrm{Ker}(\pi)$. The notion of short exact sequences of vector bundles,

$$0 \longrightarrow F \xrightarrow{\ i\ } E' \xrightarrow{\ \pi\ } E \longrightarrow 0 \tag{1.2.5}$$

is defined completely similarly. Notice that the exactness of (1.2.5) is equivalent to the exactness of the sequences between the fibers at x, for all $x \in M$. Because of that, much of the discussion on vector bundles, including terminology, is basically the same as for vector spaces and is obtained by applying it fiberwise. For instance:

- by a **right splitting** of (1.2.5) we mean a morphism $h : E \to E'$ with $\pi \circ h = \mathrm{Id}_E$
- by a **left splitting** of (1.2.5) we mean a morphism $\theta : E' \to F$ with $\theta \circ i = \mathrm{Id}_F$;

$$0 \longrightarrow F \underset{\theta}{\overset{i}{\rightleftarrows}} E' \underset{h}{\overset{\pi}{\rightleftarrows}} E \longrightarrow 0$$

Lemma 1.34. *For any short exact sequence of vector bundles (1.2.5) there are 1-1 correspondences*

$$\left\{\begin{array}{c}\text{left splittings}\\ \theta: E'\to F\end{array}\right\}\overset{\sim}{\longleftrightarrow}\left\{\begin{array}{c}\text{right splittings}\\ h: E\to E'\end{array}\right\}\overset{\sim}{\longleftrightarrow}\left\{\begin{array}{c}\text{sub-bundles } C\subset E'\\ \text{complementary to } F\end{array}\right\}\equiv\left\{\begin{array}{c}\text{sub-bundles } C\subset E' \text{ such that}\\ \pi|_C: C\to E \text{ is an isomorphism}\end{array}\right\}$$

uniquely determined by requiring $h\circ\pi + i\circ\theta = Id_E$, $C = Im(h) = Ker(\theta)$. In particular,

1). Any short exact sequence of vector bundles admits left and right splittings.
2). Any choice of splittings as above induces an isomorphism

$$E'\cong F\oplus E,$$

obtained from the direct sum decomposition $E' = F\oplus C$ combined with the isomorphism $\pi|_C: C\to E$.
3). In terms of (θ, h), the isomorphisms between E' and $F\oplus E$ are given by:

$$E'\underset{i\oplus h}{\overset{(\theta,q)}{\rightleftarrows}} F\oplus E$$

Proof. This is a very instructive exercise on linear algebra, and on checking how arguments from vector spaces extend to vector bundles! 😀

1.2.4 Quotients

Next we look at the operation that associates to a vector subspace W of a vector space V the quotient vector space V/W. To upgrade this to vector bundles, we start with a vector sub-bundle $F\subset E$ and we define E/F as the vector bundle over M with fibers

$$(E/F)_x = E_x/F_x \quad .$$

Hence, as a set,

$$E/F = \coprod_{x\in M} E_x/F_x.$$

For the smooth structure on E/F there are several possible ways to proceed. We do it here using Lemma 1.16; to apply it, we consider local frames $\mathfrak{e}_1,\dots,\mathfrak{e}_r$ of E that are adapted to F in the sense that their first k components define a local frame of F (as in (c) of Lemma 1.19). For any such local frame adapted to F, its last $(r-k)$ components give rise to a (set-theoretical) local frame of E/F

$$\mathfrak{e}_{k+1} \bmod F,\dots,\mathfrak{e}_r \bmod F.$$

Apply now Lemma 1.16 to produce a smooth structure on E/F so that all these frames become smooth. Notice that, in turn, this implies that the quotient map maps smooth sections to smooth sections:

$$\Gamma(E)\to\Gamma(E/F),\quad s\mapsto s \bmod F.$$

Since the kernel of this map is precisely $\Gamma(F)$, one obtains an algebraic description of $\Gamma(E/F)$ as

$$\Gamma(E/F)\cong\Gamma(E)/\Gamma(F)$$

(as $C^\infty(M)$-modules).

Another way to think of E/F is as "the orthogonal of F", but built without having to choose a metric (the price to pay is that, without such a choice, E/F does not sit inside E):

Exercise 1.35. Given a vector sub-bundle F of E, for any inner product g on E, the restriction of the quotient map $E \to E/F$ to the g-orthogonal $F^\perp$ induces an isomorphism of vector bundles $F^\perp \to E/F$.

Actually, this discussion is closely related to that of short exact sequences. Indeed, the quotient E/F allows one to complete any inclusion $i : F \hookrightarrow E$ into a short exact sequence

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} E/F \to 0, \tag{1.2.6}$$

and, conversely, for any short exact sequence of vector bundles, the term on the right hand side is (canonically) isomorphic to the quotient of the other two.

1.2.5 Hom-bundles; tensor products

Next we look at the operation that associates to two vector spaces V and W the vector space $\mathrm{Hom}(V,W)$ consisting of all linear maps from V to W. The outcome will be an operation

$$\mathrm{Hom}(-,-) : \mathrm{Vect}^r_{\mathbb{R}}(M) \times \mathrm{Vect}^s_{\mathbb{R}}(M) \to \mathrm{Vect}^{r\cdot s}_{\mathbb{R}}(M)$$

which associates to two vector bundles E and F over M the new vector bundle $\mathrm{Hom}(E,F)$ over M with fibers

$$\mathrm{Hom}(E,F)_x = \mathrm{Hom}(E_x, F_x).$$

And, again, we see that local frames of E and F induce a local frame of $\mathrm{Hom}(E,F)$, and then we obtain a canonical smooth structure on the hom-vector bundle. Note that a section of $\mathrm{Hom}(E,F)$ is the same thing as a morphism $u : E \to F$ of vector bundles over M; therefore, the 1-1 correspondence from Proposition 1.6 can be completed to:

$$\Gamma(\mathrm{Hom}(E,F)) \equiv \left\{ \begin{array}{c} \text{morphisms } u : E \to F \\ \text{of vector bundles over } M \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{morphisms } u_* : \Gamma(E) \to \Gamma(F) \\ \text{of } C^\infty(M)\text{-modules} \end{array} \right\}.$$

Of course, when F is the trivial vector bundle of rank 1 ($F = \mathbb{R}_M = M \times \mathbb{R}$), we recover the dual of E:

$$E^* = \mathrm{Hom}(E, \mathbb{R}_M).$$

Exercise 1.36. Consider the tautological line bundle over $\mathbb{RP}^n$ (see Example 1.17), $L = L_{\mathbb{RP}^n}$, as well as the vector bundle $L^\perp$ whose fiber over $l \in \mathbb{RP}^n$ is the orthogonal complement of l in $\mathbb{R}^{n+1}$ (see the previous Exercise). Show that one has an isomorphism of vector bundles:

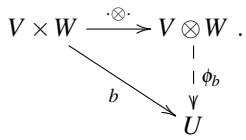
$$T(\mathbb{RP}^n) \cong \mathrm{Hom}(L, L^\perp),$$

where $T(\mathbb{RP}^n)$ is the tangent bundle of $\mathbb{RP}^n$.

We now move to the related operation of tensor products $V \otimes W$ of vector spaces. Recall that this is a new vector space that comes with a bilinear map

$$\cdot \otimes \cdot : V \times W \to V \otimes W, \quad \text{denoted } (v,w) \mapsto v \otimes w$$

which is universal in the sense that, for any other vector space U together with a bilinear map $b : V \times W \to U$, there exists a unique linear map $\phi_b : V \otimes W \to U$ such that



Since we work with finite dimensional vector spaces, the tensor product could be defined more "concretely", using duals and Hom-spaces as:

$$V \otimes W = \mathrm{Hom}(V^*, W)$$

and where, for $v \in V$, $w \in W$, the tensor $v \otimes w$ denotes the linear map

$$v \otimes w : V^* \to W, \quad \xi \mapsto \xi(v)w.)$$

Either way, one important remark is that, whenever we have a frame $\mathfrak{e}_1, \ldots, \mathfrak{e}_r$ of V, and a frame $\mathfrak{f}_1, \ldots, \mathfrak{f}_s$ of W, the collection of all possible tensor products

$$\mathfrak{e}_i \otimes \mathfrak{f}_j \in V \otimes W$$

form a frame for $V \otimes W$. In particular, $\dim V \otimes W = \dim V \cdot \dim W$.

We now have all the ingredients to proceed as before and talk about the tensor product

$$E \otimes F \to M$$

of any two vector bundles E and F over M, which is an operation

$$\boxed{- \otimes - : \mathrm{Vect}^r_{\mathbb{R}}(M) \times \mathrm{Vect}^s_{\mathbb{R}}(M) \to \mathrm{Vect}^{r \cdot s}_{\mathbb{R}}(M).}$$

Given the relationship of $\otimes$ with the Hom-operation (for vector spaces) it follows that, similarly, for any two vector bundles E and F,

$$\mathrm{Hom}(E, F) \cong E^* \otimes F, \quad E \otimes F \cong \mathrm{Hom}(E^*, F).$$

We leave it to the curious student to figure out the description of $E \otimes F$ in terms of transition functions.

1.2.6 Variations; differential forms with coefficients

Similar to taking the dual of a vector space one can consider operations of type

$$V \mapsto T^k V^*, \quad V \mapsto \Lambda^k(V^*), \quad V \mapsto S^k V^*,$$

for any integer $k \geq 1$, which associate to a vector space V the space of all k-multilinear maps $V \times \ldots \times V \to \mathbb{R}$, the skew-symmetric k-multilinear ones, and the symmetric k-multilinear ones, respectively. See subsection 4.2.1 of the bachelor notes. and, in particular, remember how one can build frames for these spaces out of frames for V. Slightly more generally, one can consider operations of type

$$(V, W) \mapsto T^k V^* \otimes W, \quad (V, W) \mapsto \Lambda^k(V^*) \otimes W, \quad (V, W) \mapsto S^k V^* \otimes W \tag{1.2.7}$$

which associate to a pair (V, W) of vector spaces of k-multilinear maps on V with values in W. All of these give rise to similar operations with vector bundles as entries. Here is a generalization of Proposition 1.6 :

Exercise 1.37. Show that, for any two vector bundles T and E over M, $k \geq 1$ integer, there is a 1-1 correspondence:

$$\left\{\begin{array}{c}\text{sections}\\ u \text{ of } \Lambda^k T^* \otimes E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}C^\infty(M)\text{-multi-linear, anti-symmetric maps}\\ u_* : \underbrace{\Gamma(T)\times\ldots\times\Gamma(T)}_{k-\text{times}} \to \Gamma(E)\end{array}\right\}$$

Notice that an instance of these constructions appeared already (implicitly) when discussing differential forms on a manifold M, which can now be described directly as sections of the exterior powers of TM:

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M).$$

The more general operations (1.2.7) allow us to talk about differential forms with coefficients in a vector bundle:

Definition 1.38. Given a vector bundle E over M, we define the space of E-valued k-differential forms on M

$$\Omega^k(M;E) = \Gamma(\Lambda^k T^* M \otimes E).$$

As for usual forms, and supported by the previous exercise, an element $\omega \in \Omega^k(M;E)$ can be thought of as a $C^\infty(M)$-multilinear antisymmetric map

$$\omega : \underbrace{\mathscr{X}(M)\times\ldots\times\mathscr{X}(M)}_{k \text{ times}} \to \Gamma(E). \tag{1.2.8}$$

Notice that also the wedge product of forms (as discussed e.g. in Section 4.2 of the bachelor notes) can be extended to this context to provide operations

$$\Omega^k(M)\times\Omega^l(M;E)\to\Omega^{k+l}(M;E), \quad \Omega^k(M;E)\times\Omega^l(M)\to\Omega^{k+l}(M;E);$$

explicitly, one can still use the standard/old formula ((4.2.3) in the bachelor notes), i.e.,

$$(\omega\wedge\eta)(X_1,\ldots,X_{k+l}) = \sum_\sigma sign(\sigma)\omega(X_{\sigma(1)},\ldots,X_{\sigma(k)})\cdot\eta(X_{\sigma(k+1)},\ldots,X_{\sigma(k+l)}), \tag{1.2.9}$$

where the sum is over all (k,l)-shuffles σ. Although this formula no longer makes sense when ω and η are both E-valued differential forms, it does make sense when one of them is E-valued and the other one $\mathbb{R}$-valued. The resulting operation makes

$$\Omega(M,E) = \bigoplus_k \Omega^k(M,E)$$

into a (left and right) module over $\Omega(M)$.

Remark 1.39 (conventions when $k=0$ or $l=0$). Similar to $\Omega^0(M) = C^\infty(M)$, one has $\Omega^0(M,E) = \Gamma(E)$. While the usual wedge product of forms, when one of the forms is of degree zero, is is just the multiplication "$\cdot$" of forms by functions, the same applies to forms with coefficients:

$$f\wedge\eta = f\cdot\eta \qquad \Big(\text{for } f\in C^\infty(M), \eta\in\Omega^l(M,E)\Big).$$

On the other hand, when wedging with a 0-form with coefficients s, one uses several notations

$$\omega\wedge s = \omega\otimes s = \omega\cdot s \qquad \Big(\text{for } \omega\in\Omega^k(M), s\in\Gamma(E)\Big), \tag{1.2.10}$$

all for the form with coefficients $(X_1,\ldots,X_k)\mapsto\omega(X_1,\ldots,X_k)\cdot s$.

Keeping in mind the fact that the spaces Ω are graded (i.e are direct sums indexed by integers) and the fact that the wedge products involved are compatible with the grading (i.e. $\Omega^k \wedge \Omega^l \subset \Omega^{k+l}$), we say that $\Omega(M)$ is a graded algebra and $\Omega(M,E)$ is a graded bimodule over $\Omega(M)$. As for the usual wedge product of forms, the left and right actions are related by[2]

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \ \ \forall\, \omega \in \Omega^k(M), \eta \in \Omega^l(M,E).$$

Also, locally, with respect to a coordinate chart $(U, \chi_1, \ldots, \chi_n)$, one can write

$$\omega = \sum_{i_1,\ldots,i_k} d\chi_{i_1} \wedge \ldots \wedge d\chi_{i_k} \otimes e^{i_1,\ldots,i_k}. \tag{1.2.11}$$

with $e^{i_1,\ldots,i_k}$ local sections of E (defined on U). Using also a local frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$ for E, we obtain expressions of type

$$\sum_{i_1,\ldots,i_k,i} f_i^{i_1,\ldots,i_k} d\chi_{i_1} \wedge \ldots \wedge d\chi_{i_k} \otimes \mathfrak{e}_i.$$

1.2.7 Complex vector bundles

So far we have concentrated on vector bundles. However, as already pointed out right from the start, Definition 1.1, complex vector bundles are handled completely similarly by replacing $\mathbb{R}$ by $\mathbb{C}$, and the entire discussion we had so far applies word by word by simply:

- replacing real vector spaces by complex ones, and ($\mathbb{R}$-)linear maps by $\mathbb{C}$-linear ones. The resulting collection of isomorphism classes of complex vector bundles of (complex) rank s will be denoted

$$\boxed{\mathrm{Vect}^s_{\mathbb{C}}(M)}$$

- $C^\infty(M)$ will become the space of $\mathbb{C}$-valued smooth functions on E,
- the inner products will become complex inner products- usually called hermitian inner products,
- the transition functions will take value in the groups $\mathrm{GL}_s(\mathbb{C})$.
- when discussing the (complex) dual F^*, the resulting transition functions (1.2.2) will use the adjoints of complex matrices (conjugate of the transpose).
- the previous tensor products $\otimes = \otimes_{\mathbb{R}}$ will be replaced by $\otimes_{\mathbb{C}}$.

In particular, the operations of taking duals, direct sums, quotients, etc- they all have a complex version.

Example 6.32' (first examples) The first examples of real vector bundles (section 1.1.5) were tangent bundles and tautological line bundles, carry over the complex setting. The tangent bundles of complex manifolds will be discussed in more detail in section 1.3.3. The complex tautological line bundle is completely analogous to the real one, discussed in Example 1.17: it is the complex line bundle $L_{\mathbb{CP}^n}$ over the complex projective space $\mathbb{CP}^n$, whose fiber above a complex line $l \in \mathbb{CP}^n$ is the lines itself, $l \subset \mathbb{C}^{n+1}$; more concretely

$$L_{\mathbb{CP}^n} = \{(l,v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in l\},$$

with the projection $L_{\mathbb{CP}^n} \to \mathbb{CP}^n$ that sends (l,v) to l.

Exercise 6.32". Prove the complex analogue of Exercise (1.36), i.e., denoting by L the tautological line bundle over $\mathbb{CP}^n$, show that one has an isomorphism of complex vector bundles:

$$T(\mathbb{CP}^n) \cong \mathrm{Hom}(L, L^\perp),$$

[2] Important: this is the first manifestation of what is known as the "graded sign rule": in an formula that involves graded elements, if two elements a and b of degrees p and q are interchanged, then the sign $(-1)^{pq}$ is introduced

where $T(\mathbb{CP}^n)$ is the tangent bundle of $\mathbb{CP}^n$. Then deduce that

$$T(\mathbb{CP}^n)\oplus\mathbb{C}\cong\underbrace{L^*\oplus\ldots\oplus L^*}_{n+1 \text{ copies}}$$

Remark 6.33". Returning to the general discussion above, the only aspect that requires extra-care is when using inner products to further identify the dual bundles, i.e. when formulating the complex version of Exercise 1.29. Indeed, if h is a hermitian inner product on the complex vector bundle F, to land in complex dual F^* we need to use the map

$$I_h : F \to F^*, \quad I_h(s_1)(s_2) = h(s_2,s_1) = \overline{h(s_1,s_2)}$$

which, in turn, is conjugate linear. There are two things that we learn here:

1. First of all, that for complex vector bundles there is yet another operation, that of conjugation:

$$\boxed{\overline{(-)} : \mathrm{Vect}^s_{\mathbb{C}}(M) \to \mathrm{Vect}^s_{\mathbb{C}}(M), \quad F \mapsto \overline{F}.}$$

   More precisely, given F, **the conjugate $\overline{F}$ of F** is the complex vector space that has the same underlying total space as F, with the only difference that we change the multiplication by complex numbers on the fiber to:

$$z\cdot v := \bar{z}v$$

   where $\bar{z}$ is the complex conjugate of z.
2. The second thing to learn here is that hermitian inner products give rise to isomorphisms

$$\boxed{F^*\cong\overline{F}}.$$

   It is interesting to know that, in general, F^* and $\overline{F}$ may fail to be isomorphic to F, as complex vector bundles. That happens already for the complex version of the tautological line bundle from Example 6.32' (though for a proof of this statement we have to wait until we discuss the first Chern class of line bundles).

On the other hand, there is an operation of complexifying real vector bundles,

$$\boxed{(-)_{\mathbb{C}} : \mathrm{Vect}^r_{\mathbb{R}}(M) \to \mathrm{Vect}^r_{\mathbb{C}}(M), \quad E \mapsto E_{\mathbb{C}}.}$$

Starting with a real vector bundle E its complexification is the complex vector bundle

$$E_{\mathbb{C}} := E\otimes\mathbb{C}$$

whose fiber above an arbitrary point $x\in M$ is $E_x\otimes\mathbb{C} = \{v\otimes 1 + w\otimes i : u,v\in E_x\}$. Alternatively, one can use the direct sum of real vector bundles and define $E_{\mathbb{C}} := E\oplus E$ in which the multiplication by complex numbers is:

$$(a+bi)\cdot(u,v) = (au-bv, av+bu) \quad \text{for } a+ib\in\mathbb{C}. \tag{1.2.12}$$

Notice that, unlike general complex vector bundles, such complexifications come with an conjugation map

$$u+iv \mapsto \overline{u+iv} := u-iv \tag{1.2.13}$$

giving rise to an isomorphism (of complex vector bundles) between $E_{\mathbb{C}}$ and $\overline{E_{\mathbb{C}}}$. He This is related to the discussion above on the difference between F and $\overline{F}$: that difference will not be seen when F arises as the complexification of a real vector bundle. Actually, one can now realise a bit better what is going on:

Exercise 1.40. Show that a complex vector bundle F is isomorphic to one of type $E_{\mathbb{C}}$ if and only if there exists an isomorphism of complex vector bundles

$$\tau : F \to \overline{F}$$

with the property that $\tau^2 = \mathrm{Id}$.

1.2.8 Complex structures on (real) vector bundles

Of course, any complex vector bundle F can be seen as a real vector bundle; when we want to emphasise that we look at F as being a real vector bundle, we use the notation $F_{\mathbb{R}}$. This describes an operation

$$(-)_{\mathbb{R}} : \mathrm{Vect}^s_{\mathbb{C}}(M) \to \mathrm{Vect}^{2s}_{\mathbb{R}}(M), \quad F \mapsto F_{\mathbb{R}}.$$

A natural question is: when does a real vector bundle E come from a complex one F? First of all, E has to have even rank. On the other hand, since F and E will be the same as real vector bundles, what is missing from E is the multiplication by $i = \sqrt{-1}$, which will be a vector bundle morphisms $J : E \to E$ such that $J^2 = -\mathrm{Id}$. Such a morphism J is called **a complex structure** on the real vector bundle E. To summarise: E comes from a complex vector bundle if and only if E admits a complex structure, and one has 1-1 correspondences:

$$\left\{\begin{array}{c}\text{complex vector bundles } F\\ \text{such that } E = F_{\mathbb{R}}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{vector bundle morphisms } J : E \to E\\ \text{such that } J^2 = -\mathrm{Id}\end{array}\right\} \equiv \left\{\begin{array}{c}\text{complex structures}\\ \text{on } E\end{array}\right\}.$$

Example 1.41. When building the complexification $E_{\mathbb{C}}$ of a real vector bundle as the direct sum $E \oplus E$, the complex structure (1.2.12) corresponds to

$$J(u,v) = (-v,u).$$

The operation of complexification can be used to encode complex structures J one a real vector bundle E differently, via certain sub-spaces of the complexification $E_{\mathbb{C}}$. This is basically a remark on vector spaces, but we chose to write it using vector bundles (the discussion is really the same).

Therefore, we start with a real vector bundle E. Now, given a complex structure $J : E \to E$ on E, we simply interpret it as an $\mathbb{R}$-linear morphism and we form its complexification,

$$J_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}, \quad J_{\mathbb{C}}(u+iv) = J(u) + iJ(v).$$

Using that $J^2_{\mathbb{C}} = -\mathrm{Id}$ and that $J_{\mathbb{C}}$ acts on complex vector spaces, it follows that $E_{\mathbb{C}}$ can be split into the i and $-i$ eigenspaces of $J_{\mathbb{C}}$:

$$E = E^{1,0} \oplus E^{0,1},$$

where the two summands are the complex vector bundles given by:

$$E^{1,0} = \{w \in E_{\mathbb{C}} : J_{\mathbb{C}}(w) = iw\} = \ldots = \{u - iJ(u) : u \in E\}, \tag{1.2.14}$$

$$E^{0,1} = \{w \in E_{\mathbb{C}} : J_{\mathbb{C}}(w) = -iw\} = \ldots = \{u + iJ(u) : u \in E\}.$$

It is clear that the two are related via the conjugation (1.2.13):

$$E^{0,1} = \overline{E^{1,0}}.$$

Furthermore, the complex vector bundle is simply (isomorphic to) E_J, which is E with the complex vector bundle structure induced by J, with the explicit isomorphism

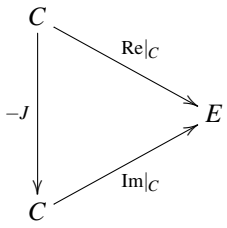
$$E_J \xrightarrow{\sim} E^{1,0}, \quad u \mapsto u - iJ(u).$$

The advantage of $E^{1,0}$ over E_J is that its properties make little reference to J- e.g., already its complex structure, comes from the fact that it is a subspace of $E_{\mathbb{C}}$. The final remark is that the complex structure J on E not only gives rise to, but it is actually determined by $E^{1,0}$.

Proposition 1.42. *The construction above, $J \mapsto E^{1,0}$, describes a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{complex structures} \\ J \textit{ on } E \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{complex sub-bundles } C \subset E_{\mathbb{C}} \\ \textit{with } E_{\mathbb{C}} = C \oplus \overline{C} \end{array} \right\}$$

Proof. To see how to recover J from C notice that we have a commutative diagram



Furthermore, a dimension counting shows that

$$\dim_{\mathbb{R}} C = 2\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} E_{\mathbb{C}} = \dim_{\mathbb{R}} E$$

and we deduce that the conditions on C imply that $\mathrm{Re} : E_{\mathbb{C}} \to E$ restricts to an isomorphism of real vector bundles $\mathrm{Re}|_C : C \to E$. Hence the diagram above provides the complete description of J. One checks directly that this construction is inverse to $J \mapsto E^{0,1}$. 😄

1.3 Geometric structures on manifolds: first examples

As we will be seeing over and over again, geometric structures on manifolds M are often introduced/described by using concepts that make sense for general vector bundles E to $E = TM$, the tangent bundle of M. Of course, that is just the starting point, and how much interesting a geometric structure is depends on the examples and the geometry that it gives rise to. For instance, just starting with an inner product on a tangent bundle will allow one to measure lengths of curves, talk about curves "of shortest length"/geodesics and, talk about "the curvature" of the space, etc. That is more than enough reason to introduce:

Definition 1.43. A **Riemannian structure (or metric)** on a manifold M is an inner product g on the tangent bundle TM. The pair (M, g) is also called a **Riemannian manifold**.

As indicate above, the importance of Riemannian metrics comes from the fact that they reveal beautiful geometric properties of the manifold, while there is an abundance of interesting examples. Actually, unlike other structures that we will be looking at, Proposition 1.11 ensures that any manifold carries a Riemannian metric. The simplest/standard example is the so-called flat Riemannian structure on $M = \mathbb{R}^m$, defined by

$$g_{\text{can}}\left(\sum_i v_i \frac{\partial}{\partial x_i}, \sum_i w_i \frac{\partial}{\partial x_i}\right) = \sum_i v_i w_i,$$

or, equivalently, defined by the condition that the canonical frame $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ is orthonormal. Or, as a section of S^2T^*M ($M = \mathbb{R}^n$),

$$g_{\text{can}} = \sum_i dx_i \cdot dx_i = \sum_i (dx_i)^2.$$

In general it is interesting to compare, locally, Riemannian metrics with the flat one on $\mathbb{R}^m$.

For other examples, it is useful to notice that inner products on vector spaces can be restricted to subspaces, hence submanifolds of Riemannian manifolds carry an induced Riemannian structure. In particular, any embedded submanifold $M \subset \mathbb{R}^n$ becomes Riemannian; it is interesting to notice here that the resulting Riemannian metric g_M on M, although is obtained by starting from the flat one on the ambient Euclidean space $\mathbb{R}^n$, is quite far from being locally isomorphic itself to the flat one g_{can} on $\mathbb{R}^m$- and comparing g_M with g_{can} is about understanding how M sits in $\mathbb{R}^n$.

We now move to other examples.

1.3.1 Parallelizations

One type of "structure" which turns out to be more interesting than it may seem at first arises by looking at global frames or, equivalently, at global trivializations, for $E = TM$.

Definition 1.44. A **parallelization** of a manifold M is a global frame $\mathfrak{e} = \{X^1, \ldots, X^m\}$ of the tangent bundle TM. We say that M is **parallelizable** if such a global frame exists.

The simplest example is, of course, $\mathbb{R}^m$ with the standard frame $\frac{\partial}{\partial x_i}$. However, even on $\mathbb{R}^m$, it is interesting to understand how other frames compare with the standard one (see below).

Exercise 1.45. Show that the tangent bundle of S^3 is trivializable. (1st hint: quaternions; 2nd hint: first do it for S^1; then write your proof using complex numbers; then go to the previous hint).

For the curious reader

Existence *The question of whether S^n is parallelizable or not can be rewritten in very down to earth terms: one is looking for $n+1$ functions*

$$F^1, \ldots, F^n \to \mathbb{R}^{n+1}$$

with the property that, for each $x \in S^n \subset \mathbb{R}^{n+1}$, vectors $F^1(x), \ldots, F^n(x)$ are linearly independent and take value in the hyperplane P_x orthogonal to x:

$$F^i(x) \in P_x := \{v \in \mathbb{R}^{n+1} : \langle v, x\rangle = 0\}.$$

This comes from the standard identification of the tangent spaces T_xS^n with $P_x \subset \mathbb{R}^{n+1}$. In the case $n = 1$ one can take

$$F^1(x_1, x_2) = (x_2, -x_1)$$

or, interpreting $\mathbb{R}^2$ as $\mathbb{C}$ ($(x_1, x_2) \mapsto x_1 + ix_2$),

$$F^1(z) = iz.$$

The previous exercise reveals that the same idea, but using quaternions instead of complex numbers, works to prove that S^3 is parallelizable. On the other hand, the 2-sphere however is not parallelizable. Actually, on S^2 there isn't even one single vector field that is nowhere zero (the "hairy ball theorem"). The core reason for that is that its "Euler characteristic" (one of the simplest topological invariants one can associate to spaces) does not vanish. Actually, a nice (but non-trivial) theorem in differential topology says that the Euler characteristic of a compact manifold vanishes if and only if it admits a no-where vanishing vector field. By the way, for the same reason, all the even dimensional spheres S^{2n} are not parallelizable.

Are the other (odd dimensional) spheres parallelizable? Yes: there is also S^7. This uses octonions (also called the Cayley algebra) instead of quaternions (S^3) and complex numbers (S^1). How general is this? In what dimensions does it work? Looking at the proofs (e.g. for S^1 and S^3) one sees that what is needed is a "normed division algebra" structure on $\mathbb{R}^{n+1}$ (in order to handle the n-sphere S^n). By that we mean a multiplication "$\cdot$" on $\mathbb{R}^{n+1}$, with unit element $1 := (1, 0, \ldots, 0)$ (but not necessarily associative- as in the case of the octonions) and satisfying the norm condition

$$|x\cdot y| = |x||y| \quad \forall\, x,y \in \mathbb{R}^{n+1},$$

where $|\cdot|$ is the standard norm. The term "division algebra" comes from the fact that the norm condition implies that the product has the "division property": $xy=0$ implies that either $x=0$ or $y=0$ (if we keep this condition, but we give up on the norm condition, we talk about division algebras). Indeed, any such operation induces

$$F^i(x) = e_i \cdot x$$

proving that S^n is parallelizable. But on which $\mathbb{R}^{n+1}$ do there exist such operations? Well: only on $\mathbb{R}$, $\mathbb{R}^2$ (complex), $\mathbb{R}^4$ (quaternions) and $\mathbb{R}^8$ (octonions)! This was known since the 19th century. It is also interesting to know such operations were considered already in the 19th century: number theory and the question of which numbers can be written as sums of two, three, etc squares. For sum of two squares, the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2+y^2)(a^2+b^2) = (xa-yb)^2+(xb+ya)^2.$$

Or, in terms of the complex numbers $z_1 = x+iy, z_2 = a_ib$:

$$|z_1 z_2| = |z_1||z_2|.$$

The search for similar "magic formulas" for sum of three squares never worked, but it did for four:

$$(x^2+y^2+z^2+t^2)(a^2+b^2+c^2+d^2) =$$

$$(xa+yb+zc+td)^2+(xb-ya-zd+tc)^2+$$

$$+(xc+yd-za-tb)^2+(xd-yc+zb-ta)^2.$$

This is governed by the quaternions and its norm equation.

Any way, returning to the spheres, we see that the trick with the multiplication can only work for S^1, S^3 and S^7. And, indeed, one can prove that there are the only parallelizable spheres! Well, S^0 as well if you insist. The proof is highly non-trivial and makes use of the machinery of Algebraic Topology.

Notice that the negative answer in the case of even dimensional spheres - due to the fact that such spheres do not even admit nowhere vanishing vector fields- comes with a very natural question: ok, but, looking at S^n, which is the largest number of linearly independent vector fields that it admits? Again, this is a simple but very non-trivial problem, whose solution (half way in the 20th century) requires again the machinery of Algebraic Topology. But here is the answer: write $n+1 = 2^{4a+r}m$ with m odd, $a \geq 0$ integer, $r \in \{0,1,2,3\}$. Then the maximal number of linearly independent vector fields on S^n is

$$r(n) = 8a+2^r-1.$$

Note that $r(n)=0$ if n is even (no nowhere vanishing vector fields on the even dimensional spheres), while the parallelizability of S^n is equivalent to $r(n)=n$, i.e. $8a+2r = 2^{4a+r}m$ which is easily seen to have the only solutions

$$m=1, a=0, r\in\{0,1,2,3\}$$

giving $n=0,2,4,8$ hence again the spheres S^0, S^1, S^3 and S^7.

The largest class of manifolds that are automatically parallelizable (because of their structure) are the Lie groups, and this can be seen as one of the reasons that S^1 and S^3 are parallelizable.

Proposition 1.46. *Any Lie group is parallelizable.*

Proof. For $g \in G$ we use the right translations $R_g : G \to G$, $R_g(a) = ag$ and their differentials at the unit $e \in G$, $(dR_g)_e : \mathfrak{g} \to T_gG$. These are linear isomorphisms because R_g are diffeomorphisms. Together, they provide an isomorphism between TG and the trivial vector bundle with fiber $\mathfrak{g}$:

$$G\times\mathfrak{g} \to TG, \quad (g,v) \mapsto (dR_g)_e(v).$$

Equivalently, choosing a basis $e^1,\dots,e^m$ of the vector space $\mathfrak{g}$,

$$V^i_g := (dR_g)_e(e^i)$$

define a frame $\mathfrak{e} := \{V^1,\dots,V^m\}$ of G. 😄

Can other spheres be made into Lie groups (besides S^1 and S^3)? The answer is no! Furthermore, while the Lie group structure for S^1 and S^3 uses complex numbers and quaternions, respectively, a similar argument for S^7 using octonions would not work because the multiplication of octonions is not associative. Nevertheless, as the division property, without associativity, is enough to conclude parallelizability (see above). Therefore S^7 arises as the border case: it is not a Lie group but, nevertheless, is parallelizable.

Exercise 1.47. Show that $S^2 \times S^1$ is parallelizable.

We now return to one of the interesting questions that we alluded to above: when is a parallelization locally isomorphic to the standard on one $\mathbb{R}^m$?

Definition 1.48. A parallelization $\mathfrak{e} = \{X^1, \dots, X^m\}$ of M is called **integrable** if around each point of M one finds a a chart (U, χ) of M such that, over U,

$$X^i = \chi^* \left(\frac{\partial}{\partial x_i} \right), \qquad 1 \leq i \leq m.$$

Searching for what characterizes integrability, it doesn't take long to discover the notion of **structure function** of the frame $\mathfrak{e}$, defined as the smooth functions $c_{i,j}^k$ on M satisfying

$$[X^i, X^j] = \sum_k c_{i,j}^k X^k.$$

Theorem 1.49. *A parallelization $\mathfrak{e}$ is integrable if and only if its structure functions vanish identically.*

Proof. The condition is $[\mathfrak{e}_i, \mathfrak{e}_j] = 0$ for all i and j; this is clearly a local condition and is satisfied by the local model, so it is necessary in order to have integrability. For the sufficiency, we use the flows Φ_i^t of the vector fields $\mathfrak{e}_i$. The condition we have implies that each two of them commute. Let $x_0 \in M$ be arbitrary and consider

$$F : \mathbb{R}^n \to M, \quad F(t_1, \dots, t_n) = \Phi_1^{t_1} \circ \dots \Phi_n^{t_n}(x_0);$$

strictly speaking, this is defined on an open neighborhood of $0 \in \mathbb{R}^n$ (corresponding to the domains of the flows). Clearly $F(0) = x_0$. Computing the partial derivatives of F at an arbitrary point $x \in M$ we find the vectors $\mathfrak{e}_i(F(x))$. In particular, by the inverse function theorem, F is a local diffeomorphism from an open containing 0 to an open containing x_0; this defines the desired local chart. 😄

There is one more comment we would like to make here: very often one prefers to work with differential forms instead of vector fields. In particular, one prefers to work with coframes instead of frames, i.e. with families of 1-forms:

$$\theta = (\theta^1, \dots, \theta^n)$$

which, at each $x \in M$, induces a basis of $T_x^* M$. Of course, the two points of view are equivalent, as one has a 1-1 correspondence $\mathfrak{e} \longleftrightarrow \theta$ between frames and coframes given by the usual duality

$$\theta^i(\mathfrak{e}_j) = \delta_{i,j}.$$

However, when one goes deeper into the theory, the parallelism between the two points of view becomes a bit less obvious. One thing to keep in mind is that the role played by the Lie derivatives of vector fields, in the dual picture of forms, is taken by the DeRham differential. For instance, for a coframe θ, the resulting structure functions are characterized by

$$d\theta^k = \sum_{i<j} c_{i,j}^k \theta^i \wedge \theta^j.$$

Exercise 1.50. Prove that, indeed, given a frame $\mathfrak{e}$ and the induced coframe θ, the structure functions of $\mathfrak{e}$ satisfies the previous equation.

Of course, in the dual picture, the local model is $(\mathbb{R}^n, dx_1, \ldots, dx_n)$, the integrability of (M, θ) is about writing (locally) $\theta^i = d\chi_i$ and the previous proposition say that this is possible if and only if all the 1-forms are closed.

Exercise 1.51. Give now another proof of the previous proposition, using the fact that, locally, all closed 1-forms are exact.

1.3.2 Foliations

Another interesting type of geometric structures on manifolds is that of "foliation". Very briefly, a p-dimensional foliation on a manifold M is a partition of M into p-dimensional submanifolds, called leave, such that, locally, that partition looks like

$$\mathbb{R}^n = \cup_{y\in\mathbb{R}^q} \mathbb{R}^p \times \{y\}, \quad (q = n-p).$$

While this notion is geometric/intuitive ("easy to picture"), it is rather hard to work with. The Frobenius theorem mentioned below says that the same notion can be approached more algebraically (infinitesimally) provided one makes use of the tangent bundle and Lie bracket of vector fields. That is the approach we take here.

From the point of view of vector bundles, the central concept here is that of vector sub-bundle. Its role is to prescribe "the tangent directions to the leaves".

Definition 1.52. A **p-dimensional distribution** on a manifold M, where $p \geq 1$ is an integer, is a rank p vector sub-bundle of the tangent bundle,

$$\mathscr{F} \subset TM.$$

An **integral** of $\mathscr{F}$ is any immersed connected submanifold $N \hookrightarrow M$ with the property that

$$T_pN = \mathscr{F}_p \quad \forall\, p \in N.$$

One says that $\mathscr{F}$ is **integrable** if for any $p \in M$ there exists such an integral with $p \in N$. In this case we also say that $\mathscr{F}$ defines a **foliation** on M, or that $(M, \mathscr{F})$ is a **foliated manifold**.

Example 1.53. On $M = \mathbb{R}^m$ one has canonical p-dimensional distribution $\mathscr{F}_{\text{can}}$ given by

$$\mathscr{F}_{\text{can},x} = \text{Span}\left\{ \left(\frac{\partial}{\partial\chi_1}\right)_x, \ldots, \left(\frac{\partial}{\partial\chi_p}\right)_x \right\}.$$

Notice that it is uniquely characterised by the fact that each of the slices

$$\mathbb{R}^p \times \{y\} \quad (\text{with } y \in \mathbb{R}^{m-p})$$

are integral submanifolds. On the other hand, $\mathscr{F}_{\text{can}}$ is also involutive.

Example 1.54. An upgrade of the previous example is obtained by starting with arbitrary submersions $\pi : M \to N$ and considering

$$\mathscr{F}_\pi := \text{Ker}\, d\pi = \{v \in TM : d\pi(v) = 0\}.$$

The regular value theorem implies that this is a $p = n - m$ dimensional distribution for which the (connected components of the) fibers of π are integral sub-manifolds. Therefore, $\mathscr{F}_p i$ defines a foliation on M. Foliations of this kind are labelled as **simple foliations**.

Example 1.55. Notice the analogy with vector fields and their integral curves. Actually, the analogy is not only about the terminology, as any vector fields $X \in \mathfrak{X}(M)$ that is nowhere vanishing vector field spans a 1-dimensional distribution:

$$\langle X \rangle \subset TM$$

and the integral curves of X become integral submanifolds of $\langle X \rangle$. We see that, already in this case, there are interesting integral submanifolds that are not necessarily embedded.

The analogy with vector bundles leads us also to the notion of **maximal integral submanifold** of a distribution $\mathscr{F}$, where maximality is with respect to inclusions that are injective immersions: $N \hookrightarrow M$ is smaller than $\widetilde{N} \hookrightarrow M$ if $N \subset \widetilde{N}$ and the inclusion $N \hookrightarrow \widetilde{N}$ is an immersion.

Next let us also point out that integrability can also be characterised in terms of charts, similarly to what appears in Definition 1.48.

Lemma 1.56. *Given a p-dimensional distribution $\mathscr{F}$ on a manifold M, the following are equivalent:*

1. *$\mathscr{F}$ is integrable.*
2. *Around any $x_0 \in M$ one can find a coordinate chart (U, χ) which identifies $\mathscr{F}|_U$ with $\mathscr{F}_{can}$, i.e.,*

$$\mathscr{F}_x = Span\left\{ \left(\frac{\partial}{\partial \chi_1}\right)_x, \dots, \left(\frac{\partial}{\partial \chi_p}\right)_x \right\} \quad \forall\, x \in U.$$

3. *Around any $x_0 \in M$ one can find a coordinate chart (U, χ) such that all the slices*

$$N_y = \chi^{-1}\left(\mathbb{R}^p \times \{y\}\right) \quad (\text{with } y \in \mathbb{R}^{n-p})$$

are integral submanifolds for $\mathscr{F}$.

The charts appearing in 2. (or, equivalently, in 3.) are called **foliated charts** for $\mathscr{F}$.

Proof. The implications $2 \Longrightarrow 3 \Longrightarrow 1$ are immediate. The proof of the implication $1 \Longrightarrow 2$ is contained in the proof of the next theorem, although it is fair to say that one can find a more direct and simple argument (exercise for the reader!). 😄

Definition 1.57. Given a distribution $\mathscr{F}$ on M, we say that $\mathscr{F}$ is **involutive** if:

$$[X, Y] \in \Gamma(\mathscr{F}) \quad \forall\ X, Y \in \Gamma(\mathscr{F}). \tag{1.3.1}$$

Example 1.58. In Example 1.54, since the sections of $\mathscr{F}_\pi$ are precisely the vector fields on M that are π-projectable to 0, and since projectability is compatible with taking Lie bracket of vector fields, it follows that $\mathscr{F}_\pi$ is involutive.

Example 1.59. Moving on to Example 1.55, a simple application of the Leibniz identity of vector fields shows that $\langle X \rangle TM$ is always integrable as well. Since line bundles are locally spanned by nowhere vanishing sections, we deduce that all 1-dimensional distributions are automatically involutive.

The discussion can be upgraded to arbitrary dimensions. Of course, the outcome will not be that involutivity if automatic. Instead, we find that involutivity is controlled by a skew-symmetric tensor on $\mathscr{F}$ (and such tensor are automatically zero on 1-dimensional spaces). We now make this more precise. Therefore, we start with a general p-dimensional distribution $\mathscr{F} \subset TM$. We introduce the quotient bundle

$$\nu := TM/\mathscr{F},$$

which is a vector bundle of rank $q = m - p$, sometimes called the normal bundle of $\mathscr{F}$. Any tangent vector on M induces an element of ν and, passing to sections, any vector field X on M induces a section of ν, denoted

$$X \bmod \mathscr{F} \in \Gamma(\nu).$$

We now introduce the curvature of $\mathscr{F}$ as the map

$$\mathrm{Curv}_{\mathscr{F}} : \Gamma(\mathscr{F}) \times \Gamma(\mathscr{F}) \to \Gamma(\nu), \quad (X,Y) \mapsto [X,Y] \bmod \mathscr{F}; \tag{1.3.2}$$

this allows us to reformulate the involutivity of $\mathscr{F}$ by an equation: the curvature of $\mathscr{F}$ vanishes. The final outcome promised above is:

Exercise 1.60. Show that Curv is $C^\infty(M)$-bilinear an skew-symmetric, and deduce that it defines a section

$$\mathrm{Curv}_{\mathscr{F}} \in \Gamma(\Lambda^2 \mathscr{F}^* \otimes \nu). \tag{1.3.3}$$

Finally, we come to the algebraic/linear characterization of integrability:

Theorem 1.61 (Frobenius). *A distribution $\mathscr{F}$ on a manifold M is integrable if and only if $\mathscr{F}$ is involutive.*

Proof. Integrability implies involutivity because the property of vector fields being tangent to a submanifold $N \subset M$ is closed under taking Lie brackets. We now prove the converse, which is by far the most difficult and interesting part of the entire discussion. Hence assume involutivity, fix x_0 and we are looking for a chart around x_0 satisfying the condition from (2). We starts with any coordinate chart (U,χ) around x_0. After eventually renumbering the coordinates, we may assume:

$$T_{x_0}M = \mathscr{F}_{x_0} \oplus \mathrm{Span}_{\mathbb{R}} \left\{ \left(\frac{\partial}{\partial \chi_{p+1}}\right)_{x_0}, \ldots, \left(\frac{\partial}{\partial \chi_n}\right)_{x_0} \right\}.$$

This implies that there exists an entire neighborhood $W \subset U$ of x_0 such that

$$T_xM = \mathscr{F}_x \oplus \mathrm{Span}_{\mathbb{R}} \left\{ \left(\frac{\partial}{\partial \chi_{p+1}}\right)_{x}, \ldots, \left(\frac{\partial}{\partial \chi_n}\right)_{x} \right\}$$

for all $x \in W$ (exercise!). From now on we may assume that $M = W \subset \mathbb{R}^m$. We consider the projection on the first p coordinates,

$$\pi : W \to \mathbb{R}^p.$$

It follows that its differential restricted to $\mathscr{F}$ induces isomorphisms

$$(d\pi)_x : \mathscr{F}_x \xrightarrow{\sim} T_{\pi(x)}\mathbb{R}^p \quad \forall\, x \in W.$$

Hence we find

$$V_x^1, \ldots, V_x^p \in \mathscr{F}_x \quad (\text{for } x \in W)$$

which are projectable (via $(d\pi)$) to

$$\left(\frac{\partial}{\partial x_1}\right)_{\pi(x)}, \ldots, \left(\frac{\partial}{\partial x_p}\right)_{\pi(x)} \in T_{\pi(x)}\mathbb{R}^p.$$

Moreover, each V^i is smooth w.r.t. $x \in W$ (show this using the smoothness of $\mathscr{F}$!). Hence we obtain vector fields $V^1, \ldots, V^p$ defined on W, spanning $\mathscr{F}$ and π-projectable to $\partial/\partial x_1$, …, $\partial/\partial x_p$. In general, the property of being π-projectable is compatible with the Lie bracket (show that!) hence the Lie brackets $[V^i, V^j]$ are π-projectable to

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

Since they also belong to $\mathscr{F}$ (the involutivity condition!), from the choice of W, it follows that $[V^i, V^j] = 0$ for all i and j. Set also $V^k = \frac{\partial}{\partial x_k}$ for $k \geq p+1$. Define now

$$h : \mathbb{R}^m \to W, \quad h(t_1,\dots,t_m) := \phi_{V^1}^{t_1}\left(\dots\left(\phi_{V^m}^{t_m}(x_0)\right)\dots\right),$$

where, strictly speaking, the domain should be some open ball $B(0,\varepsilon)$ with $\varepsilon > 0$ small enough so that the expression on the right hand side is defined. Notice that the differential of h at an arbitrary $t^0 = (t_1^0,\dots,t_m^0)$ sends

$$\left(\frac{\partial}{\partial t_j}\right)_{t^0} \mapsto \left.\frac{d}{dt_j}\right|_{t_j=t_j^0} \phi_{V^1}^{t_1^0}\left(\dots\phi_{V^m}^{t_{i-1}^0}\left(\phi_{V^m}^{t_i}\left(\phi_{V^m}^{t_{i+1}^0}\left(\dots\phi_{V^m}^{t_m^0}(x_0)\right)\right)\right)\right).$$

This involves taking the differential of the composition of the first $i-1$ flows and apply it to V^j computed at the point $\phi_{V^m}^{t_{i+1}^0}\dots\phi_{V^m}^{t_m^0}(x_0)$. This is particularly nice when:

- $t^0 = 0$: then $h(0) = x_0$ and we find that $(dh)_0 : T_0\mathbb{R}^m \to T_{x_0}W$ sends $\left(\frac{\partial}{\partial t_j}\right)_0$ to $V_{x_0}^j$ for all $1 \le j \le m$. Since $(dh)_0$ takes a basis to a basis it is a linear isomorphism and, therefore, by the inverse function theorem, after restricting h to an even smaller ball, we may assume h is a diffeomorphism.
- at arbitrary t^0 but computed for $j = 1$: then we get that dh maps $\left(\frac{\partial}{\partial t_1}\right)_{t^0} \mapsto V^1_{h(t^0)}$.
- since $V^1,\dots,V^p$ commute, so do their flows, hence the previous item actually implies that

$$\left(\frac{\partial}{\partial t_j}\right)_{t^0} \longmapsto V^j_{h(t^0)} \quad \text{for } 1 \le j \le p.$$

Defining $\widetilde{\chi}$ as the inverse of the restriction of h to a small enough ball as in the first item, the last item implies that $\frac{\partial}{\partial \widetilde{\chi}_j} = V^j$ for $1 \le j \le p$, and this is precisely the type of chart we were looking for. 😀

Remark 1.62 (more about foliations). In the setting of the theorem, and in complete analogy with flows of vector fields, through any point of M passes a unique connected integral submanifold that is maximal. Such submanifolds are called **leaves** of the foliation, and they form a partition of M. The leaves can actually be described, set theoretically, more directly, via the following equivalence relation on M: $x \sim y$ if there exists a smooth path from x to y and tangent to $\mathscr{F}$:

$$\gamma : [0,1] \to M \quad \text{with } \gamma(0) = x,\ \gamma(1) = y, \text{ such that } \dot\gamma(t) \in \mathscr{F}_{\gamma(t)}\ \forall\, t.$$

The resulting orbits (equivalence classes) are precisely the leaves of the foliation. We leave it as an exercise to describe the smooth structure on them, using foliated charts. Notice however that this has to be done with care since, as we have seen already in the case of integral curves of vector fields, the leaves may be just embedded submanifolds. On the other hand, it may be interesting to know that, after all, all the leaves will be initial submanifolds (cf. Definition 2.169 of the bachelor notes) - therefore, by Corollary 2.171 in the same notes , they admit a unique smooth structure that makes them into immersed submanifolds.

Exercise 1.63. Let $\lambda \in \mathbb{R}$. Described the leaves of the foliation on the torus $S^1 \times S^1$ induced by $\left\{\frac{\partial}{\partial\theta_1} + \lambda\frac{\partial}{\partial\theta_2}\right\}$.

Remark 1.64. The question of existence of foliations of a given codimension (where the codimension is the dimension of the ambient manifold minus the dimension of the foliation), on a given compact manifold, is a very interesting (and hard) question. Actually, the very birth of "Foliation Theory" is identified with the PhD thesis of George Reeb (1943), where he answered the question of existence of codimension one foliations on S^3. The answer was positive, and the example he found is well-known under the name of "the Reeb foliation". It arises by first realizing S^3 as obtained from two copies of the solid torus $S^1 \times D^2$ (think on the picture) by gluing them along their boundary torus $S^1 \times S^1$ (indicated on picture 1.1; see also Exercise 2.211 in the bachelor notes. In formulas, one thinks of the 3-sphere as

$$S^3 = \{(u,v) : u,v \in \mathbb{C} : |u|^2 + |v|^2 = 1\},$$

one consider the copy of the 2-torus inside S^3 given by

$$A = \{(u,v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\}$$

and the two connected components X_1 and X_2 of $S^3 \setminus A$ (described similarly to A above, but with "$\le$" and "$\ge$" instead of "="). It is not difficult to see that X_1 and X_2 are indeed solid tori. The idea is now to foliate (partition) each solid torus with a codimension one foliation which has, as one of the leaves, the boundary torus. This is done by wrapping around"planes" inside the solid torus, as indicated in picture 1.2.

After gluing, one gets a foliation on S^3. It is interesting to note that, although it is pretty clear that the result is smooth, the result can never be done analytically (the geometric phenomena of wrapping around is quite non-analytic). Actually, an old theorem of Haefliger says that S^3 does not admit any analytic codimension foliation. The theorem actually proves that compact manifolds that admit analytic codimension one foliation have infinite fundamental group.

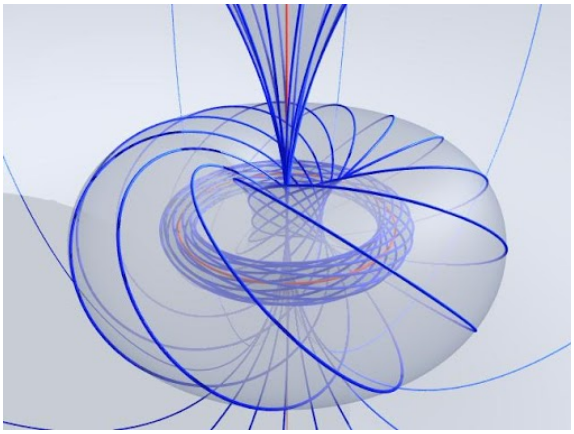


Fig. 1.1 The 3-sphere obtained by gluing two solid tori

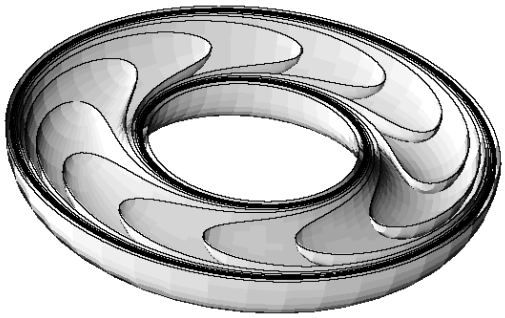


Fig. 1.2 The foliation of the solid torus

Back to the existence of codimension one foliations on manifolds, it is not so difficult to see that the even dimensional spheres cannot carry such; actually, the existence of such a foliation on a compact simply connected manifold would imply the vanishing of the Euler characteristic. So one is left with the odd dimensional spheres. The case of S^5 was answered by Lawson in 1971 and then, after the work of several people, it was proved that all odd-dimensional spheres do admit codimension one foliations. This story (about codimension one foliations on compact manifold) came to an end around 1976 with the work of Thurston who showed that, on a compact orientable manifold M such a foliation exists if and only if its Euler characteristic vanishes. The story continues with the existence of foliations of other interesting dimensions/codimensions (e.g. there is a 2-dimensional one on S^7) etc etc ... and with Foliation Theory.

1.3.3 Complex structures on manifolds

While tangent bundles TM form one important class of examples of real vector bundles, they also provide examples of complex vector bundles provided the manifold M is a complex manifold. The concept of complex manifold is defined in *complete analogy* with that of (real) manifold, by replacing the real Euclidean spaces with the complex ones and smooth maps by holomorphic maps. Hence one talks about:

- *complex m-dimensional (topological) charts* on a given set (topological space) M: homeomorphisms $\chi : U \to \Omega$ between $U \subset M$ (open) and an open $\Omega \subset \mathbb{C}^m$.
- *holomorphic compatibility* between two such charts χ_1 and χ_2: we ask that the change of coordinates $\chi_2 \circ \chi_1^{-1}$ is not only a smooth diffeomorphism, but also a holomorphic one.
- *holomorphic m-dimensional atlases*: atlases consisting of complex m-dimensional charts, each two required to be holomorphically compatible;
- *holomorphic m-dimensional structures* on M: maximal holomorphic atlases $\mathscr{C}$.
- *complex m-dimensional manifolds*: topological spaces together with aholomorphic m-dimensional structure $\mathscr{C}$ on M, and where the topological space is required to be Hausdorff and second countable.

Since opens in $\mathbb{C}^n$ can be seen as opens in $\mathbb{R}^{2n}$, and holomorphic maps between opens in $\mathbb{C}^n$ are, in particular, smooth, we deduce that: any complex m-dimensional manifold M can also be seen as a real $2m$-dimensional one. Wondering about the converse gives rise to several interesting questions:

- When is a (real) $2m$-dimensional manifold arising from a complex m-dimensional one?
- Given a (real) $2m$-dimensional manifold M, what more is needed on M to make it into an m-dimensional complex manifold?

One can be a bit more precise (at the price of becoming a bit pedantic). What we are saying is, first of all, that any holomorphic m-dimensional structure $\mathscr{C}$ on M gives rise to a smooth $2m$-dimensional one; it will be denoted $\mathrm{Re}(\mathscr{C})$ and called the smooth structure underlying the holomorphic one. Secondly, the questions we are asking are:

- Given a (real) $2m$-dimensional smooth structure $\mathscr{A}$ on M, when can one find a holomorphic m-dimensional structure $\mathscr{C}$ on M such that $\mathscr{A} = \mathrm{Re}(\mathscr{C})$?
- Given a (real) $2m$-dimensional smooth structure $\mathscr{A}$ on M, what is the extra-data on the real manifold $(M, \mathscr{A})$ that encodes a holomorphic m-dimensional structure $\mathscr{C}$ with $\mathrm{Re}(\mathscr{C}) = \mathscr{A}$.

Both questions are non-trivial and they are, of course, related. The general discussion of complex vector bundles and complex structures on real vector bundles provides a partial answer to the second question and gives rise to the following concept:

Definition 1.65. An **almost-complex structure** on a (real) manifold M is a complex structure on its tangent bundle TM- hence any vector bundle morphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{Id}$.

Continuing the discussion above, the key remark is that *any holomorphic structure on a given (real) manifold M gives rise to an almost complex structure on M*. More precisely, given a smooth manifold $(M, \mathscr{A})$, any holomorphic structure $\mathscr{C}$ on M with underlying smooth structure $\mathscr{A}$ gives rise to an almost complex structure on the tangent bundle (... of the smooth manifold $(M, \mathscr{A})$!),

$$J_{\mathscr{C}} : TM \to TM.$$

One way to describe it is to use the charts $\zeta \in \mathscr{C}$, $\zeta : U \to \Omega$, and their differentials. Since Ω is open in $\mathbb{C}^m$, $T_x\Omega$ is now canonically identified with $\mathbb{C}^m$ and, therefore, we obtain isomorphisms:

$$(d\zeta)_p : T_pM \to \mathbb{C}^m \quad (\text{for all } p \in U).$$

The complex structure on $\mathbb{C}^m$,

$$J_{\text{can}} : \mathbb{C}^m \to \mathbb{C}^m, \quad v \mapsto i \cdot v,$$

can now be transported to T_pM via these isomorphism, i.e., define

$$J_{\mathscr{C}}(v_p) := (d\zeta)_p^{-1}\left(J_{\text{can}}\left((d\zeta)_p(v_p)\right)\right) \quad (\text{for all } p \in U, v_p \in T_pM).$$

The fact that each two charts from $\mathscr{C}$ are holomorphically compatible implies that the definition above does not depend on the choice of the charts. For those that prefer an even more explicit use of local coordinates, what we are saying is that for a holomorphic chart (U,ζ) that provides the coordinates ζ_j,

$$\zeta = (\zeta_1, \ldots, \zeta_m) : U \to \Omega \subset \mathbb{C}^m,$$

writing $\zeta_j = x_j + i \cdot y_j$, one obtains coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ for the underlying smooth manifold and, w.r.t. these coordinates,

$$J_{\mathscr{C}}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J_{\mathscr{C}}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j} \tag{1.3.4}$$

The first sign that we are looking in the right direction is the following:

Lemma 1.66. *Let M be a smooth manifold and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two holomorphic structures whose underlying smooth structure is the original one. If $J_{\mathscr{C}_1} = J_{\mathscr{C}_2}$, then $\mathscr{C}_1 = \mathscr{C}_2$.*

Proof. We have to show that the change of coordinates from any chart in $\mathscr{C}_1$ to any chart in $\mathscr{C}_2$ is holomorphic. But, given the hypothesis, we see that any such change of coordinates c has the property that $(dc) \circ J_{\text{can}} = J_{\text{can}} \circ (dc)$. Writing out the components, we see that the change of coordinates is of type

$$c = (f,g) = (f_1, \ldots, f_k, g_1, \ldots, g_k) : V \to W \quad (V, W \subset \mathbb{R}^{2k} \text{ opens})$$

where the components satisfy

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial g_k}{\partial y_j}, \frac{\partial f_k}{\partial y_j} = -\frac{\partial g_k}{\partial x_j}.$$

However, these are precisely the Cauchy-Riemann equations that characterise the holomorphicity of c and, therefore, c is holomorphic. 😄

Of course, the next question is: how do we recognize which almost complex structures J are of type $J_{\mathscr{C}}$.

Definition 1.67. Given a smooth manifold M, an almost complex structure $J : TM \to TM$ is called **integrable** if it is induced by a holomorphic structure $\mathscr{C}$.

Lemma 1.68. *An almost complex structure J is integrable if and only if around any $x_0 \in M$ one can find a (real, smooth) coordinate chart, written $(U, \chi = (x_1, \ldots, x_m, y_1, \ldots, y_m))$ such that, over U,*

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Proof. The direct implication is clear from the description of $J_{\mathscr{C}}$ provided in (1.3.4). For the converse, we denote by $\mathscr{C}$ the collection of all (smooth) charts that satisfy the equations in the lemma. The remark is that, by the same reasons as in the previous proof, the change of coordinates between all charts in $\mathscr{C}$ are, by Cauchy-Riemann, holomorphic- and that gives rise to the desired holomorphic structure, 😄

The reader should compare with the notion of integrability of distributions/foliations. It is remarkable that, as in that case, the question of when an almost complex structure is integrable admits a rather simple characterization

that is linear in nature, similar to the "involutivity" condition for foliations. Actually, since Proposition 1.42 allows us to interpret almost complex structures as (certain) sub-bundles of

$$T_{\mathbb{C}}M := TM \otimes \mathbb{C},$$

a similar notion of involutivity makes sense. The following provides two facets of the resulting condition. In the statement, for a given almost complex structure $J : TM \to TM$, we will make use of the i-eigenspace of the $\mathbb{C}$-linear extension $J_{\mathbb{C}}$ of J to $T_{\mathbb{C}}M$,

$$T_J^{1,0}M = \{V \in \Gamma(T_{\mathbb{C}}M) : J(V) = i \cdot V\} \subset T_{\mathbb{C}}M.$$

We also consider the $\mathbb{C}$-blinear extension of the Lie bracket from $\Gamma(TM) = \mathfrak{X}(M)$ to $\Gamma(T_{\mathbb{C}}M) = \mathfrak{X}(M) \otimes \mathbb{C}$.

Theorem 1.69. *For any almost complex structure $J : TM \to TM$ the following are equivalent*

(1) $T_J^{1,0}M$ is involutive in the sense that

$$[X,Y] \in \Gamma(T_J^{1,0}M) \qquad \forall\ X,Y \in \Gamma(T_J^{1,0}M).$$

(2) J satisfies the following equation

$$[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0 \qquad \forall\ X,Y \in \mathfrak{X}(M). \tag{1.3.5}$$

Furthermore, for any almost complex structure J, the left hand side of (1.3.5), viewed as an operation

$$\mathscr{N}_J : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M),$$

is $C^\infty(M)$-bilinear and skew-symmetric in X and Y and, therefore, defines a tensor

$$\mathscr{N}_J \in \Gamma(\Lambda^2 T^*M \otimes TM). \tag{1.3.6}$$

The tensor $\mathscr{N}_J$ is called the **Nijenhuis tensor** associated to the almost complex structure J.

Proof. With the explicit description of $T_J^{1,0}M$ at hand,

$$T_J^{1,0}M = \{X - i \cdot J(X) : X \in TM\}$$

(cf. (1.2.14)), the equivalence between (1) and (2) a straightforward computation: we look at the bracket of vector two fields of this type,

$$[X - i \cdot J(X), Y - i \cdot J(Y)] = ([X,Y] - [J(X),J(Y)]) - i \cdot ([J(X),Y] + [X,J(Y)])$$

and the condition for this to be again type $(1,0)$ is

$$([J(X),Y] + [X,J(Y)]) = J([X,Y] - [J(X),J(Y)]).$$

Applying J and using $J^2 = -\mathrm{Id}$, we obtain (1.3.5).

For the last part the main thing to check is that, for all $f \in C^\infty(M)$, $X,Y \in \mathfrak{X}(M)$,

$$\mathscr{N}_J(X, fY) = f\mathscr{N}_J(X,Y).$$

This follows by applying repeatedly the Leibniz identity for the Lie bracket of vector fields, that is,

$$[X, fY] = f[X,Y] + L_X(f)Y,$$

and also making use of $J^2 = -\mathrm{Id}$. 😀

Theorem 1.70 (Newlander-Nirenberg). *An almost complex structure J is integrable if and only if $\mathcal{N}_J = 0$.*

Proof. too difficult to give here. 😄

For the curious reader

Existence *Also in the case of complex structures, the question of existence of a complex structure on a manifold M is a rather hard one. For instance, it seems to be an open problem the question of whether S^6 admits a complex structure. On the other hand, from all the spheres, only S^2 and S^6 admit an almost complex structure; for S^2 it comes from an honest complex structure (by viewing S^2 as a complex manifold, e.g. as $\mathbb{CP}^1$); for S^6 it follows by using, again, the octonions. Actually, there is a simple trick that allows us to relate the existence of almost complex structures to parallelizability and then, using the fact that S^1, S^3 and S^7 are the only parallelizable sphere, to conclude that S^2 and S^6 are the only spheres which admit an almost complex structure. Here is the trick:*

Exercise 1.71. *Assume that $S^n \subset \mathbb{R}^{n+1}$ admits an almost complex structure J. Interpret it as a family of linear endomorphisms*

$$J_y : P_y \to P_y \quad \text{for } y \in S^n$$

of the hyperplane $P_y = \{v \in \mathbb{R}^{n+1} : \langle v, y\rangle = 0\} \subset \mathbb{R}^{n+1}$ orthogonal to y.

Pass now to $\mathbb{R}^{n+2}$; denote by $e_0, \ldots, e_{n+1}$ its standard basis and interpret $\mathbb{R}^{n+1}$ (and all its subspaces, e.g. the P_ys) as the subspace of $\mathbb{R}^{n+2}$ via the standard inclusion

$$(y_0, \ldots, y_n) \mapsto (y_0, \ldots, y_n, 0).$$

For $x = (x_0, \ldots, x_{n+1}) \in S^{n+1} \subset \mathbb{R}^{n+2}$ different from $\pm e_{n+1}$, we denote by $p(x)$ its orthogonal projection on $\mathbb{R}^{n+1}$,

$$p(x) = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$$

and the induced element on the sphere

$$s(x) = \frac{1}{||p(x)||} p(x) \in S^n.$$

Consider the orthogonal projection onto $P_{s(x)}$,

$$pr^{\perp}_{s(x)} : \mathbb{R}^{n+1} \to P_{s(x)},\ pr^{\perp}_{s(x)}(\lambda) = \lambda - \langle \lambda, s(x)\rangle s(x)$$

and define:

$$F_i(x) := x_{n+1} e_i - x_i e_{n+1} + ||p(x)|| J_{s(x)}(pr^{\perp}_{s(x)}(e_i))$$

for all $x \in S^{n+1}$, where the very last term is set to be zero when it does not make sense (i.e. for $x = \pm e_{n+1}$). Show that $\{F^0, \ldots, F^n\}$ is a frame of S^{n+1}. Hence S^{n+1} is parallelizable.

Interesting enough, the product of any two odd dimensional spheres admits a complex structure.

1.3.4 Symplectic structures on (real) vector bundles

We now briefly discuss another interesting geometric structure that makes sense on (real) vector bundles: the skew-symmetric analogue of inner product, called symplectic structures. Their relationship with complex vector bundles will be explained in the next section.

A **symplectic vector space** is a pair (V, ω) consisting of a vector space V and a skew-symmetric bilinear map

$$\omega : V \times V \to \mathbb{R}$$

which is non-degenerate. Such a form is called a **symplectic form on the vector space** V. The existence of such a form forces V to be even dimensional (exercise, or keep on reading); this is also a first indication of the relationship with complex structures. The "standard example" is ω_{can}) on $\mathbb{R}^{2k}$ defined by

$$\omega_{\text{can}}((x,y),(x',y')) = \langle x', y\rangle - \langle x, y'\rangle, \tag{1.3.7}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^k$ and we use $\mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k$ to represent the elements of $\mathbb{R}^{2k}$ as pairs

$$(x,y) = (x_1, \ldots, x_k, y_1, \ldots, y_k).$$

More compactly, writing the standard frame of $\mathbb{R}^{2k}$ as $(e_1, \ldots, e_k, f_1, \ldots, f_k)$ and using the associated dual frame (written with upper indices) and the wedge-product operations, $\omega_{\text{can}} = f^1 \wedge e^1 + \ldots + f^k \wedge e^k$. Even more compactly, using the standard inner g_{can} product and the standard complex structure J_{can} on $\mathbb{R}^{2k}$,

$$\omega_{\text{can}}(u,v) = g_{\text{can}}(u, J_{\text{can}}(v)), \quad \forall\, u,v \in \mathbb{R}^{2k}.$$

The analogue of isometries from the setting of inner products, for symplectic vector spaces (V_i, ω_i) one talks about **symplectic maps** $A : (V_1, \omega_1) \to (V_2, \omega_2)$, by which we mean linear maps $A : V_1 \to V_2$ satisfying

$$\omega_2(A(u), A(v)) = \omega_1(u,v) \quad \forall\, u,v \in V_1.$$

The analogue of orthonormal frames w.r.t. inner products, for a symplectic vector space (V,ω), are the **symplectic frames** w.r.t. ω; interpreting frames as isomorphisms with the Euclidean space, these are simply the frames that correspond to isomorphisms between (V,ω) and $(\mathbb{R}^{2k}, \omega_{\text{can}})$. Explicitly, they are the frames of V of type

$$e = (\phi_1, \dots, \phi_k, \psi_1, \dots, \psi_k)$$

with the property that $\omega(\phi_i, \psi_i) = -\omega(\psi_i, \phi_i) = -1$ and all the other combinations are 0.

As for inner products, these notions are upgraded to vector bundles. One obtains, first of all, the notion of **symplectic vector bundle** (E,ω), or **symplectic form ω on a vector bundle** E, meaning a family $\omega = \{\omega_x\}$ of symplectic forms on the vector spaces E_x varying smoothly w.r.t. x in the sense that $\omega(s_1, s_2)$ is a smooth function whenever $s_1, s_2 \in \Gamma(E)$. The resulting map

$$\omega : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$$

and its property of being skew-symmetric $C^\infty(M)$-bilinear map encodes ω completely, and we leave it to the reader to figure out how to rephrase the non-degeneracy condition just in terms of sections. Similarly, it should be clear now what we mean by symplectic frames of a symplectic vector bundle (E,ω). What requires some thinking, but does hold true, is the analogue of (2) of Lemma 1.11: around any point in the base there exists a local symplectic frame.

Definition 1.72. An **almost-symplectic structure** on a (real) manifold M is a symplectic structure on its tangent bundle TM or, equivalently, any differential form $\omega \in \Omega^2(M)$ which is non-degenerate at each point. We also say that ω is an almost-symplectic form. It is called a **symplectic** if it is also closed.

The standard example of symplectic vector space can now be upgraded to **the standard symplectic structure** on the manifold $M = \mathbb{R}^{2k}$,

$$\omega_{\text{can}} := \sum_j dy_j \wedge dx_j \tag{1.3.8}$$

where, as above, we use the coordinates $(x_1, \dots, x_k, y_1, \dots, y_k)$ to encode the points in $\mathbb{R}^{2k}$.

Definition 1.73. Given an almost-symplectic form ω on M, a **Darboux chart** w.r.t. ω is any chart of M,

$$\chi = (\mathscr{x}_1, \dots, \mathscr{x}_m, \mathscr{y}_1, \dots, \mathscr{y}_m) : U \to \Omega \subset \mathbb{R}^{2k},$$

which provides an isomorphism between $(U, \omega|_U)$ and $(\Omega, \omega_{\text{can}}|_\Omega)$ i.e., with the property that

$$\omega|_U = \chi^* \omega_{\text{can}} = \sum_j d\mathscr{y}_j \wedge d\mathscr{x}_j,$$

We say that ω is **integrable** if around any point of M one can find a Darboux chart.

Exercise 1.74. Show that the property of being a Darboux chart is equivalent to requiring that

$$\left(\frac{\partial}{\partial \mathscr{X}_1}\right)_x, \ldots, \left(\frac{\partial}{\partial \mathscr{X}_k}\right)_x, \left(\frac{\partial}{\partial \mathscr{Y}_1}\right)_x, \ldots, \left(\frac{\partial}{\partial \mathscr{Y}_k}\right)_x$$

is a symplectic frame of (T_xM, ω_x) for all $x \in U$.

Theorem 1.75 (Darboux). *An almost symplectic form $\omega \in \Omega^2(M)$ is integrable if $d\omega = 0$ i.e., if and only if it is a symplectic form.*

Proof. The direct implication is clear since the standard symplectic form is closed. Assume now that ω is symplectic. We may assume we are on $\mathbb{R}^{2n}$ with coordinates denoted (x,y) and we work around the origin. We may also assume that $\omega = \omega_{\text{can}}$ at the origin (why?). The idea is to realize the desired diffeomorphism between open neighborhoods of the origin taking ω to ω_{can} as the flow at time $t = 1$ of a vector field. Just using vector fields does not work, but it will if we allow time-dependent vector fields. Let X_t be the time-dependent vector field we are looking for and let φ be its flow. The next idea is to connect ω to ω_{can} by a smooth family of forms, $\{\omega_t : t \in [0,1]\}$, satisfying

$$\omega_0 = \omega_{\text{can}}, \ \omega_1 = \omega$$

and make sure that $\varphi_t^* \omega_t$ stays constant or, equivalently, $\frac{d}{dt}\varphi_t^* \omega_t = 0$. Using the properties of the flow (see Exercise time-dep below) we can write this condition as follows:

$$L_{X_t}(\omega_t) + \frac{d}{dt}\omega_t = 0,$$

or, if the ω_t's are all closed and using *Cartan's magic formula* $L_X = di_X + i_X d$:

$$d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = 0.$$

That is all we have to do. Here is how one does it. First of all, one considers the most obvious family:

$$\omega_t = t\omega + (1-t)\omega_{\text{can}},$$

so that the equation we have to solve (to find X_t) becomes:

$$d(i_{X_t}\omega_t) + (\omega - \omega_{\text{can}}) = 0.$$

Note that, since $\omega - \omega_{\text{can}}$ is closed, by working in a ball around the origin, we may assume it is exact (Poincare lemma), say of type $d\eta$ for some 1-form η. Then it is enough to solve the (simpler) equation

$$i_{X_t}(\omega_t) + \eta = 0.$$

Here we deal with the operation from vector fields to 1-forms which sends a vector V to the 1-form $i_V(\omega_t)$. This we know to be a bijection (in particular surjective) if ω_t was non-degenerate. This is certainly so for $t \in \{0,1\}$, so we just have to make sure that, after eventually shrinking the neighborhood of 0 that we are working on, all the ω_t are symplectic. Here is where the assumption that ω and ω_{can} coincide at 0 comes in. They also coincide with ω_t at 0. Since ω_t is non-degenerate at 0, it will be non-degenerate in a neighborhood W_t of 0. Since we are interested in $t \in [0,1]$ and since ω_t is a continuous family (even smooth) and $[0,1]$ is compact, it follows that one can choose the same W_t for all t; call it W. If you followed the argument, you see we are done. Otherwise, start from here and read the proof backwards. 😄

Let us also point out that the standard symplectic form (1.3.8) can be globalised, provided we work in the appropriate manifolds- which are cotangent bundles T^*M of any manifold M. Similar to the tangent bundle, any coordinate chart $(\mathscr{X}_1, \ldots, \mathscr{X}_m)$ of M induces one on T^*M, denoted here $(\mathscr{X}_1, \ldots, \mathscr{X}_m, \mathscr{Y}_1, \ldots, \mathscr{Y}_m)$ of T^*M: applied to

$(x,\xi) \in T^*M$, $\mathscr{x}_j$ acts only on x, while $\mathscr{y}_j$ evaluates ξ on the tangent vector $(\partial/\partial \mathscr{x}_j)_x$. The advantage of T^*M over TM is that when looking at the expressions $\sum d\mathscr{y}_j \wedge d\mathscr{x}_j$, there will be independent of the choice of coordinates hence they will define a global 2-form

$$\omega_{\text{can}} \in \Omega^2(T^*M)$$

which is symplectic. Since the independence of the coordinates is not really enlightening and looks more like an accident, here is one way to describe the form globally (therefore providing a simple proof for the independence of the charts).

Exercise 1.76. Define the so called Liouville 1-form on T^*M, $\lambda \in \Omega^1(T^*M)$, defined as follows. For a general tangent vector to T^*M at some point $\xi \in T^*_xM$ (for some $x \in M$),

$$\mathscr{V}_\xi \in T_\xi(T^*M).$$

we map it to TM via the differential of the projection map $\pi : T^*M \to M$,

$$d\pi\left(\mathscr{V}_\xi\right) \in T_xM \quad (x = \pi(\xi))$$

and we evaluate $\xi \in T^*_xM$ on it:

$$\lambda\left(\mathscr{V}_\xi\right) := \xi\left(d\pi\left(\mathscr{V}_\xi\right)\right).$$

Check that, in the local coordinates mentioned above, $\lambda = \sum_j \mathscr{y}_j d\mathscr{x}_j$ and, therefore, ω_{can} could be defined directly as $d\lambda$.

For the curious reader

Existence *The question of which (compact) manifolds admit symplectic structure is a very hard one, especially in the case of a negative answer; this is due to the lack of (known) invariants/obstructions for symplectic structures. What one can say right away, for compact manifolds M, is that if they admit a symplectic structure, then they must be even dimensional and orientable. And a bit more: the second DeRham cohomology group $H^2(M)$ is non-zero (so this excludes all the sphere S^n with $n \neq 2$). This last remark follows from the following.*

Exercise 1.77. The **Liouville form** associated to a symplectic manifold (M,ω) is the top differential form

$$\mu_\omega := \frac{1}{k!}\omega^k = \frac{1}{k!}\underbrace{\omega \wedge \ldots \omega}_{k \text{ times}},$$

where k is half the dimension of M. Show that:

1. μ_ω is a volume form. In particular, any symplectic manifold carries an induced orientation. In particular, non-orientable manifolds cannot carry symplectic structured.
2. If M is compact then $a = [\omega] \in H^2(M)$ is non-zero. Can a^2 be zero? What is the best you can say?

Here are some more existence results/examples:

- in dimension 2, a symplectic structure is the same thing as a volume form. Hence orientable surfaces are symplectic.
- from all the spheres, only S^2 and S^6 admit an almost symplectic structure, and only S^2 admits a symplectic structure.
- the connected sum of three copies of $\mathbb{C}^2$ admits an almost symplectic structure but not a symplectic one (nor a complex one (Taubes, 1995)).
- The existence of an almost symplectic structure is equivalent to the existence of an almost complex structure (see Proposition 1.79 below).
- For any finitely presented group Γ one can find a compact 4-manifold M, which is symplectic, with fundamental group isomorphic to Γ (Gompf, 2000).
- On open manifolds (i.e. manifolds which are not compact), there is quite some flexibility- due to the so called h-principle of Gromov ("h" stands for "homotopy"). For symplectic structures it says that: starting with any almost symplectic structure on an open manifold M, one can smoothly deform it (through almost symplectic structures) into a symplectic structure.

1.3.5 Hermitian structures on complex vector bundles

We now have a closer look at the analogue of inner products in the complex context, and explain how they bring together inner products, complex structures and symplectic structures. Recall that, given a complex vector space W, a Hermitian inner product on W is a $\mathbb{R}$-bilinear map

$$h : W \times W \to \mathbb{C}$$

satisfying:

1) conjugates symmetry: $h(w_2,w_1)=\overline{h(w_1,w_2)}$ for all $w_1,w_2\in W$.
2) h is $\mathbb{C}$-linear in the first argument.
3) h is positive definite, i.e., $h(w,w)\geq 0$ for all $w\in W$ with equality only for $w=0$.

Notice that, writing

$$h=\Re(h)+i\cdot \mathrm{Im}(h),$$

then $g:=\Re(h)$ and $\omega:=\mathrm{Im}(h)$ have the following properties:

a) g is an inner product on W interpreted as a real vector space, and it satisfies $g(iu,iv)=g(u,v)$.
b) ω is a symplectic structure on W interpreted as a real vector space, and it satisfies $\omega(iu,iv)=g(u,v)$ and the stronger nondegeneracy condition that $\omega(iu,u)>0$ for all $u\neq 0$.
c) ω can be recovered from g, and the other way around, via $\omega(u,v)=g(iu,v)$. In particular, since $h=g+i\cdot\omega$, either g or ω determines h uniquely.

We deduce that there is a 1-1 correspondence

$$\left\{\begin{array}{c}\text{Hermitian inner products } h\\ \text{on the complex vector space } W\end{array}\right\}\overset{\sim}{\longleftrightarrow}\left\{\begin{array}{c}\text{inner products } g\\ \text{on the real vector space } W\\ \text{as in a) above}\end{array}\right\}\overset{\sim}{\longleftrightarrow}\left\{\begin{array}{c}\text{symplectic structures } \omega\\ \text{on the real vector space } W\\ \text{as in b) above}\end{array}\right\}$$

As for inner products on real vector spaces/real vector bundles, and As in the previous sections the entire discussion can now be upgraded to (complex) vector bundles, including the 1-1 correspondences mentioned above. As for complex structures, a slightly different perspective of obtained by starting with a real vector bundle E and looking for ways to make it into a Hermitian vector bundle.

Definition 1.78. Given a real vector bundle $E\to M$, a **Hermitian structure on the vector bundle** E is any pair (J,h) consisting of a complex structure on the vector bundle E together with a Hermitian inner product h on the corresponding complex vector bundle E_J.

The 1-1 correspondences above provide equivalent descriptions of Hermitian structures, encoded by pairs (J,g), (J,ω) and even (g,ω). The last type of pairs can be used since the formula $\omega(u,v)=g(Ju,v)$ can be used to reconstruct J. One property that is worth mentioning here, not very deep but not trivial either, is the following:

Proposition 1.79. *A real vector bundle E admits a complex structure J if and only if it admits a symplectic structure ω. Actually, any such J or ω arises from some Hermitian structure on E.*

The following provides a possible argument for the easier parts of the proposition.

Exercise 1.80. Assume that J is a complex structure the real vector bundle E and choose any inner product g on E. Then $h=g'+i\omega.$, where g' and ω are given by

$$g':=g(u,v)+g(Ju,Jv),\omega(u,v):=g(u,Jv)-g(Ju,v),$$

is a Hermitian structure on E.

For the curious reader

Existence *It is not true that an almost symplectic structure induces ("naturally") an almost complex one, or the other way. Also, while the existence of an almost complex structure is equivalent to the existence of an almost symplectic structure, anything else can happen:*

- *There are compact manifolds which admit almost complex structures, but do not admit complex or symplectic structures. This was proven for the connected sum of three copies of $\mathbb{CP}^2$ by Taubes in 1995.*
- *there are compact manifolds that admit complex structures but do not admit symplectic ones. One example is $S^3\times S^1$.*
- *There are compact manifolds that admit symplectic structures but not complex ones.*

We leave it to the reader to move now to manifolds, talk about almost Hermitian structures and wonder about the possible notions of integrability. The nicest scenario is:

Definition 1.81. A **Kähler structure** on a manifold M is a Hermitian structure on TM with the property that both the underlying almost complex structure $J : TM \to TM$ as well as the underlying almost symplectic structure $\omega \in \Omega^2(M)$ are integrable, i.e., $\mathcal{N}_J = 0$ and $d\omega = 0$.

Again, the 1-1 correspondences mentioned above, provide several equivalent way to think of Kähler structure.

Chapter 2
Connections on vector bundles (and some Riemannian geometry)

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2.1 Connections

2.1.1 The definition

Throughout this section E is a vector bundle over a manifold M. Unlike the case of smooth functions on manifolds (which are sections of the trivial line bundle!), there is no canonical way of taking derivatives of sections of (an arbitrary) E along vector fields. That is where connections come in.

Definition 2.1. A **connection** on a vector bundle E is a bilinear map ∇

$$\mathscr{X}(M)\times\Gamma(E)\to\Gamma(E),\ \ (X,s)\mapsto\nabla_X(s),$$

satisfying

$$\nabla_{fX}(s)=f\nabla_X(s),\ \nabla_X(fs)=f\nabla_X(s)+L_X(f)s,$$

for all $f\in C^\infty(M)$, $X\in\mathscr{X}(M)$, $s\in\Gamma(E)$.

There is no much distinction on working over $\mathbb{R}$ or $\mathbb{C}$; of course, in the complex case, $\mathbb{C}$-linearity is required whenever it makes sense. E.g., $\nabla_X(s)$ is required to be $\mathbb{C}$-linear in s.

Example 2.2 (trivial bundles). In the case when E is the trivial vector bundle of rank r, $M\times\mathbb{R}^r$ or $M\times\mathbb{C}^r$, one has to so-called canonical (flat) connection on the trivial bundle, denoted ∇^{can}, uniquely characterized by

$$\nabla_X(e_j^{\text{can}})=0,$$

where

$$e^{\text{can}}=\{e_1^{\text{can}},\dots,e_r^{\text{can}}\},$$

is its canonical frame (if E is not already trivialized, this says that any frame gives rise to a canonical connection). Actually, giving a connection on E is the same thing as giving an r by r matrix whose entries are 1-forms on M:

$$\omega:=(\omega_i^j)_{i,j}\in\mathscr{M}_r(\Omega^1(M)) \tag{2.1.1}$$

(where, when working over $\mathbb{C}$, the forms are $\mathbb{C}$-valued). This 1-1 correspondence is uniquely characterized by

$$\boxed{\nabla_X(e_j^{\text{can}})=\sum_{i=1}^r\omega_j^i(X)e_i^{\text{can}}}\ .$$

(work out the details!). The canonical connection corresponds to the zero matrix.

Please be aware of our conventions: for the matrix $\omega=\{\omega_j^i\}_{i,j}$, the upper indices i count the rows, while the lower ones the columns:

$$\omega=\begin{pmatrix}\omega_1^1&\dots&\omega_r^1\\ \dots&\dots&\dots\\ \omega_1^r&\dots&\omega_r^r\end{pmatrix}$$

The other convention (switching the rows and the columns) would correspond to considering the transpose matrix ${}^t(\omega)$; that convention is taken in some text-books and accounts for the sign changes between our formulas involving ω and the ones in those text-books.

Example 2.3 (tautological line bundle). We now consider the tautological line bundle $L\to\mathbb{RP}^n$ (see Example 1.17). We will make use of the embedding $L\subset\mathbb{RP}^n\times\mathbb{R}^{n+1}$. First of all, the embedding allows us to talk about "the components" s_i of arbitrary sections $s\in\Gamma(L)$, $0\le i\le n$. Hence, for a vector field $X\in\mathfrak{X}(\mathbb{R}^{n+1})$, to define $\nabla_X(s)$ it is natural to apply the usual derivatives L_X to the components. The problem is that

$$L_X(s) := (L_X(s_0), \ldots, L_X(s_n))$$

is a section of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ that may fail to be one of L. To get back to L one can use the orthogonal projections. Explicitly, the projection onto a line $l = [x_0 : x_1 : \ldots : x_n]$ is

$$\mathrm{pr}_l : \mathbb{R}^{n+1} \to l, \quad v \mapsto \frac{\langle v, x\rangle}{||x||^2}.$$

In other words, one defines $\nabla_X(s) \in \Gamma(L)$ by:

$$\nabla_X(s)(l) = \frac{x_0 \cdot L_X(s_0) + \ldots + x_n \cdot L_X(s_n)}{x_0^2 + \ldots + x_n^2} \cdot (x_0, \ldots, x_n) \quad \text{for } l = [x_0 : \ldots : x_n] \in \mathbb{RP}^n.$$

Exercise 2.4. Formulate the complex version of the previous example.

Using the same idea as in the last example one obtains an explicit connection on the tangent bundle of any embedded submanifold of an Euclidean space (with the connection depending on the embedding). More generally, you should do the following:

Exercise 2.5. Let E be a vector bundle over M, and assume that it sits inside a trivial vector bundle $M \times \mathbb{R}^k$ of some rank k; hence E_x is a vector sub-space of $\mathbb{R}^k$ for each x. We denote by

$$\mathrm{pr}_x : \mathbb{R}^k \to E_x$$

the orthogonal projection into E_x and we use the canonical connection ∇^{can} on the trivial bundle. Show that

$$\nabla_X(s)(x) := \mathrm{pr}_x(\nabla^{\mathrm{can}}_X(s)(x))$$

defines a connection on E. Similarly for complex vector bundles.

Use this and one of the theorems we discussed to conclude that any vector bundle admits a connection.

Exercise 2.6. Prove that any convex linear combination of two connections is again a connection, i.e., given ∇^1 and ∇^2 connections on E and ρ_1, ρ_2 smooth functions on M (the base manifold) such that $\rho_1 + \rho_2 = 1$, then

$$\nabla = \rho_1 \nabla^1 + \rho_2 \nabla^2$$

is also a connection. What if you have larger families? Use now a partition of unity argument to show that any vector bundle admits a connection.

The next exercise shows how connections can be produced when performing various vector bundle operations.

Exercise 2.7. Let E and E' be two vector bundles over M endowed with connections ∇ and ∇', respectively. We look at E^*, $E \oplus E'$ and $E \otimes E'$. Show that one has:

1. an induced connection ∇^* on E^*, defined by the Leibniz-type equation

$$L_X(\xi(s)) = \nabla^*_X(\xi)(s) + \xi(\nabla_X(s))$$

   for all $s \in \Gamma(E)$, $\xi \in \Gamma(E^*)$ and $X \in \mathscr{X}(M)$.
2. an induced connection $\nabla \oplus \nabla'$ on $E \oplus E'$ given by

$$(\nabla \oplus \nabla')_X(s, s') = (\nabla_X(s), \nabla'_X(s')).$$

3. an induced connection $\nabla \otimes \nabla'$ on $E \otimes E'$ given by

$$(\nabla \otimes \nabla')_X(s \otimes s') = \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s').$$

What is left out in the previous exercise is the operation of pull-back: the fact that for any smooth map $f : M \to N$, while any vector bundle $E \in \mathrm{Vect}(N)$ can be pulled back to a vector bundle $f^*E \in \mathrm{Vect}(M)$, any connection ∇ on E can be pulled back to a connection $f^*\nabla$ on f^*E. To explain this we would need to define $(f^*\nabla)_X(s)$ for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(f^*E)$. In principle, at least if s the pull-back of a section s' of E, we would like to define $(f^*\nabla)_X(s)$ to be the pull-back of $\nabla_{df(X)}(s)$; there are two problems: the last expression doesn't make sense (yet) and not all sections of f^*E are pull-backs of sections. However, we will soon overcome these problems.

2.1.2 The dual view-point

The connections can be re-interpreted by focusing more on 1-forms rather than vector fields. More precisely, the operation $(X,s) \mapsto \nabla_X(s)$ can be re-interpreted as associating to any $s \in \Gamma(E)$ the operator

$$\nabla_\cdot(s) : \mathfrak{X}(M) \to \Gamma(E), \quad X \mapsto \nabla_X(s).$$

Furthermore, the $C^\infty(M)$-linearity in X says that this defines a 1-form on M with coefficients in E; we will denote it by $d_\nabla(s)$. All together, ∇ is re-arranged into a linear operator

$$\boxed{d_\nabla : \Gamma(E) \to \Omega^1(M,E), \;\; d_\nabla(s)(X) := \nabla_X(s)}$$

that satisfies the Leibniz identity

$$\boxed{d_\nabla(fs) = f d_\nabla(s) + df \cdot s} \tag{2.1.2}$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$ (for the last term we are using the notations 1.2.10). Since ∇ can clearly be recovered from d_∇, it is not difficult to conclude that:

Proposition 2.8. *The construction $\nabla \longleftrightarrow d_\nabla$ gives a 1-1 correspondence between*

$$\left\{ \begin{array}{c} \textit{connections } \nabla \\ \textit{on } E \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{linear operators } \Gamma(E) \to \Omega^1(M,E) \\ \textit{satisfying the Leibniz (2.1.2)} \end{array} \right\}.$$

Example 2.9 (trivial bundles). The dual of Example 2.2 reads as follows. First of all, for the canonical connection on the trivial vector bundle, it is the unique one with the property that

$$d_{\nabla^{\mathrm{can}}}\left(e_j^{\mathrm{can}}\right) = 0,$$

(interpreted as saying that all the canonical sections e_j^{can} becomes flat). Secondly, for an arbitrary connection on the trivial bundle, the corresponding matrix (2.1.1) characterizing it corresponds to writing

$$\boxed{d_\nabla(e_j^{\mathrm{can}}) = \sum_i \omega_j^i \cdot e_i^{\mathrm{can}}}$$

where the "$\cdot$" is the wedge product from subsection 1.2.6, in the simple case of wedging 1-forms on M (ω_j^i) and 0-forms on M with coefficients in E (e_i).

The dual viewpoint on connections will be further discussed in subsection 2.1.6. Here is an interesting consequence:

Corollary 2.10. *Given a connection ∇ on E, when looking at a point $x \in M$, the expressions*

$$\nabla_X(s)(x) \in E_x$$

only depend on $X_x \in T_xM$ and $s \in \Gamma(E)$. In particular, one can talk about

$$\nabla_{X_x}(s) \in E_x, \qquad \text{whenever } X_x \in T_xM, \quad s \in \Gamma(E).$$

This follows from the fact that $C^\infty(M)$-linear maps $\mathfrak{X}(M) \to \Gamma(E)$ can be seen as 1-forms with values in E hence, at each $x \in M$, they give linear maps $T_xM \to E$. Notice that, with this at hand, we can now overcome one of the problems mentioned at the end of the previous section when trying to define the pull-back of connections. Indeed, given $f : M \to N$ smooth, $E \in \mathrm{Vect}(N)$ and a connection ∇ on E, and looking for $\widetilde{\nabla}$ on f^*E, the condition

$$\widetilde{\nabla}_X(f^*(s)) = f^*\left(\nabla_{df(X)}(s)\right)$$

now makes sense pointwise: it amounts to requiring, for each $x \in M$, that

$$(\widetilde{\nabla}_X(f^*(s)))(x) = \nabla_{df(X_x)}(s).$$

Recalling that any section of f^*E can be written as

$$s = f^1 \cdot f^*(s_1) + \ldots + f^k \cdot f^*(s_k) \quad \text{with } f^i \in C^\infty(M),\ s_i \in \Gamma(E),$$

you should now do the following:

Exercise 2.11. Given $f : M \to N$, $E \in \mathrm{Vect}(N)$ and a connection ∇ on E, show that there exists at most one connection $\widetilde{\nabla}$ on f^*E with the property that

$$\widetilde{\nabla}_X(f^*(s)) = f^*\left(\nabla_{df(X)}(s)\right) \tag{2.1.3}$$

for all $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. Furthermore, formulate this property from the dual point of view, using d_∇ and $d_{\widetilde{\nabla}}$, hopefully in a commutative diagram involving $\Gamma(E)$, $\Gamma(f^*E)$, $\Omega^1(N,E)$ and $\Omega^1(M,f^*E)$.

2.1.3 Locality; connection matrices

Connections are local in the sense that, for a connection ∇ and $x \in M$,

$$\nabla_X(s)(x) = 0$$

whenever $X \in \mathscr{X}(M)$, $s \in \Gamma(E)$ have the property that

$$X = 0 \text{ or } s = 0 \text{ in a neighborhood } U \text{ of } x.$$

This can be checked directly using $C^\infty(M)$-linearity in X and Leibniz, or can be derived from the remark that ∇ is a differential operator of order one in X and of order zero in f, or can be obtained by using the dual view-point (remember that 1-forms are local!); actually, at heart, these are all the same argument.

Locality implies that, for $U \subset M$ open, ∇ induces a connection ∇^U on the vector bundle $E|_U$ over U, uniquely determined by the condition

$$\nabla_X(s)|_U = \nabla^U_{X|_U}(s|_U).$$

Choosing U to be the domain of a trivialization of E, with corresponding local frame $\mathfrak{e} = \{\mathfrak{e}_1, \dots, \mathfrak{e}_r\}$, the previous discussions show that $\nabla|_U$ is uniquely determined by a matrix

$$\omega = (\omega^i_j)_{i,j} \in \mathscr{M}_r(\Omega^1(U)), \tag{2.1.4}$$

called **the connection matrix** of ∇ with respect to the local frame $\mathfrak{e}$, defined by writing out

$$\nabla_X(\mathfrak{e}_j) = \sum_{i=1}^{r} \omega^i_j(X)\mathfrak{e}_i.$$

For an arbitrary section s, writing $s|_U = \sum_i f^i \cdot e_i$, the Leibniz identity gives, over U:

$$\nabla_X(s) = \sum_i \left(L_X(f^i) \cdot e_i + \sum_j f^j \cdot \omega^i_j(X) \right) \cdot e_i \tag{2.1.5}$$

Dually, the matrix is determined by

$$d_\nabla(\mathfrak{e}_j) = \sum_i \omega^i_j \cdot \mathfrak{e}_i$$

and then, on arbitrary sections,

$$d_\nabla(s)|_U = \sum_i \left(df_i \cdot e_i + \sum_j f^j \cdot \omega^i_j \right) \cdot e_i \qquad \left(\text{where } s|_U = \sum_i f^i \cdot e_i \right).$$

Exercise 2.12. Consider the tautological line bundle $L = L_{\mathbb{CP}^1}$ over $\mathbb{CP}^1$ (see Example 6.32') and consider the connection that arise from the complex version of Exercise 2.5 (i.e. look at your solution to Exercise 2.4). Consider the chart of $\mathbb{CP}^1$ given by

$$U = \{[1 : z] : z \in \mathbb{C}\} \subset \mathbb{CP}^1$$

over which we consider the local frame of L (i.e. just a nowhere vanishing local section of L, since L is of rank one) given by

$$e([1 : z]) = e_1 + ze_2$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$. Show that

$$d_\nabla(e) = \frac{\bar{z} \cdot dz}{1 + |z|^2} e.$$

Locality can also be used to finalise the construction of pull backs of connections that was touched upon at the end of the previous two sections.

Proposition 2.13. *Let $f : M \to N$ be a smooth map, $E \in \mathrm{Vect}(N)$, ∇ is a connection on E. Then there exists a connection $f^*\nabla$ on f^*E, and is unique, with the property that*

$$d_{f^*\nabla}(f^*s) = f^* d_\nabla(s) \quad \forall\, s \in \Gamma(E). \tag{2.1.6}$$

Notice that the last condition is just the dual version of (2.1.3). Probably the most transparent way to describe $f^*\nabla$ is by describing how connection matrices of ∇ induce ones for $f^*\nabla$. We put together two things:

- the fact that for any local frame $\mathfrak{e}$ of E over some $U \subset M$, $f^*\mathfrak{e}$ is a local frame of f^*E over $f^{-1}(U)$.

- the fact that, one a frame $\mathfrak{e}$ of E over U is fixed, $\nabla|_U$ is uniquely determined by a connection matrix.

We see that the connection matrix for $f^*\nabla$ relative to $f^*\mathfrak{e}$ should be simply the pull-back of (2.1.4):

$$f^*\omega = (f^*\omega^i_j)_{i,j} \in \mathscr{M}_r(\Omega^1(f^{-1}(U))). \tag{2.1.7}$$

Proof. In principle, the last formula gives complete description of $f^*\nabla$. However, this does not quite prove the proposition yet: one would still have to prove the independence of the frames and take care of what happens on the overlaps. It is here that the uniqueness part of Exercise 2.11 comes in very handy, as the uniqueness applied over various opens takes care of both problems:

- it implies that the connection on $E|_{f^{-1}(U)}$ defined using (2.1.7) does not depend on which frame $\mathfrak{e}$ over U we use; denote it momentarily by $\widetilde{\nabla}^U$;
- for any two U and U' as above, it implies that $\widetilde{\nabla}^U$ and $\widetilde{\nabla}^{U'}$ coincide on $f^{-1}(U \cap U')$.

Hence, for $s \in \Gamma(E)$, there is one and only one $(f^*\nabla)_X(s) \in \Gamma(f^*E)$ such that, for all U as above,

$$(f^*\nabla)_X(s)|_U = \widetilde{\nabla}^U_X(s|_U). \quad \text{😀}$$

2.1.4 Derivatives of paths

Another operation that can be performed once a connection on E is chosen is that of taking derivatives of paths in E, with the outcome being still a path in E. Any path $u : I \to E$ induces a path in M, namely $\gamma = \pi \circ u$, also called **the base path of** u. We now focus more on the base path γ and look at paths u sitting above γ.

Definition 2.14. Let E be a vector bundle E over M. Given a path in M, $\gamma : I \to M$, defined over some interval $I \subset M$, by a **path in** E **above** γ we mean any path $u : I \to E$ with the property that

$$u(t) \in E_{\gamma(t)} \qquad \forall\, t \in I.$$

Example 2.15. For any $s \in \Gamma(E)$,

$$\gamma^*(s) := s \circ \gamma \tag{2.1.8}$$

is a path in E above γ. Not all paths above γ can be written in this way. However, if we allow time-dependent sections $\widetilde{s}$ of E, any u above γ can be written as

$$u(t) = \widetilde{s}(t, \gamma(t)). \tag{2.1.9}$$

Here, by a time-dependent section of E we mean a map

$$s : I \to \Gamma(E), \quad t \mapsto s_t,$$

which is smooth in the sense that $I \times M \ni (t,x) \mapsto s_t(x) \in E$ is smooth. Equivalently, s is a smooth section of the pull-back of E to $I \times M$ via the second projection. 😀

Returning to Definition 2.14, notice that $\gamma : I \to M$ can be used to pull-back E to a vector bundle γ^*E, and a path u above γ is nothing but a section

$$u \in \Gamma(\gamma^*E).$$

The resulting pull-backs of sections of E are nothing but (2.1.8). Notice that this, combined with the fact that pull-back sections span $\Gamma(\gamma^*E)$ as a $C^\infty(I)$-module, immediately implies the existence of time-dependent sections $\widetilde{s}$ as in (2.1.9). Another immediate advantage of this point of view is that we can also pull-back the connection.

Definition 2.16. Given a connection ∇ on E, **the ∇-derivative** of a path $u : I \to E$ above γ is defined as

$$\frac{\nabla u}{dt} := \nabla^{\gamma}_{\frac{d}{dt}}(u) : I \to E,$$

where $\nabla^{\gamma} = \gamma^* \nabla$ (as in Proposition 2.13).

Notice that $\frac{\nabla u}{dt}$ is again a path above γ.

Remark 2.17 (slightly different viewpoint, and explicit formulas). The ∇-derivatives also arise when analysing how much the expressions $\nabla_X(s)$ evaluated on a point x, or around it, depend on X and s. We have seen that ∇ is local: if we want to know $\nabla_X(s)$ at the point $x \in X$, then it suffices to know X and s in a neighborhood of x. Even more, the dual view-point on ∇ taught us that we only need to know X at x (and s in a neighborhood), allowing us to talk about

$$\nabla_{X_x}(s) \in E_x$$

as in Corollary 2.10. This can also be seen locally by re-writing (2.1.5) at the point x as

$$\begin{aligned}\nabla_X(s)(x) &= \sum_i \left(L_X(f^i)(x) \cdot e_i(x) + \sum_j f^j(x) \cdot \omega^i_j(X_x) \right) \cdot e_i(x) \\ &= \sum_i \left((df^i)(X_x) + \sum_j f^j(x)\, \omega^i_j(X_x) \right) \cdot \mathfrak{e}_i(x)\end{aligned}$$

This implies that $\nabla_X(s)(x)$ depends on quite a bit less than $s|U$: we only need to know s along an integral curve γ of X through x! Indeed, the last formula applied to $x = \gamma(t)$ becomes:

$$\nabla_X(s)(\gamma(t)) = \sum_i \left(\frac{df^i \circ \gamma}{dt}(t) + \sum_j f^j(\gamma(t))\, \omega^i_j \left(\frac{d\gamma}{dt}(t) \right) \right) \cdot \mathfrak{e}_i(\gamma(t))$$

This depends only on what s does on γ, i.e., on $u(t) = s(\gamma(t)) = \sum_i f^i(\gamma(t)) \cdot \mathfrak{e}_i(\gamma(t))$. Furthermore, the last expression makes sense for any path u above γ, not necessarily a pull-back $\gamma^* s$ ... and the outcome is precisely $\frac{\nabla u}{dt}$. Indeed, using the description of pull-back connections (see the discussion around Proposition 2.13) it follows right away that, for any u above γ, writing $u(t) = \sum_i u^i(t) \cdot \mathfrak{e}_j(\gamma(t))$, one has

$$\frac{\nabla u}{dt}(t) = \sum_i \left(\frac{du^i}{dt}(t) + \sum_j u^j(t)\, \omega^i_j \left(\frac{d\gamma}{dt}(t) \right) \right) \cdot \mathfrak{e}_i(\gamma(t)) \tag{2.1.10}$$

More compactly, $u = \sum_i u^i \cdot \gamma^*(\mathfrak{e}_i)$ and

$$\boxed{\frac{\nabla u}{dt} = \sum_i \left(\frac{du^i}{dt} + \sum_j u^j\, \omega^i_j \left(\frac{d\gamma}{dt} \right) \right) \cdot \gamma^*(\mathfrak{e}_i).}$$

When $u = s \circ \gamma$ for some global section s, the main property of the pull-back connection gives

$$\frac{\nabla u}{dt}(t) = \nabla_{\dot\gamma(t)}(s).$$

Exercise 2.18. For a similar formula valid for general paths u, we first write u as in (2.1.9) for some time-dependent section $\widetilde{s}$. Prove that

$$\frac{\nabla u}{dt}(t) = \left(\nabla_{\dot\gamma(t)}(s_t) + \frac{ds_t}{dt}\right)(\gamma(t)).$$

where, for each t, $s_t \in \Gamma(E)$ is the section $\widetilde{s}(t,\cdot)$.

Exercise 2.19. Notice that, for any curve $\gamma : I \to M$ in a manifold M, the derivative $\dot\gamma : I \to TM$ is a curve above γ. Assume now that $M = \mathbb{R}^n$ and we endow its tangent bundle with the canonical connection:

$$\nabla_X Y = \sum_i X(Y^i)\frac{\partial}{\partial x_i}.$$

What is then $\frac{\nabla\dot\gamma}{dt}$, for curves γ in $\mathbb{R}^n$?

2.1.5 Parallel transport

One of the main use of connections comes from the fact that a connection ∇ on E can be used to move from one fiber of E to another, along paths in the base. This is the so called parallel transport. To explain this, let us return to paths $u : I \to E$. We say that u is **parallel** (or **horizontal**) with respect to ∇ if

$$\frac{\nabla u}{dt} = 0 \quad \forall\, t \in I.$$

Proposition 2.20. *Let ∇ be a connection on the vector bundle E and $\gamma : I \to M$ a curve in M, $t_0 \in I$. Then for any $u_0 \in E_{\gamma(t_0)}$ there exists and is unique a parallel path above γ, $u : I \to E$, with $u(t_0) = u_0$.*

Proof. We can proceed locally (also because the uniqueness locally implies that the local pieces can be glued), on the domain of a local frame e which identifies E with $M \times \mathbb{R}^r$; write $u_0 = (x_0, y_0)$. By formula (2.1.10), we have to find $u^1, \dots, u^r : I \to \mathbb{R}$ satisfying

$$\frac{du^i}{dt}(t) = -\sum_j u^j(t)\omega^i_j(\dot\gamma(t)),\ u(0) = y_0. \tag{2.1.11}$$

In a matricial form (with u viewed as a column matrix), writing $A(t)$ for the matrix $-\omega(\dot\gamma(t))$, we deal with

$$\dot u(t) = A(t)u(t), \quad u(t_0) = y_0$$

and the existence and uniqueness is a standard result about first order linear ODE's $\left(u(t) = e^{\int_{t_0}^t A(s)ds}u(t_0)\right)$. 😄

Definition 2.21. Given a connection ∇ on E and a curve $\gamma : I \to M$, $t_0, t_1 \in I$, the **parallel transport** along γ (with respect to ∇) from time t_0 to time t_1 is the map

$$T_\gamma^{t_0,t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

which associates to $u_0 \in E_{\gamma(t_0)}$ the vector $u(t_1) \in E_{\gamma(t_1)}$, where u is the parallel curve above γ with $u(t_0) = u_0$.

Proposition 2.22. *With the previous notations,*

1. *each $T_\gamma^{t_0,t_1}$ is a linear isomorphism.*
2. *$T_\gamma^{t_1,t_2} \circ T_\gamma^{t_0,t_1} = T_\gamma^{t_0,t_2}$ for all $t_0,t_1,t_2 \in I$.*

Proof. For item 1. notice first that, for any two paths $u,v : I \to E$ over γ and $\lambda \in \mathbb{R}$ one has

$$\frac{\nabla(u+v)}{dt} = \frac{\nabla u}{dt} + \frac{\nabla v}{dt}, \quad \frac{\nabla(\lambda \cdot u)}{dt} = \lambda \cdot \frac{\nabla u}{dt}.$$

This implies that ($\mathbb{R}$-)linear combinations of parallel paths over γ is again parallel which, in turn, implies that each $T_\gamma^{t_0,t_1}$ is linear. It is injective because the uniqueness of ODEs implies that any solution of a linear ODE if it vanishes at one point then it vanishes everywhere.

We now move to item 2. Let $u_0 \in E$ and we want to check that

$$T_\gamma^{t_1,t_2}\left(T_\gamma^{t_0,t_1}(u_0)\right) = T_\gamma^{t_0,t_2}(u_0) \tag{2.1.12}$$

Notice that, by the very definition of parallel transport, for any parallel path $u : I \to M$ above $\gamma : I \to M$, one has

$$T_\gamma^{t_1,t_2}(u(t_1)) = u(t_2).$$

Therefore, writing $u_0 = u(t_0)$ where $u : I \to M$ is the parallel lift of γ, the left hand side of (2.1.12) is

$$T_\gamma^{t_1,t_2}\left(T_\gamma^{t_0,t_1}(u(t_0))\right) = T_\gamma^{t_1,t_2}(u(t_1)) = u(t_2) = T_\gamma^{t_0,t_2}(t_0) = T_\gamma^{t_0,t_2}(u_0)$$ 😄

Exercise 2.23. Show that Then try to guess how ∇ can be recovered from all these parallel transports (there is a natural guess!). Then prove it.

Corollary 2.24. *If M is contractible, then any vector over M is trivializable.*

Proof. Recall that contractibility means that there exists a map

$$\Gamma : M \times [0,1] \to M$$

with the property that $\Gamma_1 = \Gamma(-,1) : M \to M$ is the identity map, while $\Gamma_0 = \Gamma(-,0)$ is a constant map, call it x_0. This provides us, for each $x \in M$, a curve

$$\gamma^x := \Gamma(x,\cdot) : [0,1] \to M$$

from x_0 to x. Using parallel transport along these curves we define

$$T_\Gamma : M \times E_{x_0} \to E, \quad (x,u_0) \mapsto T_{\gamma^x}^{0,1}(u_0).$$

This is fiberwise linear isomorphism and smooth (why?), providing a trivialization of E. 😄

Exercise 2.25. On $M = S^2 \subset \mathbb{R}^3$ we consider the trivial vector bundle of rank 2, $E = S^2 \times \mathbb{R}^2$, and we denote by $\{e_1, e_2\}$ its canonical frame. Consider the connection ∇ on E whose connection matrix w.r.t. $\{e_1, e_2\}$ is

$$\omega = \begin{pmatrix} xdy & ydz \\ zdx & x^2dx + y^2dy + z^2dz \end{pmatrix},$$

where x, y, z are the coordinates in $\mathbb{R}^3$, and when writing xdy etc we mean their restriction to S^2. Please do the following:

(a) Compute $\nabla_{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}}(x^2 e_1 + xyze_2)$ (more precisely, compute the coefficients of this expression w.r.t. the basis $\{e_1, e_2\}$).
(b) Compute the connection matrix of ∇ with respect to $\{e_1, e_2\}$. Should this matrix, or at least its trace, be zero, since the vector bundle is trivial? What is going on?
(c) Compute the parallel transport $T_\gamma^{0,2\pi} : \mathbb{R}^2 \to \mathbb{R}^2$ of ∇ along $\gamma : [0, 2\pi] \to S^2$, $\gamma(t) = (\cos t, \sin t, 0)$.

Exercise 2.26. Consider the tautological line bundle $L = L_{\mathbb{CP}^1}$ and we endow it with the connection induced from its inclusion into the trivial bundle $E = \mathbb{CP}^1 \times \mathbb{C}^2$ (as in Exercise 2.5, and similar to what we did in Example 2.3). In other words,

$$\nabla : \mathfrak{X}(\mathbb{CP}^1) \times \Gamma(L) \to \Gamma(L), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}\left(\underbrace{\nabla_X^{\mathrm{flat}}(s)}_{\in \Gamma(E)} \right)$$

where we use the canonical flat connection $\nabla^{\mathrm{flat}} : \mathfrak{X}(\mathbb{CP}^1) \times \Gamma(E) \to \Gamma(E)$ and the projection map

$$\mathrm{pr} : \Gamma(E) \to \Gamma(L), \quad \mathrm{pr}(s)(l) := \mathrm{pr}_l(s(l))$$

where, for $l \in \mathbb{CP}^1$, $\mathrm{pr}_l : \mathbb{C}^2 \to l$ is the orthogonal projection w.r.t. the usual hermitian product h on $\mathbb{C}^2$:

$$\mathrm{pr}_l(z) = \frac{h(z, \xi)}{h(\xi, \xi)}\xi, \quad \text{with } \xi \in l \text{ a/any non-zero vector} \quad (h(z, z') = z_1\overline{z_1'} + z_2\overline{z_2'}).$$

1. Show that for any path paths $\gamma : I \to \mathbb{CP}^1$ and $u : I \to L$ of type

$$\gamma(t) = [z_1(t) : z_2(t)] \in \mathbb{CP}^1, \quad u(t) = a(t)e_1(\gamma(t)) + b(t)e_2(\gamma(t))$$

where z_1 and z_2 are smooth as well, and where $\{e_1, e_2\}$ is the canonical frame of E, one has

$$\frac{\nabla u}{dt}(t) = \frac{\overline{z_1} \cdot a' + \overline{z_2} \cdot b'}{|z_1|^2 + |z_2|^2}(z_1 e_1 + z_2 e_2).$$

2. For $\gamma : [0, \pi] \to \mathbb{CP}^1$, $\gamma(t) = [1 : e^{2it}]$, compute the resulting parallel transport

$$T_\gamma^{0,t} : L_{\gamma(0)} \to L_{\gamma(t)}.$$

2.1.6 Connections and DeRham-like operators

We now expand on the discussion from subsection 2.1.2, which reinterpreted connections as operators $d_\nabla : \Gamma(E) \to \Omega^1(M,E)$ or, equivalently,

$$d_\nabla : \Omega^0(M,E) \to \Omega^1(M,E).$$

This is, of course, reminiscent of the standard DeRham differential d that acts on the space $\Omega(M)$ of differential forms on M, increasing the degree by one

$$d : \Omega^*(M) \to \Omega^{*+1}(M).$$

Recall its basic properties: it satisfies the Leibniz identity:

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge d(\eta),$$

where $|\omega|$ is the degree of ω, and it is a differential (i.e. $d \circ d = 0$), one has the local description

$$\omega = \sum_{i_1,\ldots,i_p} f \cdot d\chi_{i_1} \wedge \ldots \wedge d\chi_{i_p} \longmapsto d\omega = \sum_i \sum_{i_1,\ldots,i_p} \frac{\partial f^{i_1,\ldots,i_p}}{\partial \chi_k} \cdot d\chi_i \wedge d\chi_{i_1} \wedge \ldots \wedge d\chi_{i_p},$$

and, thinking of ω as a $C^\infty(M)$-linear map one has the global decsription

$$\begin{aligned} d(\omega)(X_1,\ldots,X_{p+1}) = &\sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{p+1})) \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} L_{X_i}(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})). \end{aligned} \tag{2.1.13}$$

Let us now pass to differential forms with coefficients in a vector bundle E (see 1.2.6). The key remark here is that, while there is no canonical (i.e. free of choices) analogue of DeRham differential on $\Omega(M,E)$, connections are precisely the piece that is needed in order to define such operators. Indeed, assuming that ∇ is a connection on E, and thinking of forms $\omega \in \Omega^p(M,E)$ as $C^\infty(M)$-multilinear maps as in 1.2.8, we see that the previous formula for the DeRham differential does makes sense if we replace the Lie derivatives L_{X_i} by ∇_{X_i}. Hence one has an induced operator

$$\boxed{d_\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E).} \tag{2.1.14}$$

As in the case of DeRham operator, d_∇ satisfies the Leibniz identity

$$\boxed{d_\nabla(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge d_\nabla(\eta)} \tag{2.1.15}$$

for all $\omega \in \Omega(M)$, $\eta \in \Omega(M,E)$. Notice that, in degree zero, d_∇ acts on $\Omega^0(M,E) = \Gamma(E)$ and it coincides with the operator d_∇ previously discussed. Moreover, d_∇ defined on $\Omega^\bullet(M,E)$ is uniquely determined by what it does on the degree zero part and the fact that d_∇ satisfies the Leibniz identity.

Exercise 2.27. Fill in the details to conclude that, indeed, the operator d_∇ discussed in subsection 2.1.2 admits a linear extension (2.1.14) to arbitrary degrees, unique with the property that it satisfies the Leibniz identity (2.1.15). In other words, the 1-1 correspondence from Proposition 2.8 extends to

$$\boxed{\left\{\begin{array}{c}\text{connections } \nabla \\ \text{on } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{linear operators } \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E) \\ \text{satisfying the Leibniz identity (2.1.15)}\end{array}\right\}.}$$

As for the DeRham operator, d_∇ can be described locally, using the connection matrices. First of all, if U is the domain of a local frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$ with connection matrix ω, then one can write

$$d_\nabla(\mathfrak{e}_j) = \sum_{i=1}^{r} \omega_j^i \mathfrak{e}_i, \tag{2.1.16}$$

Assume now that U is also the domain of a coordinate chart $(U, \chi_1, \ldots, \chi_n)$. Representing $\omega \in \Omega^p(M,E)$ locally as in (1.2.11), the Leibniz identity gives the formula

$$d_\nabla(\omega) = \sum_{i_1,\ldots,i_p} (-1)^p dx_{i_1} \ldots dx_{i_p} \otimes d_\nabla(e^{i_1,\ldots,i_p}).$$

hence it suffices to describe d_∇ on sections of E. The same Leibniz formula implies that it suffices to describe d_∇ on the frame e- and that what (2.1.16) does.

2.1.7 Connections as horizontal distributions (to be improved; we skip it first)

Yet another viewpoint on connection is provided by the notion of "horizontality". Let us start with the complementary notion, that of "verticality", which is more intrinsic and does not involve connections.The entire discussion can be carried out in the more general context of "fiber bundles" which was briefly discussed in the first lecture. Recall that by a fiber bundle over M, denoted

$$\pi : P \to M,$$

we mean a manifold P together with a surjective submersion $\pi : P \to M$. Given such a fiber bundle, its fiber above a point $x \in M$ is defined as $P_x := \pi^{-1}(x)$. The regular value theorem ensures that $P_x \subset P$ are embedded submanifolds and their tangent spaces are precisely the kernel of the differential

$$(d\pi)_p : T_pP \to T_xM \qquad (x = \pi(p)). \tag{2.1.17}$$

A tangent vector $v_p \in T_pP$ is called **vertical** if it is tangent to the fiber P_x with $x = \pi(p)$; we denote by

$$T_p^{\text{vert}}P \subset T_pP$$

the subspace of such vertical tangent vectors. By a **horizontal subspace** of TP at $p \in P$ we mean any vector subspace $\mathscr{H}_p \subset T_pP$ complementary to $T_p^{\text{vert}}P$, i.e. satisfying

$$T_pP = \mathscr{H}_p \oplus T_p^{\text{vert}}P.$$

Definition 2.28. An **Ehresmann connection** on the fiber bundle $\pi : P \to M$ is a vector sub-bundle

$$\mathscr{H} \subset TP$$

consisting of horizontal subspaces $\mathscr{H}_p \subset T_pP$. The elements of $\mathscr{H}_p$ will be called horizontal (tangent) vectors, and a path $u : I \to P$ is called $\mathscr{H}$**-horizontal** if $\dot{u}$ takes values in $\mathscr{H}$.

Notice that $\mathscr{H}_p \subset T_pP$ is horizontal if and only if $(d\pi)_p|_{\mathscr{H}_p} : \mathscr{H}_p \to T_xM$ is an isomorphism. The inverses of this maps will be denoted

$$\text{hor}_p^{\mathscr{H}} : T_xM \to T_pP \quad (x = \pi(p)) \tag{2.1.18}$$

and will be called the **horizontal liftings** induced by $\mathscr{H}$; it associates to $X_x \in T_xM$ the unique horizontal tangent vector $V_p \in \mathscr{H}_p$ that projects to X_x. Without making a reference to the base points x and p, $\text{hor}^{\mathscr{H}}$ is a morphism of

vector bundles over P,

$$\mathrm{hor}^{\mathscr{H}} : \pi^*TM \to TP.$$

Exercise 2.29. Why is $\mathrm{hor}^{\mathscr{H}}$ a morphism of vector bundles?

Notice also that the horizontal lifting map determines $\mathscr{H}$: the latter is the image of the former. We deduce that there is a 1-1 correspondence:

$$\left\{ \begin{array}{c} \text{Ehresmann connections } \mathscr{H} \\ \text{on the fiber bundle } \pi : P \to M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{vector bundle morphisms hor} : \pi^*TM \to TP \\ \text{which are right inverses to } d\pi : TP \to \pi^*TM \end{array} \right\}$$

Example 2.30 (trivial bundles). To proceed locally, one first has to understand the case of trivial (product) fiber bundles. Hence let us assume $P = M \times \mathbb{R}^r$. We write elements $u \in P$ as pairs (x,y). We see that the choosing a horizontal lifting map has to be of type

$$\mathrm{hor}_{x,y} : T_xM \to T_xM \times \mathbb{R}^r, \quad v \mapsto (v, \eta_y^1(v), \ldots, \eta_y^r(v)), \tag{2.1.19}$$

where for each $i \in \{1, \ldots, r\}$ and $y \in \mathbb{R}^r$, η_y^i is a 1-form on M. An interesting situation is when each η_y^i is linear in y, i.e.,

$$\eta_y^i(-) = \sum_j y^j \cdot \eta_j^i(-) \quad \text{with} \quad \eta_j^i \in \Omega^1(M).$$

To relate Ehresmann connections to connections on vector bundles, assume now that $\pi : E \to M$ is a vector bundle and we focus on Ehresmann connections on $P = E$ which are compatible with the vector bundle structure, called "linear" bellow. The compatibility (or linearity) can be formulated in several equivalent ways. Here is probably the shortest, which is based on the idea that linearity can be controlled by the dilatation maps

$$m_\lambda : \mathbb{R}^r \to \mathbb{R}^r, \quad m_\lambda(v) = \lambda \cdot v,$$

with $\lambda \in \mathbb{R}^*$. Each is a diffeomorphism and induces a map

$$m_\lambda^* : C^\infty(\mathbb{R}^r) \to C^\infty(\mathbb{R}^r), \quad m_\lambda^*(f)(v) = f(\lambda \cdot v).$$

The following is a very good illustration of the relevance of m_λ.

Exercise 2.31. Show that a smooth function $f : \mathbb{R}^r \to \mathbb{R}$ is linear if and only if $m_\lambda^*(f) = \lambda f$.

We now use m_λ fiberwise on E, i.e. viewed as a vector bundle morphism

$$m_\lambda : E \to E.$$

An Ehresmann connection $\mathscr{H}$ on E will be called **linear** if $m_\lambda^*(\mathscr{H}) = \mathscr{H}$ or, more explicitly,

$$(dm_\lambda)_v(\mathscr{H}_v) = \mathscr{H}_{\lambda \cdot v} \quad \forall\, v \in E, \lambda \in \mathbb{R}^*.$$

Example 2.32 (trivial bundles). Continuing the previous example, we see that the connection there is linear if an only if each η_y^i is linear in y (to see this use e.g. the previous exercise):

$$\eta_y^i(-) = \sum_j y^j \cdot \eta_j^i(-) \quad \text{with} \quad \eta_j^i \in \Omega^1(M).$$

Hence linear Ehresmann connection are determined locally by 1-forms η_j^i ... which is a clear sign that linear Ehresmann connections are closely related to vector bundle connections in the sense of the previous sections. Indeed:

Theorem 2.33. *Given a vector bundle $\pi : E \to M$, there is a 1-1 correspondence*

$$\left\{\begin{array}{c}\text{linear connections}\\ \nabla \text{ on } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{Ehresmann connections on } E\\ \text{which are linear}\end{array}\right\}$$

More precisely, given ∇ on E, the corresponding Ehresmann connection on E is the unique $\mathscr{H}^{\nabla}$ with the property that the paths in E that are parallel w.r.t. ∇ coincide with those that are $\mathscr{H}^{\nabla}$-horizontal.

Proof. The property imposed on $\mathscr{H}^{\nabla}$ forces us to define, for $u_0 \in E$ sitting above $x_0 = \pi(u_0)$,

$$\mathscr{H}^{\nabla}_{u_0} = \{\dot{u}(0) : u - \text{parallel w.r.t. } \nabla \text{ and } u(0) = u_0\}.$$

To make sure that this is indeed a vector sub-bundle, and that it has the desired properties, one can proceed locally and we may assume that $E = M \times \mathbb{R}^r$. We proceed as in the proof of Proposition 2.20, with the same notations- in particular, write $u_0 = (x_0, y_0)$. By definition, $\mathscr{H}^{\nabla}_{u_0}$ consists of vectors of type

$$V_{u_0} = \left(\frac{d\gamma}{dt}(0), \frac{du}{dt}(0)\right)$$

where u is any solution of (2.1.11). Plugging the equation into V_{u_0} and denoting $v = \frac{d\gamma}{dt}(0)$, we find that $\mathscr{H}^{\nabla}_{u_0}$ consists of vectors of type

$$\left(v, -\sum_j y_0^j \omega_j^i(v)\right) \in T_{u_0}E \cong T_{x_0}M \times \mathbb{R}^r$$

i.e. it is the image of the map

$$h_{u_0} : T_{x_0}M \to T_{x_0}M \times \mathbb{R}^r, \quad v \mapsto \left(v, -\sum_j y_0^j \omega_j^i(v)\right).$$

Since this is a right inverse to the first projection (basically the horizontal lifting map), it follows that each $\mathscr{H}^{\nabla}_{u_0}$ is a horizontal subspace. Since h_{u_0} is smooth (as ω is) and of constant rank (fiberwise injective), it follows that $\mathscr{H}^{\nabla}$ is, indeed, an Ehresmann connection. The fact that paths that are parallel w.r.t. to ∇ are the same thing as those that are $\mathscr{H}^{\nabla}$-horizontal follows from the way we defined the latter (details are left as an exercise). The fact that $\mathscr{H}^{\nabla}$ is linear follows from the fact that that is a local property (w.r.t. opens in M) and from the remark we made in Example 2.32.

What is perhaps not so clear yet is why any linear Ehresmann connection $\mathscr{H}$ arise in this way. When $E = M \times \mathbb{R}^r$ this follows from Example 2.32: we build ∇ using the connection matrix $\omega = -\eta$. Notice that the uniqueness part implies that, whenever E is trivializable, $\mathscr{H}$ is induced by a ∇ (which does not depend on the choice of a trivialization). In general, given $\mathscr{H}$, for each trivializing open $U \subset M$ we find a connection ∇^U of $E|_U$ and, over the intersections $U \cap V$ of such opens, the connections coincide (again by uniqueness). This allows us to build ∇ globally such that $\mathscr{H} = \mathscr{H}^{\nabla}$. 😄

The following provides a slightly different description of the horizontal spaces. Notice that, in principle, one could use this as definition of horizontality w.r.t. ∇, including that of parallel path introduced in the previous section.

Exercise 2.34. Given a connection ∇ on the vector bundle $\pi : E \to M$, $u_0 \in E$, $x_0 = \pi(u_0)$, show that:

1. there exists $s \in \Gamma(E)$ that passes through u_0 (i.e. $s(x_0) = u_0$), such that $d_{\nabla}(s)(x_0) = 0$ (flatness at x_0).
2. for any section s as above, $\mathscr{H}^{\nabla}_{u_0}$ is precisely the image of

$$(ds)_{x_0} : T_{x_0}M \to T_{u_0}E.$$

2.2 Curvature

2.2.1 The definition

Recall that, for the standard Lie derivatives of functions along vector fields,

$$L_{[X,Y]} = L_X L_Y(f) - L_Y L_X(f).$$

Of course, this can be seen just as the definition of the Lie bracket $[X,Y]$ of vector fields but, even so, it still says something: the right hand side is a derivation on f (i.e., indeed, it comes from a vector field). The similar formula for connections fails dramatically (i.e. there are few vector bundles which admit a connection for which the analogue of this formula holds). The failure is measured by the curvature of the connection.

Definition 2.35. Given a connection ∇ on the vector bundle E, the **curvature of** ∇ is the map

$$k_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathrm{End}(\Gamma(E))$$

which associates to two vector fields $X,Y \in \mathfrak{X}(M)$ the linear map $k_\nabla(X,Y) : \Gamma(E) \to \Gamma(E)$ given by

$$k_\nabla(X,Y)s = \nabla_X \nabla_Y(s) - \nabla_Y \nabla_X(s) - \nabla_{[X,Y]}(s). \tag{2.2.1}$$

The connection is called **flat** if $k_\nabla = 0$.

The following shows the tensorial nature of the curvature.

Proposition 2.36. *For any connection ∇, the expression (2.2.1) is $C^\infty(M)$-linear in the entries $X,Y \in \mathscr{X}(M)$, $s \in \Gamma(E)$. Hence the curvature can also be interpreted as a 2-form with coefficients,*

$$k_\nabla \in \Gamma(\Lambda^2 T^*M \otimes End(E)) = \Omega^2(M; End(E)). \tag{2.2.2}$$

Proof. It follows from the properties of ∇. For instance, we have

$$\begin{aligned} \nabla_X \nabla_Y(fs) &= \nabla_X(f\nabla_Y(s) + L_Y(f)s) \\ &= f\nabla_X\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X L_Y(f)s, \end{aligned}$$

and the similar formula for $\nabla_X \nabla_Y(fs)$, while

$$\nabla_{[X,Y]}(fs) = f\nabla_{[X,Y]}(s) + L_{[X,Y]}(f)s.$$

Hence, using $L_{[X,Y]} = L_X L_Y - L_Y L_X$, we deduce that

$$k_\nabla(X,Y)(fs) = f k_\nabla(X,Y)(s),$$

and similarly the others. 😀

Exercise 2.37. This is a continuation of Exercise 2.7, hence we assume the same notations. For two finite dimensional vector spaces V and V' and $A \in \mathrm{End}(V)$ and $A' \in \mathrm{End}(V')$, we consider

$$A^* \in \mathrm{End}(V^*) \quad \text{given by } A^*(\xi)(v) = \xi(A(v)),$$

$$A \oplus A' \in \mathrm{End}(V \oplus V') \quad \text{given by } (A \oplus A')(v,v') = (A(v), A'(v')),$$
$$A \otimes A' \in \mathrm{End}(V \otimes V') \quad \text{given by } (A \otimes A')(v \otimes v') = A(v) \otimes A'(v').$$

We keep the same notations for endomorphisms of vector bundles. Show that the curvatures of ∇^*, $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ are given by

$$k_{\nabla^*}(X,Y) = -k_\nabla(X,Y)^*,$$
$$k_{\nabla\oplus\nabla'}(X,Y) = k_\nabla(X,Y) \oplus k_{\nabla'}(X,Y),$$
$$k_{\nabla\otimes\nabla'}(X,Y) = \mathrm{Id}_E \otimes k_{\nabla'}(X,Y) + k_\nabla(X,Y) \otimes \mathrm{Id}_{E'}.$$

One can express the curvature locally, with respect to a local frame $\mathfrak{e} = \{\mathfrak{e}_1, \dots, \mathfrak{e}_r\}$ of E over an open U, as

$$k_\nabla(X,Y)\mathfrak{e}_j = \sum_{j=1}^{r} k_j^i(X,Y)\mathfrak{e}_i,$$

where $k_j^i(X,Y) \in C^\infty(U)$ are smooth functions on U depending on $X, Y \in \mathscr{X}(M)$. The previous proposition implies that each k_i^j is a differential form (of degree two). Hence k_∇ is locally determined by a matrix

$$k = k(\nabla, e) := (k_j^i)_{i,j} \in \mathscr{M}_r(\Omega^2(U)),$$

called the **curvature matrix** of ∇ over U, with respect to the local frame e. Of course, one should be able to express k explicitly in terms of the connection matrix ω. Indeed, one has:

Proposition 2.38. *The curvature matrix $k = k(\nabla, e)$ and the connection one $\omega = \omega(\nabla, e)$ are related by*

$$k = d\omega + \omega \wedge \omega,$$

where $\omega \wedge \omega$ is the matrix of 2-forms given by

$$(\omega \wedge \omega)_j^i = \sum_l \omega_l^i \wedge \omega_j^l.$$

Proof. This is a rather straightforward computation:

$$\begin{aligned}
k(X,Y)(e_j) &= \nabla_X(\nabla_Y(e_j)) - \nabla_Y(\nabla_X(e_j)) - \nabla_{[X,Y]}(e_j) \\
&= \nabla_X\left(\sum_l \omega_j^l(Y)e_l\right) - \nabla_Y\left(\sum_l \omega_j^l(X)e_l\right) - \sum_i \omega_j^i([X,Y])e_i \\
&= \sum_l \Big(L_X(\omega_j^l(Y))e_l + \omega_j^l(Y)\nabla_X(e_l)\Big) - \sum_l \Big(L_Y(\omega_j^l(X))e_l + \omega_j^l(X)\nabla_Y(e_l)\Big) - \sum_i \omega_j^i([X,Y])e_i \\
&= \sum_i \Big(L_X(\omega_j^l(Y)) - L_Y(\omega_j^l(X)) - \omega_j^i([X,Y])\Big)e_i + \sum_{l,i}\Big(\omega_j^l(Y)\omega_l^i(X) - \omega_j^l(X) - \omega_l^i(Y)\Big)e_i \\
&= \sum_i d\omega_j^i(X,Y) + \sum_{l,i}(\omega_l^i \wedge \omega_j^l)(X,Y)\cdot e_i
\end{aligned}$$

hence

$$k_j^i(X,Y) = d\omega_j^i(X,Y) + (\omega_l^i \wedge \omega_j^l)(X,Y)$$

and the conclusion follows.

Corollary 2.39. *The curvature is natural w.r.t. pull-backs, i.e., for any connection ∇ on a vector bundle E over M and any smooth map $f : N \to M$, one has $k_{f^*\nabla} = f^*(k_\nabla)$: the curvature of the pull-back connection $f^*\nabla$ is obtained from the curvature of ∇ by applying pull-back along f,*

$$f^* : \Omega^2(M, End(E)) \to \Omega^2(N, f^*End(E)).$$

2.2.2 The curvature as failure of "$\frac{d^2}{dsdt} = \frac{d^2}{dtds}$"

The fact that the curvature measure the failure of the connections to satisfy "usual formulas", which is clear already from the definition, can be illustrated in several other ways. Here we focus on the standard property of functions $\gamma(t,s)$ of two variables, taking values in some $\mathbb{R}^n$: the order of taking derivatives does not matter, i.e.

$$\frac{d^2\gamma}{ds\,dt} = \frac{d^2\gamma}{dt\,ds}.$$

We explain here that the curvature of a connection shows up as the failure of such an identity for the operator $\frac{\nabla}{dt}$ induced by ∇. To explain this, assume that ∇ is a connection on a vector bundle E over M,

$$\gamma = \gamma(t,s) : I_1 \times I_2 \to M$$

is a smooth map defined on the product of two intervals $I_1, I_2 \subset \mathbb{R}$ and

$$u = u(t,s) : I_1 \times I_2 \to E$$

smooth covering γ. Applying $\frac{\nabla}{dt}$ and then $\frac{\nabla}{ds}$ one obtains a new

$$\frac{\nabla^2 u}{ds\,dt} : I_1 \times I_2 \to E$$

covering u. The next proposition explains the title of this section.

Proposition 2.40. *In the setting described above,*

$$\frac{\nabla^2 u}{ds\,dt} - \frac{\nabla^2 u}{dt\,ds} = k_\nabla\left(\frac{d\gamma}{ds}, \frac{d\gamma}{dt}\right).$$

Proof. Apply Corollary 2.39. 😄

Here is a very interesting consequence:

Corollary 2.41. *Assume that ∇ is a flat connection on a vector E, i.e. with the property that $k_\nabla = 0$. Then, for $x, y \in M$, looking at paths γ in M from x to y, the induced parallel transport*

$$T_\gamma^{0,1} : E_x \to E_y$$

only depends on the path-homotopy class of γ.

This means that, if γ_0, γ_1 are two paths which can be joined by a smooth family of paths $\{\gamma_s\}_{s\in[0,1]}$ starting at x and ending at y, then $T_{\gamma_0}^{0,1} = T_{\gamma_1}^{0,1}$. Of ocurse, the smoothness of the family means that

$$\gamma(t,s) := \gamma_s(t)$$

is a smooth map from $[0,1]\times[0,1]$ to M.

Proof. Consider a smooth family $\{\gamma_s\}_{s\in[0,1]}$ as above, encoded in $\gamma(t,s)=\gamma_s(t)$. The conditions on γ are:

$$\gamma(0,s)=x, \gamma(1,s)=y, \quad (\gamma_s \text{ starts at } x \text{ and ends at } y),$$

$$\gamma(t,0)=\gamma_0(t), \gamma(t,1)=\gamma_1(t).$$

We fix $u_0\in E_x$ and we are going to prove that $T^{0,1}_{\gamma_0}(u_0)=T^{0,1}_{\gamma_1}(u_0)$. For that, consider $u(t,s)=T^{0,t}_{\gamma_s}(u)$. By the definition of parallel transport, $\frac{\nabla}{u}dt=0$. Using the flatness of ∇ and the previous exercise we see that $v:=\frac{\nabla u}{ds}$ must satisfy

$$\frac{\nabla v}{dt}=0.$$

But, for $t=0$ and any s one has $u(0,s)=u_0$ hence $v(0,s)=0$. In other words, is we fix s, $v(\cdot,s)$ is a path in E that is parallel w.r.t. ∇ and starts at 0; by uniqueness of the parallel transport we deduce that $v=0$, i.e. $\frac{\nabla u}{ds}=0$ at all (t,s). We use this at $t=1$, where the situation is more special bacause all $u(1,s)$ take value in one single fiber- namely E_y; moreover, from the definition of $\frac{\nabla}{ds}$, we see that $\frac{\nabla u}{ds}(1,s)$ is the usual derivative of the resulting path

$$[0,1]\ni s\mapsto u(1,s)\in E_y.$$

We deduce that this last path is constant in s; in particular, $u(1,0)=u(1,1)$, i.e. $T^{0,1}_{\gamma_0}(u_0)=T^{0,1}_{\gamma_1}(u_0)$. 😄

Recall that, fixing a point $x\in M$, the path-homotopy classes of paths in M that start and end at x (loops at x) form a group- the fundamental group of M with base point x, denoted $\pi(M,x)$ (the group operation is given by the concatenation of paths). The mainfold M is called **simply connected** if all the fundamental groups are trivial.

Corollary 2.42. *If M is simply connected, then a vector bundle $E\to M$ admits a flat connection if and only if it is trvializable.*

In general, Corollary 2.41 implies that the parallel transport defines a map

$$\rho:\pi(M,x)\to GL(E_x), \quad [\gamma]\mapsto T^{0,1}_{\gamma},$$

where $GL(E_x)$ is the group of all linear isomorphisms from E_x to itself. Moreover, it is not difficult to see that this is actually a group homomorphism. In other words, ρ is a representation of the group $\pi(M,x)$ on the vector space E_x. This is called the monodromy representation of the flat connection (at the base point x).

Exercise 2.43. Show that a vector bundle is trivializable if and only if it admits a flat connection ∇ whose monodromy representation (at some base point) is trivial.

2.2.3 The curvature as failure of $d^2_\nabla=0$

Here is another illustration of the fact that the curvature measure the failure of the connections to satisfy "usual formulas". Here we focus on the basic property $d^2=0$ for the usual DeRham differential and we explain that the curvature may be seen as a measure of the failure of a similar identity, $d^2_\nabla=0$:

Proposition 2.44. *A connection ∇ is flat if and only if $d^2_\nabla=0$. In general, one has*

$$k_\nabla(X,Y)(s)=d^2_\nabla(s)(X,Y) \quad \forall\, X,Y\in\mathfrak{X}(M), s\in\Gamma(E).$$

Proof. Everything in the statement is of local nature hence we can restrict to trivialisation charts. In other words, we may assume we have a frame $\{\mathfrak{e}_1,\dots,\mathfrak{e}_r\}$, and then ∇ is encoded by a connection matrix $(\omega_i^j)_{i,j}$. We compute d_∇^2 on $\mathfrak{e}_j$ using (2.1.16) and the Leibniz identity; we find

$$\begin{aligned}
d_\nabla\left(d_\nabla(\mathfrak{e}_j)\right) &= d_\nabla\left(\sum_{i=1}^r \omega_j^i\cdot\mathfrak{e}_i\right) = \sum_{i=1}^r d_\nabla\left(\omega_j^i\cdot\mathfrak{e}_i\right)\\
&= \sum_{i=1}^r \left(d(\omega_j^i)\cdot\mathfrak{e}_i - \omega_j^i\wedge d_\nabla\mathfrak{e}_i\right)\\
&= \sum_{i=1}^r d(\omega_j^i)\cdot\mathfrak{e}_i - \sum_{i=1}^r \omega_j^i\wedge d_\nabla\mathfrak{e}_i\\
&= \sum_{i=1}^r d(\omega_j^i)\cdot\mathfrak{e}_i - \sum_{i=1}^r\sum_{l=1}^r \omega_j^i\wedge\omega_i^l\cdot\mathfrak{e}_l\\
&= \sum_{l=1}^r d(\omega_j^l)\cdot\mathfrak{e}_l - \sum_{i=1}^r\sum_{l=1}^r \omega_j^i\wedge\omega_i^l\cdot\mathfrak{e}_l\\
&= \sum_{l=1}^r \left(d\omega_j^l - \sum_{i=1}^r \omega_j^i\wedge\omega_i^l\right)\cdot\mathfrak{e}_l\\
&= \sum_{l=1}^r \left(d\omega_j^l + \sum_{i=1}^r\sum_{l=1}^r \omega_i^l\wedge\omega_j^i\right)\cdot\mathfrak{e}_l\\
&= \sum_{l=1}^r k_j^l\cdot\mathfrak{e}_l. \qquad (2.2.3)
\end{aligned}$$

It is hopefully clear (or a simple exercise) that this implies that claims in the statements. 😄

Exercise 2.45. Go back to exercise 2.12 and conclude that the corresponding local connection matrix (w.r.t. e) is given by

$$k = \frac{1}{(1+|z|^2)^2}d\bar{z}dz.$$

Remark 2.46 (A global approach). The previous lemma and the computation in the proof can be made also more precise and global- to the extent that we can basically say that "the curvature is precisely d_∇^2". This is explained below (since it is a bit more abstract, let me mention that it may be skipped at a first reading). To explain this, we first look more closely to elements

$$K\in\Omega^p(M,End(E)).$$

The wedge product formula (1.2.9) has a version when $\omega = K$ and $\eta\in\Omega^q(M,E)$:

$$(K\wedge\eta)(X_1,\dots,X_{p+q}) = \sum_\sigma sign(\sigma)K(X_{\sigma(1)},\dots,X_{\sigma(p)})(\eta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})),$$

Any such K induces a linear map

$$\hat{K}:\Omega^\bullet(M,E)\to\Omega^{\bullet+p}(M,E),\ \hat{K}(\eta) = K\wedge\eta.$$

For later use note also that the same formula for the wedge product has an obvious version also when applied to elements $K\in\Omega^p(M,End(E))$ and $K'\in\Omega^q(M,End(E))$, giving rise to operations

$$\wedge:\Omega^p(M,End(E))\times\Omega^q(M,End(E))\to\Omega^{p+q}(M,End(E)) \qquad (2.2.4)$$

which make $\Omega(M,End(E))$ into a (graded) algebra.

Exercise 2.47. Show that $\hat{K}$ is an endomorphism of the graded (left) $\Omega(M)$-module $\Omega(M,E)$ i.e., according to the graded sign rule (see the previous footnotes):

$$\hat{K}(\omega\wedge\eta)=(-1)^{pq}\omega\wedge K(\eta),$$

for all $\omega\in\Omega^q(M)$. Moreover, the correspondence $K\mapsto\hat{K}$ defines a bijection

$$\Omega^p(M,End(E))\cong End^p_{\Omega(M)}(\Omega(M,E))$$

between $\Omega^p(M,End(E))$ and the space of all endomorphisms of the graded (left) $\Omega(M)$-module $\Omega(M,E)$ which rise the degree by p.

Finally, via this bijection, the wedge operation (2.2.4) becomes the composition of operators, i.e.

$$\widehat{K\wedge K'}=\hat{K}\circ\hat{K}'$$

for all $K,K'\in\Omega(M,End(E))$.

Due to the previous exercise, we will tacitly identify the element K with the induced operator m_K. For curvature of connections we have

Proposition 2.48. *If ∇ is a connection on E, then*

$$d^2_\nabla=d_\nabla\circ d_\nabla:\Omega^\bullet(M,E)\to\Omega^{\bullet+2}(M,E)$$

is given by

$$d^2_\nabla(\eta)=k_\nabla\wedge\eta$$

for all $\eta\in\Omega^*(M;E)$, and this determines k_∇ uniquely.

Proof. First of all, d_∇ is $\Omega(M)$-linear: for $\omega\in\Omega^p(M)$ and $\eta\in\Omega^p(M,E)$,

$$\begin{aligned}d^2_\nabla(\omega\wedge\eta)&=d_\nabla(d(\omega)\wedge\eta+(-1)^p\omega\wedge d_\nabla(\eta)\\&=[d^2(\omega)\wedge\eta+(-1)^{p+1}d(\omega)\wedge d_\nabla(\eta)]+(-1)^p[d(\omega)\wedge d_\nabla(\eta)+(-1)^p\omega\wedge d^2_\nabla(\eta)\\&=\omega\wedge d_\nabla(\eta).\end{aligned}$$

Hence, by the previous exercise, it comes from multiplication by an element $k\in\Omega^2(M)$. Using the explicit Koszul-formula for d_∇ to compute d^2_∇ on $\Gamma(E)$, we see that $d^2_\nabla(s)=k_\nabla\wedge s$ for all $s\in\Gamma(E)$. We deduce that $k=k_\nabla$. 😄

2.2.4 More on connection and curvature matrices; the first Chern class

Brief recap: Recall that, given a connection ∇ on a vector bundle E, ∇ is locally determined by its connection matrices. More precisely, for any local frame

$$\mathfrak{e}=\{\mathfrak{e}_1,\dots,\mathfrak{e}_r\}$$

over some open $U\subset M$, one can write

$$\nabla_X(\mathfrak{e}_j)=\sum_{i=1}^r\omega^i_j(X)\mathfrak{e}_i$$

and ∇ is uniquely determined by the matrix

$$\omega=\omega(\nabla,\mathfrak{e}):=(\omega^j_i)_{i,j}\in\mathscr{M}_r(\Omega^1(U)).$$

Similarly, one talks about the connection matrix

$$k = k(\nabla, \mathfrak{e}) := (k^i_j)_{i,j} \in \mathscr{M}_r(\Omega^2(U)),$$

which describes the curvature of ∇ w.r.t. the local frame $\mathfrak{e}$ by

$$k(X,Y)(\mathfrak{e}_j) = \sum_{i=1}^{r} k^i_j(X,Y)\mathfrak{e}_i.$$

As we have mentioned already, the curvature matrix can be obtained from the connection matrix by the formula

$$k = d\omega + \omega \wedge \omega,$$

where $\omega \wedge \omega$ is the matrix of 2-forms given by

$$(\omega \wedge \omega)^i_j = \sum_k \omega^i_k \wedge \omega^k_j.$$

Here we would like to see how these matrices change when one changes the (local) frames.

Lemma 2.49. *Let ∇ be a connection on E and let*

$$\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}, \quad \mathfrak{e}' = \{\mathfrak{e}'_1, \ldots, \mathfrak{e}'_r\}$$

be two local frames of E over two opens U and U', respectively, with corresponding local connection matrix and curvature matrix ω and k, and ω' and k', respectively.
We also consider the transition function from the local frame $\mathfrak{e}$ to $\mathfrak{e}'$, i.e., the matrix

$$g = (g^j_i) \in \mathscr{M}_r(C^\infty(U \cap U'))$$

defined by the condition that $\mathfrak{e}'_i = \sum_{j=1}^r g^j_i \mathfrak{e}_j$, on $U \cap U'$. Then, on this overlap,

$$\omega' = g^{-1} \cdot (dg) + g^{-1} \cdot \omega \cdot g, \quad k' = g^{-1} \cdot k \cdot g.$$

Proof. Using formula (2.1.16) for d_∇ we have:

$$\begin{aligned} d_\nabla(\mathfrak{e}'_i) &= d_\nabla\left(\sum_l g^l_i \mathfrak{e}_l\right) \\ &= \sum_l d(g^l_i)\mathfrak{e}_l + \sum_{l,m} g^l_i \theta^m_l \mathfrak{e}_m, \end{aligned}$$

where for the last equality we have used the Leibniz rule and the formulas defining θ. Using the inverse matrix $g^{-1} = (\overline{g}^i_j)_{i,j}$ we change back from the frame $\mathfrak{e}$ to $\mathfrak{e}'$ by $\mathfrak{e}_j = \sum_i \overline{g}^i_j \omega_i$ and we obtain

$$d_\nabla(\mathfrak{e}'_i) = \sum_{l,j} d(g^l_i)\overline{g}^j_l \mathfrak{e}'_j + \sum_{l,m,j} g^l_i \theta^m_l \overline{g}^j_m \mathfrak{e}'_j.$$

Hence

$$(\theta')^j_i = \sum_l d(g^l_i)\overline{g}^j_l + \sum_{l,m} g^l_i \theta^m_l \overline{g}^j_m,$$

i.e. the first formula in the statement. To prove the second equation, we will use the formula (refinvariance) which expresses k in terms of θ. We have

$$d\theta' = d(g^{-1}dg + g^{-1}\theta g) = d(g^{-1})d(g) + d(g^{-1})\theta g + g^{-1}d(\theta)g - g^{-1}\theta d(g). \tag{2.2.5}$$

For $\theta' \wedge \theta'$ we find

$$g^{-1}dg \wedge g^{-1}d(g) + g^{-1}dg \wedge g^{-1}\theta g + g^{-1}\theta g \wedge g^{-1}dg + g^{-1}\theta g \wedge g^{-1}\theta g.$$

Since

$$g^{-1}dg = d(g^{-1}g) - d(g^{-1})g = -d(g^{-1})g,$$

the expression above equals to

$$-d(g^{-1}) \wedge d(g) - d(g^{-1}) \wedge \theta g + g^{-1}\theta \wedge d(g) + g^{-1}\omega \wedge \omega g.$$

Adding up with (2.2.5), the first three term cancel and we are left with:

$$k' = d\omega' + \omega' \wedge \omega' = g^{-1}(d\omega + \omega \wedge \omega)g = g^{-1}kg. \quad 😄$$

Exercise 2.50. Show that the formula $k' = g^{-1}kg$ is also a direct consequence of the fact that k_∇ is defined as a global section of $\Lambda^2 T^*M \otimes End(E)$.

Since the curvature matrix does not depend on the local frame "up to conjugation", it follows that any expression that is invariant under conjugation will produce a globally defined form on M. The simplest such expression is obtained by applying the trace:

$$Tr(k) = \sum_i k_i^i \in \Omega^2(U);$$

indeed, it does have the fundamental property that

$$Tr(gkg^{-1}) = Tr(k).$$

It follows immediately that, if k' corresponds to another local frame e' over U', then $Tr(k) = Tr(k')$ on the overlap $U \cap U'$. Hence all these pieces glue to a global 2-form on M. We obtain a 2-form

$$\boxed{Tr(k_\nabla) \in \Omega^2(M)} \tag{2.2.6}$$

with the property that, for any local frame e of E over some open U,

$$Tr(k_\nabla)|_U = Tr(k) \in \Omega^{2p}(U),$$

where $k \in M_r(\Omega^2(U))$ is the connection matrix of ∇ with respect to e.

This construction can be looked at a bit differently: for any finite dimensional vector space V one can talk about the trace map

$$Tr : \mathrm{End}(V) \to \mathbb{R} \text{ (or } \mathbb{C} \text{ in the complex case)}$$

defined on the space $\mathrm{End}(V)$ of all linear maps from V to itself; applying this to the fibers of E we find a trace map

$$Tr : \Omega^2(M, \mathrm{End}(E)) \to \Omega^2(M), \tag{2.2.7}$$

so that (2.2.6) is indeed obtained from k_∇ by applying Tr. Of course, $\Omega^\bullet(M)$ denotes the space of differential forms with coefficients in the field of scalars ($\mathbb{R}$ or $\mathbb{C}$).

Theorem 2.51. *For any vector bundle E over M,*

1. *$Tr(k_\nabla) \in \Omega^2(M)$ is a closed differential form.*
2. *The cohomology class of $Tr(k_\nabla)$ only depends on the vector bundle E and not on the connection ∇ .*
3. *In the real case, the cohomology class of $Tr(k_\nabla)$ is actually zero.*
4. *In the complex case,*
$$c_1(E) := \left[\frac{1}{2\pi i} Tr(k_\nabla)\right] \in H^2(M,\mathbb{C})$$
is actually a real cohomology class.
5. *If E and E' are isomorphic complex vector bundles over M then $c_1(E) = c_1(E')$.*
6. *The Chern class is compatible with the basic operations on vector bundles; more precisely:*
   (a) *conjugation: $c_1(\overline{E}) = -c_1(E)$.*
   (b) *direct sum: $c_1(E \oplus E') = c_1(E) + c_1(E')$.*
   (c) *tensor products: $c_1(E \otimes E') = rank(E) \cdot c_1(E') + rank(E') \cdot c_1(E)$.*
   (d) *pull-backs: $c_1(f^*E) = f^* c_1(E)$ whenever $f : N \to M$ is a smooth map and $E \in \mathrm{Vect}_{\mathbb{C}}(M)$.*

The very last item says that the first Chern class of the pull-back f^*E is obtained by applying to $c_1(E) \in H^2(M)$ the pull-back map in cohomology, $f^* : H^2(M) \to H^2(N)$.

Definition 2.52. The **first Chern class** of a complex vector bundle E over M is the DeRham cohomology class $c_1(E)$ ensured by the previous theorem.

Hence this construction can be seen as a map

$$c_1 : \mathrm{Vect}_{\mathbb{C}}(M) \to H^2(M).$$

Notice that the naturality property from the last item can be described by a commutative diagram:

$$\begin{array}{ccc} \mathrm{Vect}_{\mathbb{C}}(M) & \xrightarrow{c_1} & H^2(M) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{Vect}_{\mathbb{C}}(N) & \xrightarrow{c_1} & H^2(N) \end{array}$$

Proof. We will use the following fundamental property of the trace map:

$$Tr([A,B]) = 0$$

for any two matrices with scalar entries, $A, B \in \nabla \times \nabla$, where $[A,B] = AB - BA$. The same formula continues to hold for matrices with entries differential forms on M,

$$A \in \mathscr{M}_{r\times r}(\Omega^p(M)), B \in \mathscr{M}_{r\times r}(\Omega^q(M))$$

provided we use the graded commutators:

$$[A,B] = AB - (-1)^{pq} BA$$

(check this!). The property that $Tr(k_\nabla)$ is closed is a local property, hence can be checked locally- where we can use local frames and the associated connection and curvature matrices. Then, using $k = d\omega + \omega \wedge \omega$ and $d^2 = 0$ we find

$$dk = d(\omega) \wedge \omega - \omega \wedge d(\omega);$$

writing $d\omega = k - \omega \wedge \omega$ we find

$$dk = k \wedge \omega - \omega \wedge k = [k, \omega],$$

therefore, using the fundamental property of the trace,

$$dTr(k) = Tr(dk) = Tr([k, \omega]) = 0.$$

This closes the proof of 1. To check that the class does not depend on ∇, we assume that we have another connection ∇'. With respect to a (any) local frame over some U we obtain two connection matrices ω and ω' and similarly two curvature matrices. Consider

$$\alpha := \omega' - \omega \in \mathscr{M}_{r\times r}(\Omega^1(U)).$$

Note that the previous lemma implies that, if one changes the local frame by another one then α changes in a similar way as k: by conjugation by g. In particular, the resulting 1-forms $Tr(\alpha) \in \Omega^1(U)$ will glue to a globally defined 1-form

$$Tr(\alpha) \in \Omega^1(M).$$

(as for the curvature, this form can be interpreted more directly, but a bit more abstractly: note that the expression $\nabla'_X(s) - \nabla_X(s) \in \Gamma(E)$ is $C^\infty(M)$-linear in all entries, hence it defines a tensor, usually denoted

$$\nabla' - \nabla \in \Gamma(T^*M \otimes \mathrm{End}(E)) = \Omega^1(M, \mathrm{End}(E)).$$

With this, the previous form is just $Tr(\nabla' - \nabla)$).

We claim that $Tr(k'_{\nabla'}) - Tr(k_\nabla) = d\alpha$. Again, this can be checked locally, using a local frame and the resulting matrices. Using $\omega' = \omega + \alpha$ we compute k' and we find

$$k' = k + d\alpha + \alpha \wedge \omega + \omega \wedge \alpha + \alpha \wedge \alpha.$$

This can be re-written using the graded commutators as

$$k' = k + d\alpha + [\alpha, \omega] + \frac{1}{2}[\alpha, \alpha].$$

Applying the trace and its fundamental property we find that, indeed,

$$Tr(k') = Tr(k) + d\alpha.$$

This closes the proof of 2.

We now prove item 5. Therefore, assume that $\Phi : E' \to E$ is an isomorphism of vector bundles. On E we choose an arbitrary connection but then, on E', we build one by transporting ∇ along Φ:

$$\nabla'_X(s') := \Phi^{-1}(\nabla_X(\Phi(s)).$$

We claim that $Tr(k_{\nabla'}) = Tr(k_\nabla)$. This can be checked locally where, again, while we use an arbitrary local frame $\{e_1, \ldots, e_r\}$ for E, for E' we use $e'_j := \Phi^{-1}(e_j)$. It then follows right away that the corresponding connection matrices ω and ω' coincide, and this implies the claim made in 5.

Next, we look at the operations described in 6. The main point is that any of the operations on vector bundles that we considered can be complemented by similar operations on connections. For instance, for the pull-backs (relevant to item (d)), the pull-back connection was introduced with Proposition 2.13 and its curvature was derived in Corollary 2.39. Those clearly imply item (d). Similarly for the other operations- see Exercise 2.7 with the curvature expressions treated in Exercise 2.37. In the case of sums one also needs to notice that, for any two finite dimensional vector spaces V and V',

$$Tr : \mathrm{End}(V \oplus V') \to \mathbb{C}$$

has the property that $Tr(A \oplus A') = Tr(A) + Tr(A')$ for any $A \in \mathrm{End}(V)$, $A' \in \mathrm{End}(V')$, while for tensor products to observe that $Tr(Id_V) = \dim(V)$ and

$$Tr : \mathrm{End}(V \otimes V') \to \mathbb{C}$$

satisfies $Tr(A\otimes B) = Tr(A)\cdot Tr(B)$.

Of course, one can also proceed more "down to earth", using local frames and actual matrices. E.g., for direct sums, choosing local frames e and e' for E and E' (over some U) and computing the connection matrix corresponding to the resulting local frame $e\oplus e'$ we find the direct sum of the connection matrices of ∇ and ∇':

$$(\omega\oplus\omega') = \begin{pmatrix}\omega & 0\\ 0 & \omega'\end{pmatrix} \in \mathscr{M}_{r+r',r+r'}(\Omega^1(U)) \tag{2.2.8}$$

and similarly for the curvature matrices; applying the trace, we find that

$$Tr(k_{\nabla\oplus\nabla'}) = Tr(k_\nabla) + Tr(k_{\nabla'}). \tag{2.2.9}$$

Next, although it is not explicitly stated in the theorem, it is interesting to also consider the case of duals: start with ∇ on E and consider the dual connection ∇^* as in the first item of Exercise 2.7. Locally, while for E we use arbitrary local frames $\mathfrak{e} = \{e_1,\dots,e_r\}$, for E^* we use the dual local frames $\mathfrak{e}^* = \{e^1,\dots,e^r\}$. Writing out the definition of the connection matrices we find that $\omega(\nabla^*,e^*) = -\omega(\nabla,e)^T$ (where T stands for the transpose). It then follows that the same holds for the curvatures so that:

$$[Tr(k_{\nabla^*})] = -[Tr(k_\nabla)].$$

When E is real, since $E^*\cong E$, item 5 (the version over $\mathbb{R}$) implies right away item 3. On the other hand, when E is complex, then $E^*\cong\overline{E}$ and what this formula gives us is

$$\overline{c_1(E)} = c_1(E),$$

where the minus sign has disappeared precisely because of the factor i, whose conjugate is $-i$. This proves item 4.

We are left with item (a) in 6. In some sense that is even simpler, because a connection ∇ on E serves also as one on $\overline{E}$. The only difference in computing $c_1(\overline{E})$ comes from the way one computes the trace $Tr_V : \mathrm{End}(V)\to\mathbb{C}$ for a finite dimensional vector space, when one replaces V by its conjugate $\overline{V}$; what happens is that

$$Tr_{\overline{V}}(A) = \overline{Tr_V(A)}.$$

Due also to the presence of the $2\pi i$ we find that

$$c_1(\overline{E}) = \left[\frac{1}{2\pi i}\overline{Tr(k_\nabla)}\right] = -\overline{\left[\frac{1}{2\pi i}Tr(k_\nabla)\right]} = -\overline{c_1(E)}.$$

Since $c_1(E)$ is a real class, we deduce that $c_1(\overline{E}) = -c_1(E)$. 😀

Exercise 2.53. Consider the tautological line bundle $L = L_{\mathbb{CP}^1}$ over $\mathbb{CP}^1$. Using the computation from Exercise 2.45 show that

$$\int_{\mathbb{CP}^1} c_1(L) = -1.$$

Deduce that L is not trivializable, is not isomorphic to its conjugate and is not the complexification of any real vector bundle.

Exercise 2.54. Generalise the previous exercise to the case of the tautological line bundle $L = L_{\mathbb{CP}^n}$ over $\mathbb{CP}^n$. This time,

$$c_1(L)\in H^2(\mathbb{CP}^n)$$

cannot be integrated over $\mathbb{CP}^n$, but its n-th power can! Compute the integral and deduce that $c_1(L)$ and the same conclusions about L as above.

2.3 Connections compatible with an inner product

Very often the vector bundles one deals with come with extra structure and one is interested in connections that preserve that extra structure. A good illustration of this principle is that of vector bundles endowed with inner products.

2.3.1 Connections compatible with an inner product

Fixing now a vector bundle E together with an inner product g, there are various ways to make sense of the fact that a connection is compatible with g.

Proposition 2.55. *Given a vector bundle E endowed with an inner product g, for any connection ∇ on E the following are equivalent:*

1. *Global: for any path $\gamma : [0,1] \to M$ between x and y, the induced parallel transport $T_\gamma^{0,1}$ is an isometry*

$$T_\gamma^{0,1} : (E_x, g_x) \to (E_y, g_y).$$

2. *Infinitesimal: the following Leibniz identity holds for any $s, s' \in \Gamma(E)$ and $X \in \mathscr{X}(M)$:*

$$L_X(g(s,s')) = g(\nabla_X(s), s') + g(s, \nabla_X(s')).$$

3. *Local: for any local orthonormal frame e, the connection matrix $\omega = \omega(\nabla, e)$ is anti self-adjoint:*

$$\overline{\omega_i^j} = -\omega_j^i,$$

where the over-line denotes complex conjugation (hence, in the real case, this just means that the matrix is antisymmetric).

A connection is said to be compatible with g if it satisfies one of these equivalent conditions.

Proof. Since condition 2. is local and since one can always choose local orthonormal frames (see the exercise above), it is enough to require 2. on members of local orthonormal frames- and then one arrives at the equivalence of 2. with 3. For the equivalence with 1, note first that the condition in 2. can be re-written in terms of the derivatives $\frac{\nabla}{dt}$ induced by ∇ as follows: for any path γ in M and for any two paths $u, u' : I \to M$ above γ,

$$\frac{d}{dt} g(u,u') = g\left(\frac{\nabla u}{dt}, u'\right) + g\left((u, \frac{\nabla u'}{dt}\right).$$

Choosing $u(t) = T_\gamma^{0,t}(v)$, $u'(t) = T_\gamma^{0,t}(v')$, with $v, v' \in E_x$ ($x = \gamma(0)$), we find that $g(u,u')$ is constant hence, in particular, it takes the same values at 1 and 0:

$$g\left(T_\gamma^{0,1}(v), T_\gamma^{0,1}(v')\right) = g(v,v');$$

hence $T_\gamma^{0,1}$ is an isometry. Moreover, one can check that the last argument can be reversed to prove that 1. implies 2. (exercise!). 😀

Proposition 2.56. *Any vector bundle E admits an inner product g; for any g there exists a connection ∇ on E compatible with g.*

Proof. The existence of inner products was the subject of Lemma 1.11. Assume now that we have an arbitrary inner product g. Choose again a partition of unity but, this time, with the property that, over each U_i, one finds a local frame that is orthonormal with respect to g. Each such local frame gives rise to a connection on U_i, denoted ∇^i, which is clearly compatible with g. Define then the global connection ∇ given by

$$\nabla_X(s)(x) := \sum_i \eta_i(x)\nabla^i_{X|_{U_i}}(s|_{U_i})(x).$$

It is not difficult to see that this defines indeed a connection and a computation of $g(\nabla_X(s),s') + g(s,\nabla_X(s'))$ using the fact that each ∇^i is compatible with g (over U_i) and using that $\sum \eta_i = 1$ leads to $L_X g(s,s')$, i.e. the compatibility of ∇ with g. 😄

Exercise 2.57. Return to Theorem 2.51. Using the fact that the cohomology class of $[Tr(k_\nabla)]$ is independent of ∇ and the existence of inner products and compatible connections, find now a new proof of parts 3 and 4 of the theorem.

Exercise 2.58. Let F be a complex vector bundle over M. Show that if F can be written as the complexification of a real vector bundle over M, then $c_1(F) = 0$.

2.3.2 *Riemannian manifolds I: The Levi-Civita connection*

Recall that a Riemannian structure on a manifold M is an inner product g on its tangent bundle TM (cf. Definition 1.43). From the previous subsections we know that given a Riemannian manifold (M,g), there exist connections on TM compatible with g. However, working with TM instead of general vector bundles is a bit more special. First of all, $\Gamma(TM) = \mathscr{X}(M)$, so that a connection on TM is an operator

$$\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M).$$

Because of this, one can talk about the torsion of a connection on TM:

Definition 2.59. Given a connection ∇ on TM, the **torsion of** ∇ is the map

$$T_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad T_\nabla(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$

The connection is called **torsion free** if $T_\nabla = 0$.

The following shows the tensorial nature of the torsion.

Proposition 2.60. *For any connection ∇ on TM, $T_\nabla(X,Y)$ is $C^\infty(M)$-linear in the entries $X,Y \in \mathscr{X}(M)$. Hence the torsion can also be interpreted as a 2-form with coefficients,*

$$T_\nabla \in \Gamma(\Lambda^2 T^*M \otimes TM) = \Omega^2(M;TM),$$

This follows by a computation completely similar to (but simpler than) the one from the proof of Proposition 2.36; we leave it as an exercise.

Theorem 2.61. *For any Riemannian manifold (M,g) there exists a unique connection ∇ on TM which is compatible with g and which is torsion free. It is called the* ***Levi-Civita connection of*** *(M,g).*

Proof. This proof here (which is the standard one) is not very enlightening (hopefully we will see a much more transparent/geometric argument a bit later in the course). One just plays with the compatibility equations applied to (X,Y,Z) and their cyclic permutations, combined in such a way that most of the appearances of the ∇ disappear by using the torsion free condition. One ends up with the following identity (which, once written down, can also be checked directly):

$$2g(\nabla_X(Y),Z) = L_X(g(Y,Z)) + L_Y(g(X,Z)) - L_Z(g(X,Y)) +$$

$$+ g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X). \tag{2.3.1}$$

Hence, fixing X and Y, we see that for the expression we are looking for, $\nabla_X(Y)$, the product with respect to any Z is predetermined. Since g is an inner product, this forces the definition of $\nabla_X(Y)$ pointwise. It remains to check the identities that ensure that ∇ is a torsion free connection that is compatible with g; each such identity can be written as $E = 0$ for some expression E; to prove such an identity one proves instead that $g(E,V) = 0$ for all vector fields V; with this trick one can use the definition of ∇ and the computations become straightforward. 😄

Remark 2.62 (The Christoffel symbols). While, locally, connections ∇ on a vector bundle E are determined by connection matrices,

$$\nabla_X(\mathfrak{e}_j) = \sum_i \omega_j^i(X), \quad (\mathfrak{e} = \{\mathfrak{e}_1,\dots,\mathfrak{e}_r\} - \text{local frame of } E)$$

when looking at connections on $E = TM$, it is natural to use local frames

$$\frac{\partial}{\partial \chi^1},\dots,\frac{\partial}{\partial \chi^n}$$

induced by coordinate charts

$$(U, \chi = (\chi^1,\dots,\chi^n))$$

of M. Then the resulting 1-forms can be further expressed in terms of functions:

$$\omega_j^i = \sum_p \Gamma_{pj}^i \cdot d\chi^p,$$

where Γ_{pj}^i are smooth functions on U. These functions are called the **Christoffel symbols** of the Riemannian manifold (M,g) in the chart (U,χ); hence they are defined by the condition"

$$\boxed{\nabla_{\frac{\partial}{\partial \chi^p}}\left(\frac{\partial}{\partial \chi^j}\right) = \sum_i \Gamma_{pj}^i \frac{\partial}{\partial \chi^i}.}$$

Locally, the fact that ∇ is torsion free is equivalent to

$$\Gamma_{pj}^i = -\Gamma_{jp}^i$$

for all i,j and p. While an inner product g on TM is locally determined by the functions

$$g_{ij} := g\left(\frac{\partial}{\partial \chi^i}, \frac{\partial}{\partial \chi^j}\right),$$

the equation (2.3.1) that expresses the compatibility of ∇ with g becomes:

$$2\sum_i \Gamma_{pj}^i g_{ik} = \frac{\partial g_{jk}}{\partial x_p} + \frac{\partial g_{pk}}{\partial x_j} - \frac{\partial g_{pj}}{\partial x_k}.$$

Using the inverse (g^{ij}) of the matrix (g_{ij}) we find the following explicit formula for the Christoffel symbols in the chart (U,χ):

$$\Gamma^i_{pj} = \frac{1}{2}\sum_k \left\{ \frac{\partial g_{jk}}{\partial x_p} + \frac{\partial g_{pk}}{\partial x_j} - \frac{\partial g_{pj}}{\partial x_k} \right\} g^{ki}.$$

Exercise 2.63. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, denote by $\widetilde{\nabla}$ the corresponding Levi-Civita connection and by $\frac{\widetilde{\nabla}}{dt}$ the corresponding derivatives of paths in $T\widetilde{M}$. Assume now that $M \subset \widetilde{M}$ is a compact submanifold, we endow M with the Riemannian structure g obtained by restricting $\widetilde{g}$ to (the tangent spaces of) M and we consider the corresponding Levi-Civita connection ∇ on TM, and the operation $\frac{\nabla}{dt}$. Show that:

1. ∇ coincides with the projection onto TM, via the orthogonal projection $\mathrm{pr}_g : T\widetilde{M}|_M \to TM$ induced by g, of the restriction of $\widetilde{\nabla}$ to M, similar to what we have seen in Exercise 2.5.
2. for any path $u : I \to TM$, one has
$$\frac{\nabla u}{dt} = \mathrm{pr}_g\left(\frac{\widetilde{\nabla} u}{dt}\right).$$

Exercise 2.64. We endow S^2 with the Riemannian metric induced from $\mathbb{R}^3$, i.e., we (can) apply Exercise 2.63 to $M = S^2$, $\widetilde{M} = \mathbb{R}^3$ with the standard Riemannian structure.

1. deduce that a path $u(t) = f(t)\frac{\partial}{\partial x} + g(t)\frac{\partial}{\partial y} + h(t)\frac{\partial}{\partial z}$ in TS^2 sitting above a path $\gamma = (x(t), y(t), z(t))$ in S^2 is horizontal w.r.t. ∇ if and only of
$$\dot{f} = -x(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{g} = -y(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{h} = -z(\dot{x}f + \dot{y}g + \dot{z}h).$$
2. deduce that for any such horizontal u, its norm $||u|| = \sqrt{\langle u, u\rangle}$ is constant.

2.3.3 *Riemannian manifolds II: geodesics and the exponential map*

We have seen that the case of connections on the tangent bundle TM of a manifold M is more special: unlike the case of connections on general vector bundles, we could talk about the torsion of a connection; the vanishing of the torsion, together with the compatibility with a given inner product g on TM (i.e. a Riemannian structure on M) were conditions that implied that a Riemannian manifold (M, g) comes with a canonical connection on TM.

Similarly, while a connection ∇ on a vector bundle π over M allows us to talk about the derivative

$$\frac{\nabla u}{dt} : I \to E$$

of a path $u : I \to E$, sitting above some path $\gamma : I \to M$, in the case when $E = TM$ there is a canonical path $u_\gamma : I \to TM$ sitting above any γ: its usual derivative. In particular, one can talk about the second order derivative of $\gamma : I \to M$ (with respect to a connection ∇ on TM) defined as

$$\frac{\nabla(\dot{\gamma})}{dt} : I \to TM.$$

In particular, given a Riemannian manifold (M, g), one obtains a canonical way to talk about second derivatives of paths in M; in this situation we will use the notation

$$\boxed{\frac{D^2\gamma}{dt^2} = \frac{\nabla(\dot{\gamma})}{dt} : I \to M.}$$

Definition 2.65. Given a Riemannian manifold (M,g), a **geodesic** of (M,g) is any path $\gamma : I \to M$ with the property that $\frac{D^2\gamma}{dt^2} = 0$.

The interesting existence question for geodesics is: given any $x \in M$ (read: initial point) and $v \in T_xM$ (read: initial speed), can one find a geodesic $\gamma : I \to M$ such that $\gamma(0) = x$, $\dot\gamma(0) = v$ (where I is an interval containing 0)? Is it unique? Is there a maximal one? I.e. the same questions as for flows as vector fields; and the answers will be very similar as well. Actually, to obtain the desired statements, we just have to realize that the geodesic equations can be written as flow equations.

Proposition 2.66. *For any Riemanninan manifold (M,g), there exists a unique vector field $\mathscr{X}$ on TM such that the integral curves of $\mathscr{X}$ are precisely those of type*

$$u_\gamma(t) = (\gamma(t), \dot\gamma(t)),$$

with γ a geodesic of (M,g); in other words, $\gamma \longleftrightarrow u_\gamma$ defines a bijection

$$\left\{\begin{matrix}\text{geodesics}\\ \text{of } (M,g)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{integral curves}\\ \text{of } \mathscr{X}\end{matrix}\right\}.$$

Proof. We first look at the geodesic equations locally, in a coordinate chart (U,χ); using the description of ∇ in terms of the Christoffel symbols, we see that $\gamma = (\gamma^1,\dots,\gamma^n)$ is a geodesic if and only if

$$\frac{d^2\gamma^k}{dt^2} = -\sum_{i,j}\Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}.$$

The idea is to write this system as a first order system:

$$\begin{cases}\frac{d\gamma^k}{dt} = & y^k(t),\\ \frac{dy^k}{dt} = & -\sum_{i,j}\Gamma^k_{ij}(\gamma(t))y^i(t)y^j(t)\end{cases} \tag{2.3.2}$$

Let us formalize this. First, (U,χ) induces the chart $(TU,(\chi,y_\chi))$ for TM:

$$TU \ni \left(x, y^1\frac{\partial}{\partial\chi^1} + \dots + y^n\frac{\partial}{\partial\chi^n}\right) \overset{(\chi,y_\chi)}{\longmapsto} (\chi(x), y^1,\dots,y^n) \in \mathbb{R}^{2n}.$$

It is now clear that, in these coordinates,

$$\mathscr{X} := \sum_k y^k_\chi\frac{\partial}{\partial\chi^k} - \sum\Gamma^k_{ij}y^i_\chi y^j_\chi\frac{\partial}{\partial y^k_\chi}$$

will have the desired properties. Since the geodesic equation is global and the local integral curves of a vector field determines the vector field, it follows that different charts (U',χ') give rise to vector fields $\mathscr{X}'$ on TU' which agree with $\mathscr{X}$ on $TU \cap TU'$. Therefore $\mathscr{X}$ is globally defined and has the desired properties. 😄

The vector field that arises from the previous proposition is called the **geodesic vector field of** (M,g). Using the properties of flows of vector fields applied to $\mathscr{X}$, we find an open subset

$$\mathscr{D} \subset \mathbb{R}\times TM \ \ (\text{containing } \{0\}\times TM)$$

(the domain of the geodesic flow) and a smooth map (the geodesic flow)

$$\gamma : \mathscr{D} \to M$$

so that, for any $(x,v) \in TM$,

$$\boxed{\gamma_{x,v}(t) := \gamma(t,x,v)}$$

is the maximal geodesic with

$$\boxed{\gamma_{x,v}(0) = x,\ \dot{\gamma}_{x,v}(0) = v.}$$

It is defined on the open interval (!)

$$I_{x,v} = \{t \in \mathbb{R} : (t,x,v) \in \mathscr{D}\}.$$

Exercise 2.67. With the notations above show that, for any $c > 0$, one has

$$I_{x,cv} = \frac{1}{c} I_{x,v} \ \text{ and } \ \gamma\left(\frac{1}{c}t, x, cv\right) = \gamma(t,x,v) \ \text{ for all } t \in I_{x,v}.$$

The previous exercise indicates that the (local) information encoded in the geodesic flow is contained already in what it does at time $t = 1$. More precisely, one defines the **exponential map of** (M,g)

$$\boxed{\exp(x,v) := \gamma(1,x,v),}$$

which makes sense when (x,v) belongs to the open

$$\mathscr{U} := \{(x,v) \in TM : (1,x,v) \in \mathscr{D}\},$$

so that exp becomes a smooth map

$$\boxed{TM \supset \mathscr{U} \mapsto \exp M.}$$

With this, for all $(t,x,v) \in \mathscr{D}$, one has that $(x,tv) \in \mathscr{U}$ and

$$\boxed{\gamma(t,x,v) = \exp(x,tv).}$$

When one looks around a given point $x \in M$ it is common to consider

$$\exp_x : \mathscr{U}_x \to M,$$

the restriction of the exponential map to $\mathscr{U}_x = \mathscr{U} \cap TM$. And here is one of the fundamental properties of the exponential map:

Proposition 2.68. *For any $x \in M$,*

$$T_xM \supset \mathscr{U}_x \mapsto exp_xM$$

sends some open neighborhood of $0_x \in T_xM$ diffeomorphically into one of x in M. Actually the differential of exp_x at 0, combined with the canonical identification of $T_{0_x}(U_x)$ with T_xM, is the identity map.

Proof. By the inverse function theorem, it suffices to prove the last statement. Let $v \in T_xM$. Recall that the standard identification of T_xM with $T_{0_x}(U_x)$ interprets v as the speed at $t = 0$ of the curve $t \mapsto tv$. Therefore, it is sent by $(d\exp_x)$ to

$$\left.\frac{d}{dt}\right|_{t=0} \exp_x(tv) = \left.\frac{d}{dt}\right|_{t=0} \gamma(1,x,tv) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t,x,v) = v. \quad \text{😄}$$

Exercise 2.69. As a continuation to Exercise 2.64:

3. describe by explicit formulas all the geodesics of (S^2,g) that start at the point $(1,0,0) \in S^2$.

2.3.4 *Riemannian manifolds III: application to tubular neighborhoods*

And here is an important application of Riemannian metrics and of the exponential maps, application that is of interest outside Riemannian geometry: the existence of tubular neighborhoods.

The concept of tubular neighborhood makes sense whenever we fix a submanifold N of a manifold M; then one can talk about tubular neighborhoods of N in M which are, roughly speaking, "linear approximations of M around N". When N consists of a single point x, it is the tangent space T_xM that plays the role of such "linear approximation". The general idea is that, while the linear approximation of M around a point is a vector space, the linear approximation around a submanifold N will be a vector bundle over N.

Theorem 2.70. *Any embedded submanifold N of a manifold M admits a tubular neighborhood in M. More precisely, there exist an open $U \subset M$ containing N, a vector bundle E over N and a diffeomorphism*

$$\phi : E \to U \quad \textit{such that} \quad \phi(0_x) = x \ \ \forall\, x \in N.$$

Let us provide further insight into the theorem, by focussing on the two sides of ϕ: the submanifold N of M, and the vector bundle E over N, respectively.

Remark 2.71 (normal bundles). As we shall see below, the vector bundle in the definition is actually unique (up to isomorphisms), and canonically associated to the submanifold.

- The abstract normal bundle: Whenever $N \subset M$ is an embedded submanifold, there are two immediate vector bundles over N: the tangent bundle of N, and the restriction of the tangent bundle of M to N:

  $$TN \subset T_NM = TM|_N.$$

  The abstract normal bundle of N in M is the resulting quotient

  $$\nu_M(N) := T_NM/TN$$

  (a vector bundle over N!). For the general construction of quotients of vector bundles, please see subsection 1.2. Intuitively, the normal bundle reflects the way that N sits inside M.

- Concrete normal bundles: The abstract normal bundle is isomorphic to any complement of TN in T_NM, just that it makes sense without having to choose such a complement. Therefore, the choice of a complement can be seen as giving a concrete version of the normal bundle. In particular, whenever M is endowed with a Riemannian metric g, one obtains concrete model of the normal bundle

  $$(TN)^{\perp} := \{(x,v) \in TM : x \in N,\ g(v,w) = 0\ \forall\, w \in T_xN\}, \tag{2.3.3}$$

  isomorphic to the abstract one by the map $(x,v) \mapsto v \bmod T_xN$.

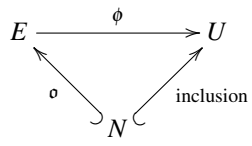
Remark 2.72 (zero sections). In general, whenever we have a vector bundle E over N, one can interpret N as a submanifold of E "as the zero-section" by identifying points in N with the zeros in the corresponding fibers:

$$N \ni x \equiv 0_x \in E_x \text{ the zero vector of the fiber } E_x.$$

With this identification in mind, the condition $\phi(0_x) = x$ in the definition of tubular neighborhoods reads: ϕ is the identity on N. More formally, one embeds N inside E via the zero-section

$$\mathfrak{o} : N \to E, \quad \mathfrak{o}(x) = 0_x,$$

and the condition $\phi(0_x) = x$ can be seen as ensuring a commutative diagram:



Exercise 2.73. Show that any section $s \in \Gamma(E)$ of a vector bundle E over N, when seen as a map $s : N \to E$, is an embedding of N in E.

Furthermore, for such embeddings $N \subset E$, the corresponding normal bundle is isomorphic to E itself. To see this, we notice there are two simple ways to produce tangent vectors to E at zeros:

- each $v \in E_x$ induces a tangent vector $\frac{d}{d}\big|_{t=0} tv \in T_{0_x}E$,
- each $w \in T_xM$ induces a tangent vector $(d\mathfrak{o})_x(w) \in T_{0_x}E$,

and, together, they fits into isomorphisms

$$E_x \oplus T_xN \xrightarrow{\sim} T_{0_x}E, \quad (v,w) \mapsto \frac{d}{d}\Big|_{t=0} tv + (d\mathfrak{o})_x(w). \tag{2.3.4}$$

Hence E appears as a concrete model for the normal bundle of N in E (just that, this time, because E was a vector bundle, this concrete model could be exhibited without extra-choices). As a curiosity, let us also point out the following consequence: any vector bundle over a manifold N can be realised as the normal bundle of N (inside a bigger manifold).

Exercise 2.74. Deduce now that, indeed, if E is as in the theorem, then E must be isomorphic (as a vector bundle over N) to the normal bundle of N in M.

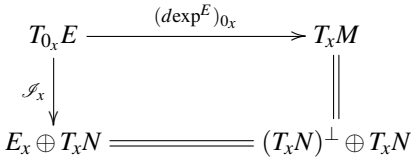
Proof (of Theorem 2.70, when N is compact (for simplicity)). Consider a Riemannian metric g on M and choose E to be the resulting concrete model (2.3.3):

$$E = (TN)^{\perp}$$

In principle, the desired diffeomorphism will be the exponential map restricted to E,

$$\exp^E : E \to M.$$

Strictly speaking, this is defined only on an open neighborhood of the zero section but, since all the arguments below are just around such "small enough" opens, we may assume that exp is defined on the entire E (just to simplify notations). Next, we claim that the differential of the exponential at zeros can be identified with the isomorphisms (2.3.4), denoted $\mathscr{I}_x$ below, in the sense that they fit in a commutative diagram:



We check the commutativity by going from the left bottom corner up, to the right and down to the right bottom corner:

- an element $v \in E_x$ is sent up to $\frac{d}{dt}tv\big|_{t=0} tv$ and then to

$$(d\exp^E)_{0_x}\left(\frac{d}{dt}\Big|_{t=0} tv\right) = \frac{d}{dt}\Big|_{t=0} \exp^E(tv) = v.$$

- an element $w \in T_xN$ is sent up to $(d\mathfrak{o})_x(w)$ and then to

$$(d\exp^E)_{0_x}(d\mathfrak{o})_x(w) = (d\exp^E \circ \mathfrak{o})_x(w) = w.$$

From the commutativity of the diagram we deduce that also the top horizontal map is an isomorphism; therefore, by the inverse function theorem, for each $x \in N$, one finds an open neighborhood V_x of 0_x in E on which $\exp^E$ becomes a diffeomorphism onto an open inside M. The different opens fit into a larger open

$$V = \cup_x V_x \subset E,$$

$\exp^E|_V$ is still a local diffeomorphism onto an open inside M, but there is no guarantee that it is injective. We now claim that there exists $\varepsilon > 0$ such that

$$B_\varepsilon := \{v \in E : ||v|| \leq \varepsilon\} \subset E$$

is inside V and $\exp^E|_{B_{\varepsilon_0}}$ is injective. The fact that $B_{\varepsilon_0} \subset V$ for some ε_0 is a version of the tube lemma from topology and we leave it as an exercise; it is here that the compactness of N is used (otherwise one has to allow ε to be a function). From now on we work only with εs that are smaller than ε_0 and we proceed by contradiction, assuming that there exists now $\varepsilon < \varepsilon_0$ such that $\exp^E|_{B_{\varepsilon_0}}$ is injective. Then we find

$$v_n, w_n \in B_{\frac{1}{n}} \text{ with } v_n \neq w_n,\ \exp^E(v_n) = \exp^E(w_n).$$

Since N is compact, so are the closed balls $\overline{B}_\varepsilon$ so, after eventually passing to convergent subsequences, we may assume that $v_n \to v$ and $w_n \to w$ with $v, w \in E$. Since $||v_n|| < \frac{1}{n}$, we find that $v = 0_x$ for some $x \in N$ and, similarly, $w = 0_y$. Passing to limit in $\exp^E(v_n) = \exp^E(w_n)$ we find that $x = y$. Therefore, both v_n as well as w_n converge to 0_x. But this is a contradiction with the fact that $\exp^E$ is a local diffeomorphism around 0_x.

Therefore the restriction of the exponential to some $B_\varepsilon \subset E$ is a diffeomorphism into an open inside M. To find a diffeomorphism defined on the entire E, one chooses any diffeomorphism $\phi : [0, \infty) \to [0, \varepsilon)$ which is the identity near $t = 0$, and one composes the exponential with the diffeomorhism

$$\hat{\phi} : E \to B_\varepsilon,\ \hat{\phi}(v) = \frac{\phi(||v||)}{||v||} \cdot v. \quad \text{😀}$$

2.4 Characteristic classes

2.4.1 The general construction

The first Chern class $c_1(E)$ is just one of the interesting invariants that one can associate to vector bundles and which can be used to distinguish vector bundle. At the heart of its construction was the trace map and its fundamental property that it is invariant under conjugation: $Tr(g\cdot k\cdot g^{-1}) = Tr(k)$ for any invertible matrix g. Generalizations of this can be obtained replacing the trace map $A \mapsto Tr(A)$ by $A \mapsto Tr(A^k)$ (for $k \geq 0$ any integer). The most general possibility is to work with arbitrary "invariant polynomials". The discussion applies both over $\mathbb{R}$ and $\mathbb{C}$, so let us denote by $\mathbb{F}$ any of the two and work over $\mathbb{F}$.

Definition 2.75. We denote by $I_r(\mathbb{F})$ the space of all functions

$$P : M_r(\mathbb{F}) \to \mathbb{F}$$

which are polynomial (in the sense that $P(A)$ is a polynomial in the entries of A), and which are invariant under the conjugation, i.e.

$$P(gAg^{-1}) = P(A)$$

for all $A \in M_r(\mathbb{F})$, $g \in GL_r(\mathbb{F})$.

Example 2.76. For each $p \geq 0$,

$$\Sigma_p : M_r(\mathbb{F}) \to \mathbb{F},\ \Sigma_p(A) = Tr(A^p)$$

is invariant. One can actually show that the elements with $0 \leq p \leq r$ generate the entire algebra $I_r(\mathbb{F})$: any $P \in I_r(\mathbb{F})$ is a polynomial combination of the Σ_p's. Even more, one has an isomorphism of algebras

$$I_r(\mathbb{F}) \cong \mathbb{F}[\Sigma_0, \Sigma_1, \ldots, \Sigma_p].$$

Example 2.77. Another set of generators are obtained using the polynomial functions

$$\sigma_p : M_r(\mathbb{F}) \to \mathbb{F}$$

defined by the equation

$$det(I + tA) = \sum_{p=0}^{r} \sigma_p(A) t^p.$$

For instance, $\sigma_1 = \Sigma_1$ is just the trace while $\sigma_p(A) = det(A)$. One can also prove that

$$I_r(\mathbb{F}) = \mathbb{F}[\sigma_0, \sigma_1, \ldots, \sigma_p].$$

Remark 2.78 (using the eigenvalues). For $P \in I_r(\mathbb{F})$, it is interesting to express $P(A)$ in terms of the eigenvalues of $x_1, \ldots, x_r$ of A (the invariance of P indicates that that is possible). For instance, for Σ_p from Example 2.76 one has

$$\Sigma_p(x_1, \ldots, x_r) = \sum_i x_i^p, \tag{2.4.1}$$

while for the σ_p from Example 2.76:

$$\sigma_p(x_1, \ldots, x_r) = \sum_{i_1 < \ldots < i_p} x_{i_1} \ldots x_{i_p}. \tag{2.4.2}$$

For general $P \in I_r(\mathbb{F})$, it can be viewed as a symmetric polynomial in $x_1, \ldots, x_r$ by evaluating it on diagonal matrices:

$$P(x_1,\dots,x_r) := P(diag(x_1,\dots,x_r)).$$

This interpretation gives rise to an isomorphism

$$I_r(\mathbb{F}) \cong Sym_{\mathbb{F}}[x_1,\dots,x_r]$$

between the algebra of invariant polynomials on $M_r(\mathbb{F})$ and the algebra of symmetric polynomials in r variables.

Remark 2.79 (Newton's formulas). The isomorphisms mentioned at the end of Examples 2.76 and 2.77 imply, in particular, that the Σ's can be expressed in terms of the σ's. With (2.4.1) and (2.4.2) at hand, it is not difficult now to find the precise formulas:

$$\Sigma_1 = \sigma_1, \quad \Sigma_2 = (\sigma_1)^2 - 2\sigma_2, \quad \Sigma_3 = (\sigma_1)^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \quad \dots$$

Let us now return to the construction of cohomology classes associated to vector bundles. In complete analogy with the trace (see the discussion proceeding Theorem 2.51), given an invariant polynomial $P \in I_r(\mathbb{F})$, of degree p, a vector bundle $E \in \mathrm{Vect}^r_{\mathbb{F}}(M)$ and a connection ∇ on E, there exists a unique differential form of degree $2p$,

$$P(E,\nabla) \in \Omega^{2p}(M)$$

with the property that, for any local frame e of E over some open U,

$$P(E,\nabla)|_U = P(k) \in \Omega^{2p}(U),$$

where $k \in M_r(\Omega^2(U))$ is the connection matrix of ∇ with respect to e. This can also be seen more globally, by first making sense of P as a map

$$\widetilde{P} : \Omega^2(M, \mathrm{End}(E)) \to \Omega^2 p(M)$$

as we did for (2.2.7) and then applying it to the curvature $k_\nabla \in \Omega^2(M, \mathrm{End}(E))$:

$$P(E,\nabla) = \widetilde{P}(k_\nabla) \in \Omega^{2p}(M).$$

The following summarizes the construction of the characteristic classes.

Theorem 2.80. *Let $P \in I_r(\mathbb{F})$ be an invariant polynomial of degree p. Then for any vector bundle $E \in \mathrm{Vect}^r_{\mathbb{F}}(M)$ and any connection ∇ on E, $P(E,\nabla)$ is a closed form and the resulting cohomology class*

$$P(E) := [P(E,\nabla)] \in H^{2p}(M)$$

does not depend on the choice of the connection ∇. It is called the P-characteristic class of E.

This theorem can be proven directly, using local connection matrices, similarly to the proof of Theorem 2.51. Here is a more global proof, which is more conceptual (and can be generalised to other settings) but which uses a heavier algebraic language.

Proof. Notice first that we may assume that $P = \Sigma_p$ for some p- that follows from the fact that these polynomials generate $I_r(\mathbb{F})$ as an algebra, and the fact that the construction

$$I_r(\mathbb{F}) \ni P \mapsto P(E,\nabla) \in \Omega^\bullet(M)$$

is compatible with the products.

We will be using the global aspects and the algebraic structure discussed in Remark 2.46. Next to that, notice that a connection ∇ on E induces a connection $\widetilde{\nabla}$ on $End(E)$, given by

$$\widetilde{\nabla}_X(T)(s) = \nabla_X(T(s)) - T(\nabla_X(s)),$$

for $X \in \mathscr{X}(M)$, $T \in \Gamma(End(E))$, $s \in \Gamma(E)$. Moreover, the induced operator

$$d_{\tilde{\nabla}} : \Omega^p(M;End(E)) \to \Omega^{p+1}(M;End(E))$$

will be given by

$$d_{\tilde{\nabla}}(K) = [d_\nabla, K] \tag{2.4.3}$$

where, as in Exercise 2.47, we identify the elements in $K \in \Omega^*(M;End(E))$ with the induced operators $\hat{K}$ and we use the graded commutators $[d_\nabla, K] = d_\nabla \circ K - (-1)^p K \circ d_\nabla$. In also follows that $d_{\tilde{\nabla}}$ satisfies the Leibniz identity with respect to the wedge product (2.2.4) on $\Omega(M,End(E))$. It also follows that $d_{\tilde{\nabla}}$ annihilates the curvature (2.2.2), property known as the Bianchi identity:

$$d_{\tilde{\nabla}}(k_\nabla) = 0. \tag{2.4.4}$$

We will also make use of the trace map as in (2.2.7), but now in arbitrary degrees:

$$Tr : \Omega^\bullet(M, \mathrm{End}(E)) \to \Omega^\bullet(M), \tag{2.4.5}$$

A simple computation shows that it has the graded trace property:

$$Tr([A,B]) = 0, \tag{2.4.6}$$

for all $A \in \Omega^p(M;End(E))$, $B \in \Omega^q(M;End(E))$, where $[A,B]$ are the graded commutators $[A,B] = A \wedge B - (-1)^{pq} B \wedge A$ and $\wedge$ is here the wedge operation (2.2.4). The other basic property of (2.4.5), whose proof is left as an exercise, is that it is compatible with $d_{\tilde{\nabla}}$ in the sense that

$$Tr \circ d_{\tilde{\nabla}} = d \circ Tr. \tag{2.4.7}$$

The fact that $\Sigma_p(k_\nabla) = Tr(k_\nabla^p) \in \Omega^{2p}(M)$ is closed follows now from this identity and (2.4.4):

$$d\left(Tr(k_\nabla^p)\right) = Tr\left(d_{\tilde{\nabla}}(k_\nabla^p)\right) = 0.$$

To prove that the resulting class

$$Tr\left(k_\nabla^p\right) \in \Omega^{2p}(M)$$

does not depend on the choice of a connection, let ∇' be another connection. The expression

$$R(X)s = \nabla'_X(s) - \nabla_X(s)$$

is then $C^\infty(M)$-linear in X and s, hence defines an element $R \in \Omega^1(M,End(E))$. A related remark is that

$$\hat{R} = d_{\nabla'} - d_\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$$

is $\Omega(M)$-linear (in the graded sense). This follows from the Leibniz rule for d_∇ and $d_{\nabla'}$. Our notation is not accidental: R and $\hat{R}$ correspond to each other by the bijection of Exercise 2.47. In what follows we will not distinguish between the two. The idea is to join the two connections by a path (of connections): for each $t \in [0,1]$ we consider

$$\nabla^t := \nabla + tR, \quad d_{\nabla^t} = d_\nabla + tR.$$

Computing the square of this operator, we find $k_{\nabla^t} = k_\nabla + t[d_\nabla, R] + t^2R^2 = k_\nabla + t d_{\tilde{\nabla}}(R) + t^2 \frac{1}{2}[R,R]$

$$k_{\nabla^t} = k_\nabla + t[d_\nabla, R] + t^2R^2 = k_\nabla + t d_{\tilde{\nabla}}(R) + t^2 \frac{1}{2}[R,R]$$

from which we deduce

$$\frac{d}{dt} k_{\nabla^t} = d_{\tilde{\nabla}}(R) + t[R,R] = d_{\tilde{\nabla}^t}(R).$$

It follows that

$$\frac{d}{dt}Tr(k_{\nabla^t}^p) = d\left(p \cdot Tr(k_{\nabla^t}^{p-1} \wedge R)\right). \tag{2.4.8}$$

Indeed, using the Leibinz identity and (2.4.6) we obtain

$$\frac{d}{dt}Tr\left(k_{\nabla^t}^p\right) = Tr\left(\sum_i k_{\nabla^t}^{i-1} \wedge d_{\tilde{\nabla}^t}(R) \wedge k_{\nabla^t}^{p-i}\right) = p \cdot Tr(k_{\nabla^t}^{p-1} \wedge d_{\tilde{\nabla}^t}(R)).$$

The same expression is obtained also by rewriting the right hand side of (2.4.8), using (2.4.7), the Leibinz identity and then the Bianchi identity (2.4.4). Hence (2.4.8) holds. Integrating from 0 to 1 we find

$$Tr\left(k_{\nabla'}^p\right) - Tr\left(k_{\nabla}^p\right) = d\left[p \cdot \int_0^1 Tr(k_{\nabla^t}^{p-1} \wedge R)\right],$$

i.e. the two terms in the left hand side differ by an exact form, hence they represent the same element in the cohomology group. 😄

Theorem 2.81. *For any $P \in I_r(\mathbb{F})$, the construction $E \mapsto P(E)$ is natural, i.e.*

1. *If $f : N \to M$ is a smooth map and*
$$f^* : H^\bullet(M) \to H^\bullet(N)$$
is the pull-back map induced in cohomology, then for the pull-back vector bundle f^*E,
$$P(f^*E) = f^*P(E).$$
2. *If two vector bundles $E, F \in \mathrm{Vect}_{\mathbb{F}}^r(M)$ are isomorphic, then $P(E) = P(F)$.*

Proof. For item 1 we claim that, if we use a connection ∇ on E and the pull-back connection $f^*\nabla$ on f^*E, then the resulting classes are related via the pull-back map $f^* : \Omega^\bullet(M) \to \Omega^\bullet(N)$:

$$P(f^*E, f^*\nabla) = f^*\left(P(E, \nabla)\right).$$

This is an equality in $\Omega^\bullet(N)$ and, to prove it, it suffices to check it on opens that cover N. We restrict our attention to opens of type $f^{-1}(U)$ where $U \subset M$ is an open over which E is trivializable. We also choose a local frame $\mathfrak{e}$ for E over U and, for f^*E over $f^{-1}(U)$, we use the pull-back frame $f^*\mathfrak{e}$. The local connection matrix w.r.t. $f^*\mathfrak{e}$ is precisely the pull-back of the connection matrix w.r.t. $\mathfrak{e}$, denoted k below; hence, over $f^{-1}(U)$, the equality to prove becomes

$$P(f^*k) = f^*P(k)$$

which follows because $f^* : \Omega^\bullet(M) \to \Omega^\bullet(N)$ is compatible with the (wedge) products.

For item 2 the argument is similar, though a bit simpler: if $\Phi : E \to F$ is an isomorphism, then any connection ∇ on E can be transported, using Φ, to one on F, denote it ∇^Φ. Furthermore, Φ takes any frame $\mathfrak{e}$ to one of F, and the connection (hence also the curvature) matrix of ∇^Φ w.r.t. $\Phi(\mathfrak{e})$ coincides with the one of ∇ w.r.t. $\mathfrak{e}$. It follows that $P(E, \nabla) = P(F, \nabla^\Phi)$ hence, in particular, $P(E) = P(F)$. 😄

Particular characteristic classes are obtained by applying the constructions of the previous section to specific polynomials. Since the polynomials Σ_p (and similarly the σ_p's) generate $I(\mathbb{F})$, the classes that they induce contain all the information that characteristic classes contain. However, it is important to allow other polynomials as well because of the properties that the resulting classes have- e.g. theior behaviour w.r.t. taking direct sums or tensor products.

2.4.2 The Chern classes $c_p : \mathrm{Vect}_{\mathbb{C}}(M) \to H^{2p}(M)$

The Chern classes correspond to $\mathbb{F} = \mathbb{C}$ and the invariant polynomials

$$c_p = \left(\frac{1}{2\pi i}\right)^p \sigma_p \in I_r(\mathbb{C}), \quad p \in \{0, 1, \ldots, r\},$$

where the bound r comes from the rank of the vector bundles. Hence they associate to complex vector bundle E rank r a cohomology class, called the p-th Chern class of E

$$c_p(E) \in H^{2p}(M) \qquad (0 \leq p \leq r,\ r = \mathrm{rank}\, E)$$

.

One also defines the total Chern class of E

$$c(E) = c_0(E) + c_1(E) + \ldots + c_r(E) \in H^{\mathrm{even}}(M);$$

it corresponds to the inhomogeneous polynomial

$$c(A) = det\left(I + \frac{1}{2\pi i}A\right).$$

The constants in front of σ_p may look a bit mysterious but should not distract the reader; to that end, let us mention that their role is to ensure that the resulting cohomology classes are real, or even integral (they come from the cohomology with integral coefficients; alternatively, when integrated over sub-manifolds, the resulting integrals are always integer numbers). Here are the main properties of the Chern classes, generalizing item 4. of Theorem 2.51.

Theorem 2.82. *The Chern classes of a complex vector bundle, priory cohomology classes with coefficients in $\mathbb{C}$, are actually real (cohomology classes with coefficients in $\mathbb{R}$). Moreover,*

1. *The total Chern class has an exponential behaviour with respect to the direct sum of vector bundles i.e., for any two complex vector bundles E and F over M,*

$$c(E \oplus F) = c(E)c(F)$$

or, component-wise,

$$c_p(E \oplus F) = \sum_{i+j=p} c_i(E)c_j(F).$$

2. *If $\overline{E}$ is the conjugated of the complex vector bundle E, then*

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

Proof. To see that the resulting classes are real the idea is to show that, for certain connections ∇, the actual forms

$$c_p(E, \nabla) \in \Omega^{2p}(M, \mathbb{C})$$

are actually real. More precisely, we claim that that is the case whenever ∇ is compatible with a Hermitian inner product on E, where we are using the rather obvious complex version of the concept of connection compatible with an inner product; for Hermitian inner products, have a look at section 1.3.5. The complex version of the last item of Proposition 2.55 tells us that $\omega^* = -\omega$, where ω^* conjugates the entries and takes the transpose. We see that, if we use such a connection, then one can compute $\sigma_p(E, \nabla)$ locally using orthonormal frames, which deliver connection matrices, and then also curvature matrices k, satisfying $k^* = -k$. Applying σ_p we obtain $\overline{\sigma_p(k)} = (-1)^p \sigma_p(k)$, hence the same formula for $\sigma_p(E, \nabla)$ and then, using the presence of i in c_p:

$$\overline{c_p(E,\nabla)} = c_p(E,\nabla).$$

Hence $c_p(E,\nabla)$ is real.

The proof of the other items is completely similar to that from the case $p = 1$, discussed in Theorem 2.51. Let us do in detail the more involved one: item 1. As in the case $p = 1$, we use the direct sum connections, the block decomposition (2.2.8) for the resulting connection matrices and similarly for the curvature matrices. Furthermore, since on block matrices

$$\det\left(I + t\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = det(I+tA)\, det(I+tB)$$

we deduce that

$$\sigma_p\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sum_{i+j=p} \sigma_i(A)\cdot\sigma_j(B),$$

then (2.2.8) is generalized to:

$$\sigma_p(k_{\nabla\oplus\nabla'}) = \sum_{i+j=p} \sigma_i(k_\nabla)\cdot\sigma_j(k_{\nabla'}),$$

and the desired identity follows. 😄

> **Exercise 2.83.** Here we look at the Chern classes of the tangent bundle of $T\mathbb{CP}^n$. We consider the negative of the first Chern class of the tautological line bundle
>
> $$a := -c_1(L) \in H^2(\mathbb{CP}^n).$$
>
> (which is non-zero, by Exercise 2.53). Show that:
>
> $$c(T\mathbb{CP}^n) = (1+a)^{n+1}.$$
>
> (Hint: Exercise 6.32").

2.4.3 The Chern character $Ch : \mathrm{Vect}_{\mathbb{C}}(M) \to H^{even}(M)$

In principle, the Chern character and the underlying classes are a re-arrangement of the Chern classes which have better algebraic properties. They correspond to the invariant polynomials

$$Ch_p = \frac{1}{p!}\left(-\frac{1}{2\pi i}\right)^p \Sigma_p \in\in I_r(\mathbb{C}), \quad p \in \mathbb{N}$$

Hence they associate to complex vector bundle E a cohomology class, called the p-th component of the Chern character of E:

$$Ch_p(E) \in H^{2p}(M).$$

They assemble together into the full Chern character of E, defined as $Ch(E) = \sum_{p\geq 0} Ch_p(E)$, which can be seen as the class obtained using the "invariant function"

$$Ch(A) = Tr\left(e^{-\frac{1}{2\pi i}A}\right)$$

(strictly speaking this is a powers series and not a polynomial; however, when evaluated on a curvature matrix it still produces a finite sum, hence it can be used like it was a polynomial). Note that $Ch(E)$ are always real cohomology classes; this follows e.g. from the similar property for the Chern classes, and the fact that the relationship between the Σ's and the σ's involve only real coefficients. Here is the main property of the Chern character.

Theorem 2.84. *The Chern character*

$$Ch : \mathrm{Vect}_{\mathbb{C}}(M) \to H^{even}(M)$$

is additive and multiplicative i.e., for any two complex vector bundles E and F over M,

$$Ch(E \oplus F) = Ch(E) + Ch(F), \;\; Ch(E \otimes F) = Ch(E)Ch(F).$$

Proof. The argument is completely similar to the one from the proof of Theorem2.82 (and Theorem 2.51)- actually the formulas become a bit simples, as Σ_p applied to a block matrix is simply the sum of Σ_p applied to each of the components. 😄

2.4.4 The Pontryagin classes $p_k : \mathrm{Vect}_{\mathbb{R}}(M) \to H^{4k}(M)$

We now pass to the case $\mathbb{F} = \mathbb{R}$. The Pontryagin classes are the analogues for real vector bundles of the Chern classes. They correspond to the polynomials

$$p_k = \left(\frac{1}{2\pi}\right)^{2k} \sigma_{2k} \in I_r(\mathbb{C}), \quad 0 \leq k \leq r/2,$$

where r comes from the rank of the vector bundles. The reason we restrict to the σ's of even degree is that the classes produced by the odd dimensional ones all vanish (see below). Hence p_k associates to a real vector bundle E of rank r a cohomology class, called the k-th Pontryagin class of E:

$$p_k(E) \in H^{4k}(M) \qquad (0 \leq k \leq r/2,\ r = \mathrm{rank}\, E)$$

.

They assemble together into the full Pontryagin class of E, defined as

$$p(E) = p_0(E) + p_1(E) + \ldots + p_{[\frac{r}{2}]} \in H^{\bullet}(M).$$

The Pontryagin class has the same property as the Chern class: for any two real vector bundles,

$$p(E \oplus F) = p(E)p(F).$$

The relationship between the two is actually much stronger (which is expected due to their definitions). To make this precise, recall that any real vector bundle E gives rise to a complex one, namely its complexification

$$E \otimes \mathbb{C} := E \otimes_{\mathbb{R}} \mathbb{C} = \{e_1 + ie_2 : e_1, e_2 \in E, \}.$$

Proposition 2.85. *For any real vector bundle E,*

$$c_l(E \otimes \mathbb{C}) = \begin{cases} (-1)^k p_k(E) & \text{if } l = 2k \\ 0 & \text{if } l = 2k+1 \end{cases}$$

Proof. When l is odd it suffices to remark that $E \otimes \mathbb{C}$ is isomorphic to its conjugation and then apply the last part of Proposition 2.82. For l even we have to go again to connections and to remark that a connection ∇ on a real vector bundle E can be complexified to give a connection $\nabla^{\mathbb{C}}$ on $E \otimes \mathbb{C}$: just extend ∇ by requiring $\mathbb{C}$-linearity. Comparing the connection matrices we immediately find that

$$c_{2k}(E\otimes\mathbb{C}) = \left(\frac{1}{i}\right)^{2k} p_k(E) = (-1)^k p_k(E). \quad \text{😄}$$

Finally, recall that one can also go from a complex vector bundle E to a real one, denoted $E_{\mathbb{R}}$, which is just E vied as a real vector bundle. One has:

Proposition 2.86. *For any complex vector bundle E,*

$$p_0(E_{\mathbb{R}}) - p_1(E_{\mathbb{R}}) + p_2(E_{\mathbb{R}}) - \ldots = (c_0(E) + c_1(E) + \ldots)(c_0(E) - c_1(E) + \ldots).$$

Proof. The main observation is that, for any complex vector bundle E, one has a canonical isomorphism

$$E_{\mathbb{R}}\otimes\mathbb{C} \cong E\oplus\overline{E}$$

defined fiberwise by

$$v + \sqrt{-1}w \mapsto (v+iw, v-iw),$$

where $\sqrt{-1}$ is "the i used to complexify $E_{\mathbb{R}}$). Hence

$$c(E_{\mathbb{R}}\otimes\mathbb{C}) = c(E)c(\overline{E})$$

and, using the formulas from Proposition 2.85 and from end of Proposition 2.82, the desired formula follows. 😄

Exercise 2.87. Return now to Exercise 2.83 and deduce that the Pontryagin class of the tangent bundle of $T\mathbb{CP}^n$ interpreted seen as a real vector bundle is

$$p(T\mathbb{CP}^n) = (1+a^2)^{n+1}.$$

Remark 2.88 (why only degree $2k$). We have mentioned that if we were to use the odd degree polynomials σ_{2k+1}, they would deliver the zero cohomology classes. This can be seen as a consequence of the second line in Proposition 2.85. However, that is a rather convoluted argument. Probably the most transparent way to prove it is to notice that, if one uses connections ∇ that are compatible with some inner product g on E, then the actual forms $\sigma_{2k+1}(E,\nabla)$ vanish. The reason is that, when computing those forms locally, one can use frames that are orthonormal w.r.t. g, which deliver connection matrices which are ske-symmetric (cf. the last item of Proposition 2.55). Then also the connection matrices k will be skew symmetric, and then the statement follows. In particular, we see the reason for the vanishing of those classes: the fact that σ_{2k+1} vanish on skew-symmetric matrices.

2.4.5 The Euler class

There are other characteristic classes which are, strictly speaking, not immediate application of the general construction discussed above, but are similar in spirit. However, they can be made part of an even more general theory, in which the starting point is not a vector bundle, but a vector bundle with extra-structure. Here we mention the Euler class, when the extra-structure is that of an orientation. More precisely: for any real oriented vector bundles E over M of rank r, the Euler class will be a cohomology class

$$e(E) \in H^r(M).$$

Again, to obtain non-vanishing classes, we restrict to even dimensional vector bundles:

$$r = 2l.$$

We have already noticed in the last remark above that characteristic classes of real vector bundles can be computed using connections compatible with an inner product, when one can use orthonormal frames which, in turn, produce

skew-symmetric connection and curvature matrices. The key observation here is that, for skew symmetric matrices of even order $2l$, $A \in \mathscr{M}_{2l}(\mathbb{R})$, $\det(A)$ is naturally a square of another expression, denoted $\text{Pfaffian}(A)$ and called the Pfaffian of A,

$$\det(A) = \text{Pfaffian}(A)^2.$$

This is a polynomial of degree l, For instance,

$$\det\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af + cd - be)^2.$$

However, Pfaffian is not invariant: the desired invariance

$$\text{Pfaffian}(gAg^{-1}) = \text{Pfaffian}(A).$$

holds only when $g \in \text{GL}_{2l}(\mathbb{R})$ has positive determinant.

And that is where the orientability comes in: because of orientability we may talk about positively oriented frames, use only such frames, and notice that for such frames the corresponding coordinate change matrices (the g from Lemma 2.49) have positive determinant. All together, this allows us to make sense of

$$\text{Pfaffian}(E, \nabla) \in \Omega^{2l}(M).$$

As before, this form is closed and the resulting cohomology class does not depend on the choice of the connection. The Euler class is

$$e(E) = \left[\left(\frac{1}{2\pi}\right)^l \text{Pfaffian}(E, \nabla)\right] \in H^{2l}(M)$$

.

Note that, since det is used in the construction of $p_l(E)$, it follows that

$$p_l(E) = e(E)^2$$

.

2.4.6 *Some applications*

Here we hint to several more applications of characteristic classes, besides that of allowing us to prove that certain vector bundles are not isomorphic. We do it through a couple of exercises pointing in some directions.

The first direction is related to the fact that for any vector bundle E one can find another one, E', such that E' is trivializable. There are several questions here. The first one is: can E' chosen to be itself trivial? The answer is yes, even if E is not trivializable (an example would by $E = TS^n$). When this happens, one says that E is **stably trivial**.

> **Exercise 2.89.** Show that if $E \in \text{Vect}_{\mathbb{R}}(M)$ is stably trivial then $p_k(E) = 0$ for all $k \neq 0$. Use this to deduce that $T\mathbb{CP}^n$ is not stably trivial.

The second question is: how small can E' be?

> **Exercise 2.90.** For real vector bundles E please do the following:
>
> 1. Show that if there exist a line bundle L such that $E \oplus L$ is trivializable then $p_l(E) = 0$ for all $l \neq 0$.
> 2. What can you say about the Pontrjagin classes of E if E has the property that there exists E' of rank $r' \leq 3$ such that $E \oplus E'$ is trivializable?

One typical situation when one arrive at trivializable sums $E \oplus E'$ is when $E = TM$ and M is embedded in an Euclidean space- then one can make use of the normal bundle as a complement. With this hint at hand, here is now an exercise:

Exercise 2.91. Show that:

1. if an n-dimensional manifold M admits an embedding in $\mathbb{R}^{n+k}$ then there exists a rank k vector bundle $\mathscr{N}$ over M such that $TM \oplus \mathscr{N}$ is trivializable.
2. $\mathbb{CP}^2$ cannot be embedded in $\mathbb{R}^5$.

Here is a little generalization of the previous exercise:

Exercise 2.92. Show that if an n-dimensional manifold M has $p_1(TM) \neq 0$, then M cannot be embedded in $\mathbb{R}^{n+1}$.

For a similar exercise but which uses the second item of Exercise 2.90 rather than the first one:

Exercise 2.93. Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.

Yet another class of applications is to the question of whether a given manifold M can be realised as the boundary of a compact orientable manifold N with boundary. For such questions one observes first that the entire theory of characteristic classes carries over, word by word, to vector bundles over manifolds with boundary. Next to that, the other key ingredient that we have on manifolds with boundary is Stokes theorem: $\int_N d\omega = \int_{\partial N} \omega$ for all $\omega \in \Omega^{n-1}(N)$.

Exercise 2.94. Do the following:

1. Show that if a manifold M of dimension $n = 4k$ can be written $M = \partial N$ with N compact orientable manifold with boundary, then $p_k(TM) = 0$.
2. Show that none of the manifolds $\mathbb{CP}^{2k}$ with $k \geq 1$ (viewed as real manifolds) can be written as the boundary of a compact orientable manifold.

Remark 2.95. While we have pointed out applications of Pontryagin classes to questions about embeddings or realizability of manifolds as boundaries, it is fair to say that there are better characteristic classes that do that: Stieffel-Withney classes. However, those fall out of the scope of our course. They are very similar to the Chern and the Pontryagin classes, but they are $\mathbb{Z}_2$-valued, hence not representable by differential forms but require the machinery of Algebraic Topology.

Chapter 3
Principal bundles (and G-structures)

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3.1 Actions

3.1.1 Recap on Lie groups and their Lie algebras

Here is a brief recap on Lie groups:

Lie groups: a Lie group G is both a group as well as a manifold, with the two structures required to be compatible- see Definition 2.92 in the bachelor notes.

Examples: keep in mind the basic examples: the circle S^1, the (real/complex) general linear groups GL_n and the various matrix subgroups like $O(n)$, $SO(n)$, $SL_n(\mathbb{R})$, $U(n)$, etc (see Example 2.97 in the bachelor notes).

Example 3.1. Similarly, the 3-sphere S^3 can be made into a (non-commutative, this time) Lie group. For that we replace $\mathbb{C}$ by the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in $\mathbb{H}$ is uniquely determined by the fact that it is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Recall also that for

$$u = x + iy + jz + kt \in \mathbb{H}$$

one defines

$$u^* = x - iy - jz - kt \in \mathbb{H}, \;\; |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

Then, the basic property $|u \cdot v| = |u| \cdot |v|$ still holds and we see that, identifying S^3 with the space of quaternionic numbers of norm 1, S^3 becomes a Lie group.

Symmetry groups: it is good to keep in mind that the main examples of subgroups are the symmetry groups of various structures on the Euclidean spaces (e.g. $O(n)$ is made of symmetries of the standard inner product on $\mathbb{R}^n$).

Closed subgroups: it may also be interesting to remember the closed subgroup theorem: any closed subgroup of a Lie group is automatically an embedded submanifold and, therefore, is a Lie groups itself (see Theorem 3.97 in the bachelor notes).

"Concrete" Lie groups: for the reader that does not feel comfortable with this generality, given that the most important examples sit naturally inside the general linear groups GL_n, and supported by the closed subgroup theorem mentioned above: you may just assume that our G's below are closed subgroups of a GL_n.

The Lie algebra: We will also use the Lie algebra of a Lie group, rather superficially for a while and then more substantially when we get to the chapter on connections on principal bundles. Recall that, given a Lie group G:

- its Lie algebra is, first of all, the vector space that is the tangent space of G at the unit:

$$\mathfrak{g} := T_e G. \tag{3.1.1}$$

  The reason we concentrate at the unit is due to the presence of left (say) translations $L_g : G \to G$, $L_g(h) = gh$ which allow us to move from e and around to arbitrary $g \in G$. In particular, each vector $u \in \mathfrak{g}$ can be translated around obtaining a vector field $\vec{u} \in \mathfrak{X}(G)$- therefore defined by

$$\vec{u}_g = (dL_g)_e(u). \tag{3.1.2}$$

- Next to being a vector space, the structure present on $\mathfrak{g}$ making it into a Lie algebra is the binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ induced from the Lie bracket of vector fields on G via the map $u \mapsto \vec{u}$, hence characterised by:

$$\overrightarrow{[u,v]} = [\vec{u}, \vec{v}].$$

Therefore, the Lie algebra of the Lie group G is the vector spaces $\mathfrak{g} := T_eG$ together with the bracket $[\cdot,\cdot]$. In general, a Lie algebra is a vector space with such a bilinear operation ("bracket") that is skew-symmetric and satisfies the Jacobi identity.

The exponential map: The Lie algebra $\mathfrak{g}$ is related to the Lie group G via the so-called exponential map

$$\exp : \mathfrak{g} \to G.$$

The key ingredient for describing it in full generality is the flow of the vector fields $\vec{u}$, $\phi^t_{\vec{u}} : G \to G$; given how $\vec{u}$ are constructed it follows that $\phi^t_{\vec{u}}(g) = g\phi^t_{\vec{u}}(e) = g\phi^1_{t\vec{u}}(e)$. In turn, this implies these flows can be recovered by what happens at $g = e$ and $t = 1$- and that is how the exponential map is defined:

$$\exp(u) := \phi^1_{\vec{u}}(e) \in G.$$

This defines $\exp : \mathfrak{g} \to G$ in general, from which the flows can be recovered as

$$\phi^t_{\vec{u}}(a) = a\exp(t\,u).$$

"Concrete" Lie groups again: For the reader that prefers to concentrate on closed subgroups $G \subset GL_n$ is a closed subgroup, the Lie algebra can be described as

$$\mathfrak{g} := \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G\ \forall\, t \in \mathbb{R}\} \subset \mathscr{M}_{n\times n},$$

the Lie bracket is the commutator of matrices, $[u,v] = u\cdot v - v\cdot u$, and exp is the usual exponential of matrices, which can now be seen as a map from $\mathfrak{g}$ to G.

The philosophy of going "infinitesimal": Lie algebras should be seen as "the infinitesimal counterpart" of Lie groups, obtained by "linearising" G and its structure (one could say that $\mathfrak{g}$ is "the linear shadow" of the group G). It is rather amazing that, for connected Lie groups G, $\mathfrak{g}$ contains almost all information of G (and, if G is simply connected, it can actually be fully recovered from $\mathfrak{g}$!). Anyway, the point of this remark is to emphasise the nature of/principle behind $\mathfrak{g}$: obtained by linearising G. Because of this, whenever encountering concepts involving Lie groups, one often tries to linearise them, discovering the "*infinitesimal counterpart* of the concept", expressed in terms of $\mathfrak{g}$ and (hopefully) easier to handle. For instance, the infinitesimal counterpart of Lie group homomorphisms $\Phi : G \to H$ are Lie algebra homomorphisms $\phi : \mathfrak{g} \to \mathfrak{h}$ (linear and compatible with the brackets), obtained from Φ as the differential at the unit. Similarly for representations and actions (see below) and many other concepts. We refer to this general philosophy also as "the Lie functor".

3.1.2 Lie groups actions

Next to Lie groups, we also need the notion of Lie group action and the resulting "category of G-manifolds".

Definition 3.2. Given a Lie group G, by a **(right) G-manifold** we mean a manifold P together with a **(right) action** of G on P, i.e., together with a smooth map

$$m : P \times G \to P, \qquad \text{also denoted} \quad m(g,p) =: p\cdot g \tag{3.1.3}$$

which satisfy the axioms for an action: for all $g,h \in G$, $p \in P$, and denoting by $e \in G$ the unit element:

$$(p\cdot g)\cdot h = p\cdot(gh), \quad p\cdot e = p.$$

A G-**equivariant** map between two G-manifolds P and Q is any smooth map $F : P \to Q$ satisfying

$$F(pg) = F(p)g \quad \forall\, p \in P, g \in G.$$

A G-**diffeomorphism** is any G-equivariant diffeomorphism. Two G-manifolds are said to be G-**diffeomorphic** if there exists a G-diffeomorphism between them.

Similarly one defines the notion of left action. However, one may choose to work only when left or only with right actions, since one can pass from one to the other by declaring:

$$g \bullet p := p \cdot g^{-1}. \tag{3.1.4}$$

Some people prefer to work with left actions, some with right, but the truth is that in some cases the right actions are more natural, and in some the left ones. Therefore, one should allow both and keep in the back of your mind the previous formula for passing from one to the other.

For any action of G on P as above, and any $p \in P$, one can talk about:

- the **isotropy** group at p:

$$G_p = \{g \in G : p \cdot g = p\} \quad \text{(a subgroup of } G\text{)}.$$

- the **orbit** through p:

$$pG := \{p \cdot g : g \in G\} \subset P \quad \text{(a subset of } P\text{)}.$$

- the **orbit space** defined as the set of all orbits:

$$P/G := \{pG : p \in P\},$$

endowed with **the quotient map**

$$\pi_{\text{can}} : P \to P/G, \quad p \mapsto G \cdot p. \tag{3.1.5}$$

Hence a point of the orbit space is an orbit (very much like in the projective space where a point of the projective space is a line). There is also an equivalence relation on P that underlies this discussion, where for $p, q \in P$:

$$p \sim q \iff q = p \cdot g \quad \text{for some } g \in G.$$

The equivalence class of $p \in P$ is precisely the orbit $p \cdot G$, hence P/G is the resulting quotient. For that reason, P/G is also called the quotient space of P modulo G.

And here are some particularly interesting properties of actions. An action is called:

- **free**: if all the isotropy groups are trivial or, more explicitly, if for $p \in P, g, h \in G$ one has:

$$p \cdot g = p \cdot h \quad \Longrightarrow \quad g = h.$$

- **transitive**: if the action has a single orbit or, more explicitly: if

$$p, q \in P \quad \Longrightarrow \quad q = p \cdot g \ \text{ for some } \ g \in G. \tag{3.1.6}$$

- **proper**: if the map

$$A : P \times G \to P \times P, \quad (p, g) \mapsto (p, p \cdot g)$$

is proper, where we recall that a map $f : M \to N$ is proper if $f^{-1}(K)$ is compact for any $K \subset N$ compact.

Notice that transitivity can also be rephrased in terms of the map A from the last item: A has to be surjective. Using Sard's theorem (see below) it follows that A is also a submersion.

For properness, we notice the following interesting sequence of implications:

$$G \text{ is compact} \quad \Longrightarrow \quad \text{proper action} \quad \Longrightarrow \quad \text{all the isotropy groups are compact}$$

Proof. The second implication follows from the fact that the preimage of a point $(p, p) \in P \times P$ by the map A from item 3 is $\{p\} \times G_p$. For the first implication notice that for any compact $K \subset P \times P$ one has

$$A^{-1}(K) = \{(p, g) : (p, pg) \in K\} \subset \mathrm{pr}_1(K) \times G$$

$$A^{-1}(K) \subset \mathrm{pr}_1(K) \times G$$

and then $A^{-1}(K)$, as a closed subset of the compact $\mathrm{pr}_1(K) \times G$, must be compact. 😄

Remark 3.3 (different contexts). Here we are working in the category of second countable smooth manifolds, but the concepts discussed in this section have analogues also other similar categories, such as that of topological spaces, that of C^k-manifolds, etc. While the analogues are rather obvious, one of the concepts above, namely transitivity of actions, requires more care. For instance:

- when working in the topological context: transitivity of actions should require A to be an open surjection.
- when working with manifolds for which the second countability axiom is removed: transitivity of actions should require A to be a surjective submersion.

For 2nd countable manifolds, these are consequences of the set-theoretical concept of transitiveness.

The infinitesimal counterpart of Lie group actions (in the sense mentioned in the previous section- see the last item there) are Lie algebra actions , or infinitesimal actions. By an **infinitesimal action** of a Lie algebra $\mathfrak{g}$ on a manifold P we mean a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{X}(P) \tag{3.1.7}$$

i.e., a linear map that is compatible with the Lie brackets we have on $\mathfrak{g}$ and the usual Lie bracket of vector fields for $\mathfrak{X}(P)$. Evaluating each $\rho(u) \in \mathfrak{X}(P)$ at points of P, one can think of ρ also as being a collection of linear maps,

$$\rho_p : \mathfrak{g} \to T_p P \qquad (p \in P).$$

Any Lie group action of G on P induces an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on P: for $u \in \mathfrak{g}$ define $\rho(u) \in \mathfrak{X}(P)$ as the vector field whose flow $\phi^t_{\rho(u)}$ is given by

$$\phi^t_{\rho(u)}(p) = p \cdot \exp(tu). \tag{3.1.8}$$

Hence, at each $p \in P$,

$$\rho_p(u) = \frac{d}{dt}\Big|_{t=0} p\cdot\exp(tu) = (dm_p)_e(u) \in T_pP,$$

where in the last formula $m_p : G \to P$ is the original action with the first variable p fixed ($m_p(g) = p\cdot$), and we take its differential at the unit $e \in G$. This makes it clear that ρ is, indeed, obtained by linearising the original action.

Two of the concepts from the last definition have an nice infinitesimal counterpart. We call an infinitesimal action:

1. **infinitesimally free**: if each $\rho_p : \mathfrak{g} \to T_pP$ is injective.
2. **infinitesimally transitive**: if each $\rho_p : \mathfrak{g} \to T_pP$ is surjective.

In general, the kernel

$$\mathfrak{g}_p := \operatorname{Ker}\rho_p \subset \mathfrak{g}$$

is called **the isotropy Lie algebra at** p. As the terminology indicates, this is precisely the Lie algebra of the isotropy group $G_p \subset G$.

Proof. To see this one uses that G_p is closed in G, which implies that it is a Lie subgroup of G whose Lie algebra consists of those $v \in \mathfrak{g}$ with the property that $\exp(tv) \in G_p$ for all t. But that property means that $\phi^t_{\rho(v)}(p) = p$ for all t, which is equivalent to $\rho(v)_p = 0$, i.e. $v \in \operatorname{Ker}\rho_p$. 😄

We deduce that, for all Lie group actions:

free action $\implies$ infinitesimally free action $\iff$ all the isotropy groups are discrete

There is a similar discussion regarding transitivity, with the main conclusion that the image of ρ_p is precisely the tangent space (at p) of the orbit $\mathcal{O}$ through p: $\operatorname{Im}(\rho_p) = T_p\mathcal{O}_p$; however, that discussion requires explaining first how the orbit is a submanifold of P, discussion which will be carried out only later. Nevertheless, we can prove now the analogue of the consequence above: we claim that, for general actions on connected manifolds P, one has

transitive action $\xrightarrow{\text{in general}}$ infinitesimally transitive action $\xrightarrow{\text{if } G\text{−connected}}$ transitive action (3.1.9)

Proof. For the direct implication we start with a transitive action. Since the map $m_p : G \to P$, $g \mapsto p\cdot g$ is smooth and surjective, by Sard's theorem, it is a submersion at some point $a \in G$. Since ρ_p is the differential of m_p at the identity, we have to move from a to e. To that end we use the right multiplication by a on G as well as on P, denoted and notice that $m_p \circ R^G_a = R^P_a \circ m_p$:

$$\begin{array}{ccc} G & \xrightarrow{m_p} & P \\ {\scriptstyle R^G_a}\downarrow & & \downarrow{\scriptstyle R^P_a} \\ G & \xrightarrow[m_p]{} & P \end{array} \qquad (3.1.10)$$

Differentiating at the identity, it follows that ρ_p is surjective as well.

Assume now that ρ_p is surjective for all p. Reasoning backwards, we see that each m_p is a submersion. In particular, the image of m_p, i.e. the orbit through p, must be open in P. But the orbits form a partition and, since arbitrary union of opens is open, the complement of each orbit is open as well. In other words, each orbit is both open as well as closed in P. Since P is connected, transitivity follows. 😄

3.1.3 Some examples

We now look at some examples. We start with 0-dimensional Lie groups, i.e. discrete groups that are at most countable.

Example 3.4. The group $G = \mathbb{Z}_2$ (a zero-dimensional Lie group!) acts on $P = \mathbb{R}$ by reflections in the origin:

$$x \cdot \widehat{k} := (-1)^k x \quad (x \in \mathbb{R},\, \widehat{k} \in \mathbb{Z}_2).$$

The action is proper (any finite group is compact!) but not free despite the fact that it has only one isotropy group that is not trivial- the one at origin. The orbit space $\mathbb{R}/\mathbb{Z}_2$ can be identified with the interval $[0,\infty)$.

Example 3.5. The group $G = \mathbb{Z}$ acts on $P = \mathbb{R}$ by translations

$$x \cdot k := x + k \quad (x \in \mathbb{R}, k \in \mathbb{Z}).$$

Although the group is not compact, the action is proper (check this!), and it clearly free as well. In this case the orbit space $\mathbb{R}/\mathbb{Z}$ can be identified with the unit circle S^1 by sending the class of x mod $\mathbb{Z}$ to $e^{2\pi i x}$.

Similarly, the group $G = \mathbb{Z}^2$ acts on $P = \mathbb{R}^2$ by translations:

$$(x,y) \cdot (m,n) = (x+m, y+n),$$

this is a free and proper action, and the resulting orbit space $\mathbb{R}^2/\mathbb{Z}^2$ can be identified with the 2-torus.

Example 3.6. Consider the action of $G = \mathbb{Z}$ on $P = S^1$ by rotations by a given angle $\theta_0 = 2\pi r_0$ where $r_0 \in \mathbb{R}$:

$$e^{2\pi i r} \cdot n := e^{2\pi i (r + n r_0)}.$$

If r_0 is not rational then the action is free, but it is not proper.

We now look at more interesting Lie groups, which are not discrete.

Example 3.7. For any Lie group G, G acts on itself both from the left as well as from the right, where the actions are given by the group multiplication itself. The actions are free, transitive and proper and the orbit spaces consist of a single point. The infinitesimal action $\rho : \mathfrak{g} \to \mathfrak{X}(G)$ corresponding to the right action sends u to $\overrightarrow{u}$ from (3.1.2).

> **Exercise 3.8.** Given a G-torsors P, as explained above, any $p \in P$ induces an equivariant diffeomorphism between $h_p : G \to P$. How does it depend on the choice of p? What happens if you try to use these diffeomorphisms to put a group structure on P?

Example 3.9. The group $G = R^*$ of nonzero real numbers (with multiplication) acts on R^{n+1} by multiplication. The action is not free since the isotropy group at the origin is the entire G. However, if we restrict the action to $P = R^{n+1} \setminus \{0\}$ the actions will be free. We leave it as an exercise to prove that the action is also proper. Notice that, in this case, the resulting quotient $P/G = \left(R^{n+1} \setminus \{0\}\right)/\mathbb{R}^*$ can be identified with the projective space $\mathbb{P}^n$.

Example 3.10. The action of R^* on $R^{n+1} \setminus \{0\}$ mentioned above restricts to an action of $G = \{-1, 1\}$ on S^n. Identifying G with $\mathbb{Z}_2$, that action becomes the reflection w.r.t. the origin:

$$x \cdot \widehat{k} := (-1)^k x \quad (x \in S^n, \widehat{k} \in \mathbb{Z}_2).$$

Since each line through the origin intersects the sphere in precisely two points x and $-x$, one obtains a canonical identification $S^n/\mathbb{Z}_2 \cong \left(R^{n+1} \setminus \{0\}\right)/\mathbb{R}^* = \mathbb{P}^n$. One advantage of the restricted action is that the properness of the action becomes more transparent (since $\mathbb{Z}_2$ is compact) and implies right away that the orbit space is compact.

Example 3.11. Consider the action of $G = S^1$ on $P = S^2$ by rotations around the OZ axis:

$$(x,y,z) \cdot (\cos\theta, \sin\theta) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z).$$

The action is proper but not free. The isotropy group at a point $p = (x,y,z) \in S^2$ is trivial except for the case when p is the north or south pole, when the isotropy group is the entire S^1. In particular, the action is not free.

The action is also not transitive. The orbits are the horizontal circles inside S^2, namely $x^2+y^2 = 1-z^2$, one for each $z \in [-1,1]$. We see that the resulting quotient S^2/S^1 can be identified with $[-1,1]$.

Example 3.12. Any Lie subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ comes with a canonical right action on $\mathbb{R}^n$- the one given by the matrix multiplication:

$$\mathbb{R}^n \times G \to \mathbb{R}^n, \quad (v,A) \mapsto v \cdot A.$$

Such actions are called *linear*. Except for when G is trivial (just the unit), they are never free since the origin is fixed by all the elements, i.e., $G_0 = G$. We also deduce that, for linear actions, properness is equivalent to the compactness of G. Notice that, interpreting matrices A as linear maps $\widehat{A}$ (see subsection 1.2.1 in the bachelor notes), G also comes with a natural left action

$$G \times \mathbb{R}^n \to \mathbb{R}^n, \quad (A,v) \mapsto \widehat{A}(v).$$

However, be aware that turning this left action into a right using the general recipe (3.1.4) is not quite the matrix multiplication but $v \bullet A := v(A^{-1})^T$. On the other hand, since $A \to (A^{-1})^T$ is a group isomorphism (from where to where?), the two right actions are "essentially the same".

Example 3.13. The linear action of $O(n+1)$ on $\mathbb{R}^{n+1}$, since it is an action by isometries, restricts to an action on $P = S^n$. This new action is both proper as well as transitive. Because the action is transitive, the isotropy groups at any two points, although different as subgroups of $O(n+1)$, they are isomorphic as Lie groups (prove this for all transitive actions!). Computing the isotropy group at the north pole p_N one finds a copy of $O(n)$ inside $O(n+1)$ sitting as:

$$O(n) \ni B \longrightarrow \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \in O(n+1).$$

Notice also that the orbit space $O(n+1)/O(n)$ can be identified with S^n via the map $A \cdot O(n) \mapsto p_N \cdot A$.

Example 3.14. For any Lie subgroup H of a Lie groups G, H acts on G via the group multiplication,

$$G \times H \to G, \quad (g,h) \mapsto gh$$

(hence, this time, $P = G$, while what was before G is now H!). This action is free and proper and the resulting quotient is the collection

$$\{g \cdot H : h \in H\}$$

of right cosets of H.

3.2 Principal bundles

3.2.1 G-torsors

When discussing vector bundles E of rank r, their fibers E_x were r-dimensional vector spaces- hence just isomorphic copies $\mathbb{R}^r$, just that the identification of E_x with $\mathbb{R}^r$ required a choice (of a frame). The situation will be similar for principal G-bundles: the fibers will be G-spaces that are just isomorphic copies of G, just that the identification with G is not canonical. That is what G-torsors are, though it is customary to adopt a more intrinsic definition:

Definition 3.15. A **G-torsor** is any G-manifold P for which the action is free and transitive.

Notice that, in this case, for $p,q \in P$ the element $g \in G$ from (3.1.6) is unique. It is called the **division of** p **by** q and is denoted by $[p:q]$. Therefore, it is the unique element of G such that

$$q = p \cdot [p:q].$$

The discussion below implies that the division map the division map $[\cdot:\cdot] : P \times P \to G$ is smooth.

Theorem 3.16. *A G-manifold P is a G-torsors if and only if it is G-diffeomorphic to G.*

Proof. Since freeness and transitiveness are preserved by G-diffeomorphisms, and they hold for G, the reverse implication follows. Assume now that P is a G-torsor. We fix $p \in P$ and consider $m_p : G \to P$. By hypothesis, this is a smooth bijection. Its differential at e is the infinitesimal action hence, by the discussion above (e.g. use (3.1.9)), it is an isomorphism. As in the previous proof (or using that $m_{pa} = m_p \circ L_a$), it follows that m_p is a local diffeomorphism at all $a \in G$, and the conclusion follows.

Example 3.17. Here is an example which makes the previous comparison more concrete. Given an r-dimensional vector space V, recall from subsection 1.1.3 that $\mathrm{Fr}(V)$ denotes the collection of all frames of V, where we think of a frame $\mathfrak{e} = (\mathfrak{e}_1, \dots, \mathfrak{e}_r)$ also as the linear isomorphism

$$\mathfrak{e} \equiv \hat{\mathfrak{e}} : \mathbb{R}^r \to V, \quad \hat{\mathfrak{e}}(v^1, \dots, v^r) = \sum v^i \mathfrak{e}_i.$$

The collection of frames comes with a natural action of GL_r from the right,

$$\mathrm{Fr}(V) \times \mathrm{GL}_r \to \mathrm{Fr}(V), \quad (\mathfrak{e}, A) \mapsto \mathfrak{e} \cdot A, \tag{3.2.1}$$

defined so that $\widehat{\mathfrak{e} \cdot A} = \hat{\mathfrak{e}} \circ \hat{A}$. Here we use our notations/conventions from the bachelor notes: Part C (Directional/partial derivatives) of the reminder on Analysis: for matrices $A = (A^i_j)_{1 \le i,j \le r} = \begin{pmatrix} A^1_1 & \dots & A^1_r \\ \dots & \dots & \dots \\ A^r_1 & \dots & A^r_r \end{pmatrix}$, $\widehat{A}$ is linear map $\hat{A} : \mathbb{R}^r \to \mathbb{R}^r$, that sends (e_j) to $\sum_{i=1}^r A^i_j e_i$. For the action (3.2.1) one obtains the explicit description

$$(\mathfrak{e} \cdot A)_i = \sum^i A^j_i \mathfrak{e}_j$$

or, in a matrix-looking form:

$$((\mathfrak{e} \cdot A)_1 \ \dots \ (\mathfrak{e} \cdot A)_r) = (\mathfrak{e}_1 \ \dots \ \mathfrak{e}_r) \cdot \begin{pmatrix} A^1_1 & \dots & A^1_r \\ \dots & \dots & \dots \\ A^r_1 & \dots & A^r_r \end{pmatrix}.$$

In this way $Fr(V)$ becomes a GL_r-torsor. Furthermore, the identification of $\mathrm{Fr}(V)$ with GL_r as torsors requires the same data as the one of V with $\mathbb{R}^r$ as vector spaces.

Remark 3.18 (different contexts). As in Remark 3.3, when working in other categories than the one of (2nd) countable manifolds, one has to use the correct concept of transitivity to ensure that the analogue of Theorem 3.16 holds true.

3.2.2 Principal bundles

We now move to principal bundles. We start with the shortest description.

Definition 3.19. Let M be a manifold and G a Lie group. A **principal G-bundle** over M,

$$\pi : P \to M,$$

consists of a G-manifold P and a surjective submersion π with the property that the action restricts to the fibers and makes each fiber into a G-torsor or, more explicitly:

(1) π is G-invariant.
(2) each fiber, when endowed with the induced action, is a G-torsor.

The reader will rightly wonder: where is the local triviality condition? As we shall see, because we work in the smooth context, that will follow automatically. For now let us point out that there are several interesting ways to pack/rephrase (1) and (2). For instance, without explicit reference to torsors:

$$\left\{\begin{matrix}(1)\\(2)\end{matrix}\right\} \iff \left\{\begin{matrix}(1')\text{ the action of } G \text{ on } P \text{ is free}\\(2')\text{ the fibers of } \pi \text{ are precisely the orbits of the action}\end{matrix}\right\}$$

Another reformulation makes use of the fibered product

$$P \times_M P = \{(p,q) \in P \times P : \pi(p) = \pi(q)\}, \tag{3.2.2}$$

which allows us to repack (1) and (2) as follows:

$$\left\{\begin{matrix}(1)\\(2)\end{matrix}\right\} \iff (3)\text{: the map } (p,g) \mapsto (p,pg) \text{ is a bijection } P \times G \xrightarrow{\sim} P \times_M P$$

The advantage of this reformulation is that it indicates a stronger property: that the map is actually a diffeomorphism. Here the smooth structure on the fibered product comes from the fact that it is an embedded submanifold of $P \times P$ (as a consequence of the fact that π is a submersion). Therefore:

$\Longrightarrow$ A principal G-bundle is the same thing as a G-manifold P together with a G-invariant surjective submersion $\pi : P \to M$ with the property that the map

$$A : P \times G \to P \times_M P, \quad (p,g) \mapsto (p,pg) \tag{3.2.3}$$

is a bijection or, by Corollary 3.24 below, a diffeomorphism.

As for vector bundles, one introduces the following terminology/notations:

- **fibers** of the principal G-bundle P: the fibers of π, $P_x := \pi^{-1}(x)$.
- **sections** of P: smooth maps $\sigma : M \to P$ satisfying $\pi \circ \sigma = \mathrm{Id}_M$.
- **local sections** of P: defined similarly, but with σ defined only over some open $U \subset M$.

- **isomorphisms** between two principal G-bundles over M, $\pi_i : P_i \to M$, $i \in \{1,2\}$: G-equivariant diffeomorphisms $F : P_1 \to P_2$ that take preserve the fibers (or, equivalently, $\pi_2 \circ F = \pi_1$).
- the **trivial principal G-bundle** over M: $M \times G$ endowed with the second projection and the action of G given by $(x,a) \cdot g := (x, ag)$.
- **trivializations** of the principal G-bundle P: isomorphisms with the trivial principal G-bundle $M \times G$. When such a trivialization exists we say that P is **trivializable**.

Example 3.20 (frame bundles). Example 3.17 can now be upgraded to one of principal GL_r-bundles by applying it fiberwise. More precisely, given a rank r vector bundle $E \in \mathrm{Vect}^r(M)$, we define the **frame bundle of** E

$$\mathrm{Fr}(E) := \{(x, \mathfrak{e}_x) : x \in M,\ \mathfrak{e}_x\text{- a frame of } E_x\},$$

endowed with the projection

$$\pi : \mathrm{Fr}(E) \to M, \quad \pi(x, \mathfrak{e}_x) = x$$

and endowed with the right action of GL_r,

$$\mathrm{Fr}(E) \times \mathrm{GL}_r \to \mathrm{Fr}(E), \quad (x, \mathfrak{e}_x) \cdot A = (x, \mathfrak{e}_x \cdot A).$$

Notice that any $\mathrm{Fr}(E)$ sits inside $\underline{\mathrm{Hom}}(M \times \mathbb{R}^r, E)$, and it is actually an open subset. Therefore, $\mathrm{Fr}(E)$ carries a natural smooth structure. With this structure, $\pi : \mathrm{Fr}(E) \to M$ is a principal GL_r-bundle. Proving this is an instructive exercise that the reader may do now, or wait until the discussion from Example 3.23. Notice that local sections of $\mathrm{Fr}(E)$ are nothing but local frames of E.

The reason that we only look at isomorphisms and not "more general morphisms" is the following:

Lemma 3.21. *Given two principal G-bundles over M, $\pi_i : P_i \to M$, $i \in \{1,2\}$, any G-equivariant $F : P_1 \to P_2$ satisfying $\pi_2 \circ F = \pi_1$ is automatically an isomorphism.*

Proof. First the injectivity of F: assume that $F(p) = F(q)$ with $p, q \in P_1$. Since F preserves the fibers, p and q must be in the same fiber and, therefore, $q = p \cdot g$ for some $g \in G$. Applying F and using its equivariance one obtains $F(q) = F(p) \cdot g$ hence, since $F(p) = F(q)$ and the action of G on P_2 is free, g must be the identity- therefore $q = p$.

Next, the surjectivity of F. Let $p_2 \in P$. Denoting $x = \pi_2(p_2)$ we choose an arbitrary element $p_1 \in P_1$ in the fiber of P_1 above x. That implies that $F(p_1)$ and p_2 are both in the fiber of P_2 above x. Therefore, $p_2 = F(p_1) \cdot g$ for some $g \in G$. Therefore $p_2 = F(p_1 \cdot g)$ is in the image of F.

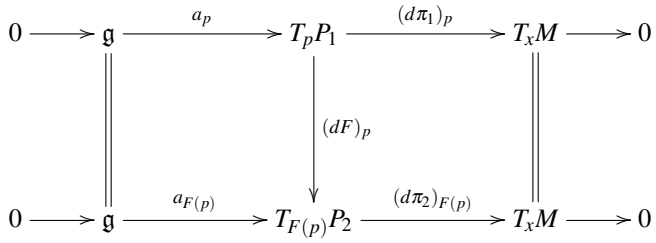
It suffices now to show that F is a local diffeomorphism or, equivalently, that

$$(dF)_p : T_p P_1 \to T_{F(p)} P_2$$

is a (linear) isomorphism for all $p \in P$. To show this we first show that for any principal G-bundle $\pi : P \to M$ and any $p \in M$, one has a short exact sequence of vector spaces:

$$0 \longrightarrow \mathfrak{g} \xrightarrow{a_p} T_p P \xrightarrow{(d\pi)_p} T_{\pi(p)} M \longrightarrow 0$$

where $\mathfrak{g} = T_e G$ and a_p is the differential at e of the map $h_p : G \to P$, $h_p(g) = g \cdot p$. This follows from the regular value theorem for π which tells us that the kernel of $(d\pi)_p$ is precisely the tangent space $T_p P_x$ of the fiber, combined with the fact that $h_p : G \to P_x$ is a diffeomorphism since P_x is a G-torsor. Applying this short exact sequence to P_1 at p, to P_2 at $F(p)$, denoting $x = \pi_1(p) = \pi_2(F(p))$, and using the chain rule one obtains a diagram in which all the squares are commutative:



The fact that the middle vertical map is an isomorphism follows now by a cute diagram chasing using the exactness of the horizontal sequences. For instance, for the injectivity, assuming $v_1 \in T_pP_1$ is in the kernel of $(dF)_p$, the commutativity of the right square tells us that v_1 is mapped to 0 also by $d\pi_1$. Hence we can write $v_1 = a_p(v)$ with $v \in \gg$. The commutativity of the diagram on the left now implies that $0 = (dF)(v_1) = a_{F(p)}(v)$, hence $v = 0$, and then $v_1 = 0$. The surjectivity is proven similarly, but one can also invoke the fact that an injective map between two finite dimensional vector spaces of the same dimension must be an isomorphism. 😄

The lemma above has an immediate consequence regarding (local) trivializations. Indeed, according to the lemma, to construct an isomorphism from $M \times G$ and P it suffices to construct a map $F : M \times G \to P$ such that $F(x,g) \in P_x$ for all x and F is G-equivariant. The last condition means that

$$F(x,g) = F(x,1) \cdot g,$$

i.e. F is uniquely determined by the values $\sigma(x) := F(x,1) \in P_x$. Therefore one obtains a 1-1 correspondence

$$\left\{\begin{array}{c}\text{(local)}\\ \text{trivializations}\\ \text{of } P\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{(local)}\\ \text{sections}\\ \text{of } P\end{array}\right\} \tag{3.2.4}$$

This is the principal bundle analogue of (1.1.5). This also clarifies one little mystery mentioned above, that local triviality was not required. Indeed, with the remark that any surjective submersion admits, around each point in the codomain, a local sections (as it follows right away from the submersion theorem- see Theorem 2.125 in the bachelor notes), Theorem we immediately deduce:

Corollary 3.22. *Any principal G-bundle is locally trivial in the sense that for any $x_0 \in M$ there exists an open $U \subset M$ containing x_0 such that $P|_U := \pi^{-1}(U) \to U$ is trivializable.*

Example 3.23 (frame bundles). In continuation of Example 3.20, notice that our habit or reinterpreting the local trivializations of a vector bundle E as local frames of E, see (1.1.5), now reads:

$$\left\{\begin{array}{c}\text{(local)}\\ \text{trivializations of } E\\ \text{(as vector bundle)}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{(local)}\\ \text{frames of } E\\ \text{(as vector bundle)}\end{array}\right\} = \left\{\begin{array}{c}\text{(local)}\\ \text{sections of } \mathrm{Fr}(E)\\ \text{(as principal bundle)}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{(local)}\\ \text{trivializations of } \mathrm{Fr}(E)\\ \text{(as principal bundle)}\end{array}\right\}$$

Explicitly: any local frame $\mathfrak{e} = \{e_1, \ldots, e_r\}$ of E over U produces the local trivialization of $\mathrm{Fr}(E)$

$$U \times \mathrm{GL}_r \to \mathrm{Fr}(E)|_U, \quad (x,A) \mapsto (x, \mathfrak{e}(x) \cdot A). \quad 😄$$

Notice that the last corollary implies that the fiberwise division map defined on the fibered product (3.2.2),

$$[\cdot : \cdot] : P \times_M P \to G,$$

(which associates to $p,q \in P$ in the same fiber the unique element $[p:q]$ of G satisfying $q = p\cdot[p:q]$) is smooth. Indeed, this is a property that can be checked locally, and it is clearly true for the trivial principal bundle. Since the map $(p,q)\mapsto(p,[q:p])$ is the inverse of (3.2.3) we deduce that:

Corollary 3.24. *For any principal G-bundle the map (3.2.3) is a diffeomorphism $P\times G \to P\times_M P$.*

Corollary 3.22 also ensures that principal G-bundles could be introduced in complete analogy with vector bundles by requiring:

(1) a manifold P together with a surjective map $\pi : P\to M$
(2) a structure of G-torsor on each fiber

$$P_x := \pi^{-1}(x) \quad (x\in M)$$

satisfying the following local triviality condition:

(LT) for each $x_0\in M$, there exists an open neighborhood U of x_0 s.t., over U, P is isomorphic to the trivial principal bundle $U\times G$ in the sense that there exists a diffeomorphism

$$h : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U\times G$$

that takes the fiber P_x to $\{x\}\times G$, G-equivariantly.

However, it is more customary to rephrase this a bit and remove any reference to G-torsors, therefore

$\Longrightarrow$ A principal G-bundle over M is the same thing as:

(1') a manifold P together with a surjective map $\pi : P\to M$
(2') an action of G on P such that π is G-invariant, i.e., $\pi(pg)=\pi(p)$ for all $p\in P$ and $g\in G$.

while with the local triviality condition stated as:

(LT') for each $x_0\in M$, there exists an open neighborhood U of x_0 and a G-equivariant diffeomorphism h making the following diagram commutative

$$\begin{array}{ccc} P|_U := \pi^{-1}(U) & \xrightarrow{\ h\ } & U\times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

Remark 3.25 (different contexts). One explanation is in order: when adopting the simpler Definition 3.19 we are really making use of the context of smooth manifolds and the 2nd countability axioms. As pointed out in remarks 3.3 and 3.18, when working in other categories then one has to be more careful- for instance the local triviality has to be imposed. In that respect, the axioms (1)-(2)-(LT) or (1')-(2')-(LT') are more useful.

As for vector bundles (see (1.1.7) and the discussion around it), principal bundles can also be encoded by transition functions. We now talk about **transition data on M with values in G**, by which we mean an open cover

$$\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}, \tag{3.2.5}$$

for some index set Λ, together with smooth functions

$$g_{\alpha,\beta} : U_\alpha\cap U_\beta \to G \tag{3.2.6}$$

satisfying, for any $\alpha,\beta,\gamma\in\Lambda$:

$$g_{\alpha,\beta}(x)\cdot g_{\beta,\gamma}(x)=g_{\alpha,\gamma}(x),\quad \forall\, x\in U_\alpha\cap U_\beta\cap U_\gamma.$$

Such a data arises whenever we start with a principal bundle $\pi: P\to M$ and an open cover $\mathscr{U}$ together with trivializations of $P|_{U_\alpha}$. These trivialisations are encoded in local sections σ_α and then, when x is in an overlap $U_\alpha\cap U_\beta$, $g_{\alpha,\beta}(x)$ arises by comparing $\sigma_\alpha(x)$ and $\sigma_\beta(x)$

$$g_{\alpha,\beta}(x):=[\sigma_\beta(x):\sigma_\alpha(x)].$$

Exercise 3.26. State and prove the analogue of Proposition 1.13.

The conclusion is, again, that modulo an appropriate notion of equivalence of equivalence of transition data, one obtains a 1-1 correspondence:

$$\left\{\begin{array}{c}\text{principal } G\text{-bundles over } M,\\ \pi: P\to M\\ \text{(up to isomorphisms)}\end{array}\right\}\overset{\sim}{\longleftrightarrow}\left\{\begin{array}{c}\text{transition data } (g_{\alpha,\beta})\\ \text{on } M \text{ with values in } G\\ \text{(up to } \sim \text{ equivalence)}\end{array}\right\} \tag{3.2.7}$$

Of course, when applied to the $G=GL_r$ and $P=\mathrm{Fr}(E)$ the frame bundle of a rank r vector bundle, one obtains precisely the transition functions of the vector bundle E. Hence this discussion shows that, up to isomorphisms, looking at rank r vector bundles is equivalent to looking at principal GL_r-bundles. A more direct/global/intrinsic description will be provided a bit later.

The following two exercises make more transparent the statement we made above, when discussing push-forwards ρ_* along group homomorphisms, namely that "in some sense, this is the analogue of the other operations on vector bundles such as direct sums, duals".

3.2.3 Smooth quotients, or ... principal bundles via free and proper actions

Let us now return to actions of Lie groups G on manifolds P and describe how, the question of understanding the quotient space P/G leads to principal bundles as well as to a perspective on principal bundles that is extremely useful in examples.

Therefore, let us start with a general action of G on P and the attempt of identifying the resulting orbit space with a more concrete S. Saying that "the orbit space P/G can be identified with S" is rather vague: one does not specify the identification (instead one just appeals to one's intuition) and one does not specify what type of identification one is talking about: as sets, as spaces, as etc?

As a set: It is clear that the minimal data is that of a bijection $\phi: P/G\to S$. Therefore, let us notice/recall that describing such a bijection is the same thing as describing a surjective map $\pi: P\to S$ whose fibers are precisely the orbits of the action or, equivalently, for any $p,p'\in P$:

$$\pi(p')=\pi(p)\Longleftrightarrow p'=p\cdot g \text{ for some } g\in G. \tag{3.2.8}$$

Of course, $\pi_\phi=\phi\circ\pi_{\mathrm{can}}$, i.e.

$$\pi(p)=\phi(p\cdot G)\quad \forall p\in P.$$

Conversely, starting from π one can use the last formula to define $\phi: P/G\to S$, and the condition (3.2.8) on π ensures that ϕ is well-defined and a bijection.

As a space: On the other hand, P/G carries a natural topology- the quotient topology-, which is the smallest topology on P/G with the property that the canonical projection (3.1.5) is continuous, explicitly defined by:

$$V\subset P/G \text{ is open in the quotient}\quad\Longleftrightarrow\quad \pi_{\mathrm{can}}^{-1}(V) \text{ is open in } P.$$

Therefore, if S is a topological space one may want to "upgrade" the identification of P/G and S to one of spaces. In other words one looks for homeomorphisms $\phi : P/G \to S$. Notice that the continuity of ϕ is equivalent to the one of the corresponding $\pi : P \to S$, but one should keep in mind that continuous bijections are not automatically homeomorphisms, so a bit of extra-work may be required.

As a manifold: Similarly, if S is a smooth manifold, one may want to further upgrade the identification to one of manifolds. There is a little problem however: what is the smooth structure on P/G? Looking for such a smooth structure, there is a natural condition to impose: that the canonical projection (3.1.5) is a submersion. Furthermore, we do know that such a smooth structure on P/G, if it exists, then it is unique- see Corollary 2.135 of section 2.4.2 in the bachelor notes. When such a smooth structure exists we say that **the quotient P/G is smooth**. We claim that:

free and proper action $\implies$ smooth quotient

More precisely, one has the following (very useful!) criteria for smoothness.

Theorem 3.27. *If an action of a Lie group G on a manifold P is free and proper then P/G is smooth. Furthermore, together with the canonical projection (3.1.5), P becomes a principal G-bundle over P/G.*

Proof. Let us start with a remark which, although will not be used in the proof, it provides some insight on how the proof was found. The proof of the result mentioned above (Corollary 2.135 of section 2.4.2 in the bachelor notes). is based on the local form of submersions (Theorem 2.125 in the bachelor notes). The interesting remark is that proof not only implies the uniqueness part of the theorem but also indicates how to proceed with constructing the smooth structure.

We will show that for any $p_0 \in P$, one finds an invariant open neighborhood U of p_0 and a smooth structure on $\pi_{\text{can}}(U)$ such that $\pi_{\text{can}}|_U : U \to \pi_{\text{can}}(U)$ is a submersion; then $\pi_{\text{can}}(U)$s can be used as domains of charts and the (smooth) compatibility of the charts will follow using again the uniqueness property.

Therefore, fix $p_0 \in P$ and look for U. We make us of the infinitesimal action $\rho_{p_0} : \mathfrak{g} \to T_{p_0}P$ and we choose a submanifold S of P containing p_0 such that

$$\rho_{p_0}(\mathfrak{g}) \oplus T_{p_0}S = T_{p_0}P.$$

It may be helpful to keep in mind that S will be important only in a neighborhood of p_0. Consider the map

$$f : S \times G \to P,\ f(p,g) = p \cdot g.$$

Computing the differential at the point (p_0, e) we find

$$(df)_{p_0,e} : T_{x_0}S \times T_eG \to T_{p_0}P, (v,\alpha) \mapsto v + \rho_{p_0}(\alpha).$$

Since this is an isomorphism, we can ensure that $(df)_{p,e}$ is an isomorphism for all $p \in S$, provided we further shrink S. Therefore f is a local diffeomorphism around all points (p,e) with $p \in S$. Furthermore, using left translations and the G-invariance of f, it follows that the same is true at all points (p,g) with $p \in S$, $g \in G$. We claim that, after eventually shrinking S even more, one may assume that the map f is injective on $S \times G$. We do that by assuming the contrary: then we find $(p_k, g_k) \neq (p_k^{'}, g_k^{'})$ in $S \times G$ such that $p_k g_k = p_k^{'} g_k^{'}$ (which do imply that $g_k \neq g_k^{'}$) and both (p_k) and $(p_k^{'})$ converge to p_0. Setting $a_k = g_k (g_k^{'})^{-1}$, we have

$$p_k^{'} = p_k a_k,\ \ a_k \neq e$$

and $(p_k a_k, p_k)$ converges to (p_0, p_0); in particular, this sequence will stay inside a compact region hence, using the properness of the action (and the fact that compacts are sequentially compact) it follows, eventually after passing to a subsequence of a_k, that (a_k) is convergent to some $a \in G$. Since both p_k and $p_k a_k = p_k^{'}$ converge to p_0, we find $p_0 \cdot a = p_0$ hence, by freeness, $a = e$. Now, we have

$$f(p_k, e) = f(p_k a_k, a_k^{-1})$$

and both (p_k, e) and $(p_k a_k, a_k^{-1})$ belong to $S \times G$ and converge to (e, p_0); moreover, we also know that f is injective (even a diffeomorphism) in a neighborhood V of (p_0, e) in $S \times G$. Hence, for k large enough, $(p_k, e) = (p_k a_k, a_k^{-1})$; in particular, $a_k = e$, which gives the desired contradiction.

Since we have arranged that $f : S \times G \to P$ is an injective local diffeomorphism, it will be a diffeomorphism into its image; denoting by U the image (necessarily an open in P!), f will identify U equivariantly with $S \times G$ and the restriction of π to U with the projection $S \times G \to S$, proving the desired existence of the smooth structure. 😄

Returning to the discussion of how to identify a quotient P/G with S, we can now upgrade the topological discussion to the smooth context. Assuming that we have already built a homeomorphism $\phi : P/G \to S$, what is left to check conclude that it is a diffeomorphism is very little: that the corresponding $\pi : P \to S$ is a submersion. More concretely, we have the following reformulation of the previous result, reformulation that is useful when we would like to make a given S into a manifold by identifying it with a quotient.

Theorem 3.28 (reformulation of Theorem 3.27). *Let S be a set and assume we have a manifold P and a surjective map*

$$\pi : P \to S$$

whose fibers coincide with the orbits of a free and proper action of a Lie group G on P. Then S can be made into a smooth manifold in one and only one way such that π is a submersion.

Furthermore, $\pi : P \to S$ becomes a principal G-bundle.

Finally, we obtain the following reformulation of the notion of principal bundle.

Corollary 3.29 (Principal G-bundles $\equiv$ manifolds endowed with a free and proper G-action).

(1) For any principal G-bundle $\pi : P \to M$ the action of G on P is free and proper.

(2) Given a manifold P endowed with a free and proper action whose orbits are the fibers of a surjective map $\pi : P \to M$, M can be made into a manifold, uniquely, so that π becomes a principal G-bundle.

Proof. Item (2) follows immediately from the previous theorem. For (1) we have to prove the properness of the map $P \times G \to P \times P$, $(p, g) \mapsto (p, pg)$. With the diffeomorphism from Corollary 3.24 at hand, this map can be identified with the inclusion $P \times_M P \hookrightarrow P \times P$- and then we are done because, in general, inclusions of closed subsets of a Hausdorff space are proper.

Example 3.30 (Quotients G/H). Start, as in Example 3.14, with a Lie group G and a closed subgroup

$$H \subset G$$

which is an embedded submanifold (hence a Lie group itself). However, in this discussion it is more convenient to consider the *left* action of H on G, namely

$$H \times G \to G, \quad (h, g) \mapsto h \cdot g \quad (\text{multiplication in } G),$$

so that the resulting quotient consists of left cosets

$$G/H = \{Ha : a \in G\}.$$

The previous result applies to the H-action on $P = G$, with the conclusion that G/H is smooth. The reason we considered the left action of H is that the resulting quotient now carries a canonical right action of G:

$$\underbrace{Ha}_{\text{in } G/H} \cdot \underbrace{g}_{\text{in } G} = \underbrace{H(ag)}_{\text{in } G/H}$$

This action is clearly transitive. The following shows that these are all the possible transititve (right) actions.

Proposition 3.31. *Any manifold P endowed with a transitive action of G is G-equivariantly diffeomorphic to one of type G/H. More precisely, choosing $p \in P$ (arbitrary) and $H = G_p$, one has a G-equivariant diffeomorphism:*

$$G/H \to P, \quad Hg \mapsto p \cdot g.$$

In particular, $G \to P$, $g \mapsto p \cdot g$ is a principal G_p-bundle.

Proof. Fix $p \in P$, $H = G_p$ and consider the resulting bijection from the statement. To conclude that it is a diffeomorphism we use the uniqueness part of the previous theorems. We are left with proving that $m_p : G \to P$ is a submersion. At the unit $e \in G$ its differential is the infinitesimal action ρ_p which is surjective as it follows from the transitivity of the action and (3.1.9). At an arbitrary $a \in G$ we can use the commutative diagram (3.1.10) again to write $m_p = R^P_a \circ m_p \circ R^G_{a^{-1}}$, and then

$$(dm_p)_a = \left(dR^P_a\right)_p \circ (dm_p)_e \circ \left(dR^G_{a^{-1}}\right)_a : T_aG \to T_{pa}P,$$

proving that m_p satisfies the submersion condition at all points $a \in G$. 😄

Example 3.32. In the list of examples above there is one that is a non-trivial transitive one: the action of $O(n+1)$ on $P = S^n$ from Example 3.13. Since the isotropy group was computed to be $O(n)$, one obtains a diffeomorphism

$$S^n \cong O(n+1)/O(n).$$

Corollary 3.33. *For any action of a Lie group G on a manifold P, any orbit $\mathcal{O} \subset P$ of the action is an immersed submanifold whose tangent space at any $p \in \mathcal{O}$ is the image of the infinitesimal action at p:*

$$T_p\mathcal{O} = Im(\rho_p : \mathfrak{g} \to T_pP).$$

Proof. Choose $p_0 \in \mathcal{O}$. Then the previous discussions imply that

$$i_{p_0} : G/G_{p_0} \to P$$

is an immersion whose image is $\mathcal{O}$. That makes $\mathcal{O}$ into an immersed submanifold and, by a translation argument, the smooth structure does not depend on the choice of p_0. For $p \in \mathcal{O}$, $T_p\mathcal{O}$ is the image of the differential of i_p. But the projection $G \to G/G_p$ is a submersion whose composition with i_p is $m_p : G \to P$, hence $T_p\mathcal{O}$ is the image of $(dm_p)_e = \rho_p$, 😄

3.2.4 Examples/constructions of principal bundles

Example 3.34 (The Hopf fibration). The Hopf fibration makes S^3 into a principal S^1-bundle over S^2: the projection map is

$$H : S^3 \to S^2, \quad H(x,y,z,t) = (x^2+y^2-z^2-t^2, 2(yz-tx), 2(ty+xz)), \tag{3.2.9}$$

while the S^1-action is

$$(x,y,z,t) \cdot (a,b) = (ax-by, ay+bx, az-bt, at+bz).$$

Or, using complex coordinates and

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}, \quad S^3 = \{(z_0,z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\},$$

one has the more compact description

$$(z_0,z_1)\cdot\lambda := (z_0\lambda, z_1\lambda),\ H(z_0,z_1) = (|z_0|^2-|z_1|^2, i\bar{z}_0 z_1). \tag{3.2.10}$$

The fact that this is a principal S^1-bundle follows by checking the conditions in Definition 3.19 (fill in the details!).

Example 3.35 (The generalized Hopf fibration). The previous example can be generalized to higher dimensions as follows, where the gain is not only the generalization itself but also more insight into the classical Hopf fibration: that the base S^2 appears because it is a copy of $\mathbb{CP}^1$ (if you think that that just complicates things, look at the projection map from S^3!). In arbitrary dimensions, S^{2n+1} into principal S^1-bundles over the complex projective spaces $\mathbb{CP}^n$. More precisely, using again complex numbers so that $S^{2n+1}\subset\mathbb{C}^{n+1}$, the projection map is

$$H_n : S^{2n+1}\to\mathbb{CP}^n,\quad H_n(z_0,\dots,z_n) = [z_0:\dots:z_n],$$

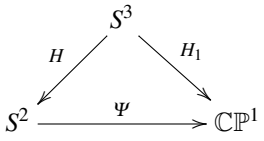
where $[z_0:\dots:z_n]$ is the complex line through the origin and the point $(z_0:\dots:z_n)$, and the S^1-action is given by multiplication of all coordinates. The fact that this is

The Hopf fibration can be "recovered" when $n=1$ as follows:

Exercise 3.36. When $n=1$ show that there exists a diffeomorphism

$$\Psi : S^2 \xrightarrow{\sim} \mathbb{CP}^1 \tag{3.2.11}$$

uniquely determined by:



$$,\quad H_1 = \Psi\circ H.$$

Example 3.37 (Circle bundles from complex line bundles). The last example can be generalised even further, making the Hopf fibrations even more transparent. The key point is the fact that given a complex line bundle $L\to M$, if we have a (Hermitian) inner produce on L we can form the circle bundle

$$S(L) := \{v\in L : h(v,v)=1\}.$$

This is a smooth submanifold of L on which $\pi : L\to M$ restricts to a submersion (this can be checked locally, e.g. using local orthonormal frames). There is also a free action of S^1 given by scalar multiplication. All together, we see that $S(L)$ becomes a principal S^1-bundle over M. This can also be seen as a variation of the frame bundle construction, but using orthonormal frames w.r.t. h which, because we are in complex rank ones, boils down to vectors of norm 1.

Notice that when applied to the tautological line bundle $L = L_{\mathbb{CP}^n}$ with its standard inner product, an element $(l,v)\in S(L)$ is uniquely determined by $v\in\mathbb{C}^{n+1}$ of norm 1, hence

$$S(L_{\mathbb{CP}^n}) = S^{2n+1}.$$

Example 3.38. Another example, closely related to the Hopf fibration, is the principal S^1-bundle

$$\pi : SO(3)\to S^2 \tag{3.2.12}$$

which associates to a matrix $A\in SO(3)$ its first column, with the right action of S^1 on $SO(3)$ given by

$$A\cdot\lambda = A\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\alpha) & \sin(\alpha)\\ 0 & -\sin(\alpha) & \cos(\alpha)\end{pmatrix}\quad \text{for } \lambda = \cos(\alpha)+i\sin(\alpha)\in S^1, \tag{3.2.13}$$

We now describe the relationship with the Hopf fibration. To that end, we will identify S^3 with the spaces of quaternionic numbers of norm 1,

$$S^3 \xrightarrow{\sim} \{u \in \mathbb{H} : |u| = 1\},\ (x,y,z,t) \mapsto x + iy + jz + kt$$

and we also identify $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\},\ (a,b,c) \mapsto ai + bj + ck.$$

For each $u \in S^3$, $A_u(v) := u^* v u$ defines a linear map $A_u : \mathbb{R}^3 \to \mathbb{R}^3$ which, in turn, is an element $A_u \in SO(3)$. One can now check that the resulting map

$$\phi : S^3 \to SO(3),\ u \mapsto A_u$$

is compatible with the principal bundles projections:

$$\begin{array}{ccc} S^3 & \xrightarrow{\quad\phi\quad} & SO(3)\ , \quad \pi \circ \phi = H. \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S^2 & \end{array}$$

However, it is not S^1-equivariant. Instead, one has

$$\phi(u\lambda) = \phi(u) \cdot \lambda^2 \ \ \forall\, u \in S^3,\ \lambda \in S^1. \tag{3.2.14}$$

This, and the conclusion on the relationship between the two fibrations, will become clearer below, once we discuss operations with principal bundles.

Exercise 3.39. Fill in the details.

Example 3.40 (Orthogonal groups as principal bundles over spheres). Let us return to the linear actions on $\mathbb{R}^{n+1}$ from Examples 3.12 and 3.13. The ones that preserve the canonical inner product restrict to actions on S^n. When the action is transitive, we end up with a principal bundle over S^n. For instance, for the group $G = O(n+1)$ we obtain

$$O(n+1) \to S^n, \quad A \mapsto p_N \cdot A = \text{the first row of } A,$$

as a principal bundle with Lie group $O(n+1)_{p_N} = O(n)$. Similarly:

- for $G = SO(n+1)$ we obtain a principal $SO(n)$-bundle $SO(n+1) \to S^n$.
- for $G = U(n+1)$ we obtain a principal $U(n)$-bundle $SU(n+1) \to S^{2n+1}$.
- for $G = U(n+1)$ we obtain a principal $U(n)$-bundle $SU(n+1) \to S^{2n+1}$.
- for $G = SU(n+1)$ we obtain a principal $SU(n)$-bundle $SU(n+1) \to S^{2n+1}$.

Notice that the first item in this list for $n = 1$ is precisely the previous example.

Example 3.41 (Stiefel manifolds). Consider the space of k-tuples of orthonormal vectors in $\mathbb{R}^{n+k}$:

$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \ldots, v_k) : v_i \in \mathbb{R}^{n+k},\ \langle v_i, v_j \rangle = \delta_{i,j}\}.$$

This is known under the name of "Stiefel manifold". One way to endow it with a smooth structure is by relating it to $O(n+k)$ and then using Theorem 3.28. We claim that there is a canonical map between the two. The key remark is that for any orthonormal matrix its rows, interpreted as vectors, form an orthonormal basis, and the same applies to the columns. In particular, for any

$$A = \begin{pmatrix} A_1^1 & \ldots & A_{n+k}^1 \\ \ldots & \ldots & \ldots \\ A_1^{n+k} & \ldots & A_{n+k}^{n+k} \end{pmatrix} \in O(n+k)$$

its first k columns

$$A_i = \begin{pmatrix} A_i^1 \\ \dots \\ A_i^{n+k} \end{pmatrix} \quad 1 \leq i \leq k,$$

interpreted as vectors,

$$A_i^T = (A_i^1, \dots, A_i^{n+k}) \in \mathbb{R}^{n+k}$$

give rise to an element in the Stiefel manifold

$$(A_1^T, \dots, A_k^T) \in V_k(\mathbb{R}^{n+k}).$$

Sending $A \in O(n+k)$ to this element defines a "projection"

$$\pi : O(n+k) \to V_k(\mathbb{R}^{n+k}).$$

We now hope to apply the Corollary 3.28 to $P = O(n+k)$ and to some group G acting on P. Furthermore, it is natural to use for G a subgroup of $O(n+k)$, with the action given by the multiplication of matrices. The next key remark here is that for any matrix $A \in O(n+k)$, its first k columns (i.e., $\pi(A)$) does not change if we multiply it by a matrix of type $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ with $B \in O(n)$, since:

$$\begin{pmatrix} A_{k,k} & A_{k,n} \\ A_{n,k} & A_{n,n} \end{pmatrix} \cdot \begin{pmatrix} I_{k,k} & 0_{k,n} \\ 0_{n,k} & B \end{pmatrix} = \begin{pmatrix} A_{k,k} & A_{k,n}B \\ A_{n,k} & A_{n,n}B \end{pmatrix}$$

where the indices indicate the number of rows and columns. This provides a clear candidate for G, namely $G = O(n)$ viewed as a subgroup of $O(n+k)$ via the "inclusion"

$$O(n) \hookrightarrow O(n+k), \quad B \mapsto \begin{pmatrix} I_{k,k} & 0_{k,n} \\ 0_{n,k} & B \end{pmatrix}.$$

Notice that the resulting right action on $O(n+k)$ is clearly free, and the compactness of the group ensures that the map is also proper. We leave it as an exercise to check now that the last condition from the corollary is satisfied. Th conclusion is that $V_k(\mathbb{R}^{n+k})$ will carry a smooth structure and, furthermore, it is the base of a principal $O(k)$-bundle

$$\pi : O(n+k) \to V_k(\mathbb{R}^{n+k}), \quad \pi(A) := (A_1^T, \dots, A_k^T). \tag{3.2.15}$$

Notice that this example also fits nicely with the class of examples from Example 3.30 and can be seen as an even higher dimensional generalization of the ones from Example 3.40. Indeed, there is an obvious action of $O(n+k)$ on $V_k(\mathbb{R}^{n+k})$ which is transitive and, computing the isotropy group at the element $(e_1, \dots, e_k) \in V_k(\mathbb{R}^{n+k})$ (the first k vectors of the canonical basis of $\mathbb{R}^{n+k}$), one finds the copy of $O(n)$ inside $O(n+k)$. Therefore, if we knew beforehand that $V_k(\mathbb{R}^{n+k})$ and the action of $O(n+k)$ on it were smooth, then we could have just applied Proposition 3.31 to deduce that (3.2.15) is a principal $O(k)$-bundle.

Example 3.42 (Grassmanians). Consider now the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$, that generalises the projective spaces (and, in that context, was mentioned already in Example 1.17):

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) = \{V \subset \mathbb{R}^{n+k} : V - k\text{-dimensional vector subspace}\}.$$

This space arises naturally as a quotient of the Stiefel manifolds from the previous example, with quotient map

$$\pi : V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k}), \ (v_1, \dots, v_k) \mapsto \mathrm{span}_{\mathbb{R}}(v_1, \dots, v_k).$$

Furthermore, this can be interpreted as the quotient map with respect to a natural action of $O(k)$ on $V_k(\mathbb{R}^{n+k})$: interpreting the elements of $B \in O(k)$ as linear isometries $B : \mathbb{R}^k \to \mathbb{R}^k$ and the elements $\phi = (v_1, \dots, v_k)$ of $V_k(\mathbb{R}^{n+k})$ as linear maps $\phi : \mathbb{R}^k \to \mathbb{R}^{n+k}$ (compatible with the inner product), the action is simply given by composition: $\phi \cdot B = \phi \circ B$. We leave it as an exercise to show that the orbits of the action are precisely the fibers of π. The fact

that the action is free and proper is clear and, therefore, by Theorem 3.29, $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ becomes a manifold and $V_k(\mathbb{R}^{n+k})$ becomes a principal $O(k)$-bundle over it.

Exercise 3.43. What do you get when $k = 1$?

Finally, we discuss two general constructions. They are nicer formulated using the notation

$$\mathrm{Prin}_G(M)$$

for the collection of isomorphism classes of principal G-bundles over a manifold M.

Example 3.44 (pull-backs). As the name suggest, fixing a Lie group G, any smooth map $f : N \to M$ gives rise to a pull-back operation

$$f^* : \mathrm{Prin}_G(M) \to \mathrm{Prin}_G(N);$$

more precisely, given a principal G-bundle $\pi : P \to M$, one forms

$$f^*(P) := N \times_M P := \{(x, p) \in N \times P : f(x) = \pi(p)\}$$

with projection the standard projection on the first coordinate, with the right action of G:

$$(x, p)g = (x, pg).$$

The construction is so that the fiber of the pull-back at an arbitrary $x \in N$ is a copy of $P_{f(x)}$:

$$f^*(P)_x = \{x\} \times P_{f(x)} \equiv P_{f(x)}.$$

To make $f^*(P)$ into a smooth manifold, notice that it sits inside the product $N \times P$ as an embedded submanifold. Checking the principal G bundle conditions is a rather easy exercise. The pull-back can also be understood in terms of transition functions as follows:

Exercise 3.45. Given a smooth map $f : N \to M$, and a transition data $(\mathscr{U} = \{U^\alpha\}_{\alpha\in\Lambda}, g_{\alpha,\beta})$ with values in G, as in (3.2.5) and (3.2.6), define the pull-back transition data:

$$f^*\mathscr{U} := \{f^{-1}(U^\alpha)\}_{\alpha\in\Lambda}, \quad f^*(g_{\alpha,\beta}) := g_{\alpha,\beta} \circ f.$$

Show that this corresponds to the operation of pulling back principal G-bundles along f.

Example 3.46 (Push forwards along group homomorphisms (changing the group)). As the name suggests. fixing now the manifold M, any Lie group homomorphism

$$\iota : H \to G$$

gives rise to a push-forward operation

$$\iota_* : \mathrm{Prin}_H(M) \to \mathrm{Prin}_G(M).$$

In some sense, this is the analogue of the other operations on vector bundles such as direct sums, duals, etc (for now this is a vague/very informative statement; it will become clearer a bit later on).

Notice that, on the other hand, the existence of such an operation is clear from the point of view of transition functions: H-valued ones become, after applying ι, G-valued ones. But, again, there is a simple description of this operation that is both global as well as explicit : for a principal H-bundle $\pi : P \to M$ we define

$$\iota_*(P) := (P \times G)/H,$$

the quotient of $P \times G$ modulo the left action of H given by $h \cdot (p,g) := (ph^{-1}, \iota(h)) \cdot g)$- so that, when working with equivalence classes $[p,g] \in \iota_*(P)$, one has

$$[ph,g] = [p, \iota(h) \cdot g] \quad (h \in H).$$

A simple check shows that the action of H on $P \times G$ is free and proper and, therefore, we can appeal again to Theorem 3.27, this time to provide $\iota_*(P)$ with a smooth structure. Endowing $\iota_*(P)$ with the projection that send the class $[p,g] \in \iota_*(P)$ to $\pi(p) \in M$, and with the G-action

$$[p,g] \cdot g' = [p, gg'],$$

we leave it as an exercise to show that $\iota_*(P)$ is, indeed, a principal G-bundle over M.

Exercise 3.47. What happens when $H = \{e\}$ is trivial? But when $G = \{e\}$ is trivial?

Example 3.48. Let us return to the discussion from Example 3.38 on the comparison between the Hopf fibration (3.2.9) and the principal bundle of Example 3.38. The puzzling non-equivariance relation (3.2.14) can no be understood more conceptually using the map $\rho : S^1 \to S^1$, $\rho(z) = z^2$ and the induced

$$\rho_* : \mathrm{Prin}_{S^1}(S^2) \to \mathrm{Prin}_{S^1}(S^2) \tag{3.2.16}$$

Indeed, in the quotient $\rho_*(S^3) = (S^3 \times S^1)/S^1$ the identifications that are made are (u,z) with $u \cdot \lambda^{-1}, \lambda^2 z)$ and, therefore, the property (3.2.14) of the map ϕ is precisely what ensures the map $[(u,z)] \mapsto \phi(u)z$ to be well defined. We deduce that (3.2.16) sends (3.2.9) to (3.2.12):

$$\rho_* \begin{pmatrix} S^3 \\ {\scriptstyle H}\downarrow \\ S^2 \end{pmatrix} = \begin{pmatrix} SO(3) \\ \downarrow{\scriptstyle \pi} \\ S^2 \end{pmatrix}.$$

The following two exercises shows that the push-forwards operation can also be seen as the analogue of the other operations on vector bundles such as direct sums or duals.

Exercise 3.49. Consider the group homomorphism

$$\rho : GL_r \to GL_r, \ \ \rho(A) = {}^t(A^{-1}).$$

Show that for any vector bundle E, the frame bundles associated to E and to its dual are related by

$$\mathrm{Fr}(E^*) \cong \rho_*(\mathrm{Fr}(E)).$$

Exercise 3.50. Show that if $\pi_i : P_i \to M$ are principal G_i-bundles, for $i \in \{1,2\}$, then "the fibered product"

$$P_1 \times_M P_2 := \{(p_1,p_2) \in P_1 \times P_2 : \pi_1(p_1) = \pi_2(p_2)\}$$

carries a natural structure of principal $G_1 \times G_2$-bundle.

Assume now that $P_i = \mathrm{Fr}(E_i)$ is the frame bundles associated to a vector bundles E_i of rank r_i (hence also $G_i = GL_{r_i}$), $i \in \{1,2\}$. Consider $r = r_1 + r_2$ and the group homomorphism

$$\rho : GL_{r_1} \times GL_{r_2} \to GL_r, \ \ (A,B) \mapsto A \oplus B := \mathrm{diag}(A,B).$$

Show that the frame bundle associated to $E_1 \oplus E_2$ is isomorphic to the push-forward of $P_1 \times_M P_2$ via ρ.

As we will see in section 3.3, a particularly interesting case of push-forwards is obtained when ι is the inclusion of a Lie subgroup H of G, which gives rise to the concept of reduction of the group- see Definition 3.56 and the comments thereafter.

3.2.5 From principal bundles to vector bundles: attaching fibers

The frame bundle construction from Example 3.20 shows how to associate principal GL_r-bundles to vector bundles of rank r. We now describe the reverse construction, which produces vector bundles out of principal GL_r-bundles. Actually, we proceed in a slightly more general setting. What we need as input is a Lie group G, a principal G-bundle

$$\pi_P : P \to M$$

as well as a representation V of G,

$$V \in \mathrm{Rep}(G),$$

i.e. a finite dimensional vector space V together with a Lie group homomorphism $\rho : G \to GL(V)$. The output will be a vector bundle

$$\pi_E : E(P,V) \to M$$

of rank equal to the dimension of V. Strictly speaking, one should be using the notation $E(P,G,V,\rho)$, but G and ρ are usually clear from the context. Of course, when returning to the 1-1 correspondence between vector bundles and principal bundles, we will apply this construction to $G = GL_r$, and ρ the canonical representation of matrices as linear maps on $V = \mathbb{R}^r$.

In this general framework, the representation ρ encodes a linear action of G on V from the left,

$$G \times V \to V, \quad (g,v) \mapsto g \cdot v := \rho(g)(v),$$

which, combined with the right action of G on P, induces a (right) action of G on $P \times V$:

$$(p,v) \cdot g := (pg, g^{-1}v).$$

We define the vector bundle obtained by **attaching to P the fiber V (w.r.t. ρ)** as the resulting quotient

$$E(P,V) = E(P,V,\rho) := (P \times V)/G = \{[p,g] : p \in P, v \in V\},$$

where $[p,v]$ denotes the element in the quotient represented by $(p,v) \in P \times G$ (so that, in $E(P,V)$, $[pg,v] = [p,gv]$ for all $p \in P$, $v \in V$, $g \in G$). Further, we endow $E(P,V)$ with the projection

$$\pi_E : E(P,V) \to M, \;\; \pi_E(p,v) = \pi_P(v).$$

and the fibers will be the vector spaces:

$$E(P,V)_x = \{[p,v] : p \in P_x, v \in V\}, \quad [p,v] + [p,w] = [p,v+w].$$

The following is a simple exercise:

Exercise 3.51. Show that there exists a unique vector space structure on the fibers $E(P,V)_x$ such that, for each $p \in P$ with $\pi(p) = x$, the bijection

$$\phi_p : V \to E(P,V)_x, \;\; v \mapsto [p,v],$$

is an isomorphism of vector spaces.

To describe the smooth structure on $E(P,V)$ one can proceed in several ways. One is to use again the theorem on the smoothness of quotients i.e., apply Theorem 3.27; for that one just needs the simple remark that, since the action of G on P is free and proper, so is the action on $P\times V$. Fixing a frame $\mathfrak{e}^V$ of V, any local section σ of P induces a local frame $\mathfrak{e}^\sigma$ of $E(P,V)$, with the i-th component defined by

$$\mathfrak{e}^\sigma_i(x) := [\sigma(x), \mathfrak{e}^V_i] \in E(P,V)_x.$$

A slightly different way to see that $E(P,V)$ is a smooth vector bundle is to notice directly (without going through local sections and without fixing a frame of V) that any local trivialization of P,

$$\Psi : P|_U \to U\times G,$$

induces an isomorphism of $E(P,V)|_U$ with the trivial vector bundle with fiber V:

$$\tilde\Psi : \tilde\pi^{-1}(U) \to U\times V,\ p\mapsto (\pi(p), g_{\Psi(p)}v),$$

where $g_{\Psi(p)}\in G$ is the second component of $\Psi(p)$.

Corollary 3.52. *For any principal G-bundle $\pi_P : P\to M$ and $V\in Rep(G)$, $\pi_E : E(P,V)\to M$ is a vector bundle.*

The following provide a few explicit examples of this construction, for our favourite principal bundles over S^2: the Hopf fibration and the group $SO(3)$.

Exercise 3.53. Here the relevant group is $G=S^1$ and we consider the canonical representation

$$r : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ \cos(\alpha)+i\sin(\alpha)\mapsto \begin{pmatrix}\cos(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \cos(\alpha)\end{pmatrix},$$

as well as "the square" of this representation:

$$r^2 : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ \cos(\alpha)+i\sin(\alpha)\mapsto \begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ -\sin(2\alpha) & \cos(2\alpha)\end{pmatrix}.$$

Consider also S^3 as a principal S^1-bundle over S^2 given by the Hopf fibration (3.2.9), and $SO(3)$ viewed as a principal S^1-bundle $\pi : SO(3)\to S^2$ as in Example 3.38. Show that:

a) $E(S^3,\mathbb{R}^2,r^2)\cong E(SO(3),\mathbb{R}^2,r)$ and they can be identified with (are isomorphic to) the tangent bundle of S^2.
b) $E(S^3,\mathbb{R}^2,r)$ can be identified with the tautological line bundle $L\to\mathbb{CP}^1$ (cf. Example 1.17); more precisely, using the diffeomorphism $\Psi : S^2\to\mathbb{CP}^1$ constructed in Exercise 3.36, $E(S^3,\mathbb{R}^2,r)$ is isomorphic to $\Psi^*(L)$.

3.2.6 *Conclusion:* $\mathrm{Vect}^r\cong \mathrm{Prin}_{GL_r}$

As we mentioned at the beginning of the previous section, the construction $E(P,V)$ is particularly interesting in the case when

$$G=GL_r,\ V=\mathbb{R}^r,\ \rho=\mathrm{Id},$$

when it associates to a principal GL_r bundle P the vector bundle $E(P,\mathbb{R}^r)$. Let us apply this to the frame bundle

$$P=\mathrm{Fr}(E)$$

of a vector bundle E of rank r over M. In this case, since an element $p \in P_x$ is the same thing as the choice of an isomorphism $i_p : \mathbb{R}^r \to E_x$, we see that that a pair $[p,v] \in E(P,\mathbb{R}^r)_x$ can be identified with $i_p(v) \in E_x$. In other words, one has an isomorphism of vector bundles over M:

$$E(P,\mathbb{R}^r) \xrightarrow{\sim} E, \quad [p,v] \mapsto i_p(v).$$

Putting everything together, we find:

Theorem 3.54. *The constructions which associate to a vector bundle E its frame bundle $Fr(E)$ and, conversely, to a principal GL_r-bundle P the vector bundle $E(P,\mathbb{R}^r)$, give rise to a 1-1 correspondence*

$$\left\{\begin{array}{c}\textit{(isomorphism classes of)}\\ \textit{rank } r \textit{ vector bundles}\\ \pi_E : E \to M\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{(isomorphism classes of)}\\ \textit{principal } GL_r\textit{-bundles}\\ \pi_P : P \to M\end{array}\right\}$$

Finally, since the spaces of sections of vector bundles carry so much information about the vector bundles themselves, for the $E(P,V)$s discussed above, it will be useful to have a more direct description, in term of the principal G-bundle P and the representation $V \in \mathrm{Rep}(G)$. This brings us to the space $C^\infty(P,V)$ of smooth functions on P with values in V, and the fact that this space comes endowed with a canonical (linear) action of G:

$$G \times C^\infty(P,V) \to C^\infty(P,V), \quad (g,f) \mapsto g \cdot f,$$

induced by the actions of G on P and V by:

$$(g \cdot f)(p) = \rho_g(f(pg)).$$

As for any action of a group, we can talk about the space of G-invariant elements:

$$C^\infty(P,V)^G := \{f \in C^\infty(P,V) : g \cdot f = f \ \ \forall\, g \in G\}.$$

The key-point is that any such G-equivariant function f induces a section of $E(P,V)$: first of all it induces

$$(\mathrm{id}, f) : P \to P \times V, \ p \mapsto (p, f(p))$$

which is equivariant (where the right action of G on $P \times V$ is precisely the one used to define $E(P,V)$); hence, passing to the quotient modulo G, it induces a map

$$s_f : M \to E(P,V).$$

Lemma 3.55. *If $E = E(P,V)$ is the associated vector bundle, then one has a bijection*

$$C^\infty(P,V)^G \xrightarrow{\sim} \Gamma(E), \quad f \mapsto s_f.$$

Proof. Sections of the trivial vector bundle $P \times V$ over P are necessarily of type (id, f) and, therefore, they are in 1-1 correspondence with smooth functions $f : P \to V$. Moreover, the equivariance of f is equivalent to s_f being equivariant, where $P \times V$ is endowed with the action of G used to define E. This implies that $\tilde{s}_f$ descends to a map

$$s_f : M \to (P \times V)/G = E,$$

which is a section of E. A minute of thinking reveals that this discussion can easily be reversed. 😀

3.3 Application of principal bundles: geometric structures on vector bundles

While frame bundles showed up above in the comparison between vector bundles and principal ones, they are also perfectly suited for handling general geometric structures (on vector bundles and, later on, on manifolds). Here we look at several examples of geometric structures (inner products, complex structures, orientations, symplectic structures, etc) and explain that, in each case, by focusing on "frames compatible with the given structure", the entire structure may be encoded in a principal bundle- leading us to the general framework.

The core of the discussion below will take place on finite dimensional vector spaces V (the fibers of the vector bundles of interest). In each of the examples, the main points specify are:

1. the structure of interest on vector spaces V,
2. the relevant notion of isomorphism,
3. the standard model (to which anything else is isomorphic to),
4. the symmetry group corresponding to the standard model, which is a subgroup of GL_n,
5. and, finally, the frames that are adapted (compatible) to the structure.

It is useful to remember that a frame $\mathfrak{e}$ of V encodes an isomorphism between $\mathbb{R}^r$ and V; the adapted frames from the last point will then correspond to isomorphisms (in the sense of 2.) with the standard model (specified at 3.).

Item 5 will produce principal bundles. To recognise them we introduce the following concept:

Definition 3.56. Given a principal L-bundle $\pi_P : P \to M$, with L a Lie group, a **reduction** of P to a subgroup

$$G \subset L,$$

also called a G**-reduction of** P, is a principal G-subbundle

$$Q \subset P$$

.

In other words, Q is required to be a submanifold of P which is stable under multiplication by elements of G (hence it inherits an action of G) and such that

$$\pi|_Q : Q \to M$$

makes Q into a principal G-bundle. This concept can also be rephrased in terms of the push-forwards operation discussed in Example 3.46, now applied to the inclusion $i : G \hookrightarrow L$. In terms of the resulting

$$i_* : \mathrm{Prin}_G(M) \to \mathrm{Prin}_L(M),$$

a G-reduction of P corresponds to writing $P = i_*(Q)$,

=

3.3.1 Inner products

Recall that an inner product on a vector space V is a symmetric bilinear map

$$g : V \times V \to \mathbb{R}$$

that is non-degenerate. An isomorphism $A : (V, g) \to (V', g')$ between two spaces endowed with inner products are, of course, the linear isometries- required to satisfy

$$g'(A(u),A(v)) = g(u,v) \quad \forall\, u,v \in V.$$

This allows us to talk, for each (V,g), about the group of symmetries of (V,g),

$$\mathrm{GL}(V,g) = \{A \in \mathrm{GL}(V) : g(A(u),A(v)) = g(u,v) \quad \forall\, u,v \in V\} \subset \mathrm{GL}(V).$$

The standard model is $\mathbb{R}^n$ with the standard inner product

$$g_{\mathrm{can}}(u,v) = \langle u,v\rangle = \sum_i u^i \cdot v^i,$$

for which the group of symmetries is the orthogonal group

$$O(n) = \mathrm{GL}(\mathbb{R}^n, g_{\mathrm{can}}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A \cdot A^T = \mathrm{Id}\}.$$

Any other (V,g) is isomorphic to the "standard model" $(\mathbb{R}^n, g_{\mathrm{can}})$; picking an isomorphism is equivalent to picking an orthonormal frame $\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ on V, i.e. satisfying:

$$g(\mathfrak{e}_i, \mathfrak{e}_j) = \delta_{i,j} \quad \forall\, i,j. \tag{3.3.1}$$

It is useful to think of them as "frames of V that are compatible with g"; they form a subspace

$$\mathrm{Fr}(V,g) \subset \mathrm{Fr}(V)$$

with the property that the action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathrm{Fr}(V)$ restricts to a free and transitive action of $O(n) \subset \mathrm{GL}_n(\mathbb{R})$ on $\mathrm{Fr}(V,g)$. Furthermore, everything holds in the smooth context, using the smooth structure on $\mathrm{Fr}(V)$ (see the previous section) and the fact that $O(n)$ is a Lie subgroups of GL_n. The conclusion is that $\mathrm{Fr}(V,g)$ is an $O(n)$-torsor.

Conversely, g can be recovered from $\mathrm{Fr}(V,g)$, since knowing the orthonormal frames w.r.t. g determines g. It follows that, on any n-dimensional vector space V, the construction $g \mapsto \mathrm{Fr}(V,g)$ provides a bijection

$$\left\{\begin{array}{c}\text{inner products } g\\ \text{on the vector space } V\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{subspaces } P \subset \mathrm{Fr}(V)\\ \text{which are } O(n)\text{- torsors}\end{array}\right\}$$

Moving to vector bundles $E \to M$, recall that an inner product on E is a family $g = \{g_x\}_{x\in M}$ of inner products on the fibers E_x varying smoothly w.r.t. x in the sense discussed in subsection 1.1.3. The previous discussion applied to the g_xs gives rise to

$$\mathrm{Fr}(E,g) = \{(x,\mathfrak{e}_x) : x \in M,\ \mathfrak{e}_x \in \mathrm{Fr}(E_x,g_x)\} \subset \mathrm{Fr}(E,g),$$

called **the orthonormal frame bundle** of (E,g). It is an instructive exercise to show that the smoothness of the family $\{g_x\}_{x\in M}$ is equivalent to the fact that $\mathrm{Fr}(E,g)$ is a smooth submanifold of $\mathrm{Fr}(E)$. It follows that:

Proposition 3.57. *For any rank n vector bundle $E \to M$ endowed with an inner product g,*

$$\pi : Fr(E,g) \to M$$

is a principal $O(n)$-bundle. Furthermore, $g \mapsto Fr(E,g)$ gives a bijection

$$\left\{\begin{array}{c}\textit{(fiberwise) inner products } g\\ \textit{on the vector bundle } E\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}O(n)\textit{-reductions } P \subset Fr(E),\\ \textit{of the frame bundle of } E\end{array}\right\}$$

3.3.2 Orientations

Recall that an orientation on a vector space V is an equivalence class $\mathfrak{o}$ of frames of V, where two frames are equivalent if the transition matrix has positive determinant. Given $(V,\mathfrak{o})$, the frames of V that are compatible with $\mathfrak{o}$ are, of course, the ones that represent $\mathfrak{o}$; they are also called positively frames, and the set of such is denoted $\mathrm{Fr}(V,\mathfrak{o}) \subset \mathrm{Fr}(V)$. An isomorphism $A : (V,\mathfrak{o}) \to (V',\mathfrak{o}')$ between two oriented vector spaces is any linear isomorphism that takes positive frames to positive ones; as above, one can talk about the resulting symmetry group $\mathrm{GL}(V,\mathfrak{o}) \subset \mathrm{GL}(V)$.

The standard model is $\mathbb{R}^n$ endowed with the orientation $\mathfrak{o}_{\mathrm{can}}$ induced by the canonical frame, with corresponding symmetry group the positive determinant group

$$\mathrm{GL}_n^+(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) > 0\}.$$

The "adapted frames" are, of course, precisely the positive frames that enter the very definition of orientations, and they are also the ones that encode the isomorphisms between $(V,\mathfrak{o})$ and $(\mathbb{R}^n,\mathfrak{o}_{\mathrm{can}})$ $(n = \dim(V))$.

Moving on, an orientation $\mathfrak{o}$ on a vector bundle $E \to M$ is a collection $\{\mathfrak{o}_x\}$ of orientations on the fibers E_x which is smooth in the following sense: around each point of M one can find a local frame of E, defined over some $U \subset M$, with the property that, at each $x \in U$, it represents $\mathfrak{o}_x$. As above, one can then talk about (local) oriented frames of E. and we introduce

$$\mathrm{Fr}(E,\mathfrak{o}) = \{(x,\mathfrak{e}_x) : x \in M,\ \mathfrak{e}_x \in \mathrm{Fr}(E_x,\mathfrak{o}_x)\} \subset \mathrm{Fr}(E)$$

As above, the construction $\mathfrak{o} \mapsto \mathrm{Fr}(E,\mathfrak{o})$ provides a bijection

$$\left\{\begin{array}{c}\text{orienations } \mathfrak{o}\\ \text{on the vector space } V\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{subspaces } P \subset \mathrm{Fr}(V)\\ \text{which are } GL_n^+\text{- torsors}\end{array}\right\}$$

Moving to vector bundles, one finds:

Proposition 3.58. *For any rank n vector bundle $E \to M$ endowed with an orientation $\mathfrak{o}$,*

$$\pi : Fr(E,\mathfrak{o}) \to M$$

is a principal GL_n^+-bundle. Furthermore, $\mathfrak{o} \mapsto Fr(E,\mathfrak{o})$ gives a bijection

$$\left\{\begin{array}{c}\textit{(fiberwise) orientations } \mathfrak{o}\\ \textit{on the vector bundle } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}GL_n^+(\mathbb{R})\textit{-reductions } P \subset Fr(E),\\ \textit{of the frame bundle of } E\end{array}\right\}$$

3.3.3 Volume elements

Recall that a volume element on a vector space V is a non-zero element $\mu \in \Lambda^r V^*$ of the top exterior power of V (hence n is the dimension of V) or, equivalently, a non-zero skew-symmetric multilinear map

$$\mu : \underbrace{V \times \ldots \times V}_{n\,\text{times}} \to \mathbb{R}.$$

The frames of V that are adapted to μ are the unit-volume ones, i.e. $\mathfrak{e} = (\mathfrak{e}_1,\ldots,\mathfrak{e}_n)$ with

$$\mu(\mathfrak{e}_1,\ldots,\mathfrak{e}_r) = 1,$$

the symmetry group of the standard module will be

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1\},$$

Exercise 3.59. Continue the discussion, with the conclusion that there is a 1-1 corresondence

$$\left\{\begin{array}{c}\text{(fiberwise) volume elements } \mu \\ \text{on the vector bundle } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\mathbb{SL}_n(\mathbb{R})\text{-reductions } P \subset \mathrm{Fr}(E), \\ \text{of the frame bundle of } E\end{array}\right\} \tag{3.3.2}$$

Notice that the fact that any volume forms gives rise to an orientation can be seen as a consequence of the inclusion

$$\mathbb{SL}_n(\mathbb{R}) \subset \mathrm{GL}_n^+(\mathbb{R}).$$

3.3.4 Complex structures

As we have already discussed, a complex structure on a (real) vector space V can be interpreted as linear map

$$J : V \to V \quad \text{satisfying} \quad J^2 = -\mathrm{Id}.$$

Isomorphisms in this setting are, of course, complex isomorphism, also interpreted as linear isomorphism $A : (V,J) \to (V',J')$ satisfying $A \circ J = J' \circ A$. The standard model is $\mathbb{C}^k$ with the usual multiplication by $i = \sqrt{-1}$ or, as a pair (V,J), it is $\mathbb{R}^{2k} = \mathbb{R}^k \oplus \mathbb{R}^k$ with the complex structure

$$J_{\mathrm{can}}(u,v) = (-v,u).$$

Using the matrix notation one has $J_{\mathrm{can}} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$ and one finds that the symmetry group of the standard model is the subgroup of $\mathrm{GL}(\mathbb{R}^{2k})$ consisting of invertible matrices of type $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. This is, of course, just **complex linear group** sitting inside the real one via

$$\mathrm{GL}_k(\mathbb{C}) \ni A + iB \longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{GL}(\mathbb{R}^{2k}, J_{\mathrm{can}}). \tag{3.3.3}$$

The frames of V that are adapted to J are the ones that correspond to isomorphisms $(\mathbb{R}^{2k}, J_{\mathrm{can}}) \to (V,J)$ or, equivalently, those of type

$$\mathfrak{e} = (\mathfrak{e}_1, \dots, \mathfrak{e}_k, \psi_1, \dots, \psi_k) \quad \text{with } \psi_j = J(\mathfrak{e}_j),$$

where $k = \frac{1}{2}\dim(V)$. They give rise to a subset $\mathrm{Fr}(V,J) \subset \mathrm{Fr}(V)$ which, since $\mathrm{GL}_k(\mathbb{C})$ sits now inside $\mathrm{GL}_{2k}(\mathbb{R})$ as a Lie subgroup, will be an embedded submanifold. As above, on any n-dimensional vector space V, the construction $J \mapsto \mathrm{Fr}(V,J)$ provides a bijection

$$\left\{\begin{array}{c}\text{complex structures } J \\ \text{on the vector space } V\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{subspaces } P \subset \mathrm{Fr}(V) \\ \text{which are } GL_k(\mathbb{C})\text{- torsors}\end{array}\right\} \qquad (n = 2k).$$

Moving to vector bundles $E \to M$ endowed with a complex structure J and applying the discussion above fiberwise, one obtains the **complex frame bundle**, which is just a copy of the frame bundle of the complex vector bundle E_J sitting inside its real counterpart:

$$\mathrm{Fr}(E,J) = \{(x,\mathfrak{e}_x) : x \in M, \mathfrak{e}_x \in \mathrm{Fr}(E_x,J_x)\} \subset \mathrm{Fr}(E).$$

Proposition 3.60. *For any rank $r = 2k$ vector bundle $E \to M$ endowed with a complex structure J,*

$$\pi : Fr(E,J) \to M$$

is a principal $GL_k(\mathbb{C})$-bundle. Furthermore, $J \mapsto Fr(E,J)$ gives a bijection

$$\left\{ \begin{array}{c} \text{complex structures } J \\ \text{on the vector bundle } E \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} GL_k(\mathbb{C})\text{-reductions } P \subset Fr(E), \\ \text{of the frame bundle of } E \end{array} \right\} \qquad (n = 2k).$$

Remark 3.61. A complex structure J on V gives rise to an orientation $\mathfrak{o}_J$ on V, namely the one induced by the complex frames. The key point is the fact that the matrices that arise when changing from one complex frame to another always have positive determinant. Put it slightly differently, and in complete analogy with the discussion on orientations induced by volume forms, the discussion is dues to an inclusion of subgroups of $GL_{2k}(\mathbb{R})$:

Exercise 3.62. Show that $\det(\mathrm{incl}(Z)) = |\det(Z)|^2$ for any $Z \in \mathrm{GL}_k(\mathbb{C})$. Then re-interpret the orientation $\mathfrak{o}_J$ associated to a complex structure using the resulting inclusion

$$\mathrm{incl}\,(\mathrm{GL}_k(\mathbb{C})) \subset \mathrm{GL}^+_{2k}(\mathbb{R}).$$

$$\left(\text{Hint for first part: there exists a matrix } X \in \mathrm{GL}_k(\mathbb{R}) \text{ such that } \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}\right)$$

3.3.5 *p-planes*

By a p-plane in a vector space V we simply mean a p-dimensional vector subspace $W \subset V$. This may seem a bit artificial at this point, but please be patient. Isomorphisms between such pairs (V,W) are, of course, the ones that preserve the p-planes. Similarly, an adapted frame of (V,W) is a frame of V with the property that its first p vectors form a frame of W.

The standard model is $(\mathbb{R}^n,\mathbb{R}^p)$, where we view $\mathbb{R}^p$ sitting inside $\mathbb{R}^n$ via the inclusion on the first components

$$\mathbb{R}^p \hookrightarrow \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}, \;\; u \mapsto (u,0);$$

and its symmetry group is $\mathrm{GL}_{p,n-p}(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ consisting of invertible matrices of type

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R})$$

with $A \in \mathrm{GL}_p(\mathbb{R})$, $C \in \mathrm{GL}_{n-p}(\mathbb{R})$ and B is an $p \times (n-p)$ matrix. As before, the discussion extends straightforwardly to vector bundles E, and one obtains a bijection

$$\left\{\begin{array}{c}\text{rank } p \text{ sub-bundles}\\ \text{of the vector bundle } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\mathrm{GL}_{p,n-p}(\mathbb{R})\text{-reductions } P \subset \mathrm{Fr}(E),\\ \text{of the frame bundle of } E\end{array}\right\}. \tag{3.3.4}$$

The discussion makes sense also over $\mathbb{C}$. A particularly interesting case is obtained when we start with an n-dimensional real vector space V, we consider its complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$, and we look at

3.3.6 Symplectic structures

Recall that the notion of symplectic structure on a vector space V is the antisymmetric versions of inner products- i.e. we talk about antisymmetric bilinear maps

$$\omega : V \times V \to \mathbb{R}.$$

Isomorphisms between (V,ω) and (V',ω') are linear isomorphisms $A : V \to V'$ satisfying

$$\omega'(A(u),A(v)) = \omega(u,v) \quad \forall\, u,v \in V.$$

The standard model is $(\mathbb{R}^{2k},\omega_{\mathrm{can}})$ where

$$\omega_{\mathrm{can}} = e^{k+1} \wedge e^1 + \ldots + e^{2k} \wedge e^k,$$

where $\{e^i\}$ is the dual of the canonical basis of $\mathbb{R}^{2k}$. Or, in terms of the standard inner product and complex structure on $\mathbb{R}^{2k}$,

$$\omega_{\mathrm{can}}(u,v) = g_{\mathrm{can}}(u, J_{\mathrm{can}}(v)), \quad \forall\, u,v \in \mathbb{R}^{2k}.$$

The corresponding symmetry is the symplectic group

$$Sp_k(\mathbb{R}) := \{M \in \mathrm{GL}_{2k}(\mathbb{R}) : M^T J_{can} M = J_{can}\}. \tag{3.3.5}$$

Given (V,ω), the isomorphisms between (V,ω) and the standard model correspond to the symplectic frames of (V,ω), i.e. those frames of V of type

$$(\mathfrak{e}_1,\ldots,\mathfrak{e}_k,\psi_1,\ldots,\psi_k) \quad (\text{in particular } n = 2k)$$

with the property that

$$\omega(\mathfrak{e}_i,\psi_i) = -1,\ \omega(\psi_i,\mathfrak{e}_i) = 1,\ \omega(\mathfrak{e}_i,\psi_j) = 0 \text{ for } i \neq j,\ \omega(\mathfrak{e}_i,\mathfrak{e}_j) = 0 = \omega(\psi_i,\psi_j).$$

They form a subspace $\mathrm{Fr}(V,\omega) \subset \mathrm{Fr}(V)$, this is an embedded submanifold (why?) and, ultimately, this construction provides a bijection:

$$\left\{\begin{array}{c}\text{symplectic forms } \omega\\ \text{on the vector space } V\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{subspaces } P \subset \mathrm{Fr}(V)\\ \text{which are } \mathrm{Sp}_k(\mathbb{R})\text{- torsors}\end{array}\right\} \qquad (n = 2k).$$

Passing to vector bundles:

Proposition 3.63. *For any rank $r = 2k$ vector bundle $E \to M$ endowed with a symplectic structure ω,*

$$\pi : Fr(E,\omega) \to M$$

is a principal $Sp_k(\mathbb{R})$-bundle. Furthermore, $\omega \mapsto Fr(E,\omega)$ gives a bijection

$$\left\{\begin{array}{c}\textit{(fiberwise) symplectic structures}\\ \textit{on the vector bundle } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}Sp_k(\mathbb{R})\textit{-reductions } P \subset Fr(E),\\ \textit{of the frame bundle of } E\end{array}\right\} \qquad (n = 2k).$$

3.3.7 Integral structures

By an integral structure on a vector space V we mean a lattice $\Lambda \subset V$, i.e. a discrete subgroup of $(V,+)$. Equivalently, Λ is of type

$$\Lambda = \mathbb{Z}\mathfrak{e}_1 + \ldots + \mathbb{Z}\mathfrak{e}_n$$

for some frame $\mathfrak{e} = (\mathfrak{e}_1,\ldots,\mathfrak{e}_n)$- and these are the frames that are adapted to Λ, also called integral frames. The standard model is $(\mathbb{R}^n,\mathbb{Z}^n)$, with symmetry group

$$\mathrm{GL}_n(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{R}).$$

Moving to vector bundles $E \to M$, a (fiberwise) integral structure on E is a collection $\{\Lambda_x\}$ of lattices in the fibers E_x which, when interpreted as a subset

$$\Lambda \subset E,$$

is smooth in the sense that around any point in M one finds a local frame e of E, over some U, such e_x is an integral frame of (E_x,Λ_x) for all $x \in U$. One obtains

$$\mathrm{Fr}(E,\Lambda) \subset \mathrm{Fr}(E).$$

Proposition 3.64. *For any rank n vector bundle $E \to M$ endowed with an integral structure Λ,*

$$\pi : Fr(E,\Lambda) \to M$$

is a principal $GL_n(\mathbb{Z})$-bundle. Furthermore, $\Lambda \mapsto Fr(E,\Lambda)$ gives a bijection

$$\left\{\begin{array}{c}\textit{integral structures } \Lambda\\ \textit{on the vector bundle } E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}GL_n(\mathbb{Z})\textit{-reductions } P \subset Fr(E),\\ \textit{of the frame bundle of } E\end{array}\right\}$$

3.3.8 Conclusion

We see that any Lie subgroup $G \subset \mathrm{GL}_n$ gives rise to a notion of "geometric structures" on n-dimensional vector spaces V/rank n vector bundles E:

- a G-**structure on the vector space** V is a subspace $P \subset \mathrm{Fr}(V)$ which is a G-torsor w.r.t. the action of GL_n on $\mathrm{Fr}(V)$ and $G \subset GL_n$).
- **a** G-**structure on the vector bundle** E is a G-reduction of the frame bundle $\mathrm{Fr}(E)$ or, equivalently, a submanifold $Q \subset \mathrm{Fr}(E)$ which is stable under the action of G and such that, when endowed with the resulting action of Q and the restriction π_P of the projection of $\mathrm{Fr}(E)$ to Q,

$$\pi_Q : Q \to M$$

is a principal G-bundle.

3.4 ... and geometric structures on manifolds

3.4.1 The general concept of G-structure

The geometric structures on manifolds that we consider here are the geometric structures in the sense of the previous section, living on tangent bundles. We use the notation $\mathrm{Fr}(M)$ for the frame bundle of TM. Hence

$$\pi : \mathrm{Fr}(M) \to M$$

is the principal GL_n-bundle whose fiber above $x \in M$ consists of all frames of the tangent space T_xM ($n = \dim M$).

Definition 3.65. Given a Lie subgroup $G \subset \mathrm{GL}_n$, a G-**structure** (or **almost** Γ_G-**structure**) on a n-dimensional manifold M is a G-reduction of the frame bundle (cf. Definition 3.56) or, more explicitly: a submanifold

$$\Sigma \subset \mathrm{Fr}(M)$$

which is G-invariant and which, when endowed with the resulting action of G and the restriction

$$\pi_\Sigma : \Sigma \to M$$

of the projection of $\mathrm{Fr}(M)$ to Σ becomes a principal G-bundle over M.

Example 3.66. Besides the examples from the previous subsection, it is instructive to consider also the case when $G = \{\mathrm{I}\}$ is the trivial group. We see that a $\{\mathrm{I}\}$-structure on M is a trivialization of the tangent bundle of M.

Example 3.67. For each Lie subgroup $G \subset \mathrm{GL}_n$ one has a "local model", which is $\mathbb{R}^n$ endowed with "the canonical G-structure"

$$\Sigma_G^{\mathrm{can}} := \left\{ \left(x,\, \sum g_{i,j} \left(\frac{\partial}{\partial x_j} \right)_x \right) \in \mathrm{Fr}(M) : \text{ the matrix } (g_{i,j}) \text{ is in } G \right\}.$$

Slightly differently: since the collection of frames of the vector space $\mathbb{R}^n$ can identified with GL_n, G itself is a G-structure on the vector space $\mathbb{R}^n$, called "the standard (vector space) G-structure" on $\mathbb{R}^n$. Viewing now $\mathbb{R}^n$ as a manifold and using the canonical identification of $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, $(\Sigma_G^{\mathrm{can}})_x$ is the standard G-structure on the vector space $\mathbb{R}^n$. In other words, under the canonical identification $\mathrm{Fr}(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathrm{GL}_n$, Σ_G^{can} becomes identified with $\mathbb{R}^n \times G$.

The following provides a simple criteria for recognising when a principal bundle corresponds to a G-structure:

Lemma 3.68. *Given a Lie group G, a principal G-bundle $\Sigma \to M$ over an n-dimensional manifold is (isomorphic to) the principal bundle associated to a G-structure on M if and only if G admits a representation*

$$\rho : G \to GL(\mathbb{R}^n)$$

which is faithful (i.e. ρ is injective) and such that, attaching to Σ the fiber $\mathbb{R}^n$ via ρ (as in subsection 3.2.5), the resulting vector bundle $E(\Sigma, \mathbb{R}^n)$ is isomorphic to TM:

$$E(\Sigma, \mathbb{R}^n) \cong TM.$$

Proof. For a G-structure Σ, the isomorphism is the same as for the frame bundle:

$$(\Sigma \times \mathbb{R}^n)/G \to TM, [\mathfrak{e}, v] \mapsto \hat{\mathfrak{e}}(v),$$

where $\hat{\mathfrak{e}} : \mathbb{R}^n \to T_xM$ is the linear isomorphism induced by a frame $\mathfrak{e} \in T_xM$. For the converse, one goes backwards, remembering that the isomorphisms $\hat{\mathfrak{e}}$ are just another way of encoding frames. That will give rise to a G-equivariant map $\Sigma \to \mathrm{Fr(TM)}$; since the map covers the identity and is an embedding in the fibers, it follows that it is an embedding- identifying Σ with its image in $\mathrm{Fr(TM)}$, i.e. with a G-structure. 😄

Another aspect that is specific to $E = TM$ and G-structures is the notion of integrability, which gives rise to one of the central problems in the theory of G-structures:

Definition 3.69. Given a G-structure Σ on M, an **chart adapted to** Σ is any coordinate chart (U, χ) of M with the property that the induced frames are adapted, i.e.,

$$\left(\left(\frac{\partial}{\partial \chi_1} \right)_x, \ldots, \left(\frac{\partial}{\partial \chi_n} \right)_x \right) \in \Sigma_x \quad \forall\, x \in U.$$

A G-structure is called **integrable** if around any point $x \in M$ one can find an adapted coordinate chart.

Example 3.70. We will look, one by one, at the various examples from the previous subsection but let us point out that while several of them are automatically integrable (for orientations and for volume forms), most of them are not. Furthermore, from the ones that are not, most of them are considered interesting only in the integrable case (symplectic, complex, p-planes/foliations) but there is also one at the other hand- that is most interesting in the non-integrable case (metrics). Even for the trivial groups G, when we talk about parallelisms $\mathfrak{e}$ (hence $\mathfrak{e}$ is a global frame), integrability is not automatic; as we shall see, it is equivalent to the fact that all the vector fields composing $\mathfrak{e}$ are pairwise commutative.

On the other hand, the canonical G-structure Σ_G^{can} is always integrable.

There is also a rather obvious notion of isomorphism between two G-structures Σ on M and Σ' on M' coming from the fact that any diffeomorphism $f : M \to M'$ has a canonical lift

$$f_* : \mathrm{Fr}(M) \to \mathrm{Fr}(M')$$

covering f, defined by the condition $\widehat{f_*(\mathfrak{e})} = df \circ \hat{\mathfrak{e}}$. More precisely, f is said to be an isomorphism between (M, Σ) and (M', Σ') if f_* takes Σ-frames to Σ'-frames:

$$(df)_x(\Sigma_x) = \Sigma_{f(x)} \quad \forall\, x \in M.$$

This also means that f^* restricts to an isomorphism of principal bundles $\Sigma \to \Sigma'$, covering f. The following is a matter of spelling out the definitions.

Corollary 3.71. *A G-structure Σ on M is integrable if and only if it is locally isomorphic to $(\mathbb{R}^n, \Sigma_G^{can})$.*

And here is yet another (slightly) different way of looking at integrable G-structures, via atlases. To that end, we introduce the following collection of diffeomorphisms between opens in $\mathbb{R}^n$:

$$\Gamma_G = \{f : U \xrightarrow{\sim} V : U, V \subset \mathbb{R}^n \text{ opens}, \mathrm{Jac_x(f)} \in \mathrm{G} \text{ for all } \mathrm{x} \in \mathrm{U}\}, \tag{3.4.1}$$

also called the pseudogroup associated to $G \subset \mathrm{GL}_n$, More conceptually, Γ_G consists of all the local automorphisms of Σ_G^{can}.

Definition 3.72. Given $G \subset \mathrm{GL}_n$ and an n-dimensional manifold M, a Γ_G-atlas for M is any smooth atlas $\mathscr{A}$ for M for which the change of coordinates belongs to Γ_G defined by (3.4.1). Two such Γ_G atlases are said to be equivalent if their union is again a Γ_G-atlas. The resulting equivalence classes of Γ_G-atlases are called **Γ_G-structures** on M.

It is clear that, given an integrable G-structure Σ, using only adapted frames (Definition 3.69) one obtains a (maximal) Γ_G-atlas. Conversely, a Γ_G-atlas induced a G-structure: the adapted frames being the canonical ones arising from charts of the resulting maximal Γ_G-atlas. Therefore:

Corollary 3.73. *There is a 1-1 correspondence between integrable G-structures and Γ_G-structures on M:*

$$\left\{\begin{array}{c}\textit{integrable}\\ G\textit{-structures on } M\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\Gamma_G\textit{-structures}\\ \textit{on } M\end{array}\right\}$$

3.4.2 $G = \{I\}$: frames and coframes

Let us briefly look at the extreme case when $G = \{I\}$ is the trivial group, briefly mentioned in Examples 3.66 and 3.70, when a G-structure is the same thing as a global frame

$$\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$$

on M. The canonical $\{I\}$-structure on $\mathbb{R}^n$ is then given by the standard global frame $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1}$. An isomorphism between two manifolds endowed with global frames, $(M, \mathfrak{e})$ and $(M, \mathfrak{e}')$, is any diffeomorphism $f : M \to M'$ with the property that

$$(df)_x(\mathfrak{e}_i(x)) = \mathfrak{e}_i'(f(x)) \quad \forall\, c \in M.$$

The integrability of $(M, \mathfrak{e})$ in the sense of $\{I\}$-structure becomes the condition that, around any point $x \in M$, one can find a coordinate chart (U, χ) such that

$$\mathfrak{e}_i = \frac{\partial}{\partial \chi_i}$$

for all i. Looking back at our discussion on parallelizations (section 1.3.1), we see that an $\{I\}$-structure is the same thing as a parallelization, and the integrability becomes the integrability induced earlier in Definition 1.48. Therefore, integrability is characterised by the v vanishing of the structure functions- see Theorem 1.49.

There is one more comment we would like to make here: very often one prefers to work with differential forms instead of vector fields. In particular, one prefers to work with coframes instead of frames, i.e. with families of 1-forms:

$$\theta = (\theta^1, \dots, \theta^n)$$

which, at each $x \in M$, induces a basis of T_x^*M. Of course, the two points of view are equivalent, as one has a 1-1 correspondence $\mathfrak{e} \longleftrightarrow \theta$ between frames and coframes given by the usual duality

$$\theta^i(\mathfrak{e}_j) = \delta_{i,j}.$$

However, when one goes deeper into the theory, the parallelism between the two points of view becomes a bit less obvious. One thing to keep in mind is that the role played by the Lie derivatives of vector fields, in the dual picture of forms, is taken by the DeRham differential. For instance, for a coframe θ, the resulting structure functions are characterized by

$$d\theta^k = \sum_{i<j} c_{i,j}^k \theta^i \wedge \theta^j.$$

Exercise 3.74. Prove that, indeed, given a frame $\mathfrak{e}$ and the induced coframe θ, the structure functions of $\mathfrak{e}$ satisfies the previous equation.

Of course, in the dual picture, the local model is $(\mathbb{R}^n, dx_1, \dots, dx_n)$, the integrability of (M, θ) is about writing (locally) $\theta^i = d\chi_i$ and the previous proposition say that this is possible if and only if all the 1-forms are closed.

Exercise 3.75. Give now another proof of the previous proposition, using the fact that, locally, all closed 1-forms are exact.

For the curious reader

Existence *Manifolds M with the property that TM is trivializable are called "parallelizable". This happens rather rarely. The situation is interesting even for the spheres; we discuss it here in a bit more detail. Note that the question of whether S^n is parallelizable or not can be rewritten in very down to earth terms: one is looking for $n+1$ functions*

$$F^1, \dots, F^n \to \mathbb{R}^{n+1}$$

with the property that, for each $x \in S^n \subset \mathbb{R}^{n+1}$, vectors $F^1(x), \dots, F^n(x)$ are linearly independent and take value in the hyperplane P_x orthogonal to x:

$$F^i(x) \in P_x := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}.$$

This comes from the standard identification of the tangent spaces T_xS^n with $P_x \subset \mathbb{R}^{n+1}$. In the case $n = 1$ one can take

$$F^1(x_1, x_2) = (x_2, -x_1)$$

or, interpreting $\mathbb{R}^2$ as $\mathbb{C}$ ($(x_1, x_2) \mapsto x_1 + ix_2$),

$$F^1(z) = iz.$$

It works quite similarly for S^3, but using quaternions. The 2-sphere however is not parallelizable. Actually, on S^2 any vector field must vanish at least one point. This is popularly known as the "hairy ball theorem", and stated as: "you can't comb a hairy ball flat without creating a cowlick"(see also the "Cyclone consequences" on Wikipedia). The mathematical reason that S^2 is not parallelizable (and actually that it does not admit any nowhere vanishing vector field) is that its "Euler characteristic" does not vanish. Actually, a nice (but non-trivial) theorem in differential topology says that the Euler characteristic of a compact manifold vanishes if and only if it admits a no-where vanishing vector field. By the way, for the same reason, all the even dimensional spheres S^{2n} are not parallelizable.

Are the other (odd dimensional) spheres parallelizable? Yes: there is also S^7. This uses octonions (also called the Cayley algebra) instead of quaternions (S^3) and complex numbers (S^1). How general is this? In what dimensions does it work? Looking at the proofs (e.g. for S^1 and S^3) we see that what we need is a "normed division algebra" structure on $\mathbb{R}^{n+1}$ (in order to handle the n-sphere S^n). By that we mean a multiplication "$\cdot$" on $\mathbb{R}^{n+1}$, with unit element $1 := (1, 0, \dots, 0)$ (but not necessarily associative- as in the case of the octonions) and satisfying the norm condition

$$|x \cdot y| = |x||y| \quad \forall\, x, y \in \mathbb{R}^{n+1},$$

where $|\cdot|$ is the standard norm. The term "division algebra" comes from the fact that the norm condition implies that the product has the "division property": $xy = 0$ implies that either $x = 0$ or $y = 0$ (if we keep this condition, but we give up on the norm condition, we talk about division algebras). Indeed, any such operation induces

$$F^i(x) = e_i \cdot x$$

proving that S^n is parallelizable. But on which $\mathbb{R}^{n+1}$ do there exist such operations? Well: only on $\mathbb{R}$, $\mathbb{R}^2$ (complex), $\mathbb{R}^4$ (quaternions) and $\mathbb{R}^8$ (octonions)! This was known since the 19th century. It is also interesting to know such operations were considered already in the 19th century: number theory and the question of which numbers can be written as sums of two, three, etc squares. For sum of two squares, the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy, z_2 = a_i b$:

$$|z_1 z_2| = |z_1||z_2|.$$

The search for similar "magic formulas" for sum of three squares never worked, but it did for four:

$$(x^2 + y^2 + z^2 + t^2)(a^2 + b^2 + c^2 + d^2) =$$

$$(xa+yb+zc+td)^2+(xb-ya-zd+tc)^2+$$
$$+(xc+yd-za-tb)^2+(xd-yc+zb-ta)^2.$$

This is governed by the quaternions and its norm equation.

Any way, returning to the spheres, we see that the trick with the multiplication can only work for S^1, S^3 and S^7. And, indeed, one can prove that there are the only parallelizable spheres! Well, S^0 as well if you insist. The proof is highly non-trivial and makes use of the machinery of Algebraic Topology.

There are a few more things that are worth mentioning here. One is that probably the largest class of manifolds that are automatically parallelizable (because of their structure) are the Lie groups. This can be seen as one of the reasons that S^1 and S^3 are parallelizable. While the Lie group structure for S^1 and S^3 uses complex numbers and quaternions, respectively, a similar argument for S^7 using octonions does not work because the multiplication of octonions is not associative (to ensure that S^7 is parallelizable, the division property is enough, but for making it into a Lie group one would need associativity). Can other spheres be made into Lie groups (besides S^1 and S^3)? No!

3.4.3 Example: Orientations

For $G=\mathrm{GL}_n^+$, G-structures can be interpreted as **orientations on** M (see Proposition 3.58 applied to $E=TM$). Notice that, in this case, the notion of adapted chart from Definition 3.69 becomes precisely the notion of positive chart for the oriented manifold $(M,\mathfrak{o})$.

Theorem 3.76. *Any orientation $\mathscr{O}$ on M, when interpreted as an GL_n^+-structure, is integrable: around any point in M one can find a positive coordinate chart (U,χ).*

Proof. We know we find a frame $\mathfrak{e}$ defined over some open U containing x_0 such that $\mathfrak{e}_x$ induces $\mathscr{O}_x$ for all $x\in U$. Choose a coordinate chart (U_0,χ) defined over some open $U_0\subset U$ containing x_0 which we may assume to be connected, such that the corresponding frame at x_0 induces the orientation $\mathscr{O}_{x_0}$. For $x\in U_0$, the matrix $A(x)$ of coordinate changes from the frame $\mathfrak{e}_x$ to the frame induced by (U_0,χ) at x, has the entries $A_{i,j}(x)$ smooth with respect to $x\in U_0$. Hence also det(A) is smooth; the determinant is also non-zero, and we know it is strictly positive at x_0. Hence it is positive on the (connected) U_0. This implies that the frame induced by (U_0,χ) at any $x\in U_0$ induces the orientation $\mathscr{O}_x$. 😄

The integrability of orientations gives rise to a slightly different way of looking at orientations on manifolds: via the choice of an oriented atlas. This is just the particular case of Corollary 3.73.

For the curious reader

Existence *Do all manifolds admit an orientation? The answer is: no but ... almost yes. On one hand, many examples are- e.g. all the spheres, or even all the simply connected manifolds, or all the complex manifolds, etc. Though there are simple examples like $\mathbb{P}^2$ that are not. But, like for $\mathbb{P}^2$ that is covered by the orientable S^2, any non-orientable manifold M can be covered by one which is orientable: one constructs the **orientation cover** of M as*

$$\tilde{M}=\{(x,\mathscr{O}_x): x\in M, \mathscr{O}_x-\textit{orientation on } T_xM\},$$

with the smooth structure defined so that the projection

$$p:\tilde{M}\to M,\quad p(x,\mathscr{O}_x)=x$$

is a local diffeomorphism, with each chart (U,χ) of M giving rise to one of $\tilde{M}$, namely $\chi\circ p$ defined on $U^+\subset\tilde{M}$ consisting of $(x,\mathscr{O}_x)$ with $x\in U$ and such that $\left(\frac{\partial}{\partial\chi_1}\right)_x,\ldots$ becomes positive w.r.t. $\mathscr{O}_x$. We leave it as an exercise to show that $\tilde{M}$ comes with a canonical orientation and becomes a 2-1 cover of M.

Why and how? (integration) One of the main uses of orientations comes from the fact that, once an orientation is fixed on a manifold M, there is an induced integration operation

$$\int_M:\Omega^n_{\mathrm{cpt}}(M)\to\mathbb{R}$$

on top-differential forms with "compact supports". Assuming that M is compact, Stokes formula implies that integration induces a linear map

$$\int_M:H^n(M)\to\mathbb{R}$$

which can actually be shown to be an isomorphism if M is connected. Further exploiting this, one may derive the following characterization for orientability: a compact connected manifold is orientable if and only if $H^n(M)$ is isomorphic to $\mathbb{R}$. See also the similar discussion for volume forms below.

This discussion may be viewed also as the starting point of an important tool of Algebraic/Differential Topology: Poincare duality. Given the compact, connected, oriented M then for any $k\in\{0,1,\ldots,n\}$, using the wedge product of forms, passing to cohomology and then using the integration gives a pairing

$$H^k(M)\times H^{n-k}(M)\to\mathbb{R},\ ([\omega],[\eta])\mapsto\int_M\omega\wedge\eta.$$

One version of the Poincare duality says that this pairing is perfect ("non-degenerate"). From this it follows for instance that, for odd dimensional M, its Euler characteristic is zero. The previous pairing is an important tool in the study of manifolds. For instance, for 4-dimensional manifolds, choosing $k = 2$, one obtains a non-degenerate bilinear form on $H^2(M)$- an algebraic invariant that tells us quite a bit about M. But we are moving to other territories here ...

3.4.4 Example: Volume forms

For $G = \mathrm{GL}_n^+$, G-structures can be interpreted as **volume forms on** M (see (3.3.2) for $E = TM$), i.e. nowhere vanishing top-degree forms $\mu \in \Omega^m(M)$. The local model is $\mathbb{R}^n$ together with

$$\mu_{\mathrm{can}} = dx_1 \wedge \ldots \wedge dx_n.$$

Theorem 3.77. *Any volume form μ on a manifold M, when interpreted as a $SL_n(\mathbb{R})$-structure, is integrable: around any point in M, one can find a coordinate chart (U, χ) such that, on U,*

$$\mu = d\chi_1 \wedge \ldots \wedge d\chi_n.$$

Proof. Start with an arbitrary chart (U, χ) around x. Writing μ in the resulting coordinates, it looks like

$$\mu = f \cdot d\chi_1 \wedge \ldots \wedge d\chi_n$$

for some smooth function f on U. Then just choose any function χ_1' near x such that $\partial\chi_1'/\partial\chi_1 = f$ and replace the old coordinates by $(\chi_1', \chi_2, \ldots, \chi_n)$. 😄

Note that the discussion on vector spaces and vector bundles implies that any volume form on M induces an orientation on M. Actually more is true when it comes to the existence question:

Exercise 3.78. Show that a manifold M admits a volume form if and only if it is orientable. (Hint: time to exercise a bit with partitions of unity!)

For the curious reader

Existence By the previous exercise, the question of existence of volume forms brings nothing new when compared with the similar question for orientations (see however below).

Why and how? A volume form μ on a compact manifolds M allows us to pass from functions to top-forms ($f \mapsto f \cdot \mu$) which, in turn, can be integrated w.r.t. the orientation induced by μ itself. In particular, and this is one of the main uses of volume forms, μ allows us to makes sense of the integral of the function $f = 1$ and, therefore, of "the volume of M". Or, without so many words, one defines the volume of M w.r.t. μ as:

$$\mathrm{Vol}_\mu(M) := \int_M \mu$$

It is remarkable that this (numerical!) invariant characterises μ uniquely up to isomorphism:

Theorem 3.79 (Moser). *If μ and μ' are two volume elements on a compact manifold M then the following are equivalent:*

1. *there exists a diffeomorphism $h : M \to M$ such that $\mu' = h^*\mu$.*
2. *$Vol_\mu(M) = Vol_{\mu'}(M)$.*

Proof. For the direct implication, note that the integration is invariant under oriented diffeomorphisms (...). Let's concentrate on the other, more difficult, implication. If you did the previous exercise, you noticed that an affine combinations of volume forms is a volume form. Hence for any $t \in [0,1]$,

$$\mu_t := t\mu + (1-t)\mu'$$

is a volume form, and it is not difficult to see that it induces the same orientation as μ. Since $\int_M \mu = \int_M \mu'$ it follows (see the relationship of integration with DeRham cohomology) that $\mu - \mu'$ is exact. We conclude that there exists a form $\eta \in \Omega^{n-1}(M)$ such that

$$\frac{d\mu_t}{dt} = d\eta$$

for all t. Moreover, since μ_t is a volume form, it is not difficult to see that the operation $\mathscr{X}(M) \to \Omega^{n-1}(M)$, $X \mapsto i_X(\eta_t)$ is an isomorphism (think first what happens for vector spaces) hence we find a vector field X_t such that

$$i_{X_t}(\mu_t) = -\eta.$$

All together, $\{X_t : t \in [0,1]\}$ form a smooth family of vector fields on M (time dependent vector field). Finally, consider the flow of this family, which can be seen as a family of diffeomorphisms

$$\varphi^t : M \to M.$$

Using the basic properties of flows, one computes:

$$\frac{d}{dt}\varphi_t^*\mu_t = \varphi_t^*(L_{X_t}(\mu_t) + \dot{\mu}_t) = \ldots = 0,$$

hence $\varphi_t^*\mu_t$ is constant with respect to t. In particular $\varphi_1^*\mu_1 = \mu_0$, i.e. $\varphi_1^*\mu = \mu'$. 😄

Remark 3.80 (more on the relationship with integration). Since the volume of M w.r.t. μ is always non-zero, we obtain that $\int_M : H^n(M) \to \mathbb{R}$ is indeed non-zero, as stated above when looking at orientability. All together, the conclusion is that for n-dimensional compact connected manifolds M, the following are equivalent:

- M is orientable.
- M admits a volume form.
- $H^n(M) \neq 0$.
- $H^n(M)$ is isomorphic to $\mathbb{R}$.

3.4.5 Example: foliations

For $G = \mathrm{GL}_{p,n-p}(\mathbb{R})$, G-structures can be interpreted as rank p sub-bundles of TM

$$\mathscr{F} \subset TM$$

(cf (3.3.4) applied to $E = TM$), i.e., p-dimensional distributions on M as discussed in the section on foliations (section 1.3.2). The integrability of $\mathscr{F}$ as a $\mathrm{GL}_{p,n-p}(\mathbb{R})$-structure (Definition 3.69) becomes precisely condition 2. from Lemma 1.56. Therefore, an integrable $\mathrm{GL}_{p,n-p}(\mathbb{R})$-structure is the same thing as a foliation in the sense of Definition 1.52.

On the other hand, the collection Γ_G from (3.4.1) consists of diffeomorphisms of type

$$\mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p \times \mathbb{R}^{n-p}, \quad (x,y) \mapsto (f(x), g(x,y)). \tag{3.4.2}$$

The resulting notions of Γ_G-atlas and Γ_G-structure (cf. Definition 3.72) are called **foliated atlas** and **foliated structure**, respectively and this is often the viewpoint that is used for introducing foliations (and it is backed up by Corollary 3.73). The overall conclusion is: a bijection

$$\left\{\begin{array}{c}\text{integrable}\\ GL_{p,n-p}(\mathbb{R})\text{-structures on } M\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{foliations}\\ \text{on } M\end{array}\right\} \tag{3.4.3}$$

3.4.6 Example: Integral affine structures

For $G = \mathrm{SL}_n(\mathbb{R})$, G-structures can be interpreted as integral structures on TM

$$\Lambda \subset TM,$$

(cf. Proposition 3.64), also called **almost integral affine structures on** M. It may be interesting to remember that Λ satisfies the smoothness condition that, around any point in M one can find an open U and a local frame $\mathfrak{e}$ over U such that, for all $x \in U$,

$$\Lambda_x = \mathrm{Span}_{\mathbb{Z}}\{\mathfrak{e}_1(x), \ldots, \mathfrak{e}_n(x)\} = \mathbb{Z}\cdot\mathfrak{e}_1(x) + \ldots + \mathbb{Z}\cdot\mathfrak{e}_n(x).$$

The integrability of Λ means that around any point of M one can find a coordinate chart (U, χ) such that

$$\Lambda_x = \mathbb{Z}\cdot\left(\frac{\partial}{\partial\chi_1}\right)_x + \ldots + \mathbb{Z}\cdot\left(\frac{\partial}{\partial\chi_n}\right)_x \tag{3.4.4}$$

for all $x \in U$. Notice the an atlas consisting of charts with this property will have some very special transition functions: they must be of type

$$F(v) = v_0 + A(v) \quad \text{with } v_0 \in \mathbb{R}^n,\ A \in \mathrm{GL}_n(\mathbb{Z}). \tag{3.4.5}$$

Such transformations of $\mathbb{R}^n$ (or between opens inside $\mathbb{R}^n$) are called **integral affine transformations**.

Definition 3.81. An **integral affine atlas** on a manifold M is any smooth atlas $\mathscr{A}$ for M for which the change of coordinates belongs are integral affine transformations. Two such atlases are said to be (integral affine equivalent) if their union is again an integral affine atlas.

A **integral affine structure** on M is an equivalence class of integral affine atlases.

Therefore, integrability gives rise to integral affine structures. Conversely, charts in an integral affine atlas induce, using (3.4.4) as definition, an integral structure Λ. Notice that all we are doing now is repeating the argument behind Corollary 3.73, hopefully a more transparent way (in this specific situation). The conclusion is the same: a bijection

$$\left\{\begin{array}{c}\text{integrable}\\ \mathrm{SL}_n(\mathbb{R})\text{-structures on } M\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{integral affine}\\ \text{structures on } M\end{array}\right\} \tag{3.4.6}$$

On the other hand, this is merely a reformulation of integrability. For a more substantial characterization of integrability, similar to what we have seen for $\{I\}$-structures or foliations (and we will see for symplectic and complex structures), we need the following two concepts, that apply whenever we have a manifold endowed with an almost integral structure Λ on TM:

- (local) **integral vector fields** of (M,Λ): (local) vector fields that take values in Λ.
- **the dual lattice**, which is the lattice $\Lambda^\vee$ in the dual of T^*M given by:

$$\Lambda_x^\vee = \{\xi \in T_x^*M : \xi(\Lambda_x) \subset \mathbb{Z}\}.$$

Theorem 3.82. *For an almost integral affine structure Λ on a manifold M the following are equivalent:*

1. *Λ is integrable (hence an integral affine structure).*
2. *any two local integral vector fields commute.*
3. *$\Lambda^\vee$ is locally spanned by closed 1-forms, i.e., any point in M admits a neighborhood U and closed 1-forms $\theta^1,\ldots,\theta^n$ on U such that $\Lambda_x^\vee = Span_{\mathbb{Z}}\{\theta_x^1,\ldots,\theta_x^n\}$.*

Proof. That 1 implies 2 comes from the fact that the condition in 2 can be checked locally and is valid for the local model. Assume now that 2 holds. Locally, we choose any frame $\{\mathfrak{e}_1,\ldots,\mathfrak{e}_n\}$ which spans Λ (as in the description of smoothness above). The condition in 2 tells us that $\mathfrak{e}$ is integrable in the sense of e-structures (see subsection 3.4.2), hence we find charts (U,χ) such that $\mathfrak{e}_i = d\chi_i$, and this implies 1. The equivalence with 3 is similar, using the dual point of view (see again subsection 3.4.2). 😄

Proof. That 1 implies 2 comes from the fact that the condition in 2 can be checked locally and is valid for the local model. Assume now that 2 holds. Locally, we choose any frame $\{\mathfrak{e}_1,\ldots,\mathfrak{e}_n\}$ which spans Λ (as in the description of smoothness above). The condition in 2 tells us that $\mathfrak{e}$ is integrable in the sense of e-structures (see subsection 3.4.2), hence we find charts (U,χ) such that $\mathfrak{e}_i = d\chi_i$, and this implies 1. The equivalence with 3 is similar, using the dual point of view (see again subsection 3.4.2). 😄

Exercise 3.83. Show that if a manifold M is orientable and admits an integral affine structure, then it admits a volume form.

3.4.7 Example: Complex structures

For $G = \mathrm{GL}_k(\mathbb{C})$ the G-structures on M can be interpreted as almost complex structures on M as introduced in Definition 1.65 (cf. (3.60)). The integrability of J from the viewpoint of $\mathrm{GL}_k(\mathbb{C})$-structures (Definition 3.69) becomes precisely the characterisation from Lemma 1.68 of the integrability of J from the point of view of complex manifolds (Definition 1.67).

On the other hand, the collection Γ_G from (3.4.1) consists of diffeomorphisms of type

$$(f,g) = (f_1,\dots,f_k,g_1,\dots,g_k) : V \to W \quad (V,W \subset \mathbb{R}^{2k} \text{ opens})$$

with the property that $(df)\circ J_{\mathrm{can}} = J_{\mathrm{can}} \circ (df)$ or, equivalently,

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial g_k}{\partial y_j}, \frac{\partial f_k}{\partial y_j} = -\frac{\partial g_k}{\partial x_j}.$$

By Cauchy-Riemann this corresponds to holomorphicity and, therefore, the resulting notions of Γ_G-atlas and Γ_G-structure (cf. Definition 3.72) are precisely the holomorphic atlases and holomorphic structures discussed at the start of the section on complex structures (section 1.3.3). The overall conclusion is a bijection

$$\left\{\begin{array}{c}\text{integrable}\\ \mathrm{GL}_k(\mathbb{C})\text{-structures on } M\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{complex}\\ \text{structures on } M\end{array}\right\} \tag{3.4.7}$$

3.4.8 Example: Symplectic structures

For $G = \mathrm{Sp}_{2k}(\mathbb{R})$ from (3.3.5), G-structure correspond to non-degenerate 2-forms on M,

$$\omega \in \Omega^2(M)$$

(cf. Proposition 3.63 applied to $E = TM$), i.e., to the almost symplectic structures from Definition 1.72.

The integrability of ω from the viewpoint of $\mathrm{Sp}_{2k}(\mathbb{R})$-structures (Definition 3.69) becomes precisely the definition of integrability from the point of view of symplectic structures (Definition 1.73). Similar to the previous discussions, one obtains a bijection

$$\left\{\begin{array}{c}\text{integrable}\\ \mathrm{Sp}_{2k}(\mathbb{R})\text{-structures on } M\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{symplectic}\\ \text{structures on } M\end{array}\right\} \tag{3.4.8}$$

Chapter 4
Connections on principal bundles

Throughout this subsection G is a Lie group and $\pi : P \to M$ is a principal G-bundle. There are several different ways of looking at connections on P. We start with the more intuitive one.

4.1 Connections

4.1.1 Connections as horizontal distributions:

A horizontal subspace of TP at $p \in P$ is a subspace

$$\mathscr{H}_p \subset T_pP$$

with the property that

$$(d\pi)_p|_{\mathscr{H}_p} : \mathscr{H}_p \to T_{\pi(p)}M \tag{4.1.1}$$

is an isomorphism.

Exercise 4.1. Show that, for a vector sub-space $\mathscr{H}_p \subset T_pP$, denoting $x = \pi(p)$ and using the vertical space $T_p^{\text{vert}}P$ from Exercise 4.22, the following are equivalent:

1. $\mathscr{H}_p$ is horizontal.
2. $\mathscr{H}_p$ is a complement of T_p^{vert} inside T_pP.
3. $(d\pi)_p : T_pP \to T_xM$ restricts to an isomorphism between $\mathscr{H}_p$ and T_xM.
4. There exists a splitting of $(d\pi)_p$, i.e. a linear map

$$h_p : T_xM \to T_pP$$

satisfying $(d\pi)_p \circ h_p = \mathrm{Id}$, such that $\mathscr{H}_p$ is the image of h_p.
(Read: h_p is the operation of "horizontal lifting" associated to $\mathscr{H}_p$).

Note that if $\mathscr{H}_p$ is a horizontal space at $p \in P$ then, for any $g \in G$,

$$R_g(\mathscr{H}_p) := \{(dR_g)_p(X_p) : X_p \in \mathscr{H}_p\} \subset T_{gp}P$$

is another horizontal subspace at gp. By distribution on P we mean a vector sub-bundle $\mathscr{H}$ of TP (hence a collection $\{\mathscr{H}_p\}_{p\in P}$ of vector sub-spaces $\mathscr{H}_p$ of T_pP, all of the same dimension, which fits together into a smooth submanifold $\mathscr{H}$ of TP).

Definition 4.2. A connection on P is a distribution $\mathscr{H}$ on P with the property that each $\mathscr{H}_p$ is a horizontal subspace of P and

$$\mathscr{H}_{pg} = R_g(\mathscr{H}_p) \quad \forall\, p \in P, g \in G.$$

While the notion of vertical vector fields on P is intrinsic (it does not depend on any choices), a connection allows us to talk about horizontal vectors on P: the ones that lie in $\mathscr{H}$ (hence one should really make a reference to $\mathscr{H}$ in the terminology- e.g. talk about $\mathscr{H}$-horizontal). Note that any vector field on M has a lift to a horizontal vector field on P, i.e. there is an operation

$$h : \mathscr{X}(M) \to \mathscr{X}(P)$$

(the horizontal lift operation associated to the connection) which associates to $X \in \mathscr{X}(M)$ the unique horizontal vector field $h(X)$ on P satisfying

$$(d\pi)_p(h(X)_p) = X_{\pi(p)}$$

for all $p \in P$. With the notations from the previous exercise, $h(X)_p := h_p(X_{\pi(p)})$.

Remark 4.3. Putting all the possible horizontal liftings together we obtain the so-called first order jet of P,

$$J^1(P) := \{(p, h_p) : p \in P, h_p \text{ is a horizontal lifting of } P \text{ at } p\}.$$

This can be seen as a (discrete for now) bundle over P (using the projection on the first coordinate) and also as a bundle over M by further composing with the projection $\pi : P \to M$. The terminology comes from the fact that this set encodes all the possible first order data (jets) associated to (local) sections of P. More precisely, given $x \in M$, any local section σ of P defined around x, induces:

- the 0-th order data at x: just the value $\sigma(x) \in P$.
- the 1-st order data at x: the value $\sigma(x)$ together with the differential of σ at p (which encodes the first order derivatives of σ at x),

$$(d\sigma)_x : T_xM \to T_{\sigma(x)}P.$$

Of course, $(d\sigma)_x$ is a horizontal lifting at $\sigma(x)$ and it is not difficult to see that

$$J^1(P) = \{(\sigma(x), (d\sigma)_x) : x \in M, \sigma - \text{local section of } P \text{ defined around } p\}.$$

Note that for a section σ of P, the resulting map

$$j^1(\sigma) : M \to J^1(P), \;\; x \mapsto (\sigma(x), (d\sigma)_x))$$

is a section of $J^1(P)$ viewed as a bundle over P. Similarly for local sections. There is a natural smooth structure on $J^1(P)$ uniquely determined by the condition that all local sections of type $j^1(\sigma)$ are smooth.

4.1.2 Connection forms

Here is a slightly different point of view on connections, not so natural, but often easier to work with. Let $\mathfrak{g}$ be the Lie algebra of G. We will use the infinitesimal action of $\mathfrak{g}$ on P (see Proposition 4.17),

$$a : \mathfrak{g} \to \mathscr{X}(P),\; a(v)_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tv),$$

and the adjoint representation of G on $\mathfrak{g}$ (see Example TO BE FILLED IN),

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \;\; g \mapsto \mathrm{Ad}_g.$$

The general discussion from subsection TO BE FILLED IN gives us the notion of G-invariant forms on P with values in $\mathfrak{g}$: those $\theta \in \Omega^k(P, \mathfrak{g})$ with the property that

$$R_g^*(\omega) = \mathrm{Ad}_{g^{-1}}(\omega)$$

for all $g \in G$.

Definition 4.4. A connection form on P is a 1-form

$$\omega \in \Omega^1(P, \mathfrak{g})$$

with the property that it is G-invariant and satisfies

$$\omega(a(v)) = v \quad \forall\, v \in \mathfrak{g}.$$

Proposition 4.5. *There is a bijection between connection 1-forms $\omega \in \Omega^1(P, \mathfrak{g})$ and connections $\mathscr{H}$ on P. The bijection associates to ω its kernels:*

$$\mathscr{H}_p := \{X_p \in T_pP : \omega_p(X_p) = 0\}.$$

Proof. For each $p \in P$ we consider the short exact sequence of vector spaces

$$\mathfrak{g} \overset{a_p}{\to} T_pP \overset{(d\pi)_p}{\to} T_{\pi(p)}M.$$

A right splitting of this sequence is a right inverse of $(d\pi)_p$; as we have noticed, such right splittings h_p correspond to horizontal subspaces $\mathscr{H}_p$. On the other hand one can also talk about left splitting of the sequence, by which we mean a left inverse of a_p or, equivalently, linear maps $\omega_p : T_pP \to \mathfrak{g}$ satisfying $\omega_p(a_p(v)) = v$ for all $v \in \mathfrak{g}$. What we need here is a simple linear algebra fact which says that, for a short exact sequence of vector space, there is a 1-1 correspondence between the choice of right splittings h_p and the choice of left splittings ω_p. Explicitly, this correspondence is described by the equation

$$a_p \circ \omega_p + h_p \circ (d\pi)_p = \mathrm{Id}.$$

After applying this at all $p \in P$, one obtains the 1-1 correspondence from the statement. Of course, one still has to check that the smoothness of ω is equivalent to the one of h and similarly for the G-invariance, but we leave these as an exercise. 😄

Exercise 4.6. Consider the interpretation of S^3 in terms of quaternions (Example 3.1); remark that, for any $u \in S^3$,

$$iu, ju, ku \in \mathbb{H} \cong \mathbb{R}^4$$

are orthogonal to u, i.e. they define vectors tangent to S^3 at u. Hence, each of these elements defines a vector field on S^3; they are denoted X_1 (using i), X_2 (using j) and X_3 (using k). For instance,

$$X_1(x,y,z,t) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \in T_{x,y,z,t}S^3 \subset T_{x,y,z,t}\mathbb{R}^4.$$

We also consider the coframe $\theta^1, \theta^2, \theta^3$ associated to the frame $\{X_1, X_2, X_3\}$, i.e. the collection of 1-forms on S^3 uniquely determined by $\theta^i(X_j) = \delta^i_j$.

1. Show that the flow of X_1 induces the action of S^1 on S^3 from the Hopf fibration (Example 3.34).
2. Compute the differentials of each θ_i and write them in terms of wedge of θ_1, θ_2 and θ_3.
3. Show that θ^1 can be interpreted as a connection on the Hopf fibration.

4.1.3 Parallel transport

As in the case of vector bundles, connections on principal bundles have associated parallel transports. Assume we fix a connection $\mathscr{H}$ on the principal G-bundle $\pi : P \to M$, with associated connection form ω. Then we say that a curve $u : I \to P$ is horizontal if

$$\dot{u}(t) \in \mathscr{H}_{\gamma(t)} \quad \forall\, t \in I,$$

where $\gamma(t) = \pi(u(t))$. Then, completely similar to Lemma 2.20, we have:

Lemma 4.7. *Let $\gamma : I \to M$ be a curve in M, $t_0 \in I$. Then for any $u_0 \in P_{\gamma(t_0)}$ there exists and is unique a horizontal path above γ, $u : I \to P$, with $u(t_0) = u_0$.*

Proof. Again, we may work locally, i.e. we may just assume that $P = M \times G$ is the trivial principal bundle. Let us now see what a connection form on this bundle looks like. First of all, the invariance condition means that, for all $(x,a) \in M \times G$, $(X_x, V_a) \in T_{x,a}(M \times G) = T_xM \times T_aG$, and for all $g \in G$,

$$\omega_{x,ag}(X_x, R_g(V_a)) = \mathrm{Ad}_{g^{-1}}(\omega_{x,a}(X_x, V_a)),$$

where we denote by $R_g : G \to G$ the right multiplication by g and also its differential $(dR_g)_a : T_aG \to T_{ag}G$. Taking $a = 1$, we see that ω is determined by all the expressions of type $\omega_{x,1}(X_x, v)$ with $1 \in G$ the unit, $v \in T_1G = \mathfrak{g}$. In turn, the condition that $\omega(a(v)) = v$ for all $v \in \mathfrak{g}$ implies that ω is uniquely determined by all expressions of type

$$\omega_{x,1}(X_x, 0) \in \mathfrak{g}$$

i.e. precisely by the restriction $\eta \in \Omega^1(M, \mathfrak{g})$ of ω to M via the inclusion $M \hookrightarrow M \times G$, $x \mapsto (x,1)$. Writing down the explicit formulas we find that the formula for ω in terms of η:

$$\omega_{x,g}(X_x, V_g) = \mathrm{Ad}_{g^{-1}}(\eta_x(X_x)) + L_{g^{-1}}(V_g),$$

where, as before, $L_{g^{-1}} : G \to G$ stands for the left multiplication by g^{-1} and also for its differential $(dL_{g^{-1}})_g : T_gG \to T_1G = \mathfrak{g}$.

A curve in $P = M \times G$ above γ can be written as

$$u(t) = (\gamma(t), g(t)).$$

The fact that it is horizontal means that $\dot{u}(t)$ takes values in the horizontal space or, equivalently, that it is killed by ω:

$$\omega_{\gamma(t),g(t)}(\dot{\gamma}(t), \dot{g}(t)) = 0 \quad \forall\, t \in I.$$

Using the previous formula for ω we see that this equation becomes

$$\dot{g}(t) = -R_{g(t)}(\eta_{\gamma(t)}(\dot{\gamma}(t))).$$

For fixed γ we need existence and uniqueness of g satisfying this equation, with an initial condition $g(t_0) = g_0$ given. We denote $v(t) = -\eta_{\gamma(t)}(\dot{\gamma}(t)$ (curve in $\mathfrak{g}$, so that the equation reads $\dot{g}(t) = R_{g(t)}(v(t)))$. At least for linear groups $G \subset GL_r$ we deal again (as in the case of vector bundles) with an ordinary ODE. For a general Lie group one can just interpret this equation as the flow equation of the time-dependent vector field $X(t,g)$ on G given by

$$X(t,g) = R_g(v(t))$$

and use the standard results about flows of time-dependent vector fields. However, one still has to make sure that the equation with the given initial condition $g(t_0) = g_0$ exists on the entire interval I on which v is defined; for that one uses the remark that, if $g(t)$ is an integral curve then so is $h(t) = g(t)a$ for all $a \in G$:

$$\dot{h}(t) = R_a(\dot{g}(t)) = R_a(R_{g(t)}(v(t)) = R_{h(t)}(v(t)).$$ 

Finally, as in the case of vector bundles, for curves $\gamma : I \to M$, $t_0, t_1 \in M$, one obtains the parallel transport (with respect to the connection) along γ:

$$T_\gamma^{t_0,t_1} : P_{\gamma(t_0)} \to P_{\gamma(t_1)}$$

which associates to $u_0 \in P_{\gamma(t_0)}$ the element $u(t_1) \in P_{\gamma(t_1)}$, where u is the horizontal path above γ with $u(t_0) = u_0$.

Exercise 4.8. Show that $T_\gamma^{t_0,t_1}$ are diffeomorphisms and are G-equivariant.

4.1.4 Curvature

Let $\mathscr{H}$ be a connection on P, with associated horizontal lifting h and connection 1-form ω. The curvature of $\mathscr{H}$ measure the failure of $\mathscr{H}$ to be involutive or, equivalently, the failure of the horizontal lifting h to preserve the Lie bracket of vector fields. However, let us introduce it in the form that is easier to work with. For that we note that the Lie bracket $[\cdot,\cdot]$ of the Lie algebra $\mathfrak{g}$ extends to a bracket

$$[\cdot,\cdot] : \Omega^1(P,\mathfrak{g}) \times \Omega^1(P,\mathfrak{g}) \to \Omega^2(P,\mathfrak{g}),$$

$$[\alpha,\beta](X,Y) := [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)].$$

Note that when $\beta = \alpha \in \Omega^1(P,\mathfrak{g})$ one obtains

$$[\alpha,\alpha](X,Y) = 2[\alpha(X),\alpha(Y)].$$

Definition 4.9. The curvature of the connection ω is defined as

$$K := d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P,\mathfrak{g}).$$

Here is the main property of the curvature (for basic forms see subsection TO BE FILLED IN):

Lemma 4.10. *The curvature is a basic form:*

$$K \in \Omega^2(P,\mathfrak{g})_{bas}.$$

Proof. The G-invariance follows from the G-invariance of ω and the fact that all the operations involved in construction K (d and the bracket) are natural (functorial), hence are respected by the action of G (give the details!). The fact that K is horizontal ... 😄

Note that the fact that Ω is horizontal implies that, in order to know Ω, it suffices to know all the expressions of type

$$K(h(X),h(Y)) \in C^\infty(P,\mathfrak{g})$$

with $X,Y \in \mathscr{X}(M)$, where $h(X) \in \mathscr{X}(M)$ is the horizontal lift of X with respect to the connection. For the following, recall that $a : \mathfrak{g} \to \mathscr{X}(P)$ denotes the induced infinitesimal action (see above).

Proposition 4.11. *The curvature can also be characterized as:*

1. *the horizontal part of $d\omega$, i.e. the horizontal form on P (with values in $\mathfrak{g}$) which coincides with $d\omega$ on horizontal vectors.*
2. *the unique horizontal form Ω with the property that, after applying the infinitesimal action $a : \mathfrak{g} \to \mathscr{X}(P)$,*
$$a(K(h(X),h(Y))) = h([X,Y]) - [h(X),h(Y)]$$
for all $X,Y \in \mathscr{X}(M)$.

Proof. We have already shown that K is horizontal and then the first part is clear from the very definition of K. For the second part, by the comments before the proposition and the injectivity of a, the uniqueness follows. Hence we just have to show that the actual curvature does have this property. For that one just compute $K(h(X),h(Y))$ using the definition of K and the fact that $\omega \circ h = 0$ and we find:

$$K(h(X),h(Y)) = -\omega([h(X),h(Y)]) = \omega(h([X,Y]) - [h(X),h(Y)]).$$

On the other hand, since $h(X)$ is π-projectable to X, we have that $[h(X),h(Y)]$ is π-projectable to $[X,Y]$ and then $h([X,Y]) - [h(X),h(Y)]$ is π-projectable to $[X,Y]-[X,Y] = 0$, i.e. it is vertical. Hence, at each point p, it is of type $a(v_p)$ with some $v_p \in \mathfrak{g}$. Plugging in the previous equation, we find that

$$K(h(X),h(Y))(p) = \omega(a(v_p)) = v_p,$$

and then the desired equation follows is just the defining property of v_p. 😀

Exercise 4.12. As a continuation of Exercise 4.6: show that the curvature of the connection defined by θ^1 is a non-zero constant (that you have to compute) times the area form σ of S^2,

$$\sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

(be aware: the forms dx, etc make sense on the entire $\mathbb{R}^3$, but we only use their restrictions to S^3. And, while the forms dx, dy and dz are linearly independent on $\mathbb{R}^3$, their restrictions to S^2 do satisfy an extra-relation: $xdx + ydy + zdz = 0$ (why?)).

4.2 Connections: vector bundles versus principal bundles

4.2.1 From principal bundles to vector bundles

While we have seen that vector bundles of rank r are in 1-1 correspondence with principal GL_r-bundles, we now show that connections on vector bundles correspond (1-1) to connections on the associated principal bundles.

In one direction (from principal to vector) one can proceed, as before, slightly more generally, by starting with any Lie group G, a principal G-bundle $\pi : P \to M$ and a representation $\rho : G \to GL(V)$. The relevant vector bundle will be $E = E(P,V)$ previously discussed. We start with a connection on the principal G-bundle P and we would like to construct a connection ∇ on E; we will encode ∇ in the operator

$$d_\nabla : \Gamma(E) \to \Omega^1(M,E).$$

Using Proposition 4.23 we see that it suffices to describe the

$$D_\nabla := h^\bullet d_\nabla (h^\bullet)^{-1} : C^\infty(P,V)^G \to \Omega^1(P,V)_{\text{bas}}.$$