Differential Geometry recap exercises: differential forms and the deRham cohomology

2026

Exercise 1 Use Mayer-Vietoris to compute the cohomology of S^1 and S^2. How does this computation generalize to computing the cohomology of S^n?

Exercise 2 In this exercise, we will show that as graded rings, we have $H^\bullet(\mathbb{CP}^n) \cong \mathbb{R}[x]/(x^{n+1})$ with x a degree 2 element.

(a) Show that the map $i : \mathbb{CP}^{n-1} \to \mathbb{CP}^n$ given by $[z_0 : \cdots : z_{n-1}] \mapsto [z_0 : \cdots : z_{n-1} : 0]$ defines an embedding.

(b) Explain that $U := \mathbb{CP}^n \setminus i(\mathbb{CP}^{n-1})$ is open and equal to

$$\{[z_0 : \cdots : z_{n-1} : 1] \in \mathbb{CP}^n\}.$$

Furthermore, we have that $U \cong \mathbb{C}^n$.

(c) Now, also consider

$$V := \{[z_0 : \cdots : z_{n-1} : z_n] \in \mathbb{CP}^n : (z_0, \ldots, z_{n-1}) \neq 0\}.$$

Show, that V is open and deformation retracts onto $i(\mathbb{CP}^{n-1})$.

(d) Show that $U \cap V$ is homotopy equivalent to S^{2n-1}.

(e) Use the Mayer-Vietoris sequence to show that

$$H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

(f) Let $p : P \to M$ be a surjective submersion. Then, for all $x \in P$, we have $T_xP/\ker(dp)_x \cong T_{p(x)}M$. Let $\alpha \in \Omega^k(P)$. Explain that if $\iota_X \alpha_x = 0$ for all $X \in \ker d_x p$, we get an induced form $\overline{\alpha}_x$ on $T_{p(x)}M$.

(g) Show that the map $\Omega^k(M) \to \Omega^k(P)$ given by

$$\alpha \mapsto p^*\alpha$$

is injective and its image is given by the $\alpha \in \Omega^k(P)$ such that $\iota_X \alpha = 0$ for all $X \in \ker dp$ and for all $x, y \in P$ with $p(x) = p(y)$, we have $\overline{\alpha}_x = \overline{\alpha}_y$.

Recall that $\mathbb{CP}^n$ as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the scaling action

$$(z_0, \ldots, z_n) \cdot \lambda \to (z_0\lambda, \ldots, z_n\lambda),$$

where $\lambda \in \mathbb{C}^\times$. Let us write $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ for the associated quotient map.

(h) Show that a form $\alpha \in \Omega^k(\mathbb{C}^{n+1} \setminus \{0\})$ is of the form $p^*\eta$ for $\eta \in \mathbb{CP}^n$ if and only if $\lambda^*\alpha = \alpha$ and $\alpha(\ker dp) = 0$.

(i) Consider the complex 2-form

$$\omega = i \sum_{j=0}^{n} \frac{dz_j \wedge d\overline{z}_j}{\|z\|^2}$$

on $\mathbb{C}^{n+1} \setminus \{0\}$. Explain that ω is actually a real 2-form and show that it is closed.

(j) Show that ω is the pullback of a closed 2-form on $\mathbb{CP}^n$.

(k) Show that ω^n defines a volume form on $\mathbb{CP}^n$ i.e. a nowhere vanishing section of $\mathcal{A}lt^{2n}T^*\mathbb{CP}^n$ (defined under exercise 1.56).

(l) Let M be an arbitrary manifold. Define the wedge product $\wedge : H^m(M) \times H^k(M) \to H^{k+m}(M) : ([\alpha], [\beta]) \to [\alpha \wedge \beta]$ where $\alpha \wedge \beta$ is the usual wedge product of differential forms. Show that this map is well-defined.

(m) Show that $H^\bullet(\mathbb{CP}^n) \cong \mathbb{R}\langle \omega, \omega^2, \ldots, \omega^n \rangle$ an isomorphism of rings. Here we endow $H^\bullet(\mathbb{CP}^n)$ with the wedge product from above and $\mathbb{R}\langle \omega, \omega^2, \ldots, \omega^n \rangle$ with the product of polynomial under assuming that $\omega^k = 0$ for $k > n$.