Differential Geometry recap exercises: submersions

2025

Exercise 1: Show that the following maps are submersions

1. $\pi : \mathbb{R}^m \to \mathbb{R}^n : (x_1, \ldots, x_m) \to (x_1, \ldots, x_n)$ with $m \geq n$;
2. $f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \to x + (y-1)^2 + z^3$;
3. $g : (0, \infty)^3 \setminus A \to \mathbb{R}^2 : (x, y, z) \to (xyz, x^2 + y^2 + z^2)$ where $A = \{(x, y, z) \,|\, x = y = z\}$;
4. $m : \mathbb{R}^2 \times \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^2 : (v, A) \to Av$, where $\mathrm{GL}_2(\mathbb{R})$ are the 2×2-invertible matrices and Av is the usual matrix multiplication;
5. $q : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1} : v \to [v]$ where $\mathbb{CP}^{n-1}$ is the complex projective space and $[v] = \{\lambda v \,|\, \lambda \in \mathbb{C} \setminus \{0\}\}$ viewed as point in $\mathbb{CP}^{n-1}$;
6. $\pi : TM \to M : (p, v) \to p$ with M a smooth manifold and TM the tangent bundle.

Exercise 2: Let M, N be smooth manifolds and $F : M \to N$ a smooth map. Show that the following are equivalent

1. F is a submersion;
2. For every $p \in M$, there exist open neighborhoods $U \subseteq M$ containing p and $V \subseteq N$ containing $F(p)$ and a smooth map $s : V \to U$ such that $F \circ s(x) = x$;
3. For every point $p \in M$ and every smooth curve $\gamma : (-1, 1) \to N$ with $\gamma(0) = F(p)$, there exists a smooth curve $\tilde{\gamma} : (-1, 1) \to M$ and an $\epsilon \in (0, 1)$ such that $F \circ \tilde{\gamma}(t) = \gamma(t)$ for all $t \in (-\epsilon, \epsilon)$.

Exercise 3: Let M be a nonempty and compact smooth manifold. Show that there does not exist a smooth submersion $F : M \to \mathbb{R}^k$ for $k > 0$. (Hint: Can you prove that submersions are open maps?)