Recap

(1)

Key word: Submersions & tangent spaces

$f : M \longrightarrow N$ submersion at $p \in M$

$\begin{cases} f : \mathbb{R}^2 \longrightarrow \mathbb{R} \\ f(x,y) = x \end{cases}$

→ if $(df)_p : T_pM \longrightarrow T_{f(p)}N$ is surjective

→ $(\forall)$ (U,x) chart of M around p
$(\forall)$ (U',x') —//— N —//— $f(p)$
} than $\underbrace{x' \circ f \circ x^{-1}}_{f_x^{x'}} : \underset{\mathbb{R}^m}{\text{open}} \longrightarrow \underset{\mathbb{R}^n}{\text{open}}$ is a submersion at $x(p)$ if $\mathrm{Jac}_{x(p)}\left(f_x^{x'}\right)$ of rank n.

→ $\exists$ (U,x)
$\exists$ (U',x') s.t. $f_x^{x'}(x_1,\dots,x_n,x_{n+1},\dots,x_m) = (x_1,\dots,x_n)$
(Submersion theorem).

(2)

$v \in T_pM$

intuitively: $C^\infty(M) := \{ f : M \longrightarrow \mathbb{R} \mid f = \text{smooth} \}$ algebra $(+,\cdot)$

① "linear approximation of M around p"
② "collection of all velocities of curves in M starting at p"



Precisely:

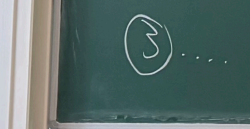
① a tgt vector at p is a derivation at p, i.e. $v : C^\infty(M) \longrightarrow \mathbb{R}$ linear s.t. $v(fg) = f(p)\,v(g) + g(p)\,v(f)$ $(\forall) f,g \in C^\infty(M)$.

② a tgt. vector at p is an equiv. class $[\gamma]_p$ of curves $\begin{cases} \gamma : (-\varepsilon,\varepsilon) \longrightarrow M \\ \gamma(0) = p \end{cases}$

where: $[\gamma_1]_p = [\gamma_2]_p \iff \gamma_1 \sim_p \gamma_2 \overset{df}{\iff}$ γ_1 and γ_2 have the same speed at $t=0$.

Notation: $[\gamma]_p =: \dfrac{d\gamma}{dt}(0)$.

③ ....



From ② → ①. Start with $[\gamma]_p$. Produce a v; call it v_γ (in coordinates).
$v_\gamma : C^\infty(M) \longrightarrow \mathbb{R}$, $v_\gamma(f) = \dfrac{d}{dt}\Big|_{t=0} f(\gamma(t)) = (f \circ \gamma)'(0)$

Thm: this is 1–1.

Vector bundles (1)

Fiber bundle: surjective submersion (*)

$\pi : \mathcal{P} \longrightarrow M$

$\mathcal{P}$: total space, M: the base

Key concepts:

- for $x \in M$: $\mathcal{P}_x := \pi^{-1}(x) = \{p \in \mathcal{P} : \pi(p) = x\}$
  the fiber above x

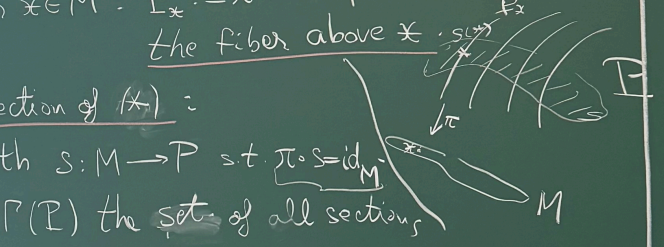
- **Section of (*)**:

any smooth $s : M \to \mathcal{P}$ s.t. $\pi \circ s = id_M$.

Denote $\Gamma(\mathcal{P})$ the set of all sections

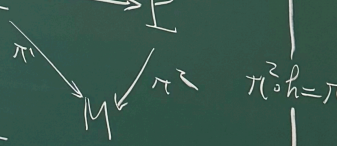
- **local sections of (*)**

any smooth $s : \underset{\text{open}}{U} \to \mathcal{P}$ s.t. $\pi \circ s = id_U$.



- **morphism between two fiber bundles** (2)

$\pi^i : \mathcal{P}^i \to M$, $i \in \{1, 2\}$ is any smooth map $h : \mathcal{P}^1 \to \mathcal{P}^2$ such that



$\pi^2 \circ h = \pi^1$

Ex 1: F = manifold : $pr_M : M \times F \to M$ called trivial bundle over M with fiber F.

Ex 2: $\pi : \mathcal{P} \to M$ f.b. $\Rightarrow$ can form its restriction to U open in M

$\mathcal{P}|_U := \pi^{-1}(U) \xrightarrow{\pi} U$

a f.b. over U.

(1)

(*)

P_x

P

M

(3)

Locally trivial fiber bundle with fiber F: any fiber bundle s.t.:

$(\forall) * \in M$, $\exists\, U \subseteq M$ open

$\exists$ isomorphism of fiber bundles.

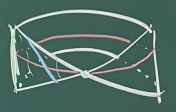
diffeo

$h : P|_U \longrightarrow U \times F$

π ↘ ↙ pr_U

U

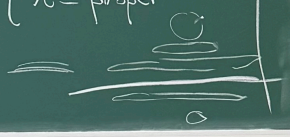
$P_* \xrightarrow{\sim} \{*\} \times F$

Ex: $P =$ 

$\downarrow$

$M =$ 

$\begin{cases} \pi : P \longrightarrow M \\ \pi = \text{proper} \end{cases}$



(2)

bundles

h map

P^2

π^2 $\pi^2 \circ h = \pi^1$

r F.

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n to U

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$\longrightarrow U$.

(4)

Variations: when F itself carries extra-"structure" & we are interested in bundles (*) where all the fibers carry that "structure".

Ex: • vector spaces ⇒ talk about bundles of vector spaces

• groups ⇒ bdles of groups.

• an action of a nice (Lie-) group G ⇒ principal G-bdles.

Vector bundles

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Vector bundles ①

Fiber bundle: surjective submersion

$\pi : \mathcal{P} \longrightarrow M$ (*)

$\mathcal{P}$: total space, M: the base

Key concepts:

- for $* \in M$: $\mathcal{P}_* := \pi^{-1}(*) = \{p \in \mathcal{P} : \pi(p) = *\}$ the fiber above $*$
- **section of (*)**:

any smooth $s : M \to \mathcal{P}$ s.t. $\pi \circ s = id_M$

Denote $\Gamma(\mathcal{P})$ the set of all sections

- **local sections of (*)**

any smooth $s : U \to \mathcal{P}$ s.t. $\pi \circ s = id_U$, $U \overset{\text{open}}{\subseteq} M$

Locally trivial fiber bundle with fiber F: ③

any fiber bundle, s.t.

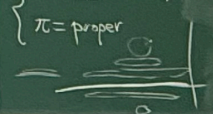
$\forall * \in M$, $\exists\, U \subseteq M$ open

$\exists$ isomorphism of fiber bundles

$h : \mathcal{P}|_U \xrightarrow{\text{diffeo}} U \times F$, $\pi = \pi_U \circ h$

$\mathcal{P}_* \xrightarrow{\sim} \{*\} \times F$

Ex: $\mathcal{P} =$ , $M =$ 

$\begin{cases} \pi : \mathcal{P} \to M \\ \pi = \text{proper} \end{cases}$ 

- **morphism between two fiber bundles**

$\pi^i : \mathcal{P}^i \to M$, $i \in \{1,2\}$ is any smooth map $h : \mathcal{P}^1 \to \mathcal{P}^2$ such that $\mathcal{P}^1 \xrightarrow{h} \mathcal{P}^2$, $\pi^2 \circ h = \pi^1$

Ex 1: F = manifold: $pr_M : M \times F \to M$ called trivial bundle over M with fiber F.

Ex 2: $\pi : \mathcal{P} \to M$ f.b. $\Rightarrow$ can form its restriction to U (U open in M)

$\mathcal{P}|_U := \pi^{-1}(U) \xrightarrow{\pi} U$ a f.b. over U.

Variations: ④

when F itself carries extra-"structure" & we are interested in bundles (*) where all the fibers carry that "structure".

Ex:

- vector spaces $\Rightarrow$ talk about bundles of vector spaces
- groups $\Rightarrow$ bdles of groups
- an nice action of a nice (Lie-) group G $\Rightarrow$ principal G-bdles.

Vector bundles

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Morphism of vecto

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Vector bundles

Vector bundle over M of rank r:
a locally trivial fiber bundle

$$\pi : E \longrightarrow M$$

with fiber $\mathbb{R}^r$ & with structures of vector spaces on each fiber E_x (for $x \in M$) s.t., in the local triviality condition:

$$E|_U \xrightarrow{h} U \times \mathbb{R}^r$$

$(x,u)+(x,v)=(x,u+v)$
$\lambda \cdot (x,u) = (x, \lambda u)$

$$E_x \longrightarrow \{x\} \times \mathbb{R}^r \text{ is linear}$$

} "a collection of vector spaces parametrised by $x \in M$, fitting together smoothly"

Morphism of vector bundles over M

$E^1 \xrightarrow{h} E^2$ s.t. $h_x = h|_{E^1_x} : E^1_x \to E^2_x$ is linear $(\forall)\, x \in M$

$\pi^1 \searrow \swarrow \pi^2$

M

} "collections $\{h_x\}_{x \in M}$ of linear maps, smooth in x"

5

5

Vector bundle over M of rank r:
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$\pi : E \longrightarrow M$
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$\searrow M \swarrow \pi^2$
$s.t. \; h_x = h|_{E^1_x} : E^1_x \to E^2_x$ is linear $(\forall) x \in M$

Special in this case:
$\Gamma(E) = \{ s : M \to E \text{ section} \}$
is a $C^\infty(M)$-module:
$\longrightarrow s_1, s_2 \in \Gamma(E)$ form $s_1 + s_2 \in \Gamma(E)$
$(s_1+s_2)(x) = s_1(x) + s_2(x)$ addition in E_x
$\longrightarrow 0 \in \Gamma(E)$ the zero section
$\longrightarrow f \in C^\infty(M), s \in \Gamma(E)$ form $f \cdot s \in \Gamma(E)$
$(f \cdot s)(x) = f(x)\, s(x)$ ($f(x) \in \mathbb{R}$, $s(x) \in E_x$)

$\Gamma(E)$, as a $C^\infty(M)$-module contains all the information about E as a vector bundle! — basic philosophy $\Rightarrow$ non-commutative geometry.

Frame: ① frame of a vector space V:
$e = \{e_1, \dots, e_r\}$ basis of V
One has:
$\{$frames e of $V\}$ $\xleftrightarrow{1-1}$ $\{$linear isomorphisms $\mathbb{R}^r \xrightarrow{\sim} V\}$
$e^{can} \longmapsto ?$

② frame of a vector bundle $E \xrightarrow{\pi} M$ the e_i s.t.
$e = \{e_1, \dots, e_r\}$
collections of sections of E s.t. $(\forall) x \in M$ $\{e_1(x), \dots, e_r(x)\}$ is a frame of E_x
One has:
$\{$frames of $E\}$ $\xleftrightarrow{1-1}$ $\{$isomorphism of vector bundles $E \simeq M \times \mathbb{R}^r\}$

③ local frame of a vector bundle $E \to M$

over some $U \subseteq M$ open:

$e = \{e_1, \dots, e_r\}$ collection of local sections $e_i : U \to E$

. . . .

Conclusion: E = locally trivial $\iff$

$\iff (\forall)\, x \in M \quad \exists\, U \subseteq M$ open & a local frame of E over U

Rk: Local frames play the role of charts of the v. bdle.

$\{e_1, \dots, e_r\}$ local frame over $U \subseteq M$

For $s \in \Gamma(E)$, over U, we can write:

$s(x) = f^1(x)\, e_1(x) + \dots + f^r(x)\, e_r(x) \quad x \in U$

$s = f^1 e_1 + \dots + f^r e_r$ over U. | s encoded in $f^1, \dots, f^r \in C^\infty(U)$

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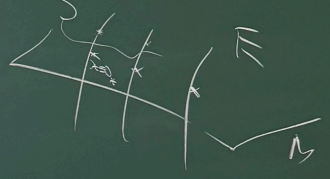
about E as a vecto

$\lambda \in S^1 \curvearrowright S^3 \subset \mathbb{C}^2 \quad (z_0, z_1) \in \mathbb{C}^2 \quad |z_0|^2 + |z_1|^2 = 1$

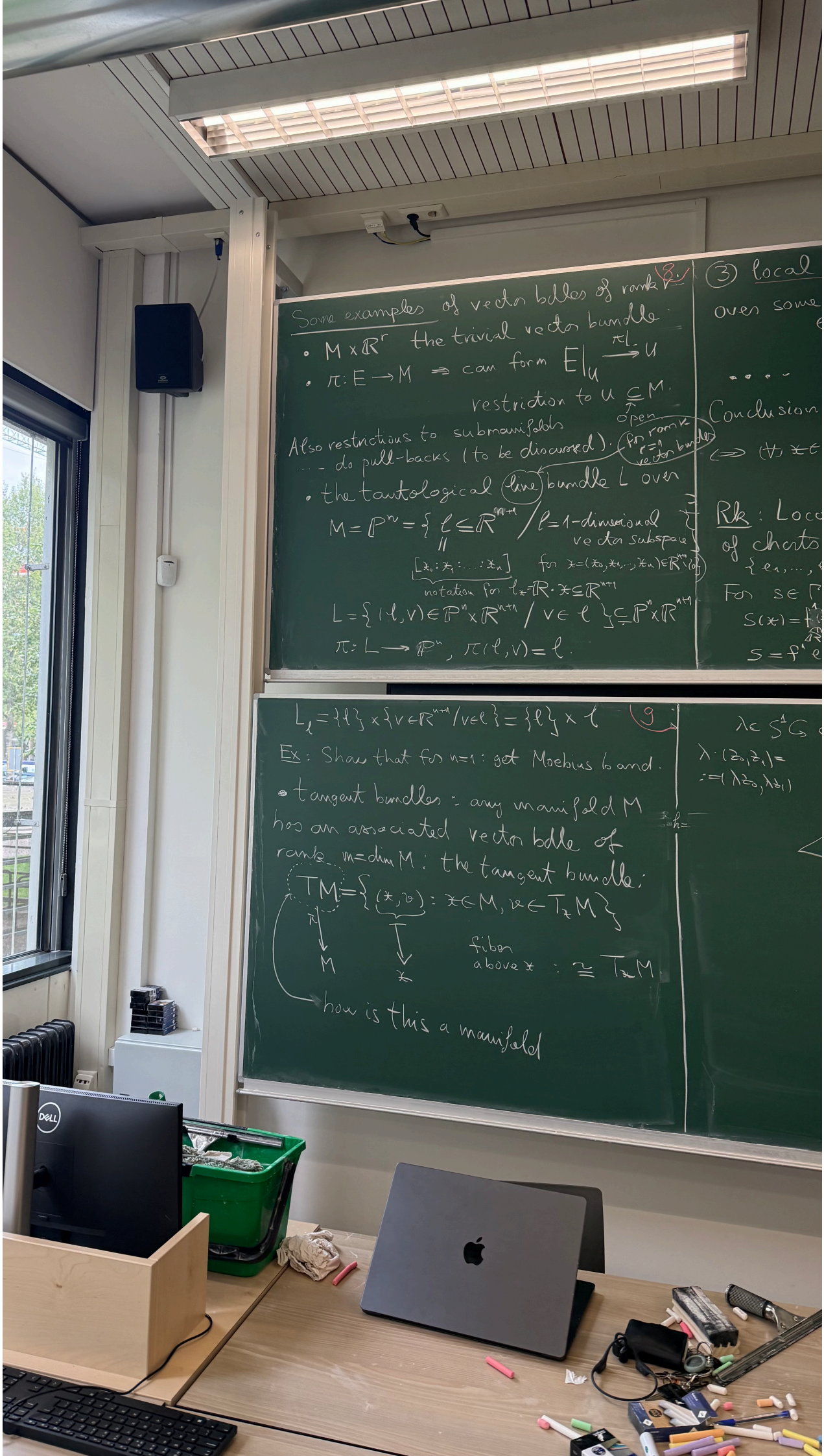
$\lambda \cdot (z_0, z_1) = (\lambda z_0, \lambda z_1)$

$S^3 \to S^2$, s

$\begin{cases} S^1 \times S^2 \xrightarrow{\sim} S^3 \\ (\lambda, x) \longmapsto \lambda s(x) \end{cases}$



Mike Rod


Some examples of vector bdles of rank
• $M \times \mathbb{R}^r$ the trivial vector bundle
• $\pi: E \to M \Rightarrow$ can form $E|_U \xrightarrow{\pi|} U$
restriction to $U \subseteq M$.
open
Also restrictions to submanifolds
... - do pull-backs (to be discussed).
(for rank $r=1$ vector bundles)
• the tautological (line) bundle L over
$M = \mathbb{P}^n = \{ \ell \subseteq \mathbb{R}^{n+1} \mid \ell = 1\text{-dimensional vector subspace} \}$
$[x_0 : x_1 : \dots : x_n]$ for $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$
notation for $\ell_x = \mathbb{R} \cdot x \subseteq \mathbb{R}^{n+1}$
$L = \{ (\ell, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1} \mid v \in \ell \} \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$
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Conclusion
$\Leftrightarrow (\forall\ x \in$
Rk: Loc
of charts
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For $s \in$
$s(x) = f$
$s = f' e$
$L_\ell = \{\ell\} \times \{v \in \mathbb{R}^{n+1} \mid v \in \ell\} = \{\ell\} \times \ell$
Ex: Show that for $n=1$: get Moebius band.
• tangent bundles: any manifold M has an associated vector bdle of rank $m = \dim M$: the tangent bundle:
$TM = \{ (x, v) : x \in M, v \in T_x M \}$
π
M
x
fiber above x : $\cong T_x M$.
how is this a manifold
$\lambda \in S^1 G$
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Some examples of vector bdles of rank r

- $M \times \mathbb{R}^r$ the trivial vector bundle
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Also restrictions to submanifolds ... - do pull-backs (to be discussed).

- the tautological (line) bundle L over $M = \mathbb{P}^n = \{ \ell \subseteq \mathbb{R}^{n+1} \mid \ell = 1\text{-dimensional vector subspace} \}$

$[x_1 : x_2 : \dots : x_n]$ notation for $\ell_x = \mathbb{R} \cdot x \subseteq \mathbb{R}^{n+1}$ for $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$

$L = \{ (\ell, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1} \mid v \in \ell \} \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$

$\pi: L \to \mathbb{P}^n$, $\pi(\ell, v) = \ell$.

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over some $U \subseteq M$ open:

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S^2

Mike Rod

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