Vector bundles (of rank r) ①

$\pi : E \longrightarrow M$ (Submersion)

$E_x := \pi^{-1}(x) \subseteq E$ for $x \in M$ all vector spaces

Local triviality: $(\forall)\, x \in M$, $\exists\, U \subseteq M$ open, $x \in U$

$\exists$ $E|_U \xrightarrow[\sim]{h} U \times \mathbb{R}^r$ fiberwise linear

π : $E|_U \to U$, pr_U : $U \times \mathbb{R}^r \to U$

$E_x \underset{h}{\simeq} \{x\} \times \mathbb{R}^r$

Morphisms of v bdles $E \to F$ over M

$E \xrightarrow{u} F$ • $\pi_F \circ u = \pi_E$ ($\Leftrightarrow$ $(\forall)\, x \in M$ u restricts to $E_x \xrightarrow{u_x} F_x$)

π_E : $E \to M$, π_F : $F \to M$

• u_x = linear $(\forall)\, x \in M$

[u = fiberwise linear]

Key words: ②

• sections $s : M \to E$, $s(x) \in E_x$ $(\forall)\, x \in M$

They form $\Gamma(E) = C^\infty(M)$-module.

"Inclusion"

$\{$ Vector bdles over $M \} \longrightarrow \{ C^\infty(M)$-modules $\}$

$E \longmapsto \Gamma(E)$

Illustration:

Prop: If E, F = v bdles over M

① any morphism $u : E \to F$ induces a $C^\infty(M)$-linear map $u_* : \Gamma E \to \Gamma F$, $s \mapsto u \circ s$

② any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ is of this type, for some! u.

• local sections: $s \in \Gamma(E|_U)$ with $U \subseteq M$ open ③ ($s : U \to E$, $s(x) \in E_x$ $(\forall)\, x \in U$)

Vector bundles (of rank r) ①

$\pi : E \to M$ (submersion)

$E_x := \pi^{-1}(x) \subseteq E$ for $x \in M$ — all vector spaces

Local triviality: $(\forall)\, x \in M$, $\exists\, U \subseteq M$ open, $x \in U$, $\exists$ $E|_U \xrightarrow{h}_{\sim} U \times \mathbb{R}^r$ fiberwise linear

π and π_U agree; $E_x \xrightarrow[h]{\sim} \{x\} \times \mathbb{R}^r$

Morphisms of v.bdles $E \to F$ over M

$E \xrightarrow{u} F$, $\pi_F \circ u = \pi_E$ ($\Leftrightarrow$ $(\forall)\, x \in M$, u restricts to $E_x \xrightarrow{u_x} F_x$)

π_E : E → M ← π_F : F

u_x = linear $(\forall)\, x \in M$

(u = fiberwise linear)

Key words ②

- sections: $s : M \to E$, $s(x) \in E_x$ $(\forall)\, x \in M$

They form a $\Gamma(E) = C^\infty(M)$-module.

"Inclusion"

$\left\{ \begin{matrix} \text{Vector} \\ \text{bdles over } M \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} C^\infty(M)\text{-} \\ \text{modules} \end{matrix} \right\}$

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- **local sections**: $s \in \Gamma(E|_U)$ with $U \subseteq M$ open ③ ($s : U \to E$, $s(x) \in E_x$ $(\forall)\, x \in U$, $U \subseteq M$)
- **frame of E**: $e = \{e_1, \dots, e_r\}$ sections $(\forall)\, x \in M$, $e_1(x), \dots, e_r(x)$ basis of E_x

  Choosing E $\Leftrightarrow$ choosing an isomorphism $E \xrightarrow[F]{\sim} M \times \mathbb{R}^r$ (as v.bdles)
- **local frames** (over some $U \subseteq M$ open): e_i only defined over U

  Hence: E = locally trivial $\Leftrightarrow$ $(\forall)\, x \in M$ $\exists$ local frame of E over some $U_x \subseteq M$ open, $x \in U_x$

Rk: Another inconvenience is: E is not a manifold right away $\rightsquigarrow$ ad-hoc arguments may be needed.

Local frames can be used to exhibit the smooth str. of E:

$\{e_1, \dots, e_r\}$ over U, U = domain of a chart of M

vector bdle of rank r

$\chi : U \xrightarrow{\sim} \Omega$ open $\subseteq \mathbb{R}^m$

$\Rightarrow E|_U \xrightarrow{e}_{\sim} U \times \mathbb{R}^r \xrightarrow[(\chi, \mathrm{id})]{\sim} \Omega \times \mathbb{R}^r$ ($\Omega \subseteq \mathbb{R}^m$ open)

Ex: The tautological line bdle over $\mathbb{P}^n$

$L = \{(\ell, v) : \ell \in \mathbb{P}^n, v \in \ell\} \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$ (submanifold)

$L \downarrow \mathbb{P}^n$, $(\ell, v) \mapsto \ell$

Local triviality? To prove: $(\forall)\, \ell \in \mathbb{P}^n$ $\exists$ local section s defined around ℓ (nowhere zero).

$L|_{U_0} \simeq \mathbb{R}^n \times \mathbb{R}$, $U_0 \subseteq \mathbb{P}^n$

$\ell = [x_0 : \dots : x_n]$

$s(\ell) = (\ell, 1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$

Reminder (bachelor): Given $M = m$-dim manif $\Rightarrow$ ④

$\Rightarrow$ for each $x \in M$: the tangent space T_xM

& ① any chart (U, x) around x induces a basis of T_xM:

$$\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_m}\right)_x \in T_xM$$

② if (U', x') is another chart then:

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \frac{\partial x'_j}{\partial x_i}(x) \cdot \left(\frac{\partial}{\partial x'_j}\right)_x$$

$U \xrightarrow{x} \Omega$ open $\subset \mathbb{R}^m$, M

But things together: $TM = \{(x, v) : x \in M, v \in T_xM\}$

... itself a manifold: each chart (U, x) of M induces one $(\tilde{U}, \tilde{x})$ of TM:

$$\tilde{U} = TM|_U := \{(x, v) \in TM : x \in U\}$$

$$\begin{cases} \tilde{x} : \tilde{U} \to \mathbb{R}^{2m} \\ \tilde{x}(x, v) = (x_1(x), \dots, x_m(x), \text{coeff}_1(v), \dots, \text{coeff}_m(v)) \end{cases}$$

$$v = \sum_{i=1}^{m} \text{coeff}_i(v) \left(\frac{\partial}{\partial x_i}\right)_x$$

⑤

Are those charts smoothly compatible?

$(U, x), (U', x')$ of M $\quad C^{x'}_{x} = x' \circ x^{-1}$: open $\subset \mathbb{R}^m \to$ open $\subset \mathbb{R}^m$

Same true for $(\tilde{U}, \tilde{x}), (\tilde{U}', \tilde{x}')$?

On because of ② $\Rightarrow$ diffeomorphism.

$\Rightarrow$ smooth $2m$-dim atlas on TM ... $\Rightarrow TM =: 2m$ manif.

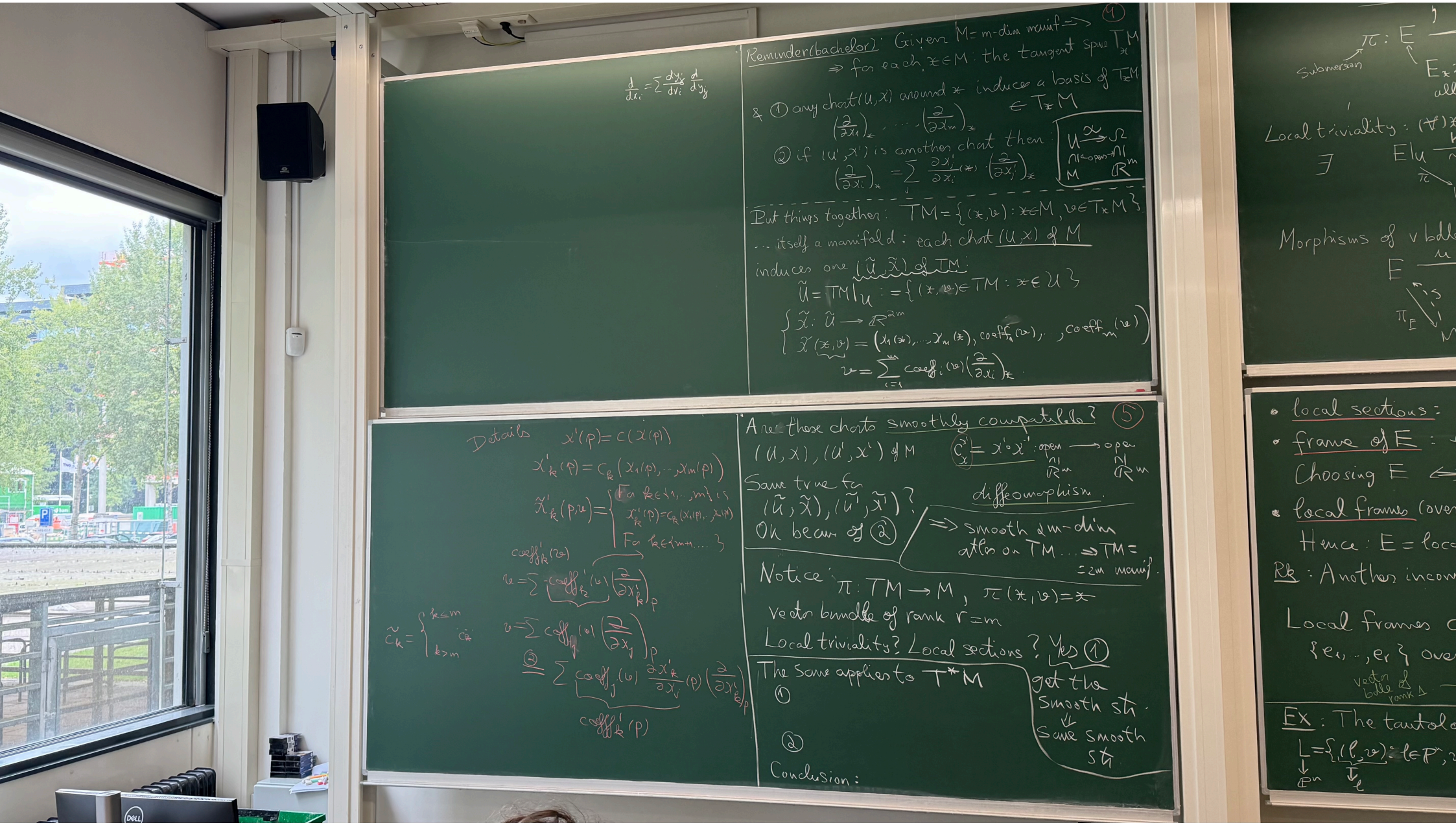
Notice: $\pi : TM \to M$, $\pi(x, v) = x$

vector bundle of rank $r = m$

Local triviality? Local sections? Yes ①

The same applies to T^*M ① ② get the smooth str. $\Rightarrow$ same smooth str.

Conclusion:


Reminder (bachelor): Given M = m-dim manif $\Rightarrow$ ④
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But things together: $TM = \{(x,v) : x \in M, v \in T_xM\}$
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$\tilde{U} = TM|_U := \{(x,v) \in TM : x \in U\}$
$\tilde{x}(x,v) = (x_1(x), \dots, x_m(x), \text{coeff}_1(v), \dots, \text{coeff}_m(v))$
$v = \sum_{i=1}^m \text{coeff}_i(v) \left(\frac{\partial}{\partial x_i}\right)_x$.
Details
$x'(p) = C(x(p))$
$x'_k(p) = C_k(x_1(p), \dots, x_m(p))$
$\tilde{x}'_k(p,v) = \begin{cases} \text{For } k \in \{1,\dots,m\}: \\ x'_k(p) = C_k(x_1(p), \dots, x_m(p)) \\ \text{For } k \in \{m+1,\dots\} \end{cases}$
$\text{coeff}'_k(v)$
$v = \sum \text{coeff}'_k(v) \left(\frac{\partial}{\partial x'_k}\right)_p$
$\tilde{C}_k = \begin{cases} C_k & k \le m \\ \cdot & k > m \end{cases}$
$v = \sum \text{coeff}_j(v) \left(\frac{\partial}{\partial x_j}\right)_p$
$= \sum \underbrace{\text{coeff}_j(v) \frac{\partial x'_k}{\partial x_j}(p)}_{\text{coeff}'_k(p)} \left(\frac{\partial}{\partial x'_k}\right)_p$
Are these charts smoothly compatible? ⑤
$(U,x), (U',x')$ of M $\quad C_x^{x'} = x' \circ x^{-1}$: open $\subset \mathbb{R}^m \to$ open $\subset \mathbb{R}^m$
Same true for $(\tilde{U},\tilde{x}), (\tilde{U}',\tilde{x}')$?
Ok because of ②
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$\Rightarrow$ smooth $2m$-dim atlas on TM ... $\Rightarrow TM$ = ... 2m manif?
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$\frac{\partial}{\partial x_i} = \sum \frac{dy_j}{dx_i} \frac{\partial}{\partial y_j}$
$\pi : E$
Submersion
E_x
Local triviality: $(\forall)$
$\exists$
$E|_U$
π_E
Morphisms of v bdle
local sections:
frame of E:
Choosing E
local frames (over
Hence: E = loc
Rk: Another incom
Local frames
$\{e_1, \dots, e_r\}$ over
vector bdle of rank Δ
Ex: The tautol
$L = \{(\ell, v) : \ell \in \mathbb{P}^1$,
$\mathbb{R}^n$
ℓ

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$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \frac{\partial x'_j}{\partial x_i}(x) \left(\frac{\partial}{\partial x'_j}\right)_x$$

$U \xrightarrow{x} \mathbb{R}^n$, $U \cap U' \neq \emptyset$, M

But things together $TM = \{(x, v) : x \in M, v \in T_xM\}$

... itself a manifold: each chart (U, x) of M induces one (which?) of TM

$\tilde{U} = TM|_U = \{(x, v) \in TM : x \in U\}$

$\tilde{x} : \tilde{U} \to \mathbb{R}^{2n}$

$\tilde{x}(x, v) = (x_1(x), \dots, x_n(x), \text{coeff}_1(v), \dots, \text{coeff}_n(v))$

$v = \sum_{i=1}^{n} \text{coeff}_i(v) \left(\frac{\partial}{\partial x_i}\right)_x$

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$(U, x), (U', x')$ of M $\quad C^\infty$ $x' \circ x^{-1}$: open $\subseteq \mathbb{R}^n \to$ open $\subseteq \mathbb{R}^n$

Same true for $(\tilde{U}, \tilde{x}), (\tilde{U}', \tilde{x}')$? Ok because of ② ⇒ smooth $2n$-dim atlas on TM. ⇒ TM = $2n$ manif.

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Notice: $\pi : TM \to M$, $\pi(x, v) = x$

vector bundle of rank $r = n$

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Conclusion:

Vector bundles (of rank r)

$\pi : E \to M$ (submersion)

$E_x := \pi^{-1}(x) \subseteq E$ for $x \in M$ all vector spaces

Local triviality: $(\forall)\, x \in M$, $\exists\, U \subseteq M$ open, $x \in U$

$E|_U \xrightarrow{h} U \times \mathbb{R}^r$ fiberwise linear

$\pi \downarrow$ $\quad \pi_U \downarrow$

U

$E_x \xrightarrow{h} \{x\} \times \mathbb{R}^r$

Morphisms of v. bdles $E \subset F$ over M

$E \xrightarrow{u} F$ • $\pi_F \circ u = \pi_E$ ($\Leftrightarrow (\forall) x \in M$ u restricts to $E_x \xrightarrow{u_x} F_x$)

π_E ↘ M ↙ π_F • u_x = linear $\forall\, x \in M$

$|u$ = fiberwise $h \circ u \circ h^{-1}|$

Key words

• sections: $s : M \to E$, $s(x) \in E_x$ $\forall x \in M$

They form $\Gamma(E) = C^\infty(M)$-module

"Inclusion"

{Vector bdles over M} ⟶ {$C^\infty(M)$-modules}

$E \longmapsto \Gamma(E)$

Illustration

Prop: $\forall\, E, F$ = v. bdles over M

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Choosing E ⟺ choosing an isomorphism $E \xrightarrow{\sim} M \times \mathbb{R}^r$ (as v. bdles)

• local frame (over some $U \subseteq M$ open): e_i only defined over U

Hence E = locally trivial ⇔ $(\forall) x \in M$ $\exists$ local frame of E over some $U \ni x$, $U \subseteq M$ open

Rk: Another inconvenience is: E is not a manifold right away & ad-hoc arguments may be needed.

Local frames can be used to exhibit the smooth str. of E

$\{e_1, \dots, e_r\}$ over U, U = domain of a chart of M ⇒ $E|_U \xrightarrow{\sim} U \times \mathbb{R}^r \xrightarrow{\sim} \Omega \times \mathbb{R}^r$ ($\Omega \subseteq \mathbb{R}^n$ open)

$\chi : U \to \Omega$ open $\subseteq \mathbb{R}^n$ $\quad (x, \mathrm{id})$

Ex: The tautological line bdle over $\mathbb{P}^n$

$L = \{(\ell, v) : \ell \in \mathbb{P}^n, v \in \ell\} \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$ submanifold

$L \xrightarrow{\pi} \mathbb{P}^n$, $\pi^{-1}(\ell) = \ell$

Local triviality? To prove: $(\forall) \ell \in \mathbb{P}^n$ $\exists$ local section defined on $U \ni \ell$, $U \subseteq \mathbb{P}^n$ around ℓ (= another word)

$L|_{U_0} \simeq \mathbb{R}^n \times \mathbb{R}$ $\quad \ell = [x_0 : \dots : x_n]$

$s(\ell) = (\ell, 1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$

(6)

Rk: More general procedure.

~~$\frac{d}{dx_i} = \frac{dy_j}{dx_i}\frac{d}{dy_j}$~~ $\quad dv_i = \frac{dy_j}{dx_i}\,dy_j$

Start with:
- M = manif, $r \in \mathbb{N}$
- collection $\{E_x\}_{x\in M}$ collection of r-dim v. spaces.

... organise: $E := \{(x,v) : x \in M, v \in E_x\}$ (set)

- open cover $\mathcal{U}$ consisting of opens $U \subset M$
- for each $U \in \mathcal{U}$ a set-theoretical local frame of E, i.e.: for each $x \in U$ a basis $e_1^U(x), \dots, e_r^U(x)$.

Condition: for $U, U' \in \mathcal{U}$, writing

$$e_i^U(x) = \sum_{j=1}^{r} \tau_{UU'}^{ij}(x)\, e_j^{U'}(x) \quad \Big| \quad \tau_{UU'}^{ij} : U \cap U' \to \mathbb{R}$$

the maps $\tau_{UU'}^{ij}$ are smooth.

Conclusion $\Rightarrow$ E a vector bundle with $e_1^U, \dots, e_r^U$ smooth local sections.

(9)

Reminder (bachelor): Given M = m-dim manif $\Rightarrow$ for each $x \in M$ the tangent space T_xM

& ① any chart (U, x) around x induces a basis of T_xM

$$\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_m}\right)_x \in T_xM$$

② if (U', x') is another chart then

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \frac{\partial x_j'}{\partial x_i}(x) \cdot \left(\frac{\partial}{\partial x_j'}\right)_x$$

$U \xrightarrow{x} \Omega$ open $\subset \mathbb{R}^m$, $U \cap U' \to \Omega \cap \Omega'$, M

Put things together: $TM = \{(x,v) : x \in M, v \in T_xM\}$

... itself a manifold: each chart (U,x) of M induces one $(\tilde{U}, \tilde{x})$ of TM:

$$\tilde{U} = TM|_U := \{(x,v) \in TM : x \in U\}$$

$$\begin{cases} \tilde{x} : \tilde{U} \to \mathbb{R}^{2m} \\ \tilde{x}(x,v) = (x_1(x), \dots, x_m(x), \text{coeff}_1(v), \dots, \text{coeff}_m(v)) \end{cases}$$

$\Gamma(TM) = \mathfrak{X}(M)$ vector fields $\quad v = \sum_{i=1}^{m} \text{coeff}_i(v) \left(\frac{\partial}{\partial x_i}\right)_x$.

(7)

Local view-point on vector bundles: $GL_r(\mathbb{R})$ = invertible

Def: A transition data on M (8)

(6)

Rk: More general procedure: $\frac{d}{dx_i} \neq \frac{d}{dy_j}$ in general

Start with:
- M = manif, $r \in \mathbb{N}$
- collection $\{E_x\}_{x \in M}$ collection of r-dim v. spaces
  - organise: $E := \{(x,v) : x \in M, v \in E_x\}$ (set)
- open cover $\mathcal{U}$ consisting of opens $U \subset M$
- for each $U \in \mathcal{U}$ a set-theoretical local frame of E, i.e.: for each $x \in U$ a basis $e_1^U(x), \dots, e_r^U(x)$.

Condition: for $U, U' \in \mathcal{U}$, writing
$$e_i^U(x) = \sum_{j=1}^{r} \tau_i^j(x) \, e_j^{U'}(x) \quad \Big| \quad \tau_i^j : U \cap U' \to \mathbb{R}$$
the maps τ_i^j are smooth.

Conclusion: $\Rightarrow$ E a vector bundle with $e_1^U, \dots, e_r^U$ smooth local sections.

Reminder(

& ① any

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But things

... itself a

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$\tilde{U} =$

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$\tilde{x}(x$

$T(TM) = \mathfrak{X}(M)$ vector fields

(7)

Local view-point on vector bundles: | $GL_r(\mathbb{R})$ = invertible $r \times r$ matrices

Start with v. bdle $\pi : E \to M$

One can choose | ① $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ an open cover of M
② on each U^α a local frame $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$ (over U^α)

Write out
$$e^\beta_{i(x)} = \sum_j (\tau^{\alpha\beta})^j_i(x) \, e^\alpha_{j(x)}, \quad (\tau^{\alpha\beta})^j_i : U_\alpha \cap U_\beta \to \mathbb{R}$$
at pts in $U_\alpha \cap U_\beta$

Write $\tau^{\alpha\beta}$ for the matrix $(\tau^{\alpha\beta})^j_i$ as entries

transition functions → $\tau^{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{M}_{r\times r}(\mathbb{R})$ ($GL_r(\mathbb{R})$)

$U_\alpha \times \mathbb{R}^r \to E \leftarrow U_\beta \times \mathbb{R}^r$

Main property: $\tau^{\alpha\beta} \cdot \tau^{\beta\delta} = \tau^{\alpha\delta}$ on $U_\alpha \cap U_\beta \cap U_\delta$ (*)

Rk: $\tau^{\alpha\alpha} = Id \Rightarrow \tau^{\alpha\beta} \cdot \tau^{\beta\alpha} = Id \Rightarrow \tau^{\beta\alpha} =$ inverse of $\tau^{\alpha\beta}$ — the cocycle property

$T^*M = \Omega^1(M)$ 1-forms

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Conclusion $\Rightarrow$ E a vector bundle with $e_1^{\mathcal{U}}, \dots, e_r^{\mathcal{U}}$ smooth local sections.

$\left.\begin{array}{l} \chi : \tilde{\mathcal{U}} \to \mathbb{R}^{2m} \\ \chi(x,v) = (x_1(x), \dots, x_m(x), \mathrm{coeff}_1(v), \dots, \mathrm{coeff}_m(v)) \end{array}\right\}$ $\sim$

$T(TM) = \mathfrak{X}(M)$ vector fields $\quad v = \sum_{i=1}^{m} \mathrm{coeff}_i(v) \left(\frac{\partial}{\partial x_i}\right)_x$.

(7)

Local view-point on vector bundles:

$GL_r(\mathbb{R})$ = invertible $r \times r$ matrices.

Start with v.bdle $\pi : E \to M$

One can choose ① $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ an open cover of M.

② on each U^α a local frame $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$ (over U^α)

Write out $e^\beta_{i(x)} = \sum_j (\tau^{\alpha\beta})^j_i(x)\, e^\alpha_{j(x)}$, $(\tau^{\alpha\beta})^j_i : U_\alpha \cap U_\beta \to \mathbb{R}$

at pts in $U_\alpha \cap U_\beta$

Write $\tau^{\alpha\beta}$ for the matrix $(\tau^{\alpha\beta})^j_i$ as entries

transition functions $\to$ $\boxed{\tau^{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{M}_{r\times r}(\mathbb{R}) \quad (GL_r(\mathbb{R}))}$ $\quad U_\alpha \times \mathbb{R}^r \hookrightarrow E \hookleftarrow U_\beta \times \mathbb{R}^r$

Main property: $\boxed{\tau^{\alpha\beta} \cdot \tau^{\beta\delta} = \tau^{\alpha\delta} \text{ on } U_\alpha \cap U_\beta \cap U_\delta}$ (*) the cocycle property

Rk: $\tau^{\alpha\alpha} = Id \Rightarrow \tau^{\alpha\beta} \cdot \tau^{\beta\alpha} = Id \Rightarrow \tau^{\beta\alpha} = (\tau^{\alpha\beta})^{-1}$ inverse of $\tau^{\alpha\beta}$

$\Gamma(T^*M) = \Omega^1(M)$ 1-forms

(8)

Def: A transition data on M with values on GL_r:

→ Open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$.

→ Smooth maps $\tau^{\alpha\beta} : U_\alpha \cap U_\beta \to GL_r$

Satisfying (*) $(\forall)\, \alpha, \beta, \delta \in \Lambda$.

(cocycle. $\Rightarrow$ $\tau^{\alpha,\alpha} = Id$, $\tau^{\beta\alpha} = (\tau^{\alpha\beta})^{-1}$ $(\forall)\, \alpha, \beta \in \Lambda$)

Ex: Any open cover $\mathcal{U} = \{U^\alpha\}$ with U^α domain of x^α

$\Rightarrow$ transition fcts: $(\tau^{\alpha\beta})^i_j = \frac{\partial x^\alpha_j}{\partial x^\beta_i}$, $\boxed{\tau^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^\beta}}$ tr. for TM.

Ex: For $T^*M \Rightarrow$ get $\frac{\partial x^\beta}{\partial x^\alpha}$.

Main points:

① to any v.bdle E one can associate tr. data. (choice dependent)

② from the transition data one can recover E: $\coprod U_\alpha \times \mathbb{R}^r / (x,v) \sim (x, \tau^{\alpha\beta}(x) \cdot v)$ for $x \in U_\alpha \cap U_\beta$, $v \in \mathbb{R}^r$

Rk: More general procedure.

Start with:
- M = manif., $r \in \mathbb{N}$
- collection $\{E_x\}_{x \in M}$ collection of r-dim $\mathbb{R}$-v. spaces
- open cover $\mathcal{U}$ consisting of opens $U \subset M$
- for each $U \in \mathcal{U}$ a set-theoretical local frame of E, i.e. for each $x \in U$ a basis $e_1^U(x), \dots, e_r^U(x)$

Condition. for $U, U' \in \mathcal{U}$, writing $e_i^U(x) = \sum_{j=1}^r \tau_i^j(x)\, e_j^{U'}(x)$ | $\tau_i^j : U \cap U' \to \mathbb{R}$

the maps τ_i^j are smooth.

Conclusion. $\Rightarrow$ E a vector bundle with $e_1^U, \dots, e_r^U$ smooth local sections.

Reminder/bachelor: Given M n-dim manif. $\Rightarrow$ (1) for each $x \in M$, the tangent space $T_x M$ & (2) any chart (U, x) around x induces a basis $\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x \in T_x M$

(2) if (U', x') is another chart then $\left(\frac{\partial}{\partial x'_i}\right)_x = \sum_j \frac{\partial x_j}{\partial x'_i}(x) \left(\frac{\partial}{\partial x_j}\right)_x$

$U \xrightarrow{\alpha} \mathbb{R}$ α smooth $\Leftrightarrow$ $\alpha \circ x^{-1} : x(U) \to \mathbb{R}$ smooth, $x(U) \subset \mathbb{R}^n$

Put things together: $TM = \{(x, v) : x \in M, v \in T_x M\}$ ... itself a manifold: each chart (U, x) of M induces one $(\tilde{U}, \tilde{x})$ of TM

$\tilde{U} = TM|_U = \{(x, v) \in TM : x \in U\}$

$\tilde{x} : \tilde{U} \to \mathbb{R}^{2n}$, $\tilde{x}(x, v) = (x_1(x), \dots, x_n(x), \mathrm{coeff}_1(v), \dots, \mathrm{coeff}_n(v))$

$\Gamma(TM) = \mathfrak{X}(M)$ vector fields $\quad X = \sum_{i=1}^n \mathrm{coeff}_i(x) \left(\frac{\partial}{\partial x_i}\right)_x$

Vector bundles (of rank r)

$\pi : E \to M$ submersion

$E_x := \pi^{-1}(\{x\}) \subset E$ for $x \in M$ call vector spaces

Local triviality: $(\forall)\, x \in M$, $\exists\, U \subset M$ open, $x \in U$, $\exists$ $E|_U \xrightarrow{h}_{\sim} U \times \mathbb{R}^r$ fiberwise linear, $E_x \xrightarrow[h]{\sim} \{x\} \times \mathbb{R}^r$

$\pi_U \circ h = \pi_E$, $\pi_U : U \times \mathbb{R}^r \to U$

Morphisms of v.bdles $E \subset F$ over M: $E \xrightarrow{u} F$, $\pi_F \circ u = \pi_E$ ($\Leftrightarrow$ $u(E_x) \subset F_x$ $\forall x \in M$), $u|_{E_x} : E_x \to F_x$ restriction of u

u_x = linear, $\forall x \in M$; $u|_E$ = fiberwise $\mathbb{R}$-linear

Key words:
- sections $s : M \to E$, $s(x) \in E_x$ $\forall x \in M$

They form $\Gamma(E) = C^\infty(M)$-module

"Inclusion" $\{\text{Vect. bdles over } M\} \to \{C^\infty(M)\text{-modules}\}$, $E \mapsto \Gamma(E)$

Illustration

Prop: $\forall$ E, F v.bdles over M

(1) any morphism $u : E \to F$ induces a $C^\infty(M)$-linear map $\Gamma E \to \Gamma F$

(2) any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ is of this type, for some! u

- **local sections**: $s \in \Gamma(E|_U)$ with $U \subset M$ open ($s : U \to E$, $s(x) \in E_x$ $\forall x \in U$, $U \xrightarrow{s} E \xrightarrow{\pi} M$, $\pi \circ s = \mathrm{id}_U$)
- **frame of E**: $e = \{e_1, \dots, e_r\}$ sections $(\forall)\, x \in M$, $e_1(x), \dots, e_r(x)$ basis of E_x

Choosing $E \Leftrightarrow$ choosing an isomorphism $E \xrightarrow{\sim} M \times \mathbb{R}^r$ (as v.bdles)

- **local frames** (over some $U \subset M$ open): e_i only defined over U

Hence: E = locally trivial $\Leftrightarrow$ $(\forall)\, x \in M$ $\exists$ local frame of E over some $U_x \subset M$ open, $x \in U_x$

Rk: Another inconvenience is: E is not a manifold right away & ad-hoc arguments may be needed.

Local frames can be used to exhibit the smooth str. of E

$\{e_1, \dots, e_r\}$ over U, U = domain of a chart of M | $\Rightarrow$ $E|_U \xrightarrow{\sim} U \times \mathbb{R}^r \xrightarrow{(x, \mathrm{id})} \Omega \times \mathbb{R}^r$, $\Omega \subset \mathbb{R}^n$ open

$x : U \to \Omega$ open $\subset \mathbb{R}^n$

vector fields of frame e

Ex: The tautological (line bdle) on $\mathbb{P}^n$

$L = \{(\ell, v) : \ell \in \mathbb{P}^n, v \in \ell\} \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$ submanifold

$L \downarrow \pi_1$ $\quad \pi_2 \downarrow \mathbb{R}^{n+1}$

Local triviality? To prove: $(\forall)\, \ell \in \mathbb{P}^n$ $\exists$ local section s around ℓ (nonzero over ℓ). $\ell = [x_0 : \dots : x_n]$, $s(\ell) = (\ell, \frac{x_0}{x_i}, \dots, \frac{x_n}{x_i})$ different on $U_i \subset \mathbb{P}^n$, $U_i \cong \mathbb{R}^n$, $L|_{U_i} \cong U_i \times \mathbb{R}^1$

Local view-point on vector bundles: ($\mathrm{GL}_r(\mathbb{R})$ = invertible $r \times r$ matrices)

Start with v.bdle $\pi : E \to M$

One can choose ① $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ an open cover of M | ② on each U^α a local frame $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$ (over U^α)

Write out $e^\beta_{i(x)} = \sum_j (\tau^{\alpha\beta})^j_i(x)\, e^\alpha_j(x)$, $(\tau^{\alpha\beta})^j_i : U_\alpha \cap U_\beta \to \mathbb{R}$ at pts in $U_\alpha \cap U_\beta$

Write $\tau^{\alpha\beta}$ for the matrix $((\tau^{\alpha\beta})^j_i)$ as entries

$\tau^{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{M}_r(\mathbb{R})$, $\mathrm{GL}_r(\mathbb{R}) \subset \mathrm{M}_r(\mathbb{R})$

$U_\alpha \times \mathbb{R}^r \supset E$, $U_\beta \times \mathbb{R}^r \supset E$

Main property: $\tau^{\alpha\beta}\, \tau^{\beta\alpha} = \tau^{\alpha\alpha}$ on $U_\alpha \cap U_\beta \cap U_\delta$ (*)

Rk: $\tau^{\alpha\alpha} = \mathrm{Id} \Rightarrow \tau^{\alpha\beta}\, \tau^{\beta\alpha} = \mathrm{Id} \Rightarrow \tau^{\beta\alpha} = (\tau^{\alpha\beta})^{-1}$ = inverse of the cocycle property

$\Gamma(T^*M) = \Omega^1(M)$ 1-forms

Def: A transition data on M with values on GL_r:
- Open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$.
- Smooth maps $\tau^{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r$

Satisfying (*) $(\forall)\, \alpha, \beta, \gamma \in \Lambda$.

(cocycl. $\Leftrightarrow$ $\tau^{\alpha\alpha} = \mathrm{Id}$, $\tau^{\beta\alpha} = (\tau^{\alpha\beta})^{-1}$ $(\forall)\, \alpha, \beta \in \Lambda$)

Ex: Any open cover $\mathcal{U} = \{U^\alpha\}$ with U^α domain of x^α

$\Rightarrow$ transition fcts: $(\tau^{\alpha\beta})^i_j = \frac{\partial x^\alpha_i}{\partial x^\beta_j}$, $\tau^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^\beta}$ tr. for TM.

Ex: For $T^*M \to$ get $\frac{\partial x^\beta}{\partial x^\alpha}$.

Main points:
① to any v.bdle E one can associate tr. data (choice dependent)
② from the transition data one can recover E: $\coprod_\alpha U_\alpha \times \mathbb{R}^r / (x, v) \sim (x, \tau^{\beta\alpha}(x) v)$ for $x \in U_\alpha \cap U_\beta$