Reminder: General procedure to produce vector bundles (over M) ①

Start with:

- collection $\{E_x\}_{x\in M}$ of r-dimensional vector spaces.

$$E := \{(x,v) : x\in M,\ v\in E_x\} \qquad \text{(set)}$$

- open cover $\mathcal{U}$ (consisting of opens $U\subseteq M$)
- for each $U\in\mathcal{U}$, a set-theoretical local frame of E over U, $\{e_1^U,\dots,e_r^U\}$

Such that, for $U,U'\in\mathcal{U}$, writing for $x\in U\cap U'$

$$e_i^U(x) = \sum_j \mathrm{coeff}_i^j(x)\, e_j^{U'}(x)$$

then the resulting

$$\mathrm{coeff}_i^j : U'\cap U' \longrightarrow \mathbb{R}$$

are smooth.

Then $\Rightarrow$ E made into a vector bundle such that all e_i^U are smooth local sections.

$E \xrightarrow{\pi_E} M$ $\quad E_x = \pi_E^{-1}(x)$

(2)

Reminder: local viewpoint on vector bundles $E \to M$:

CHOOSE
- $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ open cover of M
- e^α - frame of $E|_{U^\alpha}$, $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$

For $\alpha, \beta \in \Lambda$ compare e^α and e^β over $U^\alpha \cap U^\beta$:

$$e^\beta_i = \sum_{j=1}^{r} (\tau^{\alpha\beta})^j_i \, e^\alpha_j \qquad \text{(short: } e^\beta = e^\alpha \cdot \tau^{\alpha\beta}\text{)}$$

$\Rightarrow$ get smooth functions

$$\tau^{\alpha\beta} : U^\alpha \cap U^\beta \longrightarrow GL_r(\mathbb{R})$$

satisfying, over triple intersections $U^\alpha \cap U^\beta \cap U^\gamma$:

$$\tau^{\alpha\beta} \cdot \tau^{\beta\gamma} = \tau^{\alpha\gamma} \qquad (\forall)\, \alpha, \beta, \gamma \in \Lambda$$

} transition data

Main point 1: ~~"The"~~ "transition data contains all the info about E (but depends on choices)

E can be reconstructed!

(3)

Main point 2: Allows one to work locally and express objects on E in terms of simpler (more classical) objects.

E.g.: sections $s \in \Gamma(E)$:

⊛ locally, over U^α:

$$s|_{U_\alpha} = \sum_i f^i_\alpha \cdot e^\alpha_i \qquad \underbrace{\text{with } f^i_\alpha : U^\alpha \to \mathbb{R},\ 1 \le i \le r}_{f_\alpha = (f^1_\alpha, \dots, f^r_\alpha) : U^\alpha \to \mathbb{R}^r}$$

i.e. $s|_{U_\alpha}$ is encoded by $f_\alpha : U^\alpha \to \mathbb{R}^r$.

⊛ on overlaps $U^\alpha \cap U^\beta$:

$$(*) \qquad f^i_\alpha = \sum_{j=1}^{r} (\tau^{\alpha\beta})^i_j \, f^j_\beta \qquad (\forall)\, i \in \{1, \dots, r\}$$

$$\left\{\begin{array}{c}\text{sections} \\ s \in \Gamma E\end{array}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{array}{c}\text{collections } f^\alpha : U^\alpha \to \mathbb{R}^r \text{ for } \alpha \in \Lambda \\ \text{s.t. } (\forall)\, \alpha, \beta \in \Lambda : (*) \text{ holds.}\end{array}\right\}$$

Operations ↓	Vector bundle over M $E \xrightarrow{\pi} M$	Fiber above $x \in M$ $F_x = \pi^{-1}(x)$	Sections $s: M \to E$

Reminder local viewpoint on vector bundles $E \to M$

CHANGE
- $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ open cover of M
- e^α - frame of $E|_{U^\alpha}$, $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$

For $\alpha, \beta \in \Lambda$ compare e^α and e^β over $U^\alpha \cap U^\beta$:

$$e^\beta_j = \sum_{i=1}^r (c^{\alpha\beta})^i_j \, e^\alpha_i \quad \text{(short } e^\beta = e^\alpha \cdot c^{\alpha\beta}\text{)}$$

$\Rightarrow$ get smooth functions $c^{\alpha\beta}: U^\alpha \cap U^\beta \to GL_r(\mathbb{R})$

satisfying, over triple intersections $U^\alpha \cap U^\beta \cap U^\gamma$:

$$c^{\alpha\beta} \, c^{\beta\gamma} = c^{\alpha\gamma} \qquad (\forall \alpha, \beta, \gamma \in \Lambda)$$

} transition data

Main point 1: ~~$\mathcal{U}$~~ $(U^\alpha, c^{\alpha\beta})$ transition data contains all the info about E (but depends on choices)

can be reconstructed!

Main point 2: Allows one to work locally and explain objects on E in terms of simpler (more classical) objects

Eg.: sections $s \in \Gamma(E)$

(*) locally over U^α

$$s|_{U^\alpha} = \sum_{i=1}^r f^\alpha_i \, e^\alpha_i, \quad \text{with } f_i: U^\alpha \to \mathbb{R}, \text{ vector } f_x = (f^\alpha_1, \dots, f^\alpha_r): U^\alpha \to \mathbb{R}^r$$

$\Leftrightarrow s|_{U^\alpha}$ is encoded by $f_x: U^\alpha \to \mathbb{R}^r$

(*) on overlaps $U^\alpha \cap U^\beta$

$$(*) \quad f^\alpha_x = \sum_{j=1}^r (c^{\alpha\beta})^i_j \, f^\beta_j \qquad (\forall \alpha, \beta)$$

$$\{\text{sections } s \in \Gamma(E)\} \xrightarrow{1:1} \left\{\begin{array}{l}\text{collection } f^\alpha: U^\alpha \to \mathbb{R}^r \text{ for } \alpha \in \Lambda \\ \text{s.t. } (\forall \alpha, \beta \in \Lambda: (*) \text{ holds.}\end{array}\right\}$$

$\Gamma(E)$ the $C^\infty(M)$-module	(local) frames of E & transition functions

Reminder General procedure to produce vector bundles (over M)

Start with:
- collection $\{E_x\}_{x \in M}$ of r-dimensional vector spaces
- $E = \{(x, v): x \in M, v \in E_x\}$ (set)
- open cover $\mathcal{U}$ (consisting of opens $U \subseteq M$)
- for each $U \in \mathcal{U}$, a set-theoretical local frame of E over U, $\{e^U_1, \dots, e^U_r\}$

Such that, for $U, U' \in \mathcal{U}$, writing for $x \in U \cap U'$

$$e^{U'}_j(x) = \sum_i \text{coeff}^{UU'}_{ij}(x) \, e^U_i(x)$$

(coeff$^{UU'}_{ij}$: $U \cap U' \to \mathbb{R}$)

then the resulting coeff$^{UU'}_{ij}$ are smooth.

Then $\to E$ made into a vector bundle such that all e^U_i are smooth local sections

Operations ↓	Vector bundle over M: $E \xrightarrow{\pi_E} M$	Fiber above $x \in M$: $E_x = \pi_E^{-1}(x)$	Sections $s : M \to E$
$E \oplus F$	$= \{(e,f) \in E \times F \mid \pi_E(e) = \pi_F(f)\} \subset E \times F$ (smooth submanifold); $\pi_{E\oplus F}(e,f) = \pi_E(e) = \pi_F(f)$	$E_x \oplus F_x$	are of type $s = (s_1, s_2)$, $s_1 \in \Gamma E$, $s_2 \in \Gamma F$
$E \otimes F$	$\{(x, \xi) : x \in M, \xi \in E_x \otimes F_x\} \longrightarrow M$, $(x,\xi) \longmapsto x$	$E_x \otimes F_x$	any $s_1 \in \Gamma E$, $s_2 \in \Gamma F$ induces $s_1 \otimes s_2$ of $E \otimes F$ by: $(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x)$ (not all !)
$\underline{\mathrm{Hom}}(E,F)$ (vector bundle Hom)	$= \{(x, A) : x \in M,\ A : E_x \to F_x$ linear map$\} \to M$, $(x,A) \longmapsto x$	$\mathrm{Hom}(E_x, F_x)$ (linear maps)	
E^*	$= \mathrm{Hom}(E, \mathbb{R}_M)$ $\left(\mathbb{R}_M = \begin{array}{c} M \times \mathbb{R} \\ \downarrow \mathrm{pr}_M \\ M \end{array}\right.$ the trivial line bundle$/M$	$(E_x)^* = \mathrm{Hom}(E_x, \mathbb{R})$	

ΓE the $C^\infty(M)$-module	(local) frames of E & transition functions
$\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$	$e = \{e_1, \dots, e_r\}$ frm E, $f = \{f_1, \dots, f_s\}$ frm F $\Big\} \to$ $(e_1,0), \dots, (e_r,0), (0,f_1), \dots, (0,f_s)$ frame of $E \oplus F$
$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$	$\{e_i \otimes f_j : 1 \le i \le r,\ 1 \le j \le s\}$ rs members
$\Gamma(\underline{\mathrm{Hom}}(E,F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma E, \Gamma F)$ more explanations soon!	
	If $e = \{e_1, \dots, e_r\}$ local frame of E $\rightsquigarrow$ the dual local frame $e^* = \{e_1^*, \dots, e_r^*\}$; at any x: $\{e_1^*(x), \dots, e_r^*(x)\}$ ← basis of E_x^* is the dual basis to $\{e_1(x), \dots, e_r(x)\}$. Tr. fds for E^* = transpose of those of E

Reminder: General procedure to produce vector bundles (over M)

Operations ↓	Vector bundle over M: $E \xrightarrow{\pi_E} M$	Fiber above $x \in M$: $E_x = \pi_E^{-1}(x)$	Sections $s : M \to E$
$E \oplus F$	$= \{(e,f) \in E \times F \,/\, \pi_E(e) = \pi_F(f)\} \subseteq E \times F$ (smooth submanifold); $\pi_{E\oplus F}(e,f) = \pi_E(e) = \pi_F(f)$	$E_x \oplus F_x$	are of type $s = (s_1, s_2)$, $s_i \in \Gamma^\infty E$, $s_2 \in \Gamma^\infty F$
$E \otimes F$	$= \{(x, \xi) : x \in M, \xi \in E_x \otimes F_x\} \to M$, $(x,\xi) \mapsto x$	$E_x \otimes F_x$	any $s_1 \in \Gamma^\infty E$, $s_2 \in \Gamma^\infty F$ induces $s_1 \otimes s_2$ of $E \otimes F$ by: $(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x)$ (not all!)
$\underline{\mathrm{Hom}}(E,F)$ (vector bundle Hom)	$= \{(x, A) : x \in M,\ A : E_x \to F_x$ linear map$\} \to M$, $(x,A) \mapsto x$	$\mathrm{Hom}(E_x, F_x)$ (linear maps)	
E^*	$= \mathrm{Hom}(E, \mathbb{R}_M)$ $\left(\mathbb{R}_M = \begin{array}{c} M \times \mathbb{R} \\ \downarrow \mathrm{pr}_M \\ M \end{array}\right.$ the trivial line bdle/M	$(E_x)^* = \mathrm{Hom}(E_x, \mathbb{R})$	

Reminder local viewpoint on vector bundles $E \to M$: (2)

CHOOSE
- $\mathcal{U} = \{U^\alpha\}_{\alpha \in \Lambda}$ open cover of M
- e^α - frame of $E|_{U^\alpha}$, $e^\alpha = \{e^\alpha_1, \dots, e^\alpha_r\}$

For $\alpha, \beta \in \Lambda$ compare e^α and e^β over $U^\alpha \cap U^\beta$:

$$e^\beta_i = \sum_{j=1}^r (\tau^{\alpha\beta})^j_i \, e^\alpha_j \quad \text{(short: } e^\beta = e^\alpha \cdot \tau^{\alpha\beta}\text{)}$$

$\Rightarrow$ get smooth functions $\tau^{\alpha\beta} : U^\alpha \cap U^\beta \to GL_r(\mathbb{R})$, satisfying, over triple intersections $U^\alpha \cap U^\beta \cap U^\gamma$: $\tau^{\alpha\beta} \cdot \tau^{\beta\gamma} = \tau^{\alpha\gamma}$ $(\forall)\alpha, \beta, \gamma \in \Lambda$ — transition data.

Main point 1: ~~The~~ "transition data contains all the info about E ... choices) can be reconstructed!

Main point 2: Allows one to work locally and express objects on E in terms of simpler (more classical) objects. (3)

E.g.: sections $s \in \Gamma(E)$:

(*) locally, over U^α:

$$s|_{U_\alpha} = \sum_i f^i_\alpha \cdot e^\alpha_i \quad \text{with } f^i_\alpha : U^\alpha \to \mathbb{R},\ 1 \le i \le r$$

i.e. $s|_{U_\alpha}$ is encoded by $f_\alpha = (f^1_\alpha, \dots, f^r_\alpha) : U^\alpha \to \mathbb{R}^r$.

(*) on overlaps $U^\alpha \cap U^\beta$:

$$(*) \quad f^i_\alpha = \sum_{j=1}^r (\tau^{\alpha\beta})^i_j \, f^j_\beta \qquad (\forall) i \in \{1, \dots, r\}$$

$\{\text{sections } s \in \Gamma E\} \xleftrightarrow{1-1} \{\text{collections } f^\alpha : U^\alpha \to \mathbb{R}^r \text{ for } \alpha \in \Lambda \text{ s.t. } (\forall)\alpha, \beta \in \Lambda : (*) \text{ holds}\}$.

$\Gamma^\infty(M)$-module $\Gamma^\infty E$	(local) frames of E	$\rightsquigarrow$ transition functions
$\Gamma^\infty(E \oplus F) \cong \Gamma^\infty(E) \oplus \Gamma^\infty(F)$	$e = \{e_1, \dots, e_r\}$ for E, $f = \{f_1, \dots, f_s\}$ for F $\rightsquigarrow$ $(e_1,0), \dots, (e_r,0), (0,f_1), \dots, (0,f_s)$ frame of $E \oplus F$	
$\Gamma^\infty(E \otimes F) \cong \Gamma^\infty(E) \otimes_{C^\infty(M)} \Gamma^\infty(F)$	$\{e_i \otimes f_j : 1 \le i \le r, 1 \le j \le s\}$, rs members	
$\Gamma^\infty(\underline{\mathrm{Hom}}(E,F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma^\infty E, \Gamma^\infty F)$ more explanations soon!		
	If $e = \{e_1, \dots, e_r\}$ local frame of E $\rightsquigarrow$ the dual local frame $e^* = \{e_1^*, \dots, e_r^*\}$; at any x: $\{e_1^*(x), \dots, e_r^*(x)\}$ basis of E_x^*, the dual basis to $\{e_1(x), \dots, e_r(x)\}$	Tr. fcts for E^* = transpose of those of E

Reminder General procedure to produce vector bundles (over M) (1)

Start with:
- collection $\{E_x\}_{x \in M}$ of r-dimensional vector spaces.

$$E := \{(x, v) : x \in M, v \in E_x\} \quad \text{(set)}$$

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- for each $U \in \mathcal{U}$, e set-theoretical local frame of E over U, $\{e^U_1, \dots, e^U_r\}$

Such that, for $U, U' \in \mathcal{U}$, writing for $x \in U \cap U'$

$$e^U_i(x) = \sum_j \mathrm{coeff}^j_i(x) \, e^{U'}_j(x) \qquad \tau^U_{U'}$$

then the resulting $\mathrm{coeff}^j_i : U \cap U' \to \mathbb{R}$ are smooth.

Then $\Rightarrow$ E made into a vector bundle such that all e^U_i are smooth local sections.

$E \otimes F$ $\{(x,\xi) : x \in M, \xi \in \dots\} \to M$, $(x,\xi) \mapsto x$ (not all!)

$\underline{\mathrm{Hom}}(E,F)$ (vector bundle Hom) $= \{(x,A) : x \in M,\ A : E_x \to F_x$ linear map$\} \to M$, $(x,A) \mapsto x$ $\quad \mathrm{Hom}(E_x, F_x)$ (linear maps)

E^* $= \mathrm{Hom}(E, \mathbb{R}_M)$ $\quad \left(\mathbb{R}_M = \begin{matrix} M \times \mathbb{R} \\ \downarrow \mathrm{pr}_M \\ M \end{matrix}\right.$ the trivial line bdle $/M\Big)$ $\quad (E_x)^* = \mathrm{Hom}(E_x, \mathbb{R})$

$S^2 E^*$ $\quad S^2(E^*)$ for V = v space

(6)

Digression: some explanations:

- for V = vector space, $e_1, \dots, e_r$ = basis of V $\rightsquigarrow$ the dual basis $e_1^*, \dots, e_r^*$ (of V^*) defined by: $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.
- for V, W vector spaces: each $\xi_0 \in V^*, w_0 \in W$ gives rise to a linear map $V \to W$, $v \mapsto \xi_0(v) \cdot w_0$ | denoted $\xi_0 \otimes w_0 \in \mathrm{Hom}(V,W)$

  - $e_1, \dots, e_r$ basis of V $\rightsquigarrow$ dual $e_1^*, \dots, e_r^*$ for V^*; $f_1, \dots, f_s$ $-''-$ W $\Rightarrow$ $e_i^* \otimes f_j$ ($\in \mathrm{Hom}(V,W)$), $1 \le i \le r$, $1 \le j \le s$ is a basis of $\mathrm{Hom}(V,W)$
  - Why $\otimes$ notation? Because $\mathrm{Hom}(V,W) \simeq V^* \otimes W$, $\xi_0 \otimes w_0 \longleftarrow\!\shortmid\ \xi_0 \otimes w_0$ & similarly for vector bundles. "$V \otimes W = \mathrm{Hom}(V^*, W)$"

at any x: $\{e_1^*(x), \dots, e_r^*(x)\}$ ← basis of E_x^* is the dual basis of $e_1, \dots, e_r$ basis of V ⇒ for $i \le j$ get $t^{ij} \in S^2 V^*$ which is 1 on (e_i, e_j), 0 on rest.

$e_1^*, \dots, e_r^*$ ($e^1, \dots, e^r$)

more explicit: elements/sections $u \in \Gamma(\underline{\mathrm{Hom}}(E,F)) \equiv$ (same thing) as a morphism $u : E \to F$ of vector bundles. (7)

Each u induces a map

$$\hat{u} : \Gamma(E) \to \Gamma(F), \quad \hat{u}(s)(x) = u_x(s(x))$$

(s; $u(s)$; $\hat{u}(s)(x)$ should be in F_x; u_x linear $F_x \to F_x$; $s(x) \in E_x$)

Prop: $\hat{u}$ is $C^\infty(M)$-linear map (compat.) and any $U : \Gamma(E) \to \Gamma(F)$ $C^\infty(M)$-linear is of type $U = \hat{u}$ for some (unique) u

Given U why does u exist s.t. $U = \hat{u}$? many u in examples arise by first knowing U

want $\boxed{U(s)(x) = u_x(s(x))}$ $(\forall)\, s \in \Gamma E$, $x \in M$

Build u: for $x \in M$, what is $u_x : E_x \to F_x$?

v_0

Lemma 1: $\forall v_0 \in E_x$ we can write $v_0 = s(x)$ for some $s \in \Gamma(E)$

define $\boxed{u_x(v_0) := U(s)(x)}$

Lemma 2: does not depend on the choice of s like for 1-forms

Operations	Vector bundles over M $E \xrightarrow{\pi_E} M$	Fiber above $x \in M$ $E_x = \pi_E^{-1}(x)$	Sections $s: M \to E$
$E \oplus F$	$= \{(e,f) \in E \times F \mid \pi_E(e) = \pi_F(f)\} \subset E \times F$ (smooth submanifold)	$E_x \oplus F_x$	are of type $s = (s_1, s_2)$, $s \in \Gamma E$, $s \in \Gamma F$
$E \otimes F$	$\{(x, \xi) : x \in M, \xi \in E_x \otimes F_x\} \longrightarrow M$, $(x,\xi) \mapsto x$	$E_x \otimes F_x$	any $s \in \Gamma E$, $s \in \Gamma F$ induces $s_1 \otimes s_2$ of $E \otimes F$ by: $(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x)$ (not all!)
$\underline{\mathrm{Hom}}(E,F)$ vector bundle Hom	$= \{(x, A) : x \in M, A : E_x \to F_x$ linear map$\} \to M$, $(x,A) \mapsto x$	$\mathrm{Hom}(E_x, F_x)$ (linear maps)	
E^*	$= \mathrm{Hom}(E, \mathbb{R}_M)$ ($\mathbb{R}_M = M \times \mathbb{R} \to M$ the trivial line bdle/M)	$(E_x)^* = \mathrm{Hom}(E_x, \mathbb{R})$	
$S^2 E^*$		$S^2(E^*)$ for V v.s.	

Digression: some explanations

- for V v.s. $e_1, \dots, e_r$ basis of V $\rightsquigarrow$ the dual basis $e_1^*, \dots, e_r^*$ ($\in V^*$) defined by: $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
- for V, W v.s.: each $\xi_0 \in V^*, w_0 \in W$ gives rise to a linear map $V \to W$, $v \mapsto \xi_0(v) w_0$ | denoted $\xi_0 \otimes w_0 \in \mathrm{Hom}(V,W)$

  $\{e_1, \dots, e_r$ basis of $V \to$ dual $e_1^*, \dots, e_r^*$ for $V^*\}$ and $\{f_1, \dots, f_s \to W\} \Rightarrow e_i^* \otimes f_j \in \mathrm{Hom}(V,W)$, $1 \le i \le r$, $1 \le j \le s$ is a basis of $\mathrm{Hom}(V,W)$

  why $\otimes$ notation? Because $\mathrm{Hom}(V,W) \simeq V^* \otimes W$, $\xi_0 \otimes w_0 \longleftrightarrow \xi_0 \otimes w_0$ (similarly for vector bundles). "$V \otimes W = \mathrm{Hom}(V^*, W)$" ~~=~~

ΓE = the $C^\infty(M)$-module ΓE (4-5) $\quad$ (local) frames of E & transition functions

$\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$ $\quad$ $e = \{e_1, \dots, e_r\}$ for E, $f = \{f_1, \dots, f_s\}$ for F $\longrightarrow$ $\{(e_1,0), \dots, (e_r,0), (0,f_1), \dots, (0,f_s)\}$ frame of $E \oplus F$

$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$ $\quad$ $\{e_i \otimes f_j : 1 \le i \le r, 1 \le j \le s\}$ rs members

$\Gamma(\underline{\mathrm{Hom}}(E,F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma E, \Gamma F)$ $\quad$?

more explanations soon

If $e = \{e_1, \dots, e_r\}$ local frame of E $\rightsquigarrow$ the dual local frame $\{\underbrace{e_1^*}_{\varepsilon^1}, \dots, \underbrace{e_r^*}_{\varepsilon^r}\}$

at any x: $\{e_1^*(x), \dots, e_r^*(x)\}$ $\leftarrow$ basis of E_x^* (!) the dual

$e_1, \dots, e_r$ basis of $V \Rightarrow$ for $i \le j$ get $e^i \vee e^j \in S^2 V^*$ which is $\perp$ on (e_i, e_j), 0 on rest.

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more explicit: elements/sections $u \in \Gamma(\underline{\mathrm{Hom}}(E,F)) \equiv$ (same thing) as a morphism $u : E \to F$ of vector bundles (+)

Each u induces a map

$$\hat{u} : \underbrace{\Gamma(E)}_{S} \longrightarrow \underbrace{\Gamma(F)}_{\hat{u}(S)}, \quad \underbrace{\hat{u}(S)(x)}_{\text{should be in } F_x} = \underbrace{u_x}_{\text{linear } E_x \to F_x}\left(\overbrace{S(x)}^{\in E_x}\right)$$

Prop: $\hat{u}$ is $C^\infty(M)$-linear map and any $U : \Gamma(E) \to \Gamma(F)$ $C^\infty(M)$-linear is of type $U = \hat{u}$ for some (unique) u

(compact)

Given U why does u exist s.t. $U = \hat{u}$? many u in examples arise by first knowing U

Build u: fix $x \in M$, what is $u_x : E_x \to F_x$? Want $\boxed{U(s)(x) = u_x(s(x))}$ $(\forall)\, s \in \Gamma E$, $x \in M$

define $\boxed{u_x(v_0) := U(s)(x)}$ $\quad$ Lemma: $\forall v_0 \in E_x$ we can write $v_0 = s(x)$ for some $s \in \Gamma(E)$

Lemma 2: $\rightarrow$ does not depend on the choice of S (like for 1-forms)

S^2E^*

$S^2(E_x^*)$ for V = v. space

$S^2V^* := \{ \xi : V \times V \to \mathbb{R} \mid \xi = \text{bilinear \& symmetric} \}$

(8)

$\Lambda^k E^*$

$\Lambda^k E_x^*$ where:

$\Lambda^k V^* = \{ \xi : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R} \mid \text{multilinear skew-symmetric} \}$

Interesting concept: inner product on a vector bundle E. That is $g \in \Gamma(S^2E^*)$ (smooth in x)

which, for each x, g_x - an inner product on E_x

= collection $\{g_x\}_{x \in M}$ of inner prod. g_x on E_x, "smooth in x"

Importance: – very useful tool in proving facts

– conceptually, they are an example of "geom structures on a vector bundle"

e.g. for $E = TM$, such a g is called a Riemannian metric on $M \rightsquigarrow (M, g)$ – Riemannian manifold

Theorem: $(\forall) E \ \exists$ an inner product g on E

proof: partitions of unity

For $E = TM$

$\Gamma(\Lambda^k E^*) = \Omega^k(M)$



Operations ↓	Vector bundles over M $E \xrightarrow{\pi} M$	Fiber above $x \in M$ $E_x = \pi^{-1}(x)$	Sections $s: M \to E$
$E \oplus F$	$= \{(e,f) \in E \times F \mid \pi_E(e) = \pi_F(f)\} \subseteq E \times F$ (smooth, ...)	$E_x \oplus F_x$	are of type $s = (s_1, s_2)$, $s_1 \in \Gamma E$, $s_2 \in \Gamma F$
$E \otimes F$	$\{(x, \xi) : x \in M, \xi \in E_x \otimes F_x\} \longrightarrow M$, $(x,\xi) \mapsto x$	$E_x \otimes F_x$	every $s_1 \in \Gamma E$, $s_2 \in \Gamma F$ induces $s_1 \otimes s_2$ of $E \otimes F$ by: $(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x)$ (not all!)
$\mathrm{Hom}(E,F)$ vector bundle Hom	$= \{(x, A) : x \in M, A : E_x \to F_x$ linear map$\} \to M$, $(x,A) \mapsto x$	$\mathrm{Hom}(E_x, F_x)$ (linear maps)	
E^*	$= \mathrm{Hom}(E, \mathbb{R}_M)$ $\left(\mathbb{R}_M = \underset{M}{M \times \mathbb{R}}\right.$ the trivial line bdle/$M$$\left.\right)$	$(E_x)^* = \mathrm{Hom}(E_x, \mathbb{R})$	
$S^2 E^*$		$S^2(E_x^*)$ for V v.space; $S^2V^* := \{g : V \times V \to \mathbb{R} \mid$ bilinear & symmetric$\}$	

	⑧	$\Lambda^k E_x^*$ where: $\Lambda^k V^* = \{\xi : V \times \cdots \times V \to \mathbb{R} \mid$ multilinear, skew-symmetric$\}$

Interesting concept: inner product on a vector bundle E. That is $g \in \Gamma(S^2 E^*)$ (smooth in x)

which, for each x, g_x - an inner product on E_x

= collection $\{g_x\}_{x \in M}$ of inner prod g_x on E_x, "smooth in x"

Importance: – very useful tool in proving facts

– conceptually, they are an example of "geom structures on a vector bundle"

e.g. for $E = TM$, such a g is called a Riemannian metric on M & (M, g) — Riemannian manifold

Theorem: $(\forall) E$ $\exists$ an inner product g on E

proof: partitions of unity

$\cong$ the $C^\infty(M)$-module ΓE	(local) frames of E & transition functions
$\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$	$e = \{e_1, \dots, e_r\}$ for E, $f = \{f_1, \dots, f_s\}$ for F $\rightsquigarrow$ $(e_1,0), \dots, (e_r,0), (0,f_1), \dots, (0,f_s)$ frames of $E \oplus F$
$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$	$\{e_i \otimes f_j : 1 \le i \le r, 1 \le j \le s\}$ rs members
$\Gamma(\mathrm{Hom}(E,F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma E, \Gamma F)$ more explanations soon	?
	If $e = \{e_1, \dots, e_r\}$ local frame of E $\rightsquigarrow$ the dual local frame $e^* = \{e_1^*, \dots, e_r^*\}$; at any x $\{e_1^*(x), \dots, e_r^*(x)\}$ — basis of E_x^*, the dual $e_1, \dots, e_r$ basis of $V \Rightarrow$ for $i \le j$ get $e_i^* \wedge e_j^* \in S^2 V^*$ which is ≥ 0 on (e_i, e_i), > 0 on rest

For $E = TM$, $\Gamma(\Lambda^k E^*) = \Omega^k(M)$	if $e_1, \dots, e_r$ is a frame of V, form $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \in \Lambda^k V^*$ for $i_1 < \cdots < i_k$ — basis of $\Lambda^k V^*$