B1

From bachelor vector fields

$X \in \mathfrak{X}(M)$

maps $M \ni p \longmapsto X_p \in T_pM$, smooth.

are in 1-1 with derivations, i.e. linear

$L : C^\infty(M) \to C^\infty(M)$, $L(fg) = fL(g) + gL(f)$

Given $X \Rightarrow L_X$ given by $L_X(f) = df(X)$.

Lie bracket: $[X,Y] \in \mathfrak{X}(M)$ defined whenever $X, Y \in \mathfrak{X}(M)$ by condition

$$L_{[X,Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f)) \quad \forall f$$

Warning: be careful with saying $L_X = X$ ... and writing $[X,Y] = XY - YX$

not vector fields!!

B2

Locally, any chart (U, x) of M induces $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(U)$

and, over U, any X is a sum $\sum F_i \frac{\partial}{\partial x_i}$

Flows: $X \in \mathfrak{X}(M)$. Integral curve of X starting at $p \in M$: any $\gamma : I \to M$

$$\dot\gamma(t) = X_{\gamma(t)}, \quad \gamma(0) = p.$$

Thm: $\forall p \; \exists!$ max integral curve at p. (γ_p)

Flow of X: $\varphi_X^t : M \to M$, $p \longmapsto \gamma_p(t)$

Projectability: $\pi : M \to N$ submersion

Say $X \overset{\pi}{\rightsquigarrow} Y$ if $(d\pi)_p(X_p) = Y_{\pi(p)}$ $\forall p$

($\mathfrak{X}(M)$ $\quad$ $\mathfrak{X}(N)$)

(X is π-projectable to Y)

Useful: $X_1 \overset{\pi}{\rightsquigarrow} Y_1$, $X_2 \overset{\pi}{\rightsquigarrow} Y_2$ $\Rightarrow [X_1, X_2] \overset{\pi}{\rightsquigarrow} [Y_1, Y_2]$

B1

From bachelor: vector fields

$X \in \mathfrak{X}(M)$

maps $M \ni p \longmapsto X_p \in T_pM$, smooth

are in 1-1 with derivations, i.e. linear $\mathcal{L}: C^\infty(M) \to C^\infty(M)$, $\mathcal{L}(fg) = f\mathcal{L}(g) + g\mathcal{L}(f)$

Given $X \Rightarrow \mathcal{L}_X$ given by $\mathcal{L}_X(f) = df(X)$

Lie bracket: $[X,Y] \in \mathfrak{X}(M)$ defined whenever $X, Y \in \mathfrak{X}(M)$ by condition

$\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$ $\forall f$

Warning: be careful with saying $\mathcal{L}_X = X$ ... and writing $[X,Y] = XY - YX$ (not vector fields!!)

B2

Locally, any chart (U, x) of M induces $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(U)$

and, over U, any X is a sum $\sum F_i \frac{\partial}{\partial x_i}$

Flows: $X \in \mathfrak{X}(M)$. Integral curve of X starting at $p \in M$: any $\gamma: I \to M$

$\dot\gamma(t) = X_{\gamma(t)}$, $\gamma(0) = p$.

Thm: $\forall p$ $\exists!$ max integral curve γ_p at p.

Flow of X: $\varphi_X^t: M \to M$, $p \mapsto \gamma_p(t)$

$\frac{d}{dt}\varphi_X^t(q) = X_{\varphi_X^t(q)}$

$[X,Y] = 0 \iff \varphi_X^t \varphi_Y^s = \varphi_Y^s \varphi_X^t$ $\forall t, s$

Projectability: $\pi: M \to N$ submersion

Say $X \overset{\pi}{\rightsquigarrow} Y$ if $(d\pi)_p(X_p) = Y_{\pi(p)}$ $\forall p$ (X is π-projectable to Y)

($X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$)

Useful: $X_1 \overset{\pi}{\rightsquigarrow} Y_1$, $X_2 \overset{\pi}{\rightsquigarrow} Y_2$ $\Rightarrow [X_1, X_2] \overset{\pi}{\rightsquigarrow} [Y_1, Y_2]$

Parallelisms

Vector bundles E: a trivialization $\Leftrightarrow$ choice of a global frame

Def: Parallelism of M: $X_1, \dots, X_m \in \mathfrak{X}(M)$ a global frame of TM.

Ex: $M = S^1$: $X = \frac{\partial}{\partial \theta} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

Ex: S^2 has no parallelisms, the torus does

Ex: $S^3 \subseteq \mathbb{R}^4 \cong \mathbb{H}$, $\xi \in S^3 \Rightarrow i\xi, j\xi, k\xi$ all $\perp \xi$.
$\Rightarrow$ a parallelism.

Ex: any lie group G admits one.

pf: Use identity $e \in G$, $\mathfrak{g} = T_eG$, fix $v_1, \dots, v_m$ basis of $\mathfrak{g}$.

For $g \in G$, $R_g : G \to G$ $R_g(x) = xg$
$(dR_g)_e : \mathfrak{g} \to T_gG$ $\quad (dR_g)_e(v_i) =: X_{ig}$.
$\Downarrow$
$\{X_1, \dots, X_m\}$ parallelism.

Given $\{X_1, \dots, X_m\}$:
- a chart (U, x) is adapted if $X_i = \frac{\partial}{\partial x_i}$ over U.
- Q: when are there adapted charts around each point.

Thm: Answer yes $\Leftrightarrow$ $[X_i, X_j] = 0$ $(\forall)$ $i, j \in \{1, 2, \dots, m\}$.

pf: "$\Leftarrow$" Fix $x_0 \in M$. Use $\varphi_X^t : M \to M$ (defined near x_0) for $X \in \{X_1, \dots, X_m\}$

Define $F : \mathbb{R}^m \to M$, $(t_1, \dots, t_m) \mapsto \varphi_{X_1}^{t_1}\left(\varphi_{X_2}^{t_2}\left(\dots\left(\varphi_{X_m}^{t_m}(x_0)\right)\right)\right)$

Look at $(dF)_t : \mathbb{R}^m \to T_{F(t)}M$

$\frac{\partial}{\partial t_1} = (1, 0, \dots, 0) \mapsto \frac{\partial F}{\partial t_1}(t) = \frac{d}{dt_1}\left(\varphi_{X_1}^{t_1}\right)\left(\varphi_{X_2}^{t_2}(\dots)\right)$

$(0, 1, \dots, 0) \mapsto \boxed{\cdots} = X^2_{F(t)}$ $\qquad X^1_{F(t)}$.

At $t = 0 \Rightarrow$ the basis $\{X^1_{x_0}, \dots, X^m_{x_0}\} \Rightarrow (dF)_0$ is $\cong \Rightarrow$

IFT $\Longrightarrow$ $F|_\Omega : \Omega \to U$ a diffeo. Set $\chi := (F|_\Omega)^{-1}$.

$\Omega \subseteq \mathbb{R}^m$ open, $U \subseteq M$ open, $x_0 \in U$.

(p-dimensional) Foliations

Vector bundle E: rank p sub-bundles $F \subset E$

$\{X^1, \dots, X^m\}$ parallelism $\Longrightarrow$ $F|_\Omega : \Omega \longrightarrow U$, $\Omega \cap \mathbb{R}^m$, $U \cap M$, $\mathcal{U} := (F|_\Omega)^{-1}$.

(p-dimensional) Foliations

Vector bundle E: rank p sub-bundles $F \subseteq E$.

Def: A *p-dimensional distribution* on M is a rank p sub-bundle $\mathcal{F} \subseteq TM$.

Given $\mathcal{F}$:
- an *integral of* $\mathcal{F}$: $N \subseteq M$ submanifold, connected, $T_p N = \mathcal{F}_p$ $(\forall) p \in N$.
- call $\mathcal{F}$ *integrable* (or a Foliation) if $(\forall) x_0 \in M$ $\exists$ integral $N \subseteq M$ with $x_0 \in N$.

Ex1: $X \in \mathfrak{X}(M)$ nowhere vanishing, $\mathcal{F} := \mathrm{Span}(X) \subseteq TM$, $p = 1$. | integrals: the integral curves of X in M

Ex2: $M = \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ $(q = m - p)$, $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $1 \le i \le p$. | integrals: $\mathbb{R}^p \times \{y\}$ for $y \in \mathbb{R}^q$

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Ex: any lie group G admits one.

$\not\Rightarrow$: Use identity $e \in G$, $\mathfrak{g} = T_eG$, fix $v_1, \dots, v_m$ basis of $\mathfrak{g}$.

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$(dR_g)_e : \mathfrak{g} \to T_gG$ $\Big|$ $(dR_g)_e(v_i) =: X_g^i$.

$\{X_1, \dots, X_m\}$ parallelism

Given $\{X_1, \dots, X_m\}$: • a chart (U, x) is adapted if $X_i = \frac{\partial}{\partial x_i}$ over U.

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Thm: Answer yes $\Leftrightarrow$ $[X_i, X_j] = 0$ $\forall\, i, j \in \{1, 2, \dots, m\}$

$\Rightarrow$: "$\Leftarrow$" Fix $x_0 \in M$. Use $\varphi_X^t : M \to M$ (defined near x_0) for $X \in \{X_1, \dots, X_m\}$

Define $F : \mathbb{R}^m \to M$, $(t_1, \dots, t_m) \mapsto \varphi_{X_1}^{t_1}\left(\varphi_{X_2}^{t_2}\left(\dots\left(\varphi_{X_m}^{t_m}(x_0)\right)\right)\right)$

Look at $(dF)_t : \mathbb{R}^m \to T_{F(t)}M$

$\frac{\partial}{\partial t_1} \equiv (1, 0, \dots, 0) \mapsto \frac{\partial F}{\partial t_1}(t) = \frac{d}{dt_1}\left(\varphi_{X_1}^{t_1}\left(\varphi_{X_2}^{t_2}(\dots)\right)\right)$

$(0, 1, \dots, 0) \mapsto \bullet\bullet\bullet = X^2_{F(t)}$ $\qquad X^1_{F(t)}$.

At $t = 0 \Rightarrow$ the basis $\{X_1^{x_0}, \dots, X_m^{x_0}\} \Rightarrow (dF)_0$ is $\cong \Rightarrow$

IFT $\Rightarrow$ $F|_U : \underset{\cap}{\Omega} \to \underset{\cap}{\mathcal{U}}$ a diffeo. Set $\mathcal{U} := (F|_U)^{-1}$

$\mathbb{R}^m$ $\qquad$ M

(p-dimensional) Foliations

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Def: A p-dimensional distribution on M is a rank p sub-bundle $\mathcal{F} \subseteq TM$.

Given $\mathcal{F}$: • an integral of $\mathcal{F}$: $N \subseteq M$ submanifold, connected, $T_pN = \mathcal{F}_p$ $(\forall)\, p \in N$.

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Ex1: $X \in \mathfrak{X}(M)$ nowhere vanishing, $\mathcal{F} = \mathrm{Span}(X) \subseteq TM$, $p = 1$ $\Big|$ integrals: the integral curves of X

Ex2: $M = \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ $(q = m - p)$, $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $1 \le i \le p$ $\Big|$ integrals: $\mathbb{R}^p \times \{y\}$ for $y \in \mathbb{R}^q$

$\frac{d}{dt}\varphi_X^t(q) =$

$[X, Y] = 0 \iff \varphi_X^t \varphi_Y^s =$

Parallelisms

Vector bundle E a trivialization $\Rightarrow$ choice of a global frame

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Ex: any Lie group G admits one

$\exists$: like cotangents $\forall g \in G$, $\eta_g = T_e(\cdot, g)$, fix $\eta^1, \dots, \eta^m$ basis of $\mathfrak{g}^*$

For $g \in G$, $R_g : G \to G$ Rightward $(dR_g)_h : \mathfrak{g} \to T_g G$ | $(dR_g)_e(v^i) = X^i_g$. $\{X^1, \dots, X^m\}$ parallelism

curve $\{X^1, \dots, X^m\}$: • a chart (U, x) is adapted if $X^i = \frac{\partial}{\partial x^i}$ on U

• $\mathbb{R}$ admits adapted charts around each point

Thm: Answer yes $\iff$ $[X^i, X^j] = 0$ $\forall i, j \in \{1, \dots, m\}$

$\Rightarrow$: ✓ $\Leftarrow$: Fix $x_0 \in M$. Use $\varphi^t_X : M \to M$ for $X \in \{X^1, \dots, X^m\}$

Define $F : \mathbb{R}^m \to M$, $(t_1, \dots, t_m) \mapsto \varphi^{t_1}_{X^1}\left(\varphi^{t_2}_{X^2}\left(\dots\left(\varphi^{t_m}_{X^m}(x_0)\right)\right)\right)$

Look at $(dF)_t : \mathbb{R}^m \to T_{F(t)} M$

$\frac{\partial}{\partial t_i} = (1, 0, \dots, 0) \mapsto \frac{\partial F}{\partial t_i}(t) = \frac{d}{dt}\Big|_{t_i}\left(\varphi^{t_1}_{X^1}\left(\varphi^{t_2}_{X^2}\left(\dots\right)\right)\right) = X^i_{F(t)}$

$(0, 1, \dots, 0) \mapsto \dots = X^2_{F(t)}$

At $t = 0$ $\Rightarrow$ the basis $\{X^1_{x_0}, \dots, X^m_{x_0}\}$ $\Rightarrow$ $(dF)_0$ isom $\Rightarrow$

IFT $\Rightarrow$ $F|_\Omega : \Omega \to U$ a diffeo. Set $\mathcal{X} = (F|_\Omega)^{-1}$

$\Omega \subseteq \mathbb{R}^m$, $U \subseteq M$

(p-dimensional) Foliations

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Ex1: $X \in \mathfrak{X}(M)$ nowhere vanishing, $\mathcal{F} = \mathrm{Span}(X) \subseteq TM$, $p = 1$ | integrals: the integral curves of X

Ex2: $M = \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ ($p = n - k$), $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $i \le p$ | integrals: $\mathbb{R}^k$ right for $y \in \mathbb{R}^{n-k}$

From backwards vector fields

$X \in \mathfrak{X}(M)$

maps $M \ni p \longmapsto X_p \in T_p M$, smooth

are in $1{-}1$ with derivations $\mathcal{L} : C^\infty(M) \to C^\infty(M)$, $\mathcal{L}(fg) = f\mathcal{L}(g) + g\mathcal{L}(f)$

Given $X \Rightarrow \mathcal{L}_X$ given by $\mathcal{L}_X(f) = df(X)$

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$\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$ $\forall f$

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... and writing $[X, Y] = XY - YX$ (not vector fields!!)

$\frac{d}{dt}\varphi^t_X(q) = X_{\varphi^t_X(q)}$

$[X, Y] = 0 \iff \varphi^t_X \varphi^s_Y = \varphi^s_Y \varphi^t_X$ $\forall t, s$

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Flows: $X \in \mathfrak{X}(M)$. Integral curve of X starting at $p \in M$: any $\gamma : I \to M$ $\gamma'(t) = X_{\gamma(t)}$, $\gamma(0) = p$.

Thm: $\forall p$ $\exists!$ maximal integral curve γ_p at p

Flow of X: $\varphi^t_X : M \to M$, $p \mapsto \gamma_p(t)$

Projectability: $\pi : M \to N$ submersion. Say $X \overset{\pi}{\rightsquigarrow} Y$ if $(d\pi)_p(X_p) = Y_{\pi(p)}$ $\forall p$ (X is π-projectable to Y)

Useful: $X_1 \overset{\pi}{\rightsquigarrow} Y_1$, $X_2 \overset{\pi}{\rightsquigarrow} Y_2$ $\Rightarrow [X_1, X_2] \overset{\pi}{\rightsquigarrow} [Y_1, Y_2]$

$(d\pi)_p(X_p)=0$ $(\forall)p$. $\quad X\overset{\pi}{\sim}0,\ Y\overset{\pi}{\sim}0 \Rightarrow [X,Y]\overset{\pi}{\sim}0$

At $t=0 \Rightarrow$ the basis $\{X^1_{x_0},\dots,X^n_{x_0}\} \Rightarrow (dF)_0=\text{iso} \Rightarrow$

IFT $\Rightarrow$ $F|_\Omega : \Omega \cap \mathbb{R}^n \longrightarrow \mathcal{U} \cap M$ a diffeo. Set $\mathcal{X}:=(F|_\Omega)^{-1}$.

(p-dimensional) Foliations (3)

Vector bundle E: rank p sub-bundles $F \subset E$.

Def: A p-dimensional distribution on M is a rank p sub-bundle $\mathcal{F} \subseteq TM$.

Given $\mathcal{F}$:
- an integral of $\mathcal{F}$: $N \subseteq M$ submanifold, connected, $T_pN=\mathcal{F}_p$ $(\forall)p\in N$.
- call $\mathcal{F}$ integrable (or a Foliation) if $(\forall)x_0\in M$ $\exists$ integral $N\subseteq M$ with $x_0\in N$.

Ex1: $X\in\mathfrak{X}(M)$ nowhere vanishing, $\mathcal{F}:=\mathrm{Span}(X)\subseteq TM$, $p=1$. | integrals: the integral curves of X (in M)

Ex2: $M=\mathbb{R}^m=\mathbb{R}^p\times\mathbb{R}^q$ $(q=m-p)$, $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $1\le i\le p$. | integrals: $\mathbb{R}^p\times\{y\}$ for $y\in\mathbb{R}^q$

Def: Given $\mathcal{F}$ a foliated chart for $\mathcal{F}$: any chart (4) $\mathcal{X}: U\xrightarrow{\sim}\mathbb{R}^m=\mathbb{R}^p\times\mathbb{R}^{m-p}$ such that the slices $\mathcal{X}^{-1}(\mathbb{R}^p\times\{y\})$ are integrals of $\mathcal{F}$ $(\forall)y\in\mathbb{R}^{m-p}$.

Rk: $\Leftrightarrow$ $\mathcal{F}_x=\mathrm{Span}\left\{\left(\frac{\partial}{\partial x_1}\right)_x,\dots,\left(\frac{\partial}{\partial x_p}\right)_x\right\}$ $(\forall)x\in U$.

Def: A distribution $\mathcal{F}$ called involutive if: $X,Y\in\Gamma(\mathcal{F}) \Rightarrow [X,Y]\in\Gamma(\mathcal{F})$

Thm (Frobenius): Given $\mathcal{F}=p$ dim distr, TFAE equivalent:
(1) $\mathcal{F}=$ integrable
(2) around each $x\in M$ $\exists$ foliated chart
(3) $\mathcal{F}=$ involutive

~~pf~~: (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\overset{!}{\Longrightarrow}$ (2)

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- call $\mathcal{F}$ integrable (or a Foliation) if $(\forall) x_0 \in M$ $\exists$ integral $N \subseteq M$ with $x_0 \in N$.

Ex 1: $\begin{cases} X \in \mathfrak{X}(M) \text{ nowhere vanishing} \\ \mathcal{F} := \mathrm{Span}(X) \subseteq TM, \ p=1. \end{cases}$ | integrals: the integral curves of X in M

Ex 2: $M = \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ $(q = m-p)$, $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $1 \le i \le p$. | integrals: $\mathbb{R}^p \times \{y\}$ for $y \in \mathbb{R}^q$

Def: Given $\mathcal{F}$ $x: U \xrightarrow{\sim} \mathbb{R}$ slices x

Rk: $\Leftrightarrow$ $\mathcal{F}_*$

Def: A dis

Thm (Fro
(1) $\mathcal{F}$
(2) ∞
(3) $\mathcal{F}$

~~Pf~~: (2) $\Rightarrow$

(3')

Ex 2': Submersion $\pi: M \to N$, $\dim N = m-p$ | integrals: the fibers of π, $\pi^{-1}(y) \subseteq M$ for $y \in N$.

$\mathcal{F} = \ker(d\pi)$

involutive?

$X, Y \in \mathfrak{X}(M)$, $d\pi(X) = 0, d\pi(Y) = 0$ $\Big\}$ $\overset{?}{\Longrightarrow}$ $d\pi([X,Y]) = 0$? ?

by the RVT

$$d\pi([X,Y]) = [\underbrace{d\pi(X)}_{0}, \underbrace{d\pi(Y)}_{0}] = 0$$

$d\pi(X)$ $\quad$ $\pi_*(X)$

$p \in M$ $\quad$ $(d\pi)_p(X_p) \in T_{\pi(p)} N$

p' $\quad$ $(d\pi)_{p'}(X_{p'}) \in$ $\quad$ $\pi(p')$

$(d\pi)_p(X_p) = 0$ $(\forall) p$.

$\left.\begin{matrix} X \overset{\pi}{\sim} 0 \\ Y \overset{\pi}{\sim} 0 \end{matrix}\right\} \Rightarrow [X,Y] \overset{\pi}{\sim} 0$

Given $\{X^1, \dots, X^n$

Thm: Ans

~~Pf~~: "$\Leftarrow$" Fix

Define $F: \mathbb{R}$

Look at $(dF)_t$

$\left\{ \frac{\partial}{\partial t_1} \equiv (1, \right.$
$\left. (0, \right.$

At $t=0 \Rightarrow$ t

IFT $\Rightarrow$ F

pf of (3) ⇒ (2): Fix $x_0 \in M$. Choose (U, x_i) around x_0 s.t. ← e.g. such that $\mathcal{F}_{x_0} = \operatorname{span}\left\{\left(\frac{\partial}{\partial x_1}\right)_{x_0}, \dots, \left(\frac{\partial}{\partial x_p}\right)_{x_0}\right\}$

$$T_{x_0}M = \mathcal{F}_{x_0} \oplus \operatorname{span}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_{x_0}, \dots, \left(\frac{\partial}{\partial x_m}\right)_{x_0}\right\}$$

$\Rightarrow$ for some $W \subseteq M$ neigh of x_0

$$T_x M = \mathcal{F}_x \oplus \operatorname{span}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_x, \dots\right\} \quad (\forall)\, x \in W. \qquad (*)$$

May assume $M = W$ is a ball in $\mathbb{R}^m$, $x_0 = 0$.

Consider the projection on first p coordinates.

$$\pi : M \longrightarrow \mathbb{R}^p$$

& use

$$(d\pi)_x : T_x M \longrightarrow T_{\pi(x)}\mathbb{R}^p \ni \left(\frac{\partial}{\partial x_i}\right)_{\pi(x)}$$

$$\mathcal{F}_x \xrightarrow{\sim} \text{via } d\pi|_{\mathcal{F}}$$

$$\operatorname{Ker}(d\pi|_{\mathcal{F}}) = \mathcal{F}_x \cap \operatorname{Ker}(d\pi)_x \overset{(*)}{=} \{0\}.$$

Projectability: $\pi : M \to N$ submersion

Say $X \overset{\pi}{\rightsquigarrow} Y$ if $(d\pi)_p(X_p) = Y_{\pi(p)}$ $\forall p$

(X is π-projectable to Y)

Useful: $\left.\begin{matrix} X_1 \overset{\pi}{\rightsquigarrow} Y_1 \\ X_2 \overset{\pi}{\rightsquigarrow} Y_2 \end{matrix}\right\} \Rightarrow [X_1, X_2] \overset{\pi}{\rightsquigarrow} [Y_1, Y_2]$

$\Rightarrow$ we find $V_1, \dots, V_r \in \Gamma(\mathcal{F})$ s.t. $(d\pi)_x(V_x^i) = \left(\frac{\partial}{\partial x_i}\right)_{\pi(x)}$ $\forall x \in M$

i.e. $V^i \overset{\pi}{\rightsquigarrow} \frac{\partial}{\partial x_i}$ $\quad 1 \le i \le p$

In particular: $[V^i, V^j] \overset{\pi}{\rightsquigarrow} \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ i.e. $[V^i, V^j]$ lands in $\operatorname{Ker} d\pi$

But, by (3) $\Rightarrow [V^i, V^j]$ land in $\mathcal{F}$

$\Rightarrow [V^i, V^j] = 0$, $1 \le i, j \le p$

Set $V^k = \frac{\partial}{\partial x^k}$ for $k \in \{p+1, \dots, m\}$. Repeat the argument from the parallelization proof. ☒

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Ex 1: $\{X \in \mathfrak{X}(M) \text{ nowhere vanishing}\}$, $\mathcal{F} = \mathrm{Span}(X) \subseteq TM$, $p=1$ | integrals: the integral curves of X in M

Ex 2: $M = \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ $(q = m-p)$, $\mathcal{F}$ spanned by $\frac{\partial}{\partial x_i}$, $1 \le i \le p$ | integrals: $\mathbb{R}^p \times \{y\}$ for $y \in \mathbb{R}^q$

Def: Given $\mathcal{F}$ a foliated chart for $\mathcal{F}$: any chart $\chi : U \xrightarrow{\sim} \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p}$ such that the slices $\chi^{-1}(\mathbb{R}^p \times \{y\})$ are integrals of $\mathcal{F}$ $\forall y \in \mathbb{R}^{m-p}$.

Rk: $\iff \mathcal{F}_x = \mathrm{Span}\left\{\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_p}\right)_x\right\}$ $(\forall)\, x \in U$.

Def: A distribution $\mathcal{F}$ called involutive if: $X, Y \in \Gamma(\mathcal{F}) \Rightarrow [X,Y] \in \Gamma(\mathcal{F})$.

Thm (FROBENIUS): Given $\mathcal{F}$ p-dim distr, TFAE (equivalent):
(1) $\mathcal{F}$ = integrable
(2) around each $x \in M$ $\exists$ foliated chart
(3) $\mathcal{F}$ = involutive

Pf: (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Longrightarrow$ (2) !

Ex 2': Submersion $\pi : M \to N$, $\dim N = m-p$, $\mathcal{F} = \ker(d\pi)$ | integrals: the fibers of π, $\pi^{-1}(y) \subseteq M$ for $y \in N$.

(involutive? by the RVT)

$X, Y \in \mathfrak{X}(M)$, $d\pi(X) = 0$, $d\pi(Y) = 0$ $\overset{?}{\Longrightarrow}$ $d\pi([X,Y]) = 0$??

$$d\pi([X,Y]) = [d\pi(X), d\pi(Y)] = 0$$

$d\pi(X)$: $\pi \to N$, $p \in M$, $(d\pi)_p(X_p) \in T_{\pi(p)}N$, p', $(d\pi)_{p'}(X_{p'}) \in T_{\pi(p')}N$

$(d\pi)_p(X_p) = 0$ $(\forall)\, X$ $\Rightarrow$ $X \overset{\pi}{\to} 0$, $Y \overset{\pi}{\to} 0$ $\Rightarrow [X,Y] \overset{\pi}{\to} 0$

Given $\{X^1, \dots, X^m\}$: • a chart (U, χ) is adapted if $X^i = \frac{\partial}{\partial x^i}$ over U.
• Q: when are there adapted charts around each point.

Thm: Answer yes $\iff$ $[X^i, X^j] = 0$ $(\forall)\, i,j \in \{1, \dots, m\}$.

Pf: $\Leftarrow$: Fix $x_0 \in M$. Use $\varphi^t_X : M \to M$ for $X \in \{X^1, \dots, X^m\}$.

Define $F : \mathbb{R}^m \to M$, $(t_1, \dots, t_m) \mapsto \varphi^{t_1}_{X_1}\left(\varphi^{t_2}_{X_2}\left(\dots\left(\varphi^{t_m}_{X_m}(x_0)\right)\right)\right)$

Look at $(dF)_t : \mathbb{R}^m \to T_{F(t)}M$

$\frac{\partial}{\partial t_i} \equiv (1,0,\dots,0) \longmapsto \frac{\partial F}{\partial t_1}(t) = \frac{d}{dt_1}\left(\varphi^{t_1}_{X_1}\left(\varphi^{t_2}_{X_2}(\dots)\right)\right)$

$(0,1,\dots,0) \longmapsto \bullet\bullet\bullet = X^2_{F(t)}$

$\dots$

$X^1_{F(t)}$.

At $t=0$ $\Rightarrow$ the basis $\{X^1_{x_0}, \dots, X^m_{x_0}\}$ $\Rightarrow$ $(dF)_0$ is $\cong$ $\Rightarrow$

IFT $\Longrightarrow$ $F|_\Omega : \Omega \xrightarrow{\sim} U$, $\Omega \subseteq \mathbb{R}^m$, $U \subseteq M$, a diffeo. Set $\chi := (F|_\Omega)^{-1}$.

Pf of (3) $\Rightarrow$ (2): Fix $x_0 \in M$. Choose (U, χ) around x_0 s.t. (e.g. such that $\mathcal{F}_{x_0} = \mathrm{Span}\left\{\left(\frac{\partial}{\partial x_1}\right)_{x_0}, \dots, \left(\frac{\partial}{\partial x_p}\right)_{x_0}\right\}$)

$$T_{x_0}M = \mathcal{F}_{x_0} \oplus \mathrm{Span}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_{x_0}, \dots, \left(\frac{\partial}{\partial x_m}\right)_{x_0}\right\}$$

$\Rightarrow$ for some $W \subseteq M$ nbhd of x_0

$$T_xM = \mathcal{F}_x \oplus \mathrm{Span}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_x, \dots\right\} \quad (\forall)\, x \in W. \qquad (*)$$

May assume $M = W$ is a ball in $\mathbb{R}^m$, $x_0 = 0$.

Consider the projection on first p coordinates

$\pi : M \longrightarrow \mathbb{R}^p$

& use $(d\pi)_x : T_xM \longrightarrow T_{\pi(x)}\mathbb{R}^p \ni \left(\frac{\partial}{\partial x_i}\right)_{\pi(x)}$

$\mathcal{F}_x \xrightarrow{d\pi|_{\mathcal{F}}} T_{\pi(x)}\mathbb{R}^p$ ($\cong$)

$\ker(d\pi|_{\mathcal{F}}) = \mathcal{F}_x \cap \ker(d\pi)_x \overset{(*)}{=} \{0\}$

Projectability: $\pi : M \to N$ submersion, $X \in \mathfrak{X}(M)$. X is π-projectable if $(d\pi)_p(X_p) = (d\pi)_{p'}(X_{p'})$ whenever $\pi(p) = \pi(p')$ (X is π-projectable to $Y \in \mathfrak{X}(N)$ if $(d\pi)_p(X_p) = Y_{\pi(p)}$ $\forall p \in M$).

Useful: $X_1 \overset{\pi}{\to} Y_1$, $X_2 \overset{\pi}{\to} Y_2$ $\Rightarrow [X_1, X_2] \overset{\pi}{\to} [Y_1, Y_2]$.

$\Rightarrow$ we find $V^1, \dots, V^r \in \Gamma(\mathcal{F})$ s.t. $(d\pi)_x(V^i_x) = \left(\frac{\partial}{\partial x_i}\right)_{\pi(x)}$ $\forall x \in M$.

i.e. $V^i \overset{\pi}{\to} \frac{\partial}{\partial x_i}$ $\quad 1 \le i \le p$

In particular: $[V^i, V^j] \overset{\pi}{\to} \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$. i.e. $[V^i, V^j]$ Ranges in $\ker d\pi$ $\Rightarrow [V^i, V^j] = 0$, $1 \le i, j \le p$

But, by (3) $\rightsquigarrow [V^i, V^j]$ Ranges in $\mathcal{F}$

Set $V^k = \frac{\partial}{\partial x^k}$ for $k \in \{p+1, \dots, m\}$. Repeat the argument from the parallelization prof. ☑