①

Reminder:

Theorem (Frobenius): Let $\mathcal{F}$ = a *p-dimensional distribution* on M. The following are equivalent:

(1) $\mathcal{F}$ = *integrable*

(2) around each x_0 one finds (U, x) s.t.
$\mathcal{F}_x = \mathrm{Span}\left\{\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_p}\right)_x\right\}$ $(\forall)\, x \in U$ } *foliated chart*

(3) $\mathcal{F}$ is *involutive*, i.e.:
$[X, Y] \in \Gamma(\mathcal{F})$ $(\forall)\, X, Y \in \Gamma(\mathcal{F})$

When these hold: call $\mathcal{F}$ a *foliation* on M

Geometrically:

①'

i.e. $\mathcal{F} \subseteq TM$ vector sub-bundle of rank p

i.e.: $(\forall)\, x_0 \in M$ $\exists$ $\underset{\ni x_0}{N \subseteq M}$ *an integral of* $\mathcal{F}$

i.e. connected submanifold with: $\mathcal{F}_x = T_x N$ $(\forall)\, x \in N$.

equivalently: each slice
$N^y_0 = \chi^{-1}(\mathbb{R}^p \times \{y\})$, $y \in \mathbb{R}^{n-p}$
is an integral of $\mathcal{F}$

(in some sense: $U \xrightarrow{\chi} \mathbb{R}^n$
$\mathcal{F}|_U \xrightarrow{\sim}$ the obvious one on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$)

here we are using the Lie bracket of vector fields: $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$
& the fact that $\Gamma(\mathcal{F}) \subseteq \mathfrak{X}(M)$ (since $\mathcal{F} \subseteq TM$, $\Gamma(TM) = \mathfrak{X}(M)$)

②

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$$M = \bigcup_{L \in \mathcal{P}_{\mathcal{F}}} L$$

where "leaves" can be defined/described in various ways

$$d = \mathrm{cod}\,\mathcal{F} = \dim M - \mathrm{rank}\,\mathcal{F} = n - p$$

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② ③

Ex: On $\mathbb{R}^3$, $\mathcal{F} = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$

Ex: On $\mathbb{R}^2 \setminus \{0\}$

Ex: On T^2 ≃

Ex: On S^2: look for cod 1 foliations ... as when looking for $X \in \mathfrak{X}(S^2)$ nowhere vanishing ...

Ex: On S^3: cod 1 foliations?

Yes: Reeb's foliation

$S^3 = \{(x,y,z,t) : x^2+y^2+z^2+t^2 = 1\}$, $x^2+y^2 \geq z^2+t^2 \Rightarrow M_1 \subseteq S^3$

Reminder: COMPLEX VECTOR B

... same/similar ... story ... spa... opera...

$F^*, \oplus$ et...

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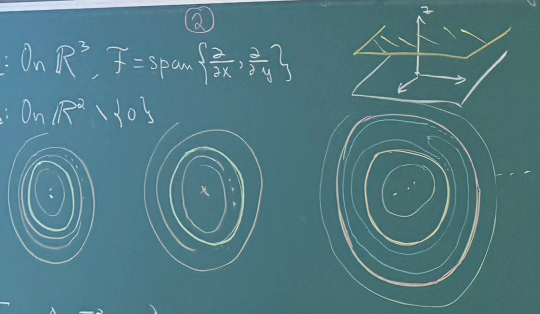
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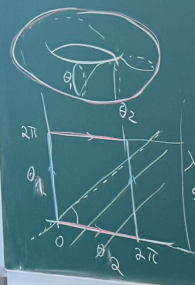
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Ex: On T^2, $\mathcal{F}^\lambda = \operatorname{span}\left\{\frac{\partial}{\partial \theta_1} + \lambda \frac{\partial}{\partial \theta_2}\right\}$, $\lambda \in \mathbb{R}$ constant.



λ gives the slope of the green lines

The leaves depend on whether $\frac{\lambda}{2\pi} \in \mathbb{Q}$ or not

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θ_1, θ_2, 2π, 0, $\frac{\pi}{2}$, 2π

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its fibers are

$E_x \otimes_{\mathbb{R}} \mathbb{C} = \{ e + f i : e, f \in E_x \}$

$\cong E_{-x} \oplus E_x$

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... same/similar ... story + same operations + one extra operation + extra care + interaction with real vector bundles

$F^*, \otimes, \mathrm{Hom}$ etc

$\mathrm{Vect}^r_{\mathbb{C}}(M) \to \mathrm{Vect}^r_{\mathbb{C}}(M)$, $F \longmapsto \overline{F}$

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Rk: $F = E_{\mathbb{C}}$ comes with a conjugation map $\tau : E_{\mathbb{C}} \longrightarrow \overline{E}_{\mathbb{C}}$, $\tau(e+fi) = e-fi$. Hence F comes with $\tau : F \to \overline{F}$ s.t. $\tau^2 = \mathrm{Id}$. (conversely, if $\tau \exists \Rightarrow \exists$ E s.t. $F \cong E_{\mathbb{C}}$)

Complex structures: I. On vector bundles (5)

Def: Given $E \in \mathrm{Vect}^r_{\mathbb{R}}(M)$, a complex structure on E is any v.b. morphism $J: E \to E$ s.t. $J \circ J = -Id_E$.

Using $J \Rightarrow$ complex mult. on E $\Rightarrow$ cplx vector bundle E_J

$$z \cdot e := xe + y\,J(e), \qquad z = x + iy$$

Ex: Start with E and look at $E \oplus E$.

Define $J: E \oplus E \longrightarrow E \oplus E$, $J(e_1, e_2) = (-e_2, e_1)$

$\Rightarrow E \oplus E$ becomes a cplx vector bundle.

$(E \oplus E)_J$

Note: this is $\cong E_{\mathbb{C}}$.

Prop: Given $E \in \mathrm{Vect}^r_{\mathbb{R}}(M)$, there is a bijection

$$\left\{\begin{array}{c}\text{complex structures} \\ J \text{ on } E\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{complex sub-bundles} \\ \mathcal{E} \subseteq E_{\mathbb{C}} \text{ such that} \\ E_{\mathbb{C}} = \mathcal{E} \oplus \overline{\mathcal{E}}\end{array}\right\}$$

($\overline{\mathcal{E}} = \tau(\mathcal{E})$)

pf/explicat: Start with J

1) complexify $J \Rightarrow J_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$, $e + if \mapsto J(e) + iJ(f)$

2) Still $J_{\mathbb{C}}^2 = -\mathrm{Id}$ ... and we are over $\mathbb{C} \Rightarrow$

$\Rightarrow E_{\mathbb{C}} = \underbrace{(\text{the } i \text{ eigenspace of } J)}_{E^{1,0}} \oplus \underbrace{(\text{the } (-i) \text{ eigenspace})}_{E^{0,1}}$

complex sub-bundles

$E^{1,0} = \{ \underbrace{\zeta}_{e+if} \in E_{\mathbb{C}} : \underbrace{J(\zeta) = i\zeta}_{J(e)+iJ(f) = -f+ie} \} = \{ e - iJ(e) : e \in E \}$

$E^{0,1} = \ldots = \ldots + \ldots$

Take $\mathcal{E} = E^{1,0}$. Note $\overline{\mathcal{E}} = E^{0,1} \Rightarrow E_{\mathbb{C}} = \mathcal{E} \oplus \overline{\mathcal{E}}$

(5)

E

ctor

E_J

1)

II on manifolds M

(7)

Def: An almost complex structure on M is a complex str. on the v. bdle TM.

$$J: TM \longrightarrow TM, \quad J^2 = -Id.$$

Ex: If M has a complex structure $\mathcal{C} \Longrightarrow$

$\Rightarrow$ gets an induce a. complex $J_{\mathcal{C}}$.

(1)

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(8)

Reminder (bachelor): Smooth structure $\mathcal{A}$ on a (top) space M:

- Look at topological charts $x : U \to \mathbb{R}^n$, $U \overset{\text{open}}{\subset} M$, $\mathbb{R}^n \overset{\text{open}}{\subset} \mathbb{R}^n$ (i.e. homeomorphisms)
- call two such charts $(U,x), (U',x')$ smoothly compatible if $C_x^{x'} := x' \circ x^{-1} : (\text{open in } \mathbb{R}^n) \to (\text{open in } \mathbb{R}^n)$, $x(U \cap U')$, is a diffeomorphism (hence smoothness required)
- Smooth atlas on M: collection $\mathcal{A}$ of topological charts, each two smoothly compatible.
- Smooth structure on M: a maximal smooth atlas.

$(M, \mathcal{A})$ smooth manifold ... M smooth manifold (...) the $(U,x) \in \mathcal{A}$ the (smooth) charts of the manif M

Reminder (bachelor): Smooth structure $\mathcal{A}$ on a (top) space M (8)

- look at topological charts $\chi : U \longrightarrow \mathbb{R}^n$, $U \subset M$ open, $\mathbb{R}^n$ open; i.e. homeomorphisms
- call two such charts $(U,\chi), (U',\chi')$ smoothly compatible if $C_\chi^{\chi'} := \chi' \circ \chi^{-1} : (\text{open in } \mathbb{R}^n) \longrightarrow (\text{open in } \mathbb{R}^n)$ is a diffeomorphism ($\chi(U \cap U')$) — hence smoothness required
- smooth atlas on M: collection $\mathcal{A}$ of topological charts, each two smoothly compatible.
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(6)

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sub-bundles

$E_{\mathbb{C}}$ such that

$\mathcal{E} \oplus \overline{\mathcal{E}}$

$\tau(\mathcal{E})$

$J(e) + iJ(f)$

$\Rightarrow$

$-i$) eigenspace)

$= 0, 1$.

$\}$

---

(9)

smooth $\longrightarrow$ holomorphic

$\mathbb{R}^m \rightsquigarrow \mathbb{C}^k$ $(m = 2k)$

$\Rightarrow$ holomorphic atlas & complex structures $\mathcal{E}$ on M

Assume M = smooth, $m = 2k$.

Try to use charts to define an a.c. structure on M.

$\chi : U \longrightarrow \mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k \cong \mathbb{C}^k$

(x, y) $\quad x = (x_1, \dots, x_k) : U \to \mathbb{R}^k$, $y = \dots$

$T_* M \xrightarrow{(d\chi)_*} T_{\chi(*)} \mathbb{C}^k$ induced by χ for $* \in U$.

J_* — complex str. on the vectn space $T_* M$ $\quad (\forall) * \in U$

$J_* \Leftarrow i$

When: $J_*^\chi = J_*^{\chi'}$ for two charts $(U,\chi), (U',\chi')$ $(\forall) * \in U \cap U'$??

Precisely when $C = \chi' \circ \chi^{-1}$ has the property $(dC)_* : T_z \mathbb{C}^k \to T_{\ast} \mathbb{C}^k$ is complex linear.

(10)

Lemma: $\mathcal{C}$: open in $\mathbb{C}^k \longrightarrow$ open $\mathbb{C}^k$ smooth. Claim:

$(dc)_*$ is $\mathbb{C}$-lin $\iff$ c = holomorphic

$\Rightarrow$: $c = (F, G)$ according $\mathbb{C}^k = \mathbb{R}^k \times \mathbb{R}^k$

$(dc)_* \neq \mathbb{C}$-lin $\therefore$ $(dc)_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}$

... the cond on F, G : $\frac{\partial}{\partial y}$ $-\frac{\partial}{\partial x}$

$$\frac{\partial F_k}{\partial x_j} = \frac{\partial G_k}{\partial y_j}, \quad \frac{\partial F_k}{\partial y_j} = -\frac{\partial G_k}{\partial x_j}$$

Cauchy-R. charact. of holomorphicity

Corollary: 1) any cplx. sth $\mathcal{C}$ induces an a.complex $J_{\mathcal{C}}$

2) given $\mathcal{C}_1, \mathcal{C}_2$ one has $\mathcal{C}_1 = \mathcal{C}_2 \iff J_{\mathcal{C}_1} = J_{\mathcal{C}_2}$

Def: An almost complex sth. J on M called **integrable** if $\exists\, \mathcal{C}$ s.t. $J = J_{\mathcal{C}}$.

(11)

Theorem (Newlander-Nirenberg):

Start with J an a.c. sth. on M and consider the corresponding

$$\mathcal{C} \subseteq T_{\mathbb{C}}M$$

Then:

J = integrable $\iff$ $\mathcal{C}$ is involutive.

$[V, W] \in \Gamma\mathcal{C}$ $\forall V, W \in \Gamma\mathcal{C}$

$\Gamma(\mathcal{C}) \subset \Gamma(T_{\mathbb{C}}M) = \{X + iY : X, Y \in \mathfrak{X}(M)\}$

$[\,,\,]$ extend the one on $\mathfrak{X}(M)$ by $\mathbb{C}$-linearity

Reminder: (1)

Theorem (Frobenius): Let $\mathcal{F}$ = a p-dimensional distribution on M. The following are equivalent:

(1) $\mathcal{F}$ = integrable

(2) around each x_0 one finds (U,x) st $\mathcal{F}_x = \mathrm{Span}\{(\frac{\partial}{\partial x_1})_x, \dots, (\frac{\partial}{\partial x_p})_x\}$ $(\forall)\, x \in U$ — foliated chart

(3) $\mathcal{F}$ is involutive, i.e. $[X,Y] \in \Gamma(\mathcal{F})$ $(\forall)\, X, Y \in \Gamma(\mathcal{F})$

Reminder (bachelor): Smooth structure $\mathcal{A}$ on a (top) space M: (2)

- look at topological charts $\chi : U \longrightarrow \mathbb{R}^n$ (i.e. homeomorphisms; U open in M, $\mathbb{R}^n$ open in $\mathbb{R}^n$)
- call two such charts $(U,\chi),(U',\chi')$ smoothly compatible if $C^{\chi'}_{\chi} := \chi' \circ \chi^{-1} : \chi(U\cap U') \longrightarrow \chi'(U\cap U')$ (open in $\mathbb{R}^n$ → open in $\mathbb{R}^n$) is a diffeomorphism (hence smoothness required)
- smooth atlas on M: collection $\mathcal{A}$ of topological charts, each two smoothly compatible
- smooth structure on M: a maximal smooth atlas

(M,d) smooth manifold ... ($\dots$) — M smooth manifold the $(U,\chi)\in\mathcal{A}$: the (smooth) charts of the manif M

Prop: Given $E \in \mathrm{Vect}^r_{\mathbb{R}}(M)$, there is a bijection (6)

$$\left\{\begin{array}{c}\text{complex structures}\\ J \text{ on } E\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{complex sub-bundles}\\ \mathcal{L} \subset E_{\mathbb{C}} \text{ such that}\\ E_{\mathbb{C}} = \mathcal{L} \oplus \overline{\mathcal{L}}\end{array}\right\}$$

($\overline{\mathcal{L}}$ = $\tau(\mathcal{L})$)

pf/explicit: Start with J

1) complexify $J \Rightarrow J_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$, $e+if \mapsto J(e)+iJ(f)$

2) Still $J_{\mathbb{C}}^2 = -\mathrm{Id}$ ... and we are over $\mathbb{C}$ $\Rightarrow$

$$E_{\mathbb{C}} = \underbrace{(\text{the } i\text{-eigenspace of } J)}_{E^{1,0}\ \text{complex sub-bundle}} \oplus \underbrace{(\text{the } (-i)\text{-eigenspace})}_{E^{0,1}}$$

$E^{1,0} = \{\, \zeta \in E_{\mathbb{C}} : J(\zeta) = i\zeta \,\} = \{\, e - iJ(e) : e \in E \,\}$ ($\zeta = e+if$; $J(e)+iJ(f) = -f+ie$)

$E^{0,1} = \dots = \{\, e + iJ(e) : e \in E \,\}$

Take $\mathcal{L} = E^{1,0}$. Note: $\overline{\mathcal{L}} = E^{0,1} \Rightarrow E_{\mathbb{C}} = \mathcal{L} \oplus \overline{\mathcal{L}}$.

smooth $\mathbb{R}^m \rightsquigarrow \mathbb{C}^k$ ($m = 2k$) $\Rightarrow$ holomorphic atlas & complex structures on M (9)

Assume M = smooth, $m = 2k$.

Try to use charts to define an a.c. structure on M.

$\chi : U \longrightarrow \mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k \cong \mathbb{C}^k$, $(x,y) \mapsto \chi = (\chi_1, \dots, \chi_k) : U \to \mathbb{R}^k$, $y = \dots$

$(d\chi)_x : T_xM \xrightarrow{\sim} T_{\chi(x)}\mathbb{C}^k$ induced by χ for $x \in U$.

J_x = complex sti on the vectn space T_xM $(\forall)\, x \in U$

$\Leftarrow (J_i)$

When: $J^{\chi}_x = J^{\chi'}_x$ for two charts $(U,\chi),(U',\chi')$ $(\forall)\, x \in U \cap U'$??

Precisely when $C = \chi' \circ \chi^{-1}$ has the property $(dC)_z : T_z\mathbb{C}^k \to T_{\tilde{z}}\mathbb{C}^k$ is complex linear.

Complex structures: I. On vector bundles (5)

Def: Given $E \in \mathrm{Vect}^r_{\mathbb{R}}(M)$, a complex structure on E is any v.b. morphism $J : E \to E$ s.t. $J \circ J = -\mathrm{Id}_E$

Using $J \Rightarrow$ complex mult. on E $\Rightarrow$ cplx vector bundle E_J

$\underset{x+iy}{z} \cdot e := xe + yJ(e)$

Ex: Start with E and look at $E \oplus E$

Define $J : E\oplus E \to E\oplus E$, $J(e_1,e_2) = (-e_2, e_1)$

$\Rightarrow E\oplus E$ becomes a cplx vector bundle.

$(E\oplus E)_J$

Note: this is $\cong E_{\mathbb{C}}$.

II. On manifolds M (7)

Def: An almost complex structure on M is a complex sti. on the v. bdle TM.

$J : TM \to TM$, $J^2 = -\mathrm{Id}$.

Ex: If M has a complex structure $\mathcal{C} \Rightarrow$ $\Rightarrow$ gets an induce a. complex $J_{\mathcal{C}}$.

Lemma: f : open in $\mathbb{C}^k \to$ open in $\mathbb{C}^k$ smooth. Claim (10)

$(dC)_z$ is $\mathbb{C}$-linear $\Longleftrightarrow$ C = holomorphic

pf: $C = (F, G)$ according $\mathbb{C}^k = \mathbb{R}^k \times \mathbb{R}^k$

$(dC)_z \in \mathbb{C}$-linear ... $(dC)_z\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}$

... the cond on F, G: $\frac{\partial}{\partial y}$ $-\frac{\partial}{\partial x}$

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Def: An almost complex sti. J on M called integrable if $\exists\, \mathcal{C}$ s.t. $J = J_{\mathcal{C}}$.

Theorem (Newlander–Nirenberg): (11)

Start with J an a.c. sti. on M, and consider the correspondence $\mathcal{C} \subset T_{\mathbb{C}}M$

Then:

J = integrable $\Longleftrightarrow$ $\mathcal{C}$ is involutive.

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