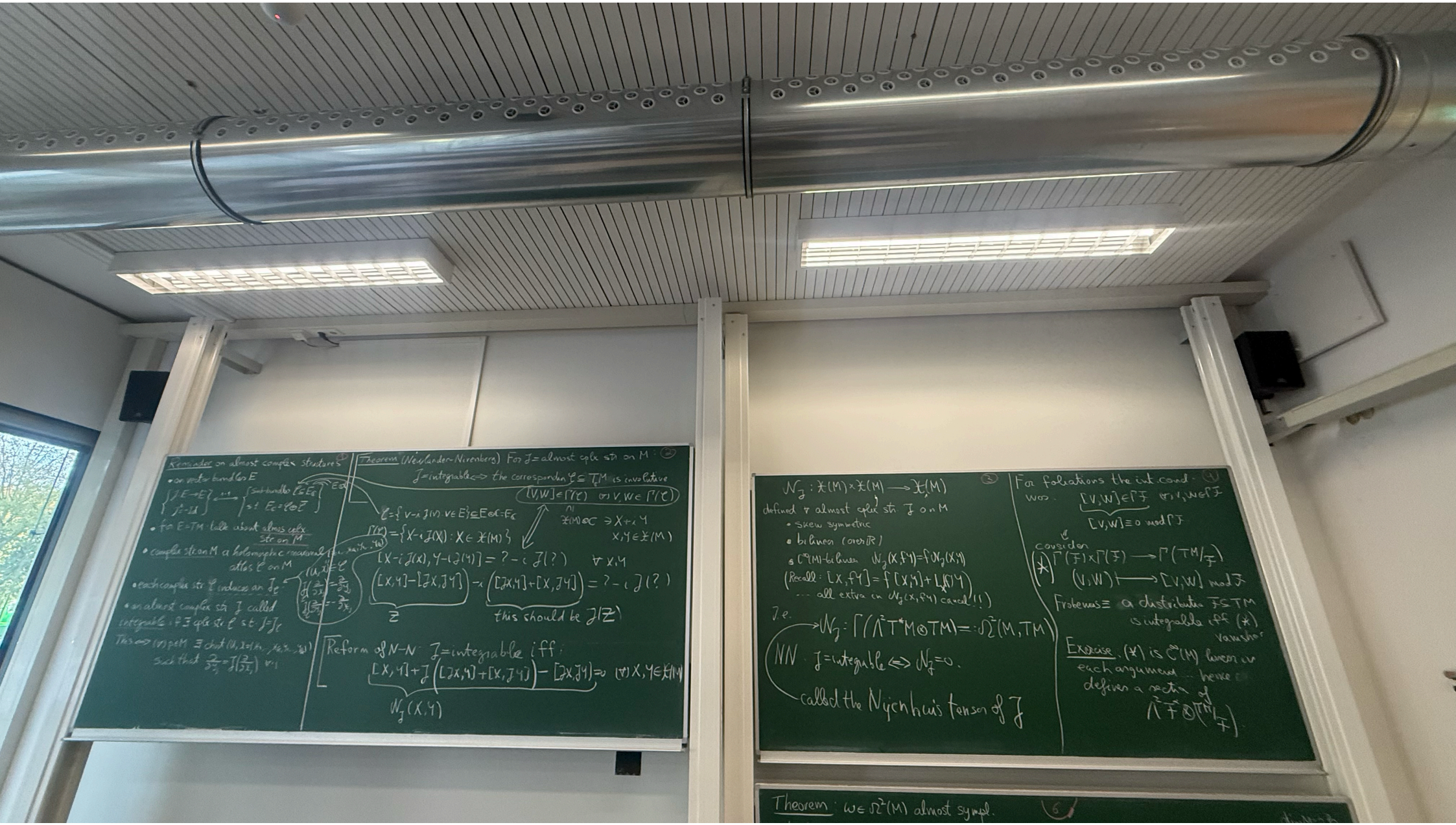

Reminder on almost complex structures
Theorem (Newlander-Nirenberg)
Reform of N-N
Theorem
For foliations the int. cond.
Exercise
Frobenius
this should be
called the Nijenhuis tensor of J

Reminder on almost complex structures ①

- on vector bundles E

$$\left\{ J: E \to E \;\middle|\; J^2 = -\mathrm{Id} \right\} \xleftrightarrow{1\text{-}1} \left\{ \text{sub-bundles } \mathcal{C} \subseteq E_{\mathbb{C}} \;\middle|\; \text{s.t. } E_{\mathbb{C}} = \mathcal{C} \oplus \overline{\mathcal{C}} \right\}$$

$E \otimes \mathbb{C} = E_{\mathbb{C}}$

$\mathcal{C} = \{ v - iJ(v) : v \in E \} \subseteq E \otimes \mathbb{C} = E_{\mathbb{C}}$

- for $E = TM$: talk about almost cplx. str. on M.
- complex str. on M: a holomorphic maximal atlas $\mathcal{C}$ on M
- each complex str. $\mathcal{C}$ induces an $J_{\mathcal{C}}$:

$(U, x_1, \dots, x_k, y_1, \dots, y_k) \in \mathcal{C}$

$J_{\mathcal{C}}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$

$J_{\mathcal{C}}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$

- an almost complex str. J called integrable if $\exists$ cplx. str. $\mathcal{C}$ s.t. $J = J_{\mathcal{C}}$.

This $\Leftrightarrow$ $(\forall)\, p \in M$ $\exists$ chart $(U, x = (x_1, \dots, x_k, y_1, \dots, y_k))$ such that $\frac{\partial}{\partial y_j} = J\left(\frac{\partial}{\partial x_j}\right)$ $\forall i, j$.

Theorem (Newlander-Nirenberg) For J = almost cplx str. on M: ②

J = integrable $\Leftrightarrow$ the corresponding $\mathcal{C} \subseteq T_{\mathbb{C}}M$ is involutive:

$$[V, W] \in \Gamma(\mathcal{C}) \quad (\forall)\, V, W \in \Gamma(\mathcal{C})$$

$\Gamma(\mathcal{C}) = \{ X - iJ(X) : X \in \mathfrak{X}(M) \}$ $\quad \mathfrak{X}(M) \otimes \mathbb{C} \ni X + iY$, $X, Y \in \mathfrak{X}(M)$

$[X - iJ(X), Y - iJ(Y)] = ? - iJ(?)$ $\quad \forall X, Y$

$$\underbrace{\left([X,Y] - [JX, JY]\right)}_{Z} - i\underbrace{\left([JX, Y] + [X, JY]\right)}_{\text{this should be } JZ} = ? - iJ(?)$$

Reform of N–N: J = integrable iff:

$$\underbrace{[X,Y] + J\left([JX,Y] + [X,JY]\right) - [JX,JY]}_{W_J(X,Y)} = 0 \quad (\forall)\, X, Y \in \mathfrak{X}(M)$$

$N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ ③

defined ∀ almost cplx sth. J on M

- skew symmetric
- bilinear (over $\mathbb{R}$)
- $C^\infty(M)$-bilinear: $N_J(X, fY) = f N_J(X,Y)$

(Recall: $[X, fY] = f[X,Y] + L_X(f)Y$)

… all extra in $N_J(X, fY)$ cancel!!).

I.e.

$$N_J : \Gamma(\Lambda^2 T^*M \otimes TM) =: \Omega^2(M, TM)$$

NN: J = integrable $\Leftrightarrow N_J = 0$.

called the Nijenhuis tensor of J

For foliations the int. cond. ④

was: $\underbrace{[V,W] \in \Gamma\mathcal{F}}$ $\forall V, W \in \Gamma\mathcal{F}$

$[V,W] \equiv 0 \mod \Gamma\mathcal{F}$.

consider

(*) $\Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \longrightarrow \Gamma(TM/\mathcal{F})$

$(V,W) \longmapsto [V,W] \mod \mathcal{F}$

Frobenius ≡ a distribution $\mathcal{F} \subseteq TM$ is integrable iff (*) vanishes.

Exercise: (*) is $C^\infty(M)$ linear in each argument … hence defines a section of

$\Lambda^2 \mathcal{F}^* \otimes (TM/\mathcal{F})$.

Structure $\mathcal{S}$ on vector bdles E	frames of E adapted to $\mathcal{S}$	when $E=TM$ it is called	Charts (U,x) adapted to $\mathcal{S}$ (5) ($\Leftrightarrow \frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}$ adapted frame)	NAME of Integrability of $\mathcal{S}$ holds ($\exists$ adapted charts)	Criteria for integrability?
global frame of E $\mathcal{S}=\{e_1,\dots,e_r\}$	e itself	a parallelism of M $\{X^1,\dots,X^m\}$	means $X^j=\frac{\partial}{\partial x_j}$ $\forall j$ ($\Leftrightarrow X^j$ the x^* of the standard $\frac{\partial}{\partial x_i}$)	flat parallelism	Yes: $\Leftrightarrow [X^j,X^k]=0$ $(\forall j,k$
inner product g on E	orthonormal frames	Riemannian metric on M	means $g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})=1$ $\forall i,j$	flat metrics	. . .
rank p sub-bundles $F\subseteq E$	those $e=\{e_1,\dots,e_m\}$ s.t. $\{e_1,\dots,e_p\}$ of F	p-dimensional distribution on M $\mathcal{F}\subseteq TM$	means: pointwise, $\mathcal{F}$ spanned by $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_p}\}$ ($\Leftrightarrow (U,\mathcal{F}_U)\simeq(\mathbb{R}^p\times\mathbb{R}^{m-p},\mathcal{F}_{flat})$)	foliation	Yes: Frobenius $[V,W]\in\Gamma\mathcal{F}$ $\forall V,W\in\Gamma\mathcal{F}$
complex structures on E, $J:E\to E$	of type $\{e_1,\dots,e_k,Je_1,\dots,Je_k\}$	almost complex structure on M $J:TM\to TM$	$x=(x_1,\dots,x_k,y_1,\dots,y_k)$ s.t. $\frac{\partial}{\partial y_j}=J(\frac{\partial}{\partial x_j})$ $(\forall j$ ($\Leftrightarrow (U,J_U)\xrightarrow[x]{\sim}(\mathbb{C}^k,J_{can})$)	complex structure on M	Yes: Newlander–Nirenberg
Symplectic structure on E, $\omega\in\Gamma(\Lambda^2E^*)$ ($\omega_p:E_p\times E_p\to\mathbb{R}$ skew, bilinear, nondeg)		almost symplectic structure on M $\omega\in\Omega^2(M)$	$x=(x_1,\dots,x_k,y_1,\dots,y_k)$ s.t. $\omega\vert_U=\sum_j dx_j\wedge dy_j$ $\mathbb{R}^4_+$	Symplectic str. on M	Yes: Darboux

Structure J on vector bdles E	frames of E adapted to J	when $E=TM$ it is called	Charts (U,x) adapted to J ($\Leftrightarrow \frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}$ adapted frame)	Integrable $\Leftrightarrow$ all p-hds $\exists$ adapted charts	Criteria for integrability?
global frame $\underline{e}$ of E ($=\{e_1,\dots,e_r\}$)	$\underline{e}$ itself	a parallelism of M $\{X^1,\dots,X^m\}$	means $X^i=\frac{\partial}{\partial x^i}$ $\forall i$ $\Leftrightarrow X^i$ the x^k of the standard $\frac{\partial}{\partial x^k}$	flat parallelism	Yes: $\Leftrightarrow [X_i,X_k]=0$ $\forall i,k$
inner product g on E	orthonormal frames	Riemannian metric on M	means $g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})=1$ $\forall i,j$	flat metrics	...
rank p sub-bundles $F\subseteq E$	those $\underline{e}=\{e_1,\dots,e_m\}$ s.t. $\{e_1,\dots,e_p\}$ of F	p-dimensional distribution on M $\mathcal{F}\subseteq TM$	means pointwise, $\mathcal{F}$ spanned by $\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^p}\}$ ($\Leftrightarrow (U,\vec{x}_U)\simeq(\mathbb{R}^p\times\mathbb{R}^{n-p},\mathcal{F}_{flat})$)	foliation	Yes: Frobenius $[V,W]\in\Gamma\mathcal{F}$ $\forall V,W\in\Gamma\mathcal{F}$
complex structure on E, $J:E\to E$	of type $\{e_1,\dots,e_n,Je_1,\dots,Je_n\}$	almost complex structure on M $J:TM\to TM$	$x=(x_1,\dots,x_n,y_1,\dots,y_n)$ s.t. $\frac{\partial}{\partial y_j}=J\left(\frac{\partial}{\partial x_j}\right)$ $\forall j$ ($\Leftrightarrow (U,\vec{x}_U)\xrightarrow{\sim}(\mathbb{C}^n,J_{can})$)	complex structure on M	Yes: Newlander–Nirenberg
Symplectic structure on E, $\omega\in\Lambda^2E^*$ ($\omega_p: E_p\times E_p\to\mathbb{R}$ skew, bilinear, wedge)		almost symplectic structure on M $\omega\in\Omega^2(M)$	$x=(x_1,\dots,x_n,y_1,\dots,y_n)$ s.t. $\omega\vert_U=\sum_i dx_i\wedge dy_i$ $\mathbb{R}^{2n}_+$	Symplectic str. on M	Yes: Darboux

Theorem: $\omega\in\Omega^2(M)$ almost sympl. Structure. Then ω = integrable (i.e., locally of type $\sum_i dx_i\wedge dy_i$) $\Leftrightarrow$ serious implication

iff $d\omega=0$

$-\omega([X,Y],Z)+\omega([X,Z],Y)-\omega([Y,Z],X)+$

$+L_X(\omega(Y,Z))-L_Y(\omega(Z,X))+L_Z(\omega(X,Y))=0$

$-(\forall X,Y,Z\in\mathfrak{X}(M)$

$D_\omega(X,Y,Z)$

$D_\omega:\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\longrightarrow C^\infty(M)$

skew, $C^\infty(M)$-multilinear

i.e. we have a 3-form

$D_\omega\in\Gamma(\Lambda^3T^*M)=\Omega^3(M)$.

This is simply $d\omega$!

$\omega=d\eta\in\Omega^1(M)$

$\dim M=2n$, M cpct

alm. sympl. $\Rightarrow$ $\mu_\omega:=\frac{1}{n!}\omega^n\in\Omega^{2n}(M)$ a volume form

$\mu_\omega=d\left(\frac{1}{n!}\eta\wedge(d\eta)^{n-1}\right)$

$\Downarrow$ Stokes

$\int_M\mu_\omega=0$

$\mathrm{Vol}(M,\mu_\omega)>0$

...tor on almost complex structures

...t bundles E

Theorem (Newlander-Nirenberg) For J = almost cplx str on M:

J = integrable $\Rightarrow$ the corre...

Theorem: $\omega \in \Omega^2(M)$ almost sympl. Structure. Then ω = integrable (i.e, locally of type $\sum_i dx_i \wedge dy_i$) iff: $d\omega = 0$

serious implication

$-\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) +$

$+ L_X(\omega(Y,Z)) - L_Y(\omega(Z,X)) + L_Z(\omega(X,Y)) = 0$

$-(\forall) X,Y,Z \in \mathfrak{X}(M)$

$D_\omega(X,Y,Z)$

$D_\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$

skew, $C^\infty(M)$-multilinear

i.e. we have a 3-form

$D_\omega \in \Gamma(\Lambda^3 T^*M) = \Omega^3(M)$.

This is simply $d\omega$!

$\omega = d\eta \in \Omega^2(M)$

alm. sympl $\Leftrightarrow$ $\mu_\omega := \frac{1}{k!}\omega^k \in \Omega^{2k}(M)$ a volume form

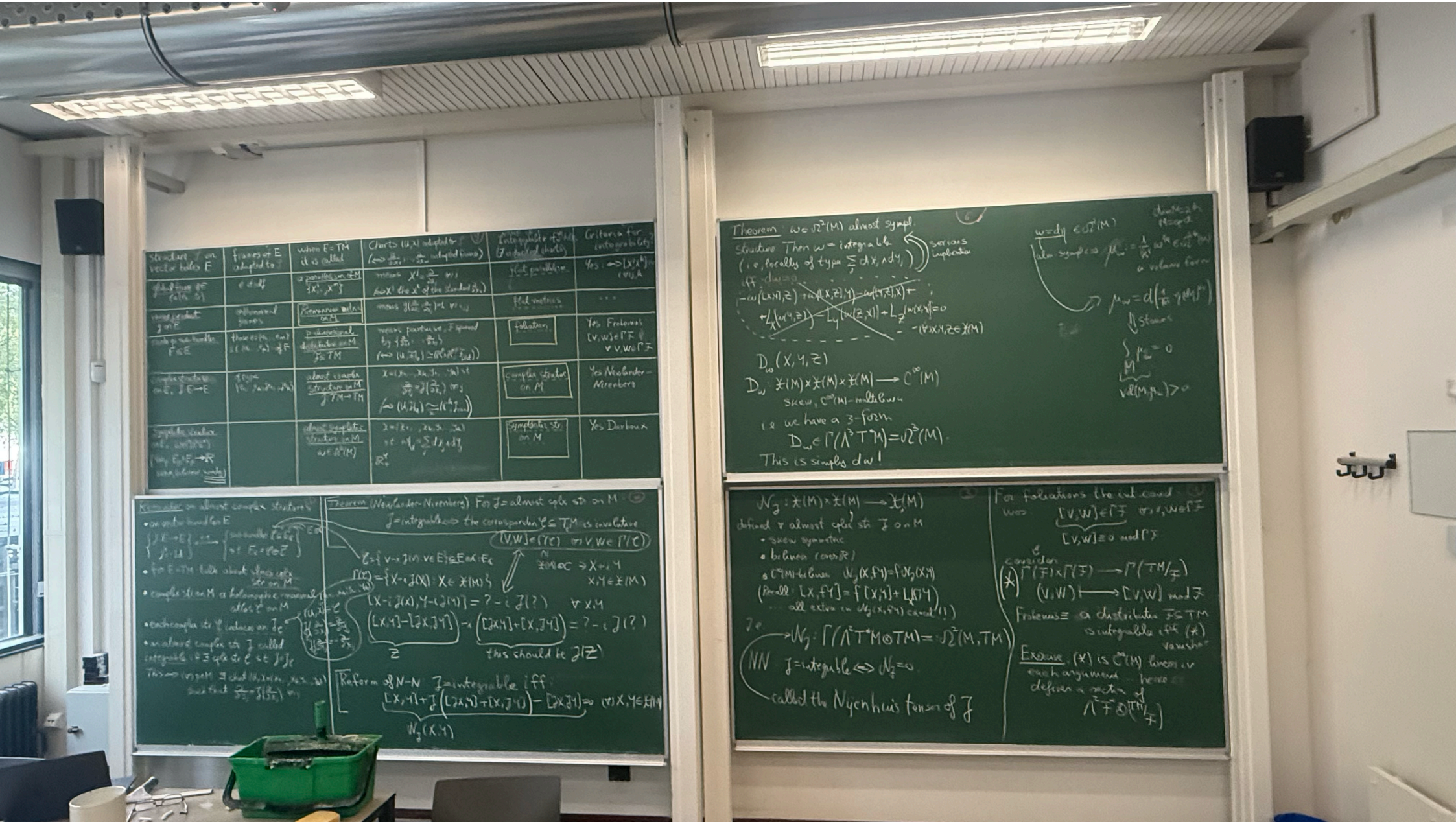
$\dim M = 2k$, M opt.

$\mu_\omega = d\left(\frac{1}{k!}\eta(d\eta)^{k-1}\right)$

Stokes

$\int_M \mu_\omega = 0$

$\mathrm{vol}(M, \mu_\omega) > 0$.


Structure ... on vector bdles E
frames of E adapted to ...
when E = TM it is called
Charts (U,x) adapted to ...
Integrability of ... ? (Frobenius)
Criteria for integrability
Riemannian metric on M
foliations
Yes: Frobenius
Yes: Newlander–Nirenberg
almost complex structure on M
complex structure on M
almost symplectic structure on M
symplectic str. on M
Yes: Darboux
Remarks on almost complex structures
Theorem (Newlander–Nirenberg) For J = almost cplx str on M:
J = integrable ⇔ the corresponding ... is involutive
[V,W] ∈ Γ(C) ∀ V,W ∈ Γ(C)
Reform of N-N
Theorem ω ∈ Ω²(M) almost sympl structure. Then ω = integrable bla
series implication
D_ω(X,Y,Z)
skew, C^∞(M)-multilinear
i.e. we have a 3-form
D_ω ∈ Γ(Λ³T*M) = Ω³(M).
This is simply dω!
N_J : X(M) × X(M) → X(M)
defined ∀ almost cplx str J on M
• skew-symmetric
• bilinear (over R)
• C^∞(M)-bilinear
called the Nijenhuis tensor of J
For foliations the integrability condition
[V,W] ∈ Γ(F) ∀ V,W ∈ Γ(F)
Frobenius ⇔ the distribution F ⊆ TM is integrable
Exercise

6

$\sum_{j=1}^{k} dx_j \wedge dy_j$

7 $\mathbb{R}^{2k}$ $x_1, \dots, x_k, y_1, \dots, y_k$

Ex: M k-dim $\Rightarrow$ T^*M (as a manifold) carries a canonical Sympl. structure $\omega_{can} \in \Omega^2(T^*M)$

Locally: if $(U, x=(x_1, \dots, x_m))$ chart of M consider the induced chart on T^*M

(T^*U, x, y^x)

ψ

$(p, \xi) \in T_p^*M$ $\xrightarrow{x}$ $x(p)$

$\xrightarrow{y^x}$ $\xi\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)$

In these coord:

$\omega_{can}|_{T^*U} = \sum dx_j \wedge dy^x_j$

Actually: ω_{can} can be described without using coordinates. See notes.

each argument ... hence defines a section of

$\Lambda^2 \mathcal{F}^* \otimes \left(TM/\mathcal{F}\right)$

Structure $\mathcal{S}$ on vector bdles E	frames of E adapted to $\mathcal{S}$	when $E=$ … it is called	$(\Leftrightarrow \frac{\partial}{\partial x_i}$
		a parallelism of M $\{X^1,\dots,X^m\}$	means X
global frame of E $e=\{e_1,\dots,e_r\}$	e itself		$(\Leftrightarrow X^j$ the x
inner product g on E	orthonormal frames	Riemannian metric on M	means $g(\frac{\partial}{\partial x}$
rank p sub-bundles $F\subseteq E$	those $e=\{e_1,\dots,e_m\}$ s.t. $\{e_1,\dots,e_p\}$ of F	p-dimensional distribution on M $\mathcal{F}\subseteq TM$	means: po by $\{\frac{\partial}{\partial x_i}$ $(\Leftrightarrow (U,\mathcal{F}$
complex structures on E, $J:E\to E$	of type $\{e_1,\dots,e_k,Je_1,\dots,Je_k\}$	almost complex structure on M $J:TM\to TM$	$X=(X_1,\dots,$ $\frac{\partial}{\partial y_i}=$ $(\Leftrightarrow (U,JI$
Symplectic structure on E, $\omega\in\Gamma(\Lambda^2E^*)$		almost symplectic structure on M	$X=(X_1,\dots$ s.t. $\omega\vert=$

(8)

- inner product g on $E\in Vect_{\mathbb{R}}(M)$
  $g_*: E_*\times E_*\to\mathbb{R}$ symmetric, bil, positive def
- symplectic structure ω on $E\in Vect_{\mathbb{R}}(M)$
  $\omega_*: E_*\times E_*\to\mathbb{R}$ skew, bil, nondegenerate
- hermitian product h on $F\in Vect_{\mathbb{C}}(M)$
  $h_*: F_*\times F_*\to\mathbb{C}$, conjugate symmetric, positive def
  $\mathbb{C}$-linear in 1st $h(v,u)=\overline{h(u,v)}$, $h(u,u)\in\mathbb{R}$
- hermitian structure on $E\in Vect_{\mathbb{R}}(M)$:
  (J,h) with $J=$ cplex str on E, h Hermitian

Interaction: $h=$ hermitian, write $h=g+i\omega$.

$\Rightarrow$ 1) $h=$ conj. symmetric $\Leftrightarrow$ $g=$ symmetric & $\omega=$ skew

2) $h=\mathbb{R}$-bilinear $\Leftrightarrow$ g & ω are bilinear

3) $h=\mathbb{C}$-linear in 1st $\Leftrightarrow$ $\omega(u,v)=-g(Ju,v)$
   $(\Leftrightarrow g(u,v)=\omega(Ju,v))$

4) $h=$ positive def. $\Leftrightarrow$ $g=$ positive def. $\Rightarrow\omega=$ nondegenerate

$F \subseteq E$ | s.t. $\{e_1, \dots, e_p\}$ of F | distribution on M $\mathcal{F} \subseteq TM$ | By $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}\}$ $(\Leftrightarrow (U, \mathcal{F}|_U) \simeq (\mathbb{R}^p \times \mathbb{R}^{n-p}, \mathcal{F}_{flat}))$ | | $\forall V, W \in \Gamma \mathcal{F}$

complex structures on E; $J: E \to E$ | of type $\{e_1, \dots, e_k, Je_1, \dots, Je_k\}$ | almost complex structure on M | $x = (x_1, \dots, x_k, y_1, \dots, y_k)$ s.t. $\frac{\partial}{\partial y} = J(\frac{\partial}{\partial x})$ $(\forall$ | complex structure on M | Yes: Newlander–

(9)

Differential k-forms on M $\quad \Omega^k(M) = \Gamma(\Lambda^k T^*M)$

$\omega \in \Omega^k(M)$ means
- for each $p \in M$ we have $\omega_p : \underbrace{T_pM \times \dots \times T_pM}_{k} \to \mathbb{R}$ skew, $\mathbb{R}$-multilin
- more globally: $\omega : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \to C^\infty(M)$ skew, $C^\infty(M)$-multilin

$\omega(X^1, \dots, X^k)(p) := = \omega_p(X^1_p, \dots, X^k_p) \in \mathbb{R}$

Locally: $\omega \in \Omega^k(M)$ can be expressed in local charts $x : U \to \Omega$ ($U \subseteq M$, $\Omega \subseteq \mathbb{R}^m$)

as

$$\omega = \sum_{\substack{i_1 < \dots < i_k \\ i_1, \dots, i_k \in \{1, 2, \dots, m\}}} F_{i_1, \dots, i_k} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$(\omega_p : E_p \times E_p \to \mathbb{R}$ skew, bilinear, nondeg)

$\omega \in \Omega^2(M)$

$\mathbb{R}^4_+$

$dx\,dy + \dots$

Differential k-forms on M, E = v.bdle $\quad \Omega^k(M;E) = \Gamma(\Lambda^k T^*M \otimes E)$

10

$C^\infty(M) = \Gamma(\underline{\mathbb{R}}_M)$ trivial line bdle over M

$\omega \in \Omega^k(M;E)$ means

- for each $p \in M$ we have $\omega_p : \underbrace{T_pM \times \dots \times T_pM}_{k} \to E_p$ skew, $\mathbb{R}$-multilin
- more globally: $\omega : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \to \Gamma(E)$ skew, $C^\infty(M)$-multilin

$\omega(X^1, \dots, X^k)(p) := \omega_p(X^1_p, \dots, X^k_p) \in \mathbb{R}$

Cheap examples: $\eta \in \Omega^k(M)$, $s \in \Gamma(E) \Rightarrow \eta \otimes s \in \Omega^k(M,E)$

Interesting examples: W_2 (for $E = TM$), curvature etc etc

$\eta \otimes s$ applied to $X^1, \dots, X^k$ is $\eta(X^1, \dots, X^k) \cdot s$

Locally: $\omega \in \Omega^k(M;E)$ can be expressed in local charts $x : U \to \mathbb{R}^n$ ($U \subseteq M$)

as

$$\omega = \sum_{\substack{i_1 < \dots < i_k \\ i_1, \dots, i_k \in \{1,2,\dots,n\}}} F_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \otimes s_{i_1,\dots,i_k}$$

one can further use a frame $\{e_1, \dots, e_r\}$ of E

$$\sum F^j_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \otimes e_j$$

Structure $\mathcal{J}$ on vector bdles E	frames of E adapted to $\mathcal{J}$	when $E=TM$ it is called	Charts (U,χ) adapted to $\mathcal{J}$ (5) ($\Leftrightarrow \frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}$ adapted frame)	NAME if Integrability of $\mathcal{J}$ holds ($\exists$ adapted charts)	Criteria for integrability?
global frame $\mathcal{J}E$ $\mathcal{E}=\{e_1,\dots,e_r\}$	e itself	a parallelism of M $\{X^1,\dots,X^m\}$	means $X^i=\frac{\partial}{\partial x_i}$ $\forall i,j$ ($\Leftrightarrow X^i$ the x^k of the standard $\frac{\partial}{\partial x_i}$)	flat parallelism	Yes: $\Leftrightarrow [X^i,X^k]=0$ $(\forall i,j,k$
inner product g on E	orthonormal frames	Riemannian metric on M	means $g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})=\delta_{ij}$ $\forall i,j$	flat metrics	...
rank p sub-bundles $F\subseteq E$	those $e=\{e_1,\dots,e_m\}$ s.t. $\{e_1,\dots,e_p\}$ - of F	p-dimensional distribution on M $\mathcal{F}\subseteq TM$	means pointwise, $\mathcal{F}$ spanned by $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_p}\}$ ($\Leftrightarrow (U,\mathcal{F}_U)\simeq(\mathbb{R}^p\times\mathbb{R}^{m-p},\mathcal{F}_{flat})$)	foliation	Yes: Frobenius $[V,W]\in\Gamma\mathcal{F}$ if $\forall V,W\in\Gamma\mathcal{F}$
complex structures on E, $\mathcal{J}:E\to E$	of type $\{e_1,\dots,e_n,\mathcal{J}e_1,\dots,\mathcal{J}e_n\}$	almost complex structure on M $\mathcal{J}:TM\to TM$	$\chi=(x_1,\dots,x_n,y_1,\dots,y_n)$ s.t. $\frac{\partial}{\partial y_j}=\mathcal{J}(\frac{\partial}{\partial x_j})$ $(\forall j$ ($\Leftrightarrow (U,\mathcal{J}_U)\xrightarrow{\chi}(\mathbb{C}^n,\mathcal{J}_{can})$)	complex structure on M	Yes: Newlander-Nirenberg
Symplectic structure on E, $\omega\in\Gamma(\Lambda^2E^*)$ ($\omega_p:E_p\times E_p\to\mathbb{R}$ skew, bilinear, nondeg)		almost symplectic structure on M $\omega\in\Omega^2(M)$	$\chi=(x_1,\dots,x_n,y_1,\dots,y_n)$ s.t. $\omega\vert_U=\sum_i dx_i\wedge dy_i$ $\mathbb{R}^{2n}_+$	symplectic str. on M	Yes: Darboux

Differential k-forms on M, $E=$ v.bdle $\Omega^k(M,E)=\Gamma(\Lambda^kT^*M\otimes E)$ (10)

$\omega\in\Omega^k(M,E)$ means: for each $p\in M$ we have $\omega_p:\underbrace{T_pM\times\dots\times T_pM}_{k}\to E_p$ skew, $\mathbb{R}$-multilinear

more globally: $\omega:\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)\to\Gamma(E)$ skew, $C^\infty(M)$-multilinear ($\omega(X^1,\dots,X^k)(p):=\omega_p(X^1_p,\dots,X^k_p)\in\mathbb{R}$)

Cheap examples: $\eta\in\Omega^k(M)$, $s\in\Gamma(E)\Rightarrow\eta\otimes s\in\Omega^k(M,E)$ (η applied to $X^1,\dots,X^k$ is $\eta(X^1,\dots,X^k)\cdot s$)

Interesting examples: ∇_g (for $E=TM$), curvature etc etc

Locally: $\omega\in\Omega^k(M,E)$ can be expressed in local charts $\chi:U\to\tilde{U}\subseteq\mathbb{R}^m$ as

$$\omega=\sum_{i_1<\dots<i_k} F_{i_1,\dots,i_k}\,dx_{i_1}\wedge\dots\wedge dx_{i_k}\otimes s_{i_1,\dots,i_k}$$

one can further use a frame $\{e_1,\dots,e_r\}$ of E

$$\sum F^j_{i_1,\dots,i_k}\,dx_{i_1}\wedge\dots\wedge dx_{i_k}\otimes e_j$$

$C^\infty(M)=\Gamma(\mathbb{R}_M)$ trivial line bdle over M

(3)

$N_{\mathcal{J}}:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ defined $\forall$ almost cplx str. $\mathcal{J}$ on M

- skew symmetric
- bilinear (over $\mathbb{R}$)
- $C^\infty(M)$-bilinear $N_{\mathcal{J}}(X,fY)=fN_{\mathcal{J}}(X,Y)$

(Recall: $[X,fY]=f[X,Y]+L_X(f)Y$ ... all extra cases $N_{\mathcal{J}}(X,fY)$ cancel!!)

i.e. $N_{\mathcal{J}}:\Gamma(\Lambda^2T^*M\otimes TM)=:\Omega^2(M,TM)$

NN: $\mathcal{J}$ integrable $\Leftrightarrow N_{\mathcal{J}}=0$.

called the Nijenhuis tensor of $\mathcal{J}$

(4)

For foliations the int. cond: wos: $\underbrace{[V,W]\in\Gamma\mathcal{F}}$ $\forall V,W\in\Gamma\mathcal{F}$

$[V,W]\equiv 0 \bmod\Gamma\mathcal{F}$.

consider (*) $\Gamma(\mathcal{F})\times\Gamma(\mathcal{F})\to\Gamma(TM/\mathcal{F})$, $(V,W)\mapsto[V,W]$ mod $\mathcal{F}$

Frobenius $\equiv$ a distribution $\mathcal{F}\subseteq TM$ is integrable iff (*) vanishes

Exercise: (*) is $C^\infty(M)$ linear in each argument ... hence defines a section of $\Lambda^2\mathcal{F}^*\otimes(TM/\mathcal{F})$.

(6)

Theorem: $\omega\in\Omega^2(M)$ almost sympl. Structure. Then $\omega=$ integrable (i.e. locally of type $\sum dx_i\wedge dy_i$) serious implication iff: $d\omega=0$

$-\omega([X,Y],Z)+\omega([X,Z],Y)-\omega([Y,Z],X)+$

$+L_X(\omega(Y,Z))-L_Y(\omega(Z,X))+L_Z(\omega(X,Y))=0$

$-(\forall X,Y,Z\in\mathfrak{X}(M)$

$D_\omega(X,Y,Z)$

$D_\omega:\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$ skew, $C^\infty(M)$-multilinear

i.e. we have a 3-form $D_\omega\in\Gamma(\Lambda^3T^*M)=\Omega^3(M)$.

This is simply $d\omega$!

(7)

$\sum_{i=1}^{k} dx_i\wedge dy_i$ $\mathbb{R}^{2k}\ni(x_1,\dots,x_k,y_1,\dots,y_k)$

Ex: M k-dim $\Rightarrow T^*M$ (as a manifold) carries a canonical sympl. structure $\omega_{can}\in\Omega^2(T^*M)$

Locally: if $(U,\chi=(x_1,\dots,x_k))$ chart of M consider the induced chart on T^*M (T^*U,x,y^x)

$(p,\xi)\in T^*M$, $x(p,\xi)=x(p)$, $y^x(p,\xi)=\xi(\frac{\partial}{\partial x_i}|_p)$

In these coord: $\omega_{can}|_{T^*U}=\sum dx_i\wedge dy^x_i$.

Actually: ω_{can} can be described without using coordinates. See notes.