CONNECTIONS ①

Def: Connection ∇ on E: $\mathbb{R}$-bilinear map

$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$, $(X, s) \longmapsto \nabla_X(s)$

s.t. $\nabla_{fX}(s) = f \nabla_X(s)$, $\nabla_X(fs) = f \nabla_X(s) + L_X(f)\, s$

$(\forall)\, X \in \mathfrak{X}(M), (\forall)\, s \in \Gamma(E), (\forall)\, f \in C^\infty(M)$ $\quad (fg)' = f'g + fg'$

Thm: on any E one can find a ∇ (... but nothing canonical & there ∃ many)

Ex 1: $E = M \times \mathbb{R}$, ∇^{can} given by $\nabla^{can}_X(g) = L_X(g)$

Ex 2: $E = M \times \mathbb{R}$, $\theta \in \Omega^1(M) \Rightarrow$ a conn. ∇^θ by: $\nabla^\theta_X(g) = L_X(g) + g\,\theta(X)$

Ex 3: $E = M \times \mathbb{R}^r$, $\{e_1^{can}, \dots, e_r^{can}\}$ by Leibniz enough to know $\nabla_X(e_j^{can}) = \sum_i \omega_j^i(X)\, e_i^{can}$

$\Rightarrow$ bijection $\left\{\begin{matrix}\text{conn. } \nabla \\ \text{on} \\ M \times \mathbb{R}^r\end{matrix}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{matrix}\text{matrices} \\ \omega = (\omega_j^i)_{1 \le i,j \le r}\end{matrix}\right\}$ with $\omega_j^i \in \Omega^1(M)$

Locality: Given ∇, ① $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, $p \in M$

$\nabla_X(s)(p)$ does not depend on X on the entire M but only on X_p; not on s but only on s around p (some neigh. of p)

① $\nabla_{X_p}(s) \in E_p$ makes sense $(\forall)\, X_p \in T_pM$, $(\forall)\, s \in \Gamma(E)$

② $(\forall)\, U \subseteq M$ open $\Rightarrow$ get a connection on $E|_U$

$\nabla^U = \nabla|_U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$

$\nabla_X(fg) \overset{①}{=} L_X(f)\, g + f\, \nabla_X(g)$

Dual viewpoint ②

$X \in \mathfrak{X}(M)$, $s \in \Gamma E$, $\nabla_X(s)$

if s fixed $\Rightarrow$ function $\mathfrak{X}(M) \to \Gamma(E)$, $X \mapsto \nabla_X(s)$, $C^\infty(M)$-linear!

i.e. a 1-form, denoted $d_\nabla(s) \in \Omega^1(M, E)$

get $d_\nabla : \Gamma(E) \to \Omega^1(M, E)$ defined by $d_\nabla(s)(X) =: \nabla_X(s)$

$\Rightarrow$ a bijection $\left\{\text{connections } \nabla \text{ on } E\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{matrix}\mathbb{R}\text{-linear maps} \\ d_\nabla : \Gamma E \to \Omega^1(M, E) \\ \text{s.t. } d_\nabla(fs) = f\, d_\nabla(s) + df \otimes s\end{matrix}\right\}$

($\Omega^1(M) \otimes \Gamma(E)$; $\Omega^1(M, E)$ given by $X \mapsto df(X)\, s$)

E.g. in Ex 3: $d_\nabla(e_j) = \sum_i \omega_j^i \otimes e_i$, $\omega_j^i \in \Omega^1(M)$

Connection matrices: Given ∇ on E ④

For any local frame $e = \{e_1, \dots, e_r\}$ of E over U

write over U: $\nabla_X(e_j) = \sum_i \omega_j^i(X)\, e_i$ $\Rightarrow$ ○

$\Rightarrow$ over U, ∇ uniquely determined by a matrix

$\omega = (\omega_j^i)_{1 \le i,j \le r}$, $\omega_j^i \in \Omega^1(U)$

the connection matrix of ∇ w.r.t. the local frame e

Rk 1: On arbitrary s, writing over U: $s = \sum_j f^j e_j$

$\Rightarrow \nabla_X(s) = \sum_j L_X(f^j)\, e_j + f^j \nabla_X(e_j)$ $\quad \left(\nabla_X(e_j) = \sum_i \omega_j^i(X)\, e_i\right)$

$$\nabla_X(s) = \sum_j \left[L_X(f^j) + \sum_i f^i\, \omega_i^j(X) \right] e_j$$

Rk 2: $d_\nabla(e_j) = \sum_i \omega_j^i \otimes e_i$, $d_\nabla(s) = \dots$

Motivation(s): $\Gamma(E) \longleftarrow C^\infty(M)$ ①

Very useful & in associating (invariants "c") to vector bundles E

s.t. $E_1 \simeq E_2 \Rightarrow c(E_1) = c(E_2)$

$E_1 \not\simeq E_2 \Leftarrow c(E_1) \neq c(E_2)$

which cannot be done here:

X

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can't even compare / move in between fibers

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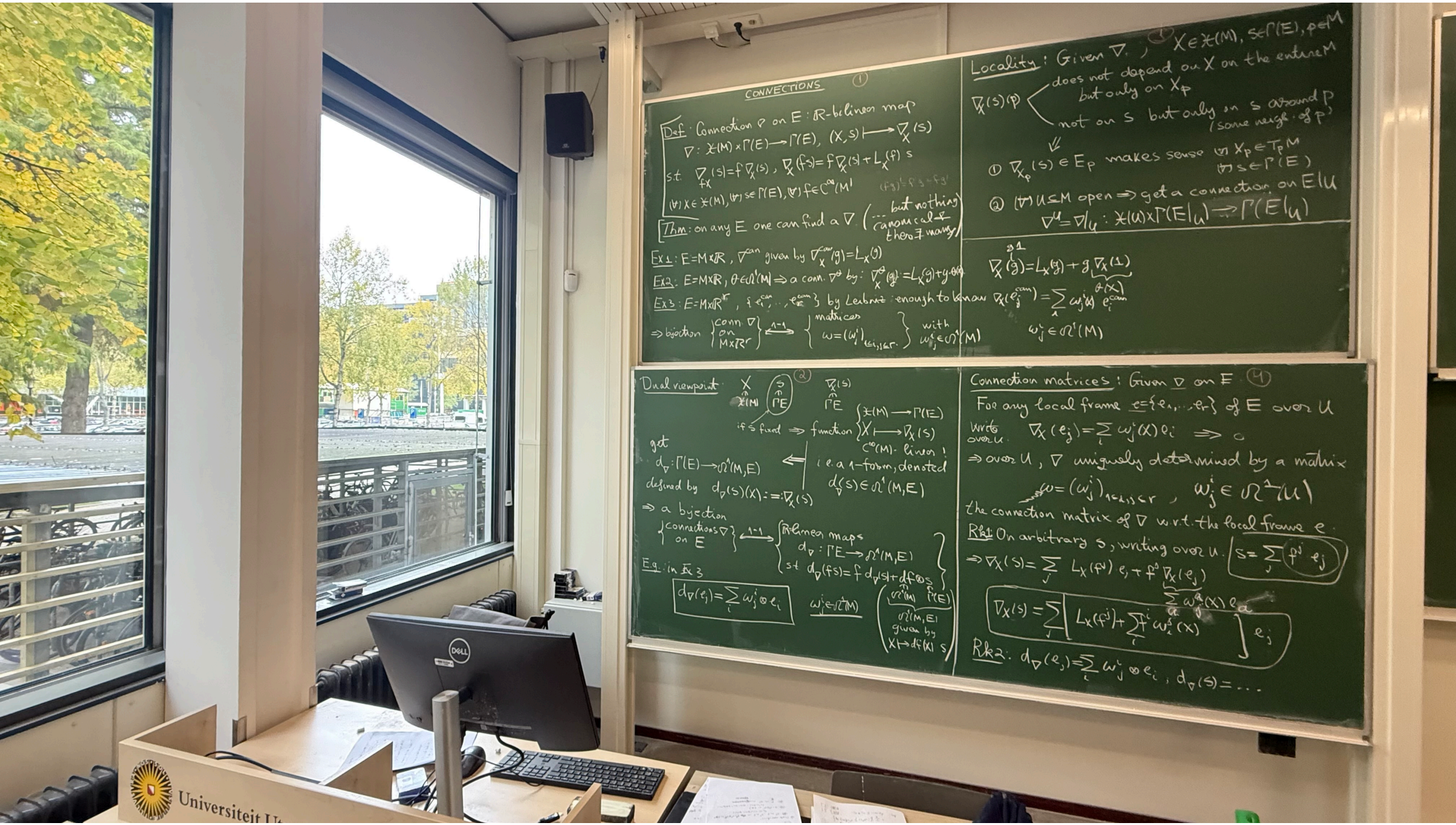
many things:

$L_X(f)$ $\quad X \in \mathfrak{X}(M)$

$df \in \Omega^1(M)$

$f = cst$

$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to$


CONNECTIONS
Def: Connection ∇ on E : ℝ-bilinear map
Thm: on any E one can find a ∇
Ex 1: E = M × ℝ, ∇^can given by ∇^can_X(g) = L_X(g)
Ex 2: E = M × ℝ, θ ∈ Ω¹(M) ⇒ a conn. ∇^θ by: ∇^θ_X(g) = L_X(g) + g θ(X)
Ex 3: E = M × ℝ^r, {e_1^can, ..., e_r^can} by Leibniz enough to know ∇_X(e_j^can) = Σ_i ω_j^i(X) e_i^can
⇒ bijection {conn. ∇ on M × ℝ^r} 1-1 {matrices ω = (ω_j^i)_{1≤i,j≤r}} with ω_j^i ∈ Ω¹(M)
Locality: Given ∇, X ∈ 𝔛(M), s ∈ Γ(E), p ∈ M
∇_X(s)(p) does not depend on X on the entire M but only on X_p
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① ∇_{X_p}(s) ∈ E_p makes sense ∀ X_p ∈ T_pM, ∀ s ∈ Γ(E)
② (∀) U ⊆ M open ⇒ get a connection on E|_U
∇^U = ∇|_U : 𝔛(U) × Γ(E|_U) → Γ(E|_U)
∇_X(fg) = L_X(f) g + f ∇_X(g)
Dual viewpoint
if s fixed ⇒ function 𝔛(M) → Γ(E), X ↦ ∇_X(s)
C^∞(M)-linear ! i.e. a 1-form, denoted d_∇(s) ∈ Ω¹(M, E)
get d_∇ : Γ(E) → Ω¹(M, E)
defined by d_∇(s)(X) := ∇_X(s)
⇒ a bijection {connections ∇ on E} 1-1 {ℝ-linear maps d_∇ : Γ(E) → Ω¹(M, E) s.t. d_∇(fs) = f d_∇(s) + df ⊗ s}
Eg. in Ex 3
d_∇(e_j) = Σ_i ω_j^i ⊗ e_i   ω_j^i ∈ Ω¹(M)
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Σ_i ω_j^i(X) e_i
∇_X(s) = Σ_j [L_X(f^j) + Σ_i f^i ω_i^j(X)] e_j
Rk 2: d_∇(e_j) = Σ_i ω_j^i ⊗ e_i ; d_∇(s) = ...

CONNECTIONS ①

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Localit

$\nabla_X(s)(p)$

① ∇_X

② (t

$\nabla_X\left(\sum_i g^i e_i\right)$

Dual viewpoint ② (Connect)

$X \in \mathfrak{X}(M)$, $s \in \Gamma E$, $\nabla_X(s) \in \Gamma E$

if s fixed $\Rightarrow$ function $\begin{cases} \mathfrak{X}(M) \longrightarrow \Gamma(E) \\ X \longmapsto \nabla_X(s) \end{cases}$ $C^\infty(M)$-linear !

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($\Omega^1(M)$ $\Gamma(E)$; $\Omega^1(M,E)$ given by $X \mapsto df(X)\, s$)

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$$\boxed{d_\nabla(e_j) = \sum_i \omega_j^i \otimes e_i} \qquad \omega_j^i \in \Omega^1(M)$$

For an

Write over U.

$\Rightarrow$ over

the con

Rk 1: On

$\Rightarrow \nabla_X($

∇_X

Rk 2

Motivation(s): $\Gamma(E)$ (1) $\longleftarrow$ $C^\infty(M)$

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$\nabla_X(fg) = L_X(f)g + f\nabla_X(g)$

$\nabla_X(g) = L_X(g) + g\nabla_X(1)$

$\nabla_X(1) = 0$

enough to know $\nabla_X(e_j^{cam}) = \sum_i \omega_j^i(X) e_i^{cam}$

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$\omega = (\omega_j^i)_{1 \le i,j \le r}$, $\omega_j^i \in \Omega^1(U)$

the connection matrix of ∇ w.r.t. the local frame e.

Rk1: On arbitrary s, writing over U. $s = \sum_j f^j e_j$

$\Rightarrow \nabla_X(s) = \sum_j L_X(f^j) e_j + f^j \nabla_X(e_j)$

$\nabla_X(e_j) = \sum_a \omega_j^a(X) e_a$

$$\nabla_X(s) = \sum_j \left[L_X(f^j) + \sum_i f^i \omega_i^j(X) \right] e_j$$

Rk2: $d_\nabla(e_j) = \sum_i \omega_j^i \otimes e_i$; $d_\nabla(s) = \dots$

Leibniz (derivation) rule/identity : $\frac{d(fg)}{dt} = \frac{df}{dt} \cdot g + f \cdot \frac{dg}{dt}$

or, more generally: $\mathcal{L}_X(a * \sim) = \mathcal{L}_X(a) * \sim + a * \mathcal{L}_X(\sim)$

where $a, \sim, \mathcal{L}_X, \mathcal{L}_X, \mathcal{L}_{X*}$: different meaning in different context

(though usually $\mathcal{L}_X$ = usual L_X or $\mathcal{L}_X = \nabla_X$ when makes sense)

Examples/constructions: (5)

① If $E \subseteq M \times \mathbb{R}^k = F$ the trivial vb of rank k.
$\Rightarrow$ a connection on E

$$\begin{array}{ccc} \mathfrak{X}(M)\times\Gamma(E) & \overset{\nabla}{\dashrightarrow} & \Gamma(E) \\ \downarrow & & \downarrow pr_E^{\perp} \\ \mathfrak{X}(M)\times\Gamma(F) & \xrightarrow{\nabla^{can}} & \Gamma(F) \end{array}$$

$\nabla_X(s) := pr_E^{\perp}(\nabla_X^{can}(s))$

Ex 1A Tautological line bdle/$\mathbb{P}^n$: $L \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$

Ex 1B $TS^n \subsetneq S^n \times \mathbb{R}^{n+1}$, TM for $M \subseteq \mathbb{R}^k$

② Simple operations

2A ∇^E on E, ∇^F on F $\Rightarrow$ $\nabla^{E\oplus F}$ on $E\oplus F$: $\nabla_X(s,s') = (\nabla_X^E(s), \nabla_X^F(s'))$ ... Leibniz

2B — " — $\Rightarrow$ $\nabla^{E\otimes F}$ on $E\otimes F$: $\nabla_X^{E\otimes F}(s_1\otimes s_2) = \nabla_X^E(s_1)\otimes s_2 + s_1\otimes\nabla_X^F(s_2)$

2C ∇ on E $\Rightarrow$ induced ∇^* on E^*

defined by: $\nabla^*: \underset{X}{\mathfrak{X}(M)} \times \underset{\xi}{\Gamma(E^*)} \to \Gamma(E^*)$

$\nabla_X^*(\xi)(s) := L_X(\xi(s)) - \xi(\nabla_X(s))$

$\nabla_X^*(\xi)\cdot s = L_X(\xi^* s) - \xi^*\nabla_X(s)$

Check: indeed ∇^* is a connection ↑ Leibniz

(6)

③ Pull-backs $\Phi: N \to M$ 2U

$Vb(M) \ni E \longmapsto \Phi^*E \in Vb(N)$, $(\Phi^*E)_x = E_{\Phi(x)}$

$\Gamma(E) \ni s \longmapsto \Phi^*(s) \in \Gamma(\Phi^*E)$, $(\Phi^*s)(x) = s(\Phi(x))$

$\left(\sum_i f_i\, \Phi^*(s_i)\right)$ ← all sections of Φ^*E are of this type; $f_i \in C^\infty(N)$, $s_i \in \Gamma(E)$

$\{e_1,\dots,e_r$ local frame of E over $U\} \longmapsto \{\Phi^*(e_1),\dots,\Phi^*(e_r)$ local frame of Φ^*E over $\Phi^{-1}(U)\}$

Now: ∇ connection on E ⇝ a connection $\Phi^*\nabla$ on Φ^*E (to do for us)

Ⓐ algebraically $\nabla' = \Phi^*\nabla$:

determined by $\nabla'_X(\Phi^*(s)) = \Phi^*(\nabla_{d\Phi(X)}(s))$ $X \in \mathfrak{X}(N)$

By Leibniz, for $S \in \Gamma(\Phi^*E)$ arbitrary, $S = \sum f_i\, \Phi^*(s_i)$

$\Rightarrow \nabla'_X(S) = \sum L_X(f_i)\Phi^*(s_i) + \sum f_i \Phi^*(\nabla_{d\Phi(X)}(s_i))$ well defined. $\nabla_X(s)$ $X \in \mathfrak{X}(N)$

Ⓑ Locally

$\{e_1,\dots,e_r\}$ of E over U

$\{\Phi^*e_1,\dots,\Phi^*e_r\}$ of Φ^*E over $\Phi^{-1}(U)$

∇ on E over U: the $\omega = (\omega_j^i)$, $\omega_j^i \in \Omega^1(U)$

The conn. matrix of $\nabla' = \Phi^*\nabla$ w.r.t. Φ^* frame $\omega' = (\omega'^i_j)$ $\omega'^i_j \in \Omega^1(\Phi^{-1}U)$

...is simply $\omega' = \Phi^*(\omega)$

Φ^*E, E, $\Phi: N \to M$, $(d\Phi)_p: T_pN \to T_{\Phi(p)}M$

$\nabla_Y(s)$ $Y \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, $U = d\Phi(X)$ $\nabla_X(s)$ $(d\Phi)_p(X_p)$

Examples/constructions: (5)

① If $E \subseteq M \times \mathbb{R}^k = F$ the trivial v.b of rank k.
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$$\begin{array}{ccc} \mathfrak{X}(M)\times\Gamma(E) & \overset{\nabla}{\dashrightarrow} & \Gamma(E) \\ \downarrow & & \uparrow pr_E^{\perp} \\ \mathfrak{X}(M)\times\Gamma(F) & \xrightarrow{\nabla^{can}} & \Gamma(F) \end{array}$$

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$$\nabla_X^*(\xi) * s = L_X(\xi * s) - \xi * \nabla_X(s)$$

Check: indeed ∇^* is a connection ... Leibniz

Leibniz (derivation) rule/identity: $\frac{d(fg)}{dt} = \frac{df}{dt}\cdot g + f\cdot\frac{dg}{dt}$

or, more generally: $\mathcal{L}_X(a * \xi) = \mathcal{L}_X(a) * \xi + a * \mathcal{L}_X(\xi)$

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$\Gamma(E) \ni s \longmapsto \phi^*(s) \in \Gamma(\phi^*E)$, $(\phi^*s)(x) = s(\phi(x))$

$\left(\sum_i f_i \cdot \phi^*(s_i) \leftarrow \text{all sections of } \phi^*E \text{ are of this type} \right)$

$f_i \in C^\infty(N)$, $s_i \in \Gamma(E)$

$\{e_1, \dots, e_r \text{ local frame of } E \text{ over } U\} \longmapsto \{\phi^*(e_1), \dots, \phi^*(e_r) \text{ local frame of } \phi^*E \text{ over } \phi^{-1}(U)\}$

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determined by $\nabla'_X(\phi^*(s)) = \phi^*\left(\nabla_{d\phi(X)}(s)\right)$ $X \in \mathfrak{X}(N)$

By Leibniz, for $s \in \Gamma(\phi^*E)$ arbitrary, $s = \sum f_i \phi^*(s_i)$

$\Rightarrow \nabla'_X(s) = \sum L_X(f_i)\phi^*(s_i) + \sum f_i \phi^*(\nabla_{d\phi(X)}(s_i))$ $\nabla_Y(s)$

well defined. $X \in \mathfrak{X}(N)$

(B) Locally:

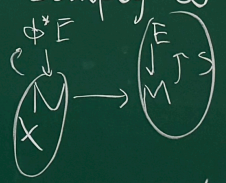
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$\phi: N \to M$

$(d\phi)_p : T_pN \to T_{\phi(p)}M$

$\nabla_Y(s)$ $Y \in \mathfrak{X}(M)$ $\nabla_X(s)$

$Y = d\phi(X)$ $s \in \Gamma(E)$ $(d\phi)_p(X_p)$

$\Rightarrow$ bijection $\{$conn. ∇ on $M\times\mathbb{R}^r\}$ $\overset{1\text{-}1}{\Longleftrightarrow}$ $\{$matrices $\omega=(\omega_j^i)_{1\le i,j\le r}\}$ with $\omega_j^i\in\Omega^1(M)$

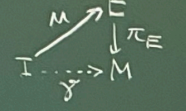
For $\omega=0$ $\Rightarrow$ resulting ∇^{can} the canonical conn

$\nabla_X(e_j)=\sum_i \omega_j^i(X)\, e_i^{can}$ $\quad$ $\omega_j^i\in\Omega^1(M)$

Paths in E & their derivatives (w.r.t. a fixed ∇ on E) (7)

Path in E: $u: I\to E$

Base path of u: $\gamma=\pi_E\circ u: I\to M$



Fix γ (for the moment): say $u: I\to E$ path above γ $(u(t)\in E_{\gamma(t)})$

$\Leftrightarrow$ section $u\in\Gamma(\gamma^*E)$

Now: $\nabla \Rightarrow$ get a connection $\nabla^\gamma:=\gamma^*\nabla$ on γ^*E

Def: The ∇-derivative of u is the new path above γ $\quad \frac{\nabla u}{dt}:=\nabla^\gamma_{\frac{d}{dt}}(u)$

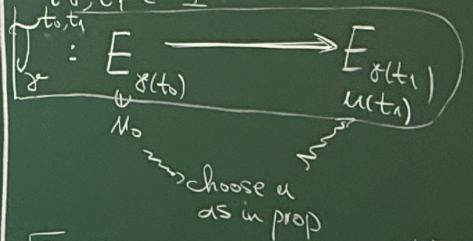
Call u parallel w.r.t. ∇ if $\frac{\nabla u}{dt}=0$ $(\forall)\, t\in I$.

Look at this locally (8)

Prop: ∇, γ given, $t_0\in I$, $u_0\in E_{\gamma(t_0)}$

$\Rightarrow \exists!\ u: I\to E$ which $\begin{cases}\text{is parallel } \frac{\nabla u}{dt}=0\\ u(t_0)=u_0\end{cases}$

$t_0, t_1\in I$



Corollary: If M = contractible $\Rightarrow$ any v.b $E\in Vb(M)$ is trivializable.

More details: next time.

CONNECTIONS ①

Def Connection ∇ on E : $\mathbb{R}$-bilinear map

$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$, $(X,s) \longmapsto \nabla_X(s)$

s.t. $\nabla_{fX}(s) = f\nabla_X(s)$, $\nabla_X(fs) = f\nabla_X(s) + L_X(f)\,s$

$(\forall)\, X \in \mathfrak{X}(M), (\forall)\, s \in \Gamma(E), (\forall)\, f \in C^\infty(M)$ $\quad (fg)' = f's + fg's$

Thm: on any E one can find a ∇ (... but nothing canonical & there ? many)

Ex 1: $E = M \times \mathbb{R}$, ∇^{can} given by $\nabla^{can}_X(g) = L_X(g)$

Ex 2: $E = M \times \mathbb{R}$, $\theta \in \Omega^1(M) \Rightarrow$ a conn ∇^θ by: $\nabla^\theta_X(g) = L_X(g) + g\,\theta(X)$

Ex 3: $E = M \times \mathbb{R}^r$, $\{e_1^{can}, \dots, e_r^{can}\}$ by Leibniz enough to know $\nabla_X(e_j^{can}) = \sum_i \omega_j^i(X)\, e_i^{can}$

$\Rightarrow$ bijection $\left\{ \begin{array}{l} \text{conn } \nabla \\ \text{on} \\ M \times \mathbb{R}^r \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{matrices} \\ \omega = (\omega_j^i)_{1 \le i,j \le r} \end{array} \right\}$ with $\omega_j^i \in \Omega^1(M)$

For $\omega = 0 \Rightarrow$ result is ∇^{can} the canonical conn

Locality: Given ∇, ① $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, $p \in M$

$\nabla_X(s)(p)$ does not depend on X on the entire M but only on X_p

not on s but only on s around p (some neigh. of p)

① $\nabla_{X_p}(s) \in E_p$ makes sense $\forall\, X_p \in T_pM$, $\forall\, s \in \Gamma(E)$

② $(\forall)\, U \subseteq M$ open $\Rightarrow$ get a connection on $E|_U$

$\nabla^U = \nabla|_U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$

$\nabla_X(fs) \overset{\text{①}}{=} L_X(f)\,s + f\,\nabla_X(s)$

$\nabla_X(s)(p) = L_X(f)(p)\,s(p) + f(p)\,\nabla_X(s)(p)$

$\nabla_X(s)(p) = L_X(f)(p)\,s(p) + f(p)\,\nabla_X(s)(p)$

$\nabla_X(s)(p) = L_X(f)(p)\,s(p) + f(p)\,\nabla_X(s)(p)$

Paths in E & their derivatives (w.r.t. a fixed ∇ on E) ⑦

Path in E : $u : I \to E$ $\qquad$ Base path of u : $\gamma = \pi_E \circ u : I \to M$

Fix γ (for the moment) : say $u : I \to E$ path above γ $(u(t) \in E_{\gamma(t)})$

Sections $u \in \Gamma(\gamma^* E)$

Now: $\nabla \Rightarrow$ get a connection $\nabla^\gamma := \gamma^*\nabla$ on $\gamma^* E$

Def: The ∇-derivative of u is the new path above γ

$\frac{\nabla u}{dt} := \nabla^\gamma_{\frac{d}{dt}}(u)$

Call u parallel w.r.t. ∇ if $\frac{\nabla u}{dt} = 0$ $(\forall)\, t \in I$.

Look at this locally ⑧

Prop: ∇, γ given, $t_0 \in I$, $u_0 \in E_{\gamma(t_0)}$

$\Rightarrow \exists!\ u : I \to E$ which $\begin{cases} \text{is parallel } \frac{\nabla u}{dt} = 0 \\ u(t_0) = u_0 \end{cases}$

$t_0, t_1 \in I$

$u : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t_1)}$, $u(t_0) \mapsto u(t_1)$

choose u as in prop

Corollary: If M = contractible $\Rightarrow$ any v.b. $E \in Vb(M)$ is trivializable

More details: next time

Examples/constructions: ⑤

① If $E \subseteq M \times \mathbb{R}^k = F$ the trivial v.b. of rank k

$\Rightarrow$ a connection on E

$\mathfrak{X}(M) \times \Gamma(E) \dashrightarrow \Gamma(E)$

$\downarrow \qquad\qquad \vdots\ pr_E^\perp$

$\mathfrak{X}(M) \times \Gamma(F) \xrightarrow{\nabla^{can}} \Gamma(F)$

$\nabla_X(s) := pr_E^\perp(\nabla^{can}_X(s))$

Ex 1A Tautological line bdle/$\mathbb{P}^n$ $L \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1}$

Ex 1B $TS^n \subsetneq S^n \times \mathbb{R}^{n+1}$, TM for $M \subseteq \mathbb{R}^k$

② Simple operations

2A ∇^E on E, ∇^F on $F \Rightarrow \nabla$ on $E \oplus F$: $\nabla_X(s,s') = (\nabla^E_X(s), \nabla^F_X(s'))$ ... Leibniz

2B $\Rightarrow \nabla^{E\otimes F}$ on $E \otimes F$ $\quad \nabla^{E\otimes F}_X(s_1 \otimes s_2) = \nabla^E_X(s_1) \otimes s_2 + s_1 \otimes \nabla^F_X(s_2)$

2C ∇ on $E \Rightarrow$ induced ∇^* on E^*

defined by: $\nabla^* : \mathfrak{X}(M) \times \Gamma(E^*) \to \Gamma(E^*)$

$\nabla^*_X(\xi)(s) := L_X(\xi(s)) - \xi(\nabla_X(s))$

$\nabla^*_X(\xi) \cdot s = L_X(\xi \cdot s) - \xi \cdot \nabla_X(s)$

Check: indeed ∇^* is a connection ... Leibniz

③ Pull-backs $\phi : N \to M \supset U$ ⑥

$Vb(M) \ni E \longmapsto \phi^*E \in Vb(N)$, $(\phi^*E)_x = E_{\phi(x)}$

$\Gamma(E) \ni s \longmapsto \phi^*(s) \in \Gamma(\phi^*E)$, $(\phi^*s)(x) = s(\phi(x))$

$\sum_i f_i\, \phi^*(s_i)$ ← all sections of ϕ^*E are of this type

$f_i \in C^\infty(N)$, $s_i \in \Gamma(E)$

$\{e_1, \dots, e_r$ local frame of E over $U\} \longmapsto \{\phi^*(e_1), \dots, \phi^*(e_r)$ local frame of ϕ^*E over $\phi^{-1}(U)\}$

Now: ∇ connection on $E \rightsquigarrow$ a connection $\phi^*\nabla$ on ϕ^*E (to define)

(A) algebraically $\nabla' = \phi^*\nabla$

determined by $\nabla'_X(\phi^*(s)) = \phi^*(\nabla_{d\phi(X)}(s))$ $\quad X \in \mathfrak{X}(N)$

By Leibniz, for $S \in \Gamma(\phi^*E)$ arbitrary, $S = \sum f_i\, \phi^*(s_i)$

$\Rightarrow \nabla'_X(S) = \sum L_X(f_i)\,\phi^*(s_i) + \sum f_i\, \phi^*(\nabla_{d\phi(X)}(s_i))$ well defined. $\quad X \in \mathfrak{X}(N)$

(B) Locally

$\{e_1, \dots, e_r\}$ of E over U

$\{\phi^*e_1, \dots, \phi^*e_r\}$ of ϕ^*E over $\phi^{-1}(U)$

∇ on E over U : the $\omega = (\omega_j^i)$, $\omega_j^i \in \Omega^1(U)$

The conn. matrix of $\nabla' = \phi^*\nabla$ w.r.t. ϕ^* frame $\omega' = (\omega'^i_j)$ $\quad \omega'^i_j \in \Omega^1(\phi^{-1}(U))$

... is simply $\omega' = \phi^*(\omega)$

$\phi : N \to M$

$(d\phi)_p : T_pN \to T_{\phi(p)}M$

$\nabla_Y(s)$ $\quad Y \in \mathfrak{X}(N)$ $\quad s \in \Gamma(E)$ $\quad (d\phi)_p(X_p)$

$Y = d\phi(X)$ $\qquad\qquad \nabla_X(s)$