# Exercises to prepare for the exam- list 1



**Exercise 1:** Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, over which we consider the vector bundle $E := TS^2$. We also consider the following

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \qquad (\star)$$

Since $S^2 \subset \mathbb{R}^3$, we can view $E$ as a sub-bundle of the trivial bundle $\widetilde{E} = S^2 \times \mathbb{R}^3$.

(a) Describe the largest $U \subset S^2$ over which $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ is a local frame of $TS^2$.
(b) Consider the similar local frame $\{\mathfrak{e}^x, \mathfrak{e}^z\}$ over the corresponding $V \subset S^2$ and compute the transition functions from $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ to $\{\mathfrak{e}^x, \mathfrak{e}^z\}$ .

**Exercise 2:** Given a morphism of vector bundles $u : E \to F$, and considering the induced map $u_* : \Gamma(E) \to \Gamma(F)$, $s \mapsto u \circ s$, show that:

- $u$ is an isomorphism of vector bundle if and only if $u_*$ is bijective;
- if $u$ is injective then so is $u_*$, but the converse is not true;
- $u$ is surjective if and only if $u_*$ is surjective.

(try to prove the first item directly, as it is easier to do so than to appeal to the other items).

**Exercise 3:** Show that:

1. If $u : E^0 \to E^1$ and $v : E^1 \to E^2$ are morphisms of vector bundles such that $\mathrm{Im}(u) = \mathrm{Ker}(v)$, prove that $\mathrm{Im}(u)$ is a vector sub-bundle of $E^1$.
2. For any exact sequence of vector bundles

$$0 \to E_0 \to E_1 \to \ldots \to E_{n-1} \to E_n \to 0$$

(hence all the arrows are vector bundle morphisms with the property that the image of one is the kernel of the next one/the kernel of one is the image of the previous one, and where 0 is the trivial vector bundle), the vector bundles

$$E_{\text{even}} := E_0 \oplus E_2 \oplus \ldots \quad \text{and} \quad E_{\text{odd}} := E_1 \oplus E_3 \oplus \ldots$$

are isomorphic.

**Exercise 4:** Prove that:

(a) The tangent bundle $TS^2$ of $S^2$ can be made into a complex vector bundle where, next to the addition and multiplication by real scalars that we already have at our disposal, we also introduce the multiplication by $i$: on $T_pS^2$ for $p = (x, y, z)$, it is given by

$$i \cdot \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right) = (cy - bz)\frac{\partial}{\partial x} + (az - cx)\frac{\partial}{\partial y} + (bx - ay)\frac{\partial}{\partial z}.$$

(b) In general, if a rank two real vector bundle $E \to M$ can be made into a complex vector bundle, then the existence of a nowhere vanishing section of $E$ implies that $E$ must be trivializable.
(c) Conclude that the hairy ball theorem is equivalent to the fact that $TS^2$ is not trivializable.
(d) Prove that the almost complex structure given above is integrable, i.e. it is induced by a holomorphic atlas on $S^2$ (a complex structure on $S^2$). Find two proofs- a direct one, and one using Newlander-Nirenberg (and try to avoid long computations).

**Exercise 5 over $\mathbb{R}$:** Recalling that $L$ is the tautological line bundle over $\mathbb{RP}^n$, prove that there exists an isomorphism of vector bundles

$$T\mathbb{RP}^n \cong \mathrm{Hom}(L, L^\perp),$$

then deduce that

$$T\mathbb{RP}^n \oplus T \cong \underbrace{L^* \oplus \ldots \oplus L^*}_{n+1 \text{ times}}.$$

Hints for the first part: investigate the tangent spaces $T_l\mathbb{RP}^n$, at an arbitrary $l \in \mathbb{RP}^n$. Keep in mind that, according to the general slogan on tangent spaces, $T_l\mathbb{RP}^n$ plays the role of "the linear approximation of $\mathbb{RP}^n$ around $l$". Now think a bit of how $\mathbb{RP}^n$ looks around $l$. Isn't a nearby line a graph of a linear map from $l$ to $l^\perp$? Doesn't the tangent space start looking like $\mathrm{Hom}(l, l^\perp)$? Turn this idea into an explicit formula. What does that mean in terms of vector bundles?

A completely similar discussion can be carried out for the complex projective space $\mathbb{CP}$. This time $T\mathbb{CP}^n$ will be treated as a complex vector bundle, and then $L$ will be the tautological (complex) line bundle over $\mathbb{CP}^n$, while the orthogonal $L^\perp$ is defined using the standard hermitian product on $\mathbb{C}^{n+1}$. Of course, $T$ will now denote the trivial complex line bundle, while $L^*$ is the *complex* dual of $L$.

**Exercise 5 over $\mathbb{C}$:** Prove that there exists an isomorphism of complex vector bundles

$$T\mathbb{CP}^n \cong \mathrm{Hom}(L, L^\perp),$$

then deduce that

$$T\mathbb{CP}^n \oplus T \cong \underbrace{\overline{L} \oplus \ldots \oplus \overline{L}}_{n+1}, \tag{0.0.1}$$

**Exercise 6:** A Lie algebroid over a manifold $M$ is a vector bundle $A$ over $M$ *together with*

- a *Lie bracket* on the space $\Gamma(A)$ of sections of $A$, i.e. a bilinear operation

  $$[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

  that is skew-symmetric and satisfies the Jacobi identity;
- a vector bundle morphism $\rho : A \to TM$ such that the following Leibniz (derivation) rule holds:

  $$[\alpha, f \cdot \beta]_A = f \cdot [\alpha, \beta]_A + L_{\rho(\alpha)}(f) \cdot \beta, \quad \forall\ \alpha, \beta \in \Gamma(A), f \in C^\infty(M),$$

  where $L_{\rho(\alpha)} : C^\infty(M) \to C^\infty(M)$ is the (derivation induced by the) the vector field $\rho(\alpha) \in \mathfrak{X}(M)$.

The map $\rho$ is called *the anchor* of $A$ and, for $\alpha \in \Gamma(A)$ we denote by $\rho(\alpha) \in \mathfrak{X}(M)$ the vector field on $M$ whose value at $x \in M$ is $\rho_x(\alpha_x)$. Also, for $x \in M$ we set

$$\mathfrak{g}_x := \mathrm{Ker}(\rho_x) \subset A_x, \quad \mathcal{F}_x := \rho_x(A_x) \subset T_xM, \tag{0.0.2}$$

and we say that $A$ is *regular* if the dimensions of these spaces do not depend on $x \in M$. Show that:

(a) if instead of giving $\rho$ we require that for any $\alpha \in \Gamma(A)$ there exists a map $D_\alpha : C^\infty(M) \to C^\infty(M)$ (necessarily unique!) such that

$$[\alpha, f \cdot \beta]_A = f \cdot [\alpha, \beta]_A + D_\alpha(f) \cdot \beta$$

for all $\beta$ and $f$, then each $D_\alpha$ is induced by a vector field $X_\alpha \in \mathfrak{X}(M)$ and, if $\mathrm{rank} A \geq 2$, then the map $\alpha \mapsto X_\alpha$ is induced by a vector bundle morphism $\rho$ as above.

(b) the map $\alpha \mapsto \rho(\alpha)$ is compatible with the usual Lie bracket $[\cdot,\cdot]_{\mathfrak{X}(M)}$ of vector fields on $M$, i.e.,

$$\big[\rho(\alpha), \rho(\beta)\big]_{\mathfrak{X}(M)} = \rho\big([\alpha, \beta]_A\big) \quad \forall\ \alpha, \beta \in \Gamma(A).$$

(c) each $\mathfrak{g}_x$ carries a Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}_x}$ such that, and uniquely determined by the condition that: for any (local) sections $\alpha, \beta$ of $A$ such that $\alpha(x), \beta(x) \in \mathfrak{g}_x$, one has

$$[\alpha, \beta]_A(x) = [\alpha(x), \beta(x)]_{\mathfrak{g}_x}.$$

(d) if $A$ is constant rank then the resulting $\mathcal{F} \subset TM$ defines a foliation on $M$.

Next, to describe an interesting class of examples of Lie algebroids, we start with an action of a connected Lie group $G$ on a manifold $M$,

$$\mathcal{A} : G \times M \to M, \quad (g, x) \mapsto A(g, x) = g \cdot x.$$

Set $\mathfrak{g} := T_eG$ and, for each $x \in M$, the differential of $\mathcal{A}^x : G \to M$, $\mathcal{A}^x(g) = gx$ at the identity element $e$ is denoted $a_x : \mathfrak{g} \to T_xM$. This is called the induced infinitesimal action at $x$ while, when varying $x$, the resulting map

$$a : \mathfrak{g} \to \mathfrak{X}(M)$$

is known under the name of the infinitesimal action associated to $\mathcal{A}$. Recall that $\mathfrak{g}$ carries a Lie bracket and $\rho$ has the basic property (that you can use without proof) that it preserves the Lie brackets. For $u \in \mathfrak{g}$ we denote by $c_u \in \Gamma(A)$ the constant section given by $u$.

(e) show that the trivial vector bundle $A := M \times \mathfrak{g}$ has a unique Lie algebroid structure whose anchor and Lie bracket satisfy

$$\rho(c_u) = a(u), \quad [c_u, c_v]_A = c_{[u,v]_{\mathfrak{g}}} \quad \forall\ u, v \in \mathfrak{g}.$$

(f) when the action is free show that $A$ is regular and each of the $\mathcal{A}^x : G \to M$ above makes $G$ into an immersed submanifold of $M$ that is a leaf of the foliation $\mathcal{F}$ induced by $A$ (item (d) above).

(g) in general, what is the relationship between the resulting $\mathfrak{g}_x$, $\mathfrak{F}_x$ from (0.0.2) and the isotropy groups and the orbits of the action:

$$G_x = \{g \in G : g \cdot x = x\}, \quad \mathcal{O}_x = \{g \cdot x : g \in G\}?$$

**Exercise 7:** Let $(M, g)$ be a Riemannian manifold and assume that $\mathcal{F} \subset TM$ is a distribution on $M$. We denote by $\mathcal{F}^\perp \subset TM$ the orthogonal of $\mathcal{F}$ with respect to $g$ and by

$$\mathrm{pr}_{\mathcal{N}} : TM \to \mathcal{F}^\perp$$

the orthogonal projection with respect to $g$. Finally, define

$$\zeta : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}^\perp), \quad \zeta(V, W) = \mathrm{pr}_{\mathcal{N}}\left([V, W]\right).$$

(a) Show that $\zeta$ is $C^\infty(M)$-linear in $V$ and $W$.
(b) Deduce that $\zeta$ may be interpreted as a section of a vector bundle over $M$ (which one?)
(c) Show that $\mathcal{F}$ defines a foliation on $M$ if and only if $\zeta = 0$.
(d) Prove that $\mathcal{F}^\perp$ is isomorphic to the normal bundle $TM/\mathcal{F}$.

**Exercise 8:** Let $\pi : E \to M$ be a vector bundle over $E$, and we consider the zero-section

$$\mathfrak{o} : M \to E, \quad x \mapsto 0_x,$$

where $0_x$ is the origin of the vector space $E_x$. First show that $\mathfrak{o}$ allows us to see $M$ as an embedded submanifold of $E$, and then prove that the corresponding normal bundle of $M$ in $E$ is isomorphic, as a vector bundle over $M$, to $E$ itself.