4

Exercise 9: Assume that $J : TM \to TM$ is an almost complex structure on a 2-dimensional manifold M. Without using the Newlander-Nirenberg theorem, but assuming that around each point in M one can find a nowhere vanishing vector field X such that $[X, JX] = 0$, prove that J is integrable.

Exercise 10: We now work over $M = S^2$, over which we consider the vector bundle $E := TS^2$. Using the fact that $S^2 \subset \mathbb{R}^3$ to embed TS^2 in the trivial bundle $S^2 \times \mathbb{R}^3$, we endow E with the resulting connection

$$\nabla : \mathfrak{X}(S^2) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_E \left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(E)} \right),$$

or, if you prefer the dual point of view, $d_\nabla : \Gamma(E) \to \Omega^1(S^2, E)$, $d_\nabla(s) = \mathrm{pr}_E(d_{\nabla^{\mathrm{flat}}}(s))$. For more details, please look at the exercise file, section 3.2. We also consider the following vector fields on S^2

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \qquad (\star)$$

and observe that $x\mathfrak{e}^x + y\mathfrak{e}^y + z\mathfrak{e}^z = 0$,

(a) Show that

$$\mathrm{pr}_E\left(\frac{\partial}{\partial x}\right) = y\mathfrak{e}^z - z\mathfrak{e}^y = \frac{\partial}{\partial x} - x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

(b) Show that

$$d_\nabla(\mathfrak{e}^x) = -x \cdot (dx \otimes \mathfrak{e}^x + dy \otimes \mathfrak{e}^y + dz \otimes \mathfrak{e}^z).$$

(c) Check that $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ is a local frame of TS^2 over $U = \{z \neq 0\}$ and compute the connection matrix ω of ∇ w.r.t. $\{\mathfrak{e}^x, \mathfrak{e}^y\}$.

(d) From the general theory it follows that for any path γ in S^2 and any path ξ in $E = TS^2$ sitting above γ, we can talk about $\frac{\nabla\xi}{dt}$, which is a new path in E above γ. Writing $\gamma(t) = (x(t), y(t), z(t))$ and denoting by $f(t)$, $g(t)$ and $h(t)$ the components of $\xi(t)$ obtained by writing

$$\xi(t) = f(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + g(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} + h(t)\left(\frac{\partial}{\partial z}\right)_{\gamma(t)}$$

prove that the corresponding components of $\frac{\nabla\xi}{dt}$ are

$$\dot{f} + x(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{g} + y(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{h} + z(\dot{x}f + \dot{y}g + \dot{z}h).$$

(e) For $\gamma(t) = (0, \cos t, \sin t)$ show that the parallel transport $T^{0,t}_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ sends $\frac{\partial}{\partial z}$ to $-\sin t \cdot \frac{\partial}{\partial y} + \cos t \cdot \frac{\partial}{\partial z}$.

Next, we consider the complex structure on $E = TS^2$ from the previous list of exercises, with

$$J\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right) = (cy - bz)\frac{\partial}{\partial x} + (az - cx)\frac{\partial}{\partial y} + (bx - ay)\frac{\partial}{\partial z}.$$

(f) Prove that ∇ is a connection also on the complex vector bundle (TS^2, J).

(g) Show that $\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ can be seen as a complex frame of (TS^2, J) over $U_x = \{y^2 + z^2 \neq 0\}$ and compute the resulting $\mathbb{C}$-valued functions f_1 and f_2 on U_x such that

$$\mathfrak{e}^y = f_1 \cdot \mathfrak{e}^x, \quad \mathfrak{e}^z = f_2 \cdot \mathfrak{e}^x$$

and explain their meaning.

(h) Show that the connection matrix of ∇, seen as a connection on the complex vector bundle (E, J), w.r.t. $\mathfrak{e}^x$, is
$$\omega = \frac{ydy + zdz}{y^2 + z^2} + x\frac{ydz - zdy}{y^2 + z^2} \cdot i \in \Omega^1(U_x, \mathbb{C}).$$

(i) Prove that the curvature matrix is, up to a sign, i times the volume form of S^2, $\mu = xdy\,dz + ydz\,dx + zdx\,dy$. (Hint: you may want to notice first that $\frac{dxdy}{z} = \frac{dydz}{x} = \frac{dzdx}{y} = \mu$, whenever the expressions are defined).

(j) Prove that the Chern class of the complex vector bundle (E, J) is non-zero.

(k) Deduce the hairy ball theorem: S^2 does not admit any nowhere vanishing vector field.

(l) You may have encountered the fact that there is a diffeomorphism $\Phi : \mathbb{CP}^1 \to S^2$, but its precise form is not important for this question. Pulling back (E, J) one obtains a complex line bundle over $\mathbb{CP}^1$,
$$\mathcal{L} := \Phi^*(E, J).$$
Show that $\mathcal{L}$ is not isomorphic to the tautological line bundle over $\mathbb{CP}^1$, or to its conjugate,

The next exercise involves the tautological complex line bundle
$$L = \left\{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in l\right\}.$$
To obtain a connection on L we use the fact that it sits inside the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ and we use the general procedure we discussed for building connections in such a setting (now over $\mathbb{C}$):
$$\nabla : \mathfrak{X}(\mathbb{CP}^n) \times \Gamma(L) \to \Gamma(L), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_L\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(E)}\right) \tag{0.0.3}$$
or, if you prefer the dual point of view,
$$d_\nabla : \Gamma(L) \to \Omega^1(\mathbb{CP}^n, L), \quad d_\nabla(s) = \mathrm{pr}(d_{\nabla^{\mathrm{flat}}}(s)).$$
In more detail, the main pieces are

- the canonical flat connection ∇^{flat} on the trivial bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$.
- the standard hermitian product on $\mathbb{C}^{n+1}$:
$$h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1} + \ldots + z_n\overline{z'_n}$$
- for each $l \in \mathbb{CP}^n$, the orthogonal projection onto l with respect to h, $\mathrm{pr}_l : \mathbb{C}^{n+1} \to l$. Hence $\mathrm{pr}_l(z)$ is the unique point in l with the property that $z - \mathrm{pr}_l(z)$ is perpendicular to l w.r.t. h; it can be described in explicit formulas by:
$$\mathrm{pr}_l(z) = \frac{h(z, \xi)}{h(\xi, \xi)}\xi, \quad \text{with } \xi \in l \text{ a/any non-zero vector.}$$
- the map induced at the level of sections and, for d_∇, on forms with coefficients:
$$\mathrm{pr}_L : \Gamma(E) \to \Gamma(L), \quad \mathrm{pr}_L : \Omega^1(\mathbb{CP}^n, E) \to \Omega^1(\mathbb{CP}^n, L).$$

Exercise 11: We consider the tautological complex line bundle L over $\mathbb{CP}^n$, $n \geq 1$. From its very definition, L is a vector sub-bundle of the trivial vector bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$. On E we consider the canonical frame

$$\mathfrak{e}_0 = (1, 0, \ldots, 0) \quad \ldots \quad \mathfrak{e}_1 = (0, 1, \ldots, 0), \quad \mathfrak{e}_n = (0, 0, \ldots, 1).$$

Furthermore, we endow L with the connection (0.0.3).

(a) working in coordinates $(z_1, \ldots, z_n)$ corresponding to the standard chart (U_0, χ_0)

$$U_0 \ni [1 : z_1 : \ldots : z_n] \xmapsto{\chi_0} (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

show that $\mathfrak{f}_0 := \mathfrak{e}_0 + z_1\mathfrak{e}_1 + \ldots + z_n\mathfrak{e}_n$ can be seen as a local frame of L over U_0.

(b) consider U_1 and $\mathfrak{f}_1$ defined similarly, and compute the transition function from $\mathfrak{f}_0$ to $\mathfrak{f}_1$.

(c) According to the general theory, for any path $u : I \to L$ sitting above some path $\gamma : I \to \mathbb{CP}^n$ one can talk about the derivative $\frac{\nabla u}{dt}$, which is again a path in L above γ. Assume now that

$$\gamma(t) = [z_0(t) : z_1(t) : \ldots : z_n(t)] \in \mathbb{CP}^n, \quad u(t) = u_0(t)\mathfrak{e}_0(\gamma(t)) + \ldots + u_n(t)\mathfrak{e}_n(\gamma(t))$$

Show that

$$\frac{\nabla u}{dt} = \frac{\overline{z_0} \cdot u_0' + \overline{z_1} \cdot u_1' + \ldots + \overline{z_n} \cdot u_n'}{|z_0|^2 + |z_1|^2 + \ldots + |z_n|^2}(z_0\mathfrak{e}_0 + z_1\mathfrak{e}_1 + \ldots + z_n\mathfrak{e}_n).$$

(d) for $n = 1$ and $\gamma : [0, \pi] \to \mathbb{CP}^1$, $\gamma(t) = [1 : e^{2it}]$, compute the resulting parallel transport

$$T_\gamma^{0,t} : L_{\gamma(0)} \to L_{\gamma(t)}.$$

(e) consider the connection matrix of ∇ w.r.t. $\mathfrak{f}_0$, which will be a $\mathbb{C}$-valued 1-form on U_0, denoted

$$\omega \in \Omega^1(U_0, \mathbb{C}).$$

Prove that ω is equal to (the pull-back via χ_0 of)

$$\frac{\overline{z_1}dz_1 + \ldots + \overline{z_n}dz_n}{1 + |z_1|^2 + \ldots + |z_n|^2} \in \Omega^1(\mathbb{C}^n, \mathbb{C}),$$

where $z_k = x_k + i \cdot y_k$ are the coordinate functions on $\mathbb{C}^n$, dz_k is the $\mathbb{C}$-valued form $dx_k + i \cdot dy_k$.

(f) Consider now the curvature matrix of ∇ w.r.t. $\mathfrak{f}_0$, denoted k. For $n = 1$ prove that k is equal to (the pull-back via χ_0 of)

$$-\frac{1}{(1 + |z|^2)^2} \cdot dz \wedge d\overline{z} \in \Omega^2(\mathbb{C}, \mathbb{C});$$

(g) for general n, show that k can be described by the following short formula:

$$k = -\partial\overline{\partial} \log(1 + |z|^2) \in \Omega^2(\mathbb{C}^n, \mathbb{C})$$

where $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$, and then show that

$$k^n = (-1)^n \frac{n!}{(1 + |z|^2)^{n+1}} \cdot dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n.$$

(h) Show that $c_1(L) \neq 0$. Actually show that all the following classes are nonzero:

$$u := c_1(L) \in H^2(\mathbb{CP}^n), \; u^2 \in H^4(\mathbb{CP}^n) \; \ldots, \; u^n \in H^{2n}(\mathbb{CP}^n)$$

(i) Show that L is not isomorphic to its conjugate $\overline{L}$.

Exercise 12: Prove that the Chern classes of $T\mathbb{CP}^n$ and the Pontrjagin classes of the underlying real vector bundle can be expressed in terms of $a := -c_1(L)$ (minus the first Chern class of the tautological line bundle) as follows:

$$c(T\mathbb{CP}^n) = (1+a)^{n+1}, \quad p(T\mathbb{CP}^n) = (1+a^2)^{n+1}.$$

Exercise 13: If a F a complex vector bundle F over M has $c_1(F) \neq 0$, prove that: Let F be a complex vector bundle over M with $c_1(F) \neq 0$. Prove that

(a) F cannot be trivialisable.
(b) F cannot be isomorphic to its conjugate $\overline{F}$.
(c) F cannot be written as the complexification of a real vector bundle over M.
(d) If F is a complex line bundle, then the underlying real vector bundle cannot be trivializable.

Exercise 14:

(a) Show that if a real vector bundle E has the property that $E \oplus L$ is trivializable for some line bundle L, then $p_l(E) = 0$ for all $l \geq 1$.
(b) If a real vector bundle E has the property that $E \oplus F$ is trivializable for some vector bundle F of rank ≤ 3, what can you say about the Pontrjagin classes of E?
(c) Show that if an n-dimensional manifold M admits an embedding in $\mathbb{R}^{n+k}$ then there exists a rank k vector bundle $\mathcal{N}$ over M such that $TM \oplus \mathcal{N}$ is trivializable.
(d) Show that $\mathbb{CP}^2$ cannot be embedded in $\mathbb{R}^5$.