Collection of exercises

Mastermath Course Differential Geometry

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- Utrecht 2026-2027 -

Utrecht University

Contents

1 Vector bundles . . . 1
- 1.1 First properties and first examples . . . 1
- 1.2 Morphisms . . . 5
- 1.3 Inside trivial bundles . . . 7
- 1.4 Transition functions . . . 9
- 1.5 More algebraically: the module of sections . . . 12
- 1.6 Constructions/operations . . . 14
  - 1.6.1 Duals . . . 14
  - 1.6.2 Sums . . . 15
  - 1.6.3 Pull-backs . . . 16
  - 1.6.4 Quotients; normal bundles . . . 17
  - 1.6.5 Hom's . . . 18
  - 1.6.6 Tensor products . . . 19
  - 1.6.7 Short exact sequences . . . 20
  - 1.6.8 More operations involving complex vector bundles . . . 22
  - 1.6.9 The tangent bundle of the projective space . . . 23

2 Some geometric structures . . . 25
- 2.1 Parallelizations . . . 25
- 2.2 Complex structures . . . 25
- 2.3 Foliations . . . 27
- 2.4 Symplectic structures . . . 29

3 Connections . . . 33
- 3.1 Some generalities and simple examples: . . . 33
- 3.2 Connections induced on sub-bundles of trivial bundles . . . 36
- 3.3 Connections on complex vector bundles . . . 39
- 3.4 Parallel transport . . . 44
- 3.5 Compatible connections . . . 47
- 3.6 Levi Civita . . . 49
- 3.7 Characteristic classes . . . 51
- 3.8 More applications of characteristic classes . . . 54

4 Principal bundles . . . 57
- 4.1 Examples . . . 57
- 4.2 Frame bundles . . . 61
- 4.3 Attaching fibers . . . 63
- 4.4 List of exercises given on December 12, 2024 . . . 65

5 Previous exams 69
5.1 Retake Diff Geom mastermath, February 12, 2026 69
5.2 Exam Diff Geom mastermath, January 15, 2026 71
5.3 Retake mastermath Diff Geom, February 6, 2024 73
5.4 Exam Differential Geometry (mastermath), January 9, 2024 77
5.5 Exam Differential Geometry, Jan 12 2022, 14.00–17.00 79
5.6 Exam Differential Geometry (mastermath) 2015/2016 83
5.7 Take-home exam for the master course "Differential Geometry" - January 4, 2014, to be handed in by February 3, 2014- 86

Chapter 1
Vector bundles

1.1 First properties and first examples

Exercise 1.1. Prove that that the projection map $\pi : E \to M$ of any vector bundle is a submersion.

Exercise 1.2. Show that any section $s \in \Gamma(E)$ of a vector bundle E over M, when seen as a map

$$s : M \to E,$$

is an embedding of M in E (an embedding of manifolds).

Exercise 1.3. Let $E \to M$ be a vector bundle. We write $\Gamma(E)$ for its space of smooth sections. Let $s, t \in \Gamma(E)$ and let $f \in C^\infty(M)$. Show that $s + t$ and $f \cdot s$ are again smooth sections. (Note that these operations are defined pointwise $(s + t)(x) = s(x) + t(x)$ and $(f \cdot s)(x) = f(x)s(x)$.

Exercise 1.4. Prove that a line bundle $L \to M$ is trivializable if and only if it admits a nowhere vanishing section.

Exercise 1.5. We first work over an arbitrary manifold M.

(a) Assuming that $\pi : L \to M$ is a line bundle over M, show that finding a trivialization of L is equivalent to finding a section of L that is nowhere vanishing.
(b) When $L = \mathbb{R}_M$, the trivial line bundle over M, describe a bijection between the sections of L and $\mathbb{R}$-valued smooth function on M.

Assume now that L is the tautological line bundle over the n-dimensional projective space $M = \mathbb{RP}^n$.

(c) Describe now a bijection between sections of L and $\mathbb{R}$-valued smooth functions on on the n-sphere, $f : S^n \to \mathbb{R}$, with the property that $f(-x) = -f(x)$ for all $x \in S^n$.
(d) Deduce that L is not trivialisable.

Exercise 1.6. Let $\pi : E \to M$ be a rank k vector bundle. Let s be a set-theoretic section of E i.e. s is a function $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. Recall that with respect to a local frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_k\}$, we obtain a decomposition

$$s(x) = f^1(x)\mathfrak{e}_1(x) + \cdots + f^k(x)\mathfrak{e}_k(x)$$

Show that s is smooth if and only if with respect to every local frame $(U, \{\mathfrak{e}_i\}_{i=1}^k)$, the coefficients $\{f^i : U \to \mathbb{R}\}$ are smooth functions.

Vector bundles often arise in physics when trying to make sense of physical quantities globally. Such physical quantities are usually defined locally, by explicit formulas written down in coordinates- and those formulas describe one, or more, functions. How to make sense of the physical quantity globally, independent of the coordinates? The problem is that functions can rarely be globalised to functions. That is where vector bundles come in, since their sections are global objects which, locally, can be represented by functions. In other words, one is now invited to interpret such physical quantities as sections of appropriate vector bundles. Now, how does one build those vector bundles? The trick is that, since in each case we are talking about concrete physical quantities, one can compute explicitly how the functions change when one changes coordinates ... and those reveal the transition functions of the vector bundle to consider ... and hopefully you will even be able to recognize the vector bundle itself, described intrinsically (so as to avoid the valid but unsatisfying "solution": define the vector bundle via its transitions). The next exercise is an illustration of the "game" described above, more mathematical.

Exercise 1.7. Let M be a manifold. We consider real-valued smooth functions $f \in C^\infty(M)$ and the main question is that of making sense globally of "the gradient of f". Recall that any chart (U, χ) of M allows us to talk about *the partial derivatives of f w.r.t. the chart* (U, χ),

$$\frac{\partial f}{\partial \chi_1}, \ldots, \frac{\partial f}{\partial \chi_m} \in C^\infty(U), \tag{1.1.1}$$

where $\frac{\partial f}{\partial \chi_i}(p)$ is defined as the i-th partial derivative of $f \circ \chi^{-1} : \chi(U) \to \mathbb{R}$ computed at the point $\chi(p)$:

$$\frac{\partial f}{\partial \chi_i}(p) = \frac{\partial f \circ \chi^{-1}}{\partial x_i}(\chi(p)) = \left.\frac{d}{dt}\right|_{t=0} \chi^{-1}(\chi_1(p), \ldots, \chi_i(p) + t, \ldots, \chi_n(p))$$

In local coordinates, "the gradient of f" should be encoded in the collection (1.1.1), arranged in a matrix.

You are asked to describe a vector bundle $E \to M$ and a section s_f of E so that, locally w.r.t. certain local frames of E, (1.1.1) are precisely the corresponding coefficients of s_f.
(Hint, recall that when moving to another chart (U', χ'), one can express the similar functions $\frac{\partial f}{\partial \chi'_i}$ in terms of the previous ones, as well as the (partial derivatives of the) change of coordinates $c = \chi' \circ \chi^{-1}$. Try to remember that formula, or deduce it. That should indicate the transition functions of E. Hopefully you will recognize E ... and then s_f.)

Exercise 1.8 (the tautological line bundle over the real projective space). First of all, I advise you to write down/look up the standard atlas $(U_0, \chi_0), \ldots, (U_n, \chi_n)$ that is used to describe the smooth structure on $\mathbb{RP}^n$. Consider now the tautological line bundle

$$L := \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in l\},$$

with the projection $\pi : L \to M$, $\pi(l, v) = l$, and with the vector space structure on its fibers given by

$$(l, v) + (l, w) = (l, v + w), \quad \text{etc.}$$

Assume that you already know that L is an embedded submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ and π is a submersion. In particular, L will be a manifold itself (with the smooth structure induced from $\mathbb{RP}^n \times \mathbb{R}^{n+1}$). Using what is discussed in the first part of the chapter :

(a) Describe, for each $i \in \{1, \ldots, n\}$, a nowhere vanishing local section s_i of L defined over U_i.
(b) Deduce that each s_i gives rise to a trivialization of $L|_{U_i}$,

$$h_i : L|_{U_i} \to U_i \times \mathbb{R},$$

and write h_i down explicitly.
(c) Using h_i and the chart $\chi_i : U_i \to \mathbb{R}^n$ (see above) construct an explicit manifold chart of L,

$$\widetilde{\chi}_i : \widetilde{U}_i \to \mathbb{R}^{n+1}, \quad \text{with domain } \widetilde{U}_i := L|_{U_i}.$$

(d) Now, assume that you didn't know that L is an embedded submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$. Use then the explicit $\widetilde{\chi}_i$ you found above to built a smooth structure on L (making it into an $n + 1$-dimensional manifold). Don't forget to check that your atlas is smooth.
(e) Continuing (e) and still no longer being allowed to use that L is an embedded submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$): show that π is a submersion and deduce that L is a vector bundle.
(f) Continuing (e) and (f): prove now that L is, indeed, an embedded submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$.

The projective space and the tautological line bundle over it makes sense almost identically also when working over $\mathbb{C}$. First of all, the complex projective space $\mathbb{CP}^n$ consists of complex lines (i.e. complex 1-dimensional subspaces) $l \subset \mathbb{C}^{n+1}$ and, denoting by $[z_0 : z_1 : \ldots : z_n]$ the line through an arbitrary nonzero vector $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$, we can write

$$\mathbb{CP}^n = \Big\{ [z_0 : z_1 : \ldots : z_n] : (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \Big\}$$

where

$$[z_0 : z_1 : \ldots : z_n] = [z'_0 : z'_1 : \ldots : z'_n] \quad \Longleftrightarrow \quad \exists\ z \in \mathbb{C} \ \text{ s.t. } \ z'_k = z \cdot z_k \ \ \forall \ \ k \in \{0, 1, \ldots, n\}.$$

As for $\mathbb{RP}^n$, also $\mathbb{CP}^n$ comes with standard charts (U_0, χ_0), (U_1, χ_1), …, (U_n, χ_n), with

$$U_0 := \Big\{ [z_0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n : z_0 \neq 0 \Big\} = \Big\{ [1 : z_1 : \ldots : z_n] : (z_1, \ldots, z_n) \in \mathbb{C}^n \Big\}, \tag{1.1.2}$$

and with $\chi_0 : U_k \to \mathbb{C}^n$ being the inverse of $(z_1, \ldots, z_n) \mapsto (1 : z_1 : \ldots : z_n]$:

$$\chi_k : U_k \to \mathbb{C}^n, \quad [z_0 : z_1 : \ldots : z_n] \longleftrightarrow \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right)$$

(and similarly for (U_k, chi_k) for all k). The difference from the real case (actually a novelty more than a difference) is the fact that the coordinate changes are holomorphic, hence $\mathbb{CP}^n$ is not only a smooth manifold, but actually a holomorphic (or complex) manifold. The tautological line bundle over $\mathbb{CP}^n$ is now

$$L = \big\{ (l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in l \big\},$$

which is a complex vector bundle of (complex) rank 1 over $\mathbb{CP}^n$ with projection $(l, z) \mapsto l$.

Exercise 1.9. Prove that the tautological line bundle over $\mathbb{CP}^n$ (see above) is a complex line bundle over $\mathbb{CP}^n$ and, at the same time, a real vector bundle of rank two.

Still as in the real case, when working with $[z_0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n$ one can always renormalise and use only $(z_0, z_1, \ldots, z_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, i.e. think of $\mathbb{CP}^n$ as

$$\mathbb{CP}^n = S^{2n+1}/S^1,$$

the quotient of S^{2n+1} modulo the action of S^1 induced by complex multiplication:

$$\lambda \cdot (z_0, z_1, \ldots, z_n) = (\lambda z_0, \lambda z_1, \ldots, \lambda z_n).$$

Exercise 1.10. Here L is still the tautological line bundle over $\mathbb{CP}^n$. Prove that for any section $s \in \Gamma(L)$ there exists and is unique a smooth function $f : S^{2n+1} \to \mathbb{C}$ such that s is given by

$$\mathbb{CP}^n \ni [z_0 : z_1 : \ldots : z_n] \longmapsto (f(z)z_0, f(z)z_1, \ldots, f(z)z_n)$$

for $z = (z_0, z_1, \ldots, z_n) \in S^{2n+1}$. Furthermore, prove that f satisfies the S^1-invariance condition:

$$f(\lambda z) = \frac{1}{\lambda} f(z) \quad \forall\ \lambda \in S^1,\ z \in S^{2n+1}. \tag{1.1.3}$$

Finally, prove that this gives rise to a linear isomorphism between $\Gamma(L)$ and the space of smooth, $\mathbb{C}$-valued functions on S^{2n+1} which are S^1-invariant:

$$\Gamma(L) \cong C^\infty(S^{2n+1})^{S^1-\mathrm{inv}}.$$

For instance, $f_0(z) = \overline{z}_0$ is such an invariant function, and that gives rise to the section

$$s_0([z_0 : z_1 : \ldots : z_n]) = \left(|z_0|^2, \overline{z}_0 z_1, \ldots, \overline{z}_0 z_n\right);$$

in turn, s_0 can be seen as a local frame of L over U_0 from (1.1.2). One can also try now to prove directly that L is not trivializable, neither as a complex nor as a real vector bundle. Characteristic classes developed later on are extremely useful in proving such statements.

The invariance condition (1.1.3) is the reason that the tautological line bundle is sometimes denoted

$$L = \mathcal{O}(-1) \tag{1.1.4}$$

Is there also an "$\mathcal{O}(1)$", where the invariance condition would be

$$g(\lambda z) = \lambda g(z) \quad \forall\ \lambda \in S^1,\ z \in S^{2n+1}? \tag{1.1.5}$$

To achieve that we use the conjugation on $\mathbb{C}$ and its obvious extension to $\mathbb{C}^{n+1}$:

$$\overline{z} := (\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_n), \quad \text{for} z = (z_0, z_1, \ldots, z_n).$$

Exercise 1.11. Over $\mathbb{CP}^n$ we consider

$$L' := \left\{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \overline{z} \in l\right\},$$

Prove that L' is a complex vector bundle over $\mathbb{CP}^n$ and its sections are in bijection with smooth function $g : S^{2n+1} \to \mathbb{C}$ satisfying the invariance condition (1.1.5).

How different is L' of the last exercise from the tautological L? Well, as real vector bundles they are isomorphic! (Why?) However, the fact that their space of sections exhibit rather different behaviour w.r.t.

complex multiplication *indicates* that the two are not isomorphic as complex vector bundles. Again, once we learn about Chern classes, this expectation can be turned into a theorem.

Next an exercise that indicates the usefulness of working over $\mathbb{C}$ even when interested in real vector bundles (much more on this to come later!).

Exercise 1.12 (the complex structure on TS^2 and the hairy ball theorem). Prove that:

(a) The tangent bundle TS^2 of S^2 can be made into a complex vector bundle where, next to the addition and multiplication by real scalars that we already have at our disposal, we also introduce the multiplication by i: on T_pS^2 for $p = (x, y, z)$, it is given by

$$i \cdot \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \right) = (cy - bz)\frac{\partial}{\partial x} + (az - cx)\frac{\partial}{\partial y} + (bx - ay).$$

(a) In general, if a rank two real vector bundle $E \to M$ can be made into a complex vector bundle, then the existence of a nowhere vanishing section of E implies that E must be trivializable.
(c) Conclude that the hairy ball theorem is equivalent to the fact that TS^2 is not trivializable.

1.2 Morphisms

Exercise 1.13. Show that for any smooth map between manifolds, $f : M \to N$, its differential df makes sense as a morphism from TM to f^*TN.

Exercise 1.14. Let $u : E \to F$ be a morphism of vector bundles. Suppose that fibrewise u is a linear isomorphism i.e. $u_x := u|_{E_x} : E_x \to F_x$ is a linear isomorphism for all $x \in M$. Show that u is an isomorphism of vector bundles i.e. u has an inverse that is also a morphism of vector bundle. In particular, show that u^{-1} is smooth.

Exercise 1.15. Let $u : F \to E$ be a morphism between two vector bundles over M, consider the restrictions to the fibers

$$u_x = u|_{F_x} : F_x \to E_x,$$

and we fix an point $x \in M$. Please do the following:

(a) Prove that if u_x is surjective then there is an entire open neighborhood U of x s.t. u_y is surjective for all $y \in U$.
(c) Then generalise this to a statement about the ranks of the linear maps u_x:

$$\operatorname{rank}_x(u) := \dim\left(\operatorname{Im}(u_x)\right),$$

namely: for any morphism u and any $x \in M$, there exists an open neighborhood U of x such that

$$\operatorname{rank}_y(u) \geq \operatorname{rank}_x(u) \quad \text{for all } y \in U.$$

(b) State and prove a similar property for the dimensions of the kernels of the maps u_x.

Exercise 1.16. Assume that we have an exact sequence of vector bundles over M,

$$E^0 \xrightarrow{u} E^1 \xrightarrow{v} E^2$$

i.e. E^0, E^1 and E^2 are vector bundles over M, u and v are morphisms of vector bundles, and they satisfy the exactness condition

$$\mathrm{Im}(u) = \mathrm{Ker}(v), \tag{1.2.1}$$

i.e. $\mathrm{Im}(u_x) = \mathrm{Ker}(v_x)$ for all $x \in M$. Prove that (1.2.1) is a (smooth) vector sub-bundle of E^1.

Exercise 1.17. Show that if F^1 and F^2 are vector sub-bundles of a vector bundle E and, at some $x \in M$,

$$E_x = F^1_x + F^2_x$$

(meaning that any vector in E_x is a sum of one from F^1_x and one from F^2_x), then this equality continue to hold at all points in an entire open neighbourhood of x.

(compare with Exercise 1.15)

Exercise 1.18. Show that any inner product g on E induces an isomorphism of vector bundles

$$I_g : E \xrightarrow{\sim} E^*,$$

uniquely characterised by the condition that $\langle I_g(s_1), s_2\rangle = g(s_1, s_2)$ for any $s_1, s_2 \in \Gamma(E)$.

1.3 Inside trivial bundles

The discussions here apply to both to real as well as to complex vector bundle. Therefore let us look at $\mathbb{K}$-vector bundles where

$$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}.$$

Exercise 1.19. The main message of this exercise is that, inside the trivial $\mathbb{K}$-bundles

$$E = M \times \mathbb{K}^l,$$

one can find very interesting/nontrival ones. Actually, any vector bundle over M can be realised as sitting inside some trivial bundle $M \times \mathbb{K}^l$ for l large enough. Here are some basic examples that you are supposed to explain in full detail:

(a) the tautological line bundle L over $\mathbb{CP}^n$; hence $M = \mathbb{CP}^n$, $\mathbb{K} = \mathbb{C}$, but what is l and what is the inclusion of L into the trivial bundle of rank l?
(b) the tangent bundle TS^n of the n-sphere naturally sits inside a trivial bundle; hence here $M = S^n$, $\mathbb{K} = \mathbb{R}$, but what is l and what is the inclusion of TS^n into the trivial bundle of rank l?

Exercise 1.20. Show that whenever we have an embedded submanifold N of a manifold M, TN is a vector sub-bundle of $TM|_N$:

$$TN \subset TM|_N.$$

On the other hand, recall that any manifold N can be embedded into some $\mathbb{R}^k$, for k large enough. Deduce that, for any manifold N, TN can be embedded in a trivial vector bundle.

Exercise 1.21. Consider the trivial vector bundle of rank 4 over S^2,

$$\widetilde{E} := S^2 \times \mathbb{R}^4,$$

with the canonical frame denoted e_1, e_2, e_3, e_4 (seen as sections of $\widetilde{E}$). We consider the four new sections σ_1, σ_2, σ_1', σ_2' of $\widetilde{E}$ that send $p = (a, b, c) \in S^2$ to:

$$\begin{cases} \sigma_1(p) = (1+a)e_1 + ce_3 + be_4 \\ \sigma_2(p) = (1+a)e_2 - be_3 + ce_4 \end{cases} \begin{cases} \sigma_1'(p) = ce_1 - be_2 + (1-a)e_3 \\ \sigma_2'(p) = be_1 + ce_2 + (1-a)e_4 \end{cases}$$

or, explicitly as elements of $S^2 \times \mathbb{R}^4$:

$$\begin{cases} \sigma_1(p) = (p, 1+a, 0, c, b) \\ \sigma_2(p) = (p, 0, 1+a, -b, c) \end{cases} \begin{cases} \sigma_1'(p) = (p, c, -b, 1-a, 0) \\ \sigma_2'(p) = (p, b, c, 0, 1-a) \end{cases}$$

Using these we define

$$E := \mathrm{Span}_{\mathbb{R}}(\sigma_1, \sigma_2, \sigma_1', \sigma_2') \subset \widetilde{E};$$

hence, at each $p = (a, b, c) \in S^2$, E_p is the linear space of $\sigma_1(p), \sigma_2(p), \sigma_1'(p), \sigma_2'(p)$. Please do the following:

(a) Show that E is a vector bundle of rank 2 (if it doesn't work in 15 min, look at the next item).
(b) Show that $\{\sigma_1, \sigma_2\}$ is a local frame of E over $S^2 \setminus \{(-1, 0, 0)\}$, and $\{\sigma_1', \sigma_2'\}$ over $S^2 \setminus \{(1, 0, 0)\}$.
(c) Compute the matrix of coordinate changes from the first frame to the second one.

(d) Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ (more precisely, (x, y, z, t) with $(x + iy, z + it)$), show that E becomes a $\mathbb{C}$-line bundle (i.e. of complex rank 1) over S^2, $\{\sigma_1\}$ becomes a $\mathbb{C}$-frame of E over S^2 minus p_S, and $\{\sigma_1'\}$ becomes a $\mathbb{C}$-frame of E over S^2 minus p_N.

Exercise 1.22. Let us focus now on complex line sub-bundles of the trivial line bundle of rank $l = k + 1$:

$$L \subset E = M \times \mathbb{C}^{k+1}$$

Given such an L:

(a) for $x \in M$, what can you say about the fiber L_x (just say the obvious, nothing fancy)?
(b) using the previous point, build now a map $f_L : M \to \mathbb{CP}^k$.
(c) what is the relationship between L, the map f_L and the tautological line bundle over $\mathbb{CP}^k$?
(d) explain why all the maps that enter this discussion (e.g. f_L) are smooth.

Next, assume that $M = S^2$ and $k = 1$, hence $E = S^2 \times \mathbb{C}^2$. Using the canonical frame $\{e_1, e_2\}$ of E, we define the sections σ, σ' of E given by

$$\begin{cases} \sigma(p) = (1 + u)e_1 + (w - iv)e_2 \\ \sigma'(p) = (w + iv)e_1 + (1 - u)e_2 \end{cases}$$

for $p = (u, v, w) \in S^2$. We also consider the complex span of these sections,

$$L := \mathrm{Span}_{\mathbb{C}}\{\sigma, \sigma'\} \subset E \quad \text{(pointwise)}.$$

(e) show that, although at each point $p = (u, v, w) \in S^2$ we used two vectors to span L_p, namely $\sigma(p)$ and $\sigma'(p)$, L is actually a complex *line* bundle.
(f) What is the relationship with the vector bundle(s) from Exercise 1.21?
(g) Use item (c) to build a map

$$\Psi : S^2 \to \mathbb{CP}^1.$$

(h) Show that this last map is actually a diffeomorphism between the manifolds S^2 and $\mathbb{CP}^1$.

Exercise 1.23. Using the previous exercise prove that

$$\Phi : \mathbb{CP}^1 \to S^2, \quad \phi([z_0 : z_1]) = \big(|z_0|^2 - |z_1|^2, 2i\overline{z}_0 z_1\big) \quad \text{for } (z_0, z_1) \in S^3$$

is a diffeomorphism. More precisely, write down explicitly the diffeomorphism $\Psi : S^2 \to \mathbb{CP}^1$ from the previous exercise, and then prove that Φ is its inverse.

1.4 Transition functions

General transition data: Recall that a "transition data on (a manifold) M with values in GL_r" (the group of invertible matrices of size r) consists of:

(TD1) an open cover $\{U^\alpha\}_{\alpha\in\Lambda}$ of M, indexed by some set Λ.
(TD2) for each $\alpha,\beta\in\Lambda$, a smooth function with values in the group GL_r of $r\times r$ invertible matrices with real entries,

$$\tau^{\alpha,\beta}: U^\alpha\cap U^\beta\to \mathrm{GL}_r$$

such that the following "cocycle condition" is satisfied: for all $\alpha,\beta,\gamma\in\Lambda$, one has

$$\tau^{\alpha,\beta}\cdot\tau^{\beta,\gamma}=\tau^{\alpha,\gamma},\quad \text{over } U^\alpha\cap U^\beta\cap U^\gamma,$$

i.e., for all $x\in U^\alpha\cap U^\beta\cap U^\gamma$, $\tau^{\alpha,\beta}(x)\cdot\tau^{\beta,\gamma}(x)=\tau^{\alpha,\gamma}(x)$ (equality of matrices).

Exercise 1.24. Show that for any such transition data one has:

(a) $\tau^{\alpha,\alpha}=\mathrm{Id}_r$ for all $\alpha\in\Lambda$, where $\mathrm{Id}_r\in\mathrm{GL}_r$ is the identity matrix.
(b) for all $\alpha,\beta\in\Lambda$, $\tau^{\beta,\alpha}=(\tau^{\alpha,\beta})^{-1}$ (inversion of matrices).

The relationship of transition data with vector bundles of rank r was two fold:

From vector bundles to transition data: To any rank r vector bundle $E\to M$ one can associate a transition data with values in GL_r as follows. The data will not be unique and will depend on the following choices:

(c1) an open cover $\{U^\alpha\}_{\alpha\in\Lambda}$ of M s.t., over each U^α, E admits a local frame.
(c2) accordingly, for each α we fix a local frame $\mathfrak{e}^\alpha$ of E over U^α.

Once these choices are be made, the transition data is give by

$$\tau^{\alpha,\beta}:=[\mathfrak{e}^\beta:\mathfrak{e}^\alpha]$$

or, more explicitly,

$$\tau^{\alpha,\beta}=\begin{pmatrix}\left(\tau^{\alpha,\beta}\right)_1^1 & \dots & \left(\tau^{\alpha,\beta}\right)_r^1\\ \dots & \dots & \dots\\ \left(\tau^{\alpha,\beta}\right)_1^r & \dots & \left(\tau^{\alpha,\beta}\right)_r^r\end{pmatrix}\quad\text{with defining equations:}\quad \mathfrak{e}_i^\beta=\sum_j\left(\tau^{\alpha,\beta}\right)_i^j\mathfrak{e}_j^\alpha.$$

Exercise 1.25. Show that for any choices of U^α and $\mathfrak{e}^\alpha$ as above, the $\tau^{\alpha,\beta}$ define a transition data on M with values in GL_r.

Different starting choices will produce very different transition data- even the indexing set Λ may change! The precise relationship between such different transition data, although can be described precisely, is difficult to work with.

From transition data to vector bundles: Conversely, starting with a transition data $\{U^\alpha,\tau^{\alpha,\beta}\}$ on a manifold M one can construct a vector bundle E. The idea is to start by putting together all the trivial bundles $U^\alpha\times\mathbb{R}^r$, i.e. consider the disjoint union of these spaces, described as

$$\{(\alpha,x,v):\alpha\in\Lambda,x\in U^\alpha,v\in\mathbb{R}^r\},$$

and define E as the quotient of this last space modulo the equivalence relation

$$(\alpha, x, v) \sim (\beta, y, w) \quad \Longleftrightarrow \quad y = x,\ w = \tau^{\beta,\alpha}(v).$$

Exercise 1.26. Show that, indeed, this is an equivalence relation.

Denoting by $[\alpha, x, v]$ the resulting equivalence classes, we can write

$$E = \{[\alpha, x, v] : \alpha \in \Lambda, x \in U^\alpha, v \in \mathbb{R}^r\}$$

and there is an obvious map π ("projection") from E to M: send $[\alpha, x, v]$ to x. Furthermore, for each $\alpha \in \Lambda$, there is also an obvious "local frame $\mathfrak{e}^\alpha$ over U^α", with components

$$\mathfrak{e}^\alpha_i : U^\alpha \to E, \quad x \mapsto [\alpha, x, e^{\text{can}}_i],$$

where $e^{\text{can}}_1, \dots, e^{\text{can}}_r$ is the usual basis of $\mathbb{R}^r$. Here we use quotes because the terms make sense only when we already know that E is a vector bundle (which is what we are trying to explain/prove here!). These "local frames" are relevant for two reasons. First of all, with the mind at the fact that local frames encode local trivializations, we are led to consider the bijections

$$h^\alpha : \pi^{-1}(U^\alpha) \to U^\alpha \times \mathbb{R}^r, \quad [(\alpha, x, v)] \mapsto (x, v_1, \dots, v_r).$$

Exercise 1.27. Use these maps h^α to build a smooth structure on E so that E becomes a smooth vector bundle for which all $\mathfrak{e}^\alpha_i$ are smooth sections.

If you already did the exercise of proving Lemma 1.16 from the notes, then you can use that lemma here; if not, then use this exercise to prove that lemma as well.

Secondly, once we know that E is a vector bundle, since it comes with the local frames $\mathfrak{e}^\alpha$, by the general theory discussed above, one obtains a transition data on M.

Exercise 1.28. Prove that the resulting transition data is precisely the transition data $\{U^\alpha, \tau^{\alpha,\beta}\}$ we started with.

Exercise 1.29. Describe now a transition data that for the tautological line bundle over $\mathbb{RP}^n$ (notice that, since $r = 1$, the transition functions now take values in $\mathbb{R}^*$!).

Small transition data: Next, let us call a transition data to be **small** if Λ consists of only two elements. Given the first exercise, a small transition data on M with values in GL_r will be the same thing as an open cover $\{U^1, U^2\}$ of M together with a smooth map

$$\tau = \tau^{1,2} : U^1 \cap U^2 \to \mathrm{GL}_r$$

(then $\tau^{2,1} = (\tau^{1,2})^{-1}$, $\tau^{1,1} = \tau^{2,2} = \mathrm{Id}$). Small data can give rise to very interesting/non-trivial vector bundles, already for $r = 1$, when $\mathrm{GL}_1 = \mathbb{R}^*$. Indeed:
Later on in the course we will be proving that any vector bundle over a contractible manifold is trivializable. Assuming you already know that, do now the following:

Exercise 1.30. Over $M = S^1$ we consider the small transition data with values in $\mathrm{GL}_1 = \mathbb{R}^*$ defined by the open cover

$$U^1 = \{(x, y) \in S^1 : y > -\frac{1}{2}\}, \quad U^2 = \{(x, y) \in S^1 : y < \frac{1}{2}\}$$

and τ given by $\tau(x, y) = \mathrm{sign}(x) = \frac{x}{|x|}$. What is the corresponding vector bundle?

Exercise 1.31. Show that any vector bundle over a sphere S^n (is isomorphic to one that) is induced by a small transition data.

Yes, the last question is not very precise; we are interested here in seeing your insight into this discussion, based on intuition rather than precise formulas. So, please use picture to explain your answer!

Exercise 1.32. Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, over which we consider the vector bundle $E := TS^2$. We also consider the following

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \qquad (\star)$$

Since $S^2 \subset \mathbb{R}^3$, we can view E as a sub-bundle of the trivial bundle $\widetilde{E} = S^2 \times \mathbb{R}^3$.

(a) Describe the largest $U \subset S^2$ over which $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ is a local frame of TS^2.
(b) Consider the similar local frame $\{\mathfrak{e}^x, \mathfrak{e}^z\}$ over the corresponding $V \subset S^2$ and compute the transition functions from $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ to $\{\mathfrak{e}^x, \mathfrak{e}^z\}$.

Exercise 1.33. We consider the tautological complex line bundle L over $\mathbb{CP}^n$. From its very definition, L is a vector sub-bundle of the trivial vector bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$. On E we consider the canonical frame

$$\mathfrak{e}_0 = (1, 0, \ldots, 0), \quad \mathfrak{e}_1 = (0, 1, \ldots, 0), \quad \mathfrak{e}_n = (0, 0, \ldots, 1).$$

We also consider the orthogonal complement of L in E,

$$L^\perp := \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\},$$

where the orthogonality $\perp$ is with respect to the standard hermitian product on $\mathbb{C}^{n+1}$:

$$h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1} + \ldots + z_n\overline{z'_n}.$$

(a) working in coordinates $(z_1, \ldots, z_n)$ corresponding to the standard chart (U_0, χ_0)

$$U_0 \ni [1 : z_1 : \ldots : z_n] \overset{\chi_0}{\longmapsto} (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

show that $\mathfrak{f}_j := -\overline{z}_j\mathfrak{e}_0 + \mathfrak{e}_j$ with $j \in \{1, \ldots, n\}$ form a local frame of $L^\perp$ over U_0.
(b) describe the similar frame $\mathfrak{f}'$ of $L^\perp$ in the similar chart (U_1, χ_1) and compute the transition functions from $\mathfrak{f}$ to $\mathfrak{f}'$.

1.5 More algebraically: the module of sections

Vector bundles can be approached also more algebraically, via the $C^\infty(M)$-module of sections that they give rise to. To explain this we start by showing that if two vector bundles E and F over M give rise to isomorphic $C^\infty(M)$-modules, then the vector bundles must be isomorphic. Actually, we show something more: any morphism of $C^\infty(M)$-modules $\Gamma(E) \to \Gamma(F)$ arises from a morphism of vector bundles.

Exercise 1.34. Assume that E and F are two vector bundles over M.

(a) Show that for any morphism of vector bundles $u : E \to F$, the map

$$u_* : \Gamma(E) \to \Gamma(F), \quad u_*(s) := u \circ s$$

is well defined and is a morphism of $C^\infty(M)$-modules.

(b) For $x \in M$ show that, for any $v \in E_x$,

(b1) there exists a section $s \in \Gamma(E)$ such that $s(x) = v$.

(b2) if s, s' are two sections as in (b1), then one can write $s' - s = f^1 \cdot s_1 + \ldots + f^k \cdot s_k$ for some sooth functions $f^1, \ldots, f^k \in C^\infty(M)$ and some sections $s_1, \ldots, s_k \in \Gamma(E)$ with the property that $f^1(x) = \ldots = f^k(x) = 0$.

(c) Show that, given a morphism of $C^\infty(M)$-modules

$$U : \Gamma(E) \to \Gamma(F),$$

there exists a morphism of vector bundles $u : E \to F$ such that $U = u_*$.

(d) Prove now that E and F are isomorphic vector bundles if and only if $\Gamma(E)$ and $\Gamma(F)$ are isomorphic $C^\infty(M)$-modules.

In a more categorical language, taking sections defines a functor

$$\Gamma : \mathcal{V}ect(M) \to \mathcal{M}od(C^\infty(M))$$

from the category of vector bundles over M to the one of $C^\infty(M)$, and this functor is fully faithful. Of course, the natural question is to identify which $C^\infty(M)$-modules arise from vector bundles. To that end, we recall (or take as definition) that, given a ring R, an R-module $\mathcal{E}$ is a *finitely generated projective R-module* if and only if there exists another R-module $\mathcal{E}'$ such that $\mathcal{E} \oplus \mathcal{E}'$ is isomorphic to a free module R^k, for some $k \in \mathbb{Z}$ (you can take this as definition).

Exercise 1.35. Show that, for any vector bundle E over M, $\Gamma(E)$ is a finitely generated projective $C^\infty(M)$-module.

Finally, you can now try to prove the following, which can be seen as a harder exercise:

Exercise 1.36. For any finitely generated projective $C^\infty(M)$-module $\mathcal{E}$ there exists a vector bundle $E \to M$ such that $\mathcal{E}$ is isomorphic to $\Gamma(E)$.

All together, one obtains an equivalence between the category of vector bundles over M and that of initely generated projective $C^\infty(M)$-modules (know as **the Serre-Swan Theorem**). Therefore, for the more algebraically oriented reader, it will always be tempting to rewrite whatever part of the theory in purely algebraic terms (instructive exercise indeed!). One should be aware however that care is required. For instance

Exercise 1.37. Prove that if a morphism $u : E \to F$ is injective then so is $u_* : \Gamma(E) \to \Gamma(F)$, but the converse is not true.

Exercise 1.38. However, it is true that, given a vector bundle morphism $u : E \to F$

$$u \text{ is surjective} \quad \Longleftrightarrow \quad u_* : \Gamma(E) \to \Gamma(F) \text{ is surjective.}$$

Here you are asked to prove the reverse implication (the direct will come a bit later). (Hint: look at Exercise 1.34).

1.6 Constructions/operations

1.6.1 Duals

We start with duals. To continue the discussion above, let us look at the corresponding modules. The pairing between E^* and E defined by evaluation, upgraded to sections, is often denoted:

$$\langle \cdot, \cdot \rangle : \Gamma(E^*) \times \Gamma(E) \to C^\infty(M);$$

hence, for $s^* \in \Gamma(E^*)$ and $s \in \Gamma(E)$, $\langle s^*, s \rangle$ is the smooth function on M given by $x \mapsto s^*_x(s_x)$.

Exercise 1.39. Given a vector bundle E over M, prove that $\Gamma(E^*)$ is isomorphic to the dual of the $C^\infty(M)$ module $\Gamma(E)$:

$$\Gamma(E^*) \cong \Gamma(E)^*.$$

Here, if you do not remember/know what dual $\mathcal{E}^*$ of an R-module $\mathcal{E}$ is (for an arbitrary commutative ring R), then define the concept yourself in such a way that the claim in the exercise holds.

Exercise 1.40. Given a vector bundle E over M and a vector sub-bundle $F \subset E$, prove that:

(a) the annihilator of F in E,

$$\mathrm{Ann}_E(F) := \Big\{ \xi \in E^* : \xi | F = 0 \Big\},$$

is a vector sub-bundle of E^*;

(b) the annihilator can be identified with the dual of the quotient of E modulo F:

$$\mathrm{Ann}_E(F) \cong (E/F)^*;$$

(c) If g is an inner product on E, what is the relationship between the annihilator and the g-orthogonal of F in E? (Hint: have a look also at Exercise 1.18).

Next, we see how "taking duals" can be applied also to morphisms (turning it into a functor).

Exercise 1.41. Given a morphism of vector bundles over M, $u : E \to F$, recall that $u_x : E_x \to F_x$ denotes the restriction of u to the fibers above $x \in M$. Each such restriction can be "dualised" to a map

$$u^T_x : F^*_x \to E^*_x, \quad \xi_x \mapsto \xi_x \circ u_x$$

and, together these define a map denoted $u^T : F^* \to E^*$ and called the transpose of u. Prove that:

(a) u^T is indeed a morphism of vector bundles.
(b) Describe the kernel and the image of u^T in terms of the ones of u.

Exercise 1.42. Let $u : E \to F$ be a morphism of vector bundles over M, and assume that the two vector bundles are endowed with inner products g_E and g_F, respectively. Prove that there exists and is unique a morphism of vector bundles Assume now that we have a second vector bundle $\pi_F : F \to M$ with fiberwise inner product g_F, and that $u : E \to F$ is a vector bundle morphism.

(a) Prove that there exists and is unique a morphism of vector bundles

$$u^* : F \to E$$

(called *the adjoint of u with respect to the fiberwise inner products g_E and g_F*) such that

$$g_F(u^*(e), f) = g_E(e, u(f)) \quad \forall\ e \in \Gamma(E), f \in \Gamma(F).$$

(b) Describe the kernel and the image of u^* in terms of the ones of u.
(c) Through a commutative diagram, describe the relationship between u^* and the transpose u^T from Exercise 1.41. Don't forget to prove that the diagram is commutative.

Exercise 1.43. Given a vector sub-bundle F of E, for any inner product g on E, the restriction of the quotient map $E \to E/F$ to the g-orthogonal $F^\perp$ induces an isomorphism of vector bundles $F^\perp \to E/F$.

1.6.2 Sums

Sums were already appearing (at least implicitly) in some of the previous exercises.

Exercise 1.44. Show that the direct sum of the tangent bundle TS^n with the trivial line bundle $S^n \times \mathbb{R}$ is isomorphic to the trivial vector bundle of rank $n+1$, $S^n \times \mathbb{R}^{n+1}$:

$$TS^n \oplus \mathbb{R}_{S^n} \cong \mathbb{R}^{n+1}_{S^n}.$$

Exercise 1.45. This is to check that "the abstract direct sum" is compatible with the concrete direct sums. To that end, assume that E^1 and E^2 are two vector sub-bundles of a vector bundle E and define the map

$$h : E^1 \oplus E^2 \to E, \quad (e^1, e^2) \mapsto e^1 + e^2.$$

Show that:

(a) h is a vector bundle morphism.
(b) h is an isomorphism if and only if, for each $x \in M$, E_x is the direct sum of E^1_x and E^2_x, i.e.,

$$E_x = E^1_x + E^2_x, \quad E^1_x \cap E^2_x = \{0\}.$$

The following draws the attention of Example 1.21 in the notes, and its first corollary.

Exercise 1.46. F be a vector sub-bundle of a vector bundle E. Choose an inner product g on E and consider the orthogonal of F with respect to g, denoted $F^\perp$; hence

$$F^\perp_x = \{v \in E_x : g(v, w) = 0\ \forall\ w \in F_x\}.$$

Prove that $F^\perp$ is a vector sun-bundle of E and $F \oplus F^\perp \cong E$.

Exercise 1.47. Show that if E admits transition functions $\tau_E^{\alpha,\beta} : U^\alpha \to \mathrm{GL}_r(\mathbb{R})$, and similarly $\tau_F^{\alpha,\beta} : U^\alpha \to \mathrm{GL}_s(\mathbb{R})$ for F, where $\mathcal{U} = \{U^\alpha\}_{\alpha\in\Lambda}$ is an open cover of M, then $E \oplus F$ admits as transition functions

$$\tau_{E\oplus F}^{\alpha,\beta} := \begin{pmatrix} \tau_E^{\alpha,\beta} & 0 \\ 0 & \tau_F^{\alpha,\beta} \end{pmatrix} : U^\alpha \cap U^\beta \to GL_{r+s}(\mathbb{R}).$$

1.6.3 Pull-backs

Exercise 1.48. Given a smooth map $f : M \to N$ and a vector bundle E over N, show that any section s of f^*E can be written as

$$s = f^1 \cdot f^*(s_1) + \ldots + f^k \cdot f^*(s_k) \quad \text{with } f^i \in C^\infty(M),\ s_i \in \Gamma(E)$$

In other words, as a $C^\infty(M)$-module, $\Gamma(f^*E)$ is generated by the sections that are pull-backs of sections of E.

(Hints: One idea is to embed your E into a trivial one, check the claims for the trivial one and then show that, if the claims hold for a vector bundle $\widetilde{E}$, then it holds for any vector sub-bundle E).

Again, if you are more algebraically minded, you may want to go on and reinterpret the previous result in terms of tensor products of modules. More precisely, next to the $C^\infty(N)$-module $\Gamma(E)$, we can turn also $C^\infty(M)$ into a module, with the module action $\bullet$ given by $a \bullet b := a \cdot (b \circ f)$, for $a \in C^\infty(M)$ and $b \in C^\infty(N)$; the two can be combined using the tensor product and, finally, one has a map

$$C^\infty(M) \otimes_{C^\infty(N)} \Gamma(E) \to \Gamma(f^*E), \quad \sum f^i \otimes s_i \mapsto \sum f^i \cdot f^*(s_i). \tag{1.6.1}$$

The exercise shows says that the map is injective, although more it is true: it is actually an isomorphism of $C^\infty(M)$-modules. Hence, for the algebraically minded student: what is f^*E? Is the vector bundle over M whose space of sections is (isomorphic to) the left hand side of (1.6.1), as $C^\infty(M)$-modules.

Exercise 1.49 (for the more algebraically minded students). Prove that the map (1.6.1) is indeed an isomorphism. Then, keeping in mind that any smooth map $f : M \to N$ induces a ring homomorphism

$$f^* : C^\infty(N) \to C^\infty(M), \quad f^*(b) = b \circ f \tag{1.6.2}$$

go on and generalise this construction even further: any ring homomorphism $\phi : S \to R$ gives rise to an operation (called induction along ϕ)

$$\mathrm{Ind}_\phi : \mathcal{M}od_S \to \mathcal{M}od_R, \quad \mathrm{Ind}_\phi(\mathcal{E}) = R \otimes_S \mathcal{E},$$

this preserves the property of modules being finitely generated projective and, when applied to (1.6.2),

$$\mathrm{Ind}_{f^*} : \mathcal{M}od_{C^\infty(N)} \to \mathcal{M}od_{C^\infty(N)}$$

sends $\Gamma(E)$ to $\Gamma(f^*E)$. Fill in the details.

1.6.4 Quotients; normal bundles

Exercise 1.50. Given a vector sub-bundle F of E, for any inner product g on E, the restriction of the quotient map $E \to E/F$ to the g-orthogonal $F^\perp$ induces an isomorphism of vector bundles

$$F^\perp \xrightarrow{\sim} E/F.$$

Exercise 1.51. Given an embedded submanifold N of a manifold M, *the normal bundle of N in M*,

$$\mathcal{N}(N, M) \to N,$$

is the vector bundle over N defined as the quotient of $TM|_N$ modulo TN. Therefore, $\mathcal{N}(N, M)$ is the vector bundle over N whose fiber above $x \in N$ is

$$\mathcal{N}_x(N, M) = T_xM/T_xN = \Big\{ v_x \bmod T_xN : v_x \in T_xM \Big\}$$

where two classes $v_x \bmod T_xN$ and $w_x \bmod T_xN$ are equal if and only if $v_x - w_x \in T_xN$. If g is a Riemannian metric on M we also define the g-normal bundle of N in M:

$$\mathcal{N}^g(N, M) := \Big\{ v_x \in T_xM : x \in N, g(v_x, w_x) = 0 \ \forall \ w_x \in T_xN \Big\}.$$

(a) Make sure that $\mathcal{N}^g(N, M)$ and $\mathcal{N}(N, M)$ are (smooth) vector bundles over N.
(b) Prove that $\mathcal{N}^g(N, M) \cong \mathcal{N}(N, M)$ for any Riemannian metric g on M.

Given an embedded submanifold N of a manifold M, the normal bundle $\mathcal{N}(N, M)$ mentioned in the previous exercise comes with the intuition/slogan that "it is the linear-like approximation of M around N". The simplest case is when when $N = \{x\}$ is a point, when $\mathcal{N}(x, M) = T_xM$ and the slogan reads: "T_xM is the linear approximation of M around x". For general N, while N is kept fixed, what can be "linearised" are the directions "normal to N". The outcome is a vector bundle over N, inside N sits as an embedded submanifold (via the zero section). Later on we will discuss the tubular neighborhood theorem which says that, indeed, one can always find a neighborhood of N in M that is diffeomorphic to $\mathcal{N}(N, M)$.

On the other hand, if M is already a vector bundle over N then according to the slogan, $\mathcal{N}(N, M)$ should be the vector bundle itself. The following exercise makes this precise:

Exercise 1.52. Assume that $\pi : E \to N$ is a vector bundle and we view N as a submanifold of $M = E$ via the zero-section $N \to E$. Prove that $\mathcal{N}(N, E)$ is isomorphic to E.

When submanifolds arise by applying globally the regular value theorem, normal bundles are trivial(izable):

Exercise 1.53. If $N = f^{-1}(y_0)$ where $f : M \to M'$ is a smooth map for which y_0 is a regular value, prove that $\mathcal{N}(N, M)$ is trivializable.

(Hint: try to describe a canonical isomorphism with $N \times T_{y_0}M$.)

1.6.5 Hom's

Exercise 1.54. So: what is the relationship between the vector bundle $\mathrm{Hom}(E,F)$ and vector bundle homomorphisms from E to F?

Exercise 1.55. Prove that for any line bundle $L \to M$, the vector bundle $\mathrm{Hom}(L,L)$ is trivializable.

For the next exercise, we will use the following notations. First of all, working in the context of (finite dimensional) vector spaces V and W and fixing

$$k \in \mathbb{N},$$

the Hom-space can be generalised by moving from linear maps from V to W to k-multilinear maps

$$u : \underbrace{V \times \ldots \times V}_{k-\text{times}} \to W. \tag{1.6.3}$$

Often it is natural to impose extra-conditions on u. When we impose symmetry, we denote by

$$\mathcal{S}ym^k(V,W)$$

the resulting space of k-multilinear symmetric maps from V to W. When $W = \mathbb{R}$ (assuming that we work with real vector spaces) we use the simpler notation $\mathcal{S}ym^k(V)$.

Exercise 1.56. Given two vector bundles E and F over M , show that

(a) Applying $\mathcal{S}ym^k$ fiberwise one obtains a vector bundle $\mathcal{S}ym^k(E,F)$ (over M).
(b) Any inner product on E can be interpreted as a section of $\mathcal{S}ym^2(E)$.
(c) In general, there is a 1-1 correspondence between sections of $\mathcal{S}ym^k(E,F)$ and $C^\infty(M)$-multilinear, symmetric maps

$$u_* : \underbrace{\Gamma(E) \times \ldots \times \Gamma(E)}_{k-\text{times}} \longrightarrow \Gamma(F).$$

Another generalization of the Hom-spaces is obtained by imposing skew-symmetry in (1.6.3); we then talk about alternating maps from V to W and we denote the resulting collection by $\mathcal{A}lt^k(V,W)$.

Another generalization of the Hom-spaces is obtained by imposing skew-symmetry in (1.6.3); we then talk about alternating maps from V to W and we denote by $\mathcal{A}lt^k(V,W)$ the resulting space; hence

$$\mathcal{A}lt^k(V,W) = \Big\{ u : \underbrace{V \times \ldots \times V}_{k-\text{times}} \to W : u \text{ is multilinear and skew-symmetric} \Big\}.$$

Again, when $W = \mathbb{R}$, we simplify the notation to $\mathcal{A}lt^k(V)$. While

$$\mathcal{A}lt^0(V,W) = \mathcal{S}ym^0(V,W) = W, \quad \mathcal{A}lt^1(V,W) = \mathcal{S}ym^1(V,W) = \mathrm{Hom}(V,W),$$

for $k \geq 2$ the spaces $\mathcal{A}lt^k(V,W)$ and $\mathcal{S}ym^k(V,W)$ are very different from each other.

Exercise 1.57. Applying $\mathcal{A}lt^k$ fiberwise show that for any two vector bundles T and F over M one obtains a vector bundle $\mathcal{A}lt^k(T,F)$ over M. Then prove that there is a 1-1 correspondence between sections of $\mathcal{A}lt^k(T,F)$ and $C^\infty(M)$-multilinear, anti-symmetric maps

$$u_* : \underbrace{\Gamma(T) \times \ldots \times \Gamma(T)}_{k-\text{times}} \longrightarrow \Gamma(F).$$

Exercise 1.58 (a bachelor-recap exercise). Recall that

(a) Check/make sure that vector fields on M are the same thing as sections of the tangent bundle, 1-forms are the same thing as sections of the cotangent bundle T^*M and, more generally, differential forms of degree k are the same thing as sections of $\Lambda^k T^*M$:

$$\mathfrak{X}(M) = \Gamma(TM), \quad \Omega^1(M) = \Gamma(T^*M) \quad \Omega^k(M) = \Gamma(\Lambda^k T^*M).$$

(b) Given a 1-form $\alpha \in \Omega^1(M)$ prove that

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M), \quad \omega(X,Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X,Y])$$

is $C^\infty(M)$-bilinear and skew-symmetric, hence it defines a 2-form $\omega \in \Omega^2(M)$.

(c) Similarly, given a 2-form $\omega \in \Omega^2(M)$ prove that

$$\begin{aligned} \eta \ : \ & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M), \\ \eta(X,Y,Z) = & -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \\ & + L_X(\omega(Y,Z)) - L_Y(\omega(X,Z)) + L_Z(\omega(X,Y)) \end{aligned}$$

defines a 3-form $\eta \in \Omega^3(M)$.

(d) When carrying out (c) for a 2-form ω obtained from (b), check that $\eta = 0$.

(e) What do all these have to do with DeRham differential?

With the previous discussion in mind, the space of differential forms on M with values in a vector bundle F (over E) is

$$\Omega^k(M,F) := \Gamma(\mathcal{A}lt^k(TM,F)).$$

Exercise 1.59. Prove that for any vector bundle morphism $A : TM \to TM$,

$$\mathcal{N}_A : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \quad \mathcal{N}_A(X,Y) = A\Big([AX,Y] + [X,AY]\Big) - A^2\big([X,Y]\big) - [AX,AY]$$

comes from a section of $\mathcal{A}lt^2(TM,TM)$, i.e. can be seen as a 2-form on M with values in TM:

$$\mathcal{N}_A \in \Omega^2(M,TM).$$

Exercise 1.60. Prove that for any vector sub-bundle $\mathcal{F}$ of TM, considering $\nu := TM/\mathcal{F}$, the map

$$\mathrm{Curv}_{\mathcal{F}} : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \to \Gamma(\nu), \quad (V,W) \mapsto [V,W] \bmod \mathcal{F},$$

comes from a section of $\mathcal{A}lt^2(\mathcal{F},\nu)$.

1.6.6 Tensor products

Now there are a couple of exercises for those that are familiar with tensor products.

Exercise 1.61 (in case you are familiar with tensor products of modules). Prove that for any vector bundles E and F over M, one has an isomorphism of $C^\infty(M)$-modules:

$$\Gamma(E_1 \otimes E_2) \cong \Gamma(E_1) \otimes_{C^\infty(M)} \Gamma(E_2).$$

(Hint: define the map going from right to left. Then make use of the fact that any vector bundle can be embedded in a trivial one.)

Exercise 1.62. Prove that for any vector bundles E and F over M, there are isomorphisms

$$\mathrm{Hom}(E,F) \cong E^* \otimes F, \quad \mathcal{S}ym^k(E,F) \cong \mathcal{S}ym^k(E) \otimes F, \quad \mathcal{A}lt^k(E,F) \cong \mathcal{A}lt^k(E) \otimes F$$

and the last vector bundles are also isomorphic to $\Lambda^k E^* \otimes F$.

Exercise 1.63. Given a manifold M, we define $\mathrm{Pic}(M)$ the collection of all isomorphism classes of complex line bundles over M. We endow $\mathrm{Pic}(M)$ with the operation induced by the tensor product of complex line bundles:

$$[L_1] \cdot [L_2] := [L_1 \otimes L_2].$$

Prove that $\mathrm{Pic}(M)$ becomes a group.

1.6.7 Short exact sequences

A **short exact sequence** of vector bundle, written as

$$0 \longrightarrow F \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } E' \longrightarrow 0, \tag{1.6.4}$$

consists of three vector bundles F, E and E' and vector bundle morphisms i and π between them such that exactness (cf. Exercise 1.16) holds at all three positions, i.e.:

$$\ker i = \{0\}, \quad \mathrm{Im}(i) = \mathrm{Ker}(\pi),\ \mathrm{Im}(\pi) = E.$$

The first condition means that i-injective, hence we may interpret F as a vector sub-bundle of E. The last condition means that π-surjective and implies that E' may be interpreted as the quotient of E modulo F. The following exercise show that one can focus either on i, or on π, or on the extremities.

Exercise 1.64. Prove that:

(a) Any surjective vector bundle morphism $\pi : E \to E'$ gives rise to a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\pi) \xrightarrow{\ \mathrm{incl}\ } E \xrightarrow{\ \pi\ } E' \longrightarrow 0\ ,$$

where "incl" is the inclusion map.

(b) Any vector bundle inclusion $i : F \to E$ gives rise to a short exact sequence

$$0 \longrightarrow F \xrightarrow{i} E \xrightarrow{\text{quot}} E/F \longrightarrow 0,$$

where "quot" is the quotient map.

(c) Any two vector bundles F and E' over a manifold M gives rise to a short exact sequence

$$0 \longrightarrow F \xrightarrow{i_1} F \oplus E' \xrightarrow{pr_2} E' \longrightarrow 0,$$

where i_1 and pr_2 are the obvious inclusion and projection, respectively.

(d) In general, any short exact sequence (1.6.4) is isomorphic to all the previous sequence from (a), (b) and (c) (in a sense that you have to make precise).

Warning: the isomorphism with the sequence from (c) is not so immediate as the other, and the reason is very simple: there is no canonical isomorphism $E \cong F \oplus E'$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & E & \xrightarrow{\pi} & E' & \longrightarrow & 0 \\ & & \| & & \sim\downarrow & & \| & & \\ 0 & \longrightarrow & F & \xrightarrow{i_1} & F \oplus E' & \xrightarrow{pr_2} & E' & \longrightarrow & 0 \end{array}$$

More insight into what is going on (and a proof for the very last part of the exercise) is provided by the concept of *splittings*, which we discuss next.

- by a **right splitting** of (1.6.4) we mean a morphism $h : E' \to E$ with $\pi \circ h = \mathrm{Id}_E$
- by a **left splitting** of (1.6.4) we mean a morphism $\theta : E \to F$ with $\theta \circ i = \mathrm{Id}_F$;

Pictorially:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & E & \xrightarrow{\pi} & E' & \longrightarrow & 0 \\ & & & \overset{\theta}{\nwarrow}\cdots & & \overset{h}{\nwarrow}\cdots & & & \end{array}$$

Exercise 1.65. Given a short exact sequence (1.6.4), prove that:

(a) For any right splitting $h : E' \to E$ there exists an unique left splitting $\theta : E \to F$ satisfying

$$h \circ \pi + i \circ \theta = \mathrm{Id}_E.$$

(b) This gives rise to a bijection between right splittings h and left splittings θ of (1.6.4).

(c) Any such pair (h, θ) of such splittings induces a isomorphism (θ, π) and $i \oplus h$, inverse to each other, as indicated in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & E & \xrightarrow{\pi} & E' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle i\oplus h}\uparrow\downarrow{\scriptstyle (\theta,\pi)} & & \| & & \\ 0 & \longrightarrow & F & \xrightarrow{i_1} & F \oplus E' & \xrightarrow{pr_2} & E' & \longrightarrow & 0 \end{array}$$

(d) Prove that left/right splittings always exist.
(Hint: see below)

And here is yet another type of interpretations of splittings, which provides more insight into the bijection from the previous exercise (and the last item).

Exercise 1.66. Given a short exact sequence (1.6.4), prove that:

(a) the choice of a right splitting $h : E' \to E$ is equivalent to the choice of a vector sub-bundle $C \subset E$ such that $\pi|_C : C \to E$ is an isomorphism. Explicitly:

$$C = \mathrm{Im}(h), \quad h = (\pi|_C)^{-1}.$$

(b) the choice of a left splitting $\theta : E \to F$ is equivalent to the choice of a vector sub-bundle $C \subset E$ that is complementary to F, i.e. such that $E = F \oplus C$. Explicitly,

$$C = \mathrm{Im}(\theta), \quad \theta = pr_F,$$

where the last map is the projection associated with the decomposition $E = F \oplus C$.

(c) for vector sub-bundles $C \subset E$ the conditions from the previous two items are equivalent:

$$\pi|_C : C \to E \text{ is an isomorphism} \quad \Longleftrightarrow \quad E = F \oplus C.$$

(d) prove (again?) that left/right splittings always exist.

Putting the last three exercises together one obtains bijections

$$\left\{\begin{array}{c}\text{left splittings}\\ \theta : E \to F\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{right splittings}\\ h : E' \to E\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{sub-bundles } C \subset E\\ \text{complementary to } F,\\ \text{i.e., s.t. } E = F \oplus C\end{array}\right\} \equiv \left\{\begin{array}{c}\text{sub-bundles } C \subset E\\ \text{s.t. } \pi|_C : C \to E'\\ \text{is an isomorphism}\end{array}\right\}$$

Exercise 1.67. Let $\pi : E \to M$ be a vector bundle. Prove that:

(a) There is a short exact sequence of vector bundles over E:

$$0 \longrightarrow \pi^*E \xrightarrow{i} TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0,$$

where, for $e \in E$,

$$i_e : E_{\pi(e)} \to T_eE$$

sends $v \in E_{\pi(e)}$ to the initial speed of $t \mapsto e + tv$ (a curve in E starting at e!).

(b) Recall that M can be seen as an embedded submanifold of E via the zero section $o : M \to E$; accordingly, the restriction of vector bundles from E to M is the pull-back operation

$$o^* : \mathrm{Vect}(E) \to \mathrm{Vect}(M).$$

Deduce that one obtains a short exact sequence of vector bundles over M:

$$0 \longrightarrow E \xrightarrow{i} o^*TE \xrightarrow{d\pi} TM \longrightarrow 0,$$

and prove that the differential of the zero section o induces a right splitting of this sequence.

(c) Do again Exercise 1.52.

1.6.8 More operations involving complex vector bundles

Exercise 1.68. Show that a complex vector bundle F is isomorphic to the complexification of a real vector bundle if and only if there exists a complex vector bundle morphism

$$\tau : F \to \overline{F}$$

with the property that $\tau^2 = \mathrm{Id}$.

Exercise 1.69. Prove that the vector bundle over $\mathbb{CP}^n$ from Exercise 1.11,

$$L' := \left\{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \overline{z} \in l\right\},$$

is isomorphic to $\overline{L}$, the conjugate of the tautological line bundle.

Exercise 1.70. This is an exercise about complex vector bundles.

(a) Show that if $u : F_1 \to F_2$ is a morphism of complex vector bundles then so is $u : \overline{F}_1 \to \overline{F}_2$ (the same u, but seen as a map between the Fs with the complex structure reversed).
(b) Show that for any complex vector bundle F, its complex dual is isomorphic to its conjugate, as complex vector bundles:
$$F^* \to \overline{F}.$$
(c) Can one conclude now that any morphism $u : F_1 \to F_2$ *induces* a morphism $u' : F_1^* \to F_2^*$? Try to describe u' directly ... what is going on?

Exercise 1.71. Show that for any complex line bundle L over a manifold M, $L \otimes \overline{L}$ is isomorphic to the trivial line bundle $M \times \mathbb{C}$.

1.6.9 The tangent bundle of the projective space

In the exercise below we try to understand/unravel the tangent bundle $T\mathbb{RP}^n$; to that end we will be working with several vector bundles over $\mathbb{RP}^n$:

- T and E: the trivial vector bundle of rank 1 and $n+1$, respectively:
$$T := \mathbb{RP}^n \times \mathbb{R}, \quad E := \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$
- L: the tautological line bundle over $\mathbb{RP}^n$- whose fiber above $l \in \mathbb{RP}^n$ is a copy of the line $l \subset \mathbb{R}^{n+1}$.
- $L^\perp$: defined similarly, but so that its fiber above $l \in \mathbb{RP}^n$ is the Euclidean orthogonal $l^\perp$ of l in $\mathbb{R}^{n+1}$.

Exercise 1.72. Prove that there exists an isomorphism of vector bundles

$$T\mathbb{RP}^n \cong \mathrm{Hom}(L, L^\perp),$$

then deduce that

$$T\mathbb{RP}^n \oplus T \cong \underbrace{L^* \oplus \ldots \oplus L^*}_{n+1 \text{ times}}.$$

Hints:

- for the first part: investigate the tangent spaces $T_l\mathbb{RP}^n$, at an arbitrary $l \in \mathbb{RP}^n$. Keep in mind that, according to the general slogan on tangent spaces, $T_l\mathbb{RP}^n$ plays the role of "the linear approximation of $\mathbb{RP}^n$ around l". Now think a bit of how $\mathbb{RP}^n$ looks around l. Isn't a nearby line

a graph of a linear map from l to $l^\perp$? Doesn't the tangent space start looking like $\mathrm{Hom}(l, l^\perp)$? Turn this idea into an explicit formula. What does that mean in terms of vector bundles?

- for the second part: keep in mind that L^* can itself be described as a Hom-bundle; and Exercise 1.55 may be useful here as well.

A completely similar discussion can be carried out for the complex projective space $\mathbb{CP}$. This time $T\mathbb{CP}^n$ will be treated as a complex vector bundle, and then L will be the tautological (complex) line bundle over $\mathbb{CP}^n$, while the orthogonal $L^\perp$ (appearing already in Exercise 1.33) is defined using the standard hermitian product on $\mathbb{C}^{n+1}$. Of course, T will now denote the trivial complex line bundle, while L^* is the *complex* dual of L.

Exercise 1.73. Prove that there exists an isomorphism of complex vector bundles

$$T\mathbb{CP}^n \cong \mathrm{Hom}(L, L^\perp),$$

then deduce that

$$T\mathbb{CP}^n \oplus T \cong \underbrace{\overline{L} \oplus \ldots \oplus \overline{L}}_{n+1}, \tag{1.6.5}$$

Chapter 2
Some geometric structures

2.1 Parallelizations

Exercise 2.1. Show that $S^2 \times S^1$ is parallelizable.

Recall from the lecture that S^1 and S^3 are parallelizable; that could be proven using complex numbers and quaternions, respectively. Furthermore, we mentioned that that the parallelizability of S^1 and S^3 can be seen as particular cases of a more general result: any Lie group is parallelizable.

Exercise 2.2. Prove that the 7-sphere S^7 is parallelizable. Can you make S^7 (with its standard smooth structure) into a Lie group?

The second part of the exercise is intended as food for thought, inviting you to think whether it is possible or not by imitating S^1 and S^3, if not than what goes wrong, then what is the actual answer to the question, etc etc. In other words you are not expected to both provide the correct answer as well as to prove it. The first part however is a precise task and requires a proper proof.

2.2 Complex structures

Exercise 2.3 (the complex structure on S^2). We now focus on $M = S^2$ and, for $p = (x, y, z) \in S^2$, define J_p by:

$$\underbrace{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}}_{\in T_pS^2} \overset{J_p}{\longmapsto} \underbrace{(cy - bz)\frac{\partial}{\partial x} + (az - cx)\frac{\partial}{\partial y} + (bx - ay)\frac{\partial}{\partial z}}_{\in T_p\mathbb{R}^3}.$$

(a) Prove that J is an almost complex structure on S^2.
(b) To prove that $\mathcal{N}_J = 0$ and, therefore, deduce that J is integrable:

(b1) Remember that TS^2 is spanned by the following vector fields

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

and compute $J\mathfrak{e}^x$ and the similar ones in terms of $\mathfrak{e}^x, \mathfrak{e}^y, \mathfrak{e}^z$.

(b2) Remember that $[\mathfrak{e}^x, \mathfrak{e}^y] = \mathfrak{e}^z$, $[\mathfrak{e}^y, \mathfrak{e}^z] = \mathfrak{e}^x$, $[\mathfrak{e}^z, \mathfrak{e}^x] = \mathfrak{e}^y$ and prove

$$[J\mathfrak{e}^x, \mathfrak{e}^y] = 0,\ [J\mathfrak{e}^x, J\mathfrak{e}^y] = -\mathfrak{e}^z,\ [\mathfrak{e}^x, J\mathfrak{e}^y] = -[J\mathfrak{e}^x, \mathfrak{e}^y] = J\mathfrak{e}^z$$

(b3) Compute $\mathcal{N}_J(\mathfrak{e}^x)$ and deduce that $\mathcal{N}_J = 0$.

(c) Show that $V := y\mathfrak{e}^z - (z-1)\mathfrak{e}^y$ commutes with JV and, using this, provide another proof that J is integrable without using the Newlander-Nirenberg theorem.

(d) Show that the stereographic projection with respect to the north pole

$$\chi : S^2 \setminus \{p_N\} \to \mathbb{R}^2, \quad \chi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

$$\left(\text{with inverse } \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \longleftarrow\!\shortmid (u,v)\right)$$

is a complex chart for (S^2, J). And in this way we obtain yet another proof that J is integrable without using the Newlander-Nirenberg theorem.

In item (b2) you will hopefully get

$$J(\mathfrak{e}^x) = -\frac{\partial}{\partial x} + x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) = z\mathfrak{e}^y - y\mathfrak{e}^z.$$

Exercise 2.4. Assume that $J : TM \to TM$ is an almost complex structure on a 2-dimensional manifold M. Without using the Newlander-Nirenberg theorem, but assuming that around each point in M one find a nowhere vanishing vector field X such that $[X, JX] = 0$, prove that J is integrable.

Exercise 2.5. Recall that an almost complex structure $J : TM \to TM$ on a manifold M makes each tangent space T_xM into a complex vector space. Recall also that the Nijenhuis tensor of J is defined by

$$\mathcal{N} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \quad \mathcal{N}(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY].$$

Assume now that J is an arbitrary almost complex structure on M. Prove that

(a) J gives rise to a 2-form with coefficients in TM, $\mathcal{N} \in \Omega^2(M, TM)$.
(b) at each $p \in M$ the resulting $\mathcal{N}_p : T_pM \times T_pM \to T_pM$ is $\mathbb{C}$-bilinear.
(c) if $V \in \mathfrak{X}(M)$ is nowhere vanishing then V and JV are linearly independent.
(d) J is automatically integrable when $\dim M = 2$.

Exercise 2.6. Short version: if S^n admits an almost complex structure then S^{n+1} is parallelizable.

And here is the longer version. Assume that $S^n \subset \mathbb{R}^{n+1}$ admits an almost complex structure J. Interpret it as a family of linear endomorphisms

$$J_y : P_y \longrightarrow P_y \quad \text{for } y \in S^n$$

of the hyperplane $P_y = \{v \in \mathbb{R}^{n+1} : \langle v, y\rangle = 0\} \subset \mathbb{R}^{n+1}$ orthogonal to y.

Pass now to $\mathbb{R}^{n+2}$; denote by $e_0, \dots, e_{n+1}$ its standard basis and interpret $\mathbb{R}^{n+1}$ (and all its subspaces, e.g. the P_ys) as the subspace of $\mathbb{R}^{n+2}$ via the standard inclusion

$$(y_0, \dots, y_n) \mapsto (y_0, \dots, y_n, 0).$$

For $x = (x_0, \dots, x_{n+1}) \in S^{n+1} \subset \mathbb{R}^{n+2}$ different from $\pm e_{n+1}$, we denote by $p(x)$ its orthogonal projection on $\mathbb{R}^{n+1}$,

$$p(x) = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$$

and the induced element on the sphere

$$s(x) = \frac{1}{||p(x)||} p(x) \in S^n.$$

Consider the orthogonal projection onto $P_{s(x)}$,

$$\mathrm{pr}^{\perp}_{s(x)} : \mathbb{R}^{n+1} \longrightarrow P_{s(x)}, \ \mathrm{pr}^{\perp}_{s(x)}(\lambda) = \lambda - \langle \lambda, s(x) \rangle s(x)$$

and define:

$$F_i(x) := x_{n+1} e_i - x_i e_{n+1} + ||p(x)|| J_{s(x)}(\mathrm{pr}^{\perp}_{s(x)}(e_i))$$

for all $x \in S^{n+1}$, where the very last term is set to be zero when it does not make sense (i.e. for $x = \pm e_{n+1}$). Show that $\{F^0, \dots, F^n\}$ is a frame of S^{n+1}. Deduce that S^{n+1} is parallelizable.

2.3 Foliations

Exercise 2.7 (foliations by fibers of submersions (simple foliations)). Given a submersion $\pi : M \to N$ prove that the distribution

$$\mathcal{F}_\pi := \mathrm{Ker} d\pi = \{X \in TM : d\pi(X) = 0\} \subset TM$$

has the following properties:

(a) it is an integrable distribution on M;
(b) it is an involutive distribution on M.

Avoid using the Frobenius theorem.

Exercise 2.8 (recap on projectability). Given a smooth map $\pi : M \to N$, a vector field $X \in \mathfrak{X}(M)$ is said to be π-projectable to a vector field $V \in \mathfrak{X}(N)$ if

$$(d\pi)_x(X_x) = V_{\pi(x)} \quad \forall \ x \in M.$$

Show that:

(a) X is π-projectable to V if and only if $L_X(\pi^*(f)) = \pi^*(L_V(f))$ for all $f \in C^\infty(N)$, where π^* is the pull-back operation on functions, and L_X is the Lie derivative operation on functions:

$$\pi^*(f) := f \circ \pi, \quad L_X(f) = X(f) = df(X).$$

(b) If X^i are π-projectable to V^i, $i \in \{1,2\}$, then $[X^1, X^2]$ is π-projectable to $[V^1, V^2]$.
(c) provide now another proof of the second item of the pervious exercise.

Exercise 2.9. Given a manifold M:

(a) Given a submanifold $L \subset M$, show that if $V^1, V^2 \in \mathfrak{X}(M)$ are two vector fields on M which, along L, are tangent to L, then so is their Lie bracket $[V^1, V^2]$.
(b) Deduce that if a distribution $\mathcal{F} \subset M$ is integrable then it must be involutive.

Exercise 2.10. Given a foliated manifold $(M, \mathcal{F})$ prove that the restriction of $\mathcal{F}$ to any leaf $L \subset M$ is precisely the tangent bundle of the leaf:

$$\mathcal{F}|_L = TL.$$

Exercise 2.11. Let $\lambda \in \mathbb{R}$. Described the leaves of the foliation on the torus $S^1 \times S^1$ induced by $\left\{ \frac{\partial}{\partial \theta_1} + \lambda \frac{\partial}{\partial \theta_2} \right\}$.

Exercise 2.12. Let $\mathcal{F} \subset TM$ be a p-dimensional distribution on a manifold M. By the Frobenius theorem, that the integrability of $\mathcal{F}$ is equivalent to the condition

$$[V, W] \equiv 0 \mod \Gamma(\mathcal{F}), \quad \forall\, V, W \in \Gamma(\mathcal{F}).$$

Show that this condition is equivalent to the vanishing of a section (that you have to describe) of a vector bundle (that you have to describe as well).

Hint: $X \mod \Gamma(\mathcal{F})$ with $X \in \mathfrak{X}(M)$ encode sections of the normal bundle $\nu := TM/\mathcal{F}$. After at least another 10-15 min of trying (and only then!): you may also want to look at some older exercises.

Exercise 2.13 (Exam 2024). Let (M, g) be a Riemannian manifold and assume that $\mathcal{F} \subset TM$ is a distribution on M. We denote by $\mathcal{F}^\perp \subset TM$ the orthogonal of $\mathcal{F}$ with respect to g and by

$$\mathrm{pr}_{\mathcal{N}} : TM \to \mathcal{F}^\perp$$

the orthogonal projection with respect to g. Finally, define

$$\mathfrak{c}_{\mathcal{F}} : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}^\perp), \quad \mathfrak{c}_{\mathcal{F}}(V, W) = \mathrm{pr}_{\mathcal{N}}\left([V, W]\right).$$

(a) Show that $\mathfrak{c}_{\mathcal{F}}$ defines a section of $\mathcal{A}lt^2(\mathcal{F}, \mathcal{F}^\perp)$;
(b) Show that $\mathcal{F}$ defines a foliation on M if and only if $\mathfrak{c}_{\mathcal{F}} = 0$.
(d) Prove that $\mathcal{F}^\perp$ is isomorphic to the normal bundle $TM/\mathcal{F}$ and, modulo this isomorphism, $\mathfrak{c}_{\mathcal{F}}$ is nothing but the curvature of $\mathcal{F}$ from Exercise 1.60.

Exercise 2.14 (retake 2026). Let M be a manifold and $\mathbb{J} : TM \to TM$ be a vector bundle morphism. Show that:

(a) the operation that associates to two vector fields $X, Y \in \mathfrak{X}(M)$ the new vector field

$$\mathcal{N}_{\mathbb{J}}(X,Y) = \mathbb{J}\Big([\mathbb{J}X, Y] + [X, \mathbb{J}Y]\Big) - \mathbb{J}^2\big([X,Y]\big) - [\mathbb{J}X, \mathbb{J}Y]$$

defines a 2-form on M with coefficients in TM, $\mathcal{N}_{\mathbb{J}} \in \Omega^2(M, TM)$;
(b) if $\mathcal{N}_{\mathbb{J}} = 0$ and the image $\mathcal{F}_{\mathbb{J}} := \mathbb{J}(TM) \subset TM$ is of constant rank, then $\mathcal{F}_{\mathbb{J}}$ defines a foliation.
(c) if, furthermore, $\mathbb{J}^3 = -\mathbb{J}$ then, for each leaf $L \subset M$ of the foliation $\mathcal{F}_{\mathbb{J}}$, the restriction $\mathbb{J}|_{TL} : TL \to TL$ defines a complex structure on L.

Exercise 2.15. Assume that $\mathcal{F}$ is a vector sub-bundle of TM and we look at "k-differential forms along $\mathcal{F}$" (for any k) by which we mean sections of the vector bundle $\Gamma(\Lambda^k \mathcal{F}^*)$. These are related to the usual k-forms on M by the obvious restriction map

$$\Omega^k(M) \longrightarrow \Gamma(\Lambda^k \mathcal{F}^*), \quad \eta \mapsto \eta|_{\mathcal{F}}$$

which, pointwise, restricts a k-multilinear skew-symmetric form on T_xM to one on $\mathcal{F}_x$. We would like talk about "DeRham differential along $\mathcal{F}$",

$$d_{\mathcal{F}} : \Gamma(\Lambda^k \mathcal{F}^*) \to \Gamma(\Lambda^{k+1} \mathcal{F}^*)$$

(defined for all k) such that it is compatible with the usual DeRham differential d in the sense that

$$d_{\mathcal{F}}(\eta|_{\mathcal{F}}) = d(\eta)|_{\mathcal{F}} \quad \forall\ \eta \in \Omega^k(M).$$

Note that, since the restriction map is surjective, $d_{\mathcal{F}}$ will be unique if it exists. Show that $d_{\mathcal{F}}$ exists if and only if $\mathcal{F}$ is involutive. (Hint: look first in low degrees. And do not forget about the Koszul formula).

2.4 Symplectic structures

Exercise 2.16. Below M is a smooth manifold, and let us recap:

(a) Given an almost complex structure $J : TM \to TM$,

$$\begin{cases} \mathcal{N}_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \\ \mathcal{N}_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] \end{cases}$$

is a skew-symmetric, $C^\infty(M)$-multilinear in each argument whose vanishing is equivalent the integrability of J (Newlander-Nirenberg theorem).
(b) Given a distribution $\mathcal{F} \subset TM$ and considering $\nu := TM/\mathcal{F}$,

$$\begin{cases} \mathrm{Curv}_{\mathcal{F}} : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \to \Gamma(\nu), \\ \mathrm{Curv}_{\mathcal{F}}(X, Y) = [X, Y] \bmod \mathcal{F} \end{cases}$$

is a skew-symmetric $C^\infty(M)$-multilinear map whose vanishing is equivalent to the integrability of $\mathcal{F}$ (Frobenius theorem).

(c) Given an almost symplectic form $\omega \in \Omega^2(M)$, describe a map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

that is skew-symmetric $C^\infty(M)$-multilinear, whose vanishing is equivalent to the fact that ω is integrable. Can you give an explicit formula for this map, involving Lie brackets and Lie derivatives, similar to the ones from the previous items?

Exercise 2.17. The **Liouville form** associated to a symplectic manifold (M, ω) is the top differential form

$$\mu_\omega := \frac{1}{k!}\omega^k = \frac{1}{k!}\underbrace{\omega \wedge \ldots \omega}_{k \text{ times}},$$

where k is half the dimension of M. Show that:

(a) μ_ω is a volume form.
(b) Deduce that any symplectic manifold carries an induced orientation. In particular, non-orientable manifolds cannot carry symplectic structures.
(c) If M is compact then $a = [\omega] \in H^2(M)$ is non-zero. In particular, compact manifolds with $H^2(M) = 0$ (like the spheres of dimension > 2) cannot carry symplectic structures.
(d) When M is compact, can a^2 be zero? What is the best you can say?

Exercise 2.18. Define the so called Liouville 1-form on T^*M, $\lambda \in \Omega^1(T^*M)$, defined as follows. For a general tangent vector to T^*M at some point $\xi \in T^*_x M$ (for some $x \in M$),

$$\mathcal{V}_\xi \in T_\xi(T^*M).$$

we map it to TM via the differential of the projection map $\pi : T^*M \to M$,

$$d\pi\left(\mathcal{V}_\xi\right) \in T_x M \quad (x = \pi(\xi))$$

and we evaluate $\xi \in T^*_x M$ on it:

$$\lambda\left(\mathcal{V}_\xi\right) := \xi\left(d\pi\left(\mathcal{V}_\xi\right)\right).$$

Check that, in the local coordinates mentioned above, $\lambda = \sum_j y_j dx_j$ and, therefore, ω_{can} could be defined directly as $d\lambda$.

Exercise 2.19. Prove that any manifold that admits a complex structure also admits an almost complex structure.

Probably the best approach to the previous exercise goes via hermitian structures. Let us start with a straightforward exercise in that direction:

Exercise 2.20. Assume that we have a complex vector bundle or, equivalently, a real vector bundle E endowed with a complex structure $J : E \to E$. Show that there is a 1-1 correspondence between hermitian structures h on (E, J) and inner products g with the property that

$$g(Ju, v) = -g(u, Jv) \quad \forall\ u, v \in E.$$

Explicitly, g is the real part of h, while h can be reconstructed as $h(u, v) = g(u, v) - ig(Ju, v)$.

The next exercise provides more insight into (as well as an actual solution to) Exercise 2.19.

Exercise 2.21.

(a) First, let's see that the existence of a complex structure J on vector bundle E implies the existence of a symplectic structure on E. To that end we assume that we a complex structure $J : E \to E$ is given, we choose a (any) metric g on E and your are asked to prove that

$$\omega(u, v) := g(u, Jv) - g(Ju, v)$$

defines a symplectic structure on E.

(b) Show that actually ω is part of a hermitian structure on E; more precisely, show that

$$h := g' + i\omega,$$

where g' is defined by $g'(u, v) = g(u, v) + g(Ju, Jv)$, together with the original J, define a hermitian structure on E.

(c) If you had more trouble with the first item than with the second one, can you explain why? After all, the second does imply the first one, no?

Exercise 2.22. Consider TS^2 with its standard complex structure, see Exercise 2.3. Prove now that TS^2 admits one and only one hermitian structure whose real part is the inner product induced from the standard inner product on $\mathbb{R}^3$ via $TS^2 \subset S^2 \times \mathbb{R}^3$.

(Hint: see Exercise 2.20).

Chapter 3
Connections

3.1 Some generalities and simple examples:

Exercise 3.1. Show that for any two connections ∇^1 and ∇^2 on a vector bundle E there exists a 1-form

$$\theta \in \Omega^1(M, \mathrm{End}(E)) \quad (\text{recall } \mathrm{End}(E) = \mathrm{Hom}(E, E))$$

such that

$$\nabla^2_X(s) = \nabla^1_X(s) + \theta(X)(s) \quad \forall\ X \in \mathfrak{X}(M), s \in \Gamma(E).$$

Then explain the statement: the space of connections on E form an affine space with underlying vector space $\Omega^1(M, \mathrm{End}(E))$ (hint: θ is basically the vector $\overrightarrow{\nabla^1\nabla^1}$).

Exercise 3.2. Prove that any convex linear combination of two connections is again a connection, i.e., given ∇^1 and ∇^2 connections on E and ρ_1, ρ_2 smooth functions on M (the base manifold) such that $\rho_1 + \rho_2 = 1$, then

$$\nabla = \rho_1\nabla^1 + \rho_2\nabla^2$$

is also a connection. What if you have larger families? Use now a partition of unity argument to show that any vector bundle admits a connection.

Exercise 3.3. Let ∇ be a connection on a vector bundle E over M. Show that one has:

(a) an induced connection ∇^* on E^*, defined by the Leibniz-type equation

$$L_X(\xi(s)) = \nabla^*_X(\xi)(s) + \xi(\nabla_X(s))$$

for all $s \in \Gamma(E)$, $\xi \in \Gamma(E^*)$ and $X \in \mathfrak{X}(M)$.

(b) an induced connection ∇^{Hom} on $\mathrm{Hom}(E, E)$, defined by the Leibniz-type equation

$$\nabla_X(T(s)) = \nabla^{\mathrm{Hom}}_X(T)(s) + T(\nabla_X(s))$$

for all $s \in \Gamma(E)$, $T \in \Gamma(\mathrm{Hom}(E, E))$ and $X \in \mathfrak{X}(M)$.

Assuming now that we have a second bundle E' endowed with a connection ∇', show that one has:

(c) an induced connection $\nabla \oplus \nabla'$ on $E \oplus E'$ given by

$$(\nabla \oplus \nabla')_X(s, s') = (\nabla_X(s), \nabla'_X(s')).$$

(d) an induced connection $\nabla \otimes \nabla'$ on $E \otimes E'$ given by

$$(\nabla \otimes \nabla')_X(s \otimes s') = \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s').$$

Exercise 3.4. Given a connection ∇ on a vector bundle E over M, show that

$$\nabla^{\text{new}}_X(u)(s_1, s_2) := u\left(\nabla_X(s_1), s_2\right) + u\left(s_1, \nabla_X(s_2)\right) - L_X\left(u(s_1, s_2)\right)$$

defines connections

$$\nabla^{\mathcal{S}ym^2}, \quad \nabla^{\mathcal{A}lt^2}$$

on the vector bundles $\mathcal{S}ym^2(E)$ and $\mathcal{A}lt^2(E)$ (from Exercise 1.56 and 1.57). Then generalise this to arbitrary degrees, and also in the presence of coefficients.

Exercise 3.5. This is a continuation of Exercise 3.3, hence we assume the same notations. For two finite dimensional vector spaces V and V' and $K \in \text{End}(V)$ and $K' \in \text{End}(V')$, we consider

$$K^* \in \text{End}(V^*) \quad \text{given by } K^*(\xi)(v) = \xi(K(v)),$$

$$\text{Ad}_K = [K, -] \in \text{End}(V) \quad \text{given by } \text{Ad}_K(T) = [K, T] = K \circ T - T \circ K,$$

$$K \oplus K' \in \text{End}(V \oplus V') \quad \text{given by } (K \oplus K')(v, v') = (K(v), K'(v')),$$

$$K \otimes K' \in \text{End}(V \otimes V') \quad \text{given by } (K \otimes K')(v \otimes v') = K(v) \otimes K'(v').$$

We keep the same notations for endomorphisms of vector bundles. Show that the curvatures of ∇^*, ∇^{Hom}, $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ from Exercise 3.3 are given by

$$k_{\nabla^*}(X, Y) = -k_\nabla(X, Y)^*,$$

$$k_{\nabla^{\text{Hom}}}(X, Y) = [k_{\nabla(X,Y)}, -],$$

$$k_{\nabla \oplus \nabla'}(X, Y) = k_\nabla(X, Y) \oplus k_{\nabla'}(X, Y),$$

$$k_{\nabla \otimes \nabla'}(X, Y) = \text{Id}_E \otimes k_{\nabla'}(X, Y) + k_\nabla(X, Y) \otimes \text{Id}_{E'}.$$

Exercise 3.6. On $M = S^2 \subset \mathbb{R}^3$ we consider the trivial vector bundle of rank 2, $E = S^2 \times \mathbb{R}^2$, and we denote by $\{e_1, e_2\}$ its canonical frame. Consider the connection ∇ on E whose connection matrix w.r.t. $\{e_1, e_2\}$ is

$$\omega = \begin{pmatrix} xdy & ydz \\ zdx & x^2dx + y^2dy + z^2dz \end{pmatrix},$$

where x, y, z are the coordinates in $\mathbb{R}^3$, and when writing xdy etc we mean their restriction to S^2. Please do the following:

(a) Compute $\nabla_{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}}(x^2 e_1 + xyz e_2)$ (more precisely, compute the coefficients of this expression w.r.t. the basis $\{e_1, e_2\}$).

(b) Compute the connection matrix of ∇ with respect to $\{e_1, e_2\}$. Shouldn't this matrix be zero, since the vector bundle is trivial? What is going on?

Exercise 3.7. We consider the trivial vector bundle of rank two over S^2,

$$E = S^2 \times \mathbb{R}^2,$$

and, as usual, we interpret the sections of E as pairs (f, g) of functions $f, g : S^2 \to \mathbb{R}$. For $\epsilon \in \mathbb{R}$ we set $\theta^\epsilon := xdy + \epsilon ydx \in \Omega^1(S^2)$ and we consider the connection ∇^ϵ on E given by

$$\nabla^\epsilon_X(f, g) = (L_X(f) + g\theta^\epsilon(X),\ f\theta^\epsilon(X) + L_X(g)) .$$

(a) what is the connection matrix of ∇^ϵ w.r.t. the canonical basis of E?
(b) what is the curvature matrix of ∇^ϵ w.r.t. the canonical basis?

Exercise 3.8. Let $E = M \times \mathbb{R}$ be the trivial line bundle over a manifold M, let $\theta \in \Omega^1(M)$ be a 1-form. Keeping in mind that $\Gamma(E) \cong C^\infty(M)$, prove that

$$\nabla^\theta : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M), \quad \nabla^\theta_X(f) = L_X(f) + f\theta(X).$$

is a connection on E and compute the curvature of ∇^θ in terms of θ.

3.2 Connections induced on sub-bundles of trivial bundles

We have seen that many vector bundles sit naturally inside a trivial bundle (and, furthermore, although "less natural" in full generality, every vector bundle can be embedded in a trivial one). The good news is that any such vector bundle comes endowed with a natural connection:

Exercise 3.9. Show that, whenever E sits inside the trivial vector bundle $\widetilde{E} = M \times \mathbb{R}^k$, the following is a connection on E:

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_E^{\mathrm{can}} \left(\underbrace{\nabla_X^{\mathrm{flat}}(s)}_{\in \Gamma(\widetilde{E})} \right) \tag{3.2.1}$$

where ∇^{flat} is the standard flat connection of the trivial bundle, and $\mathrm{pr}_E^{\mathrm{can}}$ is induced by the canonical orthogonal projection from $\mathbb{R}^k$ to the subspaces E_x.

Let us spell this out in a bit more detail, also noticing that the setting is a bit more general: whenever E sits inside a vector bundle $\widetilde{E} \to M$ that comes equipped with

- a connection $\widetilde{\nabla}$ on $\widetilde{E}$,
- an inner product g on $\widetilde{E}$

it follows that E will itself inherit connection. To that end we start by restricting $\widetilde{\nabla}$ to E:

$$\mathfrak{X}(M) \times \Gamma(E) \to \Gamma(\widetilde{E}), \quad (X, s) \mapsto \widetilde{\nabla}_X(s).$$

Then, since for each $x \in M$ we have $E_x \subset \widetilde{E}_x$ we can get back to E by using the orthogonal projection w.r.t. g, $\mathrm{pr}^g_{E_x} : \widetilde{E}_x \to E_x$, which induces a map on sections

$$\mathrm{pr}^g_E : \Gamma(\widetilde{E}) \to \Gamma(E).$$

Combining the two one now defines

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}^g_E \left(\underbrace{\widetilde{\nabla}_X(s)}_{\in \Gamma(\widetilde{E})} \right) \tag{3.2.2}$$

Exercise 3.10. Prove that, indeed, ∇ is a connection on E.

Remark 3.11 (linear algebra: orthogonal projections). For more explicit formulas, you should first think a bit about orthogonal projections

$$\mathrm{pr}^g_V : \widetilde{V} \longrightarrow V$$

onto vector subspaces $V \subset \widetilde{V}$ w.r.t. the a given inner product g on $\widetilde{V}$. By definition, for $\widetilde{v} \in \widetilde{V}$, $v := \mathrm{pr}^g_V(\widetilde{v})$ is the unique vector of V with the property that $\widetilde{v} - v$ is orthogonal to V.

Assume now that you are lucky and you do have an orthogonal basis $\{\sigma_1, \ldots, \sigma_r\}$ of V. While $v = \mathrm{pr}^g_V(\widetilde{v} \in V$ will be determined by its coefficients w.r.t. this basis, one can now go on and actually compute those coefficients from the condition that $\widetilde{v} - v$ is orthogonal to V.

Exercise 3.12. Carry out the computation described in the previous remark and conclude that

$$\mathrm{pr}^g_V(\widetilde{v}) = \frac{g(\widetilde{v}, \sigma_1)}{g(\sigma_1, \sigma_1)} \cdot \sigma_1 + \ldots + \frac{g(\widetilde{v}, \sigma_r)}{g(\sigma_r, \sigma_r)} \cdot \sigma_r. \tag{3.2.3}$$

Warning: do not prove that this formula works, but prove that the reasoning above produces this formula. In other words: you do not have to know this formula by heart, you can always deduce it!

Exercise 3.13 (the canonical connection on TS^2). We now work over $M = S^2$, over which we consider the vector bundle $E := TS^2$. Using the fact that $S^2 \subset \mathbb{R}^3$ to embed TS^2 in the trivial bundle $S^2 \times \mathbb{R}^3$, we endow E with the resulting connection

$$\nabla : \mathfrak{X}(S^2) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_E \left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(\widetilde{E})} \right),$$

or, if you prefer the dual point of view, $d_\nabla : \Gamma(E) \to \Omega^1(S^2, E)$, $d_\nabla(s) = \mathrm{pr}_E(d_{\nabla^{\mathrm{flat}}}(s))$.

We also consider the following vector fields on the sphere (as in Exercise 2.3):

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \qquad (\star)$$

(a) Show that

$$\mathrm{pr}_E\left(\frac{\partial}{\partial x}\right) = y\mathfrak{e}^z - z\mathfrak{e}^y = \frac{\partial}{\partial x} - x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

(b) Show that

$$d_\nabla(\mathfrak{e}^x) = -x \cdot (dx \otimes \mathfrak{e}^x + dy \otimes \mathfrak{e}^y + dz \otimes \mathfrak{e}^z).$$

(c) Check (again?) that $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ is a local frame of TS^2 over $U = \{z \neq 0\}$ and compute the connection matrix ω of ∇ w.r.t. $\{\mathfrak{e}^x, \mathfrak{e}^y\}$.

Exercise 3.14. Consider the rank two real vector bundle from Exercise 1.21,

$$E := \mathrm{Span}(\sigma_1, \sigma_2, \sigma_1', \sigma_2') \subset \widetilde{E} = S^2 \times \mathbb{R}^4.$$

and recall that $\{\sigma_1, \sigma_2\}$ is a local frame of E over S^2 minus $p_S = (-1, 0, 0)$, while $\{\sigma_1', \sigma_2'\}$ is a local frame of E over S^2 minus $p_N = (1, 0, 0)$. We now consider the induced connection on E,

$$\nabla : \mathfrak{X}(S^2) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}^{\mathrm{can}} \left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(\widetilde{E})} \right).$$

(a) Compute the connection matrix of ∇ w.r.t. the frame $\{\sigma_1, \sigma_2\}$.
(b) Then compute the curvature matrix w.r.t. the same frame.
(c) Compute now the connection and the curvature matrix of ∇ w.r.t. $\{\sigma_1', \sigma_2'\}$.

Exercise 3.15. Let M be a manifold endowed with a foliation $\mathcal{F}$ (interpreted as an involutive sub-bundle $\mathcal{F} \subset TM$). By an *$\mathcal{F}$-connection* on a vector bundle $\mathcal{N}$ over M we mean a bilinear map

$$\nabla : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{N}) \to \Gamma(\mathcal{N})$$

satisfying the usual identities

$$\nabla_{fV}(s) = f\nabla_V(s), \quad \nabla_V(fs) = f\nabla_V(s) + L_V(f)s$$

for all $f \in C^\infty(M)$, $s \in \Gamma(\mathcal{N})$ and (only) $V \in \Gamma(\mathcal{F})$. Given an $\mathcal{F}$-connection ∇ on $\mathcal{N}$ prove that

(a) There exists a connection $\widetilde{\nabla}$ on $\mathcal{N}$ which extends ∇ in the sense that $\nabla_V(s) = \widetilde{\nabla}_V(s)$ for all $V \in \Gamma(\mathcal{F})$ and $s \in \Gamma(\mathcal{N})$.

(b) For any leaf $L \subset M$ of $\mathcal{F}$, there exists an induced connection ∇^L on $\mathcal{N}|_L$,

$$\nabla^L : \mathfrak{X}(L) \times \Gamma(\mathcal{N}|_L) \to \Gamma(\mathcal{N}|_L),$$

which is canonical (in the sense that it does not depend on the choices one might make to describe it) and has the property that $\nabla_V(s)|_L = \nabla^L_{V|_L}(s|_L)$ for all $V \in \Gamma(\mathcal{F})$ and $s \in \Gamma(\mathcal{N})$ (note: for $V \in \Gamma(\mathcal{F})$ one has $V|_L \in \mathfrak{X}(L)$ because, at any $x \in L$, one has $V_x \in \mathcal{F}_x = T_xL$).

As an example of a vector bundle with an $\mathcal{F}$-connection we take the normal bundle $\nu = TM/\mathcal{F}$ and we endow with with the so-called "Bott connection":

$$\nabla^{\mathrm{Bott}} : \Gamma(\mathcal{F}) \times \Gamma(\nu) \to \Gamma(\nu), \quad \nabla^{\mathrm{Bott}}_V(X \bmod \mathcal{F}) := [V, X] \bmod \mathcal{F}.$$

Prove that:

(c) ∇^{Bott} is indeed an $\mathcal{F}$-connection on ν.

(d) ∇^{Bott} is $\mathcal{F}$-flat, i.e.

$$\nabla^{\mathrm{Bott}}_{[V,W]} = [\nabla^{\mathrm{Bott}}_V, \nabla^{\mathrm{Bott}}_W] \qquad \forall\, V, W \in \Gamma(\mathcal{F}).$$

(e) For any leaf L of the foliation, its normal bundle $\mathcal{N}(L, M)$, a vector bundle over L, admits a flat (usual) connection.

3.3 Connections on complex vector bundles

What was discussed in detail for real vector bundles also has a version over $\mathbb{C}$ that works pretty much the same. However, it is instructive to go through the details. First, the notion of "**connection** on a complex vector bundle E" is the same just that

$$\mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (X, s) \mapsto \nabla_X(s)$$

is now required to be $\mathbb{C}$-linear in the second argument (and still $\mathbb{R}$-linear in the first). In other words, this is nothing but a connection on E viewed as a real vector bundle, which is also required to satisfy $\nabla_X(i \cdot s) = i \cdot \nabla_X(s)$ for all X and s. Notice that this implies that the equation

$$\nabla_X(fs) = f\nabla_X(s) + L_X(f)s$$

holds for all complex valued smooth functions $f \in C^\infty(M, \mathbb{C})$. This is important when writing local formulas, when we use local frames $\mathfrak{e} = \{e_1, \ldots, e_r\}$ of the complex vector bundle E (hence, at each x, they give a basis of the complex vector space E_x). Indeed, in that case arbitrary local sections of E (over the open U where the local frame is defined) can be decomposed uniquely as $f^1 e_1 + \ldots + f^n e_n$, with $f^i \in C^\infty(U, \mathbb{C})$ (smooth $\mathbb{C}$-valued functions). Therefore, one can go on and define the local connection matrix of such a connection ∇ w.r.t. to $\mathfrak{e}$,

$$\omega = (\omega_{i,j})_{1 \leq i,j \leq r},$$

by the same formulas as in the real case,

$$\nabla_X(e_j) = \sum_i \omega_j^i(X) \cdot e_i \quad \left(\text{or } d_\nabla(e_j) = \sum_i \omega_j^i \otimes e_i \right),$$

the only difference being that each $\omega_j^i(X)$ is now a $\mathbb{C}$-valued smooth function, i.e., each ω_j^i is now a $\mathbb{C}$-valued differential form:

$$\omega_j^i \in \Omega^1(U, \mathbb{C})$$

(while usual (real) differential forms on a manifold M are forms with coefficients in the trivial real line bundle $M \times \mathbb{R}$, the $\mathbb{C}$-valued forms are forms with coefficients in the trivial complex line bundle $M \times \mathbb{C}$).

To sum up and relate to the real case:

Exercise 3.16. Assume that F is a complex vector bundle over M. Recall that F has an underlying real vector bundle $E = F_\mathbb{R}$ and, conversely, the complex vector bundle can be seen as a real vector bundle E together with an endomorphism $J : E \to E$ satisfying $J^2 = -\mathrm{Id}_E$ (encoding the multiplication by i). Show that:

(a) A (complex) connection on F is the same thing as a (real) connection ∇ on $E = F_\mathbb{R}$ satisfying

$$\nabla_X(J(s)) = J\left(\nabla_X(s)\right) \quad \forall\ X \in \mathfrak{X}(M), s \in \Gamma(E).$$

(b) Any (complex) local frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$ of F gives rise to a (real) local frame of $E = F_\mathbb{R}$:

$$\mathfrak{e}_\mathbb{R} := \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r, J(\mathfrak{e}_1), \ldots, J(\mathfrak{e}_r)\}$$

(c) Given a (complex) local frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$ of F over some open U, we now consider:

- the connection matrix of ∇ w.r.t. $\mathfrak{e}$ (an $r \times r$ matrix with entries complex 1-forms)

$$\omega = (\omega_q^p)_{1 \leq p,q \leq r} \in \mathcal{M}_r\left(\Omega^1(U, \mathbb{C})\right);$$

Furthermore, express ω in terms of its real and imaginary parts:

$$\omega = \eta + i \cdot \zeta, \quad \text{with } \eta, \zeta \in \mathcal{M}_r\left(\Omega^1(U)\right);$$

- the connection matrix of ∇ w.r.t. the local frame $\mathfrak{e}_{\mathbb{R}}$ of E:

$$\widehat{\omega} = (\widehat{\omega}_k^j)_{1 \le j,k \le 2r} \in \mathcal{M}_{2r}\left(\Omega^1(U)\right);$$

Your are asked to describe the $(2r) \times (2r)$ matrix $\widehat{\omega}$ explicitly in terms of η and ζ.

Exercise 3.17. Consider the tangent bundle TS^2 endowed with the connection ∇ from Exercise 3.13, and with the complex structure J from Exercise 2.3.

(a) Prove that ∇ is a connection on the complex vector bundle (TS^2, J) and $\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ can be seen as a complex frame of (TS^2, J) over $U_x = \{y^2 + z^2 \neq 0\}$

(b) Compute the resulting $\mathbb{C}$-valued functions f_1 and f_2 on U_x such that

$$\mathfrak{e}^y = f_1 \cdot \mathfrak{e}^x, \quad \mathfrak{e}^z = f_2 \cdot \mathfrak{e}^x.$$

(c) Show that the connection matrix of ∇ w.r.t. $\mathfrak{e}^x$ is

$$\omega = \frac{ydy + zdz}{y^2 + z^2} + x\frac{ydz - zdy}{y^2 + z^2} \cdot i \in \Omega^1(U_x, \mathbb{C}).$$

(d) Prove that the curvature matrix is i time the volume form of S^2, $\mu = xdy\,dz + ydz\,dx + zdx\,dy$.

(Hint: in (b) you should get $f_1 = \frac{-xy+iz}{y^2+z^2}$, $f_1 = \frac{-xz-iy}{y^2+z^2}$. For the computation in (d) you may want to notice first that $\frac{dxdy}{z} = \frac{dydz}{y} = \frac{dzdx}{y} = \mu$, whenever the expressions are defined).

Exercise 3.18. Consider the tangent bundle TS^2 endowed with the connection ∇ from Exercise 3.13, and with the complex structure J from Exercise 2.3.

Prove that ∇ is a connection on the complex vector bundle (TS^2, J).

(a) Prove that ∇ is a connection on the complex vector bundle (TS^2, J).

(b) Prove that $\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ can be seen as a complex frame of (TS^2, J) over $U_x = \{y^2 + z^2 \neq 0\}$ and compute the resulting $\mathbb{C}$-valued functions on U_x such that

$$\mathfrak{e}^y = f_1 \cdot \mathfrak{e}^x, \quad \mathfrak{e}^z = f_2 \cdot \mathfrak{e}^x.$$

(c) Show that the connection matrix of ∇ w.r.t. $\mathfrak{e}^x$ is

$$\omega = \frac{ydy + zdz}{y^2 + z^2} + x \cdot \frac{ydz - zdy}{y^2 + z^2} \cdot i \in \Omega^1(U_x, \mathbb{C}).$$

(d) Also compute the curvature matrix and show that it is precisely the volume form of S^2, $\mu = xdy\,dz + ydz\,dx + zdx\,dy$.

(Hint: in (b) you should get $f_1 = \frac{-xy+iz}{y^2+z^2}$, $f_1 = \frac{-xz-iy}{y^2+z^2}$. For the computation in (d) you may want to notice first that $\frac{dxdy}{z} = \frac{dydz}{y} = \frac{dzdx}{y} = \mu$).

Exercise 3.19. Show that the connection from Exercise 3.14 is a (complex) connection on E, where E is seen as a complex vector bundle as discussed in Exercise 1.21. See also Exercise 3.21 below.

Next, the construction of connections on sub-bundles of the trivial bundle from (3.2.2) works without any problem in the complex setting, with the only observation that we now use hermitian products h and, in particular, on $\mathbb{C}^k$ we use the standard hermitian product

$$h^{\text{can}}(z, z') = \sum_j z_j \overline{z'_j}, \quad \text{for } z = (z_1, \ldots, z_k),\ z' = (z'_1, \ldots, z'_k) \in \mathbb{C}^k.$$

Notice also that the formula for $pr^h(\widetilde{v})$ from Exercise 3.12 continue to hold in the complex context- hence whenever $V \subset \widetilde{V}$ is a complex vector subspace, h is a hermitian product on $\widetilde{V}$ and $\{\sigma_1, \ldots, \sigma_r\}$ is an h-orthogonal (complex) basis of V:

$$\text{pr}_V^h(\widetilde{v}) = \frac{h(\widetilde{v}, \sigma_1)}{h(\sigma_1, \sigma_1)} \cdot \sigma_1 + \ldots + \frac{h(\widetilde{v}, \sigma_r)}{h(\sigma_r, \sigma_r)} \cdot \sigma_r. \tag{3.3.1}$$

Exercise 3.20. Let V be a complex vector subspace of a hermitian vector space $(\widetilde{V}, h)$; recall that:

- $\widetilde{V}_{\mathbb{R}}$ and $V_{\mathbb{R}}$ stand for $\widetilde{V}$, respectively V, seen as real vector spaces,
- the real part of h, $g := \text{Re}(g)$, is an inner product on $\widetilde{V}_{\mathbb{R}}$,
- a frame $\mathfrak{e} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\}$ of V gives rise to the frame $\mathfrak{e}_{\mathbb{R}} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_r, i \cdot \mathfrak{e}_1, \ldots, i \cdot \mathfrak{e}_r\}$ of $V_{\mathbb{R}}$.

Prove that:

(a) $pr_V^h = pr_{V_{\mathbb{R}}}^g$;
(b) If $\mathfrak{e}$ is h-orthogonal then $\mathfrak{e}_{\mathbb{R}}$ is g-orthogonal ...
(c) ... and the resulting formula (3.3.1) using $\mathfrak{e}$ coincides with the formula (3.2.3) using $\mathfrak{e}_{\mathbb{R}}$.

Exercise 3.21 (redo Exercise 3.14 with simpler formulas). Consider again the vector bundle from Exercise 1.21, but we treat it as a complex line bundle

$$E \subset \widetilde{E} = S^2 \times \mathbb{C}^2$$

and we endow it with the resulting complex connection ∇ obtained by the general recipe.

(a) Recalling that σ_1 can be seen as a (complex) local frame of E over $S^2 \setminus (-1, 0, 0)$, compute the connection and curvature matrix of ∇ w.r.t. σ_1.
(b) What is the relationship with the connection from Exercise 3.14?
(c) Recover the (real) connection and curvature matrices from Exercise 3.14.

Let's now see some more serious implementations of the construction of connections (3.2.2) in the complex case. We give all details in the case of the tautological line bundle

$$L = \left\{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in l\right\};$$

then the resulting connection

$$\nabla : \mathfrak{X}(\mathbb{CP}^n) \times \Gamma(L) \to \Gamma(L), \quad (X, s) \mapsto \nabla_X(s) := \text{pr}_L \left(\underbrace{\nabla_X^{\text{flat}}(s)}_{\in \Gamma(E)} \right) \tag{3.3.2}$$

or, if you prefer the dual point of view,

$$d_\nabla : \Gamma(L) \to \Omega^1(\mathbb{CP}^n, L), \quad d_\nabla(s) = \mathrm{pr}(d_{\nabla^{\mathrm{flat}}}(s)),$$

makes use of the following basic pieces:

- the canonical flat connection ∇^{flat} on the trivial bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$.
- the standard hermitian product on $\mathbb{C}^{n+1}$:

$$h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1} + \ldots + z_n\overline{z'_n}$$

- for each $l \in \mathbb{CP}^n$, the orthogonal projection onto l with respect to h, $\mathrm{pr}_l : \mathbb{C}^{n+1} \to l$. Hence $\mathrm{pr}_l(z)$ is the unique point in l with the property that $z - \mathrm{pr}_l(z)$ is perpendicular to l w.r.t. h; it can be described in explicit formulas by:

$$\mathrm{pr}_l(z) = \frac{h(z,\xi)}{h(\xi,\xi)}\xi, \quad \text{with } \xi \in l \text{ a/any non-zero vector.}$$

- the map induced at the level of sections and, for d_∇, on forms with coefficients:

$$\mathrm{pr}_L : \Gamma(E) \to \Gamma(L), \quad \mathrm{pr}_L : \Omega^1(\mathbb{CP}^n, E) \to \Omega^1(\mathbb{CP}^n, L).$$

An interesting exercise is to write down ∇ locally, in terms of local connection matrices. Since L is of (complex) rank one, a local frame over an open $U \subset \mathbb{CP}^n$ is just a nowhere vanishing local section of L defined over U. We choose here U to be

$$U_0 := \{[z_0 : z_1 : \ldots : z_n] : z_0 \neq 0\},$$

i.e. the domain of the standard chart

$$\chi_0 : U_0 \to \mathbb{C}^n, \quad [1 : z_1 : \ldots : z_n] \longleftrightarrow (z_1, \ldots, z_n).$$

Over U_0 we have the local section e_0 given by that sends $[1 : z_1 : \ldots : z_n]$ to $(1, z_1, \ldots, z_n)$. We can now talk about the connection matrix of ∇ w.r.t. e, which will be a $\mathbb{C}$-valued 1-form on U, denoted

$$\omega \in \Omega^1(U_0, \mathbb{C}).$$

Exercise 3.22. You are asked to compute ω in the chart χ_0. More precisely, you are asked to prove that ω is equal to (the pull-back via χ_0 of)

$$\frac{\overline{z_1}dz_1 + \ldots + \overline{z_n}dz_n}{1 + |z_1|^2 + \ldots + |z_n|^2} \in \Omega^1(\mathbb{C}^n, \mathbb{C}),$$

where, as before, $z_k = x_k + i \cdot y_k$ stand for the coordinate functions on $\mathbb{C}^n$, $\overline{z_k} = x_k - i \cdot y_k$, $dz_k = dx_k + i \cdot dy_k$ (a $\mathbb{C}$-valued 1-form!).

The computation of the curvature matrix is a bit more involved; we organise it as follows:

Exercise 3.23. Compute now the curvature matrix k in the chart χ_0. More precisely:

(a) when $n = 1$ prove that k is equal to (the pull-back via χ_0 of)

$$-\frac{1}{(1 + |z|^2)^2} \cdot dz \wedge d\overline{z} \in \Omega^2(\mathbb{C}, \mathbb{C});$$

(b) do the computation for general n, show that k can be described by the following short formula:

$$k = -\partial\overline{\partial}\log(1+|z|^2) \in \Omega^2(\mathbb{C}^n, \mathbb{C})$$

where $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$, and then show that

$$k^n = (-1)^n \frac{n!}{(1+|z|^2)^{n+1}} \cdot dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n.$$

Exercise 3.24. Carry out the same discussion for the conjugate of the tautological line bundle

$$L' := \left\{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \overline{z} \in l\right\},$$

which we have encountered also in Exercise 1.11 and Exercise 1.69. What changes when replacing L by L'? Generalise that conclusion to general complex bundles.

Exercise 3.25. We now apply the general recipe to the orthogonal of the tautological line bundle,

$$L^\perp := \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\}$$

to construct the connection

$$\nabla : \mathfrak{X}(\mathbb{CP}^n) \times \Gamma(L^\perp) \to \Gamma(L^\perp), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_{L^\perp}\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in\Gamma(E)}\right).$$

In this exercise we will focus on the chart

$$U_0 \ni [1 : z_1 : \ldots : z_n] \xmapsto{\chi_0} (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

and we will use the coordinates $(z_1, \ldots, z_n)$ corresponding to this chart. As observed in Exercise 1.33, using the canonical frame $\{\mathfrak{e}_0, \mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$ of $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$

$$\mathfrak{f}_j := -\overline{z}_j\mathfrak{e}_0 + \mathfrak{e}_j \quad \text{with } j \in \{1, \ldots, n\}$$

form a local frame of $L^\perp$ over U_0.

(a) Show that

$$\mathrm{pr}_{L^\perp}(\mathfrak{e}_0) = -\frac{z_1}{1+|z|^2} \cdot \mathfrak{f}_1 - \ldots - \frac{z_n}{1+|z|^2} \cdot \mathfrak{f}_n.$$

(b) Compute the connection matrix ω of ∇ w.r.t. $\mathfrak{f}$.

(c) Compute the curvature matrix of ∇ w.r.t. $\mathfrak{f}$.

3.4 Parallel transport

Exercise 3.26. In the context of vector bundles $\pi : E \to M$ endowed with a connection ∇, we now discuss a coordinate free description of $\frac{\nabla u}{dt}(t)$ for paths $u : I \to E$ above some path $\gamma : I \to M$. Fix u and prove that:

(a) if u is the evaluation of section of $s \in \Gamma(E)$ along γ, i.e. if $u(t) = s(\gamma(t))$ for all $t \in I$, then

$$\frac{\nabla u}{dt}(t) = \nabla_{\dot\gamma(t)}(s) \quad \text{(evaluated at } \gamma(t)).$$

(c) while it is not always possible to find an s as above, one can always write

$$u(t) = \widetilde{s}(t, \gamma(t))$$

for some section $\widetilde{s}$ of the pull-back $\mathrm{pr}_2^* E$ of E via the second projection $\mathrm{pr}_2 : I \times M \to M$.

(d) for $\widetilde{s}$ as above, denoting by $s_t \in \Gamma(E)$ the section $\widetilde{s}(t, \cdot)$ (one for each $t \in I$), one has:

$$\frac{\nabla u}{dt}(t) = \left(\nabla_{\dot\gamma(t)}(s_t) + \frac{ds_t}{dt}\right)(\gamma(t)).$$

Exercise 3.27. Return to the setting of Exercise 3.6: the trivial vector bundle of rank 2 over $M = S^2 \subset \mathbb{R}^3$, $E = S^2 \times \mathbb{R}^2$, endowed with the connection ∇ whose connection matrix w.r.t. the canonical frame $\{e_1, e_2\}$ is

$$\omega = \begin{pmatrix} xdy & ydz \\ zdx & x^2dx + y^2dy + z^2dz \end{pmatrix}.$$

Compute the parallel transport $T_\gamma^{0,2\pi} : \mathbb{R}^2 \to \mathbb{R}^2$ of ∇ along $\gamma : [0, 2\pi] \to S^2$, $\gamma(t) = (\cos t, \sin t, 0)$.

Exercise 3.28. Return to the setting of Exercise 3.7: the trivial vector bundle of rank 2 over $M = S^2 \subset \mathbb{R}^3$, $E = S^2 \times \mathbb{R}^2$, endowed with the connection ∇^ϵ given b

$$\nabla^\epsilon_X(f, g) = \left(L_X(f) + g\theta^\epsilon(X),\ f\theta^\epsilon(X) + L_X(g)\right),$$

where $\theta^\epsilon := xdy + \epsilon ydx \in \Omega^1(S^2)$. Please do the following:

(a) when $\epsilon = 1$, prove that the resulting parallel transport along any curve $\gamma : [0, 1] \to S^2$ with the property that $\gamma(0) = \gamma(1)$ is the identity map;

(b) when $\epsilon = -1$, compute the parallel transport along the curve $\gamma(t) = (\cos t, \sin t, 0)$.

Exercise 3.29. Return to the setting of Exercise 3.8: $E = M \times \mathbb{R}$ the trivial line bundle over a manifold M, endowed with the connection

$$\nabla^\theta : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M), \quad \nabla^\theta_X(f) = L_X(f) + f\theta(X),$$

associated to a fixed (but arbitrary) 1-form $\theta \in \Omega^1(M)$. Please do the following:

(a) a path u in $E = M \times \mathbb{R}$ has two components, namely a path γ_u in M and a path f_u in $\mathbb{R}$. For $u = (\gamma_u, f_u)$, describe the components of $\frac{\nabla^\theta u}{dt}$.
(b) given a path $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1)$, show that the resulting parallel transport along γ, $T_\gamma^{0,1} : E_{\gamma(0)} \to E_{\gamma(0)}$, is the multiplication by the real number $\exp\left(-\int_{[0,1]} \gamma^*\theta\right)$.

Exercise 3.30. Consider the tautological line bundle L over $\mathbb{RP}^1$, endowed with the connection (3.2.1) coming from the inclusion $L \subset \mathbb{RP}^1 \times \mathbb{R}^2$.

(a) Consider the curve $\gamma : [0,1] \to \mathbb{RP}^1$, $\gamma(t) = [\cos \pi t, \sin \pi t]$. Since γ starts and ends at $p = [1:0]$, the parallel transport $T_\gamma^{0,1}$ along γ is a linear automorphism of L_p. Prove that

$$T_\gamma^{0,1} = -\mathrm{Id}_{L_p}.$$

(b) Recall that $\mathbb{RP}^1$ is diffeomorphic to S^1. Draw a picture of L inside $\mathbb{R}^3$, i.e. exhibit a subspace of $\mathbb{R}^3$ that is diffeomorphic to the manifold L and indicate on your drawing the fibers of L. Then, on the picture, describe the parallel transports $T_\gamma^{0,t}$. Make sure that the Moebius does show up in your picture.

Exercise 3.31. Consider $E = TS^2$, the tangent bundle of the sphere, endowed with the standard connection discussed e.g. in Exercise 3.13. From the general theory it follows that for any path γ in S^2 and any path ξ in $E = TS^2$ sitting above γ, we can talk about $\frac{\nabla \xi}{dt}$, which is a new path in E above γ. Writing $\gamma(t) = (x(t), y(t), z(t))$ and denoting by $f(t)$, $g(t)$ and $h(t)$ the components of $\xi(t)$ obtained by writing

$$\xi(t) = f(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + g(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} + h(t)\left(\frac{\partial}{\partial z}\right)_{\gamma(t)}$$

prove that the similar components of $\frac{\nabla \xi}{dt}$ are

$$\dot{f} + x(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{g} + y(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{h} + z(\dot{x}f + \dot{y}g + \dot{z}h).$$

Then, for $\gamma(t) = (0, \cos t, \sin t)$ show that the parallel transport $T_\gamma^{0,t} : E_{\gamma(0)} \to E_{\gamma(t)}$ satisfies:

$$T_\gamma^{0,t}\left(\frac{\partial}{\partial z}\right) = -\sin t \cdot \frac{\partial}{\partial y} + \cos t \cdot \frac{\partial}{\partial z}.$$

Exercise 3.32. We now return to the setting of Exercise 3.14 and Exercise 1.21:

$$E := \mathrm{Span}(\sigma_1, \sigma_2, \sigma_1', \sigma_2') \subset \widetilde{E} = S^2 \times \mathbb{R}^4,$$

endowed with the connection

$$\nabla : \mathfrak{X}(S^2) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}^{\mathrm{can}}\left(\underbrace{\nabla_X^{\mathrm{flat}}(s)}_{\in \Gamma(\widetilde{E})}\right).$$

Find a loop $\gamma : [0,1] \to S^2$, i.e. a path that starts and ends at the same point p, such that the resulting parallel transport $T_\gamma^{0,1} : E_p \to E_p$ is not the identity map (try something simple-like $(0, \cos(ct), \sin(ct))$ for some appropriate constant c).

Exercise 3.33. Let F be a complex vector bundle and let ∇ be a (complex) connection on F. One also thinks here of F as a real vector bundle E endowed with a complex structure $J : E \to E$, and of ∇ as a connection on E with the property that $\nabla_X \circ J = J \circ \nabla$ (cf. Exercise 3.16).

(a) Show that if a path $u : I \to E$ is parallel w.r.t. ∇, then so is $J \circ u$.
(b) The parallel transports w.r.t. ∇ (with ∇ viewed as a real connection on E)

$$T_\gamma^{t_0,t_1} : E_{\gamma(0)} \to E_{\gamma(1)},$$

a priori isomorphisms of real vector spaces, are actually $\mathbb{C}$-linear.

Exercise 3.34. Consider now the tautological line bundle L over $\mathbb{CP}^1$, endowed with its standard connection (3.3.2), let $p = [1:1] \in \mathbb{CP}^1$ and we consider the loop

$$\gamma : [0, 2\pi] \to \mathbb{CP}^1, \quad u(t) = [1 : e^{it}].$$

Compute the parallel transport (w.r.t. ∇)

$$T_\gamma^{0,2\pi} : L_p \to L_p$$

(you should get minus the identity map).

Exercise 3.35. Consider the tautological line bundle L over $\mathbb{CP}^1$, endowed with its standard connection (3.3.2). According to the general theory, for any path $u : I \to L$ sitting above some path $\gamma : I \to \mathbb{CP}^1$ (hence $u(t) \in L_{\gamma(t)}$), one can talk about the derivative $\frac{\nabla u}{dt}$, which is again a path in L above γ. Assume now that

$$\gamma(t) = [z_1(t) : z_2(t)] \in \mathbb{CP}^1, \quad u(t) = a(t)e_1(\gamma(t)) + b(t)e_2(\gamma(t))$$

where, as before, $\{e_1, e_2\}$ is the canonical frame of $E = \mathbb{CP}^1 \times \mathbb{C}^2$. Show that

$$\frac{\nabla u}{dt}(t) = \frac{\overline{z_1} \cdot a' + \overline{z_2} \cdot b'}{|z_1|^2 + |z_2|^2}(z_1 e_1 + z_2 e_2).$$

Then, for $\gamma : [0, \pi] \to \mathbb{CP}^1$, $\gamma(t) = [1 : e^{2it}]$, compute the resulting parallel transport

$$T_\gamma^{0,t} : L_{\gamma(0)} \to L_{\gamma(t)}.$$

3.5 Compatible connections

Let us start by rethinking a bit the concept of complex connection on a complex vector bundle F by focusing on the real objects. Therefore, the starting point is a vector bundle E together with a complex structure $J : E \to E$. Then a complex connection is reinterpreted as a (real) connection ∇ on the (real) vector bundle E that is **compatible with** J in the sense that

$$\nabla_X \circ J = J \circ \nabla_X \quad \text{for all } X \in \mathfrak{X}(M).$$

(You may have already seen this remark in Exercise 3.16). This sets up a paradigm- that of connections compatible with given structures (complex, metric, etc). With that in mind, it is instructive to first investigate a bit the compatibility condition with complex structures.

Exercise 3.36. Let E be a (real) vector bundle over M endowed with a complex structure $J : E \to E$. If ∇ is a connection on E then the following are equivalent:

(a) ∇ is compatible with J.
(b) J is flat w.r.t. the connection ∇^{Hom} induced by ∇ on $\mathrm{Hom}(E, E)$.
(c) for any $\gamma : I \to M$ and $t_0, t_1 \in I$, the resulting parallel transport

$$T_\gamma^{t_0,t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

commutes with J (i.e. it is complex linear).

(For ∇^{Hom} have a look at (b) of Exercise 3.3. Recall that flatness means $\nabla^{\mathrm{Hom}}_X(J) = 0$ for all $X \in \mathfrak{X}(M)$. You may also want to be aware of Exercise **??**).

We now move to inner products. Recall that a connection ∇ is said to be **compatible with an inner product** g is on E

$$L_X\left(g(s_1, s_2)\right) = g\left(\nabla_X(s_1), s_2\right) + g\left(s_1, \nabla_X(s_2)\right)$$

for any $X \in \mathfrak{X}(M)$, $s_1, s_2 \in \Gamma(E)$.

Exercise 3.37. Interpret the inner product g as a section of $\mathcal{S}ym^2(E)$ (from Exercise 1.56). Prove that the connection ∇ on E can be used to define a connection $\nabla^{\mathcal{S}ym}$ on $\mathcal{S}ym^2(E)$ such that the compatibility condition of ∇ and g is equivalent to requiring that g is $\nabla^{\mathcal{S}ym}$-flat.

The following equivalent characterisation of the compatibility appeared already in the lectures. You are now invited to transform the sketch of the proof given there into proper proofs; this time, you can also take advantage of the insight given by the previous exercise and the possibility of using Exercise **??**.

Exercise 3.38. Let E be a (real) vector bundle over M endowed with an inner product g. If ∇ is a connection on E then the following are equivalent:

(a) ∇ is compatible with g.
(b) for any $u_1, u_2 : I \to E$ paths above the same path $\gamma : I \to M$, one has

$$\frac{d}{dt} g\left(u_1, u_2\right) = g\left(\frac{\nabla u_1}{dt}, u_2\right) + g\left(u_1, \frac{\nabla u_2}{dt}\right).$$

(c) for any $\gamma : I \to M$ and $t_0, t_1 \in I$, the resulting parallel transport

$$T_\gamma^{t_0,t_1} : \left(E_{\gamma(t_0)}, g_{\gamma(t_0)}\right) \to \left(E_{\gamma(t_1)}, g_{\gamma(t_1)}\right)$$

is an isometry.

(d) for any local frame $\mathfrak{e} = \{e_1, \dots, e_r\}$ of E which is orthonormal (w.r.t. g) the resulting connection matrix ω (of ∇ w.r.t. $\mathfrak{e}$) is skew symmetric: $\omega_k^j = -\omega_j^k$.

Exercise 3.39. If ∇ is compatible with g then for any path $u : I \to E$ that is parallel, its length w.r.t. g,

$$||u(t)||_g := \sqrt{g(u(t), u(t))},$$

is constant.

Exercise 3.40. Adapt the previous discussion to the compatibility of connections with hermitian products.

A completely similar discussion applies to fiberwise symplectic forms ω on E: a connection ∇ on E is **compatible with** ω if

$$L_X\left(\omega(s_1, s_2)\right) = \omega\left(\nabla_X(s_1), s_2\right) + \omega\left(s_1, \nabla_X(s_2)\right)$$

for any $X \in \mathfrak{X}(M)$, $s_1, s_2 \in \Gamma(E)$. Again, seeing ω as a section of $\mathcal{A}lt^2(E)$, the compatibility condition is equivalent to the flatness of ω w.r.t. the connection $\nabla^{\mathcal{S}ym}$ induced by ∇ on $\mathcal{S}ym^2(E)$ (if you do not remember these symbols, you may want to have a look at Exercise 1.56 and Exercise 3.4). Then, as before, the compatibility condition can also be characterised in terms of parallel transports: for any $\gamma : I \to M$ and $t_0, t_1 \in I$, the resulting parallel transport

$$T_\gamma^{t_0,t_1} : \left(E_{\gamma(t_0)}, \omega_{\gamma(t_0)}\right) \to \left(E_{\gamma(t_1)}, \omega_{\gamma(t_1)}\right)$$

is a linear symplectomorphism.

Exercise 3.41. Fill in the missing details.

Exercise 3.42. Let E be a vector bundle endowed with a Hermitian structure (h, J), where we write $h = g + i \cdot \omega$ (hence g is an inner product and ω a fiberwise linear symplectic structure on E).

Given a connection ∇ on E, prove that the following are equivalent:

(a) ∇ is compatible with h;
(a) ∇ is compatible with g and ω;
(a) ∇ is compatible with g and J.

3.6 Levi Civita

Exercise 3.43. For $M = \mathbb{R}^n$ endowed with the standard Riemannian metric g^{can}, describe explicitly the Levi-Civita connection ∇, the second derivatives $\ddot{\gamma}(t) = \frac{\nabla \dot{\gamma}}{dt}$ of curves $\gamma : I \to \mathbb{R}^n$, then describe all the geodesics.

Exercise 3.44. Let M be a compact submanifold of an Euclidean space $\mathbb{R}^k$. Using the resulting embedding of the tangent bundle into the trivial bundle

$$TM \subset M \times \mathbb{R}^k,$$

and the standard metric g^{can} on $\mathbb{R}^k$, we endow TM with the resulting connection (as in Exercise 3.10) and with the Riemannian metric $g = g^{\text{can}}|_{TM}$.

a) Write down the formula for $\nabla_X(Y)$, for $X, Y \in \mathfrak{X}(M)$, in terms of ∇^{can} and the orthogonal projections, making sure that what you write actually makes sense (e.g. keep in mind that a vector *field* on M is *not* a vector field on $\mathbb{R}^k$).
b) Prove that ∇ is compatible with g.
c) Prove that ∇ is torsion free (again, make sure that the formulas you write are well-defined).
d) Finally, prove that ∇ is precisely the Levi-Civita connection of (M, g).

Note: you are allowed to use the fact that, for $M \subset \mathbb{R}^k$ compact, for any vector field X on M there exists an extension of X to $\mathbb{R}^k$, i.e. a vector field $\widetilde{X}$ on $\mathbb{R}^k$ such that $\widetilde{X}_p = X_p$ for all $p \in M$.

Exercise 3.45. We look at $M = S^2$ which we endow with the Riemannian metric g induced from $\mathbb{R}^3$ and let ∇ be the corresponding Levi-Civita connection.

(a) for paths $u(t) = f(t)\frac{\partial}{\partial x} + g(t)\frac{\partial}{\partial y} + h(t)\frac{\partial}{\partial z}$ in TS^2 sitting above a path $\gamma = (x(t), y(t), z(t))$ in S^2 write down $\frac{\nabla u}{dt}$ and deduce that, if u is horizontal, its norm $||u|| = \sqrt{\langle u, u\rangle}$ is constant.
(b) deduce that $\gamma = (x(t), y(t), z(t))$ is a geodesic of (S^2, g) if and only if

$$\ddot{x} + x(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0,\ \ddot{y} + y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0,\ \ddot{y} + y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0$$

and these imply that $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ is constant.
(c) describe by explicit formulas all the geodesics of (S^2, g) that start at the point $(1, 0, 0) \in S^2$.

You may want to be aware that this exercise belongs to a line of exercises that includes Exercise 1.32, 3.13 and 3.31.

Exercise 3.46. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, denote by $\widetilde{\nabla}$ the corresponding Levi-Civita connection and by $\frac{\widetilde{\nabla}}{dt}$ the corresponding derivatives of paths in $T\widetilde{M}$.

Assume now that $M \subset \widetilde{M}$ is a compact submanifold, we endow M with the Riemannian structure g obtained by restricting $\widetilde{g}$ to (the tangent spaces of) M and we consider the corresponding Levi-Civita connection ∇ on TM, and the operation $\frac{\nabla}{dt}$. Show that:

(a) ∇ coincides with the projection onto TM, via the orthogonal projection $\text{pr}_g : T\widetilde{M}|_M \to TM$ induced by g, of the restriction of $\widetilde{\nabla}$ to M, as in Exercise 3.10.

(b) for any path $u : I \to TM$, one has

$$\frac{\nabla u}{dt} = \mathrm{pr}_g\left(\frac{\widetilde{\nabla} u}{dt}\right).$$

Exercise 3.47. Let (M, g) be a manifold endowed with a foliation $\mathcal{F}$, and consider its g-orthogonal:

$$\mathcal{N} = \Big\{Y \in TM : g(V, Y) = 0 \ \forall\ V \in \mathcal{F}\Big\} \subset TM,$$

and its orthogonal projection $pr_{\mathcal{N}} : TM \to \mathcal{N}$ sending $X \in TM$ to the unique $Y \in \mathcal{N}$ s.t. $X - Y \in \mathcal{F}$).

(a) Using the terminology from Exercise 3.15, prove that the formula $\nabla^{\mathcal{N}}_V(Y) := pr_{\mathcal{N}}([V, Y])$ defines an $\mathcal{F}$-connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}$, which is $\mathcal{F}$-flat in the sense that

$$\nabla^{\mathcal{N}}_{[V,W]} = [\nabla^{\mathcal{N}}_V, \nabla^{\mathcal{N}}_W] \text{ for all } V, W \in \Gamma(\mathcal{F}).$$

Consider now also the Levi-Civita connection ∇^{LC} associated to g and further prove that:

(c) The operators ∇^{LC}_V with $V \in \Gamma(\mathcal{F})$ preserve $\Gamma(\mathcal{N})$ if and only if they preserve $\Gamma(\mathcal{F})$:

$$\nabla^{LC}_V(\Gamma(\mathcal{N})) \subset \Gamma(\mathcal{N}) \text{ for all } V \in \Gamma(\mathcal{F}) \quad \Longleftrightarrow \quad \nabla^{LC}_V(\Gamma(\mathcal{F})) \subset \Gamma(\mathcal{F}) \text{ for all } V \in \Gamma(\mathcal{F}).$$

(d) ∇^{LC} is an extension of $\nabla^{\mathcal{N}}$, i.e.

$$\nabla^{\mathcal{N}}_V(Y) = \nabla^{LC}_V(Y) \text{ for all } V \in \Gamma(\mathcal{F}), Y \in \Gamma(\mathcal{N})$$

if and only if ∇^{LC} preserves $\mathcal{F}$, i.e.

$$\nabla^{LC}_X(\Gamma(\mathcal{F})) \subset \Gamma(\mathcal{F}) \text{ for all } X \in \mathfrak{X}(M).$$

When the conditions from (d) hold we say that g and $\mathcal{F}$ are compatible. In that case also prove that:

(e) any geodesic γ of (M, g) with the property that $\dot{\gamma}(0) \in \mathcal{F}$ must be a leafwise curve, i.e. there exists a leaf L of M such that $\gamma(t) \in L$ for all t.

3.7 Characteristic classes

One if the basic properties of the Chern/Pontryagin classes is that they vanish if the bundle is trivialisable or, more generally, that two isomorphic vector bundles have the same classes. That gives a very useful criteria to prove that two vector bundles are not isomorphic; however, that is just the beginning!

However, for such results we need to produce characteristic classes that are non-trivial. So, the first question is: how do we prove that a cohomology class, say in degree two (like the first Chern class) is non-zero? One quite powerful tool is that of integration. Remember that whenever M is an orientable compact manifold on which we fix an orientation, we have an integral defined on the space of forms of top degree:

$$\int_M : \Omega^d(M) \to \mathbb{R} \qquad (d = \dim M),$$

with the basic property that, on exact forms, it is zero (Stokes manifold)[1]. That means that $\int_M \omega$ only depends on DeRham cohomology class $[\omega] \in H^d(M)$ and, therefore, $\int_M$ descends to a linear map

$$\int_M : H^d(M) \to \mathbb{R}.$$

This allows us to treat elements in $H^d(M)$ (top degree). What if we are interested in class es $a \in H^k(M)$ with $k < d$? It is then useful to remember that the DeRham cohomology comes with bilinear product operations (induced by the operation of wedge products of differential forms):

$$H^p(M) \times H^q(M) \to H^{p+q}(M), \quad ([\omega], [\eta]) \mapsto [\omega] \cdot [\eta] := [\omega \wedge \eta].$$

Exercise 3.48. Given a compact oriented d-dimensional manifold M show that:

(a) if $a \in H^d(M)$ has a non-zero integral, then $a \neq 0$.
(b) if $b \in H^l(M)$ has the property that $\exists\ b' \in H^{d-l}(M)$ such that $\int_M b \cdot b' \neq 0$, then $b \neq 0$.
(c) if $d = 2k$ and $c \in H^2(M)$ has the property that $\int_M c^k \neq 0$ then $c \neq 0$.

Exercise 3.49. Prove that the Chern class of the tangent bundle TS^2 endowed with its standard complex structure J from Exercise 2.3 is non-zero.

Then deduce the hairy ball theorem: S^2 does not admit any nowhere vanishing vector field.

(Hint: make use of the computation from Exercise 3.17 and then compute $\int_{S^2} c_1(TS^2, J) = 2!$)

Exercise 3.50. Prove that the Chern class of the tautological line bundle L over $\mathbb{CP}^1$ is non-zero. Then deduce that:

a) L is not trivializable.
c) L is not isomorphic, as a complex vector bundle, to its conjugate $\overline{L}$.
d) Using L and operations with vector bundles, construct a countable family of (complex) line bundles over $\mathbb{CP}^1$ with the property that any two of them are not isomorphic.

(Hint: make use of the computation from Exercise 3.23 and then compute $\int_{\mathbb{CP}^1} c_1(L) = -1!$)

[1] changing the orientation only changes the sign of the integral (assuming M is connected), which will not be relevant to our purposes: all we care is to produce nonzero integrals

The fact that the last integral is one of the explanations for the notation $L = \mathcal{O}(-1)$ mentioned already in the discussion around (1.1.4); as pointed out there, the conjugate of L deserves the name $\mathcal{O}(1)$. Actually there is an entire collection

$$\mathcal{O}(n) \quad \text{with } n \in \mathbb{Z},$$

uniquely determined by the basic requirement that $\mathcal{O}(p+q) \cong \mathcal{O}(p) \otimes \mathcal{O}(q)$ for all integers p, q. Of course, this forces $\mathcal{O}(-n)$ to be $L^{\otimes n}$ and $\mathcal{O}(n)$ to be $\overline{L}^{\otimes n}$ whenever $n \in \mathbb{N}$.

Exercise 3.51. Prove that the Chern classes of $\mathcal{O}(n)$ satisfy

$$\int_{\mathbb{CP}^1} c_1\left(\mathcal{O}(n)\right) = n \qquad \forall\ n \in \mathbb{Z}.$$

On the other hand, recall that one has a diffeomorphism $\Phi : \mathbb{CP}^1 \to S^2$ (see Exercise 1.23), hence pulling back TS^2 with its complex structure,

$$\Phi^*(TS^2, J)$$

will be a complex line bundle over $\mathbb{CP}^1$. If it is isomorphic to one of the $\mathcal{O}(n)$, prove that n must be equal to -2. Then try to prove that $n = -2$ works (by providing an explicit isomorphism).

It is now time to move to the tautological line bundle L over arbitrary $\mathbb{CP}^n$. Given the fact that we want to end up with simpler signs (in particular no signs introduced by integration), we will denote

$$a := -c_1(L) \in H^2(\mathbb{CP}^n). \tag{3.7.1}$$

Exercise 3.52. Prove that the Chern class of the tautological line bundle L over $\mathbb{CP}^n$ is non-zero. Actually prove that (3.7.1) satisfies:

$$a \neq 0, \quad a^2 \neq 0, \quad \ldots, \quad a^n \neq 0.$$

Exercise 3.53. Still working over $\mathbb{CP}^n$, we now consider the orthogonal of the tautological bundle

$$L^{\perp} := \big\{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\big\},$$

which you may have seen in Exercise 1.33 or 3.25. Express its Chern classes in terms of (3.7.1).

Remark 3.54. Notice that there are two ways to proceed: a more laborious one, in which you use the connection and the computations from Exercise 3.25, and one much simpler (but somehow "more mysterious"), based on using the basic properties of the Chern class. Both are every instructive, and the drawbacks of any of them should make you appreciate the other!

Next we turn to the relevance of the tautological line bundle L when studying the tangent bundle of $\mathbb{CP}^n$ (either as a complex or as a real vector bundle). The case $n = 1$, when $\mathbb{CP}^1 \cong S^2$, was treated in the second part of Exercise 3.51. For general $n \in \mathbb{N}$, it is useful to revise the discussion around Exercise 1.73.

Exercise 3.55. Prove that the Chern classes of $T\mathbb{CP}^n$ and the Pontrjagin classes of the underlying real vector bundle can be expressed in terms of (3.7.1) as follows:

$$c(T\mathbb{CP}^n) = (1+a)^{n+1}, \quad p(T\mathbb{CP}^n) = (1+a^2)^{n+1}.$$

3.8 More applications of characteristic classes

We have already pointed out some rather immediate applications of Chern classes. Here is recap of what we discussed, plus of couple more in the same vein.

Exercise 3.56. If a F a complex vector bundle F over M has $c_1(F) \neq 0$, prove that:

(a) F cannot be trivialisable.
(b) F cannot be isomorphic to its conjugate $\overline{F}$.
(c) F cannot be written as the complexification of a real vector bundle over M.
(d) If F is a complex line bundle, then the underlying real vector bundle cannot be trivializable.

Exercise 3.57. Let F be a complex vector bundle F of rank r over a manifold M with $H^2(M) = 0$. If $c_r(F) \neq 0$ prove E does not admit a complex line sub-bundle $L \subset E$.

Here are some more refined questions that can be answered using characteristic classes. Let us start with the basic property that *for any vector bundle E one can always find another vector bundle E' such that $E \oplus E'$ is trivializable.* An interesting question is: given a real vector bundle E, how "simple" can E' be? Here are two interesting sub-questions:

- When can one actually choose E' to be trivial? We say that E is **stably trivial** if that is possible.
- How small can E' be? Can it be of rank one, two, etc? Apply this to **the embeddability problem**.

Let us start with the first question: the stable triviality or real vector bundles. Notice that this is not such an uncommon occurance: e.g., although the tangent bundle of S^n, is almost never trivial, it is always stably trivial as pointed out in Exercise 1.44. However, while $TS^2 = T\mathbb{CP}^1$ is stably trivial, one has:

Exercise 3.58. Show that if $E \in \mathrm{Vect}_{\mathbb{R}}(M)$ is stably trivial then $p_k(E) = 0$ for all $k \neq 0$. Use this to deduce that $T\mathbb{CP}^n$ is not stably trivial for $n \geq 2$.

For the second question above, we propose the following:

Exercise 3.59. For real vector bundles E please do the following:

(a) Show that if $E \oplus L$ is trivializable for some line bundle L, then $p_l(E) = 0$ for all $l \neq 0$.
(b) If $E \oplus F$ is trivializable for some vector bundle F of rank ≤ 3, what can you say about the Pontrjagin classes of E?

One typical situation when one arrive at trivializable sums $E \oplus F$ is when $E = TM$ and M is embedded in an Euclidean space- then one can make use of the normal bundle as a complement. More precisely:

Exercise 3.60. Show that if M admits an embedding in $\mathbb{R}^{n+k}$ then there exists a rank k vector bundle $\mathcal{N}$ over M such that $TM \oplus \mathcal{N}$ is trivializable.

With this hint at hand, here is now a cool exercise:

Exercise 3.61. Show that $\mathbb{CP}^2$ cannot be embedded in $\mathbb{R}^5$.

Here is a little generalization of the previous exercise:

Exercise 3.62. Show that if an n-dimensional manifold M has $p_1(TM) \neq 0$, then M cannot be embedded in $\mathbb{R}^{n+1}$.

And here is a similar exercise, but which uses the second item of Exercise 3.59 rather than the first one. We state it in a way independent of Exercise 3.59 (but if you did do that exercise, then you can focus on the second item).

Exercise 3.63. Consider $\mathbb{CP}^4$, seen as an 8-dimensional real manifold.

(a) show that the following elements in the cohomology of $\mathbb{CP}^4$

$$p_2(T\mathbb{CP}^4), \quad \big(p_1(T\mathbb{CP}^4)\big)^2$$

have the same degree, but these elements are different from each other.

(b) deduce that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.

Next, we describe another interesting application: to the fundamental question of whether a given manifold N be written as the boundary $N = \partial M$ of a *compact* manifold with boundary M. Here it is important to notice that much of what we discussed in the course (vector bundles, tangent bundles, connections, characteristic classes, geometric structures, etc.) carries over (without any change) to manifolds with boundary. Furthermore, the Stokes theorem holds on general manifolds with boundary in the following form: if M is a compact oriented m-dimensional manifold with boundary $\partial M =: N$, then

$$\int_M d\omega = \int_N \omega|_N$$

for all forms $\omega \in \Omega^{n-1}(M)$.

Exercise 3.64.

(a) Show that if a manifold N of dimension $n = 4k$ can be written $N = \partial M$ with M compact orientable manifold with boundary, then $p_k(TN) = 0$.

(b) Deduce that the manifolds $\mathbb{CP}^{2k}$ with $k \geq 1$ (viewed as real manifolds) cannot be written as the boundary of a compact orientable manifold.

Exercise 3.65. Prove that:

(a) If a compact manifold P admits a free action of S^1 then P can be written as the boundary of a compact manifold Q (hint: it may be useful to remember the last homework. Otherwise: $\mathbb{C} \supset D^2 \supset \partial D^2 = S^1$ carries an action of S^1 and use fiber attachment).

(b for $n = 2k+1$, $\mathbb{RP}^{2k+1}$ can be written as the boundary of a compact orientable manifold.

(c) for $n = 2k+1$, $\mathbb{CP}^{2k+1}$ (see as a real manifold) can be written as the boundary of a compact orientable manifold.

Chapter 4
Principal bundles

4.1 Examples

Exercise 4.1. Consider the special orthogonal group

$$SO(3) = \{A \in GL_3(\mathbb{R}) : A \cdot A^T = I_3\}$$

Prove the following:

(a) The first column $c_1(A) \in \mathbb{R}^3$ of any matrix $A \in SO(3)$ is an element of S^2, while the second and the third columns can be interpreted as tangent vectors to S^2 at $c_1(A)$.
(b) The resulting map $c_1 : SO(3) \to S^2$ is a submersion.
(c) The following formula defines an action $\bullet$ of S^1 on $SO(3)$:

$$A \bullet z := A \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & -y & x \end{pmatrix} \quad \text{for } A \in SO(3),\ z = x + iy \in S^1$$

(d) With respect to this action, $c_1 : SO(3) \to S^2$ is a principal S^1-bundle.
(e) We now associate to this principal S^1-bundle the vector bundle E obtained by attaching the fiber $\mathbb{R}^2$ using the representation of S^1 on $\mathbb{R}^2$ given by

$$r : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

You are asked to prove that E is isomorphic to the tangent bundle TS^2.

Recall that, for the real projective spaces, $\mathbb{RP}^1$ is diffeomorphic to S^1, with the explicit diffeomorphism

$$\mathbb{RP}^1 \to S^1, \quad [x : y] \mapsto \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right). \tag{4.1.1}$$

(to see what this is about notice that it sends $[\cos\theta : \sin\theta]$ to $(\cos 2\theta, \sin 2\theta)$.) The exercise below is related to a similar discussion for $\mathbb{CP}^1$.

Exercise 4.2.

a) Show that the following defines a free and proper action of S^1 on the 3-sphere S^3:

$$(x, y, z, t) \cdot (a, b) = (ax - by, ay + bx, az - bt, at + bz)$$

b) Show that together with this action and the Hopf map

$$H : S^3 \longrightarrow S^2, \quad H(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - tx), 2(ty + xz)), \tag{4.1.2}$$

S^3 becomes a principal S^1-bundle over S^2.

(c) write everything in complex coordinates using

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}, \quad S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\},$$

and, using this, prove again (a) and (b).

d) Prove that $\mathbb{CP}^1$ and S^2 are diffeomorphic and derive an explicit diffeomorphism, similar to (4.1.1). Then compare with Exercise 1.23.

For later references, let us write down (4.1.2) in complex coordinates:

$$H : S^3 \longrightarrow S^2, \quad H(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2i\overline{z}_0 z_1).$$

The principal bundle described in the previous exercise is known as "the Hopf fibration". Notice that all its fibers are copies of S^1 (the orbits of the action), hence H makes S^3 into a collection ("bundle") of circles S^1, one for each point in S^2. May it happen that S^3 is actually $S^1 \times S^2$? No! (do you see why?). Actually the circles are bundled together in a non-trivial way, any two being intertwined. Try to imagine them! You could for instance use the stereographic projection to open up S^3, so that the circles can be pictured as circles in $\mathbb{R}^3$. I think there are also quite a lot of nice pictures and little movies on the internet ... have fun!

On the other hand, because of what happens for $\mathbb{CP}^1$, in general, the quotient map

$$H_n : S^{2n+1} \to \mathbb{CP}^n, \quad H_n(z_0, \ldots, z_n) = [z_0 : \ldots : z_n]$$

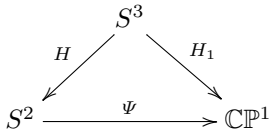
is called the (generalised) Hopf map.

Exercise 4.3. Prove that, endowing S^{2n+1} with the action of S^1 given by

$$(z_0, \ldots, z_n) \cdot \lambda = (\lambda z_0, \ldots, \lambda z_n),$$

the generalised Hopf map $H_n : S^{2n+1} \to \mathbb{CP}^n$ becomes a principal S^1-bundle.

Notice that, basically by construction, the diffeomorphisms from Exercise 4.2 identifies H_1 with the Hopf map H or, equivalently, the all fit in a commutative diagram:



, $\quad H_1 = \Psi \circ H.$

Exercise 4.4. Let $\pi : L \to M$ be a complex line bundle endowed with a hermitian product h. Using the induced norm on (the fibers of) L, $||v||_h = \sqrt{h(v, v)}$, we consider the circle bundle

$$P := \big\{ v \in L : ||v||_h = 1 \big\}, \qquad \pi_0 := \pi|_P : P \to M,$$

endowed with the action of S^1 induced by the complex multiplication $v \bullet \lambda := \lambda v$ (for $v \in P$, $\lambda \in S^1$).

(a) Prove that $\pi_0 : P \to M$ is a principal S^1-bundle.
(b) When L is the tautological line bundle over $M = \mathbb{CP}^n$, prove that the resulting principal bundle is the generalised Hopf fibration $H_n : S^{2n+1} \to \mathbb{CP}^n$.

The construction from the last exercise can also be performed backwards: build a complex line bundle out of a principal S^1-bundle; this is a simple illustration of the process of attaching fibers (if solved at the correct moment in time, this exercise comes before you learn the general construction):

Exercise 4.5. Start now with a principal S^1-bundle $\pi_0 : P \to M$ we consider the action of S^1 on the product $S^3 \times \mathbb{C}$ given by:

$$(p, z) \cdot \lambda := (p \cdot \lambda, \lambda^{-1} z) \qquad (\text{for } p \in P,\ z \in \mathbb{C},\ \lambda \in S^1)$$

Show that the resulting quotient $L(P) := (P \times \mathbb{C})/S^1$ is a complex line bundle; in detail, prove that:

(a) $L(P)$ is a smooth manifold.
(b) the map $\pi : L(P) \to M$, $[p, z] \mapsto \pi_0(p)$, is a submersion.
(c) declaring $[p, z] + [p, z'] = [p, z + z']$ and similarly for multiplication by scalars,

$$\pi : L(P) \to M$$

becomes a complex line bundle.

Then prove that:

(a) When applied to $\pi_0 : P \to M$ from the previous exercise, $L(P)$ is isomorphic to L.
(e) When applied to $c_1 : SO(3) \to S^2$ from Exercise 4.1, $L(P)$ is isomorphic to TS^2.

Notice that the last two exercises imply that the tautological line bundle over $\mathbb{CP}^n$ can be described as

$$L \cong (S^{2n+1} \times \mathbb{C})/S^1, \text{ the quotient modulo } (p, z) \cdot \lambda = (\lambda p, \lambda^{-1} z).$$

This viewpoint is very handy as it allows to describe the other canonical bundle $\mathcal{O}(n)$ mentioned in and before Exercise 3.51. Actually, if you are not happy with the construction given there in terms of tensor products, you can adopt the following as definition:

Exercise 4.6. Prove that, similarly,

$$\mathcal{O}(k) := (S^{2n+1} \times \mathbb{C})/S^1, \text{ the quotient modulo } (p, z) \cdot \lambda = (\lambda p, \lambda^k z)$$

is a complex line bundle over $S^{2n+1}/S^1 \cong \mathbb{CP}^n$, which coincides with the previously defined one.

Then generalize Exercise 1.10 by providing an isomorphism between the space of sections $\Gamma(\mathcal{O}(k))$ and the space of smooth functions $f : S^{2n+1} \to \mathbb{C}$ satisfying the S^1-invariance condition:

$$f(\lambda z) = \lambda^k f(z) \quad \forall\ \lambda \in S^1,\ z \in S^{2n+1}. \tag{4.1.3}$$

In particular, any degree k homogeneous polynomial $p \in \mathbb{C}[z_0, \ldots, z_n]$ can be seen as a section of $\mathcal{O}(k)$:

$$\mathbb{CP}^n \ni [z_0 : z_1, \ldots : z_n] \mapsto \big[(z_0, z_1, \ldots, z_n), p(z_0, z_1, \ldots, z_n)\big]$$

And here is an exercise on how to compute the Chern class of line bundles obtained from principal S^1-bundles by the above procedure.

Exercise 4.7. Let $\pi : P \to M$ be a principal S^1-bundle and consider the associated line bundle $\pi : L(P) \to M$ as discussed in Exercise 4.4 (or by applying the general theory of attaching fibers).

Assume now that we have an 1-form

$$\omega \in \Omega^1(P)$$

that satisfies the following properties:

- it is S^1-invariant, i.e. $m_\lambda^* \omega = \omega$ for any $\lambda \in S^1$, where $m_\lambda : P \to P$ is the multiplication by λ.
- on the vector field $V \in \mathfrak{X}(P)$ the whose flow is $\phi_V(p,t) = p \cdot e^{it}$, one has $\omega(V) = 1$.
  (V is called the infinitesimal generator of the action)

Show that:

(a) There exists, and is unique, $k \in \Omega^2(M)$ such that $d\omega = \pi^*(k)$.
(b) The form k is closed.
(c) the cohomology class of k, $[k] \in H^2(M)$, does not not depend on the choice of ω.
(d) up to a constant, $[k]$ is the first Chern class of $L(P)$.

Then apply this procedure to the two concrete examples mentioned above:

(e) To the Hopf fibration $H_1 : S^3 \to \mathbb{CP}^1$, when $L(P)$ is the tautological line bundle.
(f) To $c_1 : SO(3) \to S^2$ from Exercise 4.1, when $L(P) \cong TS^2$.

(Hint: for (e), consider

$$\omega := i\left(z_1 d\overline{z_1} + z_2 d\overline{z_2}\right) = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \in \Omega^1(S^3, \mathbb{R}))$$

4.2 Frame bundles

Let's repeat the construction of *the frame bundle* of a vector bundle $\pi : E \to M$ of rank r: it is

$$\mathrm{Fr}(E) = \{(x, \mathfrak{e}) : x \in M, \mathfrak{e} = (\mathfrak{e}_1, \dots, \mathfrak{e}_r) - \text{frame of } E_x\}$$

endowed with the right action of GL_r (invertible $r \times r$ matrices) given by

$$\begin{aligned}(x, \mathfrak{e}) \cdot A = (x, \mathfrak{e} \cdot A) &= \left(x, (\mathfrak{e}_1, \dots, \mathfrak{e}_r) \cdot \begin{pmatrix} A_1^1 \dots A_r^1 \\ \dots\dots\dots \\ A_1^r \dots A_r^r \end{pmatrix} \right) \\ &= \left(x, \sum_j A_1^j \mathfrak{e}_j, \dots, \sum_j A_r^j \mathfrak{e}_j \right) \end{aligned} \tag{4.2.1}$$

For the smooth structure on $\mathrm{Fr}(E)$ we make use of the fact that

$$\mathrm{Fr}(E) \subset \mathrm{Hom}(M \times \mathbb{R}^r, E),$$

where the right hand side is the Hom-vector bundle

$$\mathrm{Hom}(M \times \mathbb{R}^r, E) = \{(x, \phi) : x \in M, \phi : \mathbb{R}^r \to E_x \text{ linear}\},$$

and where the inclusion interprets $\mathfrak{e} = (\mathfrak{e}_1, \dots, \mathfrak{e}_r)$ as the linear map $\widehat{\mathfrak{e}} : \mathbb{R}^r \to E_x$ that sends the canonical basis of $\mathbb{R}^r$ to $\mathfrak{e}$ or, explicitly,

$$\widehat{\mathfrak{e}}(\lambda^1, \dots, \lambda^r) = \lambda^1 e_1 + \dots + \lambda^r \mathfrak{e}_r.$$

This realises $\mathrm{Fr}(E)$ as an open subset of $\mathrm{Hom}(M \times \mathbb{R}^r, E)$ and, therefore, as an open inside a manifold, it becomes itself a manifold.

Exercise 4.8. Please do the following:

(a) describe an action of GL_r on $\mathrm{Hom}(M \times \mathbb{R}^r, E)$ so that, when restricted to $\mathrm{Fr}(E)$, it becomes the action described above. Deduce from this that the original action of GL_r on $\mathrm{Hom}(M \times \mathbb{R}^r, E)$ is indeed a smooth action.

(b) show that

$$\pi : \mathrm{Fr}(E) \to M, \quad (x, \mathfrak{e}) \mapsto x$$

becomes a principal GL_r-bundle.

(c) carry out a similar discussion for frames of E that are orthonormal w.r.t. a given inner product g on E, giving rise to a principal $O(r)$-bundle $\mathrm{Fr}(E, g)$ *of orthonormal frames*. And similarly for complex vector bundles F endowed with a hermitian product h, giving rise to a principal $U(r)$-bundle $\mathrm{Fr}(F, h)$.

Exercise 4.9. Given a vector bundle E of rank r, show that any principal $O(r)$ sub-bundle of $\mathrm{Fr}(E)$ is of type $\mathrm{Fr}(E, g)$ for a unique inner product g on E. Similarly for hermitian products on complex vector bundles.

Exercise 4.10. We consider the orthogonal of the tautological complex line bundle L over $\mathbb{CP}^n$,

$$L^\perp := \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\},$$

we endow it with the hermitian product h obtained by restricting the standard one from $\mathbb{C}^{n+1}$ and we consider the resulting orthonormal frame bundle $\mathrm{Fr}(L^\perp, h)$. Prove that:

(a) $\mathrm{Fr}(L^\perp, h)$ is a principal $U(n)$-bundle over $\mathbb{CP}^n$.
(b) its pull-back $H^*\mathrm{Fr}(L^\perp, h)$ via the Hopf map $H : S^{2n+1} \to \mathbb{CP}^n$ is diffeomorphic to $U(n+1)$.

Exercise 4.11. Consider the orthonormal frame bundle $\mathrm{Fr}(E, g)$ of $E = TS^2$ w.r.t. the inner product induced from the standard inner product on $\mathbb{R}^3$.

(a) Prove that $\mathrm{Fr}(E, g)$ is diffeomorphic to $O(3)$.
(b) By the general theory, $\mathrm{Fr}(E, g)$ is a principal $O(2)$-bundle over S^2. Via the diffeomorphism from the previous item, one obtains a principal $O(2)$-bundle structure on $O(3)$. Describe explicitly its projection and the $O(2)$-action.

Exercise 4.12. A partial r-coframe on an n-dimensional manifold M is a collection

$$\theta = \{\theta^1, \dots, \theta^r\}$$

of 1-forms on M with the property that, at each point $x \in M$, they are linearly independent. One associates to θ:

$$P_\theta := \{(\phi_1, \dots, \phi_n) \in \mathrm{Fr}(TM) : \theta_i(\phi_j) = \delta_{i,j} \ \forall\ 1 \le i \le r, 1 \le j \le n\}.$$

Show that

(a) there exists a subgroup $G \subset GL_n$ such that the canonical action of GL_n on $\mathrm{Fr}(M)$ restricts to an action of G on P_θ, so that P_θ becomes a principal G-bundle.
(b) for the G that you found show that $\theta \mapsto P_\theta$ defines a 1-1 correspondence between partial r-coframes on M and principal G-subbundles of $Fr(M)$.

4.3 Attaching fibers

The operation of attaching fibers to principal bundles has already been mentioned (without naming it) in the construction of complex line bundles out of S^1-bundles- see Exercise 4.5. It also arises naturally when trying to recover a vector bundle out of its frame bundle.

Exercise 4.13 (continuation of Exercise 4.8). Given a rank r vector bundle E:

(d) show that the following is a smooth map

$$\mathrm{Fr}(E) \times \mathbb{R}^r \to E, \quad ((x, \mathfrak{e}), \lambda) \mapsto \lambda^1 e_1 + \ldots + \lambda^r e_r$$

(where $\mathfrak{e} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_r)$ and $\lambda = (\lambda^1, \ldots, \lambda^r)$.)

(e) describe an (right) action of GL_r on $\mathrm{Fr}(E) \times \mathbb{R}^r$ which is free and proper and which induces a diffeomorphism between the resulting quotient space and E.

(f) do the same for the orthonormal frame bundles $\mathrm{Fr}(E, g)$ and $O(r)$.

Exercise 4.14. Let P be a G-manifold (with right action), and consider a G-vector bundle

$$\pi_{\widetilde{E}} : \widetilde{E} \to P,$$

by which we mean a vector bundle together with an action of G on $\widetilde{E}$ such that $\pi_{\widetilde{E}}$ is G-equivariant and the action on $\widetilde{E}$ is fiberwise linear ($g \in G$ takes each fiber $\widetilde{E}_x$ linearly into $\widetilde{E}_{xg}$). Show that:

(a) if the action of G on P is free or proper, then so is the action of G on $\widetilde{E}$.

(b) assuming now that P is a principal G-bundle $\pi_P : P \to M$, deduce that $E := \widetilde{E}/G$ is a smooth manifold and turn E into a smooth vector bundle $\pi_E : E \to M$ (proving all the claims).

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\pi_{\widetilde{E}}} & P \\ \downarrow & & \downarrow \pi_P \\ E & \xrightarrow[\pi_E]{} & M \end{array}$$

(c) A particular case: for $V \in \mathrm{Rep}(G)$, apply the previous discussion to $\widetilde{E} := P \times V$ endowed with the diagonal action. What do you obtain?

$$\begin{array}{ccc} P \times V & \xrightarrow{pr_1} & P \\ \downarrow & & \downarrow \pi_P \\ P[V] & \xrightarrow[\pi_E]{} & M \end{array}$$

Exercise 4.15. Here the relevant group is $G = S^1$ and we consider the canonical representation

$$r : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ \cos(\alpha) + i\sin(\alpha) \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

as well as "the square" of this representation:

$$r^2 : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ \cos(\alpha) + i\sin(\alpha) \mapsto \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix}.$$

Consider also S^3 as a principal S^1-bundle over S^2 given by the Hopf fibration (Exercise 4.2), and $SO(3)$ viewed as a principal S^1-bundle $\pi : SO(3) \longrightarrow S^2$ as in Exercise 4.1. Show that:

(a) $E(S^3, \mathbb{R}^2, r^2) \cong E(SO(3), \mathbb{R}^2, r)$ and they can be identified with (are isomorphic to) the tangent bundle of S^2.
(b) $E(S^3, \mathbb{R}^2, r)$ can be identified with the tautological line bundle $L \to \mathbb{CP}^1$; more precisely, using the diffeomorphism $\Psi : S^2 \to \mathbb{CP}^1$ previously discussed (e.g. in Exercise 1.23 and 4.2), $E(S^3, \mathbb{R}^2, r)$ is isomorphic to $\Psi^*(L)$.

4.4 List of exercises given on December 12, 2024

STILL TO BE PROCESSED!

Exercise 4.16. *(2pt)* Show that a right action of a group G on a manifold M is proper if and only if for any compact $K \subset M$, $\{g \in G : Kg \cap K \neq \emptyset\} \subset G$ is compact.

Exercise 4.17. Let G be a Lie group and let $(\mathfrak{g}, [\cdot,\cdot])$ be its Lie algebra. Recall that a Lie sub-algebra of $\mathfrak{g}$ is any vector subspace $\mathfrak{h} \subset \mathfrak{g}$ with the property that $[u, v] \in \mathfrak{h}$ for all $u, v \in \mathfrak{h}$.

A foliation on G, interpreted as an involutive sub-bundle $\mathcal{F} \subset TG$, is called G-invariant if it is stable under the (differentials of the) left translations $L_g : G \to G, L_g(x) = g \cdot x$, i.e.,

$$(dL_g)_x(\mathcal{F}_x) = \mathcal{F}_{gx} \quad \forall\ g, x \in G.$$

Show that the map that associates to $\mathcal{F}$ to its fiber at the identity element e of G (hence sends $\mathcal{F}$ to $\mathcal{F}_e \subset T_eG = \mathfrak{g}$), provides a bijection:

$$\left\{\begin{array}{c} G\text{-invariant foliations} \\ \mathcal{F} \subset TG \end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c} \text{Lie subalgebras} \\ \mathfrak{h} \text{ of } \mathfrak{g} \end{array}\right\}$$

Exercise 4.18. Let M be a manifold endowed with a free action of a connected Lie group G, and consider the associated infinitesimal action

$$\rho : \mathfrak{g} \to \mathfrak{X}(M).$$

(a) Show that
$$\mathcal{F} := \{\rho_x(u) : u \in \mathfrak{g}, x \in M\} \subset TM$$
is a smooth sub-bundle of the vector bundle TM.

(b) Show that $\mathcal{F} \subset TM$ is an involutive sub-bundle of TM.

(c) Show that for each $x \in M$,
$$i_x : G \to M, \quad g \mapsto x \cdot g$$
makes G into an immersed submanifold of M, which is an integral of $\mathcal{F}$.

Exercise 4.19. Let G be a connected Lie group, with Lie algebra denoted $\mathfrak{g}$. Recall that any $v \in \mathfrak{g}$ can be left translated giving rise to a vector field $\vec{v} \in \mathfrak{X}(G)$,

$$\vec{v}_g := (dL_g)_e(v) \in T_gG, \quad (g \in G).$$

On the other hand, a vector field $V \in \mathfrak{X}(G)$ is called left invariant if $(dL_g)_x(V_x) = V_{gx}$ for all $g, x \in G$.

(a) *show that each $\vec{v}$ is left invariant and, conversely any left invariant vector field on G is of this form.*

To work with vector fields and forms on G it is useful to fix a basis $\{e_1, \ldots, e_r\}$ of the Lie algebra $\mathfrak{g}$ and consider the constants $c_{i,j}^k \in \mathbb{R}$ obtained by writing out the Lie bracket of $\mathfrak{g}$ on the basis:

$$[e_i, e_j] = \sum_{i,j,k} c_{i,j}^k e_k.$$

By left translating, we obtain r left-invariant vector fields on G

$$E_1 = \vec{v}_1, \ldots, E_r = \vec{v}_r.$$

(b) *show that* $\{E_1, \ldots, E_r\}$ *is a frame of* TG.

Passing to dual bases, we find r one-forms on G,

$$\theta^1, \ldots, \theta^r \in \Omega^1(G)$$

uniquely determined by the condition $\theta^i(E_j) = \delta_j^i$ for all i and j.

(c) *show that* $d\theta^k = \sum_{i,j} c_{i,j}^k \theta_i \wedge \theta_j$.

Next we move to connections on TG.

(d) *Show that there is a connection* ∇ *on* TG *with the property that*

$$\nabla_X(Y) = \frac{1}{2}[X, Y]$$

for all $X, Y \in \mathfrak{X}(G)$ *that are left-invariant.*

(e) *Show that* ∇ *is torsion free.*

On the other hand, since we have a frame on TG, namely $E = \{E_1, \ldots, E_r\}$ the connection ∇ is completely encoded by the corresponding connection matrix (w.r.t. E), denoted here

$$\omega = (\omega_j^i)_{1 \leq i,j \leq r} \in \mathcal{M}_r(\Omega^1(G)).$$

(f) *Show that* $\omega_j^i = \frac{1}{2} \sum_p c_{p,j}^i \theta^p$.

Next we want to bring in Riemannian metrics B on G into the discussion. We say that B is left invariant if

$$B_{gx}\,(dL_g)_x(V_x), (dL_g)_x(W_x)) = B_x(V_x, W_x)$$

for all $g, x \in G$, $V_x, W_x \in T_xG$.

(g) *Explain how an arbitrary inner product* $b : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *gives rise to a left-invariant Riemannian metric* $\vec{b}$ *on* G *which, at the unit* $e \in G$*, is precisely* b*. Conclude that this gives rise to a bijection between inner products on* $\mathfrak{g}$ *and left invariant Riemannian metrics on* G.

On the other hand, one can also talk about right-invariant metrics B, using the right translations R_g, with $R_g(x) = xg$. We say that B is bi-invariant if it is both left as well as right invariant.

(h) *Show that bi-invariant metrics* B *are in bijection with inner products* $b : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *satisfying the invariance condition*

$$b(ad_g(v), ad_g(w)) = b(v, w)$$

for all $g \in G$ *and* $v, w \in \mathfrak{g}$*, where* $ad_g : \mathfrak{g} \to \mathfrak{g}$ *is the differential at* $e \in G$ *of the map* $G \to G$*,* $x \mapsto g \cdot x \cdot g^{-1}$.

It may be interesting to know that such bi-invariant metrics often exist- e.g. they exist for all compact Lie groups G. Also, because G is connected, one can show (and you can use that without proving it) that the invariance condition on b is equivalent to the equation

$$b([u,v],w) + b(v,[u,w]) = 0 \qquad \forall\ u,v,w \in \mathfrak{g}.$$

(i) *show that a left invariant Riemannian metric B on G is bi-invariant if and only if ∇ from d) is the Levi-Civita connection associated to B.*

Next, recall that the exponential map $\exp : \mathfrak{g} \to G$ was defined so that, for each $v \in \mathfrak{g}$,

$$\gamma : \mathbb{R} \to G, \quad \gamma(t) = \exp(tv)$$

is an integral curve of $\vec{v}$ starting at $e \in G$.

(j) *Assuming that B is bi-invariant, show that each of the curves $\gamma(t) = exp(tv)$ are also geodesics of (G, B).*

(k) *So, as conclusion: what is the relationship between the exponential map from Lie theory and the exponential map for Riemannian manifolds?*

Chapter 5
Previous exams

5.1 Retake Diff Geom mastermath, February 12, 2026

Note: Please explain what you do and provide as many details as possible!!!!

Exercise 5.1 (3pts). Let M be a manifold and $\mathbb{J} : TM \to TM$ be a vector bundle morphism. Show that:

(a) the operation that associates to two vector fields $X, Y \in \mathfrak{X}(M)$ the new vector field

$$\mathcal{N}_{\mathbb{J}}(X,Y) = \mathbb{J}\Big([\mathbb{J}X,Y] + [X,\mathbb{J}Y]\Big) - \mathbb{J}^2\big([X,Y]\big) - [\mathbb{J}X,\mathbb{J}Y]$$

defines a 2-form on M with coefficients in TM, $\mathcal{N}_{\mathbb{J}} \in \Omega^2(M,TM)$;

(b) if $\mathcal{N}_{\mathbb{J}} = 0$ and the image $\mathcal{F}_{\mathbb{J}} := \mathbb{J}(TM) \subset TM$ is of constant rank, then $\mathcal{F}_{\mathbb{J}}$ defines a foliation on M.

(c) if, furthermore, $\mathbb{J}^3 = -\mathbb{J}$ then, for each leaf $L \subset M$ of the foliation $\mathcal{F}_{\mathbb{J}}$, the restriction $\mathbb{J}|_{TL} : TL \to TL$ defines a complex structure on L.

Exercise 5.2 (0.5+ 0.5+ 0.5+ 1= 2.5pts). We consider the trivial vector bundle of rank two over S^2,

$$E = S^2 \times \mathbb{R}^2,$$

and, as usual, we interpret the sections of E as pairs (f,g) of functions $f, g : S^2 \to \mathbb{R}$. For $\epsilon \in \mathbb{R}$ we set $\theta^\epsilon := xdy + \epsilon ydx \in \Omega^1(S^2)$ and we consider the connection ∇^ϵ on E given by

$$\nabla^\epsilon_X(f,g) = \left(L_X(f) + g\theta^\epsilon(X),\ f\theta^\epsilon(X) + L_X(g)\right).$$

(a) what is the connection matrix of ∇^ϵ w.r.t. the canonical basis of E?

(b) what is the curvature matrix of ∇^ϵ w.r.t. the canonical basis?

(c) when $\epsilon = 1$, prove that the resulting parallel transport along any curve $\gamma : [0,1] \to S^2$ with the property that $\gamma(0) = \gamma(1)$ is the identity map;

(d) when $\epsilon = -1$, compute the parallel transport along the curve $\gamma(t) = (\cos t, \sin t, 0)$.

Exercise 5.3 (1+1= 2pts). Consider $\mathbb{CP}^4$, seen as an 8-dimensional real manifold.

(a) show that the following elements in the cohomology of $\mathbb{CP}^4$

$$p_2(T\mathbb{CP}^4), \quad \big(p_1(T\mathbb{CP}^4)\big)^2$$

have the same degree, but these elements are different from each other.
(b) deduce that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.

Exercise 5.4 (0.5+ 0.5+ 0.5+ 0.5+ 1= 3pts). Prove the following:

(a) The first column $c_1(A) \in \mathbb{R}^3$ of any matrix $A \in SO(3)$ is an element of S^2, while the second and the third columns can be interpreted as tangent vectors to S^2 at $c_1(A)$.
(b) The resulting map $c_1 : SO(3) \to S^2$ is a submersion.
(c) The following formula defines an action $\bullet$ of S^1 on $SO(3)$:

$$A \bullet z := A \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & -y & x \end{pmatrix} \quad \text{for } A \in SO(3),\ z = x + iy \in S^1$$

(d) With respect to this action, $c_1 : SO(3) \to S^2$ is a principal S^1-bundle.
(e) We now associate to this principal S^1-bundle the vector bundle E obtained by attaching the fiber $\mathbb{R}^2$ using the representation of S^1 on $\mathbb{R}^2$ given by

$$r : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

You are asked to prove that E is isomorphic to the tangent bundle TS^2.

5.2 Exam Diff Geom mastermath, January 15, 2026

Exercise 5.5 (0.5+ 0.5+ 0.5+ 0.5+ 0.5+ 1+ 1+ 1= 5.5pts).
Let M be a manifold endowed with a Riemannian metric g and a foliation $\mathcal{F}$ (interpreted as an involutive sub-bundle $\mathcal{F} \subset TM$).

By an *$\mathcal{F}$-connection* on a vector bundle $\mathcal{N}$ over M we mean a bilinear map

$$\nabla : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{N}) \to \Gamma(\mathcal{N})$$

satisfying the usual identities $\nabla_{fV}(s) = f\nabla_V(s)$, $\nabla_V(fs) = f\nabla_V(s) + L_V(f)s$ for all $f \in C^\infty(M)$, $s \in \Gamma(\mathcal{N})$, $V \in \Gamma(\mathcal{F})$ (hence the standard notion of connection on $\mathcal{N}$ is obtained when $\mathcal{F} = TM$). Similarly, we say that ∇ is *$\mathcal{F}$-flat* if $\nabla_{[V,W]} = [\nabla_V, \nabla_W]$ for all $V, W \in \Gamma(\mathcal{F})$. Given an $\mathcal{F}$-connection ∇ on $\mathcal{N}$ prove that

(a) There exists a connection $\widetilde{\nabla}$ on $\mathcal{N}$ which extends ∇ in the sense that $\nabla_V(s) = \widetilde{\nabla}_V(s)$ for all $V \in \Gamma(\mathcal{F})$ and $s \in \Gamma(\mathcal{N})$.

(b) For any leaf $L \subset M$ of $\mathcal{F}$, there exists an induced connection ∇^L on $\mathcal{N}|_L$,

$$\nabla^L : \mathfrak{X}(L) \times \Gamma(\mathcal{N}|_L) \to \Gamma(\mathcal{N}|_L),$$

which is canonical (in the sense that it does not depend on the choices one might make to describe it) and has the property that $\nabla_V(s)|_L = \nabla^L_{V|_L}(s|_L)$ for all $V \in \Gamma(\mathcal{F})$ and $s \in \Gamma(\mathcal{N})$ (note: for $V \in \Gamma(\mathcal{F})$ one has $V|_L \in \mathfrak{X}(L)$ because, at any $x \in L$, one has $V_x \in \mathcal{F}_x = T_xL$).

Assume now that $\mathcal{N}$ is the orthogonal of $\mathcal{F}$ in TM w.r.t. g,

$$\mathcal{N} = \Big\{ Y \in TM : g(V, Y) = 0 \ \forall\ V \in \mathcal{F} \Big\} \subset TM.$$

and consider the corresponding orthogonal projection $pr_{\mathcal{N}} : TM \to \mathcal{N}$ (sending $X \in TM$ to the unique element $Y \in \mathcal{N}$ such that $X - Y \in \mathcal{F}$). Prove that

(c) The formula $\nabla^{\mathcal{N}}_V(Y) := pr_{\mathcal{N}}([V, Y])$ defines an $\mathcal{F}$-connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}$.

(d) $\nabla^{\mathcal{N}}$ is $\mathcal{F}$-flat.

(e) For any leaf L of $\mathcal{F}$, $TM|_L$ and $\mathcal{F}|_L$ have the same Pontryagin classes.

We now consider the Levi-Civita connection ∇^{LC} associated to g. Prove that:

(f) The operators ∇^{LC}_V with $V \in \Gamma(\mathcal{F})$ preserve $\Gamma(\mathcal{N})$ (i.e. $\nabla^{LC}_V(\Gamma(\mathcal{N})) \subset \Gamma(\mathcal{N})$ for all $V \in \Gamma(\mathcal{F})$) if and only if they preserve $\Gamma(\mathcal{F})$.

(g) the following are equivalent:

(g1) ∇^{LC} is an extension of $\nabla^{\mathcal{N}}$: $\nabla^{\mathcal{N}}_V(Y) = \nabla^{LC}_V(Y)$ for all $V \in \Gamma(\mathcal{F})$, $Y \in \Gamma(\mathcal{N})$
(g2) ∇^{LC} preserves $\mathcal{F}$: $\nabla^{LC}_X(\Gamma(\mathcal{F})) \subset \Gamma(\mathcal{F})$ for all $X \in \mathfrak{X}(M)$

When the equivalent conditions from the last items hold we say that g and $\mathcal{F}$ are compatible. When that happens prove that:

(h) any geodesic γ of (M, g) with the property that $\dot\gamma(0) \in \mathcal{F}$ must be a leafwise curve, i.e. there exists a leaf L of M such that $\gamma(t) \in L$ for all t.

Exercise 5.6 (0.5+ 0.5+ 1= 2pts).
Let $E = M \times \mathbb{R}$ be the trivial line bundle over a manifold M, let $\theta \in \Omega^1(M)$ be a 1-form and define the connection ∇^θ on E which, after the identification $\Gamma(E) \cong C^\infty(M)$, is given by

$$\nabla^\theta : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M), \quad \nabla^\theta_X(f) = L_X(f) + f\theta(X).$$

(a) compute the curvature of ∇^θ in terms of θ
(b) a path u in $E = M \times \mathbb{R}$ has two components, namely a path γ_u in M and a path f_u in $\mathbb{R}$. For $u = (\gamma_u, f_u)$, describe the components of $\frac{\nabla^\theta u}{dt}$.
(c) given a path $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1)$, show that the induced parallel transport, w.r.t. ∇^θ, along γ, from $t = 0$ to 1, $T^{0,1}_\gamma : E_{\gamma(0)} \to E_{\gamma(0)}$, is the multiplication by the real number $\exp\left(-\int_{[0,1]} \gamma^*\theta\right)$.

Exercise 5.7 (1+ 1+ 1+ 1= 4pts).
Prove that:

(a) If a compact manifold P admits a free action of S^1 then P can be written as the boundary of a compact manifold Q (hint: it may be useful to remember the last homework. Otherwise: $\mathbb{C} \supset D^2 \supset \partial D^2 = S^1$ carries an action of S^1 and use fiber attachment).
(b) If a manifold P of dimension $n = 4k$ can be written $P = \partial Q$ with Q compact orientable manifold with boundary, then $p_k(TP) = 0$ for all $k \geq 1$.

Consider now $P = \mathbb{CP}^n$. Recall that the tangent bundle $T\mathbb{CP}^n$ (a complex vector bundle) is related to the tautological line bundle L and the trivial complex line bundle T over $\mathbb{CP}^n$ via isomorphisms

$$T\mathbb{CP}^n \oplus T \cong \underbrace{L^* \oplus \ldots \oplus L^*}_{n+1 \text{ times}}.$$

You are allowed to use this isomorphism and the fact that $u := -c_1(L) \in H^2(\mathbb{CP}^n)$ has the property that $u^n \neq 0$; any other formulas that you may be aware from the homeworks have to be proven. Seeing $\mathbb{CP}^n$ as a real $2n$-dimensional manifold, prove now that:

(A) for $n = 2k+1$, $\mathbb{CP}^{2k+1}$ can be written as the boundary of a compact orientable manifold.
(B) for $n = 2k$, $\mathbb{CP}^{2k}$ cannot be written as the boundary of a compact orientable manifold.

Notes:

- Please explain your answers.
- The order of the exercises is random.
- The maximum mark is 10, hence you may skip some questions. If you skip one, you can still make use of it in the questions that follow.

5.3 Retake mastermath Diff Geom, February 6, 2024

Notes: Please explain your answers. The maximum mark is 10, hence you may skip some questions. If you skip one, you can still make use of it. Choose wisely.

Exercise 5.8 (6 x 0.5+1= 4pts). Let $M = S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, over which we consider the vector bundle $E := TS^2$. We also consider the following

$$\mathfrak{e}^x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathfrak{e}^y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathfrak{e}^z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \qquad (\star)$$

Since $S^2 \subset \mathbb{R}^3$, we can view E as a sub-bundle of the trivial bundle $\widetilde{E} = S^2 \times \mathbb{R}^3$. We will also use the orthogonal projection w.r.t. the standard inner product on $\mathbb{R}^3$,

$$\mathrm{pr}_E : \widetilde{E} \to E,$$

and the connection ∇ on E obtained by projecting the usual flat connection on $\widetilde{E}$:

$$\nabla : \mathfrak{X}(S^2) \times \Gamma(E) \to \Gamma(E), \quad (X,s) \mapsto \nabla_X(s) := \mathrm{pr}_E\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(\widetilde{E})}\right),$$

or, if you prefer the dual point of view, $d_\nabla : \Gamma(E) \to \Omega^1(S^2, E)$, $d_\nabla(s) = \mathrm{pr}_E(d_{\nabla^{\mathrm{flat}}}(s))$.

(a) Describe the largest $U \subset S^2$ over which $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ is a local frame of TS^2.
(b) Consider the similar local frame $\{\mathfrak{e}^x, \mathfrak{e}^z\}$ over the corresponding $V \subset S^2$ and compute the transition functions from $\{\mathfrak{e}^x, \mathfrak{e}^y\}$ to $\{\mathfrak{e}^x, \mathfrak{e}^z\}$.
(c) Show that $\mathrm{pr}_E\left(\frac{\partial}{\partial x}\right) = y\mathfrak{e}^z - z\mathfrak{e}^y = \frac{\partial}{\partial x} - x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$.
(d) Show that $d_\nabla(\mathfrak{e}^x) = -x \cdot (dx \otimes \mathfrak{e}^x + dy \otimes \mathfrak{e}^y + dz \otimes \mathfrak{e}^z)$.
(e) Compute the connection matrix ω of ∇ w.r.t. $\{\mathfrak{e}^x, \mathfrak{e}^y\}$.
(f) From the general theory it follows that for any path γ in S^2 and any path ξ in $E = TS^2$ sitting above γ, we can talk about $\frac{\nabla\xi}{dt}$, which is a new path in E above γ. Writing $\gamma(t) = (x(t), y(t), z(t))$ and denoting by $f(t)$, $g(t)$ and $h(t)$ the components of $\xi(t)$ obtained by writing

$$\xi(t) = f(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + g(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} + h(t)\left(\frac{\partial}{\partial z}\right)_{\gamma(t)}$$

prove that the similar components of $\frac{\nabla\xi}{dt}$ are

$$\dot{f} + x(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{g} + y(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{h} + z(\dot{x}f + \dot{y}g + \dot{z}h).$$

(g) For $\gamma(t) = (0, \cos t, \sin t)$ show that the parallel transport $T^{0,t}_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$, sends $\frac{\partial}{\partial z}$ to $-\sin t \cdot \frac{\partial}{\partial y} + \cos t \cdot \frac{\partial}{\partial z}$.

Exercise 5.9 (0.5+1= 1.5pts). Consider the orthonormal frame bundle $\mathrm{Fr}(E,g)$ of $E = TS^2$ w.r.t. the inner product induced from the standard inner product on $\mathbb{R}^3$.

(a) Prove that $\mathrm{Fr}(E,g)$ is diffeomorphic to $O(3)$.
(b) By the general theory, $\mathrm{Fr}(E,g)$ is a principal $O(2)$-bundle over S^2. Via the diffeomorphism from the previous item, one obtains a principal $O(2)$-bundle structure on $O(3)$. Describe explicitly its projection and the $O(2)$-action.

Exercise 5.10 (11 x 0.5= 5.5pts). Recall that an almost complex structure $J : TM \to TM$ on a manifold M makes each tangent space T_xM into a complex vector space. Recall also that the Nijenhuis tensor of J is defined by

$$\mathcal{N} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \quad \mathcal{N}(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY].$$

Assume now that J is an arbitrary almost complex structure on M. Prove that

(a) J gives rise to a 2-form with coefficients $\mathcal{N} \in \Omega^2(M, TM)$.
(b) at each $p \in M$ the resulting $\mathcal{N}_p : T_pM \times T_pM \to T_pM$ is $\mathbb{C}$-bilinear.
(c) if $V \in \mathfrak{X}(M)$ is nowhere vanishing then V and JV are linearly independent.
(d) J is automatically integrable when $\dim M = 2$.
(e) When $\dim M = 2$, under the extra assumption that around any point of M one can find a nowhere vanishing vector field V such that $[V, JV] = 0$, prove that J is integrable without using the Newlander-Nirenberg theorem.

From now on assume $M = S^2$ and, for $p = (x,y,z) \in S^2$, define J_p by:

$$\underbrace{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}}_{\in T_pS^2} \overset{J_p}{\longmapsto} \underbrace{(cy - bz)\frac{\partial}{\partial x} + (az - cx)\frac{\partial}{\partial y} + (bx - ay)\frac{\partial}{\partial z}}_{\in T_p\mathbb{R}^3}.$$

(f) Prove that J is an almost complex structure on S^2.
(g) For the vector fields $(\star)$, compute $J\mathfrak{e}^x$ and the similar ones in terms of $\mathfrak{e}^x, \mathfrak{e}^y, \mathfrak{e}^z$.
(h) Next, you may use the identities $[\mathfrak{e}^x, \mathfrak{e}^y] = \mathfrak{e}^z$, $[\mathfrak{e}^y, \mathfrak{e}^z] = \mathfrak{e}^x$, $[\mathfrak{e}^z, \mathfrak{e}^x] = \mathfrak{e}^y$. Prove now one of the the following identities

$$[J\mathfrak{e}^x, \mathfrak{e}^y] = 0, \ [J\mathfrak{e}^x, J\mathfrak{e}^y] = -\mathfrak{e}^z, \ [\mathfrak{e}^x, J\mathfrak{e}^y] = -[J\mathfrak{e}^x, \mathfrak{e}^y] = J\mathfrak{e}^z$$

(which, together with their analogues, you can use from now on).
(i) Deduce that J is integrable and $V := y\mathfrak{e}^z - (z-1)\mathfrak{e}^y$ commutes with JV.
(j) Show that the stereographic projection with respect to the north pole

$$\chi : S^2 \setminus \{p_N\} \to \mathbb{R}^2, \quad \chi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

$$\left(\text{with inverse } \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \longleftarrow\!\shortmid (u,v)\right)$$

is a complex chart for (S^2, J).
(k) Show that the connection from the first exercise is a connection on the complex vector bundle (TS^2, J).

Exercise 5.11 (1.5+ 1.5= 3pts). We endow the tangent bundle TS^2 with the complex structure from the previous exercise and consider its first Chern class, denoted

$$c_1 = c_1(TS^2) \in H^2(S^2).$$

In the previous exercise one can go on and compute the curvature matrix of ∇ (a 1×1 matrix, hence just a form), and find that the curvature k_∇ is a non-zero multiple of the area form of S^2. Using this, prove that:

(a) S^2 does not admit a nowhere vanishing vector field.

(b) There does not exist a morphism of trivial complex vector bundle over S^2, $f : S^2 \times \mathbb{C}^2 \to S^2 \times \mathbb{C}$, whose kernel is TS^2 (as a complex vector bundle).

Beaware: *This exercise can be solved even if you did not answer any of the questions from the other exercises!*

Exercise 5.12 (1.5pts). Prove that S^3, together with the right action of S^1 given by

$$(x, y, z, t) \bullet (a, b) = (ax - by, ay + bx, az - bt, at + bz),$$

and together with the map

$$H : S^3 \longrightarrow S^2, \quad H(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - tx), 2(ty + xz)),$$

is a principal S^1-bundle over S^2.

Exercise 5.13 (2x1= 2 pts). By a flag-structure on an r-dimensional vector space V we mean a sequence of vector subspaces

$$V_1 \subset V_2 \subset \ldots \subset V_{r-1} \subset V_r = V$$

where each V_k is k-dimensional.

(a) Find a group $G \subset GL_r$ such that flag-structures on V are in 1-1 correspondence with G-structures on V. What are the adapted frames?
(b) Go on and describe the resulting theory of flag-structures on manifolds, similar to the other examples we have discussed in the class.

Exercise 5.14 (7 pts). We consider the tautological complex line bundle L over $\mathbb{CP}^n$. From its very definition, L is a vector sub-bundle of the trivial vector bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$. We also consider the orthogonal complement of L in E,

$$L^\perp := \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\},$$

where the orthogonality $\perp$ is with respect to the standard hermitian product on $\mathbb{C}^{n+1}$:

$$h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1} + \ldots + z_n\overline{z'_n}.$$

Similar to what we did in the class and homeworks, but now working with $L^\perp$ instead of L, we use the standard (flat) connection ∇^{flat} on the trivial bundle E followed by the orthogonal projection on $L^\perp$ to build a connection ∇ on $L^\perp$. In more detail, we consider:

- on E: the canonical frame

$$\mathfrak{e}_0 = (1, 0, \ldots, 0), \quad \mathfrak{e}_1 = (0, 1, \ldots, 0), \quad \mathfrak{e}_n = (0, 0, \ldots, 1).$$

  and the connection ∇^{flat} defined by $\nabla^{\text{flat}}(\mathfrak{e}_j) = d_{\nabla^{\text{flat}}}(\mathfrak{e}_j) = 0$.
- for each $l \in \mathbb{CP}^n$, the orthogonal projection onto $l^\perp$ with respect to h,

$$\text{pr}_{l^\perp} : \mathbb{C}^{n+1} \to l^\perp.$$

  Hence $\text{pr}_{l^\perp}(z)$ is the unique point in $l^\perp$ with the property that $z - \text{pr}_{l^\perp}(z) \in l$. This gives rise to a morphism of vector bundles

$$\mathrm{pr}_{L^\perp} : E \to L^\perp.$$

All together, the connection ∇ on $L^\perp$ is

$$\nabla : \mathfrak{X}(\mathbb{CP}^n) \times \Gamma(L^\perp) \to \Gamma(L^\perp), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}_{L^\perp}\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in\Gamma(E)}\right),$$

or, if you prefer the dual point of view,

$$d_\nabla : \Gamma(L^\perp) \to \Omega^1(\mathbb{CP}^n, L^\perp), \quad d_\nabla(s) = \mathrm{pr}_{L^\perp}(d_{\nabla^{\mathrm{flat}}}(s)).$$

We also consider the standard charts (U_j, χ_j) of $\mathbb{CP}^n$, $0 \le j \le n$. For instance,

$$U_0 := \{[1 : z_1 : \ldots : z_n] : z_1, \ldots, z_n \in \mathbb{C}^n\}, \quad \chi_0 : U_0 \to \mathbb{C}^n,$$
$$U_0 \ni [1 : z_1 : \ldots : z_n] \overset{\chi_0}{\longmapsto} (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

(a) show that

$$\mathfrak{f}_j := -\overline{z}_j \mathfrak{e}_0 + \mathfrak{e}_j, \quad \text{for } j \in \{1, \ldots, n\}$$

form a local frame of $L^\perp$ over U_0.
(b) describe the similar frame $\mathfrak{f}'$ of $L^\perp$ over U_1 and compute the transition functions from $\mathfrak{f}$ to $\mathfrak{f}'$.
(c) Show that, in the chart (U_0, χ_0) (hence at points $[1 : z_1 : \ldots : z_n]$),

$$\mathrm{pr}_{L^\perp}(\mathfrak{e}_0) = -\frac{z_1}{1+|z|^2}\mathfrak{f}_1 - \ldots - \frac{z_n}{1+|z|^2}\mathfrak{f}_n.$$

(where $z = (z_1, \ldots, z_n)$).
(d) Compute the connection matrix ω of ∇ w.r.t. $\mathfrak{f}$, in the chart χ_0.
(e) Compute the curvature matrix of ∇ w.r.t. $\mathfrak{f}$, in the chart χ_0.
(f) Recall that the first Chern class of L is an non-zero element

$$a \in H^2(\mathbb{CP}^n).$$

Compute the Chern classes of $L^\perp$ in terms of a.
(g) Consider the orthonormal frame bundles $\mathrm{Fr}(L^\perp, h)$ associated to $L^\perp$ with the inner product induced from the h mentioned above. That is a principal $U(n)$-bundle over $\mathbb{CP}^n$. We pull it back to $S^{2n+1} \subset \mathbb{C}^n$ via the Hopf map

$$\pi : S^{2n+1} \to \mathbb{CP}^n, \quad \pi(z_0, z_1, \ldots, z_n) = [z_0 : z_1 : \ldots : z_n].$$

Show that $\pi^*\mathrm{Fr}(L^\perp, h)$ is diffeomorphic to $U(n+1)$.

Notes:

- the mark for this exam is the minimum between the number of points that you collect and 10.
- Please motivate your answers. Explain what you do. Specify what you use. Provide the details.

5.4 Exam Differential Geometry (mastermath), January 9, 2024

Exercise 5.15 (4x0.5= 2 pts). Let (M, g) be a Riemannian manifold and assume that $\mathcal{F} \subset TM$ is a distribution on M. We denote by $\mathcal{F}^\perp \subset TM$ the orthogonal of $\mathcal{F}$ with respect to g and by

$$\mathrm{pr}_{\mathcal{N}} : TM \to \mathcal{F}^\perp$$

the orthogonal projection with respect to g. Finally, define

$$\zeta : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}^\perp), \quad \zeta(V, W) = \mathrm{pr}_{\mathcal{N}}([V, W]).$$

(a) Show that ζ is $C^\infty(M)$-linear in V and W.
(b) Deduce that ζ may be interpreted as a section of a vector bundle over M. Which vector bundle is that?
(c) Show that $\mathcal{F}$ defines a foliation on M if and only if $\zeta = 0$.
(d) Prove that $\mathcal{F}^\perp$ is isomorphic to the normal bundle $TM/\mathcal{F}$.

Exercise 5.16 (1.5pts). Prove that S^3, together with the right action of S^1 given by

$$(x, y, z, t) \bullet (a, b) = (ax - by, ay + bx, az - bt, at + bz),$$

and together with the map

$$H : S^3 \longrightarrow S^2, \quad H(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - tx), 2(ty + xz)),$$

is a principal S^1-bundle over S^2.

Exercise 5.17 (2x1= 2 pts). By a flag-structure on an r-dimensional vector space V we mean a sequence of vector subspaces

$$V_1 \subset V_2 \subset \ldots \subset V_{r-1} \subset V_r = V$$

where each V_k is k-dimensional.

(a) Find a group $G \subset GL_r$ such that flag-structures on V are in 1-1 correspondence with G-structures on V. What are the adapted frames?
(b) Go on and describe the resulting theory of flag-structures on manifolds, similar to the other examples we have discussed in the class.

Exercise 5.18 (7 pts). We consider the tautological complex line bundle L over $\mathbb{CP}^n$. From its very definition, L is a vector sub-bundle of the trivial vector bundle $E = \mathbb{CP}^n \times \mathbb{C}^{n+1}$. We also consider the orthogonal complement of L in E,

$$L^\perp := \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \perp l\},$$

where the orthogonality $\perp$ is with respect to the standard hermitian product on $\mathbb{C}^{n+1}$:

$$h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1} + \ldots + z_n\overline{z'_n}.$$

Similar to what we did in the class and homeworks, but now working with $L^\perp$ instead of L, we use the standard (flat) connection ∇^{flat} on the trivial bundle E followed by the orthogonal projection on $L^\perp$ to build a connection ∇ on $L^\perp$. In more detail, we consider:

- on E: the canonical frame

$$\mathfrak{e}_0 = (1, 0, \ldots, 0), \quad \mathfrak{e}_1 = (0, 1, \ldots, 0), \quad \mathfrak{e}_n = (0, 0, \ldots, 1).$$

and the connection ∇^{flat} defined by $\nabla^{\text{flat}}(\mathfrak{e}_j) = d_{\nabla^{\text{flat}}}(\mathfrak{e}_j) = 0$.

- for each $l \in \mathbb{CP}^n$, the orthogonal projection onto $l^\perp$ with respect to h,

$$\text{pr}_{l^\perp} : \mathbb{C}^{n+1} \to l^\perp.$$

Hence $\text{pr}_{l^\perp}(z)$ is the unique point in $l^\perp$ with the property that $z - \text{pr}_{l^\perp}(z) \in l$. This gives rise to a morphism of vector bundles

$$\text{pr}_{L^\perp} : E \to L^\perp.$$

All together, the connection ∇ on $L^\perp$ is

$$\nabla : \mathfrak{X}(\mathbb{CP}^n) \times \Gamma(L^\perp) \to \Gamma(L^\perp), \quad (X, s) \mapsto \nabla_X(s) := \text{pr}_{L^\perp}\left(\underbrace{\nabla^{\text{flat}}_X(s)}_{\in \Gamma(E)}\right),$$

or, if you prefer the dual point of view,

$$d_\nabla : \Gamma(L^\perp) \to \Omega^1(\mathbb{CP}^n, L^\perp), \quad d_\nabla(s) = \text{pr}_{L^\perp}(d_{\nabla^{\text{flat}}}(s)).$$

We also consider the standard charts (U_j, χ_j) of $\mathbb{CP}^n$, $0 \le j \le n$. For instance,

$$U_0 := \{[1 : z_1 : \ldots : z_n] : z_1, \ldots, z_n \in \mathbb{C}^n\}, \quad \chi_0 : U_0 \to \mathbb{C}^n,$$
$$U_0 \ni [1 : z_1 : \ldots : z_n] \overset{\chi_0}{\longmapsto} (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

(a) show that

$$\mathfrak{f}_j := -\overline{z}_j \mathfrak{e}_0 + \mathfrak{e}_j, \quad \text{for } j \in \{1, \ldots, n\}$$

form a local frame of $L^\perp$ over U_0.

(b) describe the similar frame $\mathfrak{f}'$ of $L^\perp$ over U_1 and compute the transition functions from $\mathfrak{f}$ to $\mathfrak{f}'$.

(c) Show that, in the chart (U_0, χ_0) (hence at points $[1 : z_1 : \ldots : z_n]$),

$$\text{pr}_{L^\perp}(\mathfrak{e}_0) = -\frac{z_1}{1 + |z|^2}\mathfrak{f}_1 - \ldots - \frac{z_n}{1 + |z|^2}\mathfrak{f}_n.$$

(where $z = (z_1, \ldots, z_n)$).

(d) Compute the connection matrix ω of ∇ w.r.t. $\mathfrak{f}$, in the chart χ_0.

(e) Compute the curvature matrix of ∇ w.r.t. $\mathfrak{f}$, in the chart χ_0.

(f) Recall that the first Chern class of L is an non-zero element

$$a \in H^2(\mathbb{CP}^n).$$

Compute the Chern classes of $L^\perp$ in terms of a.

(g) Consider the orthonormal frame bundles $\text{Fr}(L^\perp, h)$ associated to $L^\perp$ with the inner product induced from the h mentioned above. That is a principal $U(n)$-bundle over $\mathbb{CP}^n$. We pull it back to $S^{2n+1} \subset \mathbb{C}^n$ via the Hopf map

$$\pi : S^{2n+1} \to \mathbb{CP}^n, \quad \pi(z_0, z_1, \ldots, z_n) = [z_0 : z_1 : \ldots : z_n].$$

Show that $\pi^*\text{Fr}(L^\perp, h)$ is diffeomorphic to $U(n+1)$.

Notes:

- the mark for this exam is the minimum between the number of points that you collect and 10.
- Please motivate your answers. Explain what you do. Specify what you use. Provide the details.

5.5 Exam Differential Geometry, Jan 12 2022, 14.00–17.00

- please do not forget to clearly write your name on each sheet of paper you hand in.
- PLEASE PROVIDE ALL DETAILS!!!!!!!!!!!!!!!!
- each question except the first one is worth 0.5 points. The final mark is the minimum between 10 and the points that you gather. Please notice that the overall total is 12 pts, so you do not have to answer all questions to achieve the maximum mark.
- regarding the questions that you do not know how to solve, or for which you do not remember the necessary concepts: just skip, and you are allowed to use them in the questions that follow. For instance, you can skip (6) but use it to solve (7), or skip (8) but use it to solve (9) and or (10), or skip (18) but use it to solve (19).

Exercise 5.19 (1pt). Prove that a line bundle $L \to M$ is trivializable if and only if it admits a nowhere vanishing section.

Exercise 5.20. Let (E, ∇^E) be a complex vector bundle with connection over a base manifold M. Recall that by the Serre–Swan theorem we can identify the sections of E^* by smooth map $\phi : \Gamma(M, E) \to C^\infty(M)$ that are $C^\infty(M)$-linear:

$$\phi(fs) = f\phi(s), \qquad \text{for } f \in C^\infty(M),\ s \in \Gamma(M, E).$$

a) Show that the following formula defines a connection ∇^{E^*} on E^*

$$(\nabla^{E^*}_X \phi)(s) := X(\phi(s)) - \phi(\nabla^E_X(s)), \qquad X \in \mathfrak{X}(M).$$

b) Let e_i be a local frame of E and let ω be the resulting connection matrix of ∇^E (allowing us to represent ∇ as $d + \omega$). For E^* we use the dual local frame f^j defined by $f^j(e_i) = \delta^j_i$; show that the resulting connection matrix of ∇^{E^*} is $-\omega^{\mathrm{tr}}$.

c) How are the curvatures of ∇^E and ∇^{E^*} related? Use this to show that

$$c_1(E^*) = -c_1(E).$$

Exercise 5.21. In this exercise we will work over the complex projective space $\mathbb{CP}^1$ consisting of complex lines $l \subset \mathbb{C}^2$ (through the origin), parametrized by pairs $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$, where the line through (z_1, z_2) is denoted $[z_1 : z_2]$.

(a) Recall that the tautological line bundle over $\mathbb{CP}^1$ is the complex rank one vector sub-bundle of the trivial vector bundle $E := \mathbb{CP}^1 \times \mathbb{C}^2$,

$$L \subset \mathbb{CP}^1 \times \mathbb{C}^2 =: E$$

consisting of pairs

$$(l, \xi) = ([z_1 : z_2], (\xi_1, \xi_2)) \in \mathbb{CP}^1 \times \mathbb{C}^2$$

with the property that $\xi \in l$ or, equivalently, $\xi_1 = z \cdot z_1$, $\xi_2 = z \cdot z_2$ for some $z \in \mathbb{C}$. We denote by $\{e_1, e_2\}$ the canonical frame of the trivial vector bundle E ($e_1(l) = (l, (1, 0))$ etc), so that an section s of L can be written as

$$s(l) = f_1(l)e_1 + f_2(l)e_2 \qquad (*)$$

with $f_1, f_2 : \mathbb{CP}^1 \to \mathbb{C}$ smooth. Show that the condition on f_1 and f_2 that ensures that s given by $(*)$ is a section of L is equivalent to the fact that

$$f_1([z_1 : z_2]) = z_1 \cdot g(z_1, z_2), \quad f_2([z_1 : z_2]) = z_2 \cdot g(z_1, z_2) \qquad (*')$$

for some $g \in C^\infty(S^3, \mathbb{C})^{S^1}$, i.e., for some smooth function $g : S^3 \to \mathbb{C}$ satisfying

$$g(z_1\lambda, z_2\lambda) = \lambda^{-1} g(z_1, z_2) \quad \forall\ (z_1, z_2) \in S^3, \lambda \in S^1.$$

(b) Show that, over the domain U of the standard chart of $\mathbb{CP}^1$

$$U = \{[z_1 : z_2] \in \mathbb{CP}^1 : z_0 \neq 0\} = \{[1 : z] : z \in \mathbb{C}\} \xrightarrow{\chi_U} \mathbb{C}, \quad [1 : z] \longrightarrow z,$$

the local section of L given by

$$e_U([1 : z]) = e_1 + ze_2$$

can be seen as a local frame of L over U and write down explicitly the corresponding trivialization of $L|_U \cong U \times \mathbb{C}$.

(c) Consider the similar local section e_V defined over $V = \{[z : 1] : z \in \mathbb{C}\}$. Compute the resulting transition function $g_{UV} : U \cap V \to GL(\mathbb{C})$.

(d) While $\Gamma(L) \subset \Gamma(E)$, there is also a "projection map"

$$\mathrm{pr} : \Gamma(E) \to \Gamma(L), \quad \mathrm{pr}(s)(l) := \mathrm{pr}_l(s(l))$$

where, for any line $l \in \mathbb{CP}^1$, $\mathrm{pr}_l : \mathbb{C}^2 \to l$ is the orthogonal projection w.r.t. the usual hermitian product h on $\mathbb{C}^2$ ($h(z, z') = z_1\overline{z_1'} + z_2\overline{z_2'}$). Show that:

$$\mathrm{pr}_l(z) = \frac{h(z, \xi)}{h(\xi, \xi)}\xi, \quad \text{with } \xi \in l \text{ a/any non-zero vector.}$$

(e) The trivial vector bundle E carries the canonical flat connection ∇^{flat},

$$\nabla^{\mathrm{flat}} : \mathfrak{X}(\mathbb{CP}^1) \times \Gamma(E) \to \Gamma(E);$$

combined with the inclusion $\Gamma(L) \subset \Gamma(E)$ and pr, it induces a connection ∇ on L:

$$\nabla : \mathfrak{X}(\mathbb{CP}^1) \times \Gamma(L) \to \Gamma(L), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(E)}\right)$$

Show that, over the domain U above, the connection matrix of L w.r.t. e_U is given by

$$\frac{\overline{z}}{1 + |z|^2} dz \in \Omega^1(U)$$

(or, more precisely, the pull-back of this formula via $\chi_U : U \to \mathbb{C}$).

(f) According to the general theory, for any path $u : I \to L$ sitting above some path $\gamma : I \to \mathbb{CP}^1$ (hence $u(t) \in L_{\gamma(t)}$), one can talk about the derivative $\frac{\nabla u}{dt}$, which is again a path in L above γ. Assume now that

$$\gamma(t) = [z_1(t) : z_2(t)] \in \mathbb{CP}^1, \quad u(t) = a(t)e_1(\gamma(t)) + b(t)e_2(\gamma(t))$$

where, as above, $\{e_1, e_2\}$ is the canonical frame of E. Show that

$$\frac{\nabla u}{dt}(t) = \frac{\overline{z_1} \cdot a' + \overline{z_2} \cdot b'}{|z_1|^2 + |z_2|^2}(z_1e_1 + z_2e_2).$$

(g) For $\gamma : [0, \pi] \to \mathbb{CP}^1$, $\gamma(t) = [1 : e^{2it}]$, compute the resulting parallel transport

$$T^{0,t}_\gamma : L_{\gamma(0)} \to L_{\gamma(t)}.$$

(h) Compute the curvature matrix of ∇ w.r.t. e_U and show that the first Chern class of L,

$$c_1(L) \in H^2(\mathbb{CP}^1)$$

is non-zero (hint: by Stokes, it suffices to show that an integral in non-zero).

(i) Show that L does not admit any section that is nowhere vanishing.

(j) Show that L is not isomorphic to its (complex) dual.

(k) As above, we use the 3-sphere S^3 interpreted as a subspace of $\mathbb{C}^2$ and we endow it with the right action of $S^1 \subset \mathbb{C}$ on S^3 given by complex multiplication:

$$(z_1, z_2) \cdot \lambda = (z_1\lambda, z_2\lambda), \qquad (**)$$

as well as the Hopf map (which you may assume without proof is a submersion)

$$H : S^3 \to \mathbb{CP}^1, \quad H(z_1, z_2) = [z_1 : z_2].$$

Show that $H : S^3 \to \mathbb{CP}^1$ is a principal S^1-bundle.

(l) Show that S^3 is isomorphic to the orthonormal frame bundle of L w.r.t. the hermitian metric h used above.

(m) Show that the vector bundle obtained by attaching to the principal S^1-bundle $H : S^3 \to \mathbb{CP}^1$ the fiber $\mathbb{C}$ with respect to the representation $\rho : S^1 \to GL(\mathbb{C})$, $\rho(\lambda)(z) = \lambda z$, is isomorphic to the tautological line bundle L:

$$E(S^3, \mathbb{C}) \cong L.$$

(n) According to the general theory one obtains a bijection

$$\Gamma(L) \cong C^\infty(S^3, \mathbb{C})^{S^1}$$

between the sections of L and functions $g : S^3 \to \mathbb{C}$ which are S^1-equivariant. Explain that this isomorphism is the one provided by item 2.

(o) In what follows, we identify the Lie algebra T_1S^1 of S^1 with $\mathbb{R}$, by identifying the vector $\frac{d}{dt}\big|_{t=0} e^{it} \in T_1S^1$ with $1 \in \mathbb{R}$. According to the general theory, the action $(**)$ of S^1 on S^3 gives rise to an infinitesimal action

$$\rho_{\mathbb{R}} : \mathbb{R} \to \mathfrak{X}(S^3).$$

Show that $\rho_{\mathbb{R}}$ sends $1 \in \mathbb{R}$ to the vector field

$$V = -y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2},$$

where we use the real coordinates corresponding to $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

(p) Deduce that a connection 1-form on the principal S^1-bundle $h : S^3 \to \mathbb{CP}^1$ is the same thing as a 1-form $\omega \in \Omega^1(S^3)$ with the property that

$$\omega(V) = 1, \quad \mathcal{L}_V(\omega) = 0.$$

(you may want to recall here that $\mathcal{L}_V(\omega)$ is related to the pull-back of ω along the flow ϕ^t_V of V; what is the formula? what is the flow of V in this case?)

(q) Show that

$$\omega := i\,(z_1 d\overline{z_1} + z_2 d\overline{z_2}) = x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 \in \Omega^1(S^3, \mathbb{R})$$

defines a connection on the principal S^1-bundle $H : S^3 \to \mathbb{CP}^1$.

(r) According to the general theory one also obtains a bijection

$$\Omega^1(\mathbb{CP}^1, L) \cong \Omega^1(S^3, \mathbb{C})_{S^1-\text{basic}}.$$

Show that it can be described as follows: start with $\theta \in \Omega^1(\mathbb{CP}^1, L)$; write θ similarly to $(*)$, i.e., use the inclusion $\Omega^1(\mathbb{CP}^1, L) \subset \Omega^1(\mathbb{CP}^1, E) = \Omega^1(\mathbb{CP}^1, \mathbb{C}^2)$ to write

$$\theta = \theta_1 e_1 + \theta_2 e_2, \quad \text{with } \theta_1, \theta_2 \in \Omega^1(\mathbb{CP}^1, \mathbb{C});$$

then the (basic) form $\eta \in \Omega^1(S^3, \mathbb{C})$ that corresponds to θ is uniquely determined by the condition analoguous to $(*')$:

$$H^*(\theta_1) = z_1 \cdot \eta, \quad H^*(\theta_2) = z_2 \cdot \eta.$$

(s) The general theory tells us that the connection ω on S^3 gives rise to a connection ∇^ω on $E(S^3, \mathbb{C}) \cong L$ determined by the commutative diagram:

$$\begin{array}{ccc} \Gamma(L) & \xrightarrow{\quad\nabla^\omega\quad} & \Omega^1(\mathbb{CP}^1, L) \\ \sim\downarrow & & \downarrow\sim \\ C^\infty(\mathbb{CP}^1, \mathbb{C})^{S^1} & \xrightarrow[d+\rho_{\mathbb{R}}(\omega)]{} & \Omega^1(\mathbb{CP}^1, \mathbb{C})_{S^1-\text{basic}} \end{array}$$

where the last horizontal map is $f \mapsto df + \rho_{\mathbb{R}}(\omega)(f)$, and $\rho_R : \mathbb{R} \to gl(\mathbb{C})$ is the infinitesimal counterpart of the representation $\rho : S^1 \to GL(\mathbb{C})$. Show that $\nabla^\omega = \nabla$.

5.6 Exam Differential Geometry (mastermath) 2015/2016

Exercise 5.22. In the lecture notes, during the discussion on symplectic forms and intrinsic torsion (more precisely, in the proof of Lemma 4.51, with a bit of change in the notation) it is claimed that for any symplectic vector space (W, ω) the sequence

$$\mathrm{Hom}(W, gl(W,\omega)) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2 W, W) \xrightarrow{\partial_\omega} \Lambda^3 W^*$$

is exact in the middle, i.e. $\mathrm{Im}(\partial) = \mathrm{Ker}(\partial_\omega)$, where

$$\partial(\xi)(u,v) = \xi(u)(v) - \xi(v)(u),$$

$$\partial_\omega(\phi)(u,v,w) = \omega(\phi(u,v),w) + \omega(\phi(v,w),u) + \omega(\phi(w,u),v).$$

To show this, let $\phi \in \mathrm{Hom}(\Lambda^2 W, W)$ such that $\partial_\omega(\phi) = 0$. To find $\xi \in \mathrm{Hom}(W, gl(W,\omega))$ such that $\phi = \partial(\xi)$:

(a) Show that for any $a, b \in \mathbb{R}$ one can find a unique $\xi_{a,b} \in \mathrm{Hom}(V, gl(V))$ such that

$$\omega(\xi_{a,b}(u)(v), w) = a \cdot \omega(\phi(u,v),w) + b \cdot \omega(\phi(u,w),v) \quad \forall\ u,v,w \in W.$$

(b) For which a and b does one have

$$\phi(u,v) = \xi_{a,b}(u)(v) - \xi_{a,b}(v)(u)\ \forall\ u,v \in W?$$

(c) For which a and b does one have $\xi_{a,b} \in \mathrm{Hom}(W, gl(W,\omega))$?

Finally, conclude that ϕ is in the image of ∂. (this exercise is worth 1.5 points)

Exercise 5.23. Let V be an n-dimensional (real) vector space, $W \subset V$ a linear subspace and $\omega_W : W \times W \to \mathbb{R}$ a linear symplectic form on W. We consider

$$GL(V,W,\omega) = \{g \in GL(V) : g(W) \subset W, g|_W \in GL(W,\omega)\}.$$

Show that:

(a) $GL(V,W,\omega)$ is a Lie group.
(b) the Lie algebra of $GL(V,W,\omega)$, viewed as a subspace of $gl(V)$, is:

$$gl(V,W,\omega) = \{A \in gl(V) : A(W) \subset W, A|_W \in gl(W,\omega)\}.$$

(c) We would like to understand the torsion space corresponding to $gl(V,W,\omega)$, $\mathcal{T} := \mathcal{T}(gl(V,W,\omega))$; recall that it is defined as the cokernel of

$$\partial : \mathrm{Hom}(V, gl(V,W,\omega)) \to \mathrm{Hom}(\Lambda^2 V, V),\ \partial(\xi)(u,v) = \xi(u)(v) - \xi(v)(u). \quad (*)$$

Consider the subspace $\mathrm{Hom}_W(\Lambda^2 V, V)$ of $\mathrm{Hom}(\Lambda^2 V, V)$ consisting of bilinear skew-symmetric maps $\phi : V \times V \to V$ with the property that, when restricted to $W \times W$, they take values in W. Define

$$\partial_\omega : \mathrm{Hom}_W(\Lambda^2 V, V) \longrightarrow \Lambda^3 W^*,$$

$$\partial_\omega(\phi)(u,v,w) = \omega(\phi(u,v),w) + \omega(\phi(v,w),u) + \omega(\phi(w,u),v).$$

Show that, for $\phi \in \mathrm{Hom}(\Lambda^2 V, V)$, the corresponding element $[\phi] \in \mathcal{T}$ is zero (i.e. ϕ is in the image of $(*)$ if and only if

$$\phi \in \mathrm{Hom}_W(\Lambda^2 V, V) \text{ and } \partial_\omega(\phi) = 0.$$

d. Remarking that $\mathrm{Hom}_W(\Lambda^2 V, V)$ is the kernel of the map

$$p_W : \mathrm{Hom}(\Lambda^2 V, V) \longrightarrow \mathrm{Hom}(\Lambda^2 W, V/W),\ p_W(\phi)(w_1, w_2) = \phi(w_1, w_2) \bmod W,$$

deduce that one has a short exact sequence of vector spaces

$$0 \longrightarrow \Lambda^3 W^* \longrightarrow \mathcal{T} \xrightarrow{\overline{p}_W} \mathrm{Hom}(\Lambda^2 W, V/W) \longrightarrow 0.$$

(this exercise is worth 2 points, with 0.5 points per item)

Exercise 5.24. Assume that $\mathcal{F}$ is a vector sub-bundle of TM and we look at "k-differential forms along $\mathcal{F}$" (for any k) by which we mean sections of the vector bundle $\Gamma(\Lambda^k \mathcal{F}^*)$. These are related to the usual k-forms on M by the obvious restriction map

$$\Omega^k(M) \longrightarrow \Gamma(\Lambda^k \mathcal{F}^*), \quad \eta \mapsto \eta|_{\mathcal{F}}$$

which, pointwise, restricts a k-multilinear skew-symmetric form on $T_x M$ to one on $\mathcal{F}_x$. We would like talk about "DeRham differential along $\mathcal{F}$",

$$d_{\mathcal{F}} : \Gamma(\Lambda^k \mathcal{F}^*) \to \Gamma(\Lambda^{k+1} \mathcal{F}^*)$$

(defined for all k) such that it is compatible with the usual DeRham differential d in the sense that

$$d_{\mathcal{F}}(\eta|_{\mathcal{F}}) = d(\eta)|_{\mathcal{F}} \quad \forall\ \eta \in \Omega^k(M).$$

Note that, since the restriction map is surjective, $d_{\mathcal{F}}$ will be unique if it exists. Show that $d_{\mathcal{F}}$ exists if and only if $\mathcal{F}$ is involutive. (Hint: look first in low degrees. And do not forget about the Koszul formula).

Exercise 5.25. An almost symplectic foliation on M is a pair $(\mathcal{F}, \omega_{\mathcal{F}})$ where:

(a) $\mathcal{F} \subset TM$ is a vector sub-bundle of M.
(b) $\omega_{\mathcal{F}} \in \Gamma(\Lambda^2 \mathcal{F}^*)$ is pointwise a linear symplectic form i.e., at each $x \in M$, ω_x (a bi-linear form on $\mathcal{F}_x$) is non-degenerate.

We say that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation if:

(c) $\mathcal{F}$ is involutive (hence a foliation).
(d) $\omega_{\mathcal{F}}$ is closed along $\mathcal{F}$ i.e., using the $d_{\mathcal{F}}$ from the previous exercise, $d_{\mathcal{F}}(\omega_{\mathcal{F}}) = 0$.

Given an almost symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$, we say that a connection ∇ on TM is compatible with $(\mathcal{F}, \omega_{\mathcal{F}})$ if:

$$\nabla_X(V) \in \Gamma(\mathcal{F}) \quad \forall\ V \in \Gamma(\mathcal{F}),$$

$$L_X(\omega_{\mathcal{F}}(V, W)) = \omega_{\mathcal{F}}(\nabla_X(V), W) + \omega_{\mathcal{F}}(V, \nabla_X(W)) \quad \forall\ X \in \mathfrak{X}(M), V, W \in \Gamma(\mathcal{F}).$$

Prove that that the existence of such a connection, which is also torsion free, implies that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation. (this exercise is worth 1 point)

Exercise 5.26. Finally, we look at (almost) symplectic foliations from the point of view of G-structures. We fix n and p and we consider almost symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$ of rank p on n-dimensional manifolds M.

(a) Describe a Lie group $G \subset GL_n$ (depending on p) such that such rank p almost symplectic foliations correspond to G-structures on M. Explain the correspondence.
(b) Show that linear G-structures on an n-dimensional vector space V correspond to pairs (W, ω) where $W \subset V$ is a p-dimensional vector subspace and ω is a linear symplectic form on W.
(c) Show that a connection ∇ on TM is compatible with $(\mathcal{F}, \omega_{\mathcal{F}})$ in the sense of the previous exercise if and only if it is compatible with the corresponding G-structure.
(d) Explain how the previous exercise is a consequence of the general properties of G-structures.
(e) Show that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation if and only if the intrinsic torsion of the corresponding G-structure vanishes.
(f) Show that the vanishing of the intrinsic torsion implies integrability. Moreover, explain the meaning of integrability in this case (what is the local model?).

(this exercise is worth 5 points: 1.5 points for each of the items c. and e., and 0.5 points for each of the other items)

Remark: The points assigned to the various items do not reflect the difficulty of the items but their importance for evaluating you knowledge (for instance, the very last item is by far the most difficult one, but it is worth only 0.5 points).

5.7 Take-home exam for the master course "Differential Geometry" - January 4, 2014, to be handed in by February 3, 2014-

Exercise 1 (about the intrinsic torsion and foliations)

The goal of this exercise is to compute the intrinsic torsion of p-dimensional distributions viewed as G-structures. See sections 1.5 and 2.5 of the lecture notes for more details on distributions and their integrated counterparts foliations. It is strongly recommended to read and understand the examples of section 5.7 as a preparation for this exercise.

The linear picture: let V be an n-dimensional vector space and $W \subset V$ a p-dimensional vector subspace. Define the Lie group $GL(V,W) = \{g \in GL(V) \mid g(W) \subset W\}$. Its Lie algebra is given by:

$$gl(V,W) = \{A \in gl(V) \mid A(W) \subset W\}$$

A linear G-structure corresponding to a choice of a subspace W is called a p-direction. The standard model of p-directions is $(V,W) = (\mathbb{R}^n, \mathbb{R}^p)$ with symmetry group $GL(\mathbb{R}^n, \mathbb{R}^p)$. Thus, a p-direction is a linear $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure.

The global picture: a $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure $\mathcal{F}$ on an n-dimensional manifold M is called a p-dimensional distribution. Fix such a structure $\mathcal{F}$. Note that we will denote by $\mathcal{F}$ both the $GL(\mathbb{R}^n, \mathbb{R}^p)$-structure (choice of frames) and the distribution (choice of subspaces of TM).

(a) Let $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a connection on M compatible with $\mathcal{F}$. Give a global description of the compatibility condition on ∇ in terms of $\Gamma(\mathcal{F})$ (i.e. ∇ is compatible with $\mathcal{F}$ if and only if ...). Recall that $\Gamma(\mathcal{F})$ is the set of vector fields tangent to the distribution $\mathcal{F}$.

(b) Recall that we have the map $\partial : \mathrm{Hom}(V, gl(V,W)) \to \mathrm{Hom}(\Lambda^2 V, V)$ and the associated torsion space $\mathcal{T}(gl(V,W))$. Prove the following linear isomorphism:

$$\mathcal{T}(gl(V,W)) \cong \mathrm{Hom}(\Lambda^2 W, V/W)$$

(c) Choose a connection ∇ compatible with $\mathcal{F}$ and compute the intrinsic torsion $\bar{T}$ of $\mathcal{F}$ (hint: use the equivariance of $\bar{T}$ together with your findings from question 2 to view $\bar{T}$ as a section of $\mathrm{Hom}(\Lambda^2 \mathcal{F}, TM/\mathcal{F})$, i.e. the vector bundle whose fiber at $x \in M$ is $\mathrm{Hom}(\Lambda^2 \mathcal{F}_x, T_x M/\mathcal{F}_x)$).

1. Prove that the intrinsic torsion vanishes if and only if $\mathcal{F}$ is involutive (hence, by the theorem of Frobenius, if and only if $\mathcal{F}$ is integrable).

Exercise 2 (about the Hopf fibration)

The next exercise discusses various descriptions and aspects of the Hopf fibration

$$\pi : S^3 \longrightarrow S^2, \quad \pi(x,y,z,t) = (x^2 + y^2 - z^2 - t^2, 2(yz - tx), 2(ty + xz)),$$

which is a principal S^1-bundle, with the S^1-action on S^3:

$$(x,y,z,t) \cdot (a,b) = (ax - by, ay + bx, az - bt, at + bz).$$

In complex coordinates, writing

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\},$$

the projection becomes

$$\pi(z_0, z_1) = (|z_0|^2 - |z_1|^2, i\bar{z}_0 z_1)$$

and the action is given by

$$(z_0, z_1) \cdot \lambda := (z_0\lambda, z_1\lambda). \tag{5.7.1}$$

We first fix some notations regarding the spaces we will be using.

- the space of quaternions:
$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$
where we recall that $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$ and, for
$$u = x + iy + jz + kt \in \mathbb{H}$$
one defines
$$u^* = x + iy + jz + kt,\ |x| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2}.$$
Note that S^3 can be interpreted now as the space of quaternionic numbers of norm 1; as a consequence, using the multiplication of quaternions, S^3 becomes a group (incidentally, note that this is isomorphic to $SU(2)$- but this will not be used below).
- the groups
$$SO(n) = \{A \in GL_n(\mathbb{R} : A\,{}^tA = \mathrm{Id},\ \det(A) = 1\}.$$
These groups will be used for $n \in \{2, 3\}$, when we use the identification and inclusion
$$S^1 \tilde{\to} SO(2) \hookrightarrow SO(3),$$
$$\cos(\alpha) + i\sin(\alpha) \tilde{\to} \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{5.7.2}$$
In particular, consider the right action of S^1 on $SO(3)$ coming from this inclusion and the multiuplication of matrices:
$$A \cdot \lambda = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{for } \lambda = \cos(\alpha) + i\sin(\alpha) \in S^1 \tag{5.7.3}$$
For $A \in SO(3)$ we denote by $A_i \in \mathbb{R}^3$ the vector corresponding to the i^{th} row of A, $i \in \{1, 2, 3\}$. Note that the condition $A\,{}^tA = \mathrm{Id}$ can be written as
$$||A_1|| = ||A_2|| = ||A_3|| = 1, \langle A_1, A_2\rangle = \langle A_2, A_3\rangle = \langle A_1, A_3\rangle = 0$$
where $||\cdot||$ and $\langle\cdot,\cdot\rangle$ are the standard Euclidean norm and inner porduct on $\mathbb{R}^3$.
- the complex projective line $\mathbb{CP}^1$, consisting of all complex lines in $\mathbb{C}^2$ through the origin (i.e. 1-dimensional complex vector subspaces of $\mathbb{C}^2$). For $(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$, we denote by
$$[z_0 : z_1] := \mathbb{C} \cdot (z_0, z_1) \in \mathbb{CP}^1$$
the induced line, so that we can write
$$\mathbb{CP}^1 = \{[z_0 : z_1] : (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}\},$$
$$[z_0 : z_1] = [z_0' : z_1'] \Longleftrightarrow \exists\ \lambda \in \mathbb{C} \text{ s.t. } z_0' = \lambda z_0, z_1' = \lambda z_1.$$
- Over $\mathbb{CP}^1$ we consider the tautological bundle E (of rank 2 as a real vector bundle) which is defined as the following sub-bundle of the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^2$:
$$E := \{(l, v) : l \in \mathbb{CP}^1, v \in l\} \subset \mathbb{CP}^1 \times \mathbb{C}^2.$$

TO DO:

(a) Check directly that $\pi : S^3 \longrightarrow S^2$ is a principal S^1-bundle.
(b) Remark that, for any $u \in S^3$,
$$iu, ju, ku \in \mathbb{H} \cong \mathbb{R}^4$$

are orthogonal to u, i.e. they define vectors tangent to S^3 at u. Use this to prove that S^3 is parallelizable. More precisely, describe explicitely the resulting vector fields X_1 (using i), X_2 (using j) and X_3 (using k) which define a parallelism of S^3. For instance,

$$X_1(x,y,z,t) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \in T_{x,y,z,t}S^3 \subset T_{x,y,z,t}\mathbb{R}^4.$$

(c) Here is a slightly different argument for the parallelizability of S^3, using the group structure that comes from multiplication of quaternions (see above). Fix $u_0 = (1,0,0,0) \in S^3$ and the tangent vectors

$$v_1 := \frac{\partial}{\partial y}(u_0), v_2 := \frac{\partial}{\partial z}(u_0), v_3 := \frac{\partial}{\partial t}(u_0) \in T_{u_0}S^3.$$

For $u = (x,y,z,t) \in S^3$, identified with the quaternion $x + iy + jz + kt$, consider the right multiplication by u:

$$R_u : S^3 \longrightarrow S^3, R_u(v) = vu,$$

its differential $(dR_u)_{u_0}$ at u_0 and

$$X_1(u) := (dR_u)_{u_0}(v_1) \in T_uS^3 \quad (uu_0 = u!)$$

and similarly X_2 and X_3. Show that these concide with the vector fields you found above.

(d) Show that the flow of X_1 induces the action (5.7.1) of S^1 on S^3.

(e) Let $\theta^1, \theta^2, \theta^3$ the associated coframe (i.e. defined by $\theta^i(X_j) = \delta^i_j$). Compute its structure equations.

(f) Show that θ^1 can be interpreted as a connection on the Hopf fibration.

(g) Show that the curvature of this connection, $K \in \Omega^2(S^2)$, is a non-zero constant (that you have to compute) times the area form σ of S^2,

$$\sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

(be aware: the forms dx, etc make sense on the entire $\mathbb{R}^3$, but we only use their restrictions to S^3. And, while the forms dx, dy and dz are linearly independent on $\mathbb{R}^3$, their restrictions to S^2 do satisfy an extra-relation: $xdx + ydy + zdz = 0$ (why?)).

(h) Show that

$$\pi_1 : S^3 \longrightarrow \mathbb{CP}^1,\ \pi_1(z_0, z_1) = [z_0 : z_1]$$

is a principal S^1-bundle, where the action of S^1 on S^3 is given by (5.7.1). What is the analogue of this in higher dimensions?

(i) Show that there exists a diffeomorphism

$$\Psi : S^2 \xrightarrow{\sim} \mathbb{CP}^1 \tag{5.7.4}$$

uniquely determined by $\pi_1 = \Psi \circ \pi$ (i.e. identifying the Hopf fibration with the one from the previous item).

(j) Show that one has a principal S^1-bundle

$$\pi_0 : SO(3) \longrightarrow S^2,$$

where the (right) action of S^1 on $SO(3)$ is the one from (5.7.3). What is the analogue of this in higher dimensions?

(k) Show that the vector bundle $SO(3)[\mathbb{R}^2, \rho]$ associated to this principal S^1-bundle and to by attaching the fiber $\mathbb{R}^2$ viewed as a representation of S^1 with

$$\rho : S^1 \xrightarrow{\sim} SO(2) \subset GL_2(\mathbb{R}) = GL(\mathbb{R}^2)$$

(see also (5.7.2)) is isomorphic to the tangent bundle of S^2.

(l) Embedd $SO(3)$ inside $\mathrm{Fr}(S^2)$ and interpret it as an S^1-structure on S^2.

(m) Identifying S^3 with the space of quaternionic numbers of norm 1,

$$S^3 \tilde{\to} \{u \in \mathbb{H} : |u| = 1\},\ (x, y, z, t) \mapsto x + iy + jz + kt$$

and $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \tilde{\to} \{v \in \mathbb{H} : v + v^* = 0\},\ (a, b, c) \mapsto ai + bj + ck,$$

show that

$$A_u(v) := u^* v u$$

defines, for each $u \in S^3$, a linear map $A_u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which, as matrix, gives

$$A_u \in SO(3).$$

(n) Consider the induced map

$$\phi : S^3 \longrightarrow SO(3),\ u \mapsto A_u.$$

Show that the projection of the Hopf fibration (see the begining) is precisely

$$\pi = \pi_0 \circ \phi.$$

(incidentally note that $\phi(u) = \phi(v)$ happens iff $v = u$ or $v = -u$- and this shows that $SO(3)$ can be identified with the real projective space $\mathbb{RP}^3$.)

(o) Show that

$$\phi(u\lambda) = \phi(u) \cdot \lambda^2 \quad \forall\ u \in S^3,\ \lambda \in S^1,$$

where $u \cdot \lambda$ uses to the action of S^1 on S^3 given by (5.7.1) and $\phi(u) \cdot \lambda^2$ uses the action of S^1 on $SO(3)$ given by (5.7.3).

(p) Consider the vector bundle $E_1 = S^3[\mathbb{R}^2, \rho]$ associated to the Hopf fibration by attaching the fiber $\mathbb{R}^2$ viewed as a representation of S^1 via ρ (see item 9) and, similarly, $E_2 = S^3[\mathbb{R}^2, \rho^2]$ defined using the square ρ^2 of the representation ρ:

$$\rho^2 : S^1 \longrightarrow GL_2(\mathbb{R}), \rho^2(\lambda) = \rho(\lambda)^2 = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix}$$

for $\lambda = \cos(\alpha) + i \sin(\alpha) \in S^1$. Show that E_2 is isomorphic to the tangent bundle of S^2 and E_1 is isomorphic to $\Psi^*(E)$ (where E is the tautological line bundle- see the begining, and for Ψ see (5.7.4)).

Consider now the tautological complex line bundle L over $\mathbb{CP}^1$. From its very definition, L is a vector sub-bundle of the trivial vector bundle $E = \mathbb{CP}^1 \times \mathbb{C}^2$. Hence, as discussed in the class, we obtain a connection on L, namely:

$$\nabla : \mathfrak{X}(\mathbb{CP}^1) \times \Gamma(L) \to \Gamma(L), \quad (X, s) \mapsto \nabla_X(s) := \mathrm{pr}\left(\underbrace{\nabla^{\mathrm{flat}}_X(s)}_{\in \Gamma(E)}\right)$$

where we are making use of the following basic pieces:

- the canonical flat connection ∇^{flat} on the trivial bundle $E = \mathbb{CP}^1 \times \mathbb{C}^2$.
- the standard hermitian product on $\mathbb{C}^2$:

$$h : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad h(z, z') = z_0\overline{z'_0} + z_1\overline{z'_1}$$

- for each $l \in \mathbb{CP}^2$, the orthogonal projection onto l with respect to h,

$$\mathrm{pr}_l : \mathbb{C}^2 \to l.$$

  Hence $\mathrm{pr}_l(z)$ is the unique point in l with the property that $z - \mathrm{pr}_l(z)$ is perpendicular to l w.r.t. h; it can be described in explicit formulas by:

$$\mathrm{pr}_l(z) = \frac{h(z, \xi)}{h(\xi, \xi)}\xi, \quad \text{with } \xi \in l \text{ a/any non-zero vector.}$$

- the map induced at the level of sections,

$$\mathrm{pr} : \Gamma(E) \to \Gamma(L).$$

This defines ∇ or, if you prefer, the dual point of view,

$$d_\nabla : \Gamma(L) \to \Omega^1(\mathbb{CP}^1, L), \quad d_\nabla(s) = \mathrm{pr}(d_{\nabla^{\mathrm{flat}}}(s)).$$

First let's compute a local connection matrix of this connection. Since L is of (complex) rank one, a local frame over an open $U \subset \mathbb{CP}^n$ is just a nowhere vanishing local section of L defined over U. We choose here U to be

$$U_0 := \{[z_0 : z_1] : z_0 \neq 0\},$$

i.e. the domain of the standard chart

$$\chi_0 : U_0 \to \mathbb{C}, \quad [1 : z] \longleftrightarrow z.$$

Over U_0 we have the local section e_0 given by that sends $[1 : z]$ to $(1, z)$. We can now talk about the connection matrix of ∇ w.r.t. e, which will be a $\mathbb{C}$-valued 1-form on U_0, denoted

$$\omega \in \Omega^1(U_0, \mathbb{C})$$

and the curvature matrix of ∇ w.r.t. e (a $\mathbb{C}$-valued 2-form on U_0), denoted

$$k \in \Omega^1(U_0, \mathbb{C})$$

Exercise 5.27.

(a) Prove that ω and k equal to (the pull-back via χ_0 of)

$$\frac{\overline{z}dz}{1+|z|^2} \in \Omega^1(\mathbb{C},\mathbb{C}), \quad \frac{1}{(1+|z|^2)^2}\cdot dz\wedge d\overline{z} \in \Omega^2(\mathbb{C},\mathbb{C}),$$

respectively, where $z = x + i\cdot y$ stand for the complex coordinate functions on $\mathbb{C}$, $\overline{z} = x - i\cdot y$, $dz = dx + i\cdot dy$ (a $\mathbb{C}$-valued 1-form!).

b) Show that $c_1(L) \in H^2(\mathbb{CP}^1)$ is nonzero (hint: use Stokes).

c) Show that L is not trivializable and is not isomorphic to its conjugate $\overline{L}$.

OLDER AND ROUGH

Exercise 5.28.

1. Show that for any distribution $\mathcal{F} \subset TM$,
$$\Omega^\bullet_{\mathcal{F}-\mathrm{bas}}(M) := \{\omega \in \Omega^\bullet(M) : L_V(\omega) = 0, i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
(where $\bullet$ stands for arbitrary degrees of forms) are preserved by DeRham differential of M.
2. Show that for any distribution $\mathcal{F} \subset TM$,
$$\Omega^k_{\mathcal{F}-\mathrm{hor}}(M) := \{\omega \in \Omega^k(M) : i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
is isomorphic to the space of sections of $\Lambda^k \nu^*$, where $\nu = TM/\mathcal{F}$.
3. If $\mathcal{F}$ is involutive, show that for all $W \in \Gamma(\mathcal{F})$, the Lie derivatives of forms along W and the pull-back of forms w.r.t. the flow ϕ^t_W of W,
$$L_W : \Omega^\bullet(M) \to \Omega^\bullet(M), \quad \left(\phi^t_W\right)^* : \Omega^\bullet(M) \to \Omega^\bullet(M),$$
they both preserve $\Omega^\bullet_{\mathcal{F}-\mathrm{hor}}(M)$.
4. Show that for any distribution $\mathcal{F} \subset TM$,
$$\Omega^\bullet_{\mathcal{F}-\mathrm{bas}}(M) := \{\omega \in \Omega^\bullet(M) : L_V(\omega) = 0, i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
(where $\bullet$ stands for arbitrary degrees of forms) are preserved by DeRham differential of M.
5. Show that a distribution $\mathcal{F} \subset TM$ is involutive if and only if
$$\Omega^\bullet_{\mathcal{F}-\mathrm{hor}}(M) := \{\omega \in \Omega^\bullet(M) : i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
are preserved by all the Lie derivatives L_W with $W \in \Gamma(\mathcal{F})$. Let $\mathcal{F} \subset TM$ be a distributions and consider, for each degree $k \in \mathbb{N}$,
$$\Omega^k_{\mathcal{F}-\mathrm{hor}}(M) := \{\omega \in \Omega^\bullet(M) : i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
$$\Omega^k_{\mathcal{F}-\mathrm{bas}}(M) := \{\omega \in \Omega^\bullet(M) : L_V(\omega) = 0, i_V(\omega) = 0 \ \forall\ V \in \Gamma(\mathcal{F})\}$$
6. Describe a linear isomorphism between $\Omega^k_{\mathcal{F}-\mathrm{hor}}(M)$ and the space of sections of $\Lambda^k \nu^*$, where $\nu = TM/\mathcal{F}$.
7. show that, in general, $\Omega^\bullet_{\mathcal{F}-\mathrm{bas}}(M)$ is preserved by the DeRham differential for M;
8. show that $\mathcal{F}$ is involutive if and only if for all $\Omega^\bullet_{\mathcal{F}-\mathrm{bas}}(M)$ is preserved by the DeRham differential for M;
9. When $\mathcal{F} = \mathrm{Ker} d\pi$ where $\pi : M \to N$ is a submersion, the pull-back of forms via π,
$$\pi^* : \Omega^\bullet(N) \to \Omega^\bullet(M),$$
takes values in $\Omega^\bullet_{\mathcal{F}_\pi-\mathrm{bas}}(M)$.
10. when π has connected fibers, then π^* becomes a linear isomorphism
$$\pi^* : \Omega^\bullet(N) \to \Omega^\bullet_{\mathcal{F}_\pi-\mathrm{bas}}(M).$$

Let E be a vector bundle over a manifold M, endowed with a connection ∇. Let $x_0 \in M$ and consider the set of loops at x_0:

$$\Omega(M, x_0) := \Big\{\gamma : [0,1] \to M : \gamma(0) = \gamma(1) = x_0\Big\}.$$

For $\gamma \in \Omega(M, x_0)$, the induced parallel transport along γ from $t = 0$ to 1 is denoted

$$T_\gamma : E_{x_0} \to E_{x_0}.$$

Given a vector bundle E over M together with a connection ∇, the parallel transport w.r.t. ∇ gives rise to a map

$$T : \Omega(M, x_0) \to GL(E_{x_0}), \quad \gamma \mapsto T_\gamma,$$

where $T_\gamma : E_{x_0} \to E_{x_0}$ is the parallel transport along from 0 to 1 along the path $t \mapsto \gamma(e^{2\pi t i})$. If $\delta : [0,1] \to M$ such that $\delta(0) = \gamma(1)$, we define the concatenation of δ and γ: Assuming that $\delta \star \gamma$ is smooth prove that

$$T_{\delta\star\gamma} = T_\delta \circ T_\gamma.$$

Exercise 5.29. Let E be a vector bundle over M, endowed with a connection ∇ and, for $\gamma : [0,1] \to M$ we consider the induced parallel transport (w.r.t. ∇ from $t = 0$ to 1), denoted

$$T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}.$$

If $\delta : [0,1] \to M$ such that $\delta(0) = \gamma(1)$, we define the concatenation of δ and γ:

$$\delta \star \gamma : [0,1] \to M, \quad (\delta \star \gamma)(t) = \begin{cases} \gamma(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \delta(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Assuming that $\delta \star \gamma$ is smooth prove that

$$T_{\delta\star\gamma} = T_\delta \circ T_\gamma.$$

and let $x_0 \in M$. We consider the set of loops at x_0,

$$\Omega(M, x_0) := \Big\{\gamma : S^1 \to M : \gamma(1) = x_0\Big\},$$

and, for $\gamma \in \Omega(M, x_0)$, the parallel transport from $t = 0$ to 1 along the path $t \mapsto \gamma(e^{2\pi t i})$ is denoted

$$T_\gamma : E_{x_0} \to E_{x_0}.$$

For $\gamma_1, \gamma_2 \in \Omega(M, x_0)$, we define their concatenation $\gamma_1 \star \gamma_2$ by Given a vector bundle E over M together with a connection ∇, the parallel transport w.r.t. ∇ gives rise to a map

$$T : \Omega(M, x_0) \to GL(E_{x_0}), \quad \gamma \mapsto\mapsto T_\gamma,$$

where $T_\gamma : E_{x_0} \to E_{x_0}$ is the parallel transport along from 0 to 1 along the path $t \mapsto \gamma(e^{2\pi t i})$.

For a manifold M and $x_0 \in M$ we consider the set of loops at x_0:

$$\Omega(M, x_0) := \Big\{\gamma : S^1 \to M : \gamma(1) = x_0\Big\}.$$

Given a vector bundle E over M together with a connection ∇, the parallel transport w.r.t. ∇ gives rise to a map

$$T : \Omega(M, x_0) \to GL(E_{x_0}), \quad \gamma \mapsto\mapsto T_\gamma,$$

where $T_\gamma : E_{x_0} \to E_{x_0}$ is the parallel transport along from 0 to 1 along the path $t \mapsto \gamma(e^{2\pi t i})$.

sending γ to the parallel transport along from 0 to 1 along the Given a connection ∇ on a line bundle $E \to M$, for any loop γ in M at a point $x_0 \in M$, i.e. for any $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x_0$, prove

that there exists a non-zero constant $c_\gamma \in \mathbb{R}$ such that

$$T^{0,1}_\gamma(v) = c_\gamma \cdot v \quad \forall\, v \in L_{x_0}.$$

Exercise 5.30. Given a connection ∇ on a line bundle $E \to M$, for any loop γ in M at a point $x_0 \in M$, i.e. for any $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x_0$, prove that there exists a non-zero constant $c_\gamma \in \mathbb{R}$ such that

$$T^{0,1}_\gamma(v) = c_\gamma \cdot v \quad \forall\, v \in L_{x_0}.$$

Exercise 5.31. In this exercise, for any manifold M and natural number r, we denote by $\mathbb{R}^r_M$ the trivial vector bundle of rank r over M, $\mathrm{pr}_M : M \times \mathbb{R}^r \to M$.

1. When $M = S^2$ show that there exists a morphism of vector bundles

$$f : S^2 \times \mathbb{R}^3 \to S^2 \times \mathbb{R}$$

   whose kernel is precisely TS^2.
2. When $M = \mathbb{CP}^1$ and L is the tautological line bundle, show that there exists another complex line bundle L' and a morphism of complex vector bundles

$$f : \mathbb{CP}^1 \times \mathbb{C}^2 \to L'$$

   whose kernel is precisely L.
3. Show that there exists no complex lin