FLOER HOMOLOGY: LECTURE 1

The aim of this series of seminars is to give an account of Floer homology. Rather than starting to introduce the general theory from the very beginning, we have chosen to adopt a different (and quite common) strategy: we will start by giving a detailed exposition on Morse homology.

Morse homology is yet another way of computing the homology of a manifold, but using tools from calculus on Banach manifolds. The techniques and steps needed to set up the Morse homology complex and to exhibit their independence on the choices of data (Morse function and generic metric), are exactly the same needed to define the Floer homology groups and show that they are canonically isomorphic to $H_*(M)$. Needless to say, the difficulties to be overcome in the Floer case are much more serious, and the theory only works (at least in its original formulation) under some strong hypothesis.

1. Introduction

All the manifolds will be closed and oriented and the maps and tensors smooth unless otherwise stated.

Let $(M, \omega)$ be a symplectic manifold and $\phi$ a Hamiltonian diffeomorphism. We expect -at least in favorable situations- to find an strictly positive integer $\kappa(M)$ depending only the topology of $M$ such that

$$\# \text{Fix}(\phi) \geq \kappa,$$

where $\kappa$ should be bigger for those $\phi$ whose fixed points are non-degenerate (i.e. when the graph of $\phi$ intersects the diagonal of $M \times M$ transversally).

Before explaining why this expectation is reasonable, let us recall what is known about equation 1 from topological fixed point theory.

Let $X$ be CW complex which is finite or retracts onto a finite one, and let $\phi: X \to X$ a continuous transformation. Then we define its Lefschetz number as follows:

$$L(\phi) = \sum_{i \in \mathbb{N}} (-1)^i \text{tr}(\phi_*: H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q}))$$

Notice that if $\phi \simeq \text{Id}$, then $L(\phi) = \chi(M)$.

**Theorem 1** (Lefschetz, 1926). In the previous situation if $L(\phi) \neq 0$, then

$$\# \text{Fix}(\phi) \geq 1$$

If $X$ is a compact manifold (possibly with boundary), the fixed points occur away from $\partial X$ and are all non-degenerate, then

$$\# \text{Fix}(\phi) \geq L(\phi)$$

Thus, if $\phi \simeq \text{Id}$

$$\# \text{Fix}(\phi) \geq \chi(M)$$

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Of course, we expect some thing much better than $\kappa = \chi(M)$ in the non-degenerate case (and $\kappa = 1$ in general).

In 1913 Poincaré stated a result along the lines of equation 1:

Let $A(r, r') = \{(x, y) \in \mathbb{R}^2 | r^2 \leq x^2 + y^2 \leq r'^2\}$, and let $\omega_0 = dx \wedge dy$ be the canonical symplectic (area) form of $\mathbb{R}^2$. The restriction of $\omega_0$ induces a symplectic structure on the annulus.

Let $\alpha$ (resp. $\beta$) be the outer (resp. inner) boundary component of the annulus. If $\phi \in \text{Diff}(M)$ with $\phi(\alpha) \subset \alpha$, then the restriction $\phi|_\alpha$ lifts to a map $f : \mathbb{R} \to \mathbb{R}$ than can be written $f(r) = r + h_\alpha(r)$, and similarly for $\phi|_\beta$.

**Theorem 2** (Poincaré-Birkhoff, 1925). With the above notation, let $\phi$ be a symplectic transformation of $(A(r, r'), \omega_0)$, sending $\alpha$ into $\alpha$ and $\beta$ into $\beta$, so that they rotate in opposite directions (more precisely $h_\alpha h_\beta < 0$). Then

$$\# \text{Fix}(\phi) \geq 2$$

**Remark 1.** The result remains valid for any symplectic form (it follows from the total area being the only symplectic invariant).

**Remark 2.** $A(r, r')$ retracts into the circle, and since $\phi$ is orientation preserving it has degree one. In other words, it is homotopic to the identity. Therefore $\tau(\phi) = \chi(A(r, r')) = 0$. So topological fixed point theory gives no information about $\# \text{Fix}(\phi)$.

**Remark 3.** The requirement on the behavior of the boundary is necessary to exclude (restrictions of) rotations, which have no fixed points. Similarly, there are examples of diffeomorphism of the annulus matching the boundary conditions of theorem 2, and without fixed points.

**Remark 4.** Observe that $2 = \beta_0(S^1) + \beta_1(S^1) = \min_{f \in C^\infty(S^1), f \text{ Morse}} \{\# \text{Crit}(f)\}$.

In the sixties Arnold made the following construction: out of two copies of $A(r, r')$ one gets $(\mathbb{T}^2, \omega)$, the torus with a symplectic structure. Assume we have two area preserving transformations as in theorem 2 and such that are compatible with the gluing, and define

$$\phi : (\mathbb{T}^2, \omega) \to (\mathbb{T}^2, \omega)$$

By Poincaré-Birkhoff, $\# \text{Fix}(\phi) \geq 4$, where

$$4 = \beta_0(\mathbb{T}^2) + \beta_1(\mathbb{T}^2) + \beta_2(\mathbb{T}^2) = \min_{f \in C^\infty(\mathbb{T}^2), f \text{ Morse}} \{\# \text{Crit}(f)\}$$

A diffeomorphism as the one described satisfies the following properties:

1. $\phi^* \omega = \omega$.
2. $\phi \simeq \text{Id}$.
3. $\phi$ preserves the center of mass, meaning that when we go to the universal cover $\mathbb{R}^2$ and write the induced periodic diffeomorphism as $\text{Id} + h$, then $\int_{\mathbb{R}^2} h = 0$.

**Definition 1.** Let $(M, \omega)$ be a symplectic manifold. A symplectomorphism $\phi$ is called Hamiltonian if $H \in C^\infty([0, 1] \times M)$ exists such that $\phi$ is the time 1 flow of $X_{H_t}$, with $X_{H_t}$ the Hamiltonian vector field of $H_t$ defined by the equation

$$i_{X_{H_t}} \omega = dH_t$$

**Remark 5.** If a symplectomorphism is Hamiltonian, the function $H$ can be chosen to belong to $C^\infty(S^1 \times M)$, $S^1 = [0, 1]/0 \sim 1$.

In the case of the torus it can be seen [1] that properties 1, 2, and 3 above are equivalent to $\phi$ being Hamiltonian.
Arnold’s conjecture 1. Let $(M, \omega)$ be a symplectic manifold, and $\phi$ a Hamiltonian diffeomorphism. Then
\[
\#\text{Fix}(\phi) \geq \min_{f \in C^\infty(M)} \{\#\text{Crit}(f)\}
\]
If all fixed points are non degenerate then
\[
\#\text{Fix}(\phi) \geq \min_{f \in C^\infty(M), f \text{ Morse}} \{\#\text{Crit}(f)\}
\]

There is a weak form of the conjectures in which the right hand side of 3 (resp. 2) is substituted by $\sum_i \beta_i(M; A)$ (resp. $\text{CL}(M)$, where $\text{CL}(M)$ is the cup length), where $A$ is any ring.

In 1973 Weinstein [11] gave a proof of Arnold’s conjecture for Hamiltonian symplectomorphisms $C^1$-close to the identity. His argument is as follows:

Consider $M \times M$ with the symplectic structure $\Omega := pr_1^* \omega - pr_2^* \omega$.

A diffeomorphism $\phi$ is symplectic if and only if its graph $\text{gr}(\phi) \subset M \times M$ is lagrangian w.r.t. to $\Omega$. The diagonal $\Delta$ is the graph of the identity, and hence a lagrangian. A small tubular neighborhood of $\Delta$ can be symplectically identified with $(T^* \Delta, -d\lambda_{\text{can}}) = (T^* M, -d\lambda_{\text{can}})$, where $\lambda_{\text{can}}$ is the Liouville 1-form. Being $C^1$-close to the identity means that $\text{gr}(\phi)$ can be identified with $s_\phi: M \to T^* M$, i.e. with a 1-form. The graph of $\phi$ being lagrangian is equivalent to $ds_\phi = 0$. Since $\phi$ is Hamiltonian, we can find $f \in C^\infty(M)$ such that $s_\phi = df$. Arnold’s conjectures, both in the degenerate and non-degenerate case, follow from $\text{Fix}(\phi) = \text{Crit}(f)$, and the fact that the non-degeneracy conditions are exactly the same.

In the late seventies the conjectures were proven for surfaces by Eliashberg. An important breakthrough came through the work of Conley and Zehnder, 1983. In [1] the conjectures were established for the standard torus $(\mathbb{T}^{2n}, \omega_0)$.

If $\phi$ is a Hamiltonian diffeomorphism and we fix $H_t: M \to M$, $t \in S^1$, a 1-periodic family of Hamiltonians giving rise to $\phi$, then
\[
\{\text{Fix}(\phi)\} \xrightarrow{1:1} \{1\text{-periodic orbits of } X_t\},
\]
where $X_t$ denotes the Hamiltonian vector field of $H_t$.

Conley and Zehnder were able to detect the contractible 1-periodic orbits through a functional, whose definition extends to any symplectic manifold $(M, \omega)$ such that
\[
\omega|_{\pi_2(M) \subset H_2(M)} = 0
\]

Let $\mathcal{LM} = \{z \in C^\infty(S^1, M) \mid z \text{ contractible}\}$. By definition, $z \in \mathcal{LM}$ is a 1-periodic orbit for $X_t$ if and only if
\[
\dot{z}(t) = X_t(z(t)) \Leftrightarrow \omega(z, \cdot) = dH_t(z)
\]

Consider the functional
\[
\mathcal{A}: \mathcal{LM} \to \mathbb{R}
\]
\[
z \mapsto -\int_{D^2} u^* \omega - \int_0^1 H_t(z(t))dt,
\]
with $u: D^2 \to M$ is any map such that $u|_{\partial D^2} = z$, where $\partial D$ is oriented counterclockwise (we orient hypersurface by putting first the outward normal).

Equation 5 implies that $\mathcal{A}$ is well defined.

To compute the first variation, let $\zeta \in z^*TM$. Then
\[
d\mathcal{A}(\zeta) = \int_0^1 \omega(\dot{z}, \zeta) - \int_0^1 dH_t(z)(\zeta)
\]

The second summand follows from permuting derivative and integral. Regarding the first, notice that once we fix a disk $u$ for $z$, we can consider for each nearby perturbation for $\epsilon$ small and in the direction of $\zeta$, a disk bounding the loop constructed...
as the union of $u$ and of the annular strip given by exponential in the direction of $\zeta$ for time in $[0, \epsilon]$. Taking the derivative of the areas is the same as taking it for the area of the strips, which is just the first term in the r.h.s. of 8 (orientation of the disk has to be taken into account to assign the right sign).

Therefore

$$z \in LM \in \text{Crit}(A) \Leftrightarrow \omega(\dot{z}, \cdot) = dH_t(z) \Leftrightarrow z \text{ is 1-periodic},$$

where the last equivalence follows from equation 6.

When $(M, \omega) = (T^2, \omega)$, one can work with its universal cover and $A$ looks a bit nicer (see [2]). It is then possible to construct an (infinite dim.) dynamical system using a standard procedure:

1. Complete $LM$ to a suitable Hilbert manifold $H$ and extend the functional $A'$: $H \to \mathbb{R}$, so that $\text{Crit}(A') = \text{Crit}(A)$.
2. Use the metric to define $-\nabla A'$ and integrate this vector field to define a dynamical system.
3. Relate critical points of $A'$ to the topology of $H$ and this to the topology/geometry of $M$.

One example of the above construction is the energy functional to study geodesics in a compact riemannian manifold, where one does Morse theory in an infinite dimensional manifold. In the case of $A$ (and eventually with $A'$) there are difficulties to do Morse theory:

- $A$ is not bounded either from above or from below, so there is no initial cell to eventually start constructing $H$ by attaching handles. This can be checked by observing that the norm of the second summand in $A$ (equation 7) is bounded independently of $z$, and the first one can be made as close as desired to $\pm\infty$. Just fix any loop $z$ s.t. $\int_{\bar{D}^2} u^* z > 0$, and take $k$ a positive integer. Consider $kz$ and approximate it by an embedded loop $z'$. Then if the approximation is small compared with the integral, we can get $A(z') \geq k/2$, $\forall k \in \mathbb{N}$ large enough. To get opposite sign (in the first summand) we just reverse the orientation of $z'$.

- The Hessian of $A$ in the critical points have positive and negative definite subspaces which are infinite dimensional. This prevents the (straightforward) assignment of an index to the critical points (see [8]).

At any rate, Conley and Zehnder for $(T^2, \omega_0)$ where able to relate the critical points of the corresponding dynamical system with the topology of the torus.

For a general symplectic manifold $(M, \omega)$ for which 5 holds one can introduce a metric $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$, with $J$ a compatible almost complex structure, and compute the “$L^2$-gradient”. If $\mu$ denotes the $L^2$-metric then by equation 8 we have:

$$\nabla_\mu A = J \dot{z} - J X_t = J \dot{z} - \nabla g X_t$$

This is not an honest gradient, just a section of a bundle over $LM$ with fibre over $z \in LM$ the Hilbert space $L^2(z^*TM)$; it is not clear how to extend $A$ to a space of “$L^2$-contractible loops”.

Still, we can think of a (in principle smooth) map $u: U \times S^1 \to M$, $U \subset \mathbb{R}$, being tangent to $-\nabla_\mu A$ as a curve $u: U \to LM$. The corresponding equation is:

When equation 5 fails to hold but the symplectic manifold is monotone ($\omega = \lambda c_1(M)$, $\lambda \in \mathbb{R}$), even though the functional is not defined one still has the 1-form given by 8 (or the functional defined in a suitable cover of $LM$). Thus it is possible to try to “do Morse theory” for this 1-form in the sense of Novikov. As a result Floer homology modules over a suitable Novikov ring can be defined (see for example [7]).
\[ \frac{\partial u}{\partial s}(t, s) = -J(u(t, s)) \left( \frac{\partial u}{\partial t}(t, s) \right) + \nabla g X_t(u(t, s)), \]

or equivalently
\[ \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} \right) - \nabla g X_t(u) = 0 \quad (10) \]

It is important to abandon the O.D.E. point of view and study solutions of the well understood P.D.E. in 10 in the appropriate Banach space of maps ([5]), thought of as “gradient lines”. In particular one can study the (bounded energy) “gradient lines” connecting critical points [3].

It is reasonable to expect the critical points and spaces of (bounded energy) gradient lines connecting them to encode some topological information of \( LM \) (and then in turn of \( M \)), because due to the work of Thom [10], Smale [9], Milnor [6] and Franks [4] (rediscovered by Witten [12]), if \( f : X \to \mathbb{R} \) is a Morse function defined in a closed oriented manifold, and \( g \) is a metric for which the stable and unstable manifolds of different critical points intersect transversely (\( g \) is generic\(^2\)), then one can compute \( H_*(X; \mathbb{Z}) \) with the cochain complex \((C_*(f, g), \partial)\) defined as follows:

\[ C_k(f, g) = \bigoplus_{y \in \text{Crit}(f), \mu(y) = k} \langle y \rangle \quad (11) \]

\[ \partial : C_k(f, g) \to C_{k-1}(f, g) \]

\[ x \mapsto \sum_{\mu(y) = k-1} n(x, y)y, \quad (12) \]

where \( \mu(y) \) denotes the index of the critical point \( y \) and \( n(x, y)y \) is the number of (negative) gradient lines connecting \( x \) and \( y \), counted with appropriate sign.

The classical proof of Franks [4] uses differential topology. Morse homology gives a different proof using Banach calculus techniques, which are the ones that can possibly be extended to the infinite dimensional setting of Floer homology.

1.1. Morse homology. Let \( f : X \to \mathbb{R} \) (\( X \) not necessarily compact and oriented) \( f \) Morse and coercive \( (f^{-1}([-\infty, a]) \) compact for all \( a \in \mathbb{R} \), and let \( g \) be a metric.

We need to analyze the spaces of gradient lines (integral curves of \( -\nabla f \), or just \( -\nabla f \) is there is no risk of confusion) connecting different critical points to conclude that the complex defined in 11 and 12 is a chain complex which computes \( H_*(X; \mathbb{Z}) \).

1.1.1. Gradient lines as zeros of a section of an appropriate Banach bundle. Gradient lines \( \gamma \) are solutions of

\[ \dot{\gamma} = -\nabla f(\gamma) \quad (13) \]

Let \( x, y \in \text{Crit}(f) \) and let \( \mathcal{B} \) be an appropriate Banach completion of the space

\[ \{ \gamma \in C^\infty(\mathbb{R}, X) | \lim_{t \to -\infty} \gamma(t) = x, \lim_{t \to +\infty} \gamma(t) = y \} \]

Let \( \mathcal{E} \to \mathcal{B} \) be a Banach bundle whose fiber over \( \gamma \in \mathcal{B} \) is an appropriate Banach linear space completing \( \gamma^*TX \). Consider the section

\(^2\)The aforementioned property of a metric relative to the intersection of stable and unstable manifolds is generic, in the sense that it holds for a residual set of (smooth) metrics, i.e. for a subset containing a countable intersection of open dense sets. Being the smooth metrics a Baire space, then this subset is dense. It is actually easy to see that this set is indeed open.
\[ F: \mathcal{B} \longrightarrow \mathcal{E} \]
\[ \gamma \longmapsto (\gamma, \dot{\gamma} + \nabla f(\gamma)), \quad (14) \]
which is seen to be smooth.

The space of (parametrized) gradient lines from \(x\) to \(y\) is
\[ \mathcal{M}(x, y) := F^{-1}(0) \quad (15) \]

1.1.2. Fredholm theory, index computations, transversality and regularity. The aim is proving that \(\mathcal{M}(x, y)\) is a manifold for a large subset of generic metrics, compute its dimension and show that any \(\gamma \in \mathcal{M}(x, y)\) is a smooth curve. One proceeds as follows:

1. Let \(\mathcal{G}\) be the Banach space of \(C^k\)-metrics in \(X\), for some \(k\). Consider \(\mathcal{E}_\mathcal{G}\) the pullback of \(\mathcal{F}\) by the projection \(\pi: \mathcal{B} \times \mathcal{G} \rightarrow \mathcal{B}\). Define the smooth section \(\mathcal{F}_\mathcal{G}: \mathcal{B} \times \mathcal{G} \longrightarrow \mathcal{E}_\mathcal{G}\)
\[ \mathcal{F}_\mathcal{G} (\gamma, g) \mapsto (\gamma, g, \dot{\gamma} + \nabla g f(\gamma)) \quad (16) \]
This section is checked to be transversal to the zero section. The implicit function theorem for Banach manifolds implies that \(\mathcal{F}^{-1}_\mathcal{G}(0) \subset \mathcal{B} \times \mathcal{G}\) is a Banach manifold, called the universal moduli space.

2. The restriction of the projection \(\pi: \mathcal{F}^{-1}_\mathcal{G}(0) \rightarrow \mathcal{G}\) is a Fredholm map of index \(\mu(x) - \mu(y)\), meaning that at each point its differential is a Fredholm map between linear Banach spaces of the aforementioned index.

The assertion about the linear map being Fredholm of the cited index is equivalent to the same assertion for the following linear map: for each fixed \(g\) and each \(\gamma \in \mathcal{M}(x, y) \subset \mathcal{B}\) consider
\[ DF_g \circ p: T\gamma \mathcal{B} \rightarrow E_{\gamma}, \]
where \(p: T_{\mathcal{F}_\mathcal{G}(\gamma)} \mathcal{B} \rightarrow E_{\gamma}\) is the linear projection parallel to the tangent space of the zero section.

3. Apply Sard-Smale’s theorem to conclude the existence of a generic subset of \(C^k\)-metrics for which \(\mathcal{F}_\mathcal{G}\) is transversal to the zero section. Show that the same happens for smooth metrics.

4. For a generic smooth metric using the properties of equation 13 conclude that any point of \(\mathcal{M}(x, y)\) is a smooth curve.

1.1.3. Compactness and gluing of broken trajectories. For generic metrics the manifolds \(\mathcal{M}(x, y)\) carry a free \(\mathbb{R}\)-action by translations in the parameter \(t\). Therefore they cannot be compact. Consider the reduced spaces of unparametrized solutions
\[ \widehat{\mathcal{M}}(x, y) := \mathcal{M}(x, y)/\mathbb{R} \]

In order to define a compactification of \(\widehat{\mathcal{M}}(x, y)\) it is necessary to understand how a sequence of points \(\gamma_n \in \widehat{\mathcal{M}}(x, y)\) may fail to have a convergent subsequence.

It can be shown that \(\widehat{\mathcal{M}}(x, y)\) can be compactified by adding broken trajectories connecting other critical points. More precisely, there exists a compactification \(\widehat{\mathcal{M}}(x, y)\) whose codimension \(r\) stratum is
\[ \bigcup_{w_1, \ldots, w_r \in \text{Crit}(f), x, w_1, \ldots, w_r, y \text{ distinct}} \widehat{\mathcal{M}}(x, w_1) \times \cdots \times \widehat{\mathcal{M}}(w_j, w_{j+1}) \times \cdots \times \widehat{\mathcal{M}}(w_r, y) \quad (17) \]
Moreover, it can be shown that $\mathcal{M}(x, y)$ is a manifold with boundary. This requires gluing analysis, which amounts to show how to approximate a broken trajectory in one of the aforementioned strata by trajectories in $\mathcal{M}(x, y)$, and in how many different ways this can be done.

1.1.4. Coherent orientations. It can be shown that the manifolds $\mathcal{M}(x, y)$ admit orientations which are coherent with their boundary structure.

With the previous three steps, one is able to define the chain complex of equations 11 and 12. The index computations giving the dimension of the spaces $\mathcal{M}(x, y)$ define a relative grading $\mu(x) - \mu(y)$ of the complex $C_*(f, g)$. The absolute grading given by the index of each critical point is a finite dimensional feature that cannot be expected to be generalized in all circumstances to the infinite dimensional setting of Floer homology [8].

The equality $\partial^2 = 0$ follows easily from the previous considerations: let $x, y \in \text{Crit}_*(f)$ such that $\mu(x) - \mu(y) = \dim \mathcal{M}(x, y) + 1 = 2$, then

$$\langle \partial^2(x), y \rangle = \sum_{z \in \text{Crit}_*(f), \mu(x) - \mu(z) = 1} n(x, z) n(z, y) y = 0,$$

because $n(x, z)n(z, y)$ are the points in $\partial \mathcal{M}(x, y)$ counted with sign.

1.1.5. Continuity method. One wants to show that all homology groups defined for pairs $(f, g)$, $g$ generic, are canonically isomorphic. This is also done through chain maps built counting solutions of equation 13, in which the metric and the Morse function may vary in 1-parameter families, giving rise to quasi-isomorphisms (chain maps which induce isomorphisms in homology). The tools are those of step 1.

Floer homology is set up using exactly the same steps, but those of course turn out to be harder and in some situations cannot be completely carried out (essentially due to problems with compactifications). It is worth mentioning that when the construction can be carried out, there is some specific generic data (which in this case is the almost complex structure $J$ and the Hamiltonians $H_t$) for which the Floer homology groups can be easily identified with the homology $H_*(M; \mathbb{Z})$ (or $\mathbb{Z}_2$ if orientations issues are neglected [3]), proving thus Arnold’s conjecture for those symplectic manifolds.

References