Cyclic cohomology and characteristic classes for foliations

(met een samenvatting in het Nederlands)

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Summary

This thesis deals with the cohomology theories and the theory of characteristic classes for leaf spaces of foliations, as well as with the interaction between the classical approach (of Grothendieck, Bott-Haefliger) and the non-commutative approach (of Connes-Moscovici) to these theories.

Leaf spaces provide a large class of examples of "singular spaces" to which standard theories do not apply directly. Grothendieck/ Bott and Haefliger overcome this problem by enlarging the category of spaces to those of etale groupoids, to which many of the classical constructions extend. Chapters 2 and 3 belong to this approach to leaf spaces. In Chapter 2 we introduce a homology theory which is in Poincare duality with Haefliger's cohomology, and we prove it has the expected properties (and these will be used in Chapter 4 when computing cyclic homology groups). In Chapter 3 we give a more geometrical (Cech-De Rham) model for Haefliger's cohomology, which allows us to geometrically construct characteristic classes for etale groupoids (hence leaf spaces) and to explain/extend Bott's formulas. These two chapters are joint work with Ieke Moerdijk.

Leaf spaces also provide a large class of examples in non-commutative geometry. From this point of view, they are modelled by their associated convolution algebras. In Chapter 4 we compute the cyclic homology of convolution algebras of etale groupoids, which is the relevant theory from the non-commutative point of view. Here we find the connection with Grothendieck-Haefliger's approach (and this is based on our homology theory of Chapter 1). Our computations extend previous computations of Brylinski, Burghelea, Connes, Karoubi, Nistor. Motivated also by the connection with the longitudinal index theory, the last sections of this chapter concentrate on the case of holonomy groupoids of foliations, and associated Chern characters.

In their approach to transversal index theorems for foliations, Connes and Moscovici have recently discovered a deep connection between the (geometrical) characteristic classes for foliations and the non-commutative ones arising, via the Chern character, in the cyclic cohomology groups computed in Chapter 4. This connection is based on a cyclic cohomology theory of a particular Hopf algebra of the geometric operators. In Chapter 5 we study this cohomology theory. First of all, we show that it applies to general Hopf algebras as an extension of the classical Lie algebra homology. Secondly, we make the connection with Cuntz-Quillen's approach to cyclic cohomology in terms of X-complexes. Also, inspired by the (rather classical) construction of characteristic classes for foliations in terms of the truncated Weil complex (recalled in the preliminaries), we describe a non-commutative version of the Weil complex. This turns out to be strongly related to Cuntz-Quillen's X-complex, and it is used to solve the problem of constructing characteristic maps associated to higher traces.

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Introduction

Geometrically, a foliation $\mathcal{F}$ on an n-dimensional manifold M is a partition into (immersed) submanifolds (called the leaves of $\mathcal{F}$), which locally looks like the partition of the Euclidean space by p-planes, $\mathbb{R}^n = \bigcup_x \mathbb{R}^p \times \{x\}$ (p is the dimension of $\mathcal{F}$, and x runs in $\mathbb{R}^{n-p}$). The space of leaves $M/\mathcal{F}$ is a standard example of a "non-commutative" or "singular" space, and a great deal of foliation theory is devoted to the study of transversal invariants and structures, which morally live on $M/\mathcal{F}$. "Morally" because $M/\mathcal{F}$, viewed as an ordinary topological space (with the quotient topology) is often very pathological. Nevertheless, forgetting its points and its topology, one can still give a good meaning to:

- the algebra of (say smooth) functions on $M/\mathcal{F}$,
- the category of sheaves on $M/\mathcal{F}$.

Hence one can extend some of the classical invariants like K-theory, sheaf cohomology, characteristic classes, i.e. precisely those which can be expressed in terms of the algebra of functions, or in terms of the category of sheaves. This is based on the right interpretation of $M/\mathcal{F}$ (as an étale groupoid rather than a space), and is the starting point for two well known approaches to leaf spaces that we will see throughout this thesis. Let us have a quick look at these approaches, as well as at their use in understanding leaf spaces (characteristic classes, and longitudinal/transversal index theorems for foliations). This will give you an impression of my interests during the last couple of years, will describe the general picture this thesis fits in, and will give me the opportunity to informally formulate some (motivating) questions.

0.0.1 Leaf spaces, transversal structures, and étale groupoids: The category *Étale* of étale groupoids is an enlargement of the category of topological spaces:

$$Top \subset \acute{E}tale \quad , \tag{0.1}$$

to which many of the classical constructions extend. An étale groupoid $\mathcal{G}$ consists not only of a space $\mathcal{G}^{(0)}$, but also of an extra space $\mathcal{G}^{(1)}$ of arrows between the points of $\mathcal{G}^{(0)}$, so that the map which assigns to an arrow its source is a local homeomorphism $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$, and the arrows satisfy certain properties (shortly summarized by asking $\mathcal{G}$ to be a groupoid, i.e. a category in which any arrow is invertible).

Étale groupoids are of great importance in understanding leaf spaces: with the extension (0.1) in mind, spaces like $M/\mathcal{F}$ can be viewed as (very well behaved) objects of *Étale* (rather than of *Top*). Moreover, the étale groupoids representing leaf spaces (known as *the reduced holonomy groupoids* of foliations), exhaust almost all étale groupoids. Let me indicate how they are naturally hidden behind the naive $M/\mathcal{F}$'s.

Choose T to be a manifold which is transversal to the foliation, and meets each leaf of $\mathcal{F}$ at least once (so that any point in $M/\mathcal{F}$ is represented by at least one point in T). Then the points in $M/\mathcal{F}$ are obtained by gluing together the points of T which lie in the same leaf. The local behavior of the gluing of two different points t_1 and t_2 of T is described by the way the transversal structure of the foliation (i.e. a neighborhood of t_1) is changing when walking from t_1 to t_2 along all possible paths. That is what is known as the holonomy of the foliation, and provides us with different arrows (*holonomy paths*) between the points of T. Hence the transversal T, together with these arrows, appears as an unraveling of the naive $M/\mathcal{F}$; they form the reduced holonomy groupoid $Hol_T(M, \mathcal{F})$ of $\mathcal{F}$, and represent $M/\mathcal{F}$ by an étale groupoid. Let us also mention that the choice of T does not affect the construction: if T' is another such transversal, then $Hol_T(M, \mathcal{F})$ and $Hol_{T'}(M, \mathcal{F})$ are isomorphic in $\acute{E}tale$ (i.e. are Morita equivalent étale groupoids).

Haefliger gives in this way a precise meaning to the notion of transversal structures/invariants of foliations, as those which are (or can be) defined in terms of the reduced holonomy groupoid. According to this, a basic example is the cohomology $H^*(M/\mathcal{F})$ of the leaf space, which is defined as the cohomology of the reduced holonomy groupoid (of course, one uses here two important properties of cohomology: it extends via (0.1) to étale groupoids, and is Morita invariant). Intuitively clear maps like the projection $\pi : M \longrightarrow M/\mathcal{F}$ are rigorously defined when working in the category of étale groupoids, and they induce maps in cohomology:

$$\pi^* : H^*(M/\mathcal{F}) \longrightarrow H^*(M) \ . \tag{0.2}$$

Also, (one way) to define transversal bundles is to require a (smooth) action of the holonomy paths on the fibers of the bundle (the guiding example is the normal bundle $\nu = TM/\mathcal{F}$ of the foliation). What about compactly supported cohomology, in Poincaré (or even Verdier) duality with $H^*(M/\mathcal{F})$, or about associating to transversal vector bundles characteristic classes living in $H^*(M/\mathcal{F})$ (rather than in $H^*(M)$)? We will see these are possible too. Notice also that there are well known properties of foliations which are naturally described without the use of the holonomy groupoid, and which turn out to be transversal. Discovering their transversal nature gives a better insight into the property itself, and often has interesting consequences. This happens for instance with the minimality of leaves [56], or even with the notion of transversal bundles.

0.0.2 Characteristic classes for foliations: Probably the shortest way to explain what a foliation is, is by specifying the bundle of vectors tangent to its leaves: it is an involutive subbundle $\mathcal{F}$ of the tangent bundle TM of M. While answering Haefliger's question of whether any subbundle of TM can be deformed into a foliation, Bott discovered his vanishing theorem, which describes an obstruction to the problem, and is the starting point in the development of characteristic classes for foliations. This theorem asserts that some of the classical invariants (namely the Pontrjagin classes) of the normal bundle $\nu = TM/\mathcal{F}$ of any foliation must vanish. Combining the remaining (non-trivial) classes, with the transgressions of the vanishing ones, new characteristic classes show up. The universal ones are organized in a group $H^*(\underline{WO}_q)$, and the construction is expressed by a characteristic map:

$$k_{\mathcal{F}}^{geom} : H^*(\underline{WO}_q) \longrightarrow H^*(M). \tag{0.3}$$

Here "geom" stands for the explicit geometric nature of the construction, in terms of curvatures of connections. Since this map is associated to the normal bundle of the foliation, which is the basic example of a transversal bundle, one expects $k_{\mathcal{F}}^{geom}$ to be a transversal invariant of the foliation, in the sense that there is a $H^*(M/\mathcal{F})$-valued version of this map (denoted by the same letter) from which one can recover the old (0.3) by composing with the pullback (0.2). One can prove this by using the relation between $k_{\mathcal{F}}^{geom}$ and a topological characteristic map (described below), but this is rather abstract. Can one give a more geometric model for $H^*(M/\mathcal{F})$ so that the $H^*(M/\mathcal{F})$-valued $k_{\mathcal{F}}^{geom}$ can be constructed explicitly? Can one see directly the vanishing theorem of Bott at this level? Can one extend these to general transversal vector bundles? We will see these are all possible.

Meanwhile, Haefliger gave a topological construction of characteristic classes, using classifying maps and an extension of Milnor's construction. He constructs an étale groupoid Γ^q (replacing the Lie group present in Milnor's construction), and a classifying space $B\Gamma^q$, so that a codimension q foliation $\mathcal{F}$ on M is classified by a map $f_{\mathcal{F}} : M \longrightarrow B\Gamma^q$. As in the case of classical principal bundles, the map $f_{\mathcal{F}}$ is unique up to homotopy, hence induces a unique map in cohomology. One gets in this way a topologically defined characteristic map:

$$k_{\mathcal{F}}^{top} := f_{\mathcal{F}}^* : H^*(B\Gamma^q) \longrightarrow H^*(M) \ . \tag{0.4}$$

It is not difficult to prove this map is a transversal invariant (in the same sense as for $k_{\mathcal{F}}^{geom}$), and we denote by the same letter its $H^*(M/\mathcal{F})$-valued version.

The two characteristic maps (0.3) and (0.4), as well as their $H^*(M/\mathcal{F})$-valued versions, are related by a universal characteristic map (conjecturally injective):

$$k : H^*(\underline{WO}_q) \longrightarrow H^*(B\Gamma^q) \ , \tag{0.5}$$

which fits into a commutative diagram:

$$\begin{tikzcd}
H^*(\underline{WO}_q) \arrow[dd, "k"'] \arrow[dr, "k_{\mathcal{F}}^{geom}"] & \\
& H^*(M/\mathcal{F}) \\
H^*(B\Gamma^q) \arrow[ur, "k_{\mathcal{F}}^{top}"'] &
\end{tikzcd}$$

The original construction of k is rather abstract (one actually proves existence [13]), but we will see that it can be defined explicitly (and, interpreting Γ^q as a "universal leaf space" for codimension q foliations, k coincides with the associated $k_{\mathcal{F}}^{geom}$).

0.0.3 Connes' non-commutative geometry and leaf spaces: Non-commutative geometry provides methods to study "non-commutative spaces", i.e. spaces whose set of points is not defined, but for which one has a good model for the algebra of continuous functions on it. According to Gelfand's theorem, giving a (locally compact) topological space X is equivalent to giving a commutative C^*-algebra (namely the algebra $C_0(X)$ of continuous functions vanishing at ∞). In many examples, although the space one has to deal with is very pathological, one can give a good meaning to its

(C^*-) algebra of continuous functions (non-commutative in general). Such spaces are called "non-commutative spaces". Many of the classical constructions in Top can be carried out in terms of $C_0(X)$, and make no use of the commutativity of this algebra; hence they can be applied to general non-commutative spaces. One has for instance the K-theory, cohomology, Chern character, elliptic operators and an index formula in this non-commutative world.

As it is clear already in the commutative case, *continuity* corresponds to the theory of C^*-algebras, and *measurability* corresponds to the theory of von Neumann algebras. Of course, in many cases the non-commutative space has also additional structure (e.g. smooth or riemannian). This means that, together with the C^*-algebra A playing the role of the algebra of continuous functions, one has also other auxiliary data. For instance *smooth structures* are represented by smooth sub-algebras, i.e. certain dense subalgebras $\mathcal{A} \subset A$ (the prototype is $C_c^\infty(M) \subset C_0(M)$). Also, a geometric space is given by a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, i.e. a triple consisting on an involutive algebra $\mathcal{A}$ of operators acting on a Hilbert space $\mathcal{H}$, and a self adjoint operator D acting on $\mathcal{H}$, satisfying certain conditions (corresponding to the ellipticity of D, the dimension of the space, etc; see [30]). The basic example is provided by the Dirac operator on a closed Riemannian manifold.

Leaf spaces are basic examples of non-commutative spaces. In general, to any smooth étale groupoid $\mathcal{G}$ one associates the smooth convolution algebra $\mathcal{A} = C_c^\infty(\mathcal{G})$ (non-commutative in general); when $\mathcal{G}$ is Hausdorff, it consists of the compactly supported smooth functions on the space of arrows $\mathcal{G}^{(1)}$, with the convolution product:

$$(f_1 * f_2)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f_1(\gamma_1)f_2(\gamma_2) \ .$$

(the sum is finite since $\mathcal{G}$ is étale, and f_1 and f_2 are compactly supported). A certain completion of it defines the C^*-algebra $A = C^*(\mathcal{G})$ of $\mathcal{G}$. This applies in particular to $\mathcal{G} = Hol_T(M, \mathcal{F})$, and we get two algebras:

$$C_c^\infty(T/\mathcal{F}) := C_c^\infty(Hol_T(M,\mathcal{F})) \subset C^*(Hol_T(M,\mathcal{F})) =: C^*(T/\mathcal{F}),$$

modeling $M/\mathcal{F}$. To eliminate the choice of a transversal T, one may work directly on the holonomy groupoid $Hol(M, \mathcal{F})$, and consider its convolution algebra (with the convolution product defined e.g. via integration with respect to a Haar system):

$$C_c^\infty(M/\mathcal{F}) := C_c^\infty(Hol(M,\mathcal{F})) \subset C^*(Hol(M,\mathcal{F})) =: C^*(M/\mathcal{F}).$$

But we emphasize that one usually reduces to complete transversals first (hence to étale groupoids). This is essentially correct by a general principle that (Morita) equivalent groupoids induce (Morita) equivalent algebras. This can be made precise for the associated C^* algebras (one can prove that $C^*(M/\mathcal{F})$ and $C^*(T/\mathcal{F})$ are stably isomorphic [60, 81]), but the case of the smooth algebras seems a bit more delicate.

0.0.4 Cyclic cohomology: Cyclic cohomology $HC^*(\mathcal{A})$ (and its periodic version $HP^*(\mathcal{A})$) is defined for any algebra $\mathcal{A}$ (usually endowed with a locally convex topology), and is the non-commutative analogue of the usual DeRham homology, while the dual theory, cyclic homology, is the analogue of the DeRham cohomology. Thus, for any manifold M, $HP_*(C_c^\infty(M)) \cong H_c^*(M)$.

In the case of foliations, the groups $HP^*(C_c^\infty(M/\mathcal{F}))$ are expected to be isomorphic to $HP^*(C_c^\infty(T/\mathcal{F}))$ (we will prove this in chapter 4). The last groups contain a part (the so called *localization at units*) which is isomorphic to $H^*(M/\mathcal{F})$ (see 4.8.3 for the precise statement). This has been shown in the Hausdorff case by Brylinski-Nistor [20] and Connes [27] (and will be proved in general in our Chapter 4). Brylinski-Nistor [20] also made partial computations of the other components, the so-called *elliptic* ones (in Chapter 4 we will complete the computation, including the *hyperbolic components*).

One can go beyond Connes' definition/properties of $HC^*(\mathcal{A})$ by means of explicit formulas, and get closer to both the derived-functor tools of homological algebra, and Cuntz's approach to K-theory ; this is precisely the Cuntz-Quillen approach (formalism) to cyclic cohomology [40]. This approach has been extremely useful in proving the general excision theorem for (bivariant) periodic cyclic cohomology [42], defining the bivariant version of cyclic cohomology, constructing the bivariant Chern-character [40, 84] compatible with excision [85], as well as proving basic properties in an unified, relatively simple fashion, etc etc.

While Connes' initial definition does not see important properties of HC^*, the situation changes substantially when it comes to motivating and using these cohomology groups. Let us recall Connes' original motivation for introducing cyclic cohomology. One knows, since Atiyah and Singer, that analytical indices (e.g. of families of elliptic operators) are elements in K-groups, and, to associate more computable invariants, one maps the K-theory classes into cohomology classes via the Chern character. This is the base of a general principle in non-commutative geometry, and cyclic homology/cohomology appear as the target of Chern characters, used in order to evaluate analytical indices (elements in K-groups). For instance, the zero-dimensional cyclic cocycles on A correspond to traces $\tau : A \longrightarrow \mathbf{C}$, and any such trace associates numbers to elements in $K_0(A)$ (from an idempotent $E = (e_{i,j}) \in M_\infty(A)$ one gets $\tau(E) = \sum \tau(e_{i,i}) \in \mathbf{C}$). The case of higher cyclic cocycles is an extension of this construction, and gives a Chern character $Ch^* : K_*(\mathcal{A}) \longrightarrow HC_*(A)$ (and similarly a Chern character Ch_* for K-homology).

The general strategy of looking at index problems in non-commutative geometry is an extension of the classical (commutative) index theorems of Atiyah-Singer. One starts with an operator D on some Hilbert space $\mathcal{H}$, on which the given algebra $\mathcal{A}$ acts, so that $(\mathcal{A}, \mathcal{H}, D)$ represents a geometric space as above. The object of study is the index of D, which can be viewed as a map:

$$Ind_D : K_0(\mathcal{A}) \longrightarrow \mathbf{Z}\ ,\ [E] \mapsto Ind(D_E).$$

The operator D defines a K-homology class, and its Chern character $Ch_*(D) \in HC^*(\mathcal{A})$ computes the index by the equality:

$$Ind_D(E) = < Ch_*(D), Ch^*(E) > \ ,\ \forall\ [E] \in K_0(A)\ ,$$

where $<\ ,\ >$ is an obvious pairing between cyclic cohomology and homology. In general, computing the cyclic homology/cohomology and giving effective formulas for $Ch_*(D)$ corresponds to solving the index problem. We refer to [29] for much more detailed and precise descriptions of these aspects, as well as many examples.

0.0.5 The longitudinal index theorem for foliations: This is a very good illustration of the power of non-commutative approach to leaf spaces. Using only differentiation along the leaves of $\mathcal{F}$ one obtains longitudinal differential operators D, which can be viewed as families of operators indexed by the leaf space. Under the condition of transversal ellipticity of D, one can define the index of D, which is an element in K-theory rather than a number. This is best illustrated in the case where the the leaves of $\mathcal{F}$ are the fibers of a submersion $M \longrightarrow B$ (hence the leaf space identifies with the base space B): in this case we are actually looking at the Atiyah-Singer index theorem for families of elliptic operators $D = \{D_l\}_{l\in B}$, and $Ind(D) \in K^0(B)$ is (morally) $Ker(D) - Coker(D)$, where $Ker(D)$ $(Coker(D))$ is the vector bundle over B whose fiber at $l \in B$ is the (finite dimensional by the ellipticity of D) vector space $Ker(D_l)$ $(Coker(D_l))$. In the case of general foliations, the base space is replaced by the leaf space, and the analytical index of D can be defined as an element $Ind_a(D) \in K_0(C^*(M/\mathcal{F}))$. Like the Atiyah-Singer index theorem for families which it extends, the longitudinal index theorem for foliations gives topological interpretations of $Ind_a(D)$, and there are various versions of this theorem. A very general version at the level of K-theory (which asserts that $Ind_a(D)$ coincides with a topologically defined index $Ind_t(D)$) was proved by Connes and Skandalis [33]. More explicit analytical invariants can be obtained if the foliation admits a transversal measure; such a measure can be viewed (as in the case of Radon measures for ordinary spaces) as a trace τ on the algebra $C^*(M/\mathcal{F})$ of functions on the leaf space, and induces a map $\tau_* : K_0(C^*(M/\mathcal{F})) \longrightarrow \mathbf{C}$. A topological formula for $\tau_*(Ind_a(D))$ is provided by Connes' index theorem for measured foliations, proved in early 80's [25]. In the general case (when the foliation is not necessarily measured) one can replace τ by a "higher trace", i.e. by cyclic cocycles (cf. our short presentation of cyclic cohomology). In other words, as it already happens in the the Atiyah-Singer context, one asks for the Chern character $Ch(Ind_a(D)) \in HC_*(C_c^\infty(M/\mathcal{F}))$ rather than for the index itself. Using the computation of $HC_*(C_c^\infty(M/\mathcal{F}))$ previously described, one gets for instance a *partial* Chern character ("localization at units" of Ch):

$$Ch^1 : K_0(C_c^\infty(M/\mathcal{F})) \longrightarrow H_c^*(M/\mathcal{F}) \ , \tag{0.6}$$

whose target is a compactly supported version of the cohomology of the leaf space. With the proper normalization of Ch^1 (one has to divide by $(2\pi)^{2m}$ in degree $2m$), the Connes' cohomological form of the longitudinal index formula can be written [29]:

$$Ch^1(Ind_a(D)) = (-1)^n \int_p \mathcal{T}(\mathcal{F}) ch(\sigma(D)) \ , \tag{0.7}$$

where $p : T\mathcal{F} \longrightarrow M/\mathcal{F}$ is the projection defined on the tangent bundle of $\mathcal{F}$ (composition of the projection $T\mathcal{F} \longrightarrow M$ with the projection $M \longrightarrow M/\mathcal{F}$), $\int_p : H_c^*(T\mathcal{F}) \longrightarrow H_c^*(M/\mathcal{F})$ is the integration along the fibers of p (all these are intuitively clear, and we will prove they become correct when working in the category of étale groupoids), $\mathcal{T}(\mathcal{F})$ is the Todd class of the complexification of $\mathcal{F}$, $ch(\sigma(D)) \in H_c^*(T\mathcal{F})$ is the (classical) Chern character of the principal symbol of D. See [86] for a proof in the case of foliations arising from flat bundles.

While Connes-Skandalis' version (at the level of K-theory) seems to be the most general form of the theorem, formula (0.7) is a quite general cohomological version (more general ones should be obtained using the complete computation of $HC_*(C_c^\infty(M/\mathcal{F}))$ and the associated Chern characters). We should also mention that there are various other longitudinal index theorems which make no use of non-commutative geometry.

For instance, following Bismut's approach to Atiyah-Singer index theorem for families, Heitsch and Lazarov [59] have recently proved a cohomological formula for certain analytical classes $\bar{Ch}(D_{\mathcal{E}}) \in H^*_{c,bas}(M/\mathcal{F})$ (which appear as the Chern character of the index bundle [59]). It involves the (rather smaller) basic cohomology groups $H^*_{c,bas}(M/\mathcal{F})$ instead of $H^*_c(M/\mathcal{F})$; in general, there is a natural map $j_b : H^*_c(M/\mathcal{F}) \longrightarrow H^*_{c,bas}(M/\mathcal{F})$, hence also an induced (smaller) basic Chern character (at units) Ch^1_{bas} defined as the composition:

$$Ch^1_{bas} : K_0(C^\infty_c(M/\mathcal{F})) \xrightarrow{Ch^1} H^*_c(M/\mathcal{F}) \xrightarrow{j_b} H^*_{c,bas}(M/\mathcal{F}) \ . \tag{0.8}$$

It is likely that Heitsch-Lazarov's $\bar{Ch}(D_{\mathcal{E}})$ is (up to normalization) $Ch^1_{bas}(Ind_a(D))$, and their theorem is related to Connes' formula (this connection, as well as the existence of Ch^1_{bas}, is conjectured in [59]).

There are various other index theorems for families indexed by a "bad space" which can be viewed as a noncommutative space. Let me mention that our discussion here is part of a general index theory for families parameterized by an étale groupoid $\mathcal{G}$, and includes the Atiyah-Singer index theorem for families (when $\mathcal{G}$ is a space), the higher index theorem of Atiyah-Connes-Moscovici (when $\mathcal{G}$ is a discrete group), the longitudinal index theorem for foliations (when $\mathcal{G}$ is the reduced holonomy groupoid), and a higher index theorem for foliations (when $\mathcal{G}$ is the reduced homotopy groupoid).

0.0.6 The transversal index theorem for foliations: This is an index theorem for leaf spaces, and is, in some sense, dual to the longitudinal index theorem. Since it involves quite a few steps (for instance it was not a priori obvious how to state the problem or, later, how to make the computation of the cohomology classes that arise), it has quite a long history, and is based on several fundamental works (some of them needed for just formulating the problem). Very briefly, in [30] it is shown that any leaf space (actually any pseudogroup of transformations on a manifold) gives rise to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and this is essentially based on the ideas of [61] of adapting the usual pseudo-differential calculus so that one does not need any assumption on the holonomy of the foliation. On the other hand, [30] also provides a local formula for $Ch_*(D)$, in terms of a certain integral $\oint$ (defined in terms of Wodzicki-type residues) . This non-commutative index formula expresses the Chern character as a finite sum of expressions of form:

$$\oint a^0[D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} |D|^{-n-2k}, \ a^i \in \mathcal{A} \tag{0.9}$$

where, for an operator T, $T^{(k)}$ denotes the k^{th} iterated commutator of D^2 with T. This formula is valid for any spectral triple satisfying a certain sumability condition. In the case of the spectral triple associated to a foliation, one still has to interpret these cohomology classes (living in $HP^*(\mathcal{A})$, $\mathcal{A} = C^\infty_c(M/\mathcal{F})$).

The last step in proving/formulating the transversal index theorem for foliations (to actually identify the cyclic cocycles entering in the index formula (0.9)) has recently been taken by Connes and Moscovici [31]. Very roughly, the situation is as follows. The cyclic cocycles appearing in (0.9) are made out by combining a trace on $\mathcal{A}$ and actions of certain geometric operators (actually *the* geometric operators) on $\mathcal{A}$:

$$\{\text{elementary operators }\} \times \mathcal{A} \longrightarrow \mathcal{A} \ , (h, a) \mapsto h(a)$$

By looking at the behavior of this action on products, one gets some kind of generalized Leibniz rules:

$$h(ab) = \sum_i h_0^i(a) h_1^i(b) \ ,$$

which dictates a Hopf algebra $\mathcal{H}_q$ constructed out of the elementary operators. This is the Connes-Moscovici Hopf algebra, and plays somehow the role of the classifying Hopf algebra for codimension q foliations: for any such foliation, one has a natural action:

$$\mathcal{H}_q \otimes \mathcal{A} \longrightarrow \mathcal{A} \ .$$

Hence, the cyclic cocycles we are interested in, are naturally associated to this action and a certain invariant trace τ on $\mathcal{A}$. One can organize all such cyclic cocycles in a new (universal) cohomology group: the cyclic cohomology of the Hopf algebra $\mathcal{H}_q$. More precisely, Connes and Moscovici define a cyclic cohomology for $\mathcal{H}_q$ such that, to any such pair $(\mathcal{A}, \tau)$ consisting of an algebra $\mathcal{A}$ on which $\mathcal{H}_q$ acts and an invariant trace τ on $\mathcal{A}$, one has an associated characteristic map:

$$k_\tau : HP^*(\mathcal{H}_q) \longrightarrow HP^*(\mathcal{A}) \ ,$$

whose image consists precisely of the cyclic cocycles on $\mathcal{A}$ which arise from the action of $\mathcal{H}_q$ and the trace τ, i.e. finite combinations of type:

$$\tau(a^0 h^1(a^1) \ \dots \ h^n(a^n)) \ , \ a^i \in \mathcal{A}, \ h^i \in \mathcal{A} \ .$$

In particular, the cyclic cocycles in the index formula are in the image of this characteristic map. Now, this discussion overlooks some important details. For instance, one has to work via Morita equivalence of algebras, and, instead of $\mathcal{A} = C_c^\infty(M/\mathcal{F})$ one has to use, in parallelism with the construction of the usual $k_{\mathcal{F}}^{geom}$, another algebra $\tilde{\mathcal{A}}$ which plays the role of the principal (frame) bundle. At the end one has to divide by the action O_q, hence the correct (non-commutative) characteristic map for foliations is:

$$k_{\mathcal{F}}^{nc} : HP^*(\mathcal{H}_q, O_q) \longrightarrow HP^*(C_c^\infty(M/\mathcal{F})) \ , \tag{0.10}$$

whose source is a relative version of the cyclic cohomology $HP^*(\mathcal{H}_q)$. When computed [31],

$$HP^*(\mathcal{H}_q, O_q) \cong H^*(\underline{WO}_q) \ ,$$

and, with the connection between $HP^*(C_c^\infty(M/\mathcal{F}))$ and $H^*(M/\mathcal{F})$ (previously described) in mind, (0.10) is a non-commutative version of $k_{\mathcal{F}}^{geom}$. The conclusion is that, the cyclic cocycles appearing in the non-commutative index formula are coming from the Gelfand-Fuchs cohomology $H^*(\underline{WO}_q)$.

There are some obvious questions left in [31], like defining the cyclic cohomology of general Hopf algebra (as defined in [31] it seemed there was a technical obstruction to this), finding the relation with Cuntz-Quillen's formalism, and constructing the characteristic maps associated to general (higher) traces. These will be the subject of our Chapter 5.

Having described the general picture, let us have a first look at the content of this thesis and the questions it answers. The title of the first chapter ("Preliminaries") is self-explanatory. Each other chapter is basically a paper which has appeared or soon will appear in specialized mathematical journals, and consists of a (finite) set of related problems. Each of these chapters starts with an extensive introduction. Here we restrict ourselves to a brief description.

0.0.7 Chapter 2: A homology theory for étale groupoids: This is a version of a joint paper with Ieke Moerdijk [35]. As previously mentioned, many of the usual constructions for spaces extend, via the inclusion $Top \subset \acute{E}tale$, to étale groupoids (hence also to leaf spaces). In this chapter we show that this is the case also for compactly supported cohomology, and the Poincaré duality (the full Verdier duality actually). We prefer to call the resulting theory the homology of étale groupoids (it also extends the usual homology of groups). The task of defining it, and proving its basic properties is more difficult than in the case of cohomology since the standard homological algebra (derived functors, Grothendieck spectral sequence, etc.) does not apply. Hence, the definitions and the proofs of our theorems (like Hochschild-Serre spectral sequence, Morita invariance, Poincaré duality) are quite explicit. We also include an appendix where we show how to handle compact supports in the non-Hausdorff case (simple examples of holonomy groupoids are non-Hausdorff). This homology theory is also essential in our computation of the cyclic cohomology of convolution algebras.

As main results of this chapter we indicate the construction of the homology (see 2.2.4), the Hochschild-Serre spectral sequence of Theorem 2.3.4, the Morita invariance of Corollary 2.3.6, the basic spectral sequences of Proposition 2.2.11, the existence of the functor $\varphi_!$ (see 2.3.2), the Verdier duality (Theorem 2.4.12), and its Corollary 2.4.13 (Poincaré duality).

From the point of view of foliations, it is precisely the groups $H_c^*(M/\mathcal{F})$ that we define. Note also that we have already (tacitly) used these groups in our presentation 0.0.5 of the longitudinal index theorem, and formula (0.7) is actually a reformulation of Connes' original formula; this reformulation makes use of the Poincaré duality for leaf spaces, the computation of the cyclic homology of the convolution algebra (proved in Chapter 4), as well as of the existence of integration over the fibers (the functors $\varphi_!$), and coincides in the bundle case with the Atiyah-Singer index formula (see Theorem 5.1 in [6]).

0.0.8 Chapter 3: *Čech* cohomology for leaf spaces, and characteristic classes: This is a version of a joint paper with Ieke Moerdijk [38]; it is a sequel of [73], and is a result of our search for more explicit models for the cohomology of leaf spaces and more geometric constructions of characteristic classes for foliations (recall e.g. that the universal map (0.5), and the $H^*(M/\mathcal{F})$-valued characteristic map $k_{\mathcal{F}}^{geom}$ are defined in a rather abstract way). Our intention is to show that the usual cohomology of étale groupoids admits a *Čech* model, which is most suitable for writing down explicit formulas for the characteristic classes associated to foliations. It is here that we give a more explicit (*Čech*-DeRham) model for $H^*(M/\mathcal{F})$, and present a simple direct proof that the geometric characteristic classes are transversal invariants. More generally, we show that any transversal vector bundle E on a foliation $(M, \mathcal{F})$ defines an explicit characteristic map:

$$k_E^{geom} : H^*(\underline{WO}_q) \longrightarrow H^*(M/\mathcal{F}) \ .$$

Our construction explains Bott's formulas for the generalized Godbillon-Vey classes, gives an explicit argument for the Bott vanishing theorem at the level of $H^*(M/\mathcal{F})$, and extends these results to any transversal bundle. Using the *Čech* model we also give another proof of Poincaré duality for étale groupoids, simpler than the original proof (of [35] or our Chapter 2). We also obtain a stronger vanishing result for Riemannian foliations. Similar $H^*(M)$-valued characteristic classes were defined by Kamber-Tondeur [64], and they are associated to what they call *foliated bundles*; from this point of view our construction shows that, if a foliated bundle is transversal, then the associated characteristic map is a transversal invariant. The formulas make sense in general, for any étale groupoid, hence we get in particular (for the étale groupoid Γ^q) an explicit description of the universal characteristic map (0.5).

0.0.9 Chapter 4: Cyclic cohomology of étale and holonomy groupoids: This is a version of [34]. In this chapter we compute the cyclic homology/cohomology of convolution algebras $C_c^\infty(\mathcal{G})$ of smooth étale groupoids and (non-reduced) holonomy groupoids. The approach uses our homology theory and its basic properties (in particular the Hochschild-Serre spectral sequence). Using the Appendix of Chapter 2, the computation is in the general setting ($\mathcal{G}$ not necessarily Hausdorff). We also explain how our methods apply to more general coefficients (i.e. to "crossed-products" by étale groupoids).

As main results of this chapter, let us mention the computations given by Theorem 4.6.2 (decomposition), Theorem 4.6.3 (localization at units), Theorem 4.6.4 (elliptic components), Theorem 4.6.5 (hyperbolic components), their duals (e.g. Theorem 4.7.4), the Poincaré duality in cyclic cohomology (Corollary 4.7.5), and the particular cases of crossed products by discrete groups (Corollary 4.10.5), orbifolds, and crossed products by Lie groups acting with discrete stabilizers. In the case of foliations we relate our computations to Haefliger's cohomology (see 4.8.3), prove that the results can be applied to the holonomy groupoid (Theorem 4.8.1 , and Theorem 4.8.2), and we give explicit descriptions of the basic Chern character (Section 4.9).

Let us also mention that our computations extend the previous results of Connes (for manifolds), Burghelea (for groups), Feigin, Nistor, Tsygan (for crossed products), Brylinski and Nistor (which have partially computed, in the Hausdorff case, the elliptic components of $HC^*(C_c^\infty(\mathcal{G}))$). In the case of foliations one gets for instance the connection between the localization of units of $HC_*(C_c^\infty(M/\mathcal{F}))$ and the groups $H_c^*(M/\mathcal{F})$ (introduced in Chapter 2); we have already used this in our presentation of the longitudinal index theorem for foliations (more precisely in the construction of the Chern character (0.6)). The basic Chern character (0.8) that we explicitly describe, gives the connection between Connes' index formula (0.7) and the Heitsch-Lazarov index theorem: $Ch_b^1(Ind_a(D))$ equals the Chern character of the index bundle (this can be easily seen by comparing the two index formulas). Of course, one would like to have a direct proof of the relation between these two (analytical) invariants, and this is our motivation for describing more explicitly the basic Chern character. We also mention that the computation of the other localizations (not just at units) should be the base for more general longitudinal index theorems.

0.0.10 Chapter 5: Cyclic cohomology of Hopf algebras: This is a version of [36]. In this chapter we show that the definition of the cyclic cohomology $HC^*(\mathcal{H}_q)$ can be extended to any Hopf algebra $\mathcal{H}$ endowed with a character δ (our notation is $HC^*_\delta(\mathcal{H})$), and we interpret this definition. This cohomology should be viewed as an extension of Lie algebra homology (which is obtained for $\mathcal{H} = U(\mathfrak{g})$, the enveloping algebra). We prove that Cuntz-Quillen's formalism applies to this theory, and we show that, inspired by the construction of the characteristic map $k^{geom}_{\mathcal{F}}$, one ends up with a non-commutative Weil complex which can be used to compute $HC^*_\delta(\mathcal{H})$. This is useful in constructing characteristic maps k_τ associated to higher traces.

The main results of this chapter are: the computation of Theorem 5.5.4 (left open in [31]), Theorem 5.5.7 (which is the starting point in our interpretation in terms of Cuntz-Quillen's formalism), the computations in Proposition 5.6.9 (for $\mathcal{H} = U_q(sl_2)$), and in Theorem 5.6.6 (for $\mathcal{H} = U(\mathfrak{g})$, which shows that this cyclic cohomology is an extension of the Lie algebra homology), the construction of the non-commutative Weil complex (section 5.7), and of the Chern-Simons transgression (Theorem 5.7.7), Theorem 5.8.9 (which expresses the relation between the Weil complex and Cuntz-Quillen's formalism), Theorem 5.8.3 (which shows that the Weil complex can be used to compute the cyclic cohomology of Hopf algebra), and Theorems 5.8.5, 5.8.7, 5.9.4 (which show how to use the Weil complex in constructing characteristic maps associated to higher traces).

Let me mention that in the simplest case $\mathcal{H} = \mathbf{C}$, one reobtains the main results of [90], while, in general, we answer some questions left in [31]: define a cyclic cohomology theory $HC^*_\delta(\mathcal{H})$ for general Hopf algebras endowed with involutive twisted antipodes, put the work of [31] into the framework of Cuntz-Quillen's formalism [40], and construct the characteristic maps associated to higher traces.

Chapter 1

Preliminaries

1.1 Groupoids

In this section we review the definition of topological groupoids, fix the notations, and mention some of the main examples.

Recall first that a *groupoid* $\mathcal{G}$ is a (small) category in which every arrow is invertible. We will write $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ for the set of objects and the set of arrows in $\mathcal{G}$, respectively, and denote the structure maps by:

$$\mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \xrightarrow{m} \mathcal{G}^{(1)} \xrightarrow{i} \mathcal{G}^{(1)} \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}^{(0)} \xrightarrow{u} \mathcal{G}^{(1)} \quad , \tag{1.1}$$

Here s and t are the source an target, m denotes composition ($m(g,h) = g \circ h$), i is the inverse ($i(g) = g^{-1}$), and, for any $x \in \mathcal{G}^{(0)}$, $u(x) = 1_x$ is the unit at x. We write $g : x \longrightarrow y$ or $x \xrightarrow{g} y$ to indicate that $g \in \mathcal{G}^{(1)}$ is an arrow with $s(g) = x$ and $t(g) = y$.

A *topological groupoid* $\mathcal{G}$ is similarly given by topological spaces $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ and by continuous structure maps as in (1.1). Our main interests will be on *smooth groupoids* (also called *Lie groupoids*). For such groupoids $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are smooth manifolds, and these structure maps are smooth; moreover, one requires s and t to be submersions, so that the fibered product $\mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)}$ in (1.1) is also a manifold.

Warning: We very often use the notation $\mathcal{G}$ for the space $\mathcal{G}^{(1)}$ of arrows.

1.1.1 Definition: A topological (smooth) groupoid $\mathcal{G}$ as above is called *étale* if the source map $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ is a local homeomorphism (local diffeomorphism). This implies that all other structure maps in (1.1) are local homeomorphisms (local diffeomorphism).

1.1.2 Germs: Any arrow $g : x \longrightarrow y$ in an étale groupoid induces a germ $\tilde{g} : (U, x) \xrightarrow{\sim} (V, y)$ from a neighborhood U of x in $\mathcal{G}^{(0)}$ to a neighborhood V of y. Indeed, we can define $\tilde{g} = t \circ \sigma$, where $x \in U \subset \mathcal{G}^{(0)}$ is so small that $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ has a section $\sigma : U \longrightarrow \mathcal{G}^{(1)}$ with $\sigma(x) = g$, and $t|_{\sigma(U)}$ is a homeomorphism into $V := t(\sigma(U))$. Then $\tilde{g} : U \xrightarrow{\sim} V$ is a homeomorphism; we will also write $\tilde{g}$ for the germ at x of this map $\tilde{g} : U \xrightarrow{\sim} V$. Note that $\tilde{1}_x$ is the identity germ, and that $\widetilde{(hg)} = \tilde{h}\tilde{g}$ if $g : x \longrightarrow y$ and $h : y \longrightarrow z$.

1.1.3 Examples of étale groupoids: (Note that in examples 4 and 5, the space $\mathcal{G}^{(1)}$ is in general not Hausdorff.)

1. Any topological space (manifold) X can be viewed as an étale (smooth) groupoid $\underline{X}$, with identity arrows only ($\underline{X}^{(0)} = X = \underline{X}^{(1)}$, etc.). We will often simply denote this groupoid by X again.

2. Any topological (Lie) group G can be viewed as a topological (smooth) groupoid with one object, G as the space of arrows and with the multiplication of G. It is étale if and only if G is discrete.

3. As a mixture of the previous examples, if a (discrete) group Γ acts from the right on a space X, one can form a groupoid $X \rtimes \Gamma$ with $(X \rtimes \Gamma)^{(0)} = X$ and $(X \rtimes \Gamma)^{(1)} = X \times \Gamma$, by taking as arrows $x \longleftarrow y$ those $\gamma \in \Gamma$ with $y = x\gamma$. This groupoid is called the translation groupoid of the action.

4. ([57, 13]) The Haefliger groupoid Γ^q has $\mathbb{R}^q$ for its space of objects. An arrow $x \longrightarrow y$ in Γ^q is a germ of a diffeomorphism $(\mathbb{R}^q, x) \longrightarrow (\mathbb{R}^q, y)$. This smooth étale groupoid and its classifying space $B\Gamma^q$ (cf. 1.1.13 below) play a central role in foliation theory, as we shell recall in 1.3.1 and 1.4.3 below.

5. (see, for example, [99, 25, 72]) For a foliation $(M, \mathcal{F})$ of codimension q, its holonomy groupoid $Hol(M, \mathcal{F})$ can be reduced to an étale groupoid $Hol_T(M, \mathcal{F})$, depending on the choice of a "complete transversal" T (i.e. a submanifold $T \subset M$ of dimension q which is transversal to the leaves and which meets every leaf at least once). We shell recall this also in 1.3.3. Two different such transversals T and T' give Morita equivalent (see 1.1.9 below) smooth étale groupoids $Hol_T(M, \mathcal{F})$ and $Hol_{T'}(M, \mathcal{F})$.

6. Any orbifold gives rise to a smooth étale groupoid. These groupoids $\mathcal{G}$ coming from orbifolds have the special property that $(s,t) : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is a proper map (see [75]). Groupoids with this property are called proper. For a proper groupoid, $\mathcal{G}^{(1)}$ is Hausdorff whenever $\mathcal{G}^{(0)}$ is.

7. If $\mathcal{G}$ is an étale groupoid, $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is a covering of $\mathcal{G}^{(0)}$, the groupoid $\mathcal{G}_\mathcal{U}$ is defined by

$$\mathcal{G}_\mathcal{U}^{(0)} = \bigcup_{\alpha \in I} U_\alpha \times \{\alpha\} \ , \ \mathcal{G}_\mathcal{U}^{(1)} = \bigcup_{\alpha,\beta \in I} (s^{-1}(U_\alpha) \cap t^{-1}(U_\beta)) \times \{\alpha, \beta\},$$

$s(x, \alpha, \beta) = (s(x), \alpha), t(x, \alpha, \beta) = (t(x), \beta)$. This applies in particular to any covering $\mathcal{U}$ of a space (manifold) M, to give a (smooth) étale groupoid $M_\mathcal{U}$. Of course, $M_\mathcal{U}$ (and $\mathcal{G}_\mathcal{U}$) is defined in general ($\mathcal{U}$ does not have to be a covering of M), but in the case where $\mathcal{U}$ is a covering, M and $M_\mathcal{U}$ are isomorphic as étale groupoids (we will explain this in 1.1.11.5 below).

8. Examples 5 and 7 above are simple examples of pullback. In general, given $f : X \longrightarrow \mathcal{G}^{(0)}$, one defines the pullback of $\mathcal{G}$ along f as the groupoid $f^*(\mathcal{G})$ whose space of objects is X, and the arrows between $x, y \in X$ are the arrows of $\mathcal{G}$ between $f(x)$ and $f(y)$.

1.1.4 Homomorphisms: Let $\mathcal{G}$ and $\mathcal{K}$ be étale groupoids. A *homomorphism* $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ is given by two continuous maps $\varphi_0 : \mathcal{K}^{(0)} \longrightarrow \mathcal{G}^{(0)}$ and $\varphi_1 : \mathcal{K}^{(1)} \longrightarrow \mathcal{G}^{(1)}$ which commute with all the structure maps in (1.1) (i. e. : $\varphi_0 s(g) = s\varphi_1(g), \varphi_1(g \circ h) = \varphi_1(g) \circ \varphi_1(h)$, etc.)

1.1.5 Actions: Let $\mathcal{G}$ be an étale groupoid. A right action of $\mathcal{G}$ on the space X consists of two continuous maps $\pi : X \longrightarrow \mathcal{G}^{(0)}$ (the moment map), $m : X \times_{\mathcal{G}^{(0)}} \mathcal{G} = \{(x, g) \in X \times \mathcal{G} : \pi(x) = t(g)\} \longrightarrow X$ (the action) such that, denoting $m(x, g) = xg$:

$$(xg)h = x(gh), x1 = x, \pi(xg) = s(g).$$

We shall call X a right $\mathcal{G}$ space with moment map π. The associated groupoid for this action, denoted $X \rtimes \mathcal{G}$, is defined as a generalization of 1.1.3.3: $(X \rtimes \mathcal{G})^{(0)} = X, (X \rtimes \mathcal{G})^{(1)} = X \times_{\mathcal{G}^{(0)}} \mathcal{G}, s(x, g) = xg, t(x, g) = x, u(x) = (x, 1), m((x, g), (y, h)) = (x, gh), i(x, g) = (xg, g^{-1})$.

There is an obvious similar notion of left $\mathcal{G}$-space. Unless specified, all $\mathcal{G}$-spaces will have the action from the right.

1.1.6 Bundles: A (right) $\mathcal{G}$-bundle over the space B consists of a $\mathcal{G}$-space E and a continuous map $p : E \longrightarrow B$ which is $\mathcal{G}$-invariant (i.e. $p(xg) = p(x)$). It is called principal if p is an open surjection and $E \times_{\mathcal{G}^{(0)}} \mathcal{G} \longrightarrow E \times_B E, (e, g) \mapsto (e, eg)$ is a homeomorphism.

1.1.7 Morphisms of Groupoids ([54, 72]): Let $\mathcal{G}$ and $\mathcal{H}$ be two groupoids. A *morphism* $P : \mathcal{G} \longrightarrow \mathcal{H}$ from $\mathcal{G}$ to $\mathcal{H}$ (or Hilsum-Skandalis map cf. [79]) consists of a space P, continuous maps (source and target): $s_P : P \longrightarrow \mathcal{G}^{(0)}, t_P : P \longrightarrow \mathcal{H}^{(0)}$, a left action of $\mathcal{G}$ on P with the moment map s_P, a right action of $\mathcal{H}$ on P with the moment map t_P, such that:

1. s_P is $\mathcal{H}$-invariant, t_P is $\mathcal{G}$-invariant;
2. the actions of $\mathcal{G}$ and $\mathcal{H}$ on P are compatible: $(gp)h = g(ph)$;
4. $s_P : P \longrightarrow \mathcal{G}^{(0)}$, as an $\mathcal{H}$-bundle with the moment map t_P, is principal.

There is an obvious smooth version of this. A nice intuitive motivation of this definition is that P can be viewed as a continuous map between the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$, described by its graph (see II.8.γ in [25]). A nice theoretical motivation is that these morphisms are exactly the topos-theoretic morphisms between the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$ viewed as toposes (i.e. between the classifying toposes of $\mathcal{G}$ and $\mathcal{H}$; see [72] for the precise statements and descriptions). The composition of two morphisms $P : \mathcal{G} \longrightarrow \mathcal{H}$, $Q : \mathcal{H} \longrightarrow \mathcal{K}$ is defined by dividing out $P \times_{\mathcal{H}^{(0)}} Q$ by the action of $\mathcal{H}$: $(p, q)h = (ph, h^{-1}q)$, and taking the obvious actions of $\mathcal{G}$ and $\mathcal{H}$. We get in this way the category of groupoids and its full subcategory of étale groupoids.

1.1.8 Example: Any continuous functor $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ can be viewed as a morphism by taking $P_\varphi = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H} = \{(c, h) \in \mathcal{G}^{(0)} \times \mathcal{H} : \varphi(c) = t(h)\}$, $s_P(c, h) = c$, $t_P(c, h) = s(h)$ and the obvious actions.

1.1.9 Morita Equivalences: Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are called *Morita equivalent* if they are isomorphic in the category of groupoids (as defined in 1.1.7). An isomorphism $P : \mathcal{G} \longrightarrow \mathcal{H}$ is called a Morita equivalence (cf. [72]).

1.1.10 Smooth context: When working in the smooth context, one has to make the usual changes in the definitions 1.1.4, 1.1.5, 1.1.6, 1.1.7, 1.1.8, 1.1.9 above: continuity is replaced by smoothness, homeomorphisms by diffeomorphisms, open surjections by surjective submersions. We will be mainly interested in smooth groupoids, although in Chapter 2 it is more natural to stay in the topological context.

1.1.11 Examples:

1. Recall ([72]) that an *essential equivalence* is a continuous functor $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ with the property that $P_\varphi = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H} = \{(c, h) \in \mathcal{G}^{(0)} \times \mathcal{H} : \varphi(c) = t(h)\} \longrightarrow \mathcal{H}^{(0)}$, $(c, h) \mapsto s(h)$ is an open surjection and the diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[d, "(s,t)"'] & \mathcal{H} \arrow[d, "(s,t)"] \\ \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \arrow[r, "\varphi\times\varphi"] & \mathcal{H}^{(0)} \times \mathcal{H}^{(0)} \end{tikzcd}$$

is a pull-back of topological spaces. It is easily seen that in this case φ induces a Morita equivalence $P_\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ (see 1.1.8). In fact we can prove (see 2.3 in [72], or [79]) that $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if there is a groupoid $\mathcal{K}$ and essential equivalences $\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}$.

2. If $(M, \mathcal{F})$ is a foliated manifold, and T is a complete transversal, there is an obvious functor $Hol_T(M, \mathcal{F}) \longrightarrow Hol(M, \mathcal{F})$ from the holonomy groupoid restricted to T into the holonomy groupoid. It is a standard simple fact that this is a Morita equivalence (cf. example 4 below).

3. If $E \longrightarrow B$ is a principal G-bundle (where G is a topological group), then the obvious projection $E \rtimes G \longrightarrow B$ (see examples 1 and 3 in 1.1.3) is a Morita equivalence.

4. Given a continuous (smooth) map $f : X \longrightarrow \mathcal{G}^{(0)}$, such that the map $X \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ is an open surjection (surjective submersion), then $\mathcal{G}$ is (smoothly) Morita equivalent to its pullback $f^*(\mathcal{G})$ (see 1.1.3.8). Indeed, the obvious homomorphism $\phi_f : f^*(\mathcal{G}) \longrightarrow \mathcal{G}$ is an essential equivalence.

5. As a particular case of the previous example we deduce that for any covering $\mathcal{U}$ of a manifold M, the associated groupoid $M_\mathcal{U}$ (see 1.1.3.7) is Morita equivalent to M. Similarly, $\mathcal{G}_\mathcal{U}$ is Morita equivalent to $\mathcal{G}$.

1.1.12 Comma Groupoids: If $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ is a continuous functor between groupoids, $d \in \mathcal{H}^{(0)}$, the *comma groupoid* d/φ is defined as follows. It has as objects pairs $(c, h) \in \mathcal{H} \times \mathcal{G}^{(0)}$ with $s(h) = d$, $t(h) = \varphi(c)$ (i.e. the space of objects is $\mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}^{(1)}$), and as morphisms from (c, h) to (c', h') those $g : c \longrightarrow c'$ in $\mathcal{G}$ with $\varphi(g)h = h'$ (i.e. the space of morphisms is $\mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{H}^{(1)}$). We have a commutative diagram:

$$\begin{tikzcd} d/\varphi \arrow[r, "\omega_d"] \arrow[d] & \mathcal{G} \arrow[d, "\varphi"] \\ \bullet \arrow[r, "d"] & \mathcal{H} \end{tikzcd}$$

where ω_d is the continuous functor which send an object (h, c) to c and a morphism g, from (h, c) to (h', c'), to g. The comma category can be viewed as the fiber of φ above d.

1.1.13 Nerve and classifying space: For an étale groupoid $\mathcal{G}$, we write $\mathcal{G}^{(n)}$ for the space of composable strings of arrows in $\mathcal{G}$:

$$x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} \dots \xleftarrow{g_n} x_n$$

For $n = 0, 1$, this agrees with the notation for the space of objects and arrows of $\mathcal{G}$, already introduced. The spaces $\mathcal{G}^{(n)}$ ($n \geq 0$) together form a simplicial space, *the nerve of* $\mathcal{G}$,

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; \mathcal{G}^{(2)} \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; \mathcal{G}^{(1)} \; \substack{\longrightarrow\\\longrightarrow} \; \mathcal{G}^{(0)} \,, \tag{1.2}$$

with the face maps $d_i : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n-1)}$ defined in the usual way:

$$d_i(g_1, ..., g_n) = \begin{cases} (g_2, ..., g_n) & \text{if } i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_1, ..., g_{n-1}) & \text{if } i = n \end{cases} .$$

Its (thick [93]) geometric realization is *the classifying space* of $\mathcal{G}$, denoted $B\mathcal{G}$. This space $B\mathcal{G}$ classifies homotopy classes of principal $\mathcal{G}$−bundles [21, 72]. A Morita equivalence $\varphi : \mathcal{H} \xrightarrow{\sim} \mathcal{G}$ induces a weak homotopy equivalence $B\mathcal{H} \xrightarrow{\sim} B\mathcal{G}$.

1.1.14 Overall assumptions: It is important to observe that in many relevant examples, the space $\mathcal{G}^{(1)}$ of arrows of an étale groupoid $\mathcal{G}$ is *non-Hausdorff*, (cf. Examples 4, and 5 in 1.1.3). However, for any space X in this paper we do assume that X has an open cover by subsets $U \subset X$ which are each paracompact, Hausdorff, locally compact, and of cohomological dimension bounded by a number d (depending on X but not on U). These assumptions hold for any (non-separated) manifold of dimension d, and in particular for each of the spaces $\mathcal{G}^{(n)}$ associated to a smooth étale groupoid.

1.2 Sheaves and cohomology

In this section we review the definition and main properties of the cohomology groups $H^n(\mathcal{G}; \mathcal{A})$ of an étale groupoid $\mathcal{G}$ with coefficients in a $\mathcal{G}$-sheaf $\mathcal{A}$. These groups have been studied by Haefliger ([55],[58]). They can also be viewed as cohomology groups of the topos of $\mathcal{G}$-sheaves (Grothendieck-Verdier) and were discussed from this point of view in [72].

1.2.1 $\mathcal{G}$-sheaves: Let $\mathcal{G}$ be an étale groupoid. A $\mathcal{G}$*-sheaf* is a sheaf $\mathcal{S}$ on the space $\mathcal{G}^{(0)}$, on which $\mathcal{G}^{(1)}$ acts continuously from the right. This means that for any arrow $g : c \longrightarrow d$ in $\mathcal{G}$, there is a morphism between stalks $\mathcal{S}_d \longrightarrow \mathcal{S}_c$, $a \mapsto ag$, satisfying the usual identities for an action. Viewing $\mathcal{S}$ as an étale space $\mathcal{S} \longrightarrow \mathcal{G}^{(0)}$ (i.e. $\mathcal{S}$ is the disjoint union of all stalks $\mathcal{S}_c$ with the germ topology) it gives a map $m : \mathcal{S} \times_{\mathcal{G}^{(0)}} \mathcal{G} \longrightarrow \mathcal{S}$, $(a, g) \mapsto ag$; the continuity of the action means that m is continuous.In other words, $\mathcal{S}$ is a right $\mathcal{G}$-space (1.1.3.6) for which the map $\mathcal{S} \longrightarrow \mathcal{G}$ is étale (a local homeomorphism).

A morphism of $\mathcal{G}$-sheaves $\mathcal{S} \longrightarrow \mathcal{S}'$ is a morphism of sheaves which commutes with the action. We will write $\underline{Sh}_s(\mathcal{G})$ for the category of all $\mathcal{G}$-sheaves of sets, and $\underline{Ab}(\mathcal{G})$ for the category of abelian $\mathcal{G}$-sheaves. These categories have convenient exactness properties: it is well known that $\underline{Sh}_s(\mathcal{G})$ is a topos, and (hence) that $\underline{Ab}(\mathcal{G})$ is an abelian category with enough injectives. If R is a ring, we write $\underline{Sh}_R(\mathcal{G})$ for the category of $\mathcal{G}$-sheaves of R-modules. Thus $\underline{Ab}(\mathcal{G}) = \underline{Sh}_{\mathbb{Z}}(\mathcal{G})$. Later, we will mostly work with the category $\underline{Sh}_{\mathbb{C}}(\mathcal{G})$ ($\underline{Sh}_{\mathbb{R}}(\mathcal{G})$) of $\mathcal{G}$-sheaves of complex (or real) vector spaces, which will simply be denoted by $Sh(\mathcal{G})$. If $P : \mathcal{G} \longrightarrow \mathcal{H}$ is a morphism of étale groupoids, it induces a functor $P^* : \underline{Sh}_s(\mathcal{H}) \longrightarrow \underline{Sh}_s(\mathcal{G})$. This construction is natural; in particular, a Morita equivalence of étale groupoids $\mathcal{G} \longrightarrow \mathcal{H}$ induces an equivalence between their categories of sheaves; see 1.1, 2.2 in [72] (or our 1.2.3).

1.2.2 Examples:

1. For any set or abelian group A the corresponding constant sheaf on $\mathcal{G}^{(0)}$ can be equipped with the trivial $\mathcal{G}$-action. We will refer to $\mathcal{G}$-sheaves of this form as constant $\mathcal{G}$-sheaves; they are simply denoted by A again.

2. The sheaf $\mathcal{A} = \mathcal{C}_{\mathcal{G}^{(0)}}$ of germs of continuous real-valued functions on $\mathcal{G}^{(0)}$ has the natural structure of a $\mathcal{G}$-sheaf: if $g : x \longrightarrow y$ in $\mathcal{G}^{(1)}$ and $\alpha \in \mathcal{A}_y$ is a germ at g, then $\alpha \cdot g$ is defined as the composition $\alpha \circ \tilde{g}$ (cf. 1.1.2). Similarly if $\mathcal{G}$ is a smooth étale groupoid, the sheaf $\Omega^n_{\mathcal{G}^{(0)}}$ of differential n-forms on $\mathcal{G}^{(0)}$ has a structure of a $\mathcal{G}$-sheaf ($n \geq 0$).

3.. The étale map $\alpha_n : \mathcal{G}^{(n)} = \{(g_1, ..., g_n) \in \mathcal{G}^{\ n} : s(g_i) = t(g_{i+1})\} \longrightarrow \mathcal{G}^{(0)}$, $(g_1, ..., g_n) \mapsto t(g_1)$ induces a $\mathcal{G}$-sheaf, denoted $\mathbb{C}[\mathcal{G}^{(n)}]$. Its stalk at $c \in \mathcal{G}^{(0)}$ is the free vector space $\mathbb{C}[\alpha_n^{-1}(c)]$ on the fibers of α_n, and the action is given by $(g_1, ..., g_n)g = (g^{-1}g_1, g_2, ..., g_n)$.

4. Let E be a sheaf on $\mathcal{G}^{(0)}$ (no action). To E we can associate a $\mathcal{G}$-sheaf $E[\mathcal{G}] = E \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} = \{(e, g) : g : x \longrightarrow y, e \in E_y\}$. The sheaf projection is the map $E[\mathcal{G}] \longrightarrow \mathcal{G}^{(0)}$ given by $(e, g) \mapsto s(g)$, while the $\mathcal{G}$-action is given by composition, $(e, g) \cdot h = (e, g \circ h)$. Sheaves (isomorphic to ones) of this form are said to be free $\mathcal{G}$-sheaves. The freeness is expressed by the adjunction property:

$$Hom_{\mathcal{G}}(E[\mathcal{G}], \mathcal{S}) = Hom_{\mathcal{G}^{(0)}}(E, \mathcal{S})$$

for any $\mathcal{G}$-sheaf $\mathcal{S}$.

5. Each of the spaces $\mathcal{G}^{(n)}$ in the nerve of $\mathcal{G}$ (cf. 1.1.13) has the structure of a $\mathcal{G}$-sheaf, with sheaf projection:

$$\varepsilon_n : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(0)}, \quad (x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} \ldots \xleftarrow{g_n} x_n) \mapsto x_n \ ,$$

and the $\mathcal{G}$-action given by composition, $(g_1, ..., g_n) \cdot h = (g_1, ..., g_n h)$. This $\mathcal{G}$-sheaf is denoted $F_{n-1}(\mathcal{G})$. For $n \geq 1$ these sheaves are free, in fact $\mathcal{G}_{n+1} = \mathcal{G}^{(n)}[\mathcal{G}]$. The system of $\mathcal{G}$-sheaves:

$$\cdots \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \ F_2(\mathcal{G}) \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \ F_1(\mathcal{G}) \ \substack{\longrightarrow\\ \longrightarrow} \ F_0(\mathcal{G}) \ , \tag{1.3}$$

has the structure of a simplicial $\mathcal{G}$-sheaf, whose stalk at $x \in \mathcal{G}^{(0)}$ is the nerve of the comma category $x/\mathcal{G}$. This stalk is a contractible simplicial set.

6. For any $\mathcal{G}$-sheaf of sets $\mathcal{S}$, one can form the free abelian $\mathcal{G}$-sheaf $\mathbb{Z}\mathcal{S}$; the stalk of $\mathbb{Z}\mathcal{S}$ at $x \in \mathcal{G}^{(0)}$ is the free abelian group on the stalk $\mathcal{S}_x$. In particular, from (1.3) we obtain a resolution:

$$\ldots \ \xrightarrow{\delta} \mathbb{Z}F_1(\mathcal{G}) \xrightarrow{\delta} \mathbb{Z}F_0(\mathcal{G}) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1.4}$$

of the constant $\mathcal{G}$-sheaf $\mathbb{Z}$, where δ is defined by the alternating sums of the face maps in (1.3).

7. If $\mathcal{G}^{(0)}$ is a topological manifold of dimension d, recall that its orientation sheaf or is given by $or(U) = H^d_c(U; \mathbb{R})^\vee$, (see e.g. [15, 63], and the Appendix for compactly supported cohomology in the case where $\mathcal{G}^{(0)}$ is non-Hausdorff). It has a natural $\mathcal{G}$-action: for any arrow $g : x \longrightarrow y$ in $\mathcal{G}$, let U_x and U_y be neighborhoods of x and y,

so small that $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ has a section σ through g with $t\circ\sigma : U_x \xrightarrow{\sim} U_y$. Then $t\circ\sigma$ induces a map $H_c^d(U_x) \xrightarrow{\sim} H_c^d(U_y)$, so also a map $H_c^d(U_x)^\vee \xrightarrow{\sim} H_c^d(U_y)^\vee$. Hence by taking germs, it gives an action $or_y \longrightarrow or_x$.

Note that if $\mathcal{G}^{(0)}$ is oriented (i.e. as a sheaf on $\mathcal{G}^{(0)}$, or is isomorphic to the constant sheaf $\mathbb{R}$), it is not necessarily constant as a $\mathcal{G}$-sheaf. When it is (i.e. when $\mathcal{G}^{(0)}$ is orientable and any arrow $g : x \longrightarrow y$ gives an orientation-preserving germ $\tilde{g}$, cf. 1.1.2) we say that $\mathcal{G}$ is *orientable*.

1.2.3 Morphisms: A homomorphism of étale groupoids $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ induces an evident functor:

$$\varphi^* : \underline{Sh}_s(\mathcal{G}) \longrightarrow \underline{Sh}_s(\mathcal{K})$$

by pullback (and similarly an exact functor $\varphi^* : \underline{Ab}(\mathcal{G}) \longrightarrow \underline{Ab}(\mathcal{K})$). This functor has a right adjoint:

$$\varphi_* : \underline{Sh}_s(\mathcal{G}) \longrightarrow \underline{Sh}_s(\mathcal{K})\ .$$

For an $\mathcal{K}$-sheaf $\mathcal{S}$, the sheaf $\varphi_*(\mathcal{S})$ on $\mathcal{G}^{(0)}$ is defined for any open set $U \subset \mathcal{G}^{(0)}$ by:

$$\varphi_*(\mathcal{S})(U) = Hom_{\mathcal{K}}(\varphi/U, \mathcal{S}).$$

Here $\varphi/U = \{(y,g) : y \in \mathcal{K}^{(0)}, g : \varphi(y) \longrightarrow x, x \in U\}$, with $\mathcal{K}$-sheaf structure given by $(y,g)h = (y, g\circ\varphi(h))$. The $\mathcal{G}$-action on this sheaf $\varphi_*(\mathcal{G})$ is defined as follows: for $\xi \in \varphi_*(\mathcal{S})_x$ and $g : x' \longrightarrow x$, let U_x be a neighborhood of x so that ξ is represented by an element $\xi \in \varphi_*(\mathcal{S})(U_x)$, and let $U_{x'}$ be so small that $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ has a section $\sigma : U_{x'} \longrightarrow \mathcal{G}^{(1)}$ through g with $t\circ\sigma(U_{x'}) \subset U_x$. Then define $\xi g \in \varphi_*(\mathcal{S})_{x'}$ to be the element represented by the morphism:

$$\tau : \varphi/U_{x'} \longrightarrow \mathcal{S}\ ,\ (y, f : \varphi(y) \longrightarrow z) \mapsto \xi(\sigma(z)\circ f).$$

These adjoint functors φ^* and φ_* together constitute a topos morphism:

$$\varphi : \underline{Sh}_s(\mathcal{K}) \longrightarrow \underline{Sh}_s(\mathcal{G}).$$

If $\varphi : \mathcal{K} \xrightarrow{\sim} \mathcal{G}$ is a Morita equivalence, then this morphism is an equivalence of categories $\underline{Sh}_s(\mathcal{K}) \cong \underline{Sh}_s(\mathcal{G})$. In fact, topos morphisms $\underline{Sh}_s(\mathcal{K}) \longrightarrow \underline{Sh}_s(\mathcal{G})$ correspond exactly to generalized morphisms $\mathcal{K} \longrightarrow \mathcal{G}$, or equivalently, to pairs of homomorphisms $\mathcal{K} \xleftarrow{\sim} \mathcal{H} \longrightarrow \mathcal{G}$.

1.2.4 Invariant sections: Let $\mathcal{S}$ be a $\mathcal{G}$-sheaf. A section $\sigma : \mathcal{G}^{(0)} \longrightarrow \mathcal{S}$ is called invariant if $\sigma(y)g = \sigma(x)$ for any arrow $g : x \longrightarrow y$ in $\mathcal{G}$. We write:

$$\Gamma_{inv}(\mathcal{G}, \mathcal{S})$$

for the set of invariant sections; it is an abelian group if $\mathcal{S}$ is an abelian sheaf. (In fact $\Gamma_{inv}(\mathcal{G}, \mathcal{S}) = \varphi_*(\mathcal{S})$ where $\varphi : \mathcal{G} \longrightarrow 1$ is the morphism into the trivial groupoid).

1.2.5 Cohomology: The category $\underline{Ab}(\mathcal{G})$ is an abelian category with enough injectives [2]; due to this fact, the cohomology of sheaves on $\mathcal{G}$ can be defined and used via elementary homological algebra tools: it is the right derived functor of $\Gamma_{inv}(\mathcal{G}, -)$.

Hence, for an abelian $\mathcal{G}$-sheaf $\mathcal{A}$, *the cohomology groups* $H^n(\mathcal{G};\mathcal{A})$ are the cohomology groups of the complex:

$$\Gamma_{inv}(\mathcal{G};\mathcal{I}^0) \longrightarrow \Gamma_{inv}(\mathcal{G};\mathcal{I}^1) \longrightarrow \dots$$

where $\mathcal{A} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$ is any resolution of $\mathcal{A}$ by injective $\mathcal{G}$-sheaves. It is obvious that a homomorphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ induces homomorphisms in cohomology:

$$\varphi^* : H^n(\mathcal{G};\mathcal{A}) \longrightarrow H^n(\mathcal{K};\varphi^*\mathcal{A}) \quad (n \geq 0) .$$

If φ is a Morita equivalence, these are isomorphisms, since $\varphi^* : \underline{Ab}(\mathcal{G}) \tilde{\longrightarrow} \underline{Ab}(\mathcal{K})$.

1.2.6 Ext functor: More generally, one can define the bi-functors $Ext^*_{\mathcal{G}}(-,-) : Sh(\mathcal{G}) \times Sh(\mathcal{G}) \longrightarrow \underline{Vs}$ (due to the applications we have in mind, we make the harmless assumption that we work over $\mathbf{C}$, so that $\underline{Vs}$ denotes the category of complex vector spaces, and $Sh(\mathcal{G}) = \underline{Sh}_{\mathbb{C}}(\mathcal{G})$ as in 1.2.1):

$$Ext^*_{\mathcal{G}}(\mathcal{A},\mathcal{B}) = R^* Hom_{Sh(\mathcal{G})}(\mathcal{A},-)(\mathcal{B}),$$

with the particular case $Ext^*_{\mathcal{G}}(\mathbb{C},-) = H^*(\mathcal{G};-)$, R^* stands for the right derived functors [62, 97].

Homological algebra provides us an alternative description of the vector spaces $Ext^p_{\mathcal{G}}(\mathcal{A},\mathcal{B})$ by means of Yoneda extensions (see [97]). For $p \geq 1$, the elements in $Ext^p_{\mathcal{G}}(\mathcal{A},\mathcal{B})$ are represented by p-extensions of $\mathcal{A}$ by $\mathcal{B}$ i.e. exact sequences in $Sh(\mathcal{G})$:

$$u : \quad 0 \longrightarrow \mathcal{B} \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_p \longrightarrow \mathcal{A} \longrightarrow 0.$$

The equivalence relation is generated by: $u \simeq \tilde{u}$ whenever there exists a morphism of complexes $u \mapsto \tilde{u}$. According to this, there is a simple description of the cup-product $Ext^p_{\mathcal{G}}(\mathcal{A},\mathcal{B}) \times Ext^q_{\mathcal{G}}(\mathcal{B},\mathcal{C}) \longrightarrow Ext^{p+q}_{\mathcal{G}}(\mathcal{A},\mathcal{C})$ for $\mathcal{A},\mathcal{B},\mathcal{C} \in Sh(\mathcal{G})$ as the concatenation of exact sequences (for the general setting of abelian categories see [97], pp. 76-80).

We have also a simple description of the cap-products: for any homological δ-functor (2.1.1 in [97]) $L_* : Sh(\mathcal{G}) \longrightarrow \underline{Vs}$ there are cap-product maps

$$L_n(\mathcal{A}) \times Ext^p_{\mathcal{G}}(\mathcal{A},\mathcal{B}) \xrightarrow{\cap} L_{n-p}(\mathcal{B}) \ ,$$

($p \geq 0, n, p$ integers, $\mathcal{A},\mathcal{B} \in Sh(\mathcal{G})$). If $p = 0$, $Ext^0_{\mathcal{G}}(\mathcal{A},\mathcal{B}) = Hom_{Sh(\mathcal{G})}(\mathcal{A},-)(\mathcal{B})$ and $\cap$ is the covariance of L_n. If $p = 1$, $u \in Ext^1_{\mathcal{G}}(\mathcal{A},\mathcal{B})$, and the cap-product by u : $- \cap u : L_n(\mathcal{A}) \longrightarrow L_{n-1}(\mathcal{B})$ is the boundary of the long exact sequence associated to (any) short exact sequence $0 \longrightarrow \mathcal{B} \longrightarrow X \longrightarrow \mathcal{A} \longrightarrow 0$ representing u. If $p \geq 2$, we iterate the case $p = 1$.

For any $\mathcal{G}$-sheaf $\mathcal{A}$ as before, there is an obvious morphism $Ext^p_{\mathcal{G}}(\mathbb{C},\mathbb{C}) \longrightarrow Ext^p_{\mathcal{G}}(\mathcal{A},\mathcal{A})$ (tensoring by $\mathcal{A}$); we get in particular an action of the cohomology on any homological δ-functor:

$$L_n(\mathcal{A}) \times H^p(\mathcal{G},\mathbb{C}) \longrightarrow L_{n-p}(\mathcal{A}) \ , \ \mathcal{A} \in Sh(\mathcal{G}) \ .$$

1.2.7 Connection with the classifying space: Any $\mathcal{G}$-sheaf $\mathcal{S}$ gives rise to a sheaf $\tilde{\mathcal{S}}$ on the classifying space $B\mathcal{G}$ and there are isomorphisms $H^*(\mathcal{G};\mathcal{S}) \simeq H^*(B\mathcal{G};\tilde{\mathcal{S}})$. This was conjectured by Haefliger and proved in [74].

1.2.8 Leray spectral sequence: For any morphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ and any $\mathcal{K}$-sheaf $\mathcal{A}$, there is a Leray spectral sequence

$$E_2^{p,q} = H^p(\mathcal{G}; R^q\varphi_*\mathcal{A}) \Longrightarrow H^{p+q}(\mathcal{K}; \mathcal{A}) \ .$$

(The $\mathcal{G}$-sheaf $R^q\varphi_*\mathcal{A}$ can be explicitly described as the sheaf associated to the presheaf $U \longrightarrow H^q(\varphi/U; \mathcal{A})$ where φ/U is the groupoid associated to the (right) action of $\mathcal{K}$ on the space $(\varphi/U)_0$ used in 1.2.3. See also [2].

1.2.9 Basic spectral sequence: Let $\mathcal{G}$ be an étale groupoid, and let $\mathcal{A}$ be a $\mathcal{G}$-sheaf. By pull-back along $\varepsilon_n : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(0)}$ (see 1.2.2.5), $\mathcal{A}$ induces a sheaf $\varepsilon_n^*(\mathcal{A})$ on $\mathcal{G}^{(n)}$ which we often simply denote by $\mathcal{A}$ again. Consider for each p and q the sheaf cohomology $H^q(\mathcal{G}^{(p)}, \mathcal{A})$ of the space $\mathcal{G}^{(p)}$. For a fixed q, these form a cosimplicial abelian group, and there is a basic spectral sequence:

$$H^pH^q(\mathcal{G}^{(*)}, \mathcal{A}) \Longrightarrow H^{p+q}(\mathcal{G}; \mathcal{A}) \ .$$

(It arises from the double complex $\Gamma(\mathcal{G}^{(p)}, \varepsilon_p^*\mathcal{I}^q)$ where $\mathcal{A} \longrightarrow \mathcal{I}^*$ is an injective resolution.)

It follows that if $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow \dots$ is any resolution by $\mathcal{G}$-sheaves $\mathcal{A}^q$ with the property that $\varepsilon_p^*(\mathcal{A}^q)$ is an acyclic sheaf on $\mathcal{G}^{(p)}$, then $H^*(\mathcal{G}; \mathcal{A})$ can be computed by the double complex

$$\Gamma(\mathcal{G}^{(p)}; \varepsilon_p^*(\mathcal{A}^q)) \ .$$

1.2.10 Čech spectral sequence: An open set $U \subset \mathcal{G}^{(0)}$ is called saturated if for any arrow $g : x \longrightarrow y$ in $\mathcal{G}$, one has $s(g) \in U$ iff $t(g) \in U$. For such a U there is an evident "full" subgroupoid $\mathcal{G}|_U \subset \mathcal{G}$, with U as space of objects. If $\mathcal{U}$ is an open cover of $\mathcal{G}^{(0)}$ by saturated opens, there is a spectral sequence:

$$\check{H}^p(\mathcal{U}; \mathcal{H}^q(\mathcal{A})) \Longrightarrow H^{p+q}(\mathcal{G}; \mathcal{A})$$

where $\mathcal{H}^q(\mathcal{A})$ is the presheaf $U \mapsto H^q(\mathcal{G}|_U; \mathcal{A}|_U)$.

1.2.11 Hypercohomology: For a cochain complex $\mathcal{A}^*$ of abelian $\mathcal{G}$-sheaves *the hypercohomology groups* $\mathbb{H}^n(\mathcal{G}; \mathcal{A}^*)$ are defined in the usual way, as the cohomology groups of the double complex $\Gamma_{inv}(\mathcal{G}; \mathcal{I}^*)$ where $\mathcal{A}^* \longrightarrow \mathcal{I}^*$ is a quasi-isomorphism into a cochain complex of injectives. (If $\mathcal{A}^*$ is concentrated in degree 0 one recovers the ordinary cohomology defined in 1.2.5). For each $q \in \mathbb{Z}$ denote by $\mathcal{H}^q(\mathcal{A}^*)$ the q-th cohomology $\mathcal{G}$-sheaf of $\mathcal{A}^*$. If $\mathcal{A}^*$ is bounded below, there is a spectral sequence for hypercohomology analogous to the one in 1.2.9:

$$H^p\mathbb{H}^q(\mathcal{G}^{(*)}; \mathcal{A}^*) \Longrightarrow \mathbb{H}^{p+q}(\mathcal{G}; \mathcal{A}^*) \ .$$

1.2.12 Internal hom: For two $\mathcal{G}$-sheaves $\mathcal{A}$ and $\mathcal{B}$ the sheaf $\underline{Hom}(\mathcal{A}, \mathcal{B})$ on $\mathcal{G}^{(0)}$ carries a natural $\mathcal{G}$-action, hence gives a $\mathcal{G}$-sheaf $\underline{Hom}_{\mathcal{G}}(\mathcal{A}, \mathcal{B})$ (or simply $\underline{Hom}(\mathcal{A}, \mathcal{B})$ again). We recall that:

$$\Gamma_{inv}(\mathcal{G}, \underline{Hom}_{\mathcal{G}}(\mathcal{A}, \mathcal{B})) = Hom(\mathcal{A}, \mathcal{B})$$

is the group of action preserving homomorphisms, i.e. morphisms in the category $\underline{Ab}(\mathcal{G})$. The derived functor of:

$$\underline{Hom}(\mathcal{A},-):\underline{Ab}(\mathcal{G})\longrightarrow\underline{Ab}(\mathcal{G})$$

will be denoted by $R^p\underline{Hom}(\mathcal{A},-)$ or by $\underline{Ext^p}(\mathcal{A},-)$.

1.2.13 Cohomology via Bar-complexes: We will need an explicit complex computing the cohomology $H^*(\mathcal{G};\mathcal{A})$ for an étale groupoid $\mathcal{G}$ and a left $\mathcal{G}$-sheaf $\mathcal{A}$ (i.e. $\mathcal{A}\in\underline{Ab}(\mathcal{G}^{op})$ cf. 1.2.1): the bar complex. This is well known (see [2, 72] or [55] for a direct approach), and is a reformulation of 1.2.9. The bar complex $B(\mathcal{G};\mathcal{A})$ is defined by:

$$B^p(\mathcal{G};\mathcal{A})=\Gamma(\mathcal{G}^{(p)};\epsilon_p^*\mathcal{A})\ .$$

In other words, an element of $B^p(\mathcal{G};\mathcal{A})$ is a continuous function f associating to a sequence $(g_1,\ldots,g_p)$ of composable arrows of $\mathcal{G}$ an element in $\mathcal{A}_{t(g_1)}$. One has a boundary operator $\delta:B^p(\mathcal{G};\mathcal{A})\longrightarrow B^{p+1}(\mathcal{G};\mathcal{A})$:

$$\delta f(g_0,g_1,\ldots,g_p)=g_0f(g_1,\ldots,g_p)-\sum_{i=0}^{p-1}(-1)^if(g_1,\ldots,g_ig_{i+1},\ldots,g_p)+(-1)^{p+1}f(g_0,g_1,\ldots,g_{p-1}).$$

Note that $B(\mathcal{G};\mathcal{A})$ can be viewed as the cochain complex underlying a cosimplicial group. We say that $\mathcal{A}$ is $\mathcal{G}$-acyclic if $\epsilon_p^*\mathcal{A}\in Sh(\mathcal{G}^{(p)})$ is acyclic for all p's. It follows from the spectral sequence of 1.2.1 that, if $\mathcal{A}$ is $\mathcal{G}$-acyclic, then $H^*(\mathcal{G};\mathcal{A})$ is computed by the bar complex $B(\mathcal{G};\mathcal{A})$. More generally, if $\mathcal{S}\in\underline{Ab}(\mathcal{G}^{op})$, then $H^*(\mathcal{G};\mathcal{S})$ is computed by the the double complex $B^\bullet(\mathcal{G};\mathcal{A}^\bullet)$ where $\mathcal{S}\longrightarrow\mathcal{A}^\bullet$ is any resolution in $\underline{Ab}(\mathcal{G}^{op})$ by $\mathcal{G}$-acyclic sheaves.

1.3 Leaf spaces, and étale groupoids

In this section we recall the definition of the holonomy groupoid, as a model for the leaf-space [57, 99, 25].

1.3.1 Foliations: Recall that a *foliation* of the manifold M is an integrable subbundle $\mathcal{F}$ of the tangent bundle of M (here integrable means that if the vector fields of M, $X,Y\in\mathcal{X}(M)$, are sections of $\mathcal{F}$, then so is their Lie bracket $[X,Y]\in\mathcal{X}(M)$). Usually we denote by q the dimension of $\mathcal{F}$ (called the codimension of the foliation), and by n the dimension of M. By the Frobenius theorem, $\mathcal{F}$ is locally of type $Ker(df)$, where $f:U\longrightarrow\mathbb{R}^q$ is a submersion defined on some open $U\subset M$. Hence we find a family $\{(U,f_U)_{U\in\mathcal{U}}\}$ of pairs (U,f_U) as before, indexed by an open cover $\mathcal{U}$ of M, such that $\mathcal{F}|_U=Ker(df_U)$.

For any $x\in U\cap V$, $U,V\in\mathcal{U}$ one has (by the implicit function theorem):

$$(f_V|_W)=\gamma_{U,V}^W\circ(f_U|_W)\ ,$$

for some $W\in\mathcal{V}(x)$ a small neighborhood, and some diffeomorphism $\gamma_{U,V}^W:f_U(W)\longrightarrow f_V(W)$. By the same theorem, $\gamma_{U,V}(x):=germ_x(\gamma_{U,V}):(\mathbb{R}^q,f_U(x))\longrightarrow(\mathbb{R}^q,f_V(x))$

depends just on $x \in U \cap V$. The basic properties of the data $\{U, f_U, \gamma_{U,V} : U \in \mathcal{U}\}$ are better expressed in the language of groupoids: they define a homomorphism of groupoids $M_{\mathcal{U}} \longrightarrow \Gamma^q$ (for the definition of Γ^q, see Example 4 in 1.1.3), or, by the Morita equivalence $M_{\mathcal{U}} \cong M$ (see 1.1.3), a morphism in *Étale* ("the classifying map of $\mathcal{F}$"):

$$f_{\mathcal{F}} : M \longrightarrow \Gamma^q \ . \tag{1.5}$$

One can recover $\mathcal{F}$ from this morphism $\{U, f_U, \gamma_{U,V} : U \in \mathcal{U}\}$, and this corresponds to the definition of a foliation in terms of *Haefliger cocycles.*

We recall that the cocycle $\{U, f_U, \gamma_{U,V} : U \in \mathcal{U}\}$ can be chosen global, i.e. such that $\gamma_{U,V}(x)$ are induced by globally defined diffeomorphisms $\gamma_{U,V} : f_U(U\cap V) \longrightarrow f_V(U\cap V)$, satisfying the cocycle condition:

$$f_V = \gamma_{U,V}\circ f_U, \quad \text{on} \quad U\cap V \ .$$

By the local form of submersions, and eventually refining $\mathcal{U}$, we may assume there exists an atlas $\{\phi_U : U \xrightarrow{\sim} \mathbb{R}^n : U \in \mathcal{U}\}$ of M, and diffeomorphisms $\psi_U : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ such that $\psi_U f_U \phi_U^{-1} = pr_2$ where $pr_2 : \mathbb{R}^n = \mathbb{R}^{n-q}\times\mathbb{R}^q \longrightarrow \mathbb{R}^q$ is the second projection. It follows that the atlas

$$\{\phi_U : U \xrightarrow{\sim} \mathbb{R}^{n-q}\times\mathbb{R}^q : U \in \mathcal{U}\ \}$$

of M has the basic property that the change of coordinates is (globally) of type:

$$\phi_U\phi_V^{-1}(x,y) = (g(x,y), h(y)) \ , \tag{1.6}$$

Since $\mathcal{F}|_U = Ker(dpr_2\circ\phi_U)$, one can rediscover $\mathcal{F}$ from this atlas. Any atlas of M satisfying $\mathcal{F}|_U = Ker(dpr_2\circ\phi_U)$, and for which the change of coordinates is locally of type (1.6) is called a foliation atlas for $\mathcal{F}$ (and this corresponds to the definition of a foliation in terms of *foliation atlases*).

The fibers $f_U^{-1}(y) = \phi_U^{-1}(\mathbb{R}^{n-q}\times y')$ (where $y' = \psi_U(y)$) of f_U are called *the plaques* of $\mathcal{F}$ in U. The change of coordinates preserves the decomposition of U into plaques, hence one can globally amalgamate these plaques into connected $(n-q)$-dimensional manifolds, called *the leaves* of $(M,\mathcal{F})$.

1.3.2 The leaf space and the holonomy pseudogroup: The *space of leaves* $M/\mathcal{F}$ of $\mathcal{F}$ is the quotient of M obtained by identifying the points which lie in the same leaf. As a topological space (with the quotient topology), it is usually badly behaved. The holonomy pseudogroup and the holonomy groupoid are basic tools in approaching such bad spaces.

By a *transversal* T of the foliation we mean an immersion $i = i_T : T \longrightarrow M$ which is transversal to $\mathcal{F}$. Such transversals should be viewed (heuristically) as manifolds parameterizing the leaves $\{L_t\}_{t\in T}$ of $(M,\mathcal{F})$ (where L_t is the leaf through $i(t)$), and defining opens in the leaf space $M/\mathcal{F}$. To parameterize the entire leaf space, one uses a *complete transversal* T, i.e. one which meets each leaf at least once.

Let T be a complete transversal. To describe the way the points in T are glued (to form points of $M/\mathcal{F}$), we need the holonomy. Given $t_0, t_1 \in T$ lying in the same leaf L, and a path $\sigma : [0,1] \longrightarrow L$ from t_0 to t_1, the translation along the plaques in a small neighborhood of $\sigma[0,1]$ defines a diffeomorphism germ $\phi_\sigma : (T, t_0) \longrightarrow (T, t_1)$. These generate a pseudogroup $\Psi = \Psi_T(M/\mathcal{F})$ on T, i.e. a family of local diffeomorphisms $T \supset Dom(\phi) \xrightarrow{\phi} \phi(Dom(\phi)) \subset T$ between opens of T, satisfying:

(i) $\phi \in \Psi \Longrightarrow \phi^{-1} \in \Psi$,

(ii) $\phi \in \Psi \Longrightarrow \phi|_U \in \Psi$ for all opens $U \subset Dom(\phi)$,

(iii) $\phi, \psi \in \Psi \Longrightarrow \phi \circ \psi \in \Psi$, whenever the composition is well defined,

(iv) If ϕ is a local diffeomorphism such that, for any $x \in dom(\phi)$ there exists $U \in \mathcal{V}(x)$ such that $\phi|_U \in \Psi$, then $\phi \in \Psi$.

The pseudogroup $\Psi_T(M/\mathcal{F})$ is called *the holonomy pseudogroup* of $(M, \mathcal{F})$ associated to the complete transversal T. Denoting by $\mathcal{O}_T(M/\mathcal{F})$ the discrete category whose objects are the opens $U \subset T$, and whose morphisms $g : U \longrightarrow V$ are the embeddings with the property that $g : U \longrightarrow g(U)$ is in $\Psi_T(M/\mathcal{F})$, $\mathcal{O}_T(M/\mathcal{F})$ should be viewed as a good model for the lattice of opens of (the "bad") $M/\mathcal{F}$. These type of embeddings g are called *holonomy embeddings* of $(M, \mathcal{F})$.

We will see that the pseudogroup $\Psi_T(M/\mathcal{F})$ does not depend on the choice of T in an essential way. Also, for particular T's, it has simpler descriptions. For instance, choosing $\{U, f_U, \gamma_{U,V}\}$, and ϕ_U as in 1.3.1, one can take $T = \coprod_{U \in \mathcal{U}} \mathbb{R}^q$, and $\Psi_T(M/\mathcal{F})$ is simply the pseudogroup of local diffeomorphisms of T generated by the $\gamma_{U,V}$'s.

1.3.3 The leaf space and the holonomy groupoid: The *holonomy groupoid* $Hol(M, \mathcal{F})$ (and its reduced versions) is the basic object for studying the leaf space $M/\mathcal{F}$. Its space of objects is M, and its morphisms describe the way the points in the same leaf are "glued". Hence, given $x, y \in M$, there exists a morphism between x and y iff they lie in the same leaf, say L. In this case, one has an equivalence relation on the set of paths $\sigma : [0, 1] \longrightarrow L$ from x to y: two such paths σ and σ' are equivalent if they induce the same germ $\phi_\sigma : (T, x) \longrightarrow (T, y)$ (see 1.3.2) for some (and then any) complete transversal T passing through x and y; this is the case, for instance, if σ and σ' are homotopic. The equivalence classes are called the holonomy paths from x to y, and they define the arrows in $Hol(M, \mathcal{F})$. Let us mention that the space of arrows of $Hol(M, \mathcal{F})$ comes equipped with a canonical structure of (usually non-Hausdorff) manifold, so that $Hol(M, \mathcal{F})$ becomes a smooth groupoid (see [99] for details). Restricting the space of objects to a complete transversal T, the resulting groupoid $Hol_T(M, \mathcal{F})$ is a (smooth) étale groupoid, and, up to an isomorphism in $\acute{E}tale$, it does not depend on the choice of T: for any two complete transversals T, T', $Hol_T(M, \mathcal{F})$, and $Hol_{T'}(M, \mathcal{F})$ are Morita equivalent (actually they are both Morita equivalent to $Hol(M, \mathcal{F})$, cf. 1.1.11).

These *reduced holonomy groupoids* $Hol_T(M, \mathcal{F})$ can be constructed directly from the holonomy pseudogroup $\Psi_T(M/\mathcal{F})$. Let us recall that, given any pseudogroup Ψ on a manifold T, one has an associated étale groupoid $\mathcal{G}(\Psi)$ whose space of objects is T, whose arrows from x to y are germs of diffeomorphisms defined by elements of Ψ, and whose space of arrows is endowed with the sheaf topology. One has:

$$Hol_T(M, \mathcal{F}) \cong \mathcal{G}(\Psi_T(M/\mathcal{F})) .$$

For the particular choice of T as at the end of 1.3.2, $Hol_T(M, \mathcal{F})$ is simply the groupoid whose space of objects is $\coprod_{U \in \mathcal{U}} \mathbb{R}^q$, and has as arrows the germs induced by the $\gamma_{U,V}$'s.

For later use, let us recall that conversely, any étale groupoid $\mathcal{G}$ defines a pseudogroup $\Psi = \Psi(\mathcal{G})$ on $\mathcal{G}^{(0)}$, namely the pseudogroup generated by the germs $\tilde{g}$ associated to arrows $g \in \mathcal{G}$ (see 1.1.2). When $\mathcal{G}$ is *effective* (in the sense that different arrows induce different germs of diffeomorphisms), one has $\mathcal{G} \cong \mathcal{G}(\Psi)$. Since all groupoids of type $\mathcal{G}(\Psi)$ (Ψ = any pseudogroup) are effective, it follows that the construction:

$$\{\text{Pseudogroups}\} \longrightarrow \{\text{Étale Groupoids}\}\ ,\ \Psi \mapsto \mathcal{G}(\Psi)$$

exhausts all the effective groupoids, and can be used to define the category of pseudogroups, so that it becomes equivalent to the full subcategory $\acute{E}tale_e$ of effective étale groupoids.

1.3.4 Warning: *When making precise statements about the leaf space, one should understand that we are working on the category of smooth étale groupoids (viewed as an enlargement of the category of manifolds), and $M/\mathcal{F}$ is just the notation for the étale holonomy groupoid $Hol_T(M,\mathcal{F})$ (defined up to an isomorphism in Étale).*

1.3.5 Transversal structures: As mentioned in the introduction, and indicated by the "warning" above, *transversal structures* are structures which live on $Hol_T(M,\mathcal{F})$ (and morally on $M/\mathcal{F}$). For instance, this is the case for the category of *transversal sheaves*, $Sh(M/\mathcal{F})$ (see 1.2.1). Hence, to give a transversal sheaf means to give a sheaf $\mathcal{A}_T$ on a (and then on any) complete transversal T, endowed with a continuous action of the holonomy. According to this general principle, the cohomology of the leaf space is simply the cohomology of the reduced holonomy groupoid:

$$H^*(M/\mathcal{F}) := H^*(Hol_T(M,\mathcal{F}))$$

(which is well defined, cf. the Morita invariance of cohomology), or, equivalently, the cohomology of the classifying space of the foliation (cf. 1.2.7). In the same way, $H^*(M/\mathcal{F};\mathcal{A})$ is defined for any transversal sheaf $\mathcal{A}$.

Let us remark also that heuristic maps like the projection:

$$\pi_{\mathcal{F}} : M \longrightarrow M/\mathcal{F}$$

have a precise meaning in $\acute{E}tale$: choosing $\{f_U, \gamma_{U,V}, \phi_U\}$ and T as at the end of 1.3.2 (see also 1.3.3), there is an obvious map $M_{\mathcal{U}} \longrightarrow Hol_T(M,\mathcal{F})$, which, together with the isomorphism (i.e. Morita equivalence of groupoids) $M_{\mathcal{U}} \cong M$, defines our $\pi_{\mathcal{F}}$. Using the obvious map $f'_{\mathcal{F}} : Hol_T(M,\mathcal{F}) \longrightarrow \Gamma^q$, the classifying map (1.5) decomposes as:

$$\begin{tikzcd} M \arrow[rr, "f_{\mathcal{F}}"] \arrow[dr, "\pi_{\mathcal{F}}"'] & & \Gamma^q \\ & M/\mathcal{F} \arrow[ur, "f'_{\mathcal{F}}"'] & \end{tikzcd}$$

1.4 Characteristic classes for foliations

In this section we recall the various constructions of the classical characteristic classes, the vanishing theorem of Bott [11], as well as the construction of the exotic characteristic classes for foliations [13].

1.4.1 Classical characteristic classes; principal bundles: In the classical picture of characteristic classes of principal G-bundles (where G is a connected Lie group), one associates to any such bundle $P \longrightarrow M$ a $H^*(M)$-valued characteristic map, whose source is the algebra of universal characteristic classes for G-bundles. A topological realization of this is:

$$k_P := f_P^* : H^*(BG) \longrightarrow H^*(M) \ , \qquad\qquad (topological)$$

where BG is the classifying space of G, and $f_P : M \longrightarrow BG$ is the classifying map of P (unique up to homotopy).

We recall an alternative geometric construction of *(topological)*, usually known as the Chern-Weil construction. Consider:

$$I(G) : = S(\mathfrak{g}^*)^G$$

the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}$ of G, graded by $|f| : = 2deg(f)$, where $deg(f)$ is the degree of f as a polynomial.

Recall also that the DG algebra $\Omega^*(P)$ comes equipped with the Lie derivatives $L_v : = L_{\tilde{v}}$ and the interior products $i_v : = i_{\tilde{v}}$ (for $v \in \mathfrak{g}$), where $\tilde{v}$ is the induced vector field on P:

$$\tilde{v}(f)(p) = \frac{d}{dt}_{|_{t=0}} f(pexp(-tv)) \ , \ \text{for } f \in C^\infty(P) \ .$$

They have the basic properties:

(i) $L_{[v,w]} = [L_v, L_w]$, $i_{[v,w]} = [i_v, L_w]$, $\forall\ v, w \in \mathfrak{g}$,

(ii) $i_v^2 = 0$, $\forall\ v \in \mathfrak{g}$,

(iii) $L_v = di_v + i_v d$, $\forall\ v \in \mathfrak{g}$.

Remark that the L_v's are the infinitesimal version of the action of G on $\Omega^*(P)$:

$$L_v(\omega) = \frac{d}{dt}_{|_{t=0}} exp(tv)\omega \ , \tag{1.7}$$

and the second relation on (i) follows from the compatibility of i_v with this action:

(i') $Ad_g(i_v) = i_{ad_g(v)}$, $\forall\ g \in G,\ v \in \mathfrak{g}$

In general, a $\mathfrak{g}$- *DG algebra* is any DG algebra Ω^*, endowed with degree 0 derivations $L_v : \Omega^* \longrightarrow \Omega^*$, and degree -1 antiderivation $i_v : \Omega^* \longrightarrow \Omega^{*-1}$, linear on $v \in \mathfrak{g}$, satisfying $(i) - (iii)$ above. A *G-DG algebra* is a Frechet DG algebra Ω^*, endowed with a (smooth) action of G (compatible with the DG algebra structure), and antiderivations $i_v : \Omega^* \longrightarrow \Omega^{*-1}$, satisfying (i'), (ii), (iii) above (where L_v's are the defined by (1.7)). Clearly, any G-DG algebra structure has an underlying $\mathfrak{g}$- DG one. One defines the space of *basic elements*:

$$\Omega^*_{basic} : = \{\omega \in \Omega^* : g\omega = \omega, i_v(\omega) = 0,\ \forall\ g \in G,\ v \in \mathfrak{g}\} \ ,$$

For instance $\Omega^*(P)_{basic} \cong \Omega^*(M)$. Also, if G is connected, one has:

$$\Omega^*_{basic} = \{\omega \in \Omega^* : L_v(\omega) = 0, i_v(\omega) = 0,\ \forall\ \ v \in \mathfrak{g}\} \ .$$

Let ∇ be a connection on the principal G-bundle P, given by a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, and form its curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P; \mathfrak{g})$. Any $f \in I(G)$ defines a closed form $f(\Omega) \in \Omega^*(P)_{basic} \cong \Omega^*(M)$. Using the DeRham model for cohomology, one gets an algebra homomorphism (the Chern-Weil homomorphism):

$$k(\nabla) : I(G) \ni f \mapsto [f(\Omega)] \in H^*(M) \ , \qquad\qquad (geometrical)$$

which does not depend on the choice of the connection. When G is compact, the two characteristic maps *(topological)* and *(geometrical)* coincide via the isomorphism:

$$H^*(BG) \cong I(G) \ .$$

Using $\Omega^*(P)$ as source of inspiration, one can define the notion of connection, and the induced Chern-Weil homomorphism $k(\nabla) : I(G) \longrightarrow H^*_{basic}(\Omega)$ for any commutative G-DG algebra Ω^*. Let us mention that such a connection can be given by a connection 1-form, i.e. a G-invariant element $\omega \in \Omega^1 \otimes \mathfrak{g}$ such that $i_v(\omega) = 1 \otimes v$, for all $v \in \mathfrak{g}$.

We now recall how *the Weil complex* [23] can be used to formalize the previous geometric constructions of the characteristic map. The idea is to interpret the connection 1-form ω, and the curvature Ω as linear maps $\Lambda^1(\mathfrak{g}^*) \longrightarrow \Omega^*(P)$, and $S^1(\mathfrak{g}^*) \longrightarrow \Omega^*(P)$, respectively. They induce an algebra map $W(\mathfrak{g}) := \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \longrightarrow \Omega^*(P)$, compatible with the Lie derivatives and interior products, and there is only one choice of a grading and a differential on $W(\mathfrak{g})$, so that the previous map is a DG map. For instance $deg(\alpha \otimes \omega) = 2deg(\alpha) + deg(\omega)$. Moreover, there is a natural G-DG algebra structure on $W(\mathfrak{g})$ (the action of G is the diagonal one, while the interior products act just on $\Lambda(\mathfrak{g}^*)$ by insertion of v). One can also define $W(\mathfrak{g})$ in terms of generators and relations. If E_a $(1 \leq a \leq dim(\mathfrak{g}))$ is a linear basis of $\mathfrak{g}$, and $f^c_{a,b}$ are the structure constants:

$$[E_a, E_b] = \sum_c f^c_{a,b} E_c \ ,$$

then $W(\mathfrak{g})$ is the free commutative DG algebra generated by symbols ρ^a of degree 1 (corresponding to the dual basis in $\mathfrak{g}^* \subset \Lambda(\mathfrak{g}^*)$), Ω^a of degree 2 (corresponding to the dual basis in $\mathfrak{g}^* \subset S(\mathfrak{g}^*)$), with the boundary:

$$\begin{aligned} d(\rho^a) &= \Omega^a - \frac{1}{2} \sum_{b,c} f^a_{b,c} \rho^b \rho^c \\ d(\Omega^a) &= \sum_{b,c} \Omega^b \rho^c \ . \end{aligned} \tag{1.8}$$

On this basis,

$$i_{E_a}(\rho_b) = \delta^a_b, \ i_{E_a}(\rho_b) = 0,$$

$$L_{E_a}(\rho_b) = \sum_c f^b_{c,a} \rho^c, \ L_{E_a}(\Omega^b) = \sum_c f^b_{c,a} \Omega^c \ .$$

The Weil algebra comes equipped with a canonical connection (given by the connection 1-form $\sum_a \rho^a \otimes E_a$), and it plays the role of classifying algebra for connections on principal G-bundles [23]: there is a $1-1$ correspondence between connections on P, and DG algebra maps $W(\mathfrak{g}) \longrightarrow \Omega(P)$ compatible with the Lie derivatives and the interior products. $W(\mathfrak{g})$ has also the basic property $W(\mathfrak{g})_{basic} \cong I(G)$. Hence, any connection ∇ on P induces a DG map:

$$k(\nabla) : W(\mathfrak{g}) \longrightarrow \Omega^*(P) \ , \tag{1.9}$$

and then a chain map:

$$k(\nabla) : I(G) \cong W(\mathfrak{g})_{basic} \longrightarrow \Omega^*(P)_{basic} \cong \Omega^*(M),$$

which coincides (in cohomology) with *(geometrical)*. The same construction applies to any commutative G-DG algebra Ω^*.

When G is not necessarily compact, *(geometrical)* does not see all the geometric characteristic classes. Passing in (1.9) to K- basic elements (K = maximal compact subgroup of G), and using the fact that $\Omega^*(P)_{K-basic} \cong \Omega^*(P/K)$ still computes $H^*(M)$ (since $P/K \longrightarrow M$ is a principal bundle with the contractible fiber G/K), one gets a better characteristic map (which still does not depend on the choice of ∇):

$$k_P^{geom} := k(\nabla)_* : H^*(W(\mathfrak{g}, K)) \longrightarrow H^*(M) \ , \tag{1.10}$$

where $W(\mathfrak{g}, K)$ denotes the subcomplex of $W(\mathfrak{g})$ consisting on K-basic elements. Denoting by

$$j : I(G) \cong W(\mathfrak{g})_{basic} \longrightarrow W(\mathfrak{g})_{K-basic} = W(\mathfrak{g}, K)$$

the obvious inclusion, the two maps *(geometrical)* and (1.10) are related by the commutative diagram:

$$\begin{tikzcd}
I(G) \arrow[dr, "k_P"] \arrow[dd, "j_*"'] & \\
& H^*(M) \\
H^*(W(\mathfrak{g}, K)) \arrow[ur, "k_P"'] &
\end{tikzcd}$$

1.4.2 Classical characteristic classes; vector bundles: In the case of real vector bundles $E \longrightarrow M$ (i.e. $G = GL_q(\mathbb{R})$), one gets the Pontrjagin classes $p_i(E) \in H^{4i}(M)$, $1 \leq q \leq [\frac{q}{4}]$, $q = dim(E)$ obtained via the characteristic map:

$$k_E : H^*(BGL_q) \cong \mathbf{C}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \longrightarrow H^*(M). \tag{1.11}$$

We briefly recall the direct geometric construction of this map. Choosing a connection ∇ on the vector bundle E, any invariant polynomials on gl_n, $f \in I_q(\mathbb{R}) := I(GL_q)$, induces a global closed form $k_f(\nabla) := f(\nabla^2) \in \Omega^*(M)$, obtained by evaluating f on the local matrix coefficients of the curvature ∇^2 of ∇. One knows that the algebra $I_q(\mathbb{R})$ is a polynomial algebra $\mathbb{R}[c_1, c_2, \ldots, c_q]$ with generators of degree $deg(c_i) = 2i$. Hence one has the induced map:

$$k(\nabla) : I_q(\mathbb{R}) = \mathbb{R}[c_1, c_2, \ldots, c_q] \longrightarrow \Omega^*(M), \ f \mapsto k_f(\nabla) \ . \tag{1.12}$$

Given two connections ∇_0, ∇_1, one can form their affine combination $t\nabla_0 + (1-t)\nabla_1$ (which is a connection on the vector bundle $pr_1^*(E)$ over $M \times \mathbb{R}$), and the associated forms $k_f(t\nabla_0+(1-t)\nabla_1) \in \Omega^*(M\times\mathbb{R})$. Defining $k_f(\nabla_0,\nabla_1) := \int_0^1 k_f(t\nabla_0+(1-t)\nabla_1) \in \Omega^*(M)$, Stokes' formula implies:

$$dk_f(\nabla_0,\nabla_1) = k_f(\nabla_0) - k_f(\nabla_1) \ ,$$

i.e., at the level of cohomology, $k(\nabla)$ does not depend on the choice of the connection. One knows that:

$$[k_f(\nabla)] = 0 \in H^*(M) \text{ for } f = c_i, \quad i = \text{odd} \ , \tag{1.13}$$

This vanishing result can be formalized in the following way (which will be useful when dealing with foliations). We introduce the commutative DG algebra WO_q generated by the symbols c_i of degree $2i$, $1 \leq i \leq q$, and the symbols h_i for odd i's, of degree $deg(h_i) = 2i-1$, $1 \leq i \leq q$:

$$\begin{aligned} WO_q &= \mathbb{R}[c_1, \ldots, c_q] \otimes E(h_1, h_3, \ldots, h_{2[\frac{q+1}{2}]-1}) \ , \\ dc_i = & \ 0, \quad dh_{2j-1} = c_{2j-1}. \end{aligned}$$

The cohomology of WO_q is obviously generated by the $p_i = c_{2i}$'s. Choosing transgressions for $k(\nabla)(c_{2i-1})$ is equivalent to extending (1.12) to a DG algebra map:

$$k(\nabla) : WO_q \longrightarrow \Omega^*(M) \ . \tag{1.14}$$

The choice of the transgressions does not affect the map induced in cohomology:

$$H^*(WO_q) \cong \mathbb{R}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \ni f \mapsto [k_f(\nabla)] \in H^*(M) \tag{1.15}$$

So far we have been working with an arbitrary system $\{c_{2i}\}$ of generators of $I_q(\mathbb{R})$. One usually works with the polynomials:

$$c_i(A) = Tr(A^i)$$

(and then one gets, up to multiplication by scalars, the components of the Chern character of $E \otimes \mathbf{C}$), or with:

$$c_i(A) = \sigma_i(x_1, \ldots, x_q) \ ,$$

where x_i are the eigenvalues of A, and $\sigma_i \in \mathbb{R}[X_1, \ldots, X_q]$ is the i^{th} fundamental symmetric polynomial (and then one gets, up to multiplication by scalars, the Chern classes of $E \otimes \mathbf{C}$). With this last choice, (1.15) and (1.11) coincide up to multiplication by scalars.

Let us also point out that one can eliminate the choice of the transgressions needed to define (1.14), by fixing a connection ∇_0 compatible with some metric on E. For this type of connections one knows that $k_f(\nabla) = 0 \in \Omega^*(M)$ for $f = c_i$ with odd i, hence we can take as definition of (1.14):

$$\begin{aligned} c_i &\mapsto k(\nabla)(c_i) \ , \\ h_{2i-1} &\mapsto k(\nabla_0,\nabla)(c_{2i-1}) \ . \end{aligned} \tag{1.16}$$

Let us also recall how this geometric construction (1.15) relates to the construction of (1.10) applied to the principal bundle associated to E. Hence $G = GL_q$, $K = O_q$, and (1.10) gives:

$$k_P : H^*(W(gl_q, O_q)) \longrightarrow H^*(M) \ . \tag{1.17}$$

One has the generators $c_i \in I_q(\mathbb{R}) \subset W(gl_q, O_q)$, and the vanishing result (1.13) translates at the level of the Weil complex into the fact that $c_{2i-1} \in W(gl_q, O_q)$ are coboundaries, which can be seen also directly by constructing transgressions h_{2i-1} for c_{2i-1} (for explicit choices, see e.g. [53]). The induced map of DG algebras $WO_q \longrightarrow W(gl_q, O_q)$ is a quasi-isomorphism, and the identification of (1.17) with (1.15) and (1.11) follows now from the sequence of isomorphism:

$$H^*(W(gl_q, O_q)) \cong H^*(WO_q) \cong \mathbb{R}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \tag{1.18}$$

1.4.3 Topological characteristic classes for foliations: Recall that one has a classifying space $B\Gamma^q$ for codimension q foliations (the classifying space of Haefliger's groupoid Γ^q; see Example 1.1.3. 4), and any such foliation $(M, \mathcal{F})$ comes with a map (unique up to homotopy) $f_{\mathcal{F}} : M \longrightarrow B\Gamma^q$ (induced by 1.5), hence a *topological characteristic map*:

$$k_{\mathcal{F}}^{top} := f_{\mathcal{F}}^* : H^*(B\Gamma^q) \longrightarrow H^*(M)\ . \tag{1.19}$$

Its target is isomorphic (via Moerdijk's result 1.2.7) to the cohomology $H^*(\Gamma^q)$ of the étale groupoid Γ^q. These groups are quite wild, and there has been done a lot of work to find good models for this cohomology, so that its interesting cohomology classes can be written down more explicitly. Let us just mention that this was Haefliger's motivation in developing the cohomology theory for étale groupoids [55, 58], and to conjecture the result explained in 1.2.7; also, the construction of the universal characteristic map (1.24) below is rather formal (see [13], page 70).

We have seen in 1.3.5 that the classifying map $f_{\mathcal{F}} : M \longrightarrow \Gamma^q$ factorizes through the projection $\pi_{\mathcal{F}} : M \longrightarrow M/\mathcal{F}$, hence $k_{\mathcal{F}}^{top}$ can be defined with values in $H^*(M/\mathcal{F})$ (as the map induced by $f'_{\mathcal{F}}$ described in 1.3.5):

$$k_{\mathcal{F}}^{top} := f_{\mathcal{F}}^* : H^*(B\Gamma^q) \longrightarrow H^*(M/\mathcal{F})\ .$$

One should see this as the topological characteristic map of the leaf space.

1.4.4 Geometric characteristic classes for foliations: The theory of geometric characteristic classes for foliations originates on the integrability problem:

> *If $E \subset TM$ is a subbundle of the tangent bundle of M, does there exists an integrable subbundle $\mathcal{F} \subset TM$ isomorphic (or homotopic) to E?*

and on Bott's vanishing theorem which asserts that the Pontrjagin classes are an obstruction to the previous problem:

> *Given a codimension q foliation $(M, \mathcal{F})$, the characteristic map of its normal bundle $\nu = TM/\mathcal{F}$:*
>
> $$k_\nu : \mathbb{R}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \longrightarrow H^*(M)$$
>
> *vanishes in degrees $> 2q$.*

Due to this vanishing phenomena, there are new (secondary) classes appearing, encoded in a new characteristic map $k_{\mathcal{F}}^{geom}$, which we will describe in parallelism with the case of vector bundles on manifolds (namely with the construction of the maps (1.15), and (1.17)). In contrast with the case of bundles, this geometrical map will not coincide with the topological version (1.19).

Let us first recall Bott's original argument for the vanishing theorem, and a first definition of $k_{\mathcal{F}}^{geom}$. Due to the fact that ν is the normal bundle of a foliation, one can choose a "special connection" ∇ (called basic in [13]). It has the property that the associated DG map (1.12) kills all the expressions on c_i, of total degree $> 2q$, hence the vanishing result.

To define the correct characteristic map for foliations, we have to take into account all the vanishing results, hence also the fact that $k(\nabla)(c_{2i-1})$ transgress. So we have to use the full map (1.14), where, to eliminate the choice of the transgressions, we fix a connection ∇_0 compatible with some metric on ν, and we use the formulas (1.16):

$$k(\nabla) : WO_q \longrightarrow \Omega^*(M),\ c_i \mapsto k(\nabla)(c_i),\ h_{2i-1} \mapsto k(\nabla_0, \nabla)(c_{2i-1})$$

The Bott vanishing implies that $k(\nabla)$ vanishes on I^{q+1}, where I is the ideal generated by the c_i's, hence $k_{\mathcal{F}}$ factors through the quotient by I^{q+1}:

$$\underline{WO}_q := WO_q/I^{q+1} \ ,$$

and the true *geometrical characteristic* map is:

$$k_{\mathcal{F}}^{geom} := k(\nabla)_* : H^*(\underline{WO}_q) \longrightarrow H^*(M). \tag{1.20}$$

Let us now recall the description of $k_{\mathcal{F}}^{geom}$ in terms of the Weil complex. Let $P \longrightarrow M$ be the principal GL_q-bundle associated associated to ν. We have seen that any connection ∇ on P induces a DG map:

$$k(\nabla) : W(gl_q, O_q) \longrightarrow \Omega^*(P/O_q), \tag{1.21}$$

which, in cohomology, induces the usual characteristic map k_ν of the vector bundle ν. Choosing a basic connection ∇, (1.21) kills curvatures in degree $> 2q$, hence factors through the quotient:

$$\underline{W}(gl_q, O_q) := W(gl_q, O_q)/I^{q+1} \ . \tag{1.22}$$

The resulting characteristic map:

$$k_{\mathcal{F}}^{geom} : H^*(\underline{W}(gl_q, O_q)) \longrightarrow H^*(M) \tag{1.23}$$

coincides with (1.20) since the quasi-isomorphisms $WO_q \longrightarrow W(gl_q, O_q)$ of 1.4.2 induce quasi-isomorphisms $\underline{WO}_q \longrightarrow \underline{W}(gl_q, O_q)$.

One knows that the two characteristic maps (1.19) and (1.20) are related by a universal characteristic map (conjecturally injective)

$$k : H^*(\underline{WO}_q) \longrightarrow H^*(B\Gamma^q) \ , \tag{1.24}$$

which fits into a commutative diagram:

$$\begin{tikzcd} H^*(\underline{WO}_q) \arrow[d, "k"'] \arrow[dr, "k_{\mathcal{F}}^{geom}"] & \\ & H^*(M) \\ H^*(B\Gamma^q) \arrow[ur, "k_{\mathcal{F}}^{top}"'] & \end{tikzcd}$$

1.4.5 The target $H^*(\underline{WO}_q)$: In contrast with the target of (1.19), the target of the geometric characteristic map (1.20) can be computed explicitly, and has various interpretations. These groups are usually called *Gelfand-Fuchs cohomology*, due to the fact that they are (canonically) isomorphic to the continuous relative cohomology $H^*(a_q, O_q)$ of the Lie algebra a_q of formal vector fields on $\mathbb{R}^q$, investigated by Fuchs and Gelfand (and this interpretation can be used in giving an alternative definition of $k_{\mathcal{F}}^{geom}$, see [14]).

Apart from the classical characteristic classes (combinations of the $[c_{2i}]$'s), there are new exotic classes. The simplest one is the Godbillon-Vey class $GV = [h_1 c_1^q] \in H^{2q+1}(\underline{WO}_q)$. More generally, the Bott-Godbillon-Vey classes $GV_\alpha = [u_1 c_{\alpha_1} \ldots c_{\alpha_t}]$ are defined for any partition $\alpha = (\alpha_1, \ldots, \alpha_t)$ of q. Given a codimension q foliation $(M, \mathcal{F})$, their images by (1.20) are denoted:

$$GV(\mathcal{F}) := k_{\mathcal{F}}^{geom}(GV) \in H^{2q+1}(M),\ GV_\alpha(\mathcal{F}) := k_{\mathcal{F}}^{geom}(GV_\alpha) \in H^{2q+1}(M)\ .$$

A (linear) basis of the space of all exotic classes is given [48] by the elements

$$h_{i_1} \ldots h_{i_s} c_1^{j_1} \ldots c_q^{j_q}\ ,$$

where $1 \leq i_1 < \ldots < i_s \leq [\frac{q+1}{2}]$ is a sequence of odd integers ($s > 0$), and j_l's are integers with the properties:

(i) $i_1 + (j_1 + 2j_2 + \ldots + qj_q \geq q + 1$,

(ii)$k < i_1$, $k = \text{odd} \Longrightarrow j_k = 0$

1.5 Cyclic cohomology

In this section we recall the various definitions of the cyclic homology/cohomology of algebras [28, 26, 45], and we recall some fundamental examples.

1.5.1 Definitions of cyclic homology: Cyclic homology/cohomology is the non-commutative version of the usual cohomology/homology of spaces (see Proposition 1.5.7). Recall that, given a unital algebra A, *the Hochschild homology* $HH_*(A)$ is defined as the homology of the complex $C_*(A) := A^{\otimes(*+1)}$, with the boundary:

$$b(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1})\,,$$

while, for defining *the cyclic homology* $HC_*(A)$, one uses the quotient $C^\lambda_*(A) := A^{\otimes(*+1)}/Im(1-\tau)$, where τ is the signed cyclic permutation:

$$\tau(a_0, a_1, \ldots, a_n) = (-1)^n(a_n, a_0, \ldots, a_{n-1}) \ .$$

The two homologies are related by a long exact sequence, *the SBI-sequence*:

$$\ldots \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \ldots$$

(see below). Apart from $C^\lambda_*(A)$, there are other complexes computing $HC_*(A)$, which are sometimes more useful. For instance, the following double complex (called *the (b,b')-complex)*, or Tsygan's double complex:

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b'"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b'"'] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-\tau"'] \arrow[d, "b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-\tau"'] \arrow[d, "b'"'] & \cdots \arrow[l, "N"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\tau"'] \arrow[d, "b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\tau"'] \arrow[d, "b'"'] & \cdots \arrow[l, "N"'] \\
A & A \arrow[l, "1-\tau"'] & A \arrow[l, "N"'] & A \arrow[l, "1-\tau"'] & \cdots \arrow[l, "N"']
\end{tikzcd}$$

(where $N = 1 + \tau + \ldots + \tau^n$ on $A^{\otimes(n+1)}$, and $b'(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, \ldots, a_ia_{i+1}, \ldots, a_n)$) still computes $HC_*(A)$, since the cohomology of its rows is zero except for the zero degree, where it is $A^{\otimes(*+1)}/Im(1-\tau)$. This double complex has an obvious (degree -2) shift operator, due to its periodicity. This operator is usually denoted by S, and is an epimorphism whose kernel is $C_*(A)$; the SBI-sequence is simply the associated long exact sequence.

Stabilizing with respect to S, i.e. using the extension (by periodicity) of this double complex to the left, one gets a periodic theory $HP_*(A)$ (with period 2), called *the periodic cyclic homology* of A. Although one usually talks about cyclic homology, it is its periodic version $HP_*(A)$ which is of maximum interest, and which is the true extension of the usual cohomology of manifolds (see Theorem 1.5.7).

Using the fact that b' is acyclic (with the contracting homotopy $s_{-1}(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n)$), one gets a new double complex computing $HC_*(A)$, usually called *the (B,b)-complex of A*:

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] \\
A^{\otimes 4} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & & \\
A & & &
\end{tikzcd}$$

where $B = (1-\tau)s_{-1}N$. The periodicity operator S is present also in this double complex.

When A is not unital, the cyclic homology is computed by the (b, b') complex, and, to form a (b, B)-complex, one needs to first adjoin a unit to A, and to reduce the resulting (b, B)-complex. Regarding the Hochschild homology, it can still be defined using the b-complex, provided A is *H-unital* (in the sense that the associated b'-complex is acyclic). This happens for instance if A has local units (e.g. for $A = C_c^{\infty}(M)$ of a manifold M).

Using the duals of the previous complexes, one gets the corresponding Hochschild, cyclic, periodic cyclic *cohomologies*, denoted $HH^*(A)$, $HC^*(A)$, $HP^*(A)$, respectively.

Useful for computations is the fact that:

$$HH_*(A) \cong Tor_{A \otimes A^{op}}(A, A) \ , \tag{1.25}$$

where A^{op} is the algebra obtained from A by reversing the multiplication: $a \cdot b := ba$. In other words, considering the functor $(-)_\sharp$ defined on the category of A-bimodules (recall that $M_\sharp := M/[M, A]$ is the quotient of M by the linear span of elements of type $xa - ax$, with $a \in A$, $x \in M$), $HH_*(A) = L^*(-)_\sharp(A)$ is computed by the complex $(B_*)_\sharp$, where $B_* \longrightarrow A$ is any resolution by projective A-bimodules. To check (1.25) it suffices to remark that, for the resolution $(A^{\otimes(*+2)}, b') \longrightarrow A$, one gets $(A^{\otimes(*+2)}, b')_\sharp = C_*(A)$.

1.5.2 Motivation: Connes' initial motivation for introducing cyclic cohomology was its use in evaluating K-theory classes. In practice, the K-theory groups are quite difficult to compute, although they contain interesting geometric elements. To overcome this problem, Connes uses the remark that any trace $\tau : A \longrightarrow \mathbf{C}$ (i.e. τ is linear, and $\tau(ab) = \tau(ba)$), extends to a trace $\tau((a_{i,j})) = \sum \tau(a_{i,i})$ on $M_\infty(A) = \cup_q M_q(A)$, and, using the definition of $K_0(A)$ in terms of idempotents, he gets a linear map:

$$\tau_* : K_0(A) \longrightarrow \mathbf{C} \ .$$

In other words, one has a pairing:

$$K_0(A) \times \{\text{traces on } A\} \longrightarrow \mathbf{C} \ ,$$

which is useful in evaluating K-theory classes. One clearly has:

$$HC^0(A) = \{\text{traces on } A\} \ ,$$

and the cyclic cocycles should be viewed as generalizations of traces (sometimes called higher traces), defined in such a way that the previous pairing extends to a pairing:

$$K_0(A) \times HP^0(A) \longrightarrow \mathbf{C} \ .$$

It associates to (the class of) an idempotent $e \in M_q(A)$, and to (the class of) a cyclic cocycle $\tau \in C_\lambda^{2k}(A)$, the number:

$$< e, \tau >: = \sum_{i_0, \ldots, i_{2k}} \tau(e_{i_0 i_1}, e_{i_1 i_2}, \ \ldots \ , e_{i_{2k-1} i_{2k}}, e_{i_{2k} i_0}) \ .$$

Actually one has a Chern character (extension of the usual one):

$$Ch : K_0(A) \longrightarrow HP_0(A) \ ,$$

and the previous pairing is made out by combining this Chern character with the obvious pairing:

$$HP_*(A) \times HP^*(A) \longrightarrow \mathbf{C} \ .$$

This type of pairings exist also at the level of K_1, and at the level of K-homology; we refer to [28] for complete and precise definitions, and for applications.

1.5.3 The topological setting: It often happens that the algebra we look at is a topological (locally convex) algebra, and to get relevant homology/cohomology groups, one has to take into account the topology. It is customary to replace the algebraic tensor product $\otimes$ by topological ones (usually the projective tensor product, see [52]). We emphasize that the choice of the tensor product is essential, and it is often dictated by the algebra we look at. The "leading principle" is: we want the Chern character (say from $K_0(A)$ to the cyclic homology of A), whose target can be computed! Just to have the Chern character, it sometimes suffices to consider the algebraic tensor product, but the algebraic theory $HC_*(A)$ is usually hard to compute, and one expects meaningful results just by making use of the topology. This happens already in the simplest example where $A = C_c^{\infty}(M)$ is the algebra of compactly supported smooth functions on a manifold; its algebraic cyclic homology is not known, and, if one changes the tensor product, one gets the cohomology of M (with compact supports) and the classical Chern character. The only thing we need for the new tensor product $\hat{\otimes}$ is that it behaves well for manifolds, i.e. satisfies $C_c^{\infty}(M)\hat{\otimes}C_c^{\infty}(N) = C_c^{\infty}(M \times N)$ (e.g. one could use the inductive [52] tensor product). There are various ways to overcome this delicate problem, but it is not in our intention to go here into details. Instead, we follow the "leading principle" above, and just mention that we can use any tensor product which behaves nicely with respect to spaces (in the sense we just explained).

1.5.4 Mixed Complexes [67]: Mixed complexes and cyclic modules are very useful when computing cyclic cohomologies. Here we recall ([45, 67, 26]) some basic definitions concerning them. We work in an abelian category $\mathcal{M}$. A *mixed complex* in $\mathcal{M}$ is a family $\{X_n : n \geq 0\}$ of objects in $\mathcal{M}$, equipped with maps of degree -1, $b : X_n \longrightarrow X_{n-1}$, maps of degree 1, $B : X_n \longrightarrow X_{n+1}$, satisfying $b^2 = B^2 = bB + Bb = 0$. A mixed complex gives rise to a first quadrant double complex in $\mathcal{M}$:

$$\begin{tikzcd}
\arrow[d, "b"'] & \arrow[d, "b"'] & \arrow[d, "b"'] & \arrow[d, "b"'] \\
X_3 \arrow[d, "b"'] & X_2 \arrow[l, "B"'] \arrow[d, "b"'] & X_1 \arrow[l, "B"'] \arrow[d, "b"'] & X_0 \arrow[l, "B"'] \\
X_2 \arrow[d, "b"'] & X_1 \arrow[l, "B"'] \arrow[d, "b"'] & X_0 \arrow[l, "B"'] & \\
X_1 \arrow[d, "b"'] & X_0 \arrow[l, "B"'] & & \\
X_0 & & &
\end{tikzcd}$$

It (or its total complex) is denoted (X_*, B, b). The Hochschild and cyclic homology of X_* are defined by $HH_*(X_*) := H_*((X_*, b))$, $HC_*(X_*) := H_*((X_*, B, b))$. There is a short exact sequence :

$$0 \longrightarrow (X_*, b) \xrightarrow{I} (X_*, B, b) \xrightarrow{S} (X_*, B, b)[-2] \longrightarrow 0,$$

where I is the inclusion on the first column and S is the shifting. Standard homological algebra gives a long exact sequence :

$$\ldots \xrightarrow{B} HH_n(X_*) \xrightarrow{I} HC_n(X_*) \xrightarrow{S} HC_{n-2}(X_*) \xrightarrow{B} HH_{n-1}(X_*) \xrightarrow{I} \ldots$$

(the SBI-sequence of X_*). Using the shift operator, the periodic cyclic homology of X_* is defined by

$$HP_*(X_*) := H_*(\varprojlim_{-r} (X_*, B, b)[-2r]).$$

1.5.5 Cyclic Objects: Usually, mixed complexes are made out of cyclic objects. Let $1 \leq r \leq \infty$. An *r-cyclic object* in $\mathcal{M}$ is a contravariant functor $X : \Lambda_r \longrightarrow \mathcal{M}$ from the generalized Connes category Λ_r (A2 in [45]). That means a simplicial object $\{X_\bullet, d_\bullet, s_\bullet\}$ in $\mathcal{M}$ together with morphisms $t_n : X_n \longrightarrow X_n$ such that:

$$d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{if } i \neq 0 \\ d_n & \text{if } i = 0 \end{cases}, \quad s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } i \neq 0 \\ t_{n+1}^2 s_n & \text{if } i = 0 \end{cases},$$

and the cyclic relation $t_n^{r(n+1)} = 1$ holds in the case $r \neq \infty$.

We define $b', b : X_n \longrightarrow X_{n-1}$, $b' = \sum_{j=0}^{n-1} (-1)^j d_j$, $b = b' + (-1)^n d_n$, $s_{-1} = s_n t_{n+1}$ the extra degeneracy (which gives a contraction for b'), $\tau_n = (-1)^n t_{n+1}$. If $r \neq \infty$, let $N = \sum_{j=0}^{(n+1)r-1} \tau_n^j$, $B = (1 - \tau_n) s_{-1} N$. Then $(X_\bullet, b, B)$ is a mixed complex; its homologies are denoted $HH_*(X_\bullet), HC_*(X_\bullet)$. They are in fact the homologies of $X_\bullet$ as a contravariant functor $\Lambda_\infty \longrightarrow \mathcal{M}$ respectively $\Lambda_r \longrightarrow \mathcal{M}$ (cf. A3.2 in [45]).

For $r = \infty$ we put $HH_*(X_\bullet) := H_*((X_\bullet, b))$.

For $r = 1$, the category Λ_1 is denoted Λ and the 1-cyclic objects are called *cyclic objects.*

1.5.6 Examples: The basic example ([45], [83]) is the ∞-cyclic vector space $A_\alpha^\natural$ associated to an unital algebra A, endowed with an endomorphism $\alpha : A_\alpha^\natural(n) := A^{\otimes(n+1)}$,

$$d_i(a_0, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) & \text{if } 0 \leq i \leq n-1 \\ (\alpha(a_n) a_0, a_1, \ldots, a_{n-1}) & \text{if } i = n \end{cases},$$

$$t(a_0, \ldots, a_n) = (\alpha(a_n), a_0, a_1, \ldots, a_{n-1}),$$

$$s_i(a_0, \ldots, a_n) = (\ldots, a_i, 1, a_{i+1}, \ldots).$$

Its Hochschild homology is denoted by $HH_*(A, \alpha)$, and its b-boundary $\sum(-1)^i d_i$ by b_α. As an extension of (1.25) (the proof presented obviously extends to this more general homology), one has:

$$HH_*(A, \alpha) \cong Tor_{A \otimes A^{op}}(A, A_\alpha), \tag{1.26}$$

where A_α is the A-bimodule with the new left action $a \cdot x := \alpha(a)x$, for $a \in A$, $x \in A_\alpha$.

If α is of order $r \neq \infty$ then $A_\alpha^\natural$ is an r-cyclic vector space; we denote by $HC_*(A;\alpha)$, $HP_*(A;\alpha)$ the corresponding homologies.

When $\alpha = id$, one gets the usual $HH_*(A), HC_*(A), HP_*(A)$, and we simplify the notation $A_{id}^\natural$ to $A^\natural$.

Algebra of smooth functions: The computation for the locally convex algebra $C^\infty(M)$ of smooth functions on a compact manifold M is the starting and motivating example [28]. Since we are in the topological setting, we use the projective [52] tensor product in the definition of the homologies. Hence the defining cyclic module $C^\infty(M)^\natural$ is $C^\infty(M)^{\hat{\otimes}(n+1)} = C^\infty(M^{n+1})$ in degree n, and the structure formulas:

$$d_i(a_0, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) & \text{if } 0 \leq i \leq n-1 \\ (a_n a_0, a_1, \ldots, a_{n-1}) & \text{if } i = n \end{cases},$$

$$t(a_0, \ldots, a_n) = (a_n, a_0, a_1, \ldots, a_{n-1}) \ ,$$

read:

$$d_i f(x_0, \ldots, x_n) = \begin{cases} f(x_0, \ldots, x_i, x_i, \ldots, x_n) & \text{if } 0 \leq i \leq n-1 \\ f(x_0, x_1, \ldots, x_n, x_0) & \text{if } i = n \end{cases}, \tag{1.27}$$

$$t f(x_0, \ldots, x_n) = f(x_1, x_2, \ldots, x_n, x_0) \ .$$

The following result is due to A. Connes [28]:

1.5.7 Theorem *(Connes): For any compact manifold M,*

$$\begin{aligned} HH_*(C^\infty(M)) &\cong \Omega^*(M) \ , \\ HP_*(C^\infty(M)) &\cong \prod_{k \equiv * \mathrm{mod} 2} H^k(M) \ . \end{aligned}$$

Group algebras: Let us briefly recall another basic computation: for the group algebra $\mathbf{C}[G]$ of a discrete group G. For any conjugacy class γ, the linear span of elements of type $(g_0, g_1, \ldots, g_n)$ with the property that $g_0 g_1 \ldots g_n \in \gamma$ is a cyclic submodule of $\mathbf{C}[G]^\natural$. It is denoted by $\mathbf{C}[G]^{\natural,\gamma}$, and the corresponding homologies by $HH_*(\mathbf{C}[G])_{(\gamma)}$, $HC_*(\mathbf{C}[G])_{(\gamma)}$, etc. (usually called the localization at γ). Clearly $\mathbf{C}[G]^\natural$ is the direct sum of the $\mathbf{C}[G]^{\natural,\gamma}$'s, with γ running over the set $< G >$ of all conjugacy classes of G. Hence:

$$HC_*(\mathbf{C}[G]) \cong \sum_{\gamma \in <G>} HC_*(\mathbf{C}[G])_{(\gamma)} \ ,$$

and similarly for HH_*, HP_*. The so called *localization at units* (for $\gamma = 1$) gives the usual group homology (with coefficients in $\mathbf{C}$):

1.5.8 Theorem *(Karoubi, Burghelea): For any discrete group G:*

$$\begin{aligned} HH_*(\mathbf{C}[G])_{(1)} &\cong H_*(G), \quad HC_*(\mathbf{C}[G])_{(1)} \cong \prod_k H_{*-2k}(G) \\ HP_*(\mathbf{C}[G])_{(1)} &\cong \prod_{k \equiv * \mathrm{mod} 2} H_*(G) \ . \end{aligned}$$

The computation of the other localizations is due to Burghelea, and is divided into two cases. *The elliptic case* is similar to the localization at units:

1.5.9 Theorem *(Burghelea): When γ is elliptic (i.e. of finite order):*

$$HP_*(\mathbf{C}[G])_{(\gamma)} \cong \prod_{k\equiv * \mathrm{mod} 2} H_*(Z_\gamma) \ ,$$

where $Z_\gamma = \{g \in G : g\gamma = \gamma g\}$.

The hyperbolic case is more complicated, and the conclusion of Burghelea's work on this case is:

1.5.10 Theorem *(Burghelea): When γ is hyperbolic (i.e. of infinite order):*

$$HH_*(\mathbf{C}[G])_{(\gamma)} \cong H_*(Z_\gamma) \ ,$$
$$HC_*(\mathbf{C}[G])_{(\gamma)} \cong H_*(N_\gamma) \ ,$$

where $N_\gamma = Z_\gamma / < \gamma >$, and the SBI-sequence identifies with the Hochschild-Serre spectral sequence (see 6.8.2 *in [97]) associated to $< \gamma > \subset Z_\gamma$.*

1.5.11 The cyclic sheaf of smooth functions: If $\mathcal{A} = \mathcal{C}^\infty_M$ is the sheaf of smooth functions on a manifold M, we define $\mathcal{A}^\natural$ as in 1.5.6 by taking into account the topology, i.e. $\mathcal{A}^\natural(n) := \Delta^*_{n+1}(\mathcal{A}^{\boxtimes(n+1)})$ where $\mathcal{A}^{\boxtimes n} := \mathcal{C}^\infty_{M^n} \in Sh(M^n)$ and $\Delta_n : M \longrightarrow M^n$ is the diagonal map. Keeping the same formulas as in 1.5.6 with $\alpha = id$, $\mathcal{A}^\natural$ is a cyclic object in $Sh(M)$. Remark that the stalk $\mathcal{A}^\natural_p$ at $p \in M$ consists of germs at $(p, \ldots, p) \in M^{n+1}$ of smooth functions $f : M^{n+1} \longrightarrow \mathbf{C}$, and the formulas for the structure maps are those in (1.27). One can look at $\mathcal{A}^\natural_p$ as a topological version of the cyclic module associated to the algebra $\mathcal{A}_p$. Similar to Theorem 1.5.7, one has a local version (which actually can be used to prove the theorem):

1.5.12 Lemma: *For any manifold M, and any $p \in M$:*

$$((\mathcal{C}^\infty_M)^\natural_p, \ b) \longrightarrow ((\Omega^*_M)_p, \ 0 \ )$$

$$(a_0, ..., a_n) \mapsto \frac{1}{n!} a_0 \ da_1 \ . \ . \ . \ da_n$$

is a quasi-isomorphism of complexes of vector spaces.

1.5.13 Twisted smooth functions: Let M be a (Hausdorff, locally compact) manifold, and $\phi : M \longrightarrow M$ a smooth proper map. One has an induced endomorphism on the locally convex algebra $C^\infty_c(M)$, $a \mapsto a^\phi := a \circ \phi$, and the associated Hochschild homology $HH_*(C^\infty_c(M), \phi)$ is computed by:

$$(C^\infty_c(M), b_\phi) : \ . \ . \ . \ \xrightarrow{b_\phi} C^\infty_c(M \times M \times M) \xrightarrow{b_\phi} C^\infty_c(M \times M) \xrightarrow{b_\phi} C^\infty_c(M) \xrightarrow{b_\phi} 0 \, ,$$

$$b_\phi(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a^\phi_n a_0, a_1, \ldots, a_{n-1}) \, .$$

Let $M^\phi = \{p \in M : \phi(p) = p\}$. For $p \in M^\phi$, we say that $germ_p(\phi)$ is *stable* if:
(*i*) M^ϕ is, locally around p, a submanifold of M with $T_pM^\phi = Ker((d\phi)_p - Id)$,
(*ii*) T_pM splits into a direct sum of T_pM^ϕ and a $(d\phi)_p$-invariant subspace .
This happens for instance if ϕ preserves a metric around p (in particular, if ϕ has finite order). We say that ϕ is stable if $germ_p(\phi)$ is stable for all $p \in M^\phi$.

The first part of Connes' result 1.5.7 has a twisted extension: for any stable ϕ:

$$HH_*(C_c^\infty(M), \phi) \cong \Omega_c^*(M^\phi) \ .$$

We will actually need a local version of this. By taking into account the topology in the sense of 1.5.11, we define the sheaf $\mathcal{A}_\phi^\natural$ on M^ϕ, whose stalk $\mathcal{A}_{\phi,p}^\natural$ at $p \in M^\phi$ is obtained by applying the twisted formulas of 1.5.6 to the endomorphism $germ_p(\phi)$: $\mathcal{A}_p \longrightarrow \mathcal{A}_p$. One denotes by b_ϕ the associated b-boundary $\sum(-1)^i d_i$. We have:

1.5.14 Proposition: *If $\phi : (M,p) \longrightarrow (M,p)$ is a stable germ of a smooth function, then:*

$$((\mathcal{C}_M^\infty)_p^\natural,\ b_\varphi\ ) \longrightarrow\ ((\mathcal{C}_{M^\varphi}^\infty)_p^\natural, b_{id}\ ) \longrightarrow\ ((\Omega_{M^\varphi}^*)_p,\ 0\ ),$$

$$(a_0, ..., a_n) \mapsto (a_0|_{M^\varphi}, ..., a_n|_{M^\varphi}) \mapsto \frac{1}{n!}\, a_0\ da_1\ .\ .\ .\ da_n|_{M^\varphi}$$

are quasi-isomorphisms of complexes of vector spaces.

Chapter 2

A homology theory for étale groupoids

2.1 Introduction

In this chapter we introduce a homology theory for étale groupoids. Étale groupoids serve as model for structures like leaf spaces of foliations, orbifolds, and orbit spaces of actions by discrete groups. In this sense, étale groupoids should be viewed as generalized spaces.

In the literature one finds, roughly speaking, two different approaches to the study of étale groupoids. One approach is based on the construction of the convolution algebras associated to an étale groupoid, in the spirit of Connes' non-commutative geometry ([25, 29]), and involves the study of cyclic and Hochschild homology and cohomology of these algebras ([20, 29]). The other approach uses methods of algebraic topology such as the construction of the classifying space of an étale groupoid and its (sheaf) cohomology groups ([13, 57, 72]).

Our motivation in this chapter is twofold. First, we want to give a more complete picture of the second approach, by constructing a suitable homology theory which complements the existing cohomology theory. Secondly, we use this homology theory as the main tool to relate the two approaches (see Theorems 4.6.3, 4.6.4, 4.6.5 of chapter 4).

Let us be more explicit: In the second approach, one defines for any étale groupoid $\mathcal{G}$ natural cohomology groups with coefficients in an arbitrary $\mathcal{G}$-equivariant sheaf. These were introduced in a direct way by Haefliger [55]. As explained in [72], they can be viewed as a special instance of the Grothendieck theory of cohomology of sites [2], and agree with the cohomology groups of the classifying space of $\mathcal{G}$ [74]. Moreover, these cohomology groups are invariant under Morita equivalences of étale groupoids. (This invariance is of crucial importance, because the construction of the étale groupoid modeling the leaf space of a given foliation involves some choices which determine the groupoid only up to Morita equivalence, cf 1.3.3). We complete this picture by constructing a homology theory for étale groupoids, again invariant under Morita equivalence, which is dual (in the sense of Verdier duality) to the existing cohomology theory. Thus, one result of our work is the extension of "the six operations of Grothendieck"[2] from spaces to leaf spaces of foliations.

Our homology theory of the leaf space of a foliation reflects some geometric properties of the foliation. For example, by integration along the fibers (leaves) it is related to the leafwise cohomology theory studied by Alvarez Lopez, Hector and others (see [1] and the references cited there). It also shows that the Ruelle-Sullivan current of a measured foliation (see [25]) lives in Hafliger's (closed) cohomology. The results of Chapter 4 imply that our homology is also the natural target for the (localized) Chern character.

The homology theory also plays a central role in explaining the relation between the sheaf theoretic and the convolution algebra approaches to étale groupoids, already referred to above. Indeed, the various cyclic homologies of étale groupoids can be shown to be isomorphic to the homology of certain associated étale groupoids; it extends the previous results of Burghelea, Connes, Feigin, Karoubi, Nistor, Tsygan. This connection explains several basic properties of the cyclic and periodic homology groups, and leads to explicit calculations (see Chapter 4). The previous work on the Baum-Connes conjecture for discrete groups, or for proper actions of discrete groups on manifolds, suggest that this homology will play a role in the Baum-Connes conjecture for étale groupoids.

From an algebraic point of view, our homology theory is an extension of the homology of groups, while from a topological point of view it extends compactly supported cohomology of spaces. In this context, we should emphasize that even in the simplest examples, the étale groupoids which model leaf spaces of foliations involve manifolds which are neither separated nor paracompact. Thus, an important technical ingredient of our work is a suitable extension of the notions related to compactly supported section of sheaves to non-separated (non-paracompact) manifolds. For example, as a special case of our results one obtains the Verdier (and Poincaré) duality for non-separated manifolds. Our notion of compactly supported sections is also used in the construction of the convolution algebra of a (non-separated) étale groupoid. We believe that this extension to non-separated spaces has a much wider use that the one in this thesis, and we have tried to give an accessible presentation of it in the appendix. The results in the appendix also play a central role in the calculation concerning the cyclic homology of étale groupoids in Chapter 4, and make it possible to extend the results of [20] for separated groupoids to the non-separated case.

We conclude this introduction with a brief outline.

In section 2.2, we present the definition of our homology theory and mention some of its immediate properties.

In section 2.3, a covariant operation $\varphi_!$ for any map φ between étale groupoids is introduced, which can intuitively be thought of as a kind of "integration along the fiber" at the level of derived categories. We then prove a Leray spectral sequence for this operation. This spectral sequence is extremely useful. For example, we will use it to prove the Morita invariance of homology. It also plays a crucial role in many calculations in Chapter 4.

In section 2.4, we prove that the operation $\mathcal{L}\varphi_!$ has a right adjoint $\varphi^!$ at the level of derived categories, thus establishing Verdier duality. The Poincaré duality between (Haefliger) cohomology and (our) homology of étale groupoids is an immediate consequence.

In an appendix, we show how to adapt the definition of the functor $\Gamma_c(X;\mathcal{A})$ (assigning to a space X and a sheaf $\mathcal{A}$ the group of compactly supported sections) in such a way that all the properties (as expressed in [16], say) can be proved without using Hausdorffness and paracompactness of the space X.

2.2 Homology/compactly supported cohomology

In this section we will introduce the homology groups $H_n(\mathcal{G};\mathcal{A})$ for any étale groupoid $\mathcal{G}$ and any $\mathcal{G}$-sheaf $\mathcal{A}$. Among the main properties to be proved will be the invariance of homology under Morita equivalence.

For any Hausdorff space X, the standard properties of the functor which assigns to a sheaf $\mathcal{S}$ its group of compactly supported sections $\Gamma_c(X;\mathcal{S})$ are well known and can be found in any book on sheaf theory. In the appendix, we show how to extend this functor to the case where X is not necessarily Hausdorff, while retaining all the standard properties. We emphasize that throughout this thesis, Γ_c will denote this extended functor.

Let us fix an étale groupoid $\mathcal{G}$. The spaces $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ (and hence the spaces $\mathcal{G}^{(n)}$ for $n \geq 0$) are assumed to satisfy the general conditions of 1.1.14, but we will not assume that $\mathcal{G}$ is Hausdorff. We write $d = \mathrm{cdim}(\mathcal{G}^{(0)})$ for the cohomological dimension of $\mathcal{G}^{(0)}$. Thus, for any $n \geq 0$ and any Hausdorff open set $U \subset \mathcal{G}^{(n)}$, the (usual) cohomological dimension of U is at most d.

2.2.1 Bar complex. Let $\mathcal{A}$ be a $\mathcal{G}$-sheaf, and assume that $\mathcal{A}$ is c-soft (cf. 2.5.1) as a sheaf on $\mathcal{G}^{(0)}$ (we will briefly say that $\mathcal{A}$ is a "c-soft $\mathcal{G}$-sheaf"). For each $n \geq 0$, consider the sheaf $\mathcal{A}_n = \tau_n^*(\mathcal{A})$ on $\mathcal{G}^{(n)}$ constructed by pull-back along $\tau_n : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(0)}$, $\tau_n(x_0 \longleftarrow \ldots \longleftarrow x_n) = x_0$. It is again a c-soft sheaf because τ_n is étale. The groups $\Gamma_c(\mathcal{G}^{(n)}, \mathcal{A}_n)$ of compactly supported sections, introduced in the Appendix, together form a simplicial abelian group:

$$B_*(\mathcal{G};\mathcal{A}) : \ \ldots \ \begin{smallmatrix}\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\end{smallmatrix} \Gamma_c(\mathcal{G}^{(2)};\mathcal{A}_2) \begin{smallmatrix}\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\end{smallmatrix} \Gamma_c(\mathcal{G}^{(1)};\mathcal{A}_1) \rightrightarrows \Gamma_c(\mathcal{G}^{(0)};\mathcal{A}_0) \tag{2.1}$$

with face maps:

$$d_i : \Gamma_c(\mathcal{G}^{(n)};\mathcal{A}_n) \longrightarrow \Gamma_c(\mathcal{G}^{(n-1)};\mathcal{A}_{n-1}) \tag{2.2}$$

defined as follows. First, for the face map $d_i : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n-1)}$ (cf. 1.1.13) there is an evident map (isomorphism in fact) $\mathcal{A}_n \longrightarrow d_i^*(\mathcal{A}_{n-1})$, whose stalk at $\vec{g} = (x_0 \xleftarrow{g_1} \ldots \xleftarrow{g_n} x_n)$ is the identity map for $i \neq 0$ and the action by $g_0 : (\mathcal{A}_n)_{\vec{g}} = \mathcal{A}_{x_0} \longrightarrow \mathcal{A}_{x_1} = d_0^*(\mathcal{A}_{n-1})_{\vec{g}}$ if $i = 0$. The map d_i in (2.2) is now obtained from this by summation along the fibres (see 2.5.12):

$$\begin{array}{ccc}
\Gamma_c(\mathcal{G}^{(n)};\mathcal{A}_n) & \xrightarrow{\quad d_i \quad} & \Gamma_c(\mathcal{G}^{(n-1)};\mathcal{A}_{n-1}) \\
\big\downarrow & & \big\uparrow{\scriptstyle \Gamma_c(\mathcal{G}^{(n-1)},\sum_{d_i})} \\
\Gamma_c(\mathcal{G}^{(n)};d_i^*(\mathcal{A}_{n-1})) & \xrightarrow{\quad \sim \quad} & \Gamma_c(\mathcal{G}^{(n-1)},(d_i)_! d_i^*\mathcal{A}_{n-1})
\end{array}$$

Using our notation 2.5.13, $B_*(\mathcal{G};\mathcal{A})$ is the chain complex associated to the simplicial vector space $n \mapsto \Gamma_c(\mathcal{G}^{(n)};\mathcal{A})$ with the structure maps:

$$d_i\langle a \mid g_1,\ldots,g_n\rangle = \begin{cases} \langle ag_1 \mid g_2,\ldots,g_n\rangle & \text{if } i = 0 \\ \langle a \mid g_1,\ldots,g_ig_{i+1},\ldots,g_n\rangle & \text{if } 1 \leq i \leq n-1 \\ \langle a \mid g_1,\ldots,g_{n-1}\rangle & \text{if } i = n \end{cases},$$

$$s_i(a \mid g_1, ..., g_n) = (a \mid ..., g_i, 1, g_{i+1}, ...),$$

The homology groups $H_n(\mathcal{G};\mathcal{A})$ are defined as the homology groups of the simplicial abelian groups (2.1), or, equivalently, as those of the associated chain complex given by the alternating sum $\delta = \sum(-1)^i d_i$.

Similarly, any bounded below chain complex $\mathcal{S}_*$ of c-soft sheaves gives rise to a double complex:

$$B_*(\mathcal{G};\mathcal{S}_*) \tag{2.3}$$

and we define *the hyperhomology* $\mathbb{H}_n(\mathcal{G};\mathcal{S}_*)$ to be the homology of the associated total complex.

2.2.2 Lemma. *Any quasi-isomorphism $\mathcal{S}_* \longrightarrow \mathcal{I}_*$ between bounded below chain complexes of c-soft $\mathcal{G}$-sheaves induces an isomorphism*

$$\mathbb{H}_n(\mathcal{G};\mathcal{S}_*) \xrightarrow{\sim} \mathbb{H}_n(\mathcal{G};\mathcal{I}_*) \ .$$

*P*roof: The spectral sequence of the double complex (2.3) takes the form

$$E^2_{p,q} = H_p H_q(\mathcal{G}_*, \mathcal{S}_*) \Longrightarrow H_{p+q}(\mathcal{G};\mathcal{S}_*) \ ,$$

where the $E^1_{p,q}$-term is the homology $\mathbb{H}_p(\mathcal{G}^{(p)};\mathcal{S}^*)$ of the complex $\Gamma_c(\mathcal{G}^{(p)};\mathcal{S}_*)$. The lemma thus follows from 2.5.7. $\square$

2.2.3 c-soft resolutions. Let $\mathcal{A}$ be an arbitrary $\mathcal{G}$-sheaf. There always exists a resolution:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \ldots \longrightarrow \mathcal{S}^d \longrightarrow 0 \tag{2.4}$$

by c-soft $\mathcal{G}$-sheaves. For example, since the category of $\mathcal{G}$-sheaves has enough injectives, one can take any injective resolution $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I}^*$ and take $\mathcal{S}^*$ to be the truncation $\tau_{\leq d}(\mathcal{I}^*)$ (softness of $\mathcal{S}^d$ then follows as in [16], p.55). Or, one can use for $\mathcal{I}^*$ the flabby Godement resolution of $\mathcal{A}$ on the space $\mathcal{G}^{(0)}$ with its natural $\mathcal{G}$-action, and truncate it. In the case of a smooth étale groupoid and working over $\mathbb{R}$, one also has the standard resolution:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \otimes \Omega^0 \longrightarrow \mathcal{A} \otimes \Omega^1 \longrightarrow \ldots$$

obtained from the $\mathcal{G}$-sheaves Ω^* of differential forms on $\mathcal{G}^{(0)}$. (Note that the last two resolutions are functorial in $\mathcal{A}$.)

Any resolution (2.4) maps into the truncated injective one. And, similarly, given two resolutions $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^*$ and $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I}^*$, there is a resolution $\mathcal{R}^*$ (e.g. the truncated injective one) and a diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{I}^* \arrow[d, dotted] \\ \mathcal{S}^* \arrow[r, dotted] & \mathcal{R}^* \end{tikzcd} \tag{2.5}$$

which commutes up to homotopy.

2.2.4 Definition of homology. Let $\mathcal{A}$ be an arbitrary $\mathcal{G}$-sheaf, and let $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \ldots \mathcal{S}^d \longrightarrow 0$ be a c-soft resolution. Then $\mathcal{S}^{-*}$ is a bounded chain complex (non-zero in degrees between $-d$ and 0), and we define *the homology groups* $H_n(\mathcal{G};\mathcal{A})$ to be $\mathbb{H}_n(\mathcal{G};\mathcal{S}^{-*})$. By 2.2.3 (see (2.5) above) and Lemma 2.2.2, this definition is independent of the choice of the resolution. Observe that

$$H_n(\mathcal{G};\mathcal{A}) = 0 \quad \text{for all } \ n < -d\ .$$

2.2.5 Extreme cases.

1. If $\mathcal{G}^{(0)}$ is a point, i.e. if $\mathcal{G}$ is a discrete group, then $H_n(\mathcal{G};\mathcal{A})$ is the usual group homology of $\mathcal{G}$.

2. If $\mathcal{G}$ is a discrete groupoid, $\mathcal{G}^{(*)}$ is a simplicial set, and $H_n(\mathcal{G};\mathcal{A})$ is the usual simplicial homology of $\mathcal{G}^{(*)}$ with twisted coefficients.

3. If $\mathcal{G}$ is a Hausdorff space X (viewed as a "trivial" groupoid, 1.1.3.1) then $H_n(\mathcal{G};\mathcal{A}) = H_c^{-n}(X;\mathcal{A})$ is the usual cohomology with compact supports (although graded differently). So the spectral sequence occurring in the proof of lemma 2.2.2 could be written as

$$H_pH_c^{-q}(\mathcal{G}^{(*)};\mathcal{A}) \Longrightarrow H_{p+q}(\mathcal{G};\mathcal{A})\ .$$

2.2.6 Long exact sequence. Any short exact sequence:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

of $\mathcal{G}$-sheaves induces a long exact sequence in homology:

$$\ldots \longrightarrow H_{n+1}(\mathcal{G};\mathcal{C}) \longrightarrow H_n(\mathcal{G};\mathcal{A}) \longrightarrow H_n(\mathcal{G};\mathcal{B}) \longrightarrow H_n(\mathcal{G};\mathcal{C}) \longrightarrow \ldots$$

The proof is standard. (The truncated Godement resolutions give a short exact sequence of resolutions $0 \longrightarrow \mathcal{S}^*(\mathcal{A}) \longrightarrow \mathcal{S}^*(\mathcal{B}) \longrightarrow \mathcal{S}^*(\mathcal{C}) \longrightarrow 0$.)

2.2.7 Compactly supported cohomology. It is sometimes more convenient to re-index the homology groups and to see them as *compactly supported cohomology groups* (see also 2.2.5.3 above). Because of this, we define:

$$H_c^n(\mathcal{G};-) = H_{-n}(\mathcal{G};-)$$

(which give a precise meaning to "$H_c^*(B\mathcal{G};\mathcal{A})$").The same applies to the functors $L_n\varphi_!$ introduced in the next paragraph: if $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ is a homomorphism, we define $R^n\varphi_! := L_{-n}\varphi_! : \underline{Ab}(\mathcal{K}) \longrightarrow \underline{Ab}(\mathcal{G})$. With these notations, Leray spectral sequence of 2.3.4 will becomes a (cohomological) spectral sequence with E^2-term $H_c^p(\mathcal{G}; R^q\varphi_!\mathcal{A}) \Longrightarrow H_c^{p+q}(\mathcal{K};\mathcal{A})$. If the fibers x/φ are oriented k-dimensional manifolds, the transgression of this spectral sequence will give "the integration along the fibers" map:

$$\int_{\text{fiber}} : H_c^*(\mathcal{G};\mathbb{R}) \longrightarrow H_c^{*-k}(\mathcal{K};\mathbb{R}).$$

2.2.8 Basic cohomology. Let $\mathcal{G}$ be a smooth étale groupoid. The space $\Omega^*_{c,basic}(\mathcal{G})$ of compactly supported basic forms is defined as the Cokernel of:

$$\Omega_c^*(\mathcal{G}^{(1)}) \stackrel{d_0-d_1}{\longrightarrow} \Omega_c^*(\mathcal{G}^{(0)}),$$

where d_0, d_1 are the maps coming from the nerve of $\mathcal{G}$. In other words, $\Omega^*_{c,basic}(\mathcal{G}) = \Gamma_{\mathcal{G}}(\Omega^*)$. The *basic compactly supported cohomology* of $\mathcal{G}$, denoted $H^*_{c,basic}(\mathcal{G})$, is defined as the cohomology of the complex $\Omega^*_{c,basic}(\mathcal{G})$ (with the differential induced by DeRham differential on $\Omega^*_c(\mathcal{G})$. There is an obvious projection from the reindexed homology (see 2.2.7):

$$j_b : H^*_c(\mathcal{G}; \mathbb{R}) \longrightarrow H^*_{c,basic}(\mathcal{G}), \tag{2.6}$$

which is an isomorphism if $\mathcal{G}$ is proper (cf. 2.3.9). In this case we also have:

$$\Omega^*_{c,basic}(\mathcal{G}) \cong \{\omega \in \Omega^*(\mathcal{G}^{(0)}) : \omega \text{ is } \mathcal{G} - \text{invariant, and } \pi(supp\,\omega) \text{ is compact in } M\,]$$

(where $\pi : \mathcal{G} \longrightarrow M$ is the projection into the quotient space of $\mathcal{G}$; see also 2.3.9). This map associates to $\omega \in \Omega^*_{c,basic}(\mathcal{G})$ the $\mathcal{G}$-invariant form $\tilde{\omega}$ on $\mathcal{G}^{(0)}$, given by:

$$\tilde{\omega}(x) = \sum_{x \xrightarrow{g} y} \omega(y) g\ .$$

2.2.9 Functoriality. Compactly supported cohomology of spaces (2.2.5.3) is covariant along local homeomorphisms and contravariant along proper maps. Analogous properties hold for homology of étale groupoids. Consider a homomorphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ between étale groupoids.

1. Suppose that φ is proper, in the sense that each $\varphi_n : \mathcal{K}^{(n)} \longrightarrow \mathcal{G}^{(n)}$ is a proper map (cf. 2.5.14). Then for any $\mathcal{G}$-sheaf $\mathcal{A}$ one obtains homomorphisms:

$$\Gamma_c(\mathcal{G}^{(n)}; \mathcal{A}_n) \longrightarrow \Gamma_c(\mathcal{K}^{(n)}; \varphi^*(\mathcal{A})_n)$$

by pullback, and hence a homomorphism:

$$\varphi^* : H_n(\mathcal{G}; \mathcal{A}) \longrightarrow H_n(\mathcal{K}; \varphi^*\mathcal{A})\ .$$

In other words, homology is contravariant along proper maps.

2. Suppose φ is étale, in the sense that each $\varphi_n : \mathcal{K}^{(n)} \longrightarrow \mathcal{G}^{(n)}$ is a local homeomorphism (it is not difficult to see that the assumption is only about φ_0). Let $\mathcal{S}$ be a c-soft $\mathcal{G}$-sheaf. For the sheaf $\mathcal{S}_n = \tau_n^*(\mathcal{S})$ on $\mathcal{G}^{(n)}$ summation along the fibers defines a homomorphism:

$$(\varphi_n)_! \tau_n^*(\varphi^*(\mathcal{S})) = (\varphi_n)_! \varphi_n^*(\tau_n^* \mathcal{S}) \longrightarrow \tau_n^*(\mathcal{S})\ ,$$

and hence a homomorphism :

$$\Gamma_c(\mathcal{K}^{(n)}; \varphi^*(\mathcal{S})_n) \longrightarrow \Gamma_c(\mathcal{G}^{(n)}; \mathcal{S}_n)\ .$$

These homomorphisms, for each $n \geq 0$, commute with the face operators (2.2). Since the functor φ^* is (always) exact and preserves c-softness (because φ is étale), this gives for each $\mathcal{G}$-sheaf $\mathcal{A}$ a homomorphism:

$$H_n(\mathcal{K}; \varphi^*\mathcal{A}) \longrightarrow H_n(\mathcal{G}; \mathcal{A})\ .$$

3. Suppose that φ is étale, and moreover suppose that for each n the square:

$$\begin{tikzcd} \mathcal{K}^{(n)} \arrow[r, "\tau_n"] \arrow[d, "\varphi_n"'] & \mathcal{K}^{(0)} \arrow[d, "\varphi_0"] \\ \mathcal{G}^{(n)} \arrow[r, "\tau_n"] & \mathcal{G}^{(0)} \end{tikzcd} \tag{2.7}$$

is a pullback. (Morphisms of this kind are exactly the projections $X \rtimes \mathcal{G} \longrightarrow \mathcal{G}$ associated to étale $\mathcal{G}$-spaces X .) For such a φ, there is an exact functor:

$$\varphi_! : \underline{Ab}(\mathcal{K}) \longrightarrow \underline{Ab}(\mathcal{G})$$

which preserves c-softness. (at the level of underlying sheaves, it is simply the functor $(\varphi_0)_! : \underline{Ab}(\mathcal{K}^{(0)}) \longrightarrow \underline{Ab}(\mathcal{G}^{(0)})$ of 2.5.9). For any c-soft $\mathcal{K}$-sheaf $\mathcal{B}$, there is a natural isomorphism:

$$\Gamma_c(\mathcal{K}_n; \mathcal{B}_n) = \Gamma_c(\mathcal{K}_n; \tau_n^{\,*}\mathcal{B}) = \Gamma_c(\mathcal{G}^{(n)}; (\varphi_n)_!\tau_n^{\,*}\mathcal{B}) = \Gamma_c(\mathcal{G}^{(n)}; \tau_n^{\,*}(\varphi_0)_!\mathcal{B}) = \Gamma_c(\mathcal{G}^{(n)}; \varphi_!(\mathcal{B}_n)),$$

for any $n \geq 0$. These yield an isomorphism

$$H_n(\mathcal{K}; \mathcal{B}) = H_n(\mathcal{G}; \varphi_!\mathcal{B}),$$

for any $\mathcal{K}$-sheaf $\mathcal{B}$.

Note that even if φ is not étale, a functor $\varphi_!$ can be defined in this way (but it is no longer exact). See also 2.3.5.4.

2.2.10 Hyperhomology. Consider any bounded below chain complex $\mathcal{A}_*$ of $\mathcal{G}$-sheaves. Let $\mathcal{A}_* \longrightarrow \mathcal{R}_*$ be a q.i. into a bounded below chain complex of c-soft $\mathcal{G}$-sheaves. (Such an $\mathcal{R}_*$ can be constructed for example by considering a resolution $\mathcal{A}_* \longrightarrow \mathcal{S}_*^0 \longrightarrow \ldots \mathcal{S}_*^d \longrightarrow 0$ as in 2.2.3 and then taking the total complex of the double complex $\mathcal{S}_q^{-p}$ $(p, q \in \mathbb{Z}, -d \leq p \leq 0)$. Define the hyperhomology $\mathbb{H}_*(\mathcal{G}; \mathcal{A}_*)$ to be the homology of the total complex associated to the double complex $B_*(\mathcal{G}; \mathcal{R}_*)$. This definition of $\mathbb{H}_*(\mathcal{G}; \mathcal{A}_*)$ does not depend on the choice of the resolution $\mathcal{R}_*$ (cf. lemma 2.2.2).

Proposition 2.2.11 *(Hyperhomology spectral sequences) Let $\mathcal{A}_*$ be a bounded below chain complex of $\mathcal{G}$-sheaves as above, and consider for each $q \in \mathbb{Z}$ the homology $\mathcal{G}$-sheaf $\mathcal{H}_q(\mathcal{A}_*)$. There are spectral sequences:*

$$E^2_{p,q} = H_p(\mathcal{G}; \mathcal{H}_q(\mathcal{A}_*)) \Longrightarrow \mathbb{H}_{p+q}(\mathcal{G}; \mathcal{A}_*) \ , \tag{2.8}$$

$$E^2_{p,q} = H_p(H_q(\mathcal{G}; \mathcal{A}_*) \Longrightarrow \mathbb{H}_{p+q}(\mathcal{G}; \mathcal{A}_*) \ . \tag{2.9}$$

*P*roof: Consider the truncated Godement resolution $0 \longrightarrow \mathcal{A}_* \longrightarrow \mathcal{S}_*^0 \longrightarrow \ldots \longrightarrow \mathcal{S}_*^d \longrightarrow 0$. It has the property that for each q, it also yields c-soft resolutions of the cycles $\mathcal{Z}_q$, the boundaries $\mathcal{B}_q$ and the homology $\mathcal{H}_q(\mathcal{A}_*)$. Write $\mathcal{C}$ for the triple complex:

$$\mathcal{C}_{p,q,r} = \Gamma_c(\mathcal{G}^{(p)}; \mathcal{S}_q^{-r}) \ ,$$

and let $\mathcal{D}$ be the double complex:

$$\mathcal{D}_{n,q} = \bigoplus_{p+q=n} \mathcal{C}_{p,q,r} \ .$$

The total complex of $\mathcal{C}$, and hence also that of $\mathcal{D}$, compute $\mathbb{H}(\mathcal{G};\mathcal{A}_*)$. Furthermore, by the property of the resolution just mentioned (and the fact that $\Gamma_c(\mathcal{G}^{(p)};-)$ preserves exact sequences of c-soft sheaves) we have for fixed p and r that:

$$H_q(\mathcal{C}_{p,*,r}) = \Gamma_c(\mathcal{G}^{(p)};\mathcal{H}_q(\mathcal{S}_*^{-r})) \ .$$

Hence, for a fixed n,

$$H_q(\mathcal{D}_{n,*}) = \bigoplus_{p+r=n} \Gamma_c(\mathcal{G}^{(p)};\mathcal{H}_q(\mathcal{S}_*^{-r})) \ .$$

But $\mathcal{H}^q(\mathcal{A}_*) \longrightarrow \mathcal{H}_q(\mathcal{S}_*^0) \longrightarrow \mathcal{H}_q(\mathcal{S}_*^1) \longrightarrow \ldots$ is a resolution of $\mathcal{H}^q(\mathcal{A}_*)$, so for a fixed q the double complex $\Gamma_c(\mathcal{G}^{(*)};\mathcal{H}_q(\mathcal{S}^{-*}))$ computes $H_*(\mathcal{G};\mathcal{H}_q(\mathcal{A}_*))$. Thus:

$$H_nH_q(\mathcal{D}_{*,*}) = H_n(\mathcal{G};\mathcal{H}_q(\mathcal{A}_*)) \ ,$$

and (2.8) is simply the spectral sequence $H_nH_q(\mathcal{D}) \Longrightarrow H_{n+q}(Tot(\mathcal{D}))$ for the double complex $\mathcal{D}$. The second spectral sequence is obvious. □

2.2.12 Cap product. For an étale groupoid $\mathcal{G}$, the Ext-groups (1.2.6) act on the homology by a *cap product*:

$$H_n(\mathcal{G};\mathcal{B}) \otimes Ext^p(\mathcal{B},\mathcal{A}) \xrightarrow{\cap} H_{n-p}(\mathcal{G};\mathcal{A}) \ . \tag{2.10}$$

This follows from the general considerations in 1.2.6. For example, for $p = 1$ an element of $Ext^1(\mathcal{B},\mathcal{A})$ can be represented by an exact sequence $0 \longleftarrow \mathcal{B} \longleftarrow \mathcal{E} \longleftarrow \mathcal{A} \longleftarrow 0$, which yields a boundary map $H_n(\mathcal{G};\mathcal{B}) \longrightarrow H_{n-1}(\mathcal{G};\mathcal{A})$ for the long exact sequence of 2.2.6. For $p > 1$, the cap product can be constructed in the same way (by decomposing a longer extension $0 \longleftarrow \mathcal{B} \longleftarrow \mathcal{E}_1 \longleftarrow \ldots \longleftarrow \mathcal{E}_n \longleftarrow \mathcal{A} \longleftarrow 0$ into short exact sequences).

In particular, when working over $\mathbb{R}$, this yields a simple description of the cap product relating homology and cohomology of étale groupoids:

$$H_n(\mathcal{G};\mathcal{B}) \otimes H^p(\mathcal{G};\mathcal{A}) \xrightarrow{\cap} H_{n-p}(\mathcal{G};\mathcal{B} \otimes_{\mathbb{R}} \mathcal{A}) \ .$$

The cap product satisfies the usual "projection formula" for a morphism $\alpha : \mathcal{C} \longrightarrow \mathcal{A}$. Explicitly, α induces $\alpha_* : H_*(\mathcal{G};\mathcal{C}) \longrightarrow H_*(\mathcal{G};\mathcal{A})$ and $\alpha_* : Ext^p(\mathcal{B};\mathcal{C}) \longrightarrow Ext^p(\mathcal{B};\mathcal{A})$, and we have for any $u \in H_n(\mathcal{G};\mathcal{B})$ and $\xi \in Ext^p(\mathcal{B};\mathcal{C})$ that:

$$\alpha_*(u \cap \xi) = u \cap \alpha_*(\xi) \ .$$

(For $p = 1$ this is just the naturality of the exact sequence 2.2.6).

2.2.13 Remark. The d^2 boundary of the hyperhomology spectral sequence (2.8):

$$\alpha^2_{p,q} : H_p(\mathcal{G};\mathcal{H}_q(\mathcal{A}_*)) \longrightarrow H_{p-2}(\mathcal{G};\mathcal{H}_{q+1}(\mathcal{A}_*))$$

is given by the cap product with an element $u_q(\mathcal{A}_*) \in Ext^2(\mathcal{H}_q(\mathcal{A}_*),\mathcal{H}_{q+1}(\mathcal{A}_*))$. Let $\pi_q : \mathcal{Z}_q(\mathcal{A}_*) \longrightarrow \mathcal{H}_q(\mathcal{A}_*)$ be the quotient map from the sheaf of cycles $\mathcal{Z}_q(\mathcal{A}_*)$. Then the extension

$$0 \longleftarrow \mathcal{H}_q(\mathcal{A}_*) \xleftarrow{\pi_q} \mathcal{Z}_q(\mathcal{A}_*) \xleftarrow{d} \mathcal{A}_{q+1} \longleftarrow \mathcal{Z}_{q+1}(\mathcal{A}_*) \longleftarrow 0$$

defines an element $v \in Ext^2(\mathcal{H}_q(\mathcal{A}_*),\mathcal{Z}_{q+1}(\mathcal{A}_*))$, and $u_q(\mathcal{A}_*)$ is $(\pi_{q+1})_*(\mathcal{A}_*)$. This is immediate from the construction of the spectral sequence (proof of 2.2.11), and the general description of the boundaries of the spectral sequence induced by a double complex.

2.2.14 Remark. Recall that a topological category $\mathcal{G}$ is said to be étale if all its structure maps are local homeomorphisms. Thus, such a category is given by maps as in (1.1), except for the absence of an inverse $i : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(1)}$. The definitions and the results of this section hold equally well for the more general context of such étale categories, and for this reason we have tried to write the proofs in such a way that they apply verbatim to this general context. The same is true for the next section, provided one takes sufficient care to define Morita equivalence for categories in the appropriate way. Because of this we need another technical tool when we deal with étale categories; it is a variant of a well known principle due to Segal (see Prop.(2.*I*) in [93]):

2.2.15 Lemma and definition: *Let $\mathcal{G}$ and $\mathcal{H}$ be étale categories. A continuous functor $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ is called a strong deformation retract of $\mathcal{H}$ if there is a continuous functor $\psi : \mathcal{H} \longrightarrow \mathcal{G}$ (called retraction) and a continuous natural transformation of functors $F : \varphi \circ \psi \longrightarrow Id_{\mathcal{H}}$ (called strong deformation retraction) such that $\psi \circ \varphi = Id_{\mathcal{G}}, F(\varphi(c)) = id_{\varphi(c)}$ for all $c \in \mathcal{G}^{(0)}$ and the maps φ, ψ, F are étale.*

In this case, for any $\mathcal{H}$-sheaf $\mathcal{A}$, φ induces an isomorphism:

$$H_*(\mathcal{G}; \varphi^*\mathcal{A}) \xrightarrow{\sim} H_*(\mathcal{H}; \mathcal{A}).$$

proof: Denote $\varphi \circ \psi$ by l. Let Φ the map induced by φ:

$$\Phi \ : B_\bullet(\mathcal{G}; \varphi^*\mathcal{A}) \longrightarrow B_*(\mathcal{H}; \mathcal{A}), \quad \Phi(a|g_1, ..., g_n) = (a|\varphi(g_1), ..., \varphi(g_n)).$$

Since φ is étale, $\varphi^* : Sh(\mathcal{H}^{(0)}) \longrightarrow Sh(\mathcal{G}^{(0)})$ preserves c-softness ([63]) so it is enough to prove that Φ is a homotopy equivalence of chain complexes, when $\mathcal{A}$ is c-soft. Define a chain map:

$$\Psi : B_*(\mathcal{H}; \varphi^*\mathcal{A}) \longrightarrow B_*(\mathcal{G}; \mathcal{A}), \quad \Psi(a|h_1, ..., h_n) = (aF(t(h_1))|\psi(h_1), ..., \psi(h_n)).$$

We have $\Psi \circ \Phi = Id$ and $\Phi \circ \Psi$ is homotopic to Id by the following homotopy:

$$h : B_*(\mathcal{H}; \varphi^*\mathcal{A}) \longrightarrow B_{*+1}(\mathcal{H}; \varphi^*\mathcal{A}), \quad h = \sum_{i=0}^{n} (-1)^i h_i,$$

$$h_i(a|h_1, ..., h_n) = (a|h_1, ..., h_i, F(s(h_i)), l(h_{i+1}), ..., l(h_n)). \ \square$$

2.3 Leray spectral sequence; Morita invariance

In this section we construct for each morphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ between étale groupoids a functor $\varphi_!$ from c-soft $\mathcal{K}$-sheaves to c-soft $\mathcal{G}$-sheaves. We derive a Leray spectral sequence for this functor (2.3.4), of which the invariance of homology under Morita equivalences will be an immediate consequence (2.3.6).

2.3.1 Comma groupoids of a homomorphism. Let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be a homomorphism of étale groupoids. For each point $x \in \mathcal{G}^{(0)}$ consider the "comma groupoid" x/φ, whose objects are the pairs $(y, g : x \longrightarrow \varphi(y))$ where $y \in \mathcal{K}^{(0)}$ and $g \in \mathcal{G}^{(1)}$. An arrow $k : (y, g) \longrightarrow (y', g')$ in x/φ is an arrow $k : y \longrightarrow y'$ in $\mathcal{K}$ such that $\varphi(k) \circ g = g'$. When equipped with the obvious fibered product topology, x/φ is again an étale groupoid.

It should be viewed as the fiber of φ above x; more exactly, there is a commutative diagram (see also 2.3.7):

$$\begin{tikzcd} x/\varphi \arrow[r, "\pi_x"] \arrow[d] & \mathcal{K} \arrow[d, "\varphi"] \\ 1 \arrow[r, "x"] & \mathcal{G} \end{tikzcd}$$

Note that an arrow $g : x \longrightarrow x'$ in $\mathcal{G}$ induces a homomorphism:

$$g^* : x'/\varphi \longrightarrow x/\varphi \tag{2.11}$$

by composition. Thus the groupoids x/φ together form a right $\mathcal{G}$-bundle of groupoids. (If $\varphi_0 : \mathcal{K}^{(0)} \longrightarrow \mathcal{G}^{(0)}$ is a local homeomorphism, then it is a $\mathcal{G}$-sheaf of groupoids.)

More generally, for any $A \subset \mathcal{G}^{(0)}$ *the comma groupoid* A/φ is defined by:

$$(A/\varphi)^{(i)} = \bigcup_{x \in A} (x/\varphi)^{(i)} \subset \mathcal{K}^{(i)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \quad , i \in \{0, 1\}$$

(with the induced topology). The nerve of A/φ consists of the spaces:

$$(A/\varphi)^{(n)} = \{(y_0 \xleftarrow{k_1} \ldots \xleftarrow{k_n} y_n, \varphi(y_n) \xleftarrow{g} x) : k_i \in \mathcal{K}^{(1)}, g \in \mathcal{G}^{(1)}, x \in A\} .$$

When $\varphi = id : \mathcal{G} \longrightarrow \mathcal{G}$, these are simply denoted by $x/\mathcal{G}$, $A/\mathcal{G}$. Dually one defines the comma groupoids $\varphi/x, \varphi/A, \mathcal{G}/x, \mathcal{G}/A$ (consisting on arrows "going into x").

2.3.2 The functors $\varphi_!$, $L_n\varphi_!$, $\mathcal{L}\varphi_!$. Let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be as above, and let $\mathcal{A}$ be a $\mathcal{K}$-sheaf. We define a simplicial $\mathcal{G}$-sheaf $B_*(\varphi; \mathcal{A})$ in analogy with the definition of the bar-complex 2.2.1. On the spaces $\mathcal{K}^{(n)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)}$ (which form the nerve of $\mathcal{G}^{(0)}/\varphi$, cf. 2.3.1) of strings of the form:

$$\varphi(y_0) \xleftarrow{\varphi(k_1)} \ldots \xleftarrow{\varphi(k_n)} \varphi(y_n) \xleftarrow{g} x$$

we define the maps:

$$\alpha_n : \mathcal{K}^{(n)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \longrightarrow \mathcal{K}^{(0)}, \ (k_1, \ \ldots k_n, g) \mapsto t(k_1) \ ,$$

$$\beta_n : \mathcal{K}^{(n)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}, \ (k_1, \ \ldots k_n, g) \mapsto s(g) \ .$$

Notice that any α_n is étale. For any $n \geq 0$ we set:

$$B_n(\varphi; \mathcal{A}) = (\beta_n)_! \alpha_n^* \mathcal{A} \ .$$

By 2.5.9, the stalk at $x \in \mathcal{G}^{(0)}$ is described by:

$$B_n(\varphi; \mathcal{A})_x = \Gamma_c(\beta_n^{-1}(x); \alpha_n^* \mathcal{A}) = B_n(x/\varphi; \pi_x^* \mathcal{A}) \ . \tag{2.12}$$

This gives us the (stalk-wise) definition of the simplicial structure on $B_n(\varphi; \mathcal{A})$. To check the continuity, let us just remark that the boundaries can be described globally. Indeed, using the maps:

$$d_i : \mathcal{K}^{(n)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} = (\mathcal{G}^{(1)}/\varphi)^{(n)} \longrightarrow \mathcal{K}^{(n-1)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} = (\mathcal{G}^{(1)}/\varphi)^{(n-1)}$$

coming from the nerve of $\mathcal{G}^{(0)}/\varphi$ (see 1.1.13, 2.3.1), we have $\beta_n = \beta_{n-1}d_i$ (for all $0 \leq i \leq n$) and there are evident maps $\alpha_n^*\mathcal{A} \longrightarrow d_i^*\alpha_{n-1}^*\mathcal{A}$ (compare to the definition of (2.2)); the boundaries of $B_*(\varphi;\mathcal{A})$ are in fact:

$$(\beta_n)_!\alpha_n^*\mathcal{A} = (\beta_{n-1})_!(d_i)_!\alpha_n^*\mathcal{A} \longrightarrow (\beta_{n-1})_!(d_i)_!d_i^*\alpha_{n-1}^*\mathcal{A} \longrightarrow (\beta_{n-1})_!\alpha_{n-1}^*\mathcal{A}\ .$$

To describe the action of $\mathcal{G}$ on $B_*(\varphi;\mathcal{A})$, let $g : x \longrightarrow x'$ be an arrow in $\mathcal{G}$. The homomorphism (2.11) induces an obvious map $B_*(x'/\varphi;\pi_{x'}^*\mathcal{A}) \longrightarrow B_*(x/\varphi;\pi_x^*\mathcal{A})$ which, via (2.12), is the action by g ($: B_*(\varphi;\mathcal{A})_{x'} \longrightarrow B_*(\varphi;\mathcal{A})_x$).

If $\mathcal{S}$ is a c-soft $\mathcal{K}$-sheaf, $\mathcal{L}\varphi_!\mathcal{S}$ is defined as the chain complex of $\mathcal{G}$-sheaves (associated to the simplicial complex) $B_*(\varphi;\mathcal{S})$. If $\mathcal{S}_*$ is a bounded below chain complex of c-soft $\mathcal{K}$-sheaves, define $\mathcal{L}\varphi_!\mathcal{S}_*$ as the total complex of $B_*(\varphi;\mathcal{S}_*)$. For an arbitrary $\mathcal{K}$-sheaf $\mathcal{A}$, $\mathcal{L}\varphi_!\mathcal{A}$ is defined to be $B_*(\varphi;\mathcal{S}^{-*})$ where $\mathcal{S}^*$ is a resolution of $\mathcal{A}$ as in 2.2.4. More generally, we define $\mathcal{L}\varphi_!\mathcal{A}_*$ for any bounded below chain complex of $\mathcal{K}$-sheaves using a resolution $\mathcal{A}_* \longrightarrow \mathcal{R}_*$ as in 2.2.10. As in the case of homology (cf. 2.2.4) , we see that $\mathcal{L}\varphi_!$ is well defined up to quasi-isomorphism; in particular, the "derived functors":

$$L_n\varphi_!(-) = \mathcal{H}_n(\mathcal{L}\varphi_!(-)) : \underline{Ab}(\mathcal{K}) \longrightarrow \underline{Ab}(\mathcal{G})$$

are well defined up to isomorphism. For $n = 0$ we simply denote $L_0\varphi_! : \underline{Ab}(\mathcal{K}) \longrightarrow \underline{Ab}(\mathcal{G})$ by $\varphi_!$.

Proposition 2.3.3 *For any $x \in \mathcal{G}^{(0)}$, there are isomorphisms:*

$$(L_n\varphi_!(\mathcal{A}))_x \cong H_n(x/\varphi;\pi_x^*\mathcal{A}) \quad \text{for all} \quad x \in \mathcal{G}^{(0)}\ . \tag{2.13}$$

Proof: This is an immediate consequence of relation (2.12), and the fact that α_n^*'s preserve c-softness since they are induced by étale maps. □

Theorem 2.3.4 *("Leray-Hochschild-Serre spectral sequence") For any homomorphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ between étale groupoids and any $\mathcal{K}$-sheaf $\mathcal{A}$ there is a natural spectral sequence:*

$$E^2_{p,q} = H_p(\mathcal{G};L_q\varphi_!\mathcal{A}) \Longrightarrow H_{p+q}(\mathcal{K};\mathcal{A})\ .$$

Proof: The spectral sequence follows from an isomorphism:

$$\mathbb{H}_*(\mathcal{G};\mathcal{L}\varphi_!\mathcal{A}) \cong H_*(\mathcal{K};\mathcal{A}) \tag{2.14}$$

and 2.2.11 applied to $\mathcal{L}\varphi_!(\mathcal{A})$.

To prove (2.14) we consider the double complex $\mathcal{C}_{p,q}(\mathcal{A}) = B_p(\mathcal{G};B_q(\varphi;\mathcal{A}))$ and we show that there are maps $\mathcal{C}_{0,q}(\mathcal{A}) \longrightarrow B_q(\mathcal{K};\mathcal{A})$, functorial in $\mathcal{A}$, such that the augmented complex

$$\ldots \longrightarrow \mathcal{C}_{2,q}(\mathcal{A}) \longrightarrow \mathcal{C}_{1,q}(\mathcal{A}) \longrightarrow \mathcal{C}_{0,q}(\mathcal{A}) \longrightarrow B_q(\mathcal{K};\mathcal{A}) \longrightarrow 0 \tag{2.15}$$

is acyclic for any c-soft $\mathcal{K}$-sheaf $\mathcal{A}$.

Using the diagram:

$$\begin{tikzcd}
\mathcal{K}^{(q)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(p+1)} \arrow[r, "u"] \arrow[d, "v"'] & \mathcal{G}^{(p)} \arrow[d, "\tau_p"] \\
\mathcal{K}^{(q)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \arrow[r, "\beta_q"] \arrow[d, "w"'] \arrow[dr, "\alpha_q"] & \mathcal{G}^{(0)} \\
\mathcal{K}^{(q)} \arrow[r, "\tau_q"] & \mathcal{K}^{(0)}
\end{tikzcd}$$

where $\alpha_q, \beta_q, \tau_q, \tau_p$ are those defined before, v, w are the projections into the first components, u is the projection into the last components and $\gamma_q = wv$, we have by the general properties of the Appendix:

$$\begin{aligned}
\mathcal{C}_{p,q}(\mathcal{A}) &= \Gamma_c(\mathcal{G}^{(p)}; \tau_p^*(\beta_q)_! \alpha_q^* \mathcal{A}) \\
&= \Gamma_c(\mathcal{G}^{(p)}; u_! v^* \alpha_q^* \mathcal{A}) \\
&= \Gamma_c(\mathcal{K}^{(q)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(p+1)}; v^* \alpha_q^* \mathcal{A}) \\
&= \Gamma_c(\mathcal{K}^{(q)}; (\gamma_q)_! v^* \alpha_q^* \mathcal{A}) \\
&= \Gamma_c(\mathcal{K}^{(q)}; (\gamma_q)_! (\gamma_q)^* \tau_q^* \mathcal{A}) \ , \\
B_q(\mathcal{K}; \mathcal{A}) &= \Gamma_c(\mathcal{K}^{(q)}; \tau_q^* \mathcal{A}) \ .
\end{aligned}$$

Via these equalities, the augmented chain complex (2.15) commies from an augmented simplicial sheaf on $\mathcal{K}^{(q)}$ whose stalk at $x \xleftarrow{k_1} \ \ldots \ \xleftarrow{k_q} y$ has the form:

$$\ldots \longrightarrow \bigoplus_{\varphi(y) \xleftarrow{f} x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} x_2} \mathcal{A}_x \longrightarrow \bigoplus_{\varphi(y) \xleftarrow{f} x_0 \xleftarrow{g_1} x_1} \mathcal{A}_x \longrightarrow \bigoplus_{\varphi(y) \xleftarrow{f} x_0} \mathcal{A}_x \longrightarrow \mathcal{A}_x \ .$$

This is in fact the augmented bar complex computing the homology of the (contractible, discrete) category $\mathcal{G}/\varphi(y)$ with constant coefficients $\mathcal{A}_x$. In particular it is acyclic (with the usual contraction $(f, g_1, \ \ldots, g_n; a) \mapsto (1, f, g_1, \ \ldots, g_n; a)$). $\square$

2.3.5 Remarks and examples.

1). From the construction of the spectral sequence, and Remark 2.2.13, it follows that the boundary at the level of $E^2_{p,q}$ is of type:

$$d^2_{p,q} = - \cap u_q : E^2_{p,q} \longrightarrow E^2_{p-2,q+1} \text{ with } u_q \in Ext_{\mathcal{G}}(L_q \varphi_! \mathcal{A}, L_{q+1} \varphi_! \mathcal{A})$$

constructed as in 2.2.13 (and replacing $\mathcal{A}_*$ by $\mathcal{L}\varphi_!(\mathcal{A})$).

2). The isomorphism (2.14) is actually a consequence of the quasi-isomorphism $\mathcal{L}\varphi_! pt_! = pt_!$ (where pt is the map into the trivial groupoid); this is a particular case of the naturality property $\mathcal{L}\varphi_! \mathcal{L}\psi_! = \mathcal{L}(\varphi \circ \psi)_!$ ("up to quasi-isomorphism"), which can be proved in an analogous way. Compare to [97].

3). If $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ is étale , $\mathcal{S} \in \underline{Ab}(\mathcal{K})$, then there is no need of c-soft resolutions to define $\mathcal{L}\varphi_! \mathcal{S}$. Indeed, the condition on φ implies that the maps β_n defined in 2.3.2 are étale, so there is a quasi-isomorphism $\mathcal{L}\varphi_! \mathcal{S} \simeq B_*(\varphi; \mathcal{S})$.

4). Let $\varphi : \mathcal{H} \longrightarrow \mathcal{G}$ be a morphism for which all the squares in (2.7) are pullbacks. Recall that in this case, the functor $(\varphi_0)_! : \underline{Ab}(\mathcal{K}^{(0)}) \longrightarrow \underline{Ab}(\mathcal{G}^{(0)})$ "extends" to a functor $\varphi_! : \underline{Ab}(\mathcal{K}) \longrightarrow \underline{Ab}(\mathcal{G})$, making the diagram:

$$\begin{tikzcd}
\underline{Ab}(\mathcal{K}) \arrow[r, "\text{forget}"] \arrow[d, "\varphi_!"'] & \underline{Ab}(\mathcal{K}^{(0)}) \arrow[d, "(\varphi_0)_!"] \\
\underline{Ab}(\mathcal{G}) \arrow[r, "\text{forget}"] & \underline{Ab}(\mathcal{G}^{(0)})
\end{tikzcd}$$

commute. This simple minded functor of 2.2.9 agrees (up to quasi-isomorphism) with the functor $\mathcal{L}\varphi_!$, described in 2.3.2. Indeed, for such a morphism φ and a point $x \in \mathcal{G}^{(0)}$

the comma groupoid x/φ is a space (or more precisely, equivalent to the groupoid corresponding to a space, cf. 1.1.3.1). In this case, the spectral sequence 2.3.4 degenerates for c-soft sheaves $\mathcal{B}$ (but not for arbitrary sheaves). If φ is moreover étale, it does always degenerate, and yields the isomorphism already proved in 2.2.9.3.

Corollary 2.3.6 *("Morita invariance") For any Morita equivalence $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ and any $\mathcal{G}$-sheaf $\mathcal{A}$ there is a natural isomorphism*

$$H_p(\mathcal{G}; \mathcal{A}) \cong H_p(\mathcal{K}; \varphi^*\mathcal{A}).$$

Proof: Theorem 2.3.4 gives a spectral sequence $H_p(\mathcal{G}; L_q\varphi_!\varphi^*\mathcal{A}) \Longrightarrow H_{p+q}(\mathcal{K}; \varphi^*\mathcal{A})$. By (2.13) the stalk of $L_q\varphi_!\varphi^*\mathcal{A}$ at a point $x \in \mathcal{G}^{(0)}$ computes the homology of the nerve of x/φ. If φ is a Morita equivalence, this nerve is a contractible simplicial set. Thus, the spectral sequence degenerates to give an isomorphism:

$$H_p(\mathcal{G}; L_0\varphi_!\varphi^*\mathcal{A}) \cong H_p(\mathcal{K}; \varphi^*\mathcal{A}) \ .$$

It thus suffices to observe that the $\mathcal{G}$-sheaf $L_0\varphi_!\varphi^*\mathcal{A}$ is isomorphic to $\mathcal{A}$ itself. $\square$

2.3.7 Fibered products of groupoids. For homomorphisms $\varphi : \mathcal{H} \longrightarrow \mathcal{G}$ and $\psi : \mathcal{K} \longrightarrow \mathcal{G}$, their fibered product $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$:

$$\begin{tikzcd} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} \arrow[r, "q"] \arrow[d, "p"'] & \mathcal{K} \arrow[d, "\psi"] \\ \mathcal{H} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

is constructed as follows. The space of objects is the space $\mathcal{H}^{(0)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{K}^{(0)}$ of triples (y, g, z) where $y \in \mathcal{H}^{(0)}, z \in \mathcal{K}^{(0)}$ and $g : \varphi(y) \longrightarrow \psi(z)$ in $\mathcal{G}$. An arrow $(y, g, z) \longrightarrow (y', g', z')$ is a pair of arrows $h : y \longrightarrow y'$ in $\mathcal{H}$ and $k : z \longrightarrow z'$ in $\mathcal{K}$ such that $g' \circ \varphi(h) = \psi(k) \circ g$. The groupoid $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is again étale if $\mathcal{G}, \mathcal{H}, \mathcal{K}$ are. This notion of fibered product is the appropriate one for groupoids and (generalized) morphisms described in 1.1.6 and 1.1.9. In particular, if $\psi : \mathcal{K} \longrightarrow \mathcal{G}$ is a Morita equivalence, then so is $p : \mathcal{H} \times_{\mathcal{G}} \mathcal{K} \longrightarrow \mathcal{H}$.

Proposition 2.3.8 *(Change-of-base formula) Consider a fibered product $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ of étale groupoids as in 2.3.7. For any (c-soft) $\mathcal{K}$-sheaf $\mathcal{S}$, there is a canonical quasi-isomorphism:*

$$\varphi^* \ \mathcal{L}\varphi_!(\mathcal{S}) \simeq \mathcal{L}p_!q^*(\mathcal{S}) \ .$$

Proof: For $y \in \mathcal{H}^{(0)}$, the comma groupoid y_0/p is Morita equivalent to the comma groupoid $\varphi(y_0)/\psi$, (by a Morita equivalence $y_0/p \longrightarrow \varphi(y_0)/\psi$ which is continuous in y_0 and which respects the action by $\mathcal{H}$). Using this observation, the proposition follows in a straightforward way from 2.3.6 and 2.5.11. $\square$

2.3.9 Orbifolds. As we have already mentioned in 1.1.3.5, orbifolds are characterized by étale groupoids which are proper. Let $\mathcal{G}$ be such a groupoid. The "leaf space" M of $\mathcal{G}$ (i.e. the space obtained from $\mathcal{G}^{(0)}$ dividing out by the equivalence relation $x \sim y$

iff there is an arrow in $\mathcal{G}$ from x into y), will be a Hausdorff space; it is the underlying space of the orbifold induced by $\mathcal{G}$ [75]. The obvious projection $\pi : \mathcal{G} \longrightarrow M$ induces a spectral sequence:

$$H_p(M; L_q\pi_!\mathcal{A}) \Longrightarrow H_{p+q}(\mathcal{G}; \mathcal{A}),$$

for any $\mathcal{A} \in \underline{Ab}(\mathcal{G})$. The stalk of $L_q\pi_!$ at $x \in M$ is:

$$(L_q\pi_!)_x \cong H_q(G_{\tilde{x}}; \mathcal{A}_{\tilde{x}}),$$

where $\tilde{x} \in \mathcal{G}^{(0)}$ is any lift of x, and $G_{\tilde{x}}$ is the (finite) group $\{\gamma \in \mathcal{G}^{(1)} : s(\gamma) = t(\gamma) = \tilde{x}\}$ (this follows from 2.3.3 and the Morita equivalence $x/\pi \sim G_{\tilde{x}}$.

In particular, for $\mathcal{A} \in \underline{Sh}_{\mathbb{R}}(\mathcal{G})$, the spectral sequence degenerates and gives an isomorphism:

$$H_*(\mathcal{G}; \mathcal{A}) \cong H_c^{-*}(M; \pi_!\mathcal{A}). \tag{2.16}$$

This also shows that the "co-invariants functor":

$$\Gamma_{\mathcal{G}}(-) := H_0(\mathcal{G}; -) : \underline{Sh}_{\mathbb{R}}(\mathcal{G}) \longrightarrow \underline{Vs}$$

is left exact and that $H_c^*(\mathcal{G}; -)$ (see 2.2.7) are the right derived functors of $\Gamma_{\mathcal{G}}$.

2.3.10 Relations to cyclic cohomology. As an illustration, we now briefly describe how the Hochschild and the cyclic homology of the convolution algebra can be described in terms of our sheaf homology. We refer to Chapter 4 for details.

Let $\mathcal{G}$ be a (not necessarily Hausdorff) smooth groupoid. We introduce the groupoid of loops $\Omega(\mathcal{G}) = B^{(0)} \rtimes_{\mathcal{G}^{(0)}} \mathcal{G}$, where $B^{(0)} = \{\gamma \in \mathcal{G}^{(1)} : s(\gamma) = t(\gamma)\}$ is the space of loops with the $\mathcal{G}$-action given by conjugation $(\gamma, g) \mapsto g^{-1}\gamma g$. It has $B^{(0)}$ as space of objects, while the arrows between two loops γ and γ' are all the arrows g from $s(\gamma)$ to $s(\gamma')$ with the property $\gamma = g^{-1}\gamma' g$. This groupoid appears in the computation of the cyclic/Hochschild cohomology of the convolution algebra $\mathcal{C}_c^\infty(\mathcal{G})$ associated to the groupoid $\mathcal{G}$ (where, again, in the non-Hausdorff case one uses the adapted Γ_c of the Appendix to define $\mathcal{C}_c^\infty(\mathcal{G})$). The Hochschild homology can be expressed in terms of our homology, as:

$$HH_*(\mathcal{C}_c^\infty(\mathcal{G})) = \mathbb{H}_*(\Omega(\mathcal{G}); \mathcal{A}_{tw}^\natural),$$

where $\mathcal{A}$ is the sheaf of smooth functions, and $\mathcal{A}_{tw}^\natural$ is the associated simplicial sheaf defined in 4.5.6 of Chapter 4. Using Connes' category Λ, and its version Λ_∞ (see 1.5.5), these homology groups are isomorphic to $H_*(\Lambda_\infty \times \Omega(\mathcal{G}); \mathcal{A}_{tw}^\natural)$. We remark that $\mathcal{G} \subset \Omega(\mathcal{G})$ as units, and we get, as "localization at units":

$$HH_*(\mathcal{C}_c^\infty(\mathcal{G}))_{[1]} = \bigoplus_{p+q=n} H_p(\mathcal{G}; \Omega^q) \ .$$

In order to describe the cyclic homology of the convolution algebra in terms of our homology theory, one introduces a new category $\Lambda_\infty \wedge \Omega(\mathcal{G})$ which is the étale category obtained from $\Lambda_\infty \times \Omega(\mathcal{G})$ by imposing the relations $(t_n^{n+1}, id_\gamma) = (id_{[n]}, \gamma)$, for all $\gamma \in B^{(0)}, n \geq 0$. Then $\mathcal{A}_{tw}^\natural$ is a sheaf on $\Lambda_\infty \wedge \Omega(\mathcal{G})$; Proposition 4.6.1 and Proposition 4.5.7 (see also the definition in 4.4.1) will imply:

$$HC_*(\mathcal{C}_c^\infty(\mathcal{G})) = H_*(\Lambda_\infty \wedge \Omega(\mathcal{G}); \mathcal{A}_{tw}^\natural).$$

In cyclic homology, the previous "localization at units" becomes:

$$HP_i(\mathcal{C}_c^\infty(\mathcal{G}))_{[1]} = \prod_k H_{i+2k}(\mathcal{G}) \quad , i \in \{0,1\}.$$

More generally, any $\mathcal{G}$-invariant subspace $\mathcal{O} \subset B^{(0)}$ defines a groupoid $\Omega_{\mathcal{O}}(\mathcal{G}) = \mathcal{O} \rtimes \mathcal{G}^{(0)} \subset \Omega(\mathcal{G})$, and the localized Hochschild and cyclic homology (indicated by the subscript $\mathcal{O}$). When $\mathcal{O}$ is elliptic (i.e. $ord(\gamma) < \infty$, for all $\gamma \in \mathcal{O}$), it will be shown in Theorem 4.6.4 that:

$$HP_i(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \prod_k H_{i+2k}(\Omega_{\mathcal{O}}(\mathcal{G})) \quad , \ i \in \{0,1\}. \tag{2.17}$$

2.4 Verdier duality

In this section all sheaves are sheaves of $\mathbb{R}$- modules, i.e. real vector spaces (we can actually use any field of characteristic 0), and Hom and $\otimes$ are all over $\mathbb{R}$. We will establish a Verdier type duality for the functor $\mathcal{L}\varphi_!$ (i.e. $\varphi_!$ viewed at the level of the derived categories) and an associated functor $\varphi^!$ to be described, by extending one of the standard treatments [63] to étale groupoids. (But our presentation is self-contained.) As a special case, we will obtain a Poincaré duality between the (Haefliger) cohomology of étale groupoids described in Section 2 and the homology theory (Section 4).

2.4.1 Tensor products. As a preliminary remark, we observe the following properties of tensor products over $\mathbb{R}$. First, if $\mathcal{A}$ is a c-soft sheaf on a space Y and $\mathcal{B}$ is any other sheaf, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is again c-soft. Moreover, for the constant sheaf associated to a vector space V we have $\Gamma_c(Y; \mathcal{A} \otimes V) = \Gamma_c(Y; \mathcal{A}) \otimes V$ (cf. 2.5.6). It follows by comparing the stalks that for a map $f : Y \longrightarrow X$, also:

$$f_!(\mathcal{A} \otimes \varphi^*\mathcal{B}) = f_!(\mathcal{A}) \otimes \mathcal{B}$$

for any sheaf $\mathcal{B}$ on X (see [16, 63]). These properties extend to a morphism $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ of étale groupoids: for a c-soft $\mathcal{K}$-sheaf $\mathcal{A}$ and any $\mathcal{G}$-sheaf $\mathcal{B}$, there is an isomorphism:

$$\varphi_!(\mathcal{A} \otimes \varphi^*\mathcal{B}) = \varphi_!(\mathcal{A}) \otimes \mathcal{B} \ .$$

2.4.2 The sheaves $\mathbb{R}[V]$. Let us fix an étale groupoid $\mathcal{K}$. Any open set $V \subset \mathcal{K}^{(0)}$ gives a free $\mathcal{K}$-sheaf (see 1.2.2.3) of sets $\tilde{V}$, given by the étale map $s : t^{-1}(V) \longrightarrow \mathcal{K}^{(0)}$ and the $\mathcal{K}$-action defined by composition. Let $\mathbb{R}[V]$ be the free $\mathbb{R}$-module on this $\mathcal{K}$-sheaf $\tilde{V}$. So $\mathbb{R}[V]$ is a $\mathcal{K}$-sheaf of vector spaces, and for any other such $\mathcal{K}$-sheaf $\mathcal{B}$ we have:

$$Hom_{\mathcal{K},\mathbb{R}}(\mathbb{R}[V], \mathcal{B}) = Hom_{\mathcal{K}}(\tilde{V}; \mathcal{B}) = Hom_{\mathcal{K}^{(0)}}(V, \mathcal{B}) = \Gamma(V; \mathcal{B}) \tag{2.18}$$

(These four occurrences of $\mathcal{B}$ denote $\mathcal{B}$ as a $\mathcal{K}$-sheaf of vector spaces, as a $\mathcal{K}$-sheaf of sets, and (twice) as a sheaf on $\mathcal{K}^{(0)}$, respectively.)

There is a natural morphism:

$$e = e_V : \mathcal{K}/V \longrightarrow \mathcal{K} \tag{2.19}$$

of étale groupoids (of the kind described in 2.2.9.3), and $\mathbb{R}[V]$ can also be obtained from the constant sheaf $\mathbb{R}$ on $\mathcal{K}/V$ as:

$$\mathbb{R}[V] = e_!(\mathbb{R}) . \tag{2.20}$$

From this point of view, the mapping properties (2.18) follow by the adjunction between $e_!$ and e^*, together with the Morita equivalence $\mathcal{K}/V \simeq V$ (where V is viewed as a trivial groupoid, 1.1.3.3).

If $V, W \subset \mathcal{K}^{(0)}$ are open sets and $\sigma : V \longrightarrow \mathcal{K}^{(1)}$ is a section of $s : \mathcal{K}^{(1)} \longrightarrow \mathcal{K}^{(0)}$ such that $t \circ \sigma(V) \subset W$, then composition with σ gives a morphism $\mathbb{R}[V] \longrightarrow \mathbb{R}[W]$. In this sense, the construction is functorial in V.

Lemma 2.4.3 *For any $\mathcal{K}$-sheaf of vector spaces $\mathcal{A}$ there is an exact sequence of the form:*

$$\bigoplus_j \mathbb{R}\,[V_j] \longrightarrow \bigoplus_i \mathbb{R}\,[V_i] \longrightarrow \mathcal{A} \longrightarrow 0 .$$

Proof: It suffices to prove that any $\mathcal{K}$-sheaf can be covered by $\mathcal{K}$-sheaves of the form $\mathbb{R}\,[V]$, and this is clear from (2.18). $\square$

2.4.4 The sheaves $\mathcal{S}_V$. Let $\mathcal{S}$ be any c-soft $\mathcal{K}$-sheaf. We write $\mathcal{S}_V$ for the sheaf $\mathcal{S} \otimes \mathbb{R}\,[V]$. Note that:

$$\mathcal{S}_V = \mathcal{S} \otimes e_!(\mathbb{R}) = e_!(e^*(\mathcal{S}) \otimes \mathbb{R}) = e_! e^*(\mathcal{S})$$

(see 2.4.1). In particular, $\mathcal{S}_V$ is again c-soft, and has the following mapping properties:

$$Hom_{\mathcal{K}}(\mathcal{S}_V, \mathcal{A}) \;=\; Hom_{\mathcal{K}}(\mathbb{R}\,[V],\ \underline{Hom}(\mathcal{S}, \mathcal{A})) = \Gamma(V,\ \underline{Hom}(\mathcal{S}, \mathcal{A})) = Hom_V(\mathcal{S}|_V, \mathcal{A}|_V). \tag{2.21}$$

Now suppose $V = \bigcup V_i$ is an open cover. We claim that the associated sequence

$$\ldots \longrightarrow \bigoplus \mathcal{S}_{V_{i_0 i_1}} \longrightarrow \bigoplus \mathcal{S}_{V_{i_0}} \longrightarrow \mathcal{S}_V \longrightarrow 0 \tag{2.22}$$

is exact. To see this, it suffices to prove that the sequence obtained by homming into any injective $\mathcal{K}$-sheaf $\mathcal{I}$,

$$0 \longrightarrow Hom_{\mathcal{K}}(\mathcal{S}_V, \mathcal{I}) \longrightarrow Hom_{\mathcal{K}}(\bigoplus \mathcal{S}_{V_{i_0}}, \mathcal{I}) \longrightarrow \ \ldots$$

is exact. This is clear from the mapping properties (2.21).

2.4.5 The sheaves $\varphi_!(\mathcal{S}_V)$. From now on let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be a homomorphism between étale groupoids. For an open set $V \subset \mathcal{K}^{(0)}$, φ induces a map $\varphi_V : V \longrightarrow \mathcal{G}$, which fits into a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\sim", "i"'] \arrow[d, "\varphi_V"'] & \mathcal{K}/V \arrow[d, "e"] \\ \mathcal{G} & \mathcal{K}. \arrow[l, "\varphi"'] \end{tikzcd}$$

(where i is the canonical Morita equivalence). Thus, for any c-soft $\mathcal{K}$-sheaf $\mathcal{S}$, we have:

$$\varphi_!(\mathcal{S}_V) = \varphi_! e_! e^*(\mathcal{S}) = (\varphi_V)_!(\mathcal{S}|_V) \ . \tag{2.23}$$

Notice that the groupoid x/φ_V is a space (1.1.3.3) for any object $x \in \mathcal{G}^{(0)}$; this and the general description of $\varphi_!$ (see (2.13)) give a simple description of the stalks of $\varphi_!(\mathcal{S}_V)$. It follows from this description and the corresponding fact for spaces that $\varphi_!$ maps the exact sequence (2.22) into an exact sequence:

$$\ldots \longrightarrow \bigoplus \varphi_!(\mathcal{S}_{V_{i_0 i_1}}) \longrightarrow \bigoplus \varphi_!(\mathcal{S}_{V_{i_0}}) \longrightarrow \varphi_!(\mathcal{S}_V) \longrightarrow 0 \ . \tag{2.24}$$

2.4.6 The $\mathcal{K}$-sheaves $\varphi^!(\mathcal{S}, \mathcal{I})$. Again, let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be any homomorphism between étale groupoids, let $\mathcal{S}$ be a c-soft $\mathcal{K}$-sheaf, and let $\mathcal{I}$ be an injective $\mathcal{G}$-sheaf. Define for each open set $V \subset \mathcal{K}^{(0)}$:

$$\varphi^!(\mathcal{S}, \mathcal{I})(V) = Hom_{\mathcal{G}}(\varphi_!(\mathcal{S}_V), \mathcal{I}) \ .$$

We claim that this defines a sheaf $\varphi^!(\mathcal{S}, \mathcal{I})$ on $\mathcal{K}^{(0)}$. Indeed, for an inclusion $V \subset W$ there is an evident map $\varphi^!(\mathcal{S}, \mathcal{I})(W) \longrightarrow \varphi^!(\mathcal{S}, \mathcal{I})(V)$ induced by the map $\mathcal{S}_V \longrightarrow \mathcal{S}_W$. And for a covering $V = \bigcup V_i$, the sheaf property follows from the injectivity of $\mathcal{I}$ together with the exact sequence (2.24). Furthermore, this sheaf $\varphi^!(\mathcal{S}, \mathcal{I})$ carries a natural $\mathcal{K}$-action: for any arrow $k : y \longrightarrow z$ in $\mathcal{K}$, let W_y and W_z be neighborhoods of y and z so small that $s : \mathcal{K}^{(1)} \longrightarrow \mathcal{K}^{(0)}$ has a section σ through k with $t \circ \sigma : W_y \longrightarrow W_z$. Then σ gives a map $\mathbb{R}[W_y] \longrightarrow \mathbb{R}[W_z]$ and hence $\mathcal{S}_{W_y} \longrightarrow \mathcal{S}_{W_z}$. By composition, one obtains a map $\varphi^!(\mathcal{S}, \mathcal{I})(W_z) \longrightarrow \varphi^!(\mathcal{S}, \mathcal{I})(W_y)$, and hence by taking germs an action $(-) \cdot k : \varphi^!(\mathcal{S}, \mathcal{I})_z \longrightarrow \varphi^!(\mathcal{S}, \mathcal{I})_y$.

Proposition 2.4.7 *(Duality formula) Let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be a morphism of étale groupoids. For any injective $\mathcal{G}$-sheaf $\mathcal{I}$, any c-soft $\mathcal{K}$-sheaf $\mathcal{S}$ and any other $\mathcal{K}$-sheaf $\mathcal{A}$, there is a natural isomorphism of abelian groups:*

$$Hom_{\mathcal{K}}(\mathcal{A}, \varphi^!(\mathcal{S}, \mathcal{I})) \ \cong \ Hom_{\mathcal{G}}(\varphi_!(\mathcal{A} \otimes \mathcal{S}), \mathcal{I}) \ .$$

In particular, $\varphi^!(\mathcal{S}, \mathcal{I})$ is again injective.

Proof: By 2.4.3 and the fact that $\varphi_!$ is right exact on sequences of c-soft sheaves, it suffices to define a natural isomorphism:

$$Hom_{\mathcal{K}}(\mathbb{R}[V], \varphi^!(\mathcal{S}, \mathcal{I})) \cong Hom_{\mathcal{G}}(\varphi_!(\mathbb{R}[V] \otimes \mathcal{S}), \mathcal{I}) \ .$$

But, using (2.18) and the definitions:

$$Hom_{\mathcal{K}}(\mathbb{R}[V], \varphi^!(\mathcal{S}, \mathcal{I})) \cong \Gamma(V, \varphi^!(\mathcal{S}, \mathcal{I})) \cong Hom_{\mathcal{G}}(\varphi_!(\mathcal{S}_V), \mathcal{I}) \cong Hom_{\mathcal{G}}(\varphi_!(\mathbb{R}[V] \otimes \mathcal{S}), \mathcal{I})$$

□

As for spaces [63], one can state and prove a somewhat stronger version of 2.4.7, using the "internal hom" (see 1.2.12):

Proposition 2.4.8 *(Duality formula, strong form) For any $\varphi, \mathcal{A}, \mathcal{S}$ and $\mathcal{I}$ as in 2.4.7 there is a natural isomorphism of $\mathcal{G}$-sheaves*

$$\varphi_* \underline{Hom}_{\mathcal{K}}(\mathcal{A}, \varphi^!(\mathcal{S}, \mathcal{I})) \ \cong \ \underline{Hom}_{\mathcal{G}}(\varphi_!(\mathcal{A} \otimes \mathcal{S}), \mathcal{I}) \ .$$

Proof: It suffices to prove that for any $\mathcal{G}$-sheaf $\mathcal{B}$ there is an isomorphism:

$$Hom_{\mathcal{G}}(\mathcal{B}, \underline{Hom}_{\mathcal{K}}(\mathcal{A}, \varphi^{!}(\mathcal{S}, \mathcal{I}))) \cong Hom_{\mathcal{G}}(\mathcal{B}, \underline{Hom}_{\mathcal{G}}(\varphi_{!}(\mathcal{A} \otimes \mathcal{S}), \mathcal{I})) \ ,$$

natural in $\mathcal{B}$. This is immediate from 2.4.7 and 2.4.1:

$$\begin{aligned} Hom_{\mathcal{G}}(\mathcal{B}, \varphi_* \underline{Hom}_{\mathcal{K}}(\mathcal{A}, \varphi^{!}(\mathcal{S}, \mathcal{I}))) &= Hom_{\mathcal{K}}(\varphi^* \mathcal{B}, \underline{Hom}_{\mathcal{K}}(\mathcal{A}, \varphi^{!}(\mathcal{S}, \mathcal{I}))) \\ &= Hom_{\mathcal{K}}(\varphi^*(\mathcal{B}) \otimes \mathcal{A}, \varphi^{!}(\mathcal{S}, \mathcal{I})) \\ &= Hom_{\mathcal{G}}(\varphi_{!}(\varphi^*(\mathcal{B}) \otimes \mathcal{A} \otimes \mathcal{S}), \mathcal{I}) \\ &= Hom_{\mathcal{G}}(\mathcal{B} \otimes \varphi_{!}(\mathcal{A} \otimes \mathcal{S}), \mathcal{I}). \ \square \end{aligned}$$

2.4.9 Remark. Let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be an étale morphism such that each of the squares in (2.7) is a pull-back. Thus $\mathcal{K} = E \rtimes \mathcal{G}$ for some étale $\mathcal{G}$-space E. Then $\varphi_{!}$ has a simple description as in (2.2.9.3), and is left adjoint to φ^*. Thus:

$$Hom_{\mathcal{G}}(\varphi_{!}(\mathcal{A} \otimes \mathcal{S}), \mathcal{I}) = Hom_{\mathcal{K}}(\mathcal{A} \otimes \mathcal{S}, \varphi^* \mathcal{I}) = Hom_{\mathcal{K}}(\mathcal{A}, \underline{Hom}_{\mathcal{K}}(\mathcal{S}, \varphi^* \mathcal{I})).$$

Since this holds for any $\mathcal{A}$, proposition 2.4.7 implies that for such a φ,

$$\varphi^{!}(\mathcal{S}, \mathcal{I}) = \underline{Hom}_{\mathcal{K}}(\mathcal{S}, \varphi^* \mathcal{I})) \ \ (\ = \underline{Hom}_{\mathcal{G}}(\varphi_{!} \mathcal{S}, \mathcal{I}). \ \ )$$

2.4.10 Duality for complexes. We now extend these isomorphisms to (co-) chain complexes. It will be convenient to work with *chain* complexes for $\mathcal{A}$ and $\mathcal{S}$ and *cochain* complexes for $\mathcal{I}$ in 2.4.7, 2.4.8. Thus, we will use the following convention: if $\mathcal{A}$ is a chain complex and $\mathcal{B}$ is a cochain complex, $Hom(\mathcal{A}, \mathcal{B})$ is the *cochain* complex defined by:

$$Hom(\mathcal{A}, \mathcal{B})^n = \prod_{p+q=n} Hom(\mathcal{A}_p, \mathcal{B}^q) \ .$$

Recall for later use that if $\mathcal{B}^*$ is injective and bounded below, then for any quasi-isomorphism of chain complexes $\mathcal{A}_* \longrightarrow \mathcal{C}_*$ the map $Hom(\mathcal{A}_*, \mathcal{B}^*) \longrightarrow Hom(\mathcal{C}_*, \mathcal{B}^*)$ is again a quasi-isomorphism (by a standard "mapping cone" argument it is enough to prove the assertion for $\mathcal{C}_* = 0$; in this case remark that $Hom(\mathcal{A}_*, \mathcal{B}^*)$ is the total complex of a double cochain complex whose rows $Hom(\mathcal{A}_*, \mathcal{B}^p)$ are acyclic by the injectivity of $\mathcal{B}^p$) .

Similarly, for a bounded below chain complex $\mathcal{S}_*$ of sheaves as in 2.4.7 we define the cochain complex $\varphi^{!}(\mathcal{S}_*, \mathcal{I}^*)$ by:

$$\varphi^{!}(\mathcal{S}_*, \mathcal{I}^*) = \prod_{p+q=n} \varphi^{!}(\mathcal{S}_p, \mathcal{I}^q) \ .$$

With these conventions, 2.4.7 gives an isomorphism of cochain complexes

$$Hom_{\mathcal{K}}(\mathcal{A}_*, \varphi^{!}(\mathcal{S}_*, \mathcal{I}^*)) \cong Hom_{\mathcal{G}}(\varphi_{!}(\mathcal{A}_* \otimes \mathcal{S}_*), \mathcal{I}^*) \ , \tag{2.25}$$

for any cochain complex $\mathcal{I}^*$ of injective $\mathcal{G}$-sheaves, and any bounded below chain complexes $\mathcal{A}_*$ and $\mathcal{S}_*$ of $\mathcal{K}$-sheaves with $\mathcal{S}_*$ c-soft. There is also an obvious "strong" version of (2.25):

$$\varphi_* \underline{Hom}_{\mathcal{K}}(\mathcal{A}_*, \varphi^{!}(\mathcal{S}_*, \mathcal{I}^*) \cong \underline{Hom}_{\mathcal{G}}(\varphi_{!}(\mathcal{A}_* \otimes \mathcal{S}_*), \mathcal{I}^*) \ .$$

2.4.11 The functor $\varphi^!(\mathcal{I}^*)$. Now let $d = cohdim(\mathcal{K}^{(0)})$, and fix a resolution:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^0 \longrightarrow \ \ldots \ \longrightarrow \mathcal{S}^d \longrightarrow 0$$

of the constant sheaf $\mathbb{R}$ by c-soft $\mathcal{K}$-sheaves (cf. 2.2.3). For a cochain complex $\mathcal{I}^*$ of injective $\mathcal{G}$-sheaves define:

$$\varphi^!(\mathcal{I}^*) = \varphi^!(\mathcal{S}^{-*}, \mathcal{I}^*) \ .$$

Then $\varphi^!$ is adjoint to $\varphi_!$ in the derived category:

Theorem 2.4.12 *(Adjointness) Let $\varphi : \mathcal{K} \longrightarrow \mathcal{G}$ be a morphism between étale groupoids. For any bounded below chain complex $\mathcal{A}_*$ of c-soft $\mathcal{K}$-sheaves and any bounded below cochain complex $\mathcal{I}^*$ of injective $\mathcal{G}$-sheaves there is a natural quasi-isomorphism:*

$$Hom_{\mathcal{K}}(\mathcal{A}_*, \varphi^!(\mathcal{I}^*)) \simeq Hom_{\mathcal{G}}(\mathcal{L}\varphi_!(\mathcal{A}_*), \mathcal{I}^*) \ .$$

(there is also a " strong" version derived from 2.4.8).

Proof: Denote by $\mathcal{F}_*$ the free resolution of the constant sheaf obtained by tensoring (1.4) (for $\mathcal{K}$ instead of $\mathcal{G}$) by $\mathbb{R}$. Since $\mathcal{A}_* \otimes \mathcal{F}_* \longrightarrow \mathcal{A}_*$ is a quasi-isomorphism, using (2.25) with $\mathcal{A}_* \otimes \mathcal{F}_*$ instead of $\mathcal{A}_*$ and $\mathcal{S}_* = \mathcal{S}^{-*}$, the fact that $\varphi^!(\mathcal{I}^*)$ are injective (cf. 2.4.7) and the general remark in 2.4.10 we get a quasi-isomorphism:

$$Hom_{\mathcal{K}}(\mathcal{A}_*, \varphi^!(\mathcal{S}_*, \mathcal{I}^*) \simeq Hom_{\mathcal{G}}(\varphi_!(\mathcal{A}_* \otimes \mathcal{F}_* \otimes \mathcal{S}_*), \mathcal{I}^*) \ . \tag{2.26}$$

Remark that for any bounded below chain complex $\mathcal{B}_*$ of c-soft $\mathcal{K}$-sheaves we have a quasi-isomorphism followed by an isomorphism: $\mathcal{L}\varphi_!\mathcal{B}_* \simeq \mathcal{L}\varphi_!(\mathcal{B}_* \otimes \mathcal{F}_*) = \varphi_!(\mathcal{B}_* \otimes \mathcal{F}_*)$. This for $\mathcal{B}_* = \mathcal{A}_* \otimes \mathcal{S}_*$ and the quasi-isomorphism $\mathcal{A}_* \simeq \mathcal{A}_* \otimes \mathcal{S}_*$ give $\mathcal{L}\varphi_!\mathcal{A}_* \simeq \varphi_!(\mathcal{A}_* \otimes \mathcal{F}_* \otimes \mathcal{S}_*)$. Using this, the fact that $\mathcal{I}^*$'s are injective and (2.26), the statement of the theorem follows easily. □

Poincaré duality follows in the usual way:

Theorem 2.4.13 *(Poincaré duality) Let $\mathcal{K}$ be an étale groupoid, and suppose that $\mathcal{K}^{(0)}$ is a topological manifold of dimension d. Let or be the orientation $\mathcal{K}$-sheaf (1.2.2.6). There is a natural isomorphism:*

$$H^{p+d}(\mathcal{K}; or) \cong H_p(\mathcal{K}; \mathbb{R})^\vee \qquad (p \in \mathbb{Z}).$$

Proof: Let $\mathcal{G} = 1$ be the trivial groupoid. In 2.4.12, let $\mathcal{I}^*$ be the complex $\mathbb{R}$ concentrated in degree $-p$, and let $\mathcal{A}_*$ be the complex $\mathcal{A}_i = \mathcal{S}^{-i}$ (as in 2.4.11). As $\mathcal{A}_*$ is quasi-isomorphic to $\mathbb{R}$, the complex on the right of the quasi-isomorphism in 2.4.12 has:

$$H^0(Hom(\varphi_!(\mathcal{A}_*), \mathcal{I}^*) = H_p(\mathcal{K}; \mathbb{R})^\vee \ .$$

Now consider the left hand side of the quasi-isomorphism of 2.4.12, $Hom(\mathcal{A}_*, \varphi^!(\mathcal{I}^*))$ in this special case. Note first that $\varphi^!(\mathbb{R})$ is quasi-isomorphic to the orientation sheaf concentrated in degree $-d$:

$$\varphi^!(\mathbb{R}) \simeq or\,[d] \ . \tag{2.27}$$

Indeed, for any open set $V \subset \mathcal{K}^{(0)}$, $\varphi^!(\mathbb{R})(V) = Hom(\Gamma_c(V; \mathcal{S}^{-*}); \mathbb{R})$ (see (2.23)) is the complex which computes $H_c^{-*}(V; \mathbb{R})^\vee$, and the argument for (2.27) is just like the one for spaces. Thus, for $\mathcal{I}^* = \mathbb{R}\,[p]$ and $\mathcal{A}_* = \mathcal{S}^{-*} \simeq \mathbb{R}\,[0]$,

$$H^0(Hom_{\mathcal{K}}(\mathcal{A}_*, \varphi^!(\mathcal{I}^*)) = H^0\Gamma_{inv}(\mathcal{K}, \varphi^!(\mathcal{I}^*)) = H^{p+d}(\mathcal{K}, or) \ . \ \square$$

2.5 Appendix: Compact supports in non-Hausdorff spaces

In this section we explain how the usual notions concerning compactness and sheaves on Hausdorff spaces extend to our more general context (see 1.1.14). For basic definitions and facts for sheaves on Hausdorff spaces, we refer the reader to any of the standard sources [49, 63, 16].

2.5.1 c-soft sheaves. Let X be a space satisfying the general assumptions in 1.1.14. An abelian sheaf $\mathcal{A}$ on X is said to be *c-soft* if for any Hausdorff open $U \subset X$ its restriction $\mathcal{A}|_U$ is a c-soft sheaf on U in the usual sense (see e.g. [16, 49, 63]). By the same property for Hausdorff spaces, it follows that c-softness is a local property, i.e., a sheaf $\mathcal{A}$ is c-soft iff there is an open cover $X = \bigcup U_i$ such that each $\mathcal{A}|_U$ is a c-soft sheaf on $\mathcal{A}$.

2.5.2 The functor Γ_c. Let $\mathcal{A}$ be a c-soft sheaf on X and let $\mathcal{A}'$ be its Godement resolution (i.e. $\mathcal{A}'(U) = \Gamma(U_{\text{discr}}; \mathcal{A})$ is the set of all (not necessarily continuous) sections, for any open $U \subset X$). For any Hausdorff open set $W \subset X$, let $\Gamma_c(W, \mathcal{A})$ be the usual set of compactly supported sections. If $W \subset U$, there is an evident homomorphism, "extension by 0" $\Gamma_c(W, \mathcal{A}) \longrightarrow \Gamma_c(U, \mathcal{A}) \subset \Gamma(U, \mathcal{A}')$. For any (not necessarily Hausdorff) open set $U \subset X$, we define $\Gamma_c(U, \mathcal{A})$ to be the image of the map:

$$\bigoplus_W \Gamma_c(W, \mathcal{A}) \longrightarrow \Gamma(U, \mathcal{A}') \ ,$$

where W ranges over all Hausdorff open subsets $W \subset U$.

Observe that $\Gamma_c(U, \mathcal{A})$ so defined is evidently functorial in $\mathcal{A}$, and that for any inclusion $U \subset U'$ we have an "extension by zero" monomorphism:

$$\Gamma_c(U, \mathcal{A}) \longrightarrow \Gamma_c(U', \mathcal{A}) \ .$$

The following lemma shows that in the definition of $\Gamma_c(U, \mathcal{A})$ it is enough to let W range over a Hausdorff open cover of U; in particular, it shows that the definition agrees with the usual one if U itself is Hausdorff.

Lemma 2.5.3 *Let $\mathcal{A}$ be a c-soft sheaf on X. For any open cover $U = \bigcup W_i$ where each W_i is Hausdorff, the sequence:*

$$\bigoplus_i \Gamma_c(W_i, \mathcal{A}) \longrightarrow \Gamma_c(U, \mathcal{A}) \longrightarrow 0$$

is exact.

Proof: It suffices to show that for any Hausdorff open $W \subset U$, the map

$$\bigoplus_i \Gamma_c(W \bigcap W_i, \mathcal{A}) \longrightarrow \Gamma_c(W, \mathcal{A})$$

is surjective. This is well known (see e.g. [49]). $\square$

This lemma is in fact a special case of the following Proposition ("Mayer-Vietoris"):

Proposition 2.5.4 *Let $X = \bigcup_i U_i$ be an open cover indexed by an ordered set I, and let $\mathcal{A}$ be a c-soft sheaf on X. Then there is a long exact sequence:*

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} \Gamma_c(U_{i_0 i_1}, \mathcal{A}) \longrightarrow \bigoplus_{i_0} \Gamma_c(U_{i_0}, \mathcal{A}) \longrightarrow \Gamma_c(X, \mathcal{A}) \longrightarrow 0 \tag{2.28}$$

Here $U_{i_0 \ldots i_n} = U_{i_0} \cap \ldots \cap U_{i_n}$, as usual. (There is of course a similar exact sequence if I is not ordered.)

Proof: The proposition is of course well known in the case where X is a paracompact Hausdorff space. We first reduce the proof to the case where each of the U_i is Hausdorff, as follows. Let $X = \bigcup_{j \in J} W_j$ be a cover by Hausdorff open sets, and consider the double complex:

$$C_{p,q} = \bigoplus \Gamma_c(W_{j_0 \ldots j_p} \cap U_{i_0 .. i_q}, \mathcal{A}) \ ,$$

where the sum is over all $j_0 < \ldots < j_p$, $i_0 < \ldots < i_q$. For a fixed $p \geq 0$, the column $C_{p,*}$ is a sum of exact Mayer-Vietoris sequences for the Hausdorff open sets $W_{j_0 \ldots j_p}$, augmented by $C_{p,-1} = \bigoplus_{j_0 < \ldots < j_p} \Gamma_c(W_{j_0 \ldots j_p}, \mathcal{A})$. Keeping the notation $U_{i_0 .. i_q} = X = W_{j_0 \ldots j_p}$ if $q = -1 = p$, we observe that for a fixed $q \geq -1$, the row $C_{*,q}$ is a sum of Mayer-Vietoris sequences for the spaces $U_{i_0 .. i_q}$ with respect to the open covers $\{W_j \cap U_{i_0 .. i_q}\}$. So, if the proposition would hold for covers by Hausdorff sets, each row $C_{*,q}$ ($q \geq -1$) is also exact. By a standard double complex argument it follows that the augmentation column $C_{-1,*}$ is also exact, and this column is precisely the sequence in the statement of the proposition. This shows that it suffices to show the proposition in the special case where each U_i is Hausdorff.

So assume each $U_i \subset X$ is Hausdorff. Observe first that exactness of the sequence (2.28) at $\Gamma_c(X, \mathcal{A})$ now follows by Lemma 2.5.3. To show exactness elsewhere, consider for each finite subset $I_0 \subset I$ the space $U^{I_0} = \bigcup_{i \in I_0} U_i$ and the subsequence:

$$\ldots \longrightarrow \bigoplus_{i_0<i_1 \text{ in } I_0} \Gamma_c(U_{i_0 i_1}, \mathcal{A}) \longrightarrow \bigoplus_{i_0 \text{ in } I_0} \Gamma_c(U_{i_0}, \mathcal{A}) \longrightarrow \Gamma_c(U^{I_0}, \mathcal{A}) \longrightarrow 0 \tag{2.29}$$

of (2.28). Clearly (2.28) is the directed union of the sequences of the form (2.29), where $I_0 \subset I$ ranges over all finite subsets of I. So exactness of (2.28) follows from exactness of each such sequence of the form (2.29). Thus, it remains to prove the proposition in the special case of a *finite* cover $\{U_i\}$ of X by Hausdorff open sets.

So assume $X = U_1 \cup \ldots \cup U_n$ where each U_i is Hausdorff. For $n = 1$, there is nothing to prove. For $n = 2$, the sequence has the form

$$0 \longrightarrow \Gamma_c(U_1 \cap U_2, \mathcal{A}) \longrightarrow \Gamma_c(U_1, \mathcal{A}) \bigoplus \Gamma_c(U_2, \mathcal{A}) \longrightarrow \Gamma_c(U_1 \cap U_2, \mathcal{A}) \longrightarrow 0 \ .$$

This sequence is exact at $\Gamma_c(X, \mathcal{A})$ by 2.5.3, and evidently exact at other places. Exactness for $n = 3$ can be proved using exactness for $n = 2$. Indeed, consider the following diagram, whose upper two rows are the sequences for $n = 2, 3$ and whose third row is constructed by taking vertical Cokernels, so that all columns are exact (we delete the sheaf $\mathcal{A}$ from the notation)(compare to pp. 187 in [15]):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & \Gamma_c(U_{12}) \arrow[r] \arrow[d] & \Gamma_c(U_1) \oplus \Gamma_c(U_2) \arrow[r] \arrow[d] & \Gamma_c(U_1 \cup U_2) \arrow[r] \arrow[d] & 0 \\
\Gamma_c(U_{123}) \arrow[r] \arrow[d] & \oplus_{1\leq i<j\leq 3}\Gamma_c(U_{ij}) \arrow[r] \arrow[d] & \Gamma_c(U_1) \oplus \Gamma_c(U_2) \oplus \Gamma_c(U_3) \arrow[r] \arrow[d] & \Gamma_c(U_1 \cup U_2 \cup U_3) \arrow[r] \arrow[d] & 0 \\
\Gamma_c(U_{123}) \arrow[r] \arrow[d] & \Gamma_c(U_{13}) \oplus \Gamma_c(U_{23}) \arrow[r] \arrow[d] & \Gamma_c(U_3) \arrow[r, "\pi"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

To show that the middle row is exact, it thus suffices to prove that the lower row is exact. This row can be decomposed into a Mayer-Vietoris sequence for the case $n = 2$, already shown to be exact,

$$0 \longrightarrow \Gamma_c(U_{123}) \longrightarrow \Gamma_c(U_{13}) \oplus \Gamma_c(U_{23}) \longrightarrow \Gamma_c(U_3 \cap (U_1 \cup U_2)) \longrightarrow 0$$

and the sequence:

$$0 \longrightarrow \Gamma_c(U_3 \cap (U_1 \cup U_2)) \longrightarrow \Gamma_c(U_3) \longrightarrow C \longrightarrow 0 \ .$$

The exactness of the latter sequence is easily proved by a diagram chase, using exactness of the right-hand column.

An identical argument will show that the exactness for a cover by $n + 1$ opens follows from exactness for one by n opens, so the proof is completed by induction. □

Proposition 2.5.4 is our main tool for transfering standard facts from sheaf theory on Hausdorff spaces to the non-Hausdorff case, as illustrated by the following corollaries.

Corollary 2.5.5 *Let $Y \subset X$ be a closed subspace, and let $\mathcal{A}$ be a c-soft sheaf on X. There is an exact sequence*

$$0 \longrightarrow \Gamma_c(X - Y, \mathcal{A}) \stackrel{i}{\longrightarrow} \Gamma_c(X, \mathcal{A}) \stackrel{r}{\longrightarrow} \Gamma_c(Y, \mathcal{A}) \longrightarrow 0$$

(i is extension by zero, r is the restriction).

Proof: This (including the fact that the map r is well defined) follows by elementary homological algebra from the fact that the Corollary holds for Hausdorff spaces, by using 2.5.4 for a cover of X by Hausdorff open sets U_i and for the induced covers of Y by $\{U_i \cap Y\}$ and $X - Y$ by $\{U_i - Y\}$. □

Corollary 2.5.6 *For a family $\mathcal{A}_i$ of c-soft sheaves on X the direct sum $\oplus\mathcal{A}_i$ is again c-soft, and:*

$$\Gamma_c(X, \oplus\mathcal{A}_i) \cong \oplus\Gamma_c(X, \mathcal{A}_i) \ .$$

In particular, when working over $\mathbb{R}$, we have for any c-soft sheaf $\mathcal{S}$ of $\mathbb{R}$-vector spaces and any vector space V that the tensor product $\mathcal{S} \otimes_{\mathbb{R}} V$ (here V is the constant sheaf) is again c-soft, and the familiar formula:

$$\Gamma_c(X, \mathcal{S} \otimes_{\mathbb{R}} V) \cong \Gamma_c(X, \mathcal{S}) \otimes_{\mathbb{R}} V \ . \tag{2.30}$$

Corollary 2.5.7 *Let $\mathcal{A}_* \longrightarrow \mathcal{B}_*$ be a quasi-isomorphism between chain complexes of c-soft sheaves on X. Then:*

$$\Gamma_c(X, \mathcal{A}_*) \longrightarrow \Gamma_c(X, \mathcal{B}_*)$$

is again a quasi-isomorphism.

Proof: By a "mapping cone argument" we may assume that $\mathcal{B}_* = 0$. In other words, we have to show that $\Gamma_c(X, \mathcal{A}_*)$ is acyclic whenever $\mathcal{A}_*$ is. This follows from the Mayer-Vietoris sequence 2.5.4 together with the Hausdorff case.

(We remark that it is necessary to assume that the chain complexes are bounded below if X does not have locally finite cohomological dimension, as in 1.1.14). □

The following Corollary is included for application in chapter 4.

Corollary 2.5.8 *Let $Y \subset X$ be a closed subspace, and let $\theta : X \longrightarrow \mathbb{R}$ be a continuous map such that $\theta^{-1}(0) = Y$. Let $\mathcal{A}$ be a c-soft sheaf on X. Then for any $\alpha \in \Gamma_c(X, \mathcal{A})$,*

$$\alpha|_Y = 0 \ \text{ iff } \ \exists\ \varepsilon > 0 : \quad \alpha|_{\theta^{-1}(-\varepsilon,\varepsilon)} = 0$$

(here $\alpha|_Y$ is the restriction $r(\alpha)$ as in 2.5.5).

Proof: For $\varepsilon \geq 0$, write $Y_\varepsilon = \{x \in X : |\theta(x)| \leq \varepsilon\}$, and for each open $U \subset X$ write

$$\Gamma_c^\varepsilon(U, \mathcal{A}) = \{\alpha \in \Gamma_c(U, \mathcal{A}) : \alpha|_{U \cap Y_\varepsilon} = 0\} \ .$$

It suffices to show that:

$$\bigoplus_{\varepsilon > 0} \Gamma_c^\varepsilon(X, \mathcal{A}) \longrightarrow \Gamma_c^0(X, \mathcal{A})$$

is epi. Let $\{U_i\}$ be a cover of X by Hausdorff open sets, and consider the diagram:

$$\begin{tikzcd}
\bigoplus_{i,\varepsilon>0} \Gamma_c^\varepsilon(U_i, \mathcal{A}) \arrow[r, "u"] \arrow[d] & \bigoplus_i \Gamma_c^0(U_i, \mathcal{A}) \arrow[r, "\sim"] \arrow[d, "\pi"] & \bigoplus_i \Gamma_c(U_i - Y, \mathcal{A}) \arrow[d, "\pi'"] \\
\bigoplus_{\varepsilon>0} \Gamma_c^\varepsilon(X, \mathcal{A}) \arrow[r, "v"] & \Gamma_c^0(X, \mathcal{A}) \arrow[r, "\sim"] & \Gamma_c(X - Y, \mathcal{A})
\end{tikzcd}$$

where the isomorphisms on the right come from 2.5.5. We wish to show that v is epi. Since u is epi by the Hausdorff case, it suffices to show that π is epi, or, equivalently, that π' is epi. This is indeed the case by 2.5.4. □

It is quite clear that using c-soft resolutions one can define compactly supported cohomology $H_c^*(X, \mathcal{A})$ for any $\mathcal{A} \in \underline{Ab}(X)$. In particular, we get an extension $H_c^0(X, -)$ of $\Gamma_c(X, -)$ to all sheaves; this extension is still denoted by $\Gamma_c(X, -)$.

Proposition 2.5.9 *Let $f : Y \longrightarrow X$ be a continuous map. There is a functor $f_! : \underline{Ab}(Y) \longrightarrow \underline{Ab}(X)$ with the following properties:*

(i) For any open $U \subset X$ and any $\mathcal{B} \in \underline{Ab}(Y)$, $\Gamma_c(U, f_!\mathcal{B}) = \Gamma_c(f^{-1}(U), \mathcal{B})$.

(ii) For any point $x \in X$ and any $\mathcal{B} \in \underline{Ab}(Y)$, $f_!(\mathcal{B})_x = \Gamma_c(f^{-1}(x), \mathcal{B})$.

(iii) $f_!$ is left exact and maps c-soft sheaves into c-soft sheaves.

(iv) For any fibered product

$$\begin{tikzcd} Z \times_X Y \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "e"] & X \end{tikzcd}$$

along an étale map e and for any c-soft $\mathcal{B} \in \underline{Ab}(Y)$, there is a canonical isomorphism

$$q_!p^*\mathcal{B} \cong e^*f_!\mathcal{B} \ .$$

(see 2.5.11 below for the case where e is not étale).

Proof: Of course the proposition is well known in the Hausdorff case. For the more general case, recall first from [18] the correspondence for any *Hausdorff* space Z between c-soft sheaves $\mathcal{S}$ on Z and flabby cosheaves $\mathcal{C}$ on Z, given by:

$$\Gamma_c(W, \mathcal{S}) = \mathcal{C}(W) \tag{2.31}$$

(natural with respect to the opens $W \subset Z$). Given the cosheaf $\mathcal{C}$, the stalk of the corresponding sheaf $\mathcal{S}$ at a point $z \in Z$ is given by the exact sequence:

$$0 \longrightarrow \mathcal{C}(Z - z) \longrightarrow \mathcal{C}(Z) \longrightarrow \mathcal{S}_z \longrightarrow 0 \ . \tag{2.32}$$

We use this correspondence in the construction of $f_!$. (However, see remark 2.5.10 below for a description of $f_!$ which doesn't use this correspondence).

We discuss first the construction of $f_!$ on c-soft sheaves. Let $\mathcal{B} \in \underline{Ab}(Y)$ be c-soft. First, assume X is Hausdorff. Let $\mathcal{B}$ be a c-soft sheaf on Y, and define a cosheaf $\mathcal{C} = c(\mathcal{B})$ by $\mathcal{C}(U) = \Gamma_c(f^{-1}(U), \mathcal{B})$. Note that $\mathcal{C}$ is indeed a flabby cosheaf, by 2.5.4. By the correspondence (2.31), there is a c-soft sheaf $\mathcal{S}$ on X, uniquely determined up to isomorphism by the identity $\Gamma_c(U, \mathcal{S}) = \mathcal{C}(U)$ for any open $U \subset X$. Thus, if X is Hausdorff, we can define $f_!\mathcal{B}$ to be this sheaf $\mathcal{S}$.

In the general case, cover X by Hausdorff opens U_i, and define in this way for each i a c-soft sheaf $\mathcal{S}_i$ on U_i by:

$$\Gamma_c(V, \mathcal{S}_i) = \Gamma_c(f^{-1}(V), \mathcal{B}) \ . \tag{2.33}$$

Then (again by the equivalence between sheaves and cosheaves) there is a canonical isomorphism $\theta_{ij} : \mathcal{S}_j|_{U_{ij}} \longrightarrow \mathcal{S}_i|_{U_{ij}}$ satisfying the cocycle condition. Therefore the sheaves $\mathcal{S}_i$ patch together to a sheaf $\mathcal{S}$ on X, uniquely determined up to isomorphism by the condition that $\mathcal{S}|_{U_i} = \mathcal{S}_i$ (by an isomorphism compatible with θ_{ij}). Thus we can define $f_!\mathcal{B}$ to be $\mathcal{S}$.

We prove the properties $(i)-(iv)$ in the statement of the proposition for $\mathcal{B} \in \underline{Ab}(Y)$ c-soft. Property (i) clearly holds for an open set U contained in some U_i, by (2.33).

For general U, property (i) then follows by the Mayer-Vietoris sequence. Next, identity (2.32) yields for any point $x \in X$ an exact sequence:

$$0 \longrightarrow \Gamma_c(Y - f^{-1}(x),\ \mathcal{B}) \longrightarrow \Gamma_c(Y, \mathcal{B}) \longrightarrow f_!(\mathcal{B})_x \longrightarrow 0\ ,$$

and hence, by 2.5.5 the isomorphism (ii). of the Proposition. Finally, (iv) is clear from the local nature of the construction of $f_!$.

For general $\mathcal{A} \in \underline{Ab}(Y)$ we define $f_!(\mathcal{A}) \in \underline{Ab}(X)$ as the kernel of the map $f_!(\mathcal{S}^0) \longrightarrow f_!(\mathcal{S}^1)$ where $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \ldots$ is a c-soft resolution of $\mathcal{A}$ (from the first part it follows that it is well defined up to isomorphisms). The properties (i) and (ii) are now immediate consequences of the definition and of the previous case. Using 2.5.7 and (ii) it easily follows that $f_!$ transforms acyclic complexes of c-soft sheaves on $\underline{Ab}(Y)$ into acyclic complexes on $\underline{Ab}(X)$. This immediately implies that $\varphi_!$ is left exact. $\square$

2.5.10 Remark. We outline an alternative construction and proof of Proposition 2.5.9, which does not use the correspondence between sheaves and cosheaves. This construction will be used in the proof of 2.5.11 below. We will assume that $\mathcal{B}$ is c-soft and X is Hausdorff. (As in the proof of 2.5.9, the construction of $f_!$ for general X is then obtained by glueing the constructions over a cover by Hausdorff opens $U_i \subset X$.)

So, let $\mathcal{B}$ be a c-soft sheaf on Y. For any open set $V \subset Y$, denote by $\mathcal{B}_V$ the sheaf on Y obtained by extending $\mathcal{B}|_V$ by zero. Thus $\mathcal{B}_V$ is evidently c-soft, and $\Gamma_c(Y, \mathcal{B}_V) = \Gamma_c(V, \mathcal{B})$. Moreover, an inclusion $V \subset W$ induces an evident map $\mathcal{B}_V \hookrightarrow \mathcal{B}_W$.

Now let $Y = \bigcup W_i$ be a cover by Hausdorff open sets. This cover induces a long exact sequence:

$$\ldots \longrightarrow \bigoplus_{i_0 < i_1} \mathcal{B}_{W_{i_0 i_1}} \longrightarrow \bigoplus_{i_0} \mathcal{B}_{W_{i_0}} \longrightarrow \mathcal{B} \longrightarrow 0$$

of c-soft sheaves on Y. By Corollary 2.5.7, the functor $\Gamma_c(Y, -)$ applied to this long exact sequence again yields an exact sequence, and this is precisely the Mayer-Vietoris sequence of 2.5.4. For each $i_0, \ldots, i_n$ let $f_{i_0,\ldots,i_n} : W_{i_0,\ldots,i_n} \longrightarrow X$ be the restriction of f; this is a map between Hausdorff spaces, so we have $(f_{i_0,\ldots,i_n})_!(\mathcal{B}_{W_{i_0,\ldots,i_n}})$ defined as usual. Define $f_!(\mathcal{B})$ as the cokernel fitting into a long exact sequence:

$$\ldots \longrightarrow \bigoplus_{i_0 < i_1} (f_{i_0 i_1})_!(\mathcal{B}_{W_{i_0 i_1}}) \longrightarrow \bigoplus_{i_0} (f_{i_0})_!(\mathcal{B}_{W_{i_0}}) \longrightarrow f_!(\mathcal{B}) \longrightarrow 0\ . \tag{2.34}$$

For $x \in X$, we have $(f_{i_0})_!(\mathcal{B}_{W_{i_0}})_x = \Gamma_c(f^{-1}(x) \cap W_{i_0}; \mathcal{B})$ by the Hausdorff case. So taking stalks of the long exact sequence in (2.34) at x and using the Mayer-Vietoris sequence 2.5.4 for the space $f^{-1}(x)$ we find $f_!(\mathcal{B})_x = \Gamma_c(f^{-1}(x), \mathcal{B})$ as in 2.5.9 (ii). Property 2.5.9 (i) is proved in a similar way (using 2.5.5).

The functor $f_!$ can be extended to the derived category $D(Y)$ by taking a c-soft resolution $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \ldots$ and defining $\mathcal{R}f_!(\mathcal{A})$ as the complex $f_!(\mathcal{S}^*)$. Up to quasi-isomorphisms, this complex is well defined and does not depend on the resolution $\mathcal{S}^*$, by 2.5.5. (In this way, we obtain in fact a well defined functor $\mathcal{R}f_! : D(Y) \longrightarrow D(X)$ at the level of derived categories, which is sometimes simply denoted by $f_!$ again). In particular, $\mathcal{H}^*(\mathcal{R}\varphi_!(\mathcal{A}))$ gives in fact the right derived functors $R^* f_!$ of $f_!$.

Lemma 2.5.11 *For any pullback diagram:*

$$\begin{tikzcd} Z \times_X Y \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "e"] & X \end{tikzcd}$$

and any sheaf $\mathcal{B}$ on Y, there is a canonical quasi-isomorphism:

$$(\mathcal{R}q_!)p^*\mathcal{B} \simeq e^*(\mathcal{R}f_!)\mathcal{B} \ .$$

Proof: Using Mayer-Vietoris for covers of X and Z by Hausdorff open sets, it suffices to consider the case where X and Z are both Hausdorff. Clearly it also suffices to prove the lemma in the special case where $\mathcal{B}$ is c-soft.

Let $Y = \bigcup W_i$ as in 2.5.10, so that $f_!(\mathcal{B})$ fits into a long exact sequence (2.34) of c-soft sheaves on X. Applying the exact functor e^* to this sequence and using the lemma in the Hausdorff case, one obtains a long exact sequence of the form:

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} q_!p^*(\mathcal{B}|_{W_{i_0i_1}}) \longrightarrow \bigoplus_{i_0} q_!p^*(\mathcal{B}|_{W_{i_0}}) \longrightarrow e^*f_!(\mathcal{B}) \longrightarrow 0 \ . \qquad (2.35)$$

Now let $p^*(\mathcal{B}) \longrightarrow \mathcal{S}^*$ be a c-soft resolution over the pullback $Z \times_X Y$. Then for any open $U \subset Y$, $\mathcal{S}^*_{p^{-1}(W)}$ is a c-soft resolution of $p^*(\mathcal{B}_W)$, so $q_!(\mathcal{S}^*_{p^{-1}(W)})$ is a c-soft resolution of $q_!p^*(\mathcal{B})$. The lemma now follows by comparing the sequence (2.35) to the defining sequence

$$\ldots \longrightarrow \bigoplus_{i_0<i_1} q_!(\mathcal{S}_{p^{-1}W_{i_0i_1}}) \longrightarrow \bigoplus_{i_0} q_!(\mathcal{S}_{p^{-1}W_{i_0}}) \longrightarrow q_!(\mathcal{S}) \longrightarrow 0$$

for $q_!(p^*(\mathcal{B})) \stackrel{def}{=} q_!(\mathcal{S})$. $\square$

2.5.12 $f_!$ **on étale maps.** Let $f : Y \longrightarrow X$ be an étale map, i.e. a local homeomorphism. It is well known that the pullback functor $f^* : \underline{Ab}(X) \longrightarrow \underline{Ab}(Y)$ has an exact left-adjoint $f_! : \underline{Ab}(Y) \longrightarrow \underline{Ab}(X)$, described on the stalks by $f_!(\mathcal{B})_x = \oplus_{y\in f^{-1}(x)}\mathcal{B}_y$. This construction agrees with the one in 2.5.9. In particular, for étale f, the counit of the adjunction defines a map:

$$\Sigma_f : f_!f^*(\mathcal{A}) \longrightarrow \mathcal{A} \ ,$$

"summation along the fiber", for any sheaf $\mathcal{A}$ on X.

2.5.13 Notation: Let $\mathcal{A} \in Sh(X), \mathcal{B} \in Sh(Y)$ be c-soft sheaves. Most of the maps we are going to deal with are of type:

$$(\alpha, f)_* : \Gamma_c(X;\mathcal{A}) \longrightarrow \Gamma_c(Y,\mathcal{B}) \ , \ ((\alpha, f)_*u)(y) = \sum_{x\in f^{-1}(y)} \alpha_x(u(x)) \in \mathcal{B}_y \quad (y \in Y),$$

for some étale map $f : X \longrightarrow Y$, and some morphism of sheaves $\alpha : \mathcal{A} \longrightarrow f^*\mathcal{B}$. In other words, $(\alpha, f)_*$ is the composition:

$$\Gamma_c(X;\mathcal{A}) \xrightarrow{\alpha_*} \Gamma_c(X;f^*\mathcal{B}) \stackrel{2.5.9.3}{\simeq} \Gamma_c(Y, f_!f^*\mathcal{B}) \xrightarrow{int_*} \Gamma_c(Y,\mathcal{B}),$$

where $int := \Sigma_f : f_! f^* \mathcal{B} \longrightarrow \mathcal{B}$ is, at the stalk at $y \in Y$ (see also 2.5.9):

$$\bigoplus_{x \in f^{-1}(y)} \mathcal{B}_y \longrightarrow \mathcal{B}_y, \qquad \sum_{x \in f^{-1}(y)} b_x x \mapsto \sum b_x.$$

Rather than writing the explicit formula for $(\alpha, f)_*$, we prefer to briefly indicate the maps f and α using the notation:

$$\Gamma_c(X, \mathcal{A}) \longrightarrow \Gamma_c(Y, \mathcal{B}) \ , \ (a|x) \mapsto (\alpha(a)|f(x))$$

$(x \in X, a \in \mathcal{A}_x)$.

2.5.14 $f_!$ **on proper maps.** Define a map $f : Y \longrightarrow X$ between (non-necessarily Hausdorff) spaces to be *proper* if:

(i) the diagonal $Y \longrightarrow Y \times_X Y$ is closed.

(ii) for any Hausdorff open $U \subset X$ and any compact $K \subset U$, the set $f^{-1}(K)$ is compact.

It is easy to see that if f is proper then $f_! = f_*$, as in the Hausdorff case. Furthermore, for any c-soft sheaf $\mathcal{A}$ on X, there is a natural map $\Gamma_c(X, \mathcal{A}) \longrightarrow \Gamma_c(Y, f^*\mathcal{A})$ defined by pullback, as in the Hausdorff case.

2.5.15 Remark: Although this does not simplify matters, one could theoretically interpret some of the constructions and results of this Appendix as follows. First, observe that for *Hausdorff* groupoids, the results in Sections 1-6 of the paper can be based on the usual definition of Γ_c and are independent of the Appendix. Now, any non-separated manifold (or sufficiently nice space, cf. 1.1.14) X can be viewed as a trivial groupoid (1.1.3.1), which is Morita equivalent to the Hausdorff étale groupoid $\mathcal{G}$ defined from an open cover $\{U_i\}$ of X by Hausdorff open sets, by taking $\mathcal{G}^{(0)}$ to be the disjoint sum of the U_i, and $\mathcal{G}^{(1)} = \mathcal{G}^{(0)} \times_X \mathcal{G}^{(0)}$.

Chapter 3

$\check{C}ech$ cohomology for leaf spaces, and characteristic classes

3.1 Introduction

In this Chapter we introduce a $\check{C}ech$ cohomology for leaf spaces, and we show that it is a natural object for the theory of characteristic classes for foliations (Bott vanishing theorem, secondary classes, etc.). The idea is quite simple: we look at transversals to the foliations as defining opens in the leaf space; small opens $U \subset M$ project to transversals, while the inclusions $U \subset V$ $(\subset M)$ to holonomy embeddings (see 1.3.2) between transversals. The $\check{C}ech$ complexes are defined by viewing the holonomy embeddings as describing "inclusions" in the leaf space.

We emphasize that the main feature of our $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ is that they are good places for writing down formulas. All the constructions we describe are explicit/geometric, rather an extension of the usual constructions from manifolds (as in [15]) to "spaces" like $M/\mathcal{F}$. This is explained in the first two sections. The price to pay is that, a priori, our $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ is not conceptually well behaved. We remove this inconvenience in the last section, where we show that it is isomorphic to the "usual" cohomology $H^*(M/\mathcal{F})$. The last groups have been studied before, and they appear from three different sources as candidates to the cohomology of the leaf space: from Connes's non-commutative geometry approach ([29]), in which the leaf space is modeled by its non-commutative algebra "$C^{\infty}_{c}(M/\mathcal{F})$" of smooth functions, and $H^*(M/\mathcal{F})$ is the (localization at units of the) cohomology of the algebra $HC^*_{\mathrm{per}}(\mathcal{C}^{\infty}_{c}(M/\mathcal{F}))$ (cf. Chapter 4), from the topos-theoretic approach ([72]) in which the leaf space is modeled by the associated category $Sh(M/\mathcal{F})$ of sheaves, and $H^*(M/\mathcal{F})$ are the right derived functors of the invariant-section functor (described explicitly by Haefliger in terms of bar-complexes), and from Bott-Haefliger's work on characteristic classes, in which $H^*(M/\mathcal{F})$ appears as the cohomology of the classifying space of the foliation. That these three approaches give the same result is a consequence of the work in [20, 34, 35, 74] and our Chapter 4. In this light, our $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ should be viewed as a geometric model of these cohomologies. In this chapter, it is not our intention to go into details concerning these different approaches. Instead, we remark that, "at the limit", the $\check{C}ech$ complexes $\check{C}^*(\mathcal{U}, \mathcal{A})$ give the bar-complexes of Bott-Haefliger (described in 1.2.13) which can be used to define $H^*(M/\mathcal{F}; \mathcal{A})$, and, using this remark, we prove that the two cohomologies are isomorphic. A consequence of this theorem is the the Poincaré duality $H^*(M/\mathcal{F}) \cong H^*_c(M/\mathcal{F}; or)^\vee$ for any foliation, with a proof very similar to the usual proof for manifolds ([15]). Actually we give a different proof of our Poincaré duality 2.4.13 of Chapter 2. Another consequence is that, since the constructions make sense for any étale groupoid, we get in the case of Haefliger's Γ^q, an explicit construction of the universal characteristic map $k : H^*(\underline{WO}_q) \longrightarrow H^*(B\Gamma^q)$.

3.2 Transversal structures

Let $(M, \mathcal{F})$ be a codimension q foliation. As described in 1.3.5, transversal structures of the foliations are structures which morally live on the leaf space $M/\mathcal{F}$, or, more precisely, which can be expressed in terms of the holonomy groupoid. In this section we look at some simple examples that we need in this chapter: transversal bundles and cohomology.

3.2.1 Foliated bundles: Recall [64] that a *foliated principal G-bundle* (over the foliation $(M, \mathcal{F})$) consists of a principal G-bundle $\pi : P \longrightarrow M$, endowed with an equivariant foliation $\tilde{\mathcal{F}}$, which is a lift of $\mathcal{F}$ to P, and which is transversal to the fibers of π.

This means that the foliation $\tilde{\mathcal{F}}$ on P comes equipped with a continuous action of G ($\tilde{\mathcal{F}}_p \longrightarrow \tilde{\mathcal{F}}_{pg}$ for $p \in P$, $g \in G$), such that:

(i) $T^{ver}_p(P) \cap \tilde{\mathcal{F}}_p = 0$, for all $p \in P$,

(ii) $(d\pi)_p : \tilde{\mathcal{F}}_p \longrightarrow \mathcal{F}_{\pi(p)}$ is an isomorphism, for all $p \in P$.

Recall that $T^{ver}_p(P) := T_p(P_x) = Ker((d\pi)_p) \subset T_pP$, where $x = \pi(p)$, $P_x = \pi^{-1}(x)$.

As remarked in [64], $\tilde{\mathcal{F}}$ can be viewed as a partial connection on the principal G-bundle $\pi : P \longrightarrow M$ (in the sense that the usual condition $T_p(P) = T^{ver}_p(P) \oplus \tilde{\mathcal{F}}_p$ for connections is replaced by the weaker condition (i)). Let us remark that, as in the classical case of connections, one still has the notion of parallel transport, with the only difference that it is defined only for longitudinal curves (i.e. curves which lie inside leaves of $\mathcal{F}$): for any such curve $\alpha : I \longrightarrow M$ ($I = [0, 1]$), there is an induced map:

$$P_\alpha : P_{\alpha(0)} \longrightarrow P_{\alpha(1)} ,$$

constructed as follows. For $p \in P_{\alpha(0)}$, there is a unique horizontal lift $\tilde{\alpha} : I \longrightarrow P$ of α starting at p (i.e. such that $\tilde{\alpha}'(t) \in \tilde{\mathcal{F}}_{\alpha(t)}$ for all $t \in I$, $\tilde{\alpha}(0) = p$), and $P_\alpha(p) := \tilde{\alpha}(1)$.

3.2.2 Transversal bundles: A *transversal principal G-bundle* is a foliated principal G-bundle P with the property that its parallel transport depends only on the holonomy classes of curves. With the obvious notion of morphisms, we form the category $Bun_G(M/\mathcal{F})$ of transversal principal G-bundles.

An alternative definition is to require P to be a principal G-bundle on M, endowed with a (smooth) action of the holonomy of the foliation, compatible with the action of G (see Proposition 3.2.3 below). More generally, one can construct the category $Bun_G(\mathcal{G})$ of principal G-bundles on $\mathcal{G}$, for any (smooth) groupoid $\mathcal{G}$ (not necessarily étale): an object is a principal G-bundle $\pi : P \longrightarrow \mathcal{G}^{(0)}$, endowed with an action

$\mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} P \longrightarrow P$, $(\gamma, p) \mapsto \gamma \cdot p$, such that $(\gamma p)g = \gamma(pg)$ for all $g \in G$, $\gamma \in \mathcal{G}$, $p \in P$. One has for any $x \in \mathcal{G}^{(0)}$:

$$P_{s(\gamma)} \ni p \ \mapsto \ \gamma \cdot p \in P_{t(\gamma)} \ . \tag{3.1}$$

Proposition 3.2.3 *Let $\pi : P \longrightarrow M$ be a principal G-bundle endowed with a (smooth) action of the holonomy of $\mathcal{F}$ (i.e. $P \in Bun_G(Hol(M, \mathcal{F}))$). Then, defining $\tilde{\mathcal{F}} \subset TP$ by:*

$$\tilde{\mathcal{F}}_p := \{(\frac{d}{dt})_{t=0}(\alpha_t \cdot p) : \alpha \text{ is a longitudinal curve starting at } \pi(p)\}$$

(where α_t is the curve $s \mapsto \alpha(ts)$), P becomes a transversal principal G-bundle, whose parallel transport is given by:

$$P_\alpha : P_{\alpha(0)} \longrightarrow P_{\alpha(1)}, \ p \ \mapsto \ \alpha \cdot p \ .$$

Moreover, this construction induces an isomorphism of categories:

$$Bun_G(Hol(M, \mathcal{F})) \ \cong \ Bun_G(M/\mathcal{F}) \ .$$

proof: To prove that $\tilde{\mathcal{F}}_p$ is a foliation on P, which is an equivariant lift of $\mathcal{F}$, we may work locally and assume that $M = \mathbb{R}^q \times \mathbb{R}^{n-q}$ is endowed with the standard codimension q foliation, $P = \mathbb{R}^q \times \mathbb{R}^{n-q} \times G$. The action (3.1) of the holonomy on P is of type:

$$P_{(x,y)} \longrightarrow P_{(x,z)}, \ (x, y, g) \mapsto (x, z, u(x, z)^{-1}u(x, y)g),$$

for some smooth function $u : \mathbb{R}^n \longrightarrow G$. For $p = (x_0, y_0, g_0) \in P$ we then have:

$$\tilde{\mathcal{F}}_p = \{(\frac{d}{dt})_{t=0}(x_0, \alpha(t), u(x_0, \alpha(t))^{-1}u(x_0, y_0)g) : \alpha : I \longrightarrow \mathbb{R}^{n-q}, \alpha(0) = y_0\}.$$

On the other hand, the submersion:

$$f : P = \mathbb{R}^q \times \mathbb{R}^{n-q} \times G \longrightarrow \mathbb{R}^q \times G, \ (x, y, g) \mapsto (x, u(x, g)g) \ ,$$

defines a foliation $\tilde{\mathcal{F}}'$ on P whose leaf through p is:

$$\{(x_0, y, u(x_0, y)^{-1}u(x_0, y_0)g_0 : \ y \in \mathbb{R}^p\},$$

hence $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}'$, and this clearly shows the assertions we wanted to prove.

Globally now, for any curve α (as in the statement):

$$(d\pi)_p((\frac{d}{dt})_{t=0}(\alpha_t \cdot p)) \ = \ \alpha'(0),$$

hence $T^{ver}_p(P) \cap \tilde{\mathcal{F}}_p = 0$. It also shows that the horizontal lift (at p) of α is $t \mapsto \alpha_t \cdot p$, hence the parallel transport along α is $P_\alpha(p) = \alpha \cdot p$ (and this depends just on the holonomy class of α).

Of course one also has some smoothness problems (like $t \mapsto \alpha_t$ is smooth, the smoothness of the action of the holonomy on P corresponds to the smoothness of $\tilde{\mathcal{F}}$, etc), but these are straightforward (see [99] for the description of local charts in the holonomy groupoid). $\square$

As for sheaves, any Morita equivalence $\mathcal{G} \cong \mathcal{H}$ induces an equivalence of categories $Bun_G(\mathcal{G}) \cong Bun_G(\mathcal{H})$. In particular, for any complete transversal T, one has:

$$Bun_G(M/\mathcal{F}) \cong Bun_G(Hol_T(M, \mathcal{F})), \ P \mapsto P|_T \ .$$

The usual description of principal bundles in terms of transition functions extends from bundles over manifolds to bundles over étale groupoids. To see this, we need the embedding category of [73].

3.2.4 The embedding category [73]: Let $\mathcal{G}$ be any étale groupoid, and let $\mathcal{U}$ be a fixed basis for the open sets of $\mathcal{G}^{(0)}$. Define a discrete category $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ as follows. Its objects are opens $U \in \mathcal{U}$, and an arrow $U \longrightarrow V$ is a section $\sigma : U \longrightarrow \mathcal{G}^{(1)}$ of $s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ with the property that $t \circ \sigma : U \longrightarrow \mathcal{G}^{(0)}$ is an embedding of U in V. We write $\sigma : U \longrightarrow V$ and refer to σ as a *"$\mathcal{G}$-embedding"*. Note the distinction between the arrow σ and the actual embedding $t \circ \sigma$. Composition of two such $\mathcal{G}$-embeddings $\sigma : U \longrightarrow V$ and $\tau : V \longrightarrow W$ is defined in terms of the composition of $\mathcal{G}$, as:

$$(\tau \circ \sigma)(x) = \tau(\sigma(x)) \cdot \sigma(x) \ .$$

In [73] it was proved that the classifying space $B\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ has the same weak homotopy type as $B\mathcal{G}$ (provided the basic open sets are all contractible). Thus, $B\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ can be viewed as a CW-complex model for the "bad" (possible non-Hausdorff) classifying space $B\mathcal{G}$. Remark also that, if T is a complete transversal, and $\mathcal{U}$ is a basis for the open sets of T, then $\mathcal{O}_{\mathcal{U}}(Hol_T(M, \mathcal{F}))$ coincides with the discrete category $\mathcal{O}_{\mathcal{U}}(M/\mathcal{F})$ described in 1.3.2.

As in the case of bundles over manifolds, any $P \in Bun_G(\mathcal{G})$ has associated transition functions. Indeed, choosing a basis $\mathcal{U}$ by trivialisation charts of P, any $\mathcal{G}$-embedding $\sigma : U \longrightarrow V$ induces (using the action of $\mathcal{G}$ on P) a G-equivariant map $P_U \longrightarrow P_V$, $p \mapsto \sigma(\pi(p)) \cdot p$, i.e., using the trivialisation charts $P_U \cong U \times G$, one gets:

$$g_\sigma : U \longrightarrow G \ ,$$

associated to any $\mathcal{G}$-embedding σ.

Proposition 3.2.5 *The transition functions $\{g_\sigma\}$ satisfy the cocycle relation:*

$$g_{\tau\sigma} = g_\tau \circ \sigma \cdot g_\sigma \ ,$$

whenever τ and σ are composable. Conversely, any such system $\{g_\sigma : \sigma \in \mathcal{O}_{\mathcal{U}}(\mathcal{G})\}$ is the system of transition functions of a principal bundle $P \in Bun_G(\mathcal{G})$.

proof: The cocycle relation is straightforward. For the converse, one defines P as the quotient of $\coprod_{U \in \mathcal{U}} \{U\} \times U \times G$ by the equivalence relation:

$$(U, x, g) \sim (V, x, g_{U,V}\, g), \quad x \in U, \ g \in G \ ,$$

whenever $U \subset V$, where $g_{U,V}$ is the transition function associated to the obvious $\mathcal{G}$ embedding $i_{U,V} : U \longrightarrow V$. The action of G on P is simply the multiplication on the last component, the projection $\pi : P \longrightarrow \mathcal{G}^{(0)}$ just keeps the x, and the action of $\mathcal{G}$ on

P is defined as follows. For an arrow $\gamma : x \longrightarrow y$ of $\mathcal{G}$, and an element $p \in P_x$, one may choose $U \in \mathcal{U}$ containing x, such that $p = [U, x, g]$ for some $g \in G$, and U is the domain of a $\mathcal{G}$-embedding $\sigma : U \longrightarrow V$ with $\sigma(x) = \gamma$; then:

$$\gamma \cdot [U, x, g] := [V, y, g_\sigma(x)g] \ .$$

To check that this is well defined, assume that $p = [U', x, g']$, for some other $U' \in \mathcal{U}$, domain of a $\mathcal{G}$-embedding $\sigma' : U' \longrightarrow V'$ with $\sigma'(x) = \gamma$. Since $\mathcal{U}$ is a basis, we may assume $U' \subset U$, $V' \subset V$, and $\sigma|_{U'} = \sigma'$. Then $g' = g_{U',U}(x)^{-1}g$, and $g_{V',V}(y)g_{\sigma'}(x) = g_\sigma(x)g_{U',U}(x)$, hence:

$$[V', y, g_{\sigma'}(x)g'] = [V, y, g_{V',V}(y)g_{\sigma'}g_{U',U}(x)^{-1}g] = [V, y, g_\sigma(x)g] \ .$$

It is straightforward to check now that $\mathcal{U}$ is a cover by domains of trivialisation charts of P, and the associated transition functions are precisely the g_σ's. $\square$

3.2.6 Cohomologies of the leaf space: As mentioned in 1.3.5, we define the cohomology $H^*(M/\mathcal{F})$ of the leaf space as the cohomology of the holonomy groupoid (or its reduced version). By our computations in Chapter 4 this fits well with the "non-commutative approach" to leaf spaces (which tells us that the right cohomology is the cyclic cohomology of the convolution algebra), and, by Moerdijk's result [74], also with Bott's approach via classifying spaces. Another cohomology associated to the leaf space is *the basic cohomology* (see e.g. [91]) $H^*_{bas}(M/\mathcal{F})$, which is defined as the cohomology of the complex $\Omega^*_{basic}(M/\mathcal{F})$ of basic forms:

$$\Omega^p_{bas}(M/\mathcal{F}) = \{\omega \in \Omega^p(M) : \ i_v(\omega) = 0, L_v(\omega) = 0, \ \forall v \in \Gamma(\mathcal{F})\}.$$

In general, the two cohomologies are related by a map $i : H^*_{bas}(M/\mathcal{F}) \longrightarrow H^*(M/\mathcal{F})$ which will be described also in the next sections. Let us just mention that, composing with the map induced by the projection $\pi : M \longrightarrow M/\mathcal{F}$, one gets the obvious map induced by the inclusion $j : \Omega^*_{bas}(M/\mathcal{F}) \hookrightarrow \Omega^*(M)$:

$$\begin{tikzcd} H^*_{bas}(M/\mathcal{F}) \arrow[r, "i"] \arrow[dr, "j"'] & H^*(M/\mathcal{F}) \arrow[d, "\pi^*"] \\ & H^*(M) \end{tikzcd}$$

In general i is not an isomorphism. While $H^*_{bas}(M/\mathcal{F})$ is finite dimensional in many cases, $H^*(M/\mathcal{F})$ is much larger. Nevertheless $H^*(M/\mathcal{F})$ has many advantages over $H^*_{bas}(M/\mathcal{F})$. Let us mention that in [64] the construction of the geometric characteristic map (1.20) of the normal bundle is extended to any foliated vector bundle E:

$$k^{geom}_E : H^*(\underline{WO}_q) \longrightarrow H^*(M) \ .$$

We will see that the map naturally factorizes through $H^*(M/\mathcal{F})$ (while it is known that it does not factorize through $H^*_{bas}(M/\mathcal{F})$ unless $\mathcal{F}$ is riemannian; see also Corollary 3.4.11). Also, $H^*(M/\mathcal{F})$ satisfies Poincaré duality in full generality by our results in Chapter 2 (see Section 3.6 for a simpler proof).

A similar discussion holds for the case of compactly supported cohomology. First of all, using the definition in 2.2.7 and the Morita invariance (Corollary 2.3.6), we define:

$$H_c^*(M/\mathcal{F}) := H_c^*(Hol_T(M,\mathcal{F}))\ .$$

Using the integration over $\pi : M \longrightarrow M/\mathcal{F}$ one gets (in the orientable case) a map:

$$\int_{\mathcal{F}} : H_c^*(M) \longrightarrow H_c^{*-p}(M/\mathcal{F})$$

where p is the dimension of the foliation.
Haefliger [56] also introduced the compactly supported basic cohomology $H^*_{c,bas}(M/\mathcal{F})$ (see also 2.2.8 and Section 3.6), together with an integration over the leaves map:

$$\int_{\mathcal{F}} : H_c^*(M) \longrightarrow H^{*-p}_{c,bas}(M/\mathcal{F}).$$

The two integrations are connected via a natural morphism j_b in a commutative diagram:

$$\begin{tikzcd} H_c^*(M) \arrow[r, "\int_{\mathcal{F}}"] \arrow[dr, "\int_{\mathcal{F}}"'] & H_c^{*-p}(M/\mathcal{F}) \arrow[d, "j_b"] \\ & H^{*-p}_{c,bas}(M/\mathcal{F}) \end{tikzcd}$$

3.3 *Čech* cohomology of the leaf space

3.3.1 Čech-DeRham complex: Let $(M,\mathcal{F})$ be a codimension q foliation, let $\mathcal{U}$ be a basis of opens of a complete transversal T, and let $\mathcal{O}_{\mathcal{U}}(M/\mathcal{F})$ be the associated discrete category of holonomy embeddings (hence the arrows $g : U \longrightarrow V$ are the embeddings with the property that $germ_x(g) : (T,x) \longrightarrow (T,g(x))$ is a holonomy germ, for all $x \in U$; see 1.3.2 and 3.2.4). Recall that we view it as a good replacement of the lattice of opens of the leaf space (motivated e.g. by Proposition 3.2.5). Inspired by this, we introduce the Čech complexes:

$$\check{C}^*_{\mathcal{U}}(M/\mathcal{F},\Omega^p) : \prod_{U_0}\Omega^p(U_0) \stackrel{\delta}{\longrightarrow} \prod_{U_0 \stackrel{\sigma_1}{\longrightarrow} U_1}\Omega^p(U_0) \stackrel{\delta}{\longrightarrow} \prod_{U_0 \stackrel{\sigma_1}{\longrightarrow} U_1 \stackrel{\sigma_2}{\longrightarrow} U_2}\Omega^p(U_0) \stackrel{\delta}{\longrightarrow} \ \ldots\ , \tag{3.2}$$

where the product is over the strings of composable arrows of $\mathcal{O}_{\mathcal{U}}(M/\mathcal{F})$, with the boundary:

$$\begin{aligned} \delta(\omega)(\sigma_1,\ \ldots,\sigma_{p+1}) &= \sigma_1^*\omega(\sigma_2,\ \ldots,\sigma_{p+1})+ \\ &+\textstyle\sum_{i=1}^{p}(-1)^i\omega(\sigma_1,\ \ldots,\sigma_{i+1}\sigma_i,\ \ldots,\sigma_{p+1}) + (-1)^{p+1}\omega(\sigma_1,\ \ldots,\sigma_p). \end{aligned} \tag{3.3}$$

The Čech-DeRham complex (relative to $\mathcal{U}$) is the resulting double complex

$$(\check{C}^*_{\mathcal{U}}(M/\mathcal{F},\Omega^*),\delta,d)\ ,$$

where d is DeRham differential. We keep the same notation for its total complex, with sign convention for the total boundary: $D = d + (-1)^s\delta$ on $C^s(M/\mathcal{F}, \Omega^t)$. As in the case of manifolds [15], we have a product:

$$(\omega \cdot \eta)(\sigma_1, \ldots, \sigma_{p+p'}) = (-1)^{pp'}\omega(\sigma_1, \ldots, \sigma_p) \cdot \sigma_1^* \ldots \sigma_p^* \eta(\sigma_{p+1}, \ldots, \sigma_{p+p'}).$$

which makes $(\check{C}^*_{\mathcal{U}}(M/\mathcal{F}, \Omega^*), \delta, d)$ into a bigraded differential algebra.

We denote by $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ the resulting cohomology. Unless specified otherwise, the forms we consider are complex-valued forms, and the coefficients are $\mathbf{C}$.

3.3.2 Basic properties: Let us remark that the cohomology of the complex (3.2) is, in degree 0, the space $\Omega^p_{bas}(M/\mathcal{F})$ of basic forms of the foliations (see 3.2.6). A first consequence of this remark is an obvious morphism of complexes $\Omega^*_{bas}(M/\mathcal{F}) \longrightarrow \check{C}^*(\mathcal{U}, \Omega^*)$, hence also a map in cohomology (compare to 3.2.6):

$$i : H^p_{bas}(M/\mathcal{F}) \longrightarrow \check{H}^*_{\mathcal{U}}(M/\mathcal{F}). \tag{3.4}$$

Another consequence is that the cohomology of (3.2) does not depend, in degree 0, neither on the choice of the transversal T, nor of the basis $\mathcal{U}$. As we will see in section 3.6, this is true in any degree. More precisely, for any two coverings $\mathcal{U}, \mathcal{U}'$ (of the complete transversals T, T'), we form the covering $\mathcal{U} \coprod \mathcal{U}'$ (of $T \coprod T'$), and the obvious maps: $\check{C}^*_{\mathcal{U}}(M/\mathcal{F}, \Omega^p) \longleftarrow \check{C}^*_{\mathcal{U}\coprod\mathcal{U}'}(M/\mathcal{F}, \Omega^p) \longrightarrow \check{C}^*_{\mathcal{U}'}(M/\mathcal{F}, \Omega^p)$ induce isomorphisms in cohomology. So, a consequence of the discussion in Section 3.6 is the following:

Corollary 3.3.3 *The cohomology $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ does not depend on the choice of $\mathcal{U}$ or of the complete transversal T. It is simply denoted by $\check{H}^*(M/\mathcal{F})$.*

Let us also remark the existence of a pull-back map (compare to 3.2.6):

$$\pi^* : \check{H}^*(M/\mathcal{F}) \longrightarrow H^*(M), \tag{3.5}$$

constructed as follows. We choose a Haefliger cocycle $(D, f_D, \gamma_{D,D'})$ of the foliation as in 1.3.1, and we construct $T = \coprod_{D\in\mathcal{D}} \mathbb{R}^q$ as in 1.3.2. We choose any basis $\mathcal{U}$ of T containing all the opens $f_D(D \cap D')$. Associating to any inclusion $D \subset D'$ the Γ-embedding $\gamma_{D,D'} : f_D(D) \tilde{\longrightarrow} f_{D'}(D) \hookrightarrow f_{D'}(D')$, and using the pull-back of forms along f_D's, there is an obvious chain map $\check{C}^*_{\mathcal{U}}(M/\mathcal{F}, \Omega^*) \longrightarrow \check{C}^*_{\mathcal{D}}(M, \Omega^*)$, which serves as definition of (3.5).

Clearly, the composition of (3.4) and (3.5):

$$H^p_{bas}(M/\mathcal{F}) \xrightarrow{i} \check{H}^*_{\mathcal{U}}(M/\mathcal{F}) \xrightarrow{\pi^*} H^*(M)$$

coincides with the usual morphism coming from the inclusion $j : \Omega^*_{bas}(\mathcal{F}) \hookrightarrow \Omega^*(M)$ (compare to 3.2.6).

Another consequence of the freedom of choosing arbitrary $\mathcal{U}$ (cf. 3.3.3), is that one can compute $\check{H}^*(M/\mathcal{F})$ by the simpler Čech complexes:

$$\check{C}^*_{\mathcal{U}}(M/\mathcal{F}) : \prod_{U_0} \mathbf{C} \xrightarrow{\delta} \prod_{U_0 \xrightarrow{\sigma_1} U_1} \mathbf{C} \xrightarrow{\delta} \prod_{U_0 \xrightarrow{\sigma_1} U_1 \xrightarrow{\sigma_2} U_2} \mathbf{C} \xrightarrow{\delta} \ldots,$$

provided we choose a contractible basis $\mathcal{U}$. For arbitrary (constant) coefficients A, there is a similar complex $\check{C}^*_{\mathcal{U}}(M/\mathcal{F}; A)$ whose cohomology does not depend on the choice of the contractible basis (cf. Section 3.6 again), and is denoted $\check{H}^*(M/\mathcal{F}; A)$.

3.3.4 Explicit formulas: Once Theorem 3.6.6 is proved, many of the properties of $\check{H}^*(M/\mathcal{F})$ are known since Bott-Haefliger's work on foliations, and are consequences of abstract (but non-trivial) universal properties. Our aim is to show that, using our quite intuitive model for the cohomology of the leaf space, all these properties (and some others) can be deduced in a naive explicit way, as an extension of the usual constructions on manifolds, to "spaces" like $M/\mathcal{F}$. This is best illustrated by the next sections.

Let us start here with a quite trivial example: the obstruction to the transverse orientability of the foliation. We choose a contractible basis $\mathcal{U}$ by trivialisation charts for the tangent bundle of T, and, for any arrow $\sigma : U \longrightarrow V$ in $\mathcal{O}_{\mathcal{U}}(M/\mathcal{F})$, consider its differential $J_\sigma : U \longrightarrow GL_q$ (the transition functions of the normal bundle $\nu = \tau_M/\mathcal{F}$ of the foliation). We define *the first Stieffel-Whitney class* $w_1(M/\mathcal{F}) \in \check{H}^*(M/\mathcal{F}; \mathbf{Z}_2)$ represented by the cocycle $w_1(\sigma) = sign(J_\sigma)$. By construction:

Proposition 3.3.5 *The foliation is transversally orientable if and only if*

$$w_1(M/\mathcal{F}) \in \check{H}^*(M/\mathcal{F}; \mathbf{Z}_2)$$

vanishes.

Another simple example is the obstruction to the existence of a transversal connection on a given transversal principal G-bundle P, i.e. a connection on P invariant under holonomy. Of course, the existence of such a connection is a strong requirement. Let $\mathcal{U}$ be as before, and let $\{g_\sigma\}$ be the transition functions of P (see 3.2.5). For any arrow $\sigma : U \longrightarrow V$ in $\mathcal{O}_{\mathcal{U}}(M/\mathcal{F})$ we form the 1-form $\eta^G_\sigma := g^*_\sigma \eta^G \in \Omega^1(U, \mathfrak{g})$, where η^G is the Maurer-Cartan form of G. This can be viewed as a closed cocycle in $\check{C}^*(\mathcal{U}; \Omega^1(-, \mathfrak{g}))$ (where $U \mapsto \Omega^1(U, \mathfrak{g})$ is considered with the action of arrows σ given by $\Omega^1(V, \mathfrak{g}) \longrightarrow \Omega^1(U, \mathfrak{g})$, $\omega \mapsto g_\sigma \sigma^*(\omega) g_\sigma^{-1}$). Its class is denoted by $a(P) \in \check{H}^1(M/\mathcal{F}; \Omega^1(-, \mathfrak{g}))$ and is analogous to the Atiyah-Molino class of [77].

Proposition 3.3.6 *The transversal G-bundle P admits a transversal connection if and only if $a(P)$ vanishes.*

proof: Locally, a connection ∇ is given by connection 1-forms which are of type $\omega(\nabla_U) = \omega^0_U + \pi^*_U \omega_U$, where $\omega_U \in \Omega^1(U, \mathfrak{g})$ is any 1-form, ω^0_U is the canonical connection form, and $\pi_U : P_U \longrightarrow U$. The compatibility condition means $\omega_V = g_\sigma \sigma^* \omega_U g_\sigma^{-1} + \eta^G_\sigma$, which shows that $a(P)$ must be trivial. □

3.3.7 The universal complex: Let us now remark that there is a universal complex, obtained as an analogue of (3.2) with the only difference that the arrows σ run through all embeddings $\mathbb{R}^q \longrightarrow \mathbb{R}^q$. It is suggestively denoted by $\check{C}^*(\Gamma^q, \Omega^*)$; one has:

$$\check{C}^*(\Gamma^q, \Omega^p) : \ \Omega^p(\mathbb{R}^q) \stackrel{\delta}{\longrightarrow} \prod_{\mathbb{R}^q \xrightarrow{\sigma_1} \mathbb{R}^q} \Omega^p(\mathbb{R}^q) \stackrel{\delta}{\longrightarrow} \prod_{\mathbb{R}^q \xrightarrow{\sigma_1} \mathbb{R}^q \xrightarrow{\sigma_2} \mathbb{R}^q} \Omega^p(\mathbb{R}^q) \stackrel{\delta}{\longrightarrow} \ \ldots\ , \qquad (3.6)$$

The resulting cohomology, denoted by $\check{H}^*(\Gamma^q)$, is the group of universal (topological) characteristic classes (see 1.4.3). This because, from [93] (or from the more general result of section 3.6) $\check{H}^*(\Gamma^q) \cong H^*(B\Gamma^q)$. We emphasize that the assertion on universal characteristic classes should be understood here by the obvious maps:

$$f^*_{\mathcal{F}} : \check{H}^*(\Gamma^q) \longrightarrow \check{H}^*_{\mathcal{U}}(M/\mathcal{F})$$

associated to any foliation $\mathcal{F}$ (of course, one has to choose charts $U \cong \mathbb{R}^q$ first, but $f^*_{\mathcal{F}}$ does not depend on the choice). All the formulas we write have their universal analogue in $\check{C}^*(\Gamma^q, \Omega^*)$. Because of this it is interesting to pass from $\check{C}^*(\Gamma^q, \Omega^*)$ to the simpler $\check{C}^*(\Gamma^q)$. Recall that the inclusion $i : \check{C}^*(\Gamma^q) \longrightarrow \check{C}^*(\Gamma^q, \Omega^*)$ induces isomorphisms in cohomology, by the Poincaré Lemma.

Lemma 3.3.8 *A closed k-cocycle in the Čech DeRham complex:*

$$u = u_0 + u_1 + \ldots + u_k, \quad u_s \in \check{C}^s(\Gamma^q, \Omega^{k-s})$$

represents the same class in $\check{H}^*(\Gamma^q)$ *as the k-cocycle* $\tilde{u}$ *in the Čech complex* $\check{C}^*(\Gamma^q)$*, given by:*

$$\tilde{u}(\sigma_1, \ldots, \sigma_k) = \sum_{s=0}^{k} (-1)^{k(s-1)+\frac{s(s-1)}{2}} \int_{I_{\sigma_1, \ldots, \sigma_s}} u_{k-s}(\sigma_{s+1}, \ldots, \sigma_k).$$

Here, $I_{\sigma_1, \ldots, \sigma_s}$ *is the s-cube:*

$$I_{\sigma_1, \ldots, \sigma_s}(t_1, \ldots, t_s) = \sigma_s(\sigma_{s-1}(\ldots\sigma_3(\sigma_2(\sigma_1(0)t_1)t_2)\ldots)t_{s-1})t_s.$$

proof: This is a computation involving the explicit formulas in the Poincaré Lemma. Denote $C^{s,t} = \check{C}^s(\Gamma^q, \Omega^t)$, $C^s = \check{C}^s(\Gamma^q)$. To keep track of the formulas, we introduce some operators. First of all, the one involved in the Poincaré Lemma $H : C^{s,t} \longrightarrow C^{s,t-1}$, $Hu(\sigma_1, \ldots, \sigma_s) = \int_0^1 h^*(u(\sigma_1, \ldots, \sigma_s))$, where $h(z, v) = zv$ $(z \in [0, 1])$. It gives a contraction along t:

$$dH + Hd = 1 \text{ on } C^{s,t}, t > 0, \; Hd + ir = 1, \text{ on } C^{s,0} \tag{3.7}$$

where $r : C^{s,0} \longrightarrow C^s$, $ru(\sigma_1, \ldots, \sigma_s) = u(\sigma_1, \ldots, \sigma_s)(0)$, and i is the inclusion. H does not commute with d; one has $\delta H - H\delta = (-1)^{s+1}C = A - B : C^{s,t} \longrightarrow C^{s+1,t-1}$, where C is defined by the previous equality, $(Au)(\sigma_1, \ldots, \sigma_{s+1}) = \int_0^1 a^*_{\sigma_1} u(\sigma_2, \ldots, \sigma_{s+1})$, $(Bu)(\sigma_1, \ldots, \sigma_{s+1}) = \int_0^1 b^*_{\sigma_1} u(\sigma_2, \ldots, \sigma_{s+1})$, $a_\sigma(x, z) = \sigma(x)z$, $b_\sigma(x, z) = \sigma(xz)$ $(z \in [0, 1])$. A direct computation shows that:

$$dC + Cd = 0, \text{ on } C^{s,t}, t > 0, \; Cd = iL \text{ on } C^{s,0} \tag{3.8}$$

where $L : C^{s,0} \longrightarrow C^{s+1}$, $(Lu)(\sigma_1, \ldots, \sigma_{s+1}) = u(\sigma_2, \ldots, \sigma_{s+1})(\sigma_1(0))$.

Now, we construct inductively $\tilde{u}_s \in C^{s,k-s}$ with the property that $\tilde{u}_s + \sum_{j=s+1}^{k} u_j$ is a new cocycle which differs from u by a boundary. We choose $\tilde{u}_0 = u_0$. If $\tilde{u}_j$, $1 \leq j \leq s$ have been constructed, $s + 1 < k$, then, subtracting $D((-1)^s H\tilde{u}_s)$, we may choose $\tilde{u}_{s+1} = u_{s+1} - \delta((-1)^s H\tilde{u}_s) = u_{s+1} - (-1)^s(H\delta\tilde{u}_s + (-1)^{s+1}C\tilde{u}_s) = u_{s+1} - (-1)^s H((-1)^s du_{s+1}) + C\tilde{u}_s = dHu_{s+1} + C\tilde{u}_s$. Similarly, if $s + 1 = k$, one gets $\tilde{u}_k = ir(u_k) + C\tilde{u}_{k-1}$. Since $u_0 = dHu_0$, we get that u is cobordant to $\tilde{u}_k = \sum_{s=1}^{k} C^s dHu_{k-s} + ir(u_k) \in C^{k,0}$. On the other hand, one has (by the previous formulas): $C^s dHu_{k-s} = iLB^{s-1}Hu_{k-s}$, up to the sign appearing in the statement, and the final formula follows from the remark:

$$LB^{s-1}Hu_{k-s}(\sigma_1, \ldots, \sigma_k) = \int_{I_{\sigma_1, \ldots, \sigma_s}} u_{k-s}(\sigma_{s+1}, \ldots, \sigma_k). \; \square$$

3.4 Characteristic classes: explicit formulas in $M/\mathcal{F}$

Inspired by the usual construction of characteristic classes of principal (or vector) bundles on manifolds in the Čech-DeRham complex [13], we will construct in the next section the Chern-Weil homomorphism:

$$k_P : S(\mathfrak{g}^*)^G \longrightarrow \check{H}^*(M/\mathcal{F}) \tag{3.9}$$

for any transversal principal G-bundle P, and, more generally, the secondary characteristic classes (compare to our preliminaries in 1.4), described by an exotic Chern-Weil characteristic map:

$$k_P : H^*(\underline{W}(\mathfrak{g}, K)) \longrightarrow \check{H}^*(M/\mathcal{F}). \tag{3.10}$$

In the case of (the principal GL_q-bundle associated to) a transversal vector bundle E, one gets:

$$k_E^{geom} : H^*(\underline{WO}_q) \longrightarrow \check{H}^*(M/\mathcal{F}).$$

Of particular interest is the case of the normal bundle of $\mathcal{F}$, when the resulting map (denoted by $k_{\mathcal{F}}^{geom}$) encodes the secondary characteristic classes, and, composed with (3.5), gives the usual [13, 58, 64] characteristic homomorphism (1.20) described in 1.4.4.

In this section we briefly describe the case of vector bundles, with emphasis on explicit formulas.

3.4.1 Chern-Weil for transversal vector bundles: Let us first describe the closed cocycles representing the Chern classes of any transversal vector bundle E. We fix an invariant polynomial f on $\mathfrak{gl}_k$ (where k is the dimension of E), and let d be its degree. We consider a complete transversal T, and a basis $\mathcal{U}$ of T. We choose an arbitrary local connection, that is, a collection $\nabla = \{\nabla(U) : U \in \mathcal{U}\}$ of connections on the restrictions E_U. For any holonomy embedding $\sigma : U \longrightarrow V$ (see 1.3.2), denote by $\nabla(\sigma)$ the connection on E_U obtained by using the pull-back connection $\sigma^*\nabla(V)$ on σ^*E_V, and the isomorphism $E_U \cong \sigma^*E_V$ induced by the action of the holonomy on E.

Recall the Chern-Simons construction of characteristic forms. If ∇_i, $0 \leq i \leq p$ are $p+1$ connections on a vector bundle $F \in Vb(M)$ over a manifold M, $t_0\nabla_0 + \ldots + t_p\nabla_p$ denotes the affine combinations of the given ∇_i's, which is a connection on $\pi_p^*F \in Vb(M \times \mathbf{\Delta}^p)$ ($\pi_p : M \times \mathbf{\Delta}^p \longrightarrow M$ is the projection). Applying our invariant polynomial f to the (local) curvature matrix $(t_0\nabla_0 + \ldots + t_p\nabla_p)^2$, the resulting form is a globally defined form on $M \times \mathbf{\Delta}^p$, which, integrated over $\mathbf{\Delta}^p$, gives a new form, denoted

$$(-1)^p k(\nabla_0, \ldots, \nabla_p)(f) \in \Omega^{2d-p}(M) \ .$$

Stokes' formula implies:

$$dk(\nabla_0, \ldots, \nabla_p)(f) = \sum_{i=0}^{p}(-1)^i k(\nabla_0, \ldots, \widehat{\nabla_i}, \ldots, \nabla_p)(f) \ . \tag{3.11}$$

Returning to our vector bundle $E \in Vb(M/\mathcal{F})$, we define $k_E(f) \in \check{C}^*(\mathcal{U}, \Omega^*)$ as the cocycle of total degree $2d$ defined by:

$$k_E(f)(\sigma_1, \ldots, \sigma_p) = k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ldots, \nabla(\sigma_p...\sigma_2\sigma_1))(f) \in \Omega^{2d-p}(U_0).$$

(U_0 is the domain of σ_1).

Lemma 3.4.2 *The cocycle $k_E(f) \in \check{C}^*(\mathcal{U}, \Omega^*)$ is closed, and its cohomology class $k_E(f) \in \check{H}^{2d}_{\mathcal{U}}(M/\mathcal{F})$ does not depend on the choice of the local connections.*

This follows from Stokes' formula above, and is a particular case of Theorem 3.5.3 proved in the next section. The resulting homomorphism:

$$k_E : I_q(\mathbb{R}) \longrightarrow \check{H}^*_{\mathcal{U}}(M/\mathcal{F})$$

plays the role of the Chern-Weil homomorphism of the transversal vector bundle E. We shall see that it is an algebra homomorphism. Composed with (3.5), it gives the usual Chern-Weil homomorphism of E as a vector bundle on M (i.e. *(geometrical)* of the preliminary section 1.4, for $G = GL_k$). The starting point for the construction of secondary classes is the following strong form of Bott vanishing theorem, at the level of $M/\mathcal{F}$ (not just M):

Proposition 3.4.3 *(Bott vanishing): For any transversal vector bundle E, k_E vanishes in degrees $> 2q$.*

proof: We look in each bi-degree (s, t), and we remark that the components of $k_E(f)$ are zero unless:

$$s + t = 2d, s \leq d, t \leq q \quad (d = deg(f)).$$

Indeed, the forms $k_E(f)(\sigma_1, \ldots, \sigma_s) \in \Omega^t(U_0)$ vanish for $s > d$ since in the integration appearing in the definition of k_f, the forms on t_i's have degree at most $d < s$ (see also our explicit formulas in 3.5.1). On the other hand, $\Omega^t(U_0) = 0$ for $t > q$. $\square$

Remark: It is useful to have the vanishing information at the level of the bi-graded algebra $\check{C}^*(\mathcal{U}, \Omega^*)$. For this, let us remark that the previous argument shows that any product of elements in this (non-commutative) algebra, which contains elements k_{f_i}, where f_i are invariant polynomials with $\sum deg(f_i) > q$, vanishes.

3.4.4 Formulas for the Godbillon-Vey classes in the leaf space: Let us now specialize to the case of the normal bundle $\nu = TM/\mathcal{F}$ of the foliation. The previous constructions define the characteristic map:

$$k_\nu : \mathbb{R}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \longrightarrow \check{H}^*_{\mathcal{U}}(M/\mathcal{F}),$$

which composed with (3.5) induces the usual $H^*(M)$-valued characteristic map of ν. Here $p_i = c_{2i}$, where c_i is, as in 1.4.2, a system of generators of $Inv(\mathfrak{gl}_q)$.

For explicit formulas, let us choose a basis $\mathcal{U}$ by trivialisation charts (as in the construction of (3.5) for instance), and let $J_\sigma : U \longrightarrow GL_q$ denote the Jacobian of $\sigma : U \longrightarrow V$ (any holonomy embedding). They are the associated transition functions of the transversal bundle ν. Locally, we choose the trivial connections. The corresponding $\nabla(\sigma)$ are given by the connection 1-forms:

$$\omega_\sigma := J_\sigma^{-1} dJ_\sigma \in \Omega^1(U; \mathfrak{gl}_q),$$

for $\sigma : U \longrightarrow V$. Then the Chern character $Ch_\nu \in \check{C}^*(\mathcal{U}, \Omega^*)$ is given by:

$$(\sigma_1, \ldots, \sigma_p) \mapsto (-1)^p \int_{t_0+t_1+\ldots+t_p \leq 1} exp(\,(t_1\omega_{\sigma_1} + t_2\omega_{\sigma_2\sigma_1} + \ldots + t_p\omega_{\sigma_p\ldots\sigma_2\sigma_1})^2\,) dt_0 dt_1 \ldots dt_p\,.$$

The first Chern class $C_1 = C_1(\nu) \in \check{C}^*(\mathcal{U}, \Omega^*)$, i.e. the one induced by the trace-polynomial, should be cohomologically trivial (since ν is real). And, indeed,

$$C_1^{(1,1)}(\sigma) = Tr(J_\sigma^{-1} dJ_\sigma),\ C_1^{(0,2)} = C_1^{(2,0)} = 0\ ,$$

and we can see that $C_1 = D(U_1)$, where:

$$U_1^{(0,1)} = 0,\ U_1^{(1,0)}(\sigma) = log(\mid det(J_\sigma)\mid).$$

The vanishing theorem (more precisely $C_1^{q+1} = 0$) gives us the first exotic class (Godbillon-Vey) of the leaf space: $U_1 C_1^q \in \check{C}^*(\mathcal{U}, \Omega^*)$ (see 1.4.5) is a new closed cocycle, whose class does not depend on the choice of U_1, and is denoted by $GV(\mathcal{F}) = [U_1 C_1^q] \in \check{H}^{2q+1}_{\mathcal{U}}(M/\mathcal{F})$. It pulls-backs via (3.5) to the usual Godbillon-Vey class of the foliation (see 1.4.5). We have:

Corollary 3.4.5 *$GV(\mathcal{F}) \in \check{H}^{2q+1}(M/\mathcal{F})$ is represented in the Čech- DeRham complex by the closed cocycle $gv_{\mathcal{F}}$ living in bi-degree $(q+1, q)$:*

$$gv_{\mathcal{F}}(\sigma_1,\ \ldots, \sigma_{q+1}) = log(\mid det(J_\sigma)\mid)\sigma_1^* Tr(\omega_{\sigma_2})\sigma_1^*\sigma_2^* Tr(\omega_{\sigma_3})\ \ldots\ \sigma_1^* ... \sigma_q^* Tr(\omega_{\sigma_{q+1}}).$$

All the formulas make sense in the universal complex $\check{C}(\Gamma^q, \Omega^*)$ of the previous section; using Lemma 3.3.8, one gets the formulas in the simpler complex $\check{C}(\Gamma^q)$. For instance, for the case $q = 1$:

Corollary 3.4.6 *The universal Godbillon-Vey class $GV \in H^3(B\Gamma^1) \cong \check{H}^3(\Gamma^1)$ is represented in $\check{C}(\Gamma^1)$ by the cocycle:*

$$\tilde{gv}_1(\sigma_1, \sigma_2, \sigma_3) = \int_0^{\sigma_1(0)} log(\mid \sigma_2'(t)\mid) \frac{\sigma_3''(\sigma_2(t))}{\sigma_3'(\sigma_2(t))} \sigma_2'(t) dt.$$

3.4.7 Bott's formulas: Similarly we can write down explicit formulas for the Bott-Godbillon-Vey classes (see 1.4.5):

$$GV_\alpha(\mathcal{F}) = [U_1 k_\nu(c_{\alpha_1})\ \ldots\ k_\nu(c_{\alpha_l})] \in \check{H}^{2q+1}_{\mathcal{U}}(M/\mathcal{F}),$$

(and its universal versions), where $c_i(A) = Tr(A^i)$, and $\alpha = (\alpha_1,\ \ldots, \alpha_l)$ is any partition of q. Again, the previous classes do not depend on the explicit choice of the transgression U_1 of C_1, and it is not difficult to see that they are given by the following cocycles which explain the formulas discovered by Bott in [12]:

Corollary 3.4.8 *The Bott-Godbillon-Vey class $GV_\alpha(\mathcal{F}) \in \check{H}^{2q+1}(M/\mathcal{F})$ is represented in the Čech- DeRham complex by the closed cocycle gv_α living in bi-degree $(q+1, q)$:*

$$gv_\alpha(\sigma_1, \ldots, \sigma_{q+1}) = log(\mid det(J_{\sigma_1})\mid) \cdot \sigma_1^* \{Tr[\ \omega_{\sigma_2} \cdot \sigma_2^*(\omega_{\sigma_3}) \cdot\ \ldots\ (\sigma_{(\alpha_1-1)} ... \sigma_2)^*(\omega_{\sigma_{\alpha_1}})]\ \} \cdot$$

$$(\sigma_{\alpha_1} ... \sigma_2\sigma_1)^* \{Tr[\ \omega_{\sigma_{(\alpha_1+1)}} \cdot \sigma^*_{(\alpha_1+1)}(\omega_{(\alpha_1+2)}) \cdot\ \ldots\ (\sigma_{(\alpha_1+\alpha_2-1)} ... \sigma_{(\alpha_1+1)})^*(\omega_{\sigma_{\alpha_2}})]\ \} \cdot\ \ldots$$

This obviously implies:

Corollary 3.4.9 *If $\mathcal{F}$ is an SL_q-foliation, then all its Bott-Godbillon-Vey classes vanish.*

3.4.10 The case of riemannian foliations: In the case of riemannian foliations, one knows [70] that the Bott vanishing theorem (in $H^*(M)$) can be strengthened: k_ν vanishes in degrees $> q$. We also mention that in general (when $\mathcal{F}$ is not necessarily riemannian), the basic cohomology (see (3.4)) is not enough for constructing all these characteristic classes. In the light of these facts, the following result is quite suggestive (although it is obvious):

Corollary 3.4.11 *If $(M, \mathcal{F})$ admits a transversal connection (e.g. if $\mathcal{F}$ is riemannian), then $k_{\mathcal{F}} : Inv(\mathfrak{gl}_q) \longrightarrow \check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ vanishes in degrees $> q$, and factors, via (3.4), through the basic cohomology $H^p_{bas}(M/\mathcal{F})$.*

$$\begin{tikzcd} Inv(gl_q) \arrow[r, "riemannian\ case"] \arrow[dr, "k_\nu"] \arrow[ddr, "k_\nu"'] & H^*_{bas}(M/\mathcal{F}) \arrow[d, "i"] \\ & H^*(M/\mathcal{F}) \arrow[d, "\pi^*"] \\ & H^*(M) \end{tikzcd}$$

This is actually true for the full characteristic map $k_{\mathcal{F}}$ defined on $H^*(\underline{WO}_q)$.

3.5 The Chern-Weil homomorphism for leaf spaces

In this section we construct the Chern-Weil homomorphism (3.9) (as well as its exotic version (3.10)) of any transversal principal G-bundle.

3.5.1 The Chern-Simons construction: The Chern-Simons construction for vector bundles (described in 3.4.1) extends to principal G-bundles over a manifold M, or, more generally, to commutative G-DG algebras (see 1.4.1). We can keep ourselves at the algebraic level since the forms involved on the products by the standard simplices $\mathbf{\Delta}^p$ have just polynomial coefficients, and the integration can be performed formally. More precisely, we consider the algebraic De Rham complex $\Omega(p)$ of $\mathbf{\Delta}^p := \{(t_0, \ldots, t_{p-1}) : t_i \geq 0, \sum t_i \leq 1\}$ (usual forms with polynomial coefficients). One has:

$$\Omega(1) : 0 \longrightarrow \mathbb{R}[t] \stackrel{d}{\longrightarrow} \mathbb{R}[t]dt \longrightarrow 0 \ ,$$

and $\Omega(p) = \Omega(1)^{\otimes p}$. Using the inclusions:

$$f_i : \mathbf{\Delta}^{p-1} \longrightarrow \mathbf{\Delta}^p, \ (t_0, \ldots, t_{p-2}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-2})$$

for $0 \leq i \leq p-1$, and:

$$f_p : \mathbf{\Delta}^{p-1} \longrightarrow \mathbf{\Delta}^p, \ (t_0, \ldots, t_{p-2}) \mapsto (t_0, \ldots, t_{p-2}, 1 - t_0 - \ldots - t_{p-2}),$$

we get boundary maps $\delta_i : \Omega(p) \longrightarrow \Omega(p-1)$ for $0 \leq i \leq p$. In terms of the coordinate functions $t_0, \ldots, t_{p-1}$ of $\Omega(p)$:

$$t_j \stackrel{\delta_i}{\mapsto} \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

for $0 \leq i \leq p-1$, while, for $i = p$:

$$t_j \stackrel{\delta_p}{\mapsto} \begin{cases} t_j & \text{if } j \leq p-1 \\ 1 - t_0 - \ldots - t_{p-1} & \text{if } j = p-1 \end{cases}$$

One can see $\Omega(*)$ as a simplicial object in the category of DG algebras. The integration over $\mathbf{\Delta}^p$ defines a map:

$$\int_{\mathbf{\Delta}^p} : \Omega(p) \longrightarrow \mathbb{R} \ ,$$

which is non-zero only on the degree p-part of $\Omega(p)$. Hence, viewing $\mathbb{R}$ as the complex with $\mathbb{R}$ concentrated in degree 0, $\int_{\mathbf{\Delta}^p}$ has degree $-p$. It is not a chain map; instead, one has the Stokes formula:

$$\int_{\mathbf{\Delta}^p} d(\omega) = \sum_{i=0}^{p} (-1)^{i+1} \int_{\mathbf{\Delta}^{p-1}} \delta_i(\omega).$$

We recall that we use the well known sign conventions (e.g., for tensor product of maps: $(f \otimes g)(a \otimes b) = (-1)^{deg(g) deg(a)} f(a) \otimes g(b)$, and for commutators: $[f, g] = fg - (-1)^{deg(f)} gf$). For any chain complex Ω^* we still denote by $\int_{\mathbf{\Delta}^p}$ the degree $-p$ map:

$$Id \otimes \int_{\mathbf{\Delta}^p} : \Omega \otimes \Omega(p) \longrightarrow \Omega \ ,$$

and by δ_i the chain map $Id \otimes \delta_i : \Omega(p) \otimes \Omega \longrightarrow \Omega(p-1) \otimes \Omega$. We then have Stokes formula:

$$[\int_{\mathbf{\Delta}^p}, d] \ = \ \sum_{i=0}^{p} (-1)^{i+1} \int_{\mathbf{\Delta}^{p-1}} \delta_i \ .$$

Let now Ω be a commutative G-DG algebra, and assume we are given connections ∇_i on Ω, $0 \leq i \leq p$, described by the connection 1-forms $\omega_i \in \Omega^1 \otimes \mathfrak{g}$. We form the new connection $t_0 \nabla_0 + \ldots + t_{p-1} \nabla_{p-1} + (1 - t_0 - \ldots - t_{p-1}) \nabla_p$ on $\Omega \otimes \Omega(p)$, described by the connection 1-form:

$$\omega_0 \otimes t_0 + \ldots + \omega_{p-1} \otimes t_{p-1} + \omega_p \otimes (1 - t_0 - \ldots - t_{p-1}) \in \Omega^1 \otimes \mathfrak{g} \otimes \mathbb{R}[t] \subset (\Omega \otimes \Omega(p))^1 \otimes \mathfrak{g}.$$

Combining the homomorphism induced by the affine combination of the connections ∇_i:

$$k(t_0 \nabla_0 + \ldots + t_{p-1} \nabla_{p-1} + (1 - t_0 - \ldots - t_{p-1}) \nabla_p) : W(\mathfrak{g}) \longrightarrow \Omega \otimes \Omega(p). \qquad (3.12)$$

with the integration map, we define the degree $-p$ map from $W(\mathfrak{g})$ into Ω:

$$k(\nabla_0, \ldots, \nabla_p) := (-1)^p \int_{\mathbf{\Delta}^p} k(t_0 \nabla_0 + \ldots + t_{p-1} \nabla_{p-1} + (1 - t_0 - \ldots - t_{p-1}) \nabla_p) \ .$$

Stokes' formula implies:

$$[k(\nabla_0, \ldots, \nabla_p), d] = \sum_{i=0}^{p} (-1)^i k(\nabla_0, \ldots, \widehat{\nabla_i}, \ldots, \nabla_p) \ . \qquad (3.13)$$

3.5.2 Chern-Weil for leaf spaces: Let now P be a transversal principal G-bundle for the foliation $(M, \mathcal{F})$, T any transversal, and $\mathcal{U}$ a basis of T. We consider the double complex $\check{C}(\mathcal{U}, \Omega^*(P_U))$, analogous to *Čech*-De Rham complex, with the only difference that we replace the DG algebras $\Omega^*(U)$ by the G-DG algebras $\Omega^*(P_U)$ (for $U \in \mathcal{U}$), and, for a holonomy embedding $\sigma : U \longrightarrow V$, we use the pull-back $\sigma^* : \Omega^*(P_V) \longrightarrow \Omega^*(P_U)$ along the induced map $P_U \longrightarrow P_V$, $p \mapsto \sigma(\pi(p)) \cdot p$. With the induced L_v's and i_v's, $\check{C}(\mathcal{U}, \Omega^*(P_U))$ becomes a G-DG algebra (see 1.4.1), which is non-commutative in general.

We choose a family $\nabla = \{\nabla(U)\}_{U \in \mathcal{U}}$ of local connections ($\nabla(U)$ of P_U), and, for $\sigma : U \longrightarrow V$ any holonomy embedding, denote by $\nabla(\sigma)$ the connection on P_U obtained from $\nabla(V)$ and the isomorphism $P_U \cong (P_V)|_{\sigma(U)}$ (the last isomorphism is the one induced by σ and the action of the holonomy on P). Define:

$$k_\nabla : W(\mathfrak{g}) \longrightarrow \check{C}_{\mathcal{U}}(M/\mathcal{F}, \Omega(P_U)) \ , \tag{3.14}$$

$$k_\nabla(w)(\sigma_1, \ \ldots, \sigma_p) = k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ \ldots, \nabla(\sigma_p ... \sigma_2\sigma_1))(w) \ .$$

This is not an algebra homomorphism, but it is compatible with the interior products and the Lie derivatives (the action of G). In particular, passing to basic elements (see 1.4.1), it induces:

$$k_\nabla : I(G) \longrightarrow \check{C}_{\mathcal{U}}(M/\mathcal{F}, \Omega^*) \ . \tag{3.15}$$

3.5.3 Theorem: *The previous (3.14) is a chain map, and the induced*

$$k_P : I(G) \longrightarrow \check{H}^*(M/\mathcal{F}) \tag{3.16}$$

is an algebra homomorphism which does not depend on the choice of the local connections $\nabla(U)$. Moreover, composed with $\pi^* : \check{H}^*(M/\mathcal{F}) \longrightarrow H^*(M)$ (see (3.5)), it gives the usual Chern-Weil homomorphism of P, viewed as a bundle on M (see 1.4.1).

proof: For $w \in W(\mathfrak{g})$ we compute the total boundary D of $k(\nabla_0, \ldots, \nabla_p)(w)$:

$$\begin{aligned} D(k_\nabla(w))(\sigma_1, \ \ldots, \sigma_p) \ &= \ \sigma_1^* k_\nabla(w)(\sigma_2, \ \ldots, \sigma_p) \\ &+ \ \sum_{i=1}^{p-1} (-1)^i k_\nabla(w)(\sigma_1, \ \ldots, \sigma_{i+1}\sigma_i, \ \ldots, \sigma_p) + \\ &+ \ (-1)^p k_\nabla(w)(\sigma_1, \ \ldots, \sigma_{p-1}) \\ &+ \ (-1)^p d(\ k_\nabla(w)(\sigma_1, \ \ldots, \sigma_p) \ ) \end{aligned}$$

By the definition of k_∇ this is:

$$\begin{aligned} &= \ \sigma_1^* \{ k(\nabla(U_1), \nabla(\sigma_2), \nabla(\sigma_3\sigma_2), \ \ldots, \nabla(\sigma_p ... \sigma_3\sigma_2))(w) \} + \\ &+ \ \sum_{i=1}^{p-1} (-1)^i k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ \ldots, \nabla(\widehat{\sigma_i \ldots \sigma_2\sigma_1}), \ \ldots, \nabla(\sigma_p ... \sigma_2\sigma_1))(w) \\ &+ \ (-1)^p k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ \ldots, \nabla(\sigma_{p-1} ... \sigma_2\sigma_1))(w) + \\ &+ \ (-1)^p d \{ k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ \ldots, \nabla(\sigma_p ... \sigma_2\sigma_1))(w) \ \} \ , \end{aligned}$$

(where "$\widehat{*}$" stands for omission) which, by the basic formula (3.13) equals:

$$k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ \ldots, \nabla(\sigma_p ... \sigma_2\sigma_1))(d(w)) = k_\nabla(d(w))(\sigma_1, \ \ldots, \sigma_p).$$

This proves that (3.14) is a chain map.

To prove that k_∇ is an algebra homomorphism at the level of cohomology, we show that it factorizes as:

$$\begin{tikzcd} W(\mathfrak{g}) \arrow[rr, "k_\nabla"] \arrow[dr, "k"'] & & \check{C}_{\mathcal{U}}(\mathcal{G}, \Omega(P_U)) \\ & \tilde{W}(\mathfrak{g}) \arrow[ur, "\tilde{k}_\nabla"'] & \end{tikzcd}$$

where $\tilde{W}(\mathfrak{g})$ is a (co)simplicial replacements of $W(\mathfrak{g})$ with the main property that any local connection $\nabla = \{\nabla(U)\}_{U\in\mathcal{U}}$ induces a natural map $\tilde{k}_\nabla$ of G-DG algebras, and one has a map k (depending just on G), which induces an algebra isomorphism at the level of basic cohomology.

To start with, we recall the definition of the simplicial Weil algebra $\tilde{W}(\mathfrak{g})$ of [64]. We have the (semi)simplicial Lie algebra $\{\mathfrak{g}^{\oplus(n+1)}, d_i\}$ ("semi" stands for the fact that we consider just face maps), whose boundaries are:

$$d_i : \mathfrak{g}^{\oplus(n+1)} \longrightarrow \mathfrak{g}^{\oplus n},\ (v_0, \ldots, v_n) \mapsto (v_0, \ldots, \widehat{v_i}, \ldots, v_n),$$

and then $\tilde{W}(\mathfrak{g})$ is the induced (semi)cosimplicial object of G-DG algebras:

$$W(\mathfrak{g}) \rightrightarrows W(\mathfrak{g}\oplus\mathfrak{g}) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} W(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}} \cdots$$

We denote by $\tilde{W}(\mathfrak{g})$ also the associated total complex, which has a natural structure of G-DG algebra, with the product coming from the cosimplicial structure and the product on each component. That means, as in the case of $\check{C}_{\mathcal{U}}(\mathcal{G}, \Omega)$, that it is given by $a \cdot b := \lambda_{p,p'}(a)\eta_{p,p'}(b)$, where:

$$\lambda_{p,p'} := \delta_{p+p'} \ldots \delta_{p+2}\delta_{p+1} : \tilde{W}(\mathfrak{g})(p) \longrightarrow \tilde{W}(\mathfrak{g})(p+p'),$$

$$\eta_{p,p'} := \delta_0 \ldots \delta_0 : \tilde{W}(\mathfrak{g})(p') \longrightarrow \tilde{W}(\mathfrak{g})(p+p')\ ,\ (p \text{ times})\ .$$

Remark that any string of arrows $(\sigma_1, , \ldots, \sigma_p)$ induces a connection on the G^{p+1}-DG algebra $\Omega^*(P_{U_0})$, as the sum of the G-connections $\nabla(U_0)$, $\nabla(\sigma_1)$, $\nabla(\sigma_2\sigma_1)$, ..., $\nabla(\sigma_p \ldots \sigma_2\sigma_1)$. Hence we have DG algebra maps $W(\mathfrak{g}^{\oplus(p+1)}) \longrightarrow \Omega^*(P_{U_0})$, associated to any string of p arrows; combined, they define a G-DG algebra map $\tilde{k}_\nabla : \tilde{W}(\mathfrak{g}) \longrightarrow \check{C}_{\mathcal{U}}(\mathcal{G}, \Omega(P_U))$. On the other hand, each $W(\mathfrak{g}^{\oplus(p+1)})$ is a G-DG algebra endowed with $p+1$ (tautological) connections ∇_i^t, $0 \le i \le p$ hence induced degree $-p$ maps:

$$k(\nabla_0^t, \ldots, \nabla_p^t) : W(\mathfrak{g}) \longrightarrow W(\mathfrak{g}^{\oplus(p+1)})\ ,$$

which combined define a chain map (compatible with the Lie derivatives and the interior products):

$$k : W(\mathfrak{g}) \longrightarrow \tilde{W}(\mathfrak{g})\ ,$$

$$k(w)_{p,n-p} := k(\nabla_0^t, \ldots, \nabla_p^t)(w) \in W(\mathfrak{g}^{\oplus(p+1)})^{n-p},\ \text{for } w \in W(\mathfrak{g})^n\ .$$

Remark the similarity between the definition of k_∇ and the one of k (actually one can do this type of construction for any semi-cosimplicial object $\{C(p)\}_{p\ge0}$ of G-DG algebras, endowed with certain connections; e.g. it would be enough to start with a connection on $C(0)$). It is not difficult to see that $k_{\nabla} = \tilde{k}_{\nabla} \circ k$, hence it suffices to show that k induces an algebra map:

$$k : I(G) \longrightarrow H^*(\tilde{W}(\mathfrak{g})).$$

One has an obvious projection $\pi : \tilde{W}(\mathfrak{g}) \longrightarrow W(\mathfrak{g})$ which is a left inverse of k, and which is a map of G- DG algebras. So it suffices to show that $k \circ \pi$ is homotopic to the identity, via a homotopy which restricts to basic elements, and this is one of the main properties of $\tilde{W}(\mathfrak{g})$ presented in [64].

To see that the definition of k_P does not depend on the choice of the local connection remark that, for any other $\nabla' = \{\nabla'(U)\}_{U \in \mathcal{U}}$, one has a homotopy between k_{∇} and $k_{\nabla'}$ given by:

$$h(w)(\sigma_1, \ldots, \sigma_p) =$$

$$= \sum_{i=0}^{p} (-1)^i k(\nabla(\sigma_i \ldots \sigma_2 \sigma_1), \ldots, \nabla(\sigma_p \ldots \sigma_2 \sigma_1), \nabla'(U_0), \nabla'(\sigma_1), \ldots, \nabla'(\sigma_i \ldots \sigma_2 \sigma_1)).$$

□

3.5.4 The exotic Chern-Weil homomorphism: The vanishing theorem holds in general for any transversal principal G-bundle: the chain map (3.14) vanishes on I^{q+1} (see the proof of the next theorem). Dividing out by I^{q+1}, and by K-basic elements we get (compare with the case of vector bundles) a chain map $k_{\nabla} : \underline{W}(\mathfrak{g}, K) \longrightarrow \check{C}_{\mathcal{U}}(M/\mathcal{F}, \Omega(P_U/K))$, and the exotic version of the Chern-Weil homomorphism:

$$k_P^{geom} : H^*(\underline{W}(\mathfrak{g}, K)) \longrightarrow \check{H}^*(M/\mathcal{F}) . \tag{3.17}$$

3.5.5 Theorem: *For any transversal principal G-bundle P, the associated Chern-Weil homomorphism (3.16) vanishes in degrees $> 2q$, and the associated exotic map (3.17) is an algebra homomorphism which does not depend on the choice of the local connections ∇_U. Moreover, composed with $\pi^* : \check{H}^*(M/\mathcal{F}) \longrightarrow H^*(M)$ (see (3.5)), it gives the exotic Chern-Weil homomorphism of the foliated bundle P (as defined in [64]).*

proof: To check the vanishing assertion, choose $w \in I^{q+1}$ of total degree k, and of polynomial degree $2d$ (hence $d > q$, and $k \geq 2d$) and look at the induced element:

$$\omega = k(\nabla(U_0), \nabla(\sigma_1), \nabla(\sigma_2\sigma_1), \ldots, \nabla(\sigma_p \ldots \sigma_2\sigma_1))(w) \in \Omega^{k-p}(P_{U_0}).$$

By the general properties of the Chern-Simons construction, it vanishes for $p > d$. On the other hand, $i_{v_1} \ldots i_{v_{k-2d+1}}(\omega) = 0$ for any vertical vector fields v_i (since $w \in S^{2d}(\mathfrak{g}^*) \otimes \Lambda^{k-2d}(\mathfrak{g}^*)$ has this property), and $i_{X_1} \ldots i_{X_{q+1}}(\omega) = 0$ for any horizontal vector fields X_i. It follows that $\omega = 0$ also when $(k - 2d) + q < deg(\omega) = k - p$, i.e. when $p < 2d - q$. Since $d < 2d - q$, it follows that for any p at least one of the inequalities $p < 2d - q$ or $p > d$ holds, hence we are done.

That k_P^{geom} does not depend on the choice of ∇ follows as in the proof of Theorem 3.5.3. The same applies to the proof that it is an algebra homomorphism, with the only mention that, as we replace $W(\mathfrak{g})$ by its truncation $\underline{W}(\mathfrak{g})$, we have to replace $W(\mathfrak{g}^{\oplus(p+1)})$ by its quotient $\underline{W}(\mathfrak{g}^{\oplus(p+1)})$ by $I(p)^{q+1}$ where $I(p)$ is the ideal generated by horizontal elements of positive degree (e.g. $I(0) = I$ previously used). □

3.6 *Čech*-DeRham theorem and Poincaré duality

The main result of this section is the following *Čech*-DeRham type theorem for leaf space:

Theorem 3.6.1 *For any foliation $(M, \mathcal{F})$ and any covering $\mathcal{U}$ which form a basis for the opens of a complete transversal M, the cohomology of the Čech-DeRham complex $\check{C}^*_{\mathcal{U}}(M/\mathcal{F}; \Omega^*)$ is isomorphic to $H^*(M/\mathcal{F})$.*

Using this (and its analogue with compact supports) we obtain a new proof of the Poincaré duality for leaf-spaces, proved in the previous chapter (Theorem 2.4.13). We emphasize that, since Theorem 3.6.1 can be viewed as a Mayer-Vietoris argument, this new proof of Poincaré duality is rather an extension of the usual proof [15] from manifolds to leaf-spaces (or étale groupoids).

Theorem 3.6.2 *(Poincaré duality) For any codimension q foliation $(M, \mathcal{F})$:*

$$H^*(M/\mathcal{F}; \mathcal{O}) \cong H_c^{q-*}(M/\mathcal{F}; \mathbb{R})^{\vee} \ ,$$

where $\mathcal{O}$ is the transversal orientation sheaf.

It is not difficult to see that all the constructions and the results of this chapter can be done in full generality, for any étale groupoid. Actually, to prove the last two theorems (and stronger versions with coefficients), it is more natural to work in this generality. Let me mention that this also provides a geometric model for $H^*(B\Gamma^q)$, which avoids the non-Hausdorffness of Γ^q (compare to De Rham model constructed in [55]).

3.6.3 *Čech* **cohomology for étale groupoids:** Let $\mathcal{G}$ be an étale groupoid, and $\mathcal{U}$ a basis for the opens in $\mathcal{G}^{(0)}$. For any abelian $\mathcal{G}$-sheaf $\mathcal{A}$, we write $\Gamma(-; \mathcal{A})$ for the contravariant functor from the embedding category $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ (see 3.2.4) to abelian groups defined on objects by $U \mapsto \Gamma(U; \mathcal{A})$. For $\sigma : U \longrightarrow V$ the corresponding map $\sigma^* : \Gamma(V; \mathcal{A}) \longrightarrow \Gamma(U; \mathcal{A})$ is given by:

$$\sigma^*(\alpha)(x) = \alpha(t\sigma(x)) \cdot \sigma(x) \quad (x \in U) \ .$$

Replacing the holonomy embeddings by $\mathcal{G}$-embeddings, there is an obvious *Čech* complex $\check{C}_{\mathcal{U}}(\mathcal{G}, \mathcal{A})$ defined in analogy with (3.2), for any $\mathcal{A}$. Of course, one can define $\check{C}^*_{\mathcal{U}}(\mathcal{G}, \mathcal{M})$ similarly, for any contravariant functor $\mathcal{M}$ from $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ to abelian groups; remark that it is the standard complex computing $H^*(\mathcal{O}_{\mathcal{U}}(\mathcal{G}); \mathcal{M})$, the cohomology of the discrete category $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ with coefficients in $\mathcal{M}$.

When $\mathcal{A}$ is locally acyclic (i.e. $A\mid_U \in Ab(U)$ is acyclic for any $U \in \mathcal{U}$), we define $\check{H}^*_{\mathcal{U}}(\mathcal{G}; \mathcal{A})$ as the cohomology of $\check{C}^*_{\mathcal{U}}(\mathcal{G}, \mathcal{A})$; in general, we choose a resolution $\mathcal{A} \longrightarrow \mathcal{I}^*$ by locally acyclic sheaves (e.g. use one of the standard resolutions), and define $\check{H}^*_{\mathcal{U}}(\mathcal{G}; \mathcal{A})$ as the cohomology of the double complex $\check{C}^*_{\mathcal{U}}(\mathcal{G}, \mathcal{I}^*)$. For instance, when working over $\mathbb{C}$ we can use the standard resolution by forms, and we get a $\check{C}ech - DeRham$ complex computing $\check{H}^*_{\mathcal{U}}(\mathcal{G})$ (when we omit the coefficients, we simply mean $\mathbb{C}$). In particular, one clearly has:

$$\check{H}^*_{\mathcal{U}}(M/\mathcal{F}) \cong \check{H}^*_{\mathcal{U}}(Hol_T(M, \mathcal{F})) \ .$$

We also note that, by standard arguments, the definition does not depend on the choice of the resolution. From the spectral sequence associated to the double complex $\check{C}^*_{\mathcal{U}}(\mathcal{G}, \mathcal{I}^*)$ we get:

Corollary 3.6.4 *For any $\mathcal{G}$-sheaf $\mathcal{A}$ there is a spectral sequence:*

$$E_2^{s,t} = H^s(\mathcal{O}_{\mathcal{U}}(\mathcal{G}); H^t(-; \mathcal{A})) \Longrightarrow \check{H}^{s+t}_{\mathcal{U}}(\mathcal{G}; \mathcal{A}) \ .$$

Similarly, using the covariant functor $\Gamma_c(-; \mathcal{A}) : U \mapsto \Gamma_c(U; \mathcal{A})$ on $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$, we consider the standard complex computing the homology of $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ with coefficients:

$$\check{C}^{\mathcal{U}}_*(\mathcal{G}, \mathcal{A}) : \ \ldots \xrightarrow{\delta} \bigoplus_{U_0 \xrightarrow{\sigma_1} U_1 \xrightarrow{\sigma_2} U_2} \Gamma_c(U_0; \mathcal{A}) \xrightarrow{\delta} \bigoplus_{U_0 \xrightarrow{\sigma_1} U_1} \Gamma_c(U_0; \mathcal{A}) \xrightarrow{\delta} \bigoplus_{U_0} \Gamma_c(U_0; \mathcal{A}) \ .$$

$$\begin{aligned} \delta(a, \sigma_1, \ \ldots, \sigma_p) &= ((\sigma_1)_* a, \sigma_2, \ \ldots, \sigma_p) + \\ &+ \textstyle\sum_{i=1}^{p-1} (-1)^i (a, \sigma_1, \ \ldots, \sigma_{i+1}\sigma_i, \ \ldots, \sigma_p). \end{aligned}$$

We can view it also as a cochain complex $\check{C}^*_{c,\mathcal{U}}(\mathcal{G}, \mathcal{A})$, whose component in degree $-n$ is the product over strings of n arrows. If $\mathcal{A}$ is c-soft, define $\check{H}^*_{c,\mathcal{U}}(\mathcal{G}; \mathcal{A})$ as the cohomology of this cochain complex; in general, we use a resolution again.

Since we can use bounded resolutions we have an analog of Corollary 3.6.4:

Corollary 3.6.5 *For any $\mathcal{G}$- sheaf $\mathcal{A}$ there is a spectral sequence:*

$$E_2^{s,t} = H_{-s}(\mathcal{O}_{\mathcal{U}}(\mathcal{G}); H^t_c(-; \mathcal{A})) \Longrightarrow \check{H}^{s+t}_{c,\mathcal{U}}(\mathcal{G}; \mathcal{A}) \ .$$

The *Čech* versions are isomorphic to the usual cohomology (see 1.2.5) and compactly supported cohomology (see 2.2.7) of the groupoid:

Theorem 3.6.6 *For any $\mathcal{G}$-sheaf $\mathcal{A}$ and any basis $\mathcal{U}$, one has:*

$$\begin{aligned} \check{H}^*_{\mathcal{U}}(\mathcal{G}; \mathcal{A}) &\cong H^*(\mathcal{G}; \mathcal{A}) \ , \\ \check{H}^*_{c,\mathcal{U}}(\mathcal{G}; \mathcal{A}) &\cong H^*_c(\mathcal{G}; \mathcal{A}) \ . \end{aligned}$$

Before proving this theorem, observe that together with Corollary 3.6.5 we get a new proof of Poincaré duality for étale groupoids (Theorem 2.4.13 in Chapter 2).

Corollary 3.6.7 (Poincaré duality) *For any (smooth) étale groupoid $\mathcal{G}$:*

$$H^*(\mathcal{G}; \mathcal{O}) \cong H^{*-q}_c(\mathcal{G}; \mathbb{R})^\vee \ .$$

proof: We choose an basis $\mathcal{U}$ of contractible opens. Using Corollary 3.6.5 for $\mathcal{A} = \mathbb{R}$, and the fact that we are working over $\mathbb{R}$ we have:

$$(H_{q-*}(\mathcal{G}; \mathbb{R}))^\vee \cong H_*(\mathcal{O}_{\mathcal{U}}(\mathcal{G}); H^q_c(-; \mathbb{R}))^\vee \cong H^*(\mathcal{O}_{\mathcal{U}}(\mathcal{G}); H^q_c(-; \mathbb{R})^\vee),$$

Since the orientation sheaf $\mathcal{O}$ is locally constant, with $\mathcal{O}(U) = H^q_c(-; \mathbb{R})^\vee$, it suffices to use Theorem 3.6.6. $\square$

Proof of Theorem 3.6.6: We prove the second isomorphism (the other one is similar). Since both groups are defined via resolutions, it suffices to show that, if $\mathcal{A}$ is c-soft, then there is a functorial (on $\mathcal{A}$) quasi-isomorphism between the two complexes $\check{C}^{\mathcal{U}}_*(\mathcal{G},\mathcal{A})$ and $B_*(\mathcal{G};\mathcal{A})$ (see 2.2.1) computing the cohomologies in the statement.

We first introduce the auxiliary bisimplicial space S. For $s,t \geq 0$, let $S_{s,t}$ be the space of strings:

$$U_0 \xleftarrow{\sigma_1} \ \ldots \ \xleftarrow{\sigma_s} U_s \xleftarrow{g} x_0 \xleftarrow{h_1} \ \ldots \ \xleftarrow{h_t} x_t \ , \tag{3.18}$$

where $\sigma_1, \ldots, \sigma_s$ are arrows in $\underline{Emb}(\mathcal{G})$, while $g, h_1, \ldots, h_t$ are arrows in $\mathcal{G}$ and "$U_s \xrightarrow{g} x_0$" indicates that $t(g) \in U_s$. The topology of $S_{s,t}$ is that of the disjoint sum (over all strings $U_0 \longleftarrow \ldots \longleftarrow U_s$) of fibered products $U_s \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(t+1)}$. The bisimplicial structure of S is defined in terms of the nerves of $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ (for s) and of $\mathcal{G}$ (for t).

The $\mathcal{G}$-sheaf $\mathcal{A}$ induces for each s and t a sheaf $\mathcal{A}_{s,t}$ on $S_{s,t}$ by pullback along the map $S_{s,t} \longrightarrow \mathcal{G}^{(0)}$ sending the string (3.18) to x_0. Write:

$$\mathcal{C}_{s,t}(\mathcal{A}) = \Gamma_c(S_{s,t};\mathcal{A}_{s,t}) \ . \tag{3.19}$$

Then $\mathcal{C}(\mathcal{A})$ is a bisimplicial abelian group. (Its bisimplicial structure is the one implicit in the formulas (3.20), (3.21) below). We also write $\mathcal{C}(\mathcal{A})$ for the associated double chain complex. The conclusion follows now in a relatively straightforward way, by observing that $\mathcal{C}_{s,t}(\mathcal{A})$ is acyclic in each of the s- and q- direction separately, as we now explain.

First, for fixed s, we can conveniently rewrite (3.19) as:

$$\mathcal{C}_{s,t}(\mathcal{A}) = \bigoplus_{U_0 \longleftarrow \ldots \longleftarrow U_s} \Gamma_c((\mathcal{G}/U_s)^{(t)};\pi_s^*(\mathcal{A})) \tag{3.20}$$

Here $\mathcal{G}/U_s$ is the comma category of arrows $x \xrightarrow{g} y \in U_s$ (see 2.3.1), while $\pi_s : \mathcal{G}/U_s \longrightarrow \mathcal{G}$ is the obvious projection functor. Observe that $\mathcal{G}/U_s$ is Morita equivalent to the space U_s. For a fixed $U_0 \longleftarrow \ldots \longleftarrow U_s$, the complex $\Gamma_c((\mathcal{G}/U_s)^{(t)};\pi_s^*(\mathcal{A}))$ (along t) computes the homology $H_t(\mathcal{G}/U_s;\pi_s^*\mathcal{A})$. So by Morita invariance (2.3.6) and the fact that $\mathcal{A}$ is c-soft,

$$H_t(\mathcal{G}/U_s;\pi_s^*\mathcal{A}) = H_t(U_s;\mathcal{A}|_{U_s}) = \begin{cases} \Gamma_c(U_s;\mathcal{A}) & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Furthermore, $H_0(\mathcal{C}_{s,*}(\mathcal{A}))$ is precisely the complex:

$$\bigoplus_{U_0 \longleftarrow \ldots \longleftarrow U_s} \Gamma_c(U_s;\mathcal{A})$$

computing the homology of the category $\mathcal{O}_{\mathcal{U}}(\mathcal{G})$, i.e.:

$$H_s H_0(\mathcal{C}(\mathcal{A})) = H_s(\mathcal{O}_{\mathcal{U}}(\mathcal{G});\ \Gamma_c(\mathcal{A})) \ . \tag{3.21}$$

Now let us fix t. Writing $\theta_{s,t} : S_{s,t} \longrightarrow \mathcal{G}^{(t)}$ for the projection, we can rewrite (3.20) as

$$\mathcal{C}_{s,t}(\mathcal{A}) = \Gamma_c(\mathcal{G}^{(s)};(\theta_{s,t})_!(\mathcal{A}_{s,t})) \ . \tag{3.22}$$

At a point $(x_0 \longrightarrow \ . \ . \ . \ \longrightarrow x_t)$ in $\mathcal{G}^{(t)}$, the stalk of $(\theta_{s,t})_!(\mathcal{A}_{s,t})$ is:

$$\bigoplus_{U_0 \longleftarrow . \ . \ . \ \longleftarrow U_s} \quad \bigoplus_{g : x_0 \longrightarrow y \in U_s} \mathcal{A}_{x_0} \tag{3.23}$$

We claim that this is an acyclic complex in s. To see this, write $x_0/\mathcal{O}_{\mathcal{U}}(\mathcal{G})$ for the category whose objects are pairs $(U, x_0 \overset{g}{\longleftarrow} y \in U_s)$ and whose arrows $(U,g) \overset{\sigma}{\longrightarrow} (V,h)$ are $\mathcal{G}$-embeddings $\sigma : U \longrightarrow V$ such that $\sigma(tg)\cdot g = h$. Since $\mathcal{G}$ is an étale groupoid, we have:

$$x_0/\mathcal{O}_{\mathcal{U}}(\mathcal{G}) = \lim_{x_0 \in W} W/\mathcal{O}_{\mathcal{U}}(\mathcal{G}) \tag{3.24}$$

is a co-limit of comma groupoids. But comma categories have contractible classifying spaces, hence zero homology with constant coefficients. Thus the complex (3.23) is acyclic, because it computes $\lim_W(W/\mathcal{O}_{\mathcal{U}}(\mathcal{G}); \mathcal{A}_{x_0})$.

This proves that for a fixed t, the chain complex $(\theta_{s,t})_!(\mathcal{A}_{s,t})$ is an acyclic complex of c-soft sheaves on $\mathcal{G}_s$. By (3.22) and 2.5.7, it follows that, for fixed t, the complex $\mathcal{C}_{s,t}(\mathcal{A})$ is exact in the s- variable, and:

$$H_s(\mathcal{C}_{*,t}(\mathcal{A})) = \begin{cases} \Gamma_c(\mathcal{G}^{(t)}; \mathcal{A}) & \text{if } s = 0 \\ 0 & \text{if } s \neq 0 \end{cases} .$$

Then $H_t H_0(\mathcal{C}_{*,t}(\mathcal{A})) = H_t(\mathcal{G}; \mathcal{A})$, and, comparing this last identity to (3.21), we have constructed quasi-isomorphisms

$$B_*(\mathcal{G}; \mathcal{A}) \ \overset{\sim}{\longleftarrow} \ \mathcal{C}_{*,*}(\mathcal{A}) \ \overset{\sim}{\longrightarrow} \ \check{C}^{\mathcal{U}}_*(\mathcal{G}, \mathcal{A}) \tag{3.25}$$

which are functorial with respect to $\mathcal{A}$. □

3.6.8 Characteristic maps for étale groupoids: As in Section 3.3, we have the *Čech − DeRham* (double) complex $\check{C}^*_{\mathcal{U}}(\mathcal{G}, \Omega^*)$ and all the constructions we have described for $M/\mathcal{F}$ work for any étale groupoid $\mathcal{G}$. For instance, we get a Chern-Weil homomorphism:

$$k_P^{geom} : I(G) \longrightarrow H^*(\mathcal{G}) \tag{3.26}$$

for any principal bundle $P \in Bun_G(\mathcal{G})$, the vanishing theorem still holds, and we have an induced exotic Chern-Weil homomorphism $k_P^{geom} : H^*(\underline{W}(\mathfrak{g}, K)) \longrightarrow H^*(\mathcal{G})$. In particular, for any vector bundle $E \in Vect(\mathcal{G})$ the induced characteristic map k_E : $\mathbb{R}[p_1, p_2, \ldots, p_{[\frac{q}{4}]}] \longrightarrow H^*(\mathcal{G})$ vanishes in degrees $> 2q$ (q is the dimension of $\mathcal{G}^{(0)}$), and has an induced exotic version:

$$k_E^{geom} : H^*(\underline{WO}_q) \longrightarrow H^*(\mathcal{G}) \ .$$

When E is the tangent bundle, and $\mathcal{G}$ is the reduced holonomy groupoid, one recovers (1.20).

Chapter 4

Cyclic cohomology of étale and holonomy groupoids

4.1 Introduction

In the general picture of non-commutative geometry, cyclic homology plays the role of compactly supported de Rham cohomology and is the target of the Chern character. The dual theory is cyclic cohomology, which plays the role of closed de Rham homology. The pairing between these two is an important tool in performing numerical computations of K-theory classes (indices).

Often the non-commutative space we have to deal with is an orbit space of an étale groupoid; in particular, any étale groupoid can be viewed as such a non-commutative space. This fits in with Grothendieck's idea of what a "generalized space" is [2, 72], and includes examples like leaf spaces of foliations, orbit spaces of group actions on manifolds, orbifolds. To say what the groups $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))$ look like is an important step in solving index problems [8, 29, 31, 85] and in understanding the connection between the topology and the analysis of "leaf spaces" (here we have in mind in particular the Baum-Connes assembly map [7]).

The computation of $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))$ was started by Connes for the case where $\mathcal{G} = M$ is a manifold [28], Burghelea and Karoubi for the case where $\mathcal{G} = G$ is a group [22, 65] and by Feigin, Tsygan, and Nistor for crossed products by groups [45], [83] (see the preliminary section 1.5). The general strategy is to decompose these homology groups as direct sums of localized homologies; there are two different kinds of components, which behave differently. Following the terminology introduced in [22], these are called elliptic and hyperbolic components. Usually the hyperbolic ones are more difficult to compute and involve in a deeper way the combinatorics of the groupoid.

In the general setting of smooth étale groupoids the results were partially extended by Brylinski and Nistor [20]: for a *Hausdorff* étale groupoid $\mathcal{G}$, the localized homologies $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ are defined for any invariant closed-open set $\mathcal{O}$ of loops; the *elliptic* components are computed in terms of double complexes; in particular the localization at units is related to the homology of the classifying space $B\mathcal{G}$. In the case of holonomy groupoids of foliations $(M, \mathcal{F})$, one first reduces to the étale setting by restricting to complete transversals; it is expected that $HC_*(\mathcal{C}_c^\infty(Hol(M, \mathcal{F})))$ is isomorphic to the cyclic homology of the reduced étale groupoid $Hol_T(M, \mathcal{F})$. There are some important questions left:

1) Compute the hyperbolic components;

2) Remove the Hausdorffness condition (simple examples coming from foliations are non-Hausdorff);

3) Find a way to book-keep the computations; in particular give a more conceptual proof and a more conceptual meaning of the results.

4) In the case of foliations, prove that by reducing the étale setting one does not loose information on the cyclic homology of the full holonomy groupoid.

In this chapter we answer all the questions above, among some others. The main tool we use is the homology theory for étale groupoids introduced in Chapter 2; in particular the results are stated in terms of this homology (we refer to 2.3.10 of Chapter 2 to an overview of the main results). This leads to various models (DeRham, Alexander-Spanier, Čech, etc) for representing cyclic cocycles. As immediate consequences we derive the Morita invariance, the Poincaré duality and the functoriality of the cyclic homology of étale groupoids.

The approach and the results of this chapter owe a great deal to the previous work of several authors, especially Brylinski [20], Burghelea [22], and Nistor [20, 83]. The computation we give for the localization at units is, beyond the formalism, the same as the one given in [20]; to the same paper we owe the important idea of reduction to loops (proposition 4.6.1). The method for computing the other localizations are inspired by the initial work of Burghelea [22]; the difficulty is that the topological arguments (at the level of classifying spaces) used in that paper do not work in this generality any more. An older idea [72] that working with classifying toposes (i.e. sheaves) might be easier than working with classifying spaces (and this was pointed out, for the first time in our context, in the same paper [20]) becomes essential for us. With this in mind, our job is to replace the classifying spaces used by Burghelea by suitable étale categories (for instance, in the case of groups, the diagram in the proof of Prop. 1.8 in [22], is obtained from the diagram in the proof of our Lemma 4.4.4 taking the classifying spaces), and the topological arguments by a suitable algebraic-topological formalism (long exact sequences and spectral sequences for homology of étale groupoids).

It is important to point out that our definition of compactly supported forms on non-Hausdorff manifolds (see Chapter 2, the Appendix) is related to, but not the same as the one given by Connes (section 6 in [25]). Ours has basic properties, like the existence of a de Rham differential, which are not shared by Connes' (as remarked in the introduction of [20]); it is also the right object for extending Poincaré-duality to non-Hausdorff manifolds (Theorem 2.4.13 of Chapter 2). For this reason we expect it to be useful also in other problems which deal with foliations with non-Hausdorff graph.

We conclude this introduction with an outline.

In section 4.2 we introduce the homology of groupoids with coefficients in cyclic sheaves; more generally, given a cyclic groupoid (i.e. a groupoid with an action of $\mathbb{Z}$ on it, see 4.3.1) we consider twisted cyclic sheaves (for which the usual identity $t^{(n+1)} = 1$ is replaced by $t^{(n+1)}$ =the action of the generator $1 \in \mathbb{Z}$, see 4.4.1). In section 4.3 and 4.4 we prove the main technical results concerning these homologies like the Gysin sequence (Prop. 4.3.4), the Feigin-Nistor-Tsygan spectral sequences (Prop. 4.4.7 and 4.4.8); at the end we derive as a simple consequence the Eilenberg-Zilber-type spectral sequence for cyclic objects which is one of the main results in [47]. The older approaches to cyclic homology of crossed products by (discrete) groups can not be directly extended to the setting of étale groupoids; section 4.5 uses the cyclic groupoids to overcome this problem. In particular we are able to extend the Feigin-Tsygan-Nistor spectral sequences [45, 83], and Nistor's description of the S boundary [83]. See Theorem 4.5.11 and Theorem 4.5.12 .

In section 4.6-4.10 we come down to earth with more concrete applications; here is a list of them:

4) For smooth étale groupoids we extend the old results of Burghelea proving that the elliptic components $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ are computed by the homology of the normalizer $\mathcal{N}_{\mathcal{O}}$ of $\mathcal{O}$ (see Theorem 4.6.3 and Theorem 4.6.4);

5) For hyperbolic components $HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$, we describe a $H^*(\mathcal{N}_{\mathcal{O}})$-module structure which identifies S in the SBI-sequence with the product by an element $e_{\mathcal{O}} \in H^2(\mathcal{N}_{\mathcal{O}})$; in particular we get a vanishing condition for $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ (which extends a similar result of Burghelea [22] and Nistor [83]). For stable $\mathcal{O}$'s we also give a more concrete description of $HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$. See Theorem 4.6.5;

6) In section4.7 we show how the methods apply to *co*homology. In particular we get that the pairing between HP_* and HP^* is a Poincaré-duality pairing, so it is highly non-trivial. See 4.7.4, 4.7.5 .

7) For group-actions on manifolds we get the old results for the elliptic components [8, 20], and a new description of the hyperbolic ones (see Corollary 4.10.5). Our computations give (as expected), and motivate the Chern character for discrete group actions, defined by Baum-Connes in [8]. Moreover, we extend these to action of Lie groups with discrete stabilizers (see 4.10.7).

8) For foliations we prove that the cyclic homology is a well defined invariant of the leaf space of the foliation, in the sense that the process of reducing to the setting of smooth étale groupoids does not depend on the choice of the complete transversal (see Theorem 4.8.1). Moreover, we prove that the resulting groups are also isomorphic to the cyclic homology of the full holonomy groupoid (see Theorem 4.8.2). At units we rediscover the (topological) exotic characteristic classes for foliations (see 4.8.3, 4.10.1).

9) An instance of the Chern character in cyclic homology is the basic Chern character. We describe it directly (see 4.9.6, 4.9.8). In the case of orbifolds, our computations of the cyclic homology (see 4.10.9, 4.10.10), and our description of the Chern character (see 4.10.11) makes the connection between the non-commutative approach to orbifolds, and the Chern character defined by Kawasaki (in a direct way) in his work on index theorems for orbifolds [69].

4.2 Cyclic $\mathcal{G}$ -sheaves

4.2.1 Cyclic sheaves: Let $\mathcal{G}$ be an étale groupoid. We work in the abelian category of sheaves of (real or complex) vector spaces of $\mathcal{G}$ (see 1.2.1); by a *cyclic $\mathcal{G}$-sheaf* we mean a cyclic object (see 1.5.5) $\mathcal{A}_*$ in $Sh(\mathcal{G})$. The associated homologies, as defined in 1.5.5, are denoted by:

$$\widetilde{HH}_*(\mathcal{A}_*), \widetilde{HC}_*(\mathcal{A}_*), \widetilde{HP}_*(\mathcal{A}_*) \in Sh(\mathcal{G}) \ .$$

4.2.2 Examples:

1. As an extension of 1.5.6, any sheaf of complex algebras $\mathcal{A} \in Sh(\mathcal{G})$ defines a cyclic $\mathcal{G}$-sheaf $\mathcal{A}^{\natural}$ having the stalk at $c \in \mathcal{G}^{(0)}$: $(\mathcal{A}^{\natural})_c = (\mathcal{A}_c)^{\natural}$. If $\mathcal{G}$ is smooth and $\mathcal{A} = \mathcal{C}^{\infty}_{\mathcal{G}^{(0)}}$, then we reserve the notation $\mathcal{A}^{\natural}$ for the cyclic $\mathcal{G}$-sheaf defined by taking into account the topology (see 1.5.11).

2. The standard resolution of $\mathbb{C}$ on $Sh(\mathcal{G})$ [35, 58, 72] (see 1.2.2.3):

$$\ldots \longrightarrow \mathbb{C}[\mathcal{G}^{(3)}] \longrightarrow \mathbb{C}[\mathcal{G}^{(2)}] \longrightarrow \mathbb{C}[\mathcal{G}^{(1)}] \; (\longrightarrow \mathbb{C} \longrightarrow 0),$$

comes from the simplicial structure on the nerve of $\mathcal{G}$:

$$d_i(g_0, \ldots, g_n) = \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 \leq i \leq n-1 \\ (g_0, \ldots, g_{n-1}) & \text{if } i = n \end{cases},$$

and s_i's inserts units (as in 2.2.1). It inherits a structure of cyclic $\mathcal{G}$-sheaf by (compare to 7.4.5. in [71]):

$$t(g_0, \ldots, g_n) = (g_0 g_1 \ldots g_n, (g_1 \ldots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

4.2.3 Definition: *If $\mathcal{A}_*$ is a cyclic $\mathcal{G}$-sheaf, define its Hochschild and cyclic hyperhomology by $HH_*(\mathcal{G}; \mathcal{A}_*) = \mathbb{H}_*(\mathcal{G}; (\mathcal{A}_*, b))$, $HC_*(\mathcal{G}; \mathcal{A}_*) = \mathbb{H}_*(\mathcal{G}; (\mathcal{A}_*, B, b))$ (compare to [65, 66]). If $\mathcal{A}_n$ is c-soft for all n, define $HP_*(\mathcal{G}; \mathcal{A}_*) = \mathbb{H}_*(\mathcal{G}; \lim_{\leftarrow}(\mathcal{A}_*, B, b)[-2r])$ (in the general case one can define HP_* using resolutions; see [98]).*

This is an extension of the definition given by Karoubi for groups (section II in [65]).

4.2.4 SBI-sequences: From the general considerations in 1.5.4, there is a long exact sequence in $Sh(\mathcal{G})$:

$$\ldots \xrightarrow{B} \widetilde{HH}_n(\mathcal{A}_*) \xrightarrow{I} \widetilde{HC}_n(\mathcal{A}_*) \xrightarrow{S} \widetilde{HC}_{n-2}(\mathcal{A}_*) \xrightarrow{B} \widetilde{HH}_{n-1}(\mathcal{A}_*) \xrightarrow{I} \ldots,$$

and, using 2.2.6, a long exact sequence of vector spaces:

$$\ldots \xrightarrow{B} HH_n(\mathcal{G}; \mathcal{A}_*) \xrightarrow{I} HC_n(\mathcal{G}; \mathcal{A}_*) \xrightarrow{S} HC_{n-2}(\mathcal{G}; \mathcal{A}_*) \xrightarrow{B} HH_{n-1}(\mathcal{G}; \mathcal{A}_*) \xrightarrow{I} \ldots.$$

4.2.5 First spectral sequences: Using the first spectral sequence of 2.2.11 we get two spectral sequences with E^2-terms:

$$H_p(\mathcal{G}; \widetilde{HH}_q(\mathcal{A}_*)) \Longrightarrow HH_{p+q}(\mathcal{G}; \mathcal{A}_*) \text{ and } H_p(\mathcal{G}; \widetilde{HC}_q(\mathcal{A}_*)) \Longrightarrow HC_{p+q}(\mathcal{G}; \mathcal{A}_*).$$

4.2.6 Second spectral sequences: Using the second spectral sequence of 2.2.11 we get two spectral sequences with E^2-terms::

$$HH_p(H_q(\mathcal{G}; \mathcal{A}_*)) \Longrightarrow HH_{p+q}(\mathcal{G}; \mathcal{A}_*) \text{ and } HC_p(H_q(\mathcal{G}; \mathcal{A}_*)) \Longrightarrow HC_{p+q}(\mathcal{G}; \mathcal{A}_*).$$

4.2.7 Lemma: *If the morphism $f : \mathcal{A}_* \longrightarrow \tilde{A}_*$ of cyclic $\mathcal{G}$-sheaves induces a quasi-isomorphism $f : (\mathcal{A}_*, b) \longrightarrow (\tilde{A}_*, \tilde{b})$ of complexes of sheaves, then it induces isomorphisms:*

$$HH_*(\mathcal{G}; \mathcal{A}_*) \simeq HH_*(\mathcal{G}; \tilde{A}_*), \; HC_*(\mathcal{G}; \mathcal{A}_*) \simeq HC_*(\mathcal{G}; \tilde{A}_*).$$

proof: this is a consequence of 2.2.2, 4.2.4, 4.2.5 and comparison-theorem for spectral sequences (compare to 2.5.2 in [98]). □

4.2.8 Assume that $\mathcal{G}$ is an étale groupoid and $\mathcal{A}_*$ is a cyclic $\mathcal{G}$-sheaf such that any $\mathcal{A}_n$ is c-soft. From the definition we see that $HH_*(\mathcal{G};\mathcal{A}_*)$ is computed by the bi-simplicial vector space $B_*(\mathcal{G};\mathcal{A}_*)$; so, from the Eilenberg-Zilber theorem [97], it is computed by its diagonal, i.e. by the simplicial vector space:

$$\mathcal{C}(\mathcal{G};\mathcal{A}_*):\ \ldots \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \Gamma_c(\mathcal{G}^{(2)};\mathcal{A}_2)\ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \Gamma_c(\mathcal{G}^{(1)};\mathcal{A}_1)\ \substack{\longrightarrow\\ \longrightarrow}\ \Gamma_c(\mathcal{G}^{(0)};\mathcal{A}_0)\,,$$

$$d_i(a \mid g_1,\ldots,g_n) = \begin{cases} (d_0(a)g_1 \mid g_2,\ldots,g_n) & \text{if } i=0\\ (d_i(a) \mid g_1,\ldots,g_ig_{i+1},\ldots,g_n) & \text{if } 1\le i\le n-1\\ (d_n(a) \mid g_1,\ldots,g_{n-1}) & \text{if } i=n \end{cases}\ ,$$

$$s_i(a \mid g_1,\ldots,g_n) = (s_i(a) \mid \ldots,g_i,1,g_{i+1},\ldots)\ .$$

Combining the cyclic structure of $\mathcal{A}_*$ with the one on the nerve of $\mathcal{G}$ (see [22] , pag. 358), we define the following cyclic structure on $\mathcal{C}(\mathcal{G};\mathcal{A}_*)$:

$$t(a \mid g_1,\ldots,g_n) = (t(a)g_1\ldots g_n|(g_1\ldots g_n)^{-1},g_1,\ldots,g_{n-1}).$$

The following is a particular case of 4.4.6 in the next sub-section:

4.2.9 Lemma: *If $\mathcal{A}_*$ is a cyclic $\mathcal{G}$-sheaf such that any $\mathcal{A}_n$ is c-soft, then $HH_*(\mathcal{G};\mathcal{A}_*)$, $HC_*(\mathcal{G};\mathcal{A}_*)$, $HP_*(\mathcal{G};\mathcal{A}_*)$ are computed by the cyclic vector space $\mathcal{C}(\mathcal{G};\mathcal{A}_*)$.*

4.3 Cyclic Groupoids ; Gysin sequences

In this subsection we extend Burghelea's definition of cyclic groupoids to this topological setting, give some important examples, and describe the Gysin sequence, which will be an important tool in our computations of cyclic homology (due to the fact that the SBI sequence is, after all, a Gysin sequence). We also extend the definitions and the results of the previous subsection.

4.3.1 Cyclic categories: We call *cyclic category* an étale category $\mathcal{G}$ endowed with an action of the cyclic group $\mathbb{Z}$; by this we mean there is given a continuous map $\theta:\mathcal{G}^{(0)}\longrightarrow\mathcal{G}^{(1)},\ c\mapsto\theta_c$ such that:

1. $\theta_c\in Aut(c)$, for all $c\in\mathcal{G}^{(0)}$;
2. $g\theta_c=\theta_d g$, for all $g:c\to d$ in $\mathcal{G}$.

A morphism between two cyclic categories $(\mathcal{G},\theta),(\mathcal{H},\tau)$ is a continuous functor $f:\mathcal{G}\longrightarrow\mathcal{H}$ such that $f(\theta_c)=\tau_{f(c)}\quad\forall\ c\in\mathcal{G}^{(0)}$.

For discrete groupoids this agrees with the old definition given by Burghelea ([22], page 358). In general, there is an action of $\mathbb{Z}$ on the space of arrows: the generator acts as $\mathcal{G}^{(1)}\longrightarrow\mathcal{G}^{(1)}$, $g\mapsto\theta_{t(g)}g$. The localization of $(\mathcal{G},\theta)$, denoted $\mathcal{G}_{(\theta)}$, is obtained from $\mathcal{G}$ by imposing the relations $\theta_c=id_c\quad,\forall\ c\in\mathcal{G}^{(0)}$; to be more precise about the topology, put $\mathcal{G}^{(0)}_{(\theta)} := \mathcal{G}^{(0)}$, $\mathcal{G}^{(1)}_{(\theta)} := \mathcal{G}^{(1)}/\mathbb{Z}$ with the obvious structure maps. It is not difficult to see that $\mathcal{G}_{(\theta)}$ is still an étale category.

We call $(\mathcal{G},\theta)$ elliptic if $ord(\theta_c) < \infty \quad , \forall\ c \in \mathcal{G}^{(0)}$. We call it hyperbolic if for any $g : c \longrightarrow d$ in $\mathcal{G}$, the equality $g\theta_c^n = g$ holds just for $n = 0$ (in particular $ord(\theta_c) = \infty \quad , \forall\ c \in \mathcal{G}^{(0)}$; if $\mathcal{G}$ is a groupoid, this is the only condition).

If $(\mathcal{G},\theta)$, $(\mathcal{G}',\theta')$ are cyclic categories, so is $(\mathcal{G}\times\mathcal{G}',\theta\times\theta')$. If $(\mathcal{G},\theta)$ is a cyclic category then so is $(\mathcal{G},\theta^{-1})$, provided that the map $\mathcal{G}^{(0)} \longrightarrow \mathcal{G}^{(1)}$, $c \mapsto \theta_c^{-1}$ is continuous; in this case, for any other cyclic category $(\mathcal{G}',\theta')$, the localization of $(\mathcal{G},\theta^{-1})\times(\mathcal{G}',\theta')$ is denoted by $\mathcal{G}\wedge\mathcal{G}'$.

4.3.2 Examples: (see 1.5.5 for the definition of Λ_r)

1. if $\mathcal{G}$ is an étale category, then $(\mathcal{G}, id)$ is a cyclic category and $\mathcal{G}_{(id)} = \mathcal{G}$.
2. (Λ_∞, T), where $T([n]) := t_n^{n+1}$, is a hyperbolic cyclic category with $(\Lambda_\infty)_{(T)} = \Lambda$.
3. If G is a group, $g \in Center(G)$ then (G,g) is a cyclic category with $G_{(g)} = G/ < g >$.
4. If C_r is a cyclic group generated by γ, $ord(\gamma) = r+1$ then $(C_r)_{(\gamma)}$ =trivial and $\Lambda_\infty \wedge C_r = \Lambda_r \quad , \forall\ \ 0 \le r \le \infty$.
5. For any cyclic category $(\mathcal{G},\theta)$: $\mathcal{G}\wedge * = \mathcal{G}_{(\theta)}$, $\mathcal{G}\wedge\mathbb{Z} = \mathcal{G}$.

4.3.3 Lemma: *Let $(\mathcal{G},\theta)$ be a hyperbolic cyclic category, $\varphi : \mathcal{G} \longrightarrow \mathcal{G}_{(\theta)}$ the projection functor. Then for any $c \in \mathcal{G}^{(0)}$, the functor:*

$$\mathbb{Z} =< \theta_c > \ \tilde{\longrightarrow} c/\varphi,$$

which sends the single object $*$ of $\mathbb{Z}$ to (id_c, c) and $n \in \mathbb{Z}$ to $\theta^n : (id_c, c) \longrightarrow (id_c, c)$ is a strong deformation retract of d/φ. Moreover, it is a Morita equivalence if $\mathcal{G}$ is a groupoid.

proof: Choose a set-theoretic map $\sigma : \mathcal{G}^{(1)}_{(\theta)} \longrightarrow \mathcal{G}^{(1)}$ which is a retract of φ and:

$$s(\sigma(h)) = s(h), \quad t(\sigma(h)) = t(h), \quad \sigma(c) = c, \quad \forall\ h \in \mathcal{G}^{(1)}_{(\theta)}, c \in \mathcal{G}^{(0)}.$$

We get a map $n : \mathcal{G} \longrightarrow \mathbb{C}$ uniquely determined by the equality:

$$g = \theta^{n(g)}\sigma(\varphi(g)), \quad \forall\ g \in \mathcal{G}^{(1)}.$$

Recall (2.3.1) that the discrete category c/φ has as objects pairs (h,d) with $d \in \mathcal{G}^{(0)}, h : c \longrightarrow d$ a morphism in $\mathcal{G}_{(\theta)}$ and as morphisms from (h,d) to $(h\ ', d\ ')$ those morphisms $g : d \longrightarrow d\ '$ in $\mathcal{G}$ with the property $\varphi(g)h = h\ '$. Define the retraction $\psi : d/\varphi \longrightarrow \mathbb{Z}$ by sending a morphism $g : (h,d) \longrightarrow (h\ ', d\ ')$ (in c/φ) to $n(g\sigma(h))$. The deformation retraction $F : \psi \circ \varphi \longrightarrow Id$ is defined as follows: to an object (h,d) in c/φ it associates the morphism $F(h,d) = \sigma(h) : \psi(\varphi(h,d)) = (id_c, c) \longrightarrow (h,d)$ in c/φ. That φ is an essential equivalence when $\mathcal{G}$ is a groupoid is obvious. □

Proposition 4.3.4 : *If $(\mathcal{G},\theta)$ is a hyperbolic cyclic category, then for any $\mathcal{A} \in Sh(\mathcal{G}_{(\theta)})$ there is a long exact sequence:*

$$\ldots \longrightarrow H_n(\mathcal{G};\mathcal{A}) \longrightarrow H_n(\mathcal{G}_{(\theta)};\mathcal{A}) \xrightarrow{d} H_{n-2}(\mathcal{G}_{(\theta)};\mathcal{A}) \longrightarrow H_{n-1}(\mathcal{G};\mathcal{A}) \longrightarrow \ldots \quad .$$

Here the boundary is of type $d = - \cap e(\mathcal{G},\theta)$, the cap product by some cohomology class $e(\mathcal{G},\theta) \in H^2(\mathcal{G}_{(\theta)};\mathbb{C})$ which does not depend on $\mathcal{A}$ and is called the Euler class of $(\mathcal{G},\theta)$. Moreover, $e(\mathcal{G},\theta)$ has the naturality property in the following sense: for any morphism $f : (\mathcal{G},\theta) \longrightarrow (\mathcal{H},\tau)$ of hyperbolic cyclic categories, $f^*e(\mathcal{H},\tau) = e(\mathcal{G},\theta)$.

proof: The Hochschild-Serre spectral sequence for φ (cf. Theorem 2.3.4, see also 2.2.14), $E^2_{p,q} = H_p(\mathcal{G}_{(\theta)}; L_q\varphi_!\mathcal{A}) \Longrightarrow H_{p+q}(\mathcal{G};\mathcal{A})$, has:

$$(L_q\varphi_!\mathcal{A})_c = H_q(c/\varphi;\mathcal{A}) = H_q(\mathbb{Z};\mathcal{A}_c) \ .$$

The last equality follows from Proposition 2.3.3, the previous lemma and Lemma 2.2.15 (or Morita invariance 2.3.6 if $\mathcal{G}$ is a groupoid). Also it is continuous with respect to $c \in \mathcal{G}^{(0)}$. So we get $L_q\varphi_!\mathcal{A} = \mathcal{A}$ if $q \in \{0,1\}$, and 0 otherwise, and this implies the long exact sequence.

From Remark 2.3.5.1 we know its boundary is of type $d = -\cap e(\mathcal{A})$ for some $e(\mathcal{A}) \in Ext_{\mathcal{G}_{(\theta)}}(\mathcal{A},\mathcal{A})$. Recall (see 2.2.12) that the action of $H^2(\mathcal{G}_{(\theta)};\mathbb{C}) = Ext_{\mathcal{G}_{(\theta)}}(\mathbb{C},\mathbb{C})$ on $H_*(\mathcal{G}_{(\theta)};\mathcal{A})$ is defined using the morphism: $Ext_{\mathcal{G}_{(\theta)}}(\mathbb{C},\mathbb{C}) \longrightarrow Ext_{\mathcal{G}_{(\theta)}}(\mathcal{A},\mathcal{A}), u \mapsto u \otimes \mathcal{A}$. So it is enough to prove that $e(\mathcal{A}) = e(\mathbb{C}) \otimes \mathcal{A}$. Recall also that the spectral sequence we used is obtained (see the proof of Theorem 2.3.4) from the equality: $H_*(\mathcal{G};\mathcal{A}) = \mathbb{H}_*(\mathcal{G}_{(\theta)}; \mathcal{L}_*(\mathcal{A}))$, where $\mathcal{L}_*(\mathcal{A})) = \mathcal{L}\varphi_!(\mathcal{A})$ is a chain complex in $Sh(\mathcal{G}_{(\theta)})$ which can be described as follows (we spell out the general definition of $\mathcal{L}\varphi_!$ in 2.3.2). Define:

$$X_n = \{(g_1,...,g_{n+1}) : (g_1,...,g_n) \in \mathcal{G}^{(n)}, g_{n+1} \in \mathcal{G}^{(1)}_{(\theta)}, s(g_n) = t(g_{n+1})\},$$

and the maps $\alpha, \beta : X_n \longrightarrow \mathcal{G}^{(0)}, \alpha(g_1,...,g_{n+1}) = t(g_1), \beta(g_1,...,g_{n+1}) = s(g_{n+1})$. Then $\mathcal{L}_n(\mathcal{A}) = \beta_!\alpha^*\mathcal{A}$ has the stalk at $c \in \mathcal{G}^{(0)}$:

$$(\mathcal{L}_n(\mathcal{A}))_c = \bigoplus_{d \in \mathcal{G}^{(0)}} \mathcal{A}_d \otimes \mathbb{C}[\alpha^{-1}(d) \cap \beta^{-1}(c)],$$

and the action of $\mathcal{G}_{(\theta)}$ and the boundaries are:

$$(a, g_1, ..., g_{n+1})g = (a, g_1, ..., g_{n+1}g),$$

$$d_n(a, g_1, ..., g_{n+1}) = (ag_1, ..., g_{n+1}) + \sum_{i=1}^{n} (-1)^i (a, g_1, ..., g_i g_{i+1}, ..., g_{n+1}).$$

We get the following representative for $e(\mathcal{A})$:

$$0 \longrightarrow \mathcal{A} \xrightarrow{j} Coker(d_1) \xrightarrow{d_0} \mathcal{L}_0(\mathcal{A}) \xrightarrow{\epsilon} \mathcal{A} \longrightarrow 0,$$

where $\epsilon(a, g_1) = ag_1, j(a) = \widehat{(a,\theta,1)}$. Denoting this extension by $u(\mathcal{A})$, remark there is a map:

$$\mathcal{L}_*(\mathcal{A}) \longrightarrow \mathcal{L}_*(\mathbb{C}) \otimes \mathcal{A}, \quad (a, g_1, ..., g_{n+1}) \mapsto (ag_1...g_{n+1}, g_1, ..., g_{n+1})$$

which induces a map $u(\mathcal{A}) \longrightarrow u(\mathbb{C}) \otimes \mathcal{A}$. This proves $e(\mathcal{A}) = e(\mathbb{C}) \otimes \mathcal{A}$.

Denote by $\tilde{\mathcal{L}}_*(\mathbb{C}), \tilde{u}(\mathbb{C})$ the analogous constructions for $(\mathcal{H},\tau)$ and define a chain-map:

$$\mathcal{L}_*(\mathbb{C}) \longrightarrow f^*\tilde{\mathcal{L}}_*(\mathbb{C}), \quad (g_1, ..., g_{n+1}) \mapsto (f(g_1), ..., f(g_{n+1})).$$

It induces a map of extensions $u(\mathbb{C}) \longrightarrow f^*\tilde{u}(\mathbb{C})$ and this proves the naturality. $\square$

4.3.5 Remark: Applying 4.3.4 to (Λ_∞, T) (see 4.3.2) we get the usual SBI-sequence for cyclic vector-spaces as the Gysin-sequence for the projection $\Lambda_\infty \longrightarrow \Lambda$ and the S-boundary as an Euler class $e(\Lambda_\infty, T) \in H^2(\Lambda;\mathbb{C})$. Our description of S in terms of extensions (as described in the previous proof) is very close to the one given in [84], pp. 565.

4.4 Cyclic Homology of sheaves on cyclic groupoids

4.4.1 Cyclic sheaves on cyclic groupoids: Let $(\mathcal{G}, \theta)$ be a cyclic category. A *θ-cyclic $\mathcal{G}$-sheaf* is a ∞-cyclic object $\mathcal{A}_*$ in $Sh(\mathcal{G})$ (i.e. a contravariant functor $\Lambda_\infty \longrightarrow Sh(\mathcal{G})$ cf. 1.5.5) such that, for any $c \in \mathcal{G}^{(0)}$, the morphism $(t_{c,n})^{n+1} : (\mathcal{A}_n)_c \longrightarrow (\mathcal{A}_n)_c$ coincides with the action of θ_c. In other words, a θ-cyclic object is a $\Lambda_\infty \wedge \mathcal{G}$-sheaf. Define:

$$HH_*(\mathcal{G}, \theta; \mathcal{A}_*) := H_*(\Lambda_\infty \times \mathcal{G}; \mathcal{A}_*)\ ,\ HC_*(\mathcal{G}, \theta; \mathcal{A}_*) := H_*(\Lambda_\infty \wedge \mathcal{G}; \mathcal{A}_*).$$

If $\mathcal{G}$ is an étale groupoid, and $\theta = id$, this agrees with the earlier definition of cyclic $\mathcal{G}$-sheaves (4.2.1) and their homologies (4.2.3).

4.4.2 ***SBI*-sequence**: Since $\Lambda_\infty \times \mathcal{G}$ is always hyperbolic, 4.3.4 applies, so for any θ-cyclic $\mathcal{G}$-sheaf $\mathcal{A}_*$ there is a long exact sequence:

$$\ldots \longrightarrow HH_n(\mathcal{G}, \theta; \mathcal{A}_*) \longrightarrow HC_n(\mathcal{G}, \theta; \mathcal{A}_*) \longrightarrow HC_{n-2}(\mathcal{G}, \theta; \mathcal{A}_*) \longrightarrow HH_{n-1}(\mathcal{G}, \theta; \mathcal{A}_*)$$

4.4.3 Remark: If $(\mathcal{G}, \theta)$ is a cyclic category and $\mathcal{A}_*$ is a θ-cyclic $\mathcal{G}$-sheaf, then $HH_*(\mathcal{G}, \theta; \mathcal{A}_*) = \mathbb{H}_*(\mathcal{G}; (\mathcal{A}_*, b))$ (one way to see this is by applying the Hochschild-Serre spectral sequence 2.3.4 to the projection $\Lambda_\infty \times \mathcal{G} \longrightarrow \mathcal{G}$). This implies that the analogue of 4.2.7 holds.

4.4.4 Lemma: *If $(\mathcal{G}, \theta)$ is a cyclic category and $\mathcal{A}_*$ is a θ-cyclic $\mathcal{G}$-sheaf, then the ∞-cyclic vector space $n \mapsto H_q(\mathcal{G}; \mathcal{A}_n)$ is cyclic for every q, and there are spectral sequences:*

$$E^2_{p,q} = HH_p(H_q(\mathcal{G}; \mathcal{A}_*)) \Longrightarrow HH_{p+q}(\mathcal{G}, \theta; \mathcal{A}_*)\ ,$$

$$E^2_{p,q} = HC_p(H_q(\mathcal{G}; \mathcal{A}_*)) \Longrightarrow HC_{p+q}(\mathcal{G}, \theta; \mathcal{A}_*)\ .$$

proof: Consider the diagram with columns coming from cyclic categories:

$$\begin{array}{cccccccc}
\Lambda_\infty \times * = & \Lambda_\infty & \xleftarrow{\pi_1} & \Lambda_\infty \times \mathcal{G} & \xrightarrow{\pi_2} & \mathcal{G} & = * \times \mathcal{G} \\
& \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & \\
\Lambda_\infty \wedge * = & \Lambda & \xleftarrow{\pi_1'} & \Lambda_\infty \wedge \mathcal{G} & \xrightarrow{\pi_2'} & \mathcal{G}_{(\theta)} & = * \wedge \mathcal{G}
\end{array}$$

Here $\pi_1, \pi_2, \pi_1', \pi_2'$ are the projections. For any integer $n \geq 0$ we get from 4.3.3 strong deformation retracts:

$$[n]/\pi \tilde{\leftarrow} \mathbb{Z},\ \ [n]/\pi_1 = ([n]/\Lambda_\infty) \times \mathcal{G} \tilde{\leftarrow} \mathcal{G},\ \ [n]/\pi_1' = ([n]/\pi) \wedge \mathcal{G} \tilde{\leftarrow} \mathbb{Z} \wedge \mathcal{G} = \mathcal{G}.$$

The Hochschild-Serre spectral sequences induced by π_1 and π_1' together with Lemma 2.2.15 give the desired spectral sequences. $\square$

4.4.5 The associated cyclic module: Assume that $(\mathcal{G}, \theta)$ is a cyclic groupoid and $\mathcal{A}_*$ is a θ-cyclic $\mathcal{G}$-sheaf such that each $\mathcal{A}_n$ is c-soft. From 4.4.3 we see that $HH_*(\mathcal{G}, \theta; \mathcal{A}_*)$ is computed by the bi-simplicial vector space $B_*(\mathcal{G}; \mathcal{A}_*)$; so, as in 4.2.8, it is computed by the simplicial vector space $\mathcal{C}(\mathcal{G}; \mathcal{A}_*)$. Define $\mathcal{C}(\mathcal{G}, \theta; \mathcal{A}_*)$ as the cyclic vector space having $\mathcal{C}(\mathcal{G}; \mathcal{A}_*)$ as underlying simplicial vector space and the following cyclic structure:

$$t(a \mid g_1, ..., g_n) = (t(a)\theta^{-1} g_1 ... g_n | (g_1 ... g_n)^{-1}\theta, g_1, ..., g_{n-1}).$$

4.4.6 Lemma and definition: *If $(\mathcal{G}, \theta)$ is a cyclic groupoid and $\mathcal{A}_*$ is a θ-cyclic $\mathcal{G}$-sheaf such that each $\mathcal{A}_n$ is c-soft, then $HH_*(\mathcal{G}, \theta; \mathcal{A}_*)$, $HC_*(\mathcal{G}, \theta; \mathcal{A}_*)$ are computed by the cyclic vector space $\mathcal{C}(\mathcal{G}, \theta; \mathcal{A}_*)$. In this case, define $HP_*(\mathcal{G}, \theta; \mathcal{A}_*) = HP_*(\mathcal{C}(\mathcal{G}, \theta; \mathcal{A}_*))$.*

proof: On the standard resolution of $\mathbb{C}$ in $Sh(\mathcal{G})$ (see 4.2.2), considered as a simplicial $\mathcal{G}$-sheaf, we define a structure of θ-cyclic $\mathcal{G}$-sheaf by:

$$t(g_0, ..., g_n) = (\theta^{-1} g_0 g_1 ... g_n, (g_1 ... g_n)^{-1}\theta, g_1, ..., g_{n-1}).$$

Denote it by $\mathcal{B}_* \in Sh(\Lambda_\infty \wedge \mathcal{G})$, and define $\mathcal{L}_* \in Sh(\Lambda_\infty \wedge \mathcal{G})$ by $\mathcal{L}_n = \mathcal{A}_n \otimes \mathcal{B}_n$ with the structure maps given by the tensor-product of the structure maps of $\mathcal{B}_*$ and $\mathcal{A}_*$. It is a standard fact that the projection $\mathcal{L}_* \longrightarrow \mathcal{A}_*$ induces a quasi-isomorphism $(\mathcal{L}_*, b) \simeq (\mathcal{A}_*, b)$ (it follows, for instance, from the fact that $\mathcal{B}_*$ is a resolution of $\mathbb{C}$ and from the Eilenberg-Zilber theorem); the "SBI-trick" (see 4.4.3) implies that $HC_*(\mathcal{G}, \theta; \mathcal{L}_*) \xrightarrow{\sim} HC_*(\mathcal{G}, \theta; \mathcal{A}_*)$. From this and 4.4.4 applied to the θ-cyclic $\mathcal{G}$-sheaf $\mathcal{L}_*$, we get a spectral sequence with: $E^2_{p,q} = HC_p(H_q(\mathcal{G}; \mathcal{L}_*)) \Longrightarrow HC_{p+q}(\mathcal{G}, \theta; \mathcal{A}_*)$. It is enough to make a straightforward remark: $H_q(\mathcal{G}; \mathcal{L}_n) = 0$, for all $q \geq 1$ (all $\mathcal{L}_n$'s are free, [35]) and for $q = 0$, the cyclic vector space $n \mapsto H_0(\mathcal{G}; \mathcal{L}_n)$ is in fact $\mathcal{C}(\mathcal{G}, \theta; \mathcal{A}_*)$. □

Proposition 4.4.7 *(elliptic case): Let $(\mathcal{G}, \theta)$ be an elliptic cyclic groupoid, $\mathcal{A}_*$ a θ-cyclic $\mathcal{G}$-sheaf. For any integer n put $\mathcal{A}_{(\theta),n} := \varphi_!(\mathcal{A}_n) \in Sh(\mathcal{G}_{(\theta)})$, where $\varphi : \mathcal{G} \longrightarrow \mathcal{G}_{(\theta)}$ is the projection. Then $\mathcal{A}_{(\theta),*}$ is a cyclic $\mathcal{G}_{(\theta)}$-sheaf and:*

$$HH_*(\mathcal{G}, \theta; \mathcal{A}_*) = HH_*(\mathcal{G}_{(\theta)}; \mathcal{A}_{(\theta),*}) \quad , \quad HC_*(\mathcal{G}, \theta; \mathcal{A}_*) = HC_*(\mathcal{G}_{(\theta)}; \mathcal{A}_{(\theta),*}),$$

and the analogue for HP_*.

proof: Assume for simplicity that each $\mathcal{A}_n$ is c-soft (in general we work with c-soft resolutions). From Lemma 4.3.3 we have the Morita equivalences $< \theta_c > \xrightarrow{\sim} c/\varphi$, $\forall c \in \mathcal{G}^{(0)}$; Proposition 2.3.3 gives $(L_q\varphi_!\mathcal{A}_n)_c =$
$H_q(< \theta_c >; (\mathcal{A}_n)_c) = 0$, $\forall q \neq 0$ and $(\mathcal{A}_{(\theta),n})_c = Coinv_{\theta_c}((\mathcal{A}_n)_c)$. In particular, the ∞-cyclic object $\mathcal{A}_{(\theta),*}$ is cyclic. Also the spectral sequence of φ degenerates; this ensures that the obvious projection of bi-simplicial vector spaces $\{B_p(\mathcal{G}; \mathcal{A}_q) : p, q \geq 0\} \longrightarrow \{B_p(\mathcal{G}_{(\theta)}; \mathcal{A}_{(\theta),q}) : p, q \geq 0\}$ is a quasi-isomorphism on the $q = constant$ columns. By the Eilenberg-Zilber theorem, it is a quasi-isomorphism between their diagonals, i.e. the projection $\mathcal{C}(\mathcal{G}, \theta; \mathcal{A}_*) \longrightarrow \mathcal{C}(\mathcal{G}_{(\theta)}, id; \mathcal{A}_{(\theta),*})$ of cyclic vector spaces induces isomorphism between their Hochschild (hence also cyclic) homology; then Lemma 4.4.6 ends the proof. □

Proposition 4.4.8 *(hyperbolic case): Let $(\mathcal{G}, \theta)$ be a hyperbolic cyclic groupoid, $\mathcal{A}_*$ a θ-cyclic $\mathcal{G}$-sheaf. Then $\widetilde{HH}_*(\mathcal{A}_*) \in Sh(\mathcal{G})$ are in fact $\mathcal{G}_{(\theta)}$-sheaves, and there are spectral sequences:*

$$E^2_{p,q} = H_p(\mathcal{G}; \widetilde{HH}_q(\mathcal{A}_*)) \Longrightarrow HH_*(\mathcal{G}, \theta; \mathcal{A}_*) \quad ,$$

$$E^2_{p,q} = H_p(\mathcal{G}_{(\theta)}; \widetilde{HH}_q(\mathcal{A}_*)) \Longrightarrow HC_*(\mathcal{G}, \theta; \mathcal{A}_*).$$

Moreover, $HC_*(\mathcal{G}, \theta; \mathcal{A}_*)$ are modules over the ring $H^*(\mathcal{G}_{(\theta)}; \mathbb{C})$ and S in the SBI sequence is the (cap-) product by the Euler class $e(\mathcal{G}, \theta) \in H^2(\mathcal{G}_{(\theta)}; \mathbb{C})$.

proof: That θ acts trivially on $\widetilde{HH}_*(\mathcal{A}_*)$ follows by using the homotopy between id and θ , $h : \mathcal{A}_* \longrightarrow \mathcal{A}_{*+1}$ given by:

$$h_n = s_{-1}(1 + \tau + ... + \tau^n) : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1} .$$

From the proof of Lemma 4.3.3, we get strong deformation retracts:

$$c/\pi_2 = \Lambda_\infty \times (c/\mathcal{G}) \tilde{\leftarrow} \Lambda_\infty, \quad c/\pi_2' = \Lambda_\infty \wedge (c/\varphi) \tilde{\leftarrow} \Lambda_\infty \wedge < \theta_c >= \Lambda_\infty.$$

Now the spectral sequences follow from the Hochschild-Serre spectral sequences 2.3.4 induced by π_2, π_2' (see the diagram in the proof of 4.4.4), and Lemma 2.2.15. The last part follows from Proposition 4.3.4 applied to $\Lambda_\infty \times \mathcal{G}$ and the isomorphism $H^*(\Lambda_\infty \wedge \mathcal{G}; \mathbb{C}) \simeq H^*(\mathcal{G}_{(\theta)}; \mathbb{C})$. The last isomorphism follows from the spectral sequence 1.2.8 induced by π_2' in cohomology. $\Box$

4.4.9 Eilenberg-Zilber Theorem in the cyclic case: As an application we give a new proof of the spectral sequence for cylindrical objects which is one of the main results in [47]. Recall ([47], pp. 164) that a cylindrical vector space is a functor $C_{*,*} : \Lambda_\infty \wedge \Lambda_\infty \longrightarrow \underline{Vs}$. Its diagonal is naturally a cyclic vector space.

4.4.10 Corollary: *If $C_{*,*}$ is a bi-cyclic vector space which is cylindrical, then the ∞-cyclic vector space $n \mapsto H_q(C_{n,*})$ is cyclic for any $q \geq 0$, and there are spectral sequences:*

$$E^2_{p,q} = HH_p(H_q(C_{*,*})) \Longrightarrow HH_{p+q}(diag(C_{*,*})),$$
$$E^2_{p,q} = HC_p(H_q(C_{\bullet\bullet})) \Longrightarrow HC_{p+q}(diag(C_{*,*})).$$

proof: With the same method as in the proof of Lemma 4.4.6 we see that the homologies $HH_*(\Lambda_\infty, T; C_{*,*})$, $HC_*(\Lambda_\infty, T; C_{*,*})$ are computed by the cyclic module $diag(C_{*,*})$. It suffices to apply Lemma 4.4.4 to (Λ_∞, T). $\Box$

4.5 Cyclic Homology of crossed products by étale groupoids

In this subsection we introduce the cyclic homology of crossed products by étale groupoids, and, relating them with the cyclic homology of cyclic groupoids, we derive the Feigin-Tsygan-Nistor spectral sequences. These will be essential in the next subsection when we will treat a particular case of crossed products, namely convolution algebras.

4.5.1 Recall that if G is a discrete group acting on an unital algebra A, the cross-product algebra $A \rtimes G$ is $A \otimes \mathbb{C}[G]$ with the convolution product $(a, g)(b, h) = ((ah)b, gh)$. We see that the induced cyclic vector space $(A \rtimes G)^\natural$ (see 1.5.6) has:

$$(A \rtimes G)^\natural(n) = A^{\otimes(n+1)} \otimes \mathbb{C}[G^{n+1}],$$

while the cyclic structure is given by the formulas (see also [83]):

$$d_i(a_0, ..., a_n | g_0, ..., g_n) = \begin{cases} (a_0, ..., (a_i g_{i+1}) a_{i+1}, ..., a_n | g_0, ..., g_i g_{i+1}, ..., g_n) & \text{if } 0 \leq i \leq n-1 \\ ((a_n g_0) a_0, a_1, ..., a_{n-1} | g_n g_0, g_1, ..., g_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(a_0, ..., a_n|g_0, ..., g_n) = (..., a_i, 1, a_{i+1}, ...|..., g_i, 1, g_{i+1}, ...),$$
$$t(a_0, ..., a_n|g_0, ..., g_n) = (a_n, a_0, ..., a_{n-1}|g_n, g_0, ..., g_{n-1}).$$

4.5.2 Lemma and definition: *Let $\mathcal{G}$ be an étale groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff, $\mathcal{A}$ a c-soft $\mathcal{G}$-sheaf of complex algebras. If $u, v \in \Gamma_c(\mathcal{G}; s^*\mathcal{A})$, then the following formula:*

$$(u * v)(g) = \sum_{g_1 g_2 = g} (u(g_1)g_2)v(g_2) \quad \forall g \in \mathcal{G},$$

gives a well defined element $u * v \in \Gamma_c(\mathcal{G}; s^*\mathcal{A})$. The resulting algebra, $(\Gamma_c(\mathcal{G}; s^*\mathcal{A}), *)$ is called the crossed product of $\mathcal{A}$ and $\mathcal{G}$ and is denoted $\mathcal{A} \rtimes_{alg} \mathcal{G}$.

proof: The element $u * v \in \Gamma_c(\mathcal{G}; s^*\mathcal{A})$ is obtained via the composition of:

$$\Gamma_c(\mathcal{G}; s^*\mathcal{A}) \times \Gamma_c(\mathcal{G}; s^*\mathcal{A}) \longrightarrow \Gamma_c(\mathcal{G} \times \mathcal{G}; s^*\mathcal{A} \boxtimes s^*\mathcal{A}) \longrightarrow$$
$$\longrightarrow \Gamma_c(\mathcal{G} \times_{\mathcal{G}^{(0)}} \mathcal{G}; s^*\mathcal{A} \boxtimes s^*\mathcal{A}|_{\mathcal{G} \times_{\mathcal{G}^{(0)}} \mathcal{G}}) \longrightarrow \Gamma_c(\mathcal{G}; s^*\mathcal{A})$$

where the first map is the obvious one, the second is the restriction to $\mathcal{G} \times_{\mathcal{G}^{(0)}} \mathcal{G}$ (which is closed in $\mathcal{G} \times \mathcal{G}$ since $\mathcal{G}^{(0)}$ is Hausdorff), and the third is:

$$(a, b \mid g_1, g_2) \mapsto ((ag_2)b, g_1 g_2). \quad \square$$

4.5.3 Examples:

1. If G is a discrete group acting on an algebra A, then $A \rtimes_{alg} G$ is the usual crossed product (described in 4.5.1).

2. If $\mathcal{G}$ is a smooth étale groupoid, $\mathcal{A} = \mathcal{C}^\infty_{\mathcal{G}^{(0)}}$, then $\mathcal{A} \rtimes_{alg} \mathcal{G} = \mathcal{C}^\infty_c(\mathcal{G})$ is the convolution algebra of $\mathcal{G}$ (we use that $\mathcal{C}^\infty_c(\mathcal{G}) = \Gamma_c(\mathcal{G}; \mathcal{C}^\infty_{\mathcal{G}})$ and $\mathcal{C}^\infty_{\mathcal{G}} = s^*\mathcal{C}^\infty_{\mathcal{G}^{(0)}}$ since s is étale). We will come back to this example in the next subsection.

4.5.4 The cyclic object $(\mathcal{A} \rtimes \mathcal{G})^\natural$: Let $\mathcal{G}$ be an étale groupoid, $\mathcal{A}$ a c-soft $\mathcal{G}$-sheaf of complex unital algebras. Consider *Burghelea's space* [20]: $B^{(n)} = \{(\gamma_0, \gamma_1, ..., \gamma_n) \in \mathcal{G}^{(n+1)} : t(\gamma_0) = s(\gamma_n)\}$, and $\sigma_n : B^{(n)} \longrightarrow (\mathcal{G}^{(0)})^{n+1}, \sigma_n(\gamma_0, ..., \gamma_n) = (s(\gamma_0), ..., s(\gamma_n))$. Define the cyclic vector space $(\mathcal{A} \rtimes \mathcal{G})^\natural$ by:

$$(\mathcal{A} \rtimes \mathcal{G})^\natural(n) = \Gamma_c(B^{(n)}; \sigma_n^* \mathcal{A}^{\boxtimes(n+1)}).$$

For the structure maps, we keep the same formulas as in 4.5.1. Its homologies are denoted $HH_*(\mathcal{A} \rtimes \mathcal{G})$, $HC_*(\mathcal{A} \rtimes \mathcal{G})$, $HP_*(\mathcal{A} \rtimes \mathcal{G})$.

This definition is suitable for the main examples: the discrete case 4.5.1, and the smooth case (see Proposition 4.6.1). Here are some further motivations. First of all, $(\mathcal{A} \rtimes \mathcal{G})^\natural$ is the maximal object for which the formulas in 4.5.1 make sense; this allows us to use our methods in this general setting. A way to think of $(\mathcal{A} \rtimes \mathcal{G})^\natural$ is as follows (and this is in fact the main motivation). Usually our objects are endowed with topologies; in particular, the relevant cyclic homology of $\mathcal{A} \rtimes_{alg} \mathcal{G}$ uses the topology (i.e. certain topological tensor product [52] is used for defining $(\mathcal{A} \rtimes_{alg} \mathcal{G})^\natural$). We have inclusions:

$$(\mathcal{A} \rtimes_{alg} \mathcal{G})^{\otimes(n+1)} \hookrightarrow \Gamma_c(\mathcal{G}^{n+1}; s^*_{n+1}(\mathcal{A}^{\boxtimes(n+1)})) \;\;, \;\; (s_{n+1} = s \times ... \times s \; ; (n+1) \; times)$$

and the vector spaces $\Gamma_c(\mathcal{G}^{n+1}; s^*_{n+1}(\mathcal{A}^{\boxtimes(n+1)}))$ are the best candidates for defining $(\mathcal{A} \rtimes_{alg} \mathcal{G})^\natural$ by taking into account the topology. From the first half of 2.5.9.2 we get (compare to Proposition 4.1 in [20]):

4.5.5 Lemma: $(\mathcal{A}\rtimes\mathcal{G})^{\natural}(n) = \Gamma_c(\mathcal{G}^{n+1}; s_{n+1}^{*}(\mathcal{A}^{\boxtimes(n+1)}))/\{u : u\,|_{B^{(n)}} = 0\}$.

So our definition can be viewed as a "topological normalization" of the cyclic vector space associated to $\mathcal{A}\rtimes_{alg}\mathcal{G}$. Of course, in practice one more step is needed: to prove that passing to this "topological normalization" does not change the cyclic homology (see for instance Prop. 4.6.1).

And as a final motivation for our choice and notation, let us just remark that the usual formulas for the Chern-character (see e.g. [29, 66, 71] or our preliminary section 1.5) define a Connes-Chern-Karoubi character:

$$Ch : K_{*}^{alg}(\mathcal{A}\rtimes_{alg}\mathcal{G}) \longrightarrow HP_{*}(\mathcal{A}\rtimes\mathcal{G}).$$

Because of these we do believe that the results (or better: the methods) we describe here may be useful to a larger extent.

4.5.6 The cyclic groupoid $\Omega(\mathcal{G})$; the twisted sheaves $\mathcal{A}^{\natural}_{tw}$: An important role in understanding $(\mathcal{A}\rtimes\mathcal{G})^{\natural}$ (even in the discrete case 4.5.1) is played by the $\mathcal{G}$-space of loops $B^{(0)} = \{\gamma \in \mathcal{G}^{(1)} : s(\gamma) = t(\gamma)\}$; the action of $\mathcal{G}$ is given by:

$$B^{(0)} \times_{\mathcal{G}^{(0)}} \mathcal{G} \longrightarrow B^{(0)}\ ,\ (\gamma, g) \mapsto g^{-1}\gamma g.$$

Let $\Omega(\mathcal{G}) := B^{(0)}\rtimes\mathcal{G}$ be the associated groupoid (see 1.1.5); it is a cyclic groupoid with the cyclic structure θ defined by $\theta(\gamma) := (\gamma, \gamma)$. Define a θ-cyclic $\Omega(\mathcal{G})$-sheaf $\mathcal{A}^{\natural}_{tw}$ by $\mathcal{A}^{\natural}_{tw} := s^{*}(\mathcal{A}^{\natural})$ where $s : B^{(0)} \longrightarrow \mathcal{G}^{(0)}$ is the restriction of the source map, and $\mathcal{A}^{\natural} = \mathcal{A}^{\otimes(*+1)}$ was defined in 4.2.2; the formulas for the structure maps are defined in such a way that, at the stalk at $\gamma \in B^{(0)}$ with $s(\gamma) = c$, the ∞-cyclic vector space $(\mathcal{A}^{\natural}_{tw})_{\gamma}$ is $\mathcal{A}^{\natural}_{c,\sigma_{\gamma}}$, i.e. the one associated to the action of γ on the algebra $\mathcal{A}_c$ (cf. 1.5.6). Compare to (9), (10) in [20].

Proposition 4.5.7 : *$HH_{*}(\mathcal{A}\rtimes\mathcal{G}) = HH_{*}(\Omega(\mathcal{G}), \theta; \mathcal{A}^{\natural}_{tw})$, and similarly for cyclic and periodic cyclic homology.*

proof: The $n-$th nerve of $\Omega(\mathcal{G})$ is:

$$\Omega(\mathcal{G})^{(n)} = \{(\gamma|\gamma_1, ..., \gamma_n) :\ (\gamma, \gamma_1) \in \Omega(\mathcal{G})^{(1)}, (\gamma_1, ..., \gamma_n) \in \mathcal{G}^{(n)}\}$$

(here "|" is just a notation in order to separate the loops from the usual arrows). The isomorphism of vector spaces:

$$\Gamma_c(B^{(n)}; \sigma_n^{*}\mathcal{A}^{\boxtimes(n+1)}) \simeq \Gamma_c(\Omega(\mathcal{G})^{(n)}; \mathcal{A}^{\natural}_{tw}),$$

$$(a_0, ..., a_n|\gamma_0, ..., \gamma_n) \mapsto (a_0\gamma_1...\gamma_n\gamma_0, a_1\gamma_2...\gamma_n\gamma_0, \ldots, a_n\gamma_0|\gamma_1...\gamma_n\gamma_0, \gamma_1, \ldots, \gamma_n),$$

(compare to 4.1 in [20]) gives an isomorphism of cyclic vector spaces:

$$(\mathcal{A}\rtimes\mathcal{G})^{\natural} \simeq \mathcal{C}(\Omega(\mathcal{G}), \theta; \mathcal{A}^{\natural}_{tw}),$$

so it is enough to use Lemma 4.4.6. $\square$

4.5.8 The groupoids $\mathcal{Z}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}}$; the twisted sheaves $\mathcal{A}^{\natural}_{\mathcal{O}}$: For $\mathcal{O} \subset B^{(0)}$, $\mathcal{G}$-invariant we define the cyclic groupoid $\mathcal{Z}_{\mathcal{O}} := \mathcal{O} \rtimes \mathcal{G}$ (the restriction of $\Omega(\mathcal{G})$ to $\mathcal{O}$). Its localization $(\mathcal{Z}_{\mathcal{O}})_{(\theta)}$ (i.e. obtained from $\mathcal{Z}_{\mathcal{O}}$ by imposing the relations $(\gamma, 1) = (\gamma, \gamma)$, $\forall\, \gamma \in \mathcal{O}$; see 4.3.1) is denoted by $\mathcal{N}_{\mathcal{O}}$. These two groupoids play the role of the centralizer and normalizer of $\mathcal{O}$ (see also section 4.10). We define a θ-cyclic $\mathcal{Z}_{\mathcal{O}}$-sheaf $\mathcal{A}^{\natural}_{\mathcal{O}}$ in such a way that, stalkwise:

$$(\mathcal{A}^{\natural}_{\mathcal{O}})_{\gamma} = \mathcal{A}^{\natural}_{c,\sigma_{\gamma}}, \quad \forall\, \gamma \in \mathcal{O} \quad (c = s(\gamma))$$

(see 1.5.6). In other words, $\mathcal{A}^{\natural}_{\mathcal{O}}$ is the restriction of $\mathcal{A}^{\natural}_{tw}$ to $\mathcal{O}$. Define also the cyclic $\mathcal{N}_{\mathcal{O}}$ -sheaf $\mathcal{A}^{\natural}_{(\mathcal{O})} := (\mathcal{A}^{\natural}_{\mathcal{O}})_{(\theta)}$ (with the notations of Proposition 4.4.7); the stalk of $\mathcal{A}^{(n)}_{(\mathcal{O})}$ at $\gamma \in \mathcal{O}$ is $Coinv_{<\gamma>}(\mathcal{A}_c^{\otimes(n+1)})$.

4.5.9 Localization: For $\mathcal{O}$ as before, we can define the cyclic vector space $(\mathcal{A} \rtimes \mathcal{G})^{\natural}_{\mathcal{O}}$ as in 4.5.4 with the only difference we replace $B^{(n)}$ by $B^{(n)}_{\mathcal{O}} = \{(\gamma_0, \gamma_1, ..., \gamma_n) \in B^{(n)} : \gamma_0\gamma_1...\gamma_n \in \mathcal{O}\}$. Denote its homologies by $HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}}, HC_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}}, HP_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}}$. As in 4.5.7 we have:

$$HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} = HH_*(\mathcal{Z}_{\mathcal{O}}, \theta; \mathcal{A}^{\natural}_{\mathcal{O}}) \ , \ HC_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} = HC_*(\mathcal{Z}_{\mathcal{O}}, \theta; \mathcal{A}^{\natural}_{\mathcal{O}}) \ ,$$

and the analogue for HP_*. If $\mathcal{O}$ is open in $B^{(0)}$, there are "extension by 0" maps: $HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} \longrightarrow HH_*(\mathcal{A} \rtimes \mathcal{G})$; if $\mathcal{O}$ is closed in $B^{(0)}$, there are "restriction" maps: $HH_*(\mathcal{A} \rtimes \mathcal{G}) \longrightarrow HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}}$. The same discussion applies to HC_*, HP_*. The following is obvious:

Proposition 4.5.10 *If $B^{(0)} = \bigcup \mathcal{O}$ is a $\mathcal{G}$-invariant disjoint open covering of $B^{(0)}$, then:*

$$HH_*(\mathcal{A} \rtimes \mathcal{G}) = \bigoplus_{\mathcal{O}} HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}}, \ HC_*(\mathcal{A} \rtimes \mathcal{G}) = \bigoplus_{\mathcal{O}} HC_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}},$$

and the analogue for HP_*.

4.5.11 Elliptic case: We call $\mathcal{O} \subset B^{(0)}$ elliptic if it is $\mathcal{G}$-invariant and $ord(\gamma) < \infty$, for all $\gamma \in \mathcal{O}$. From Proposition 4.5.7, Proposition 4.4.7, and the spectral sequences in 4.2.5 and 4.2.6 we get (compare to A6.2 in [45], section 2 in [83]):

Theorem *(elliptic case): If $\mathcal{O}$ is elliptic, then $\mathcal{A}^{\natural}_{(\mathcal{O})}$ is a cyclic $\mathcal{N}_{\mathcal{O}}$-sheaf and $HH_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} = HH_*(\mathcal{N}_{\mathcal{O}}; \mathcal{A}^{\natural}_{(\mathcal{O})})$, $HC_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} = HC_*(\mathcal{N}_{\mathcal{O}}; \mathcal{A}^{\natural}_{(\mathcal{O})})$, $HP_*(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}} = HP_*(\mathcal{N}_{\mathcal{O}}; \mathcal{A}^{\natural}_{(\mathcal{O})})$. In particular, there are Feigin-Tsygan-Nistor type spectral sequences:*

$$E^2_{p,q} = H_p(\mathcal{N}_{\mathcal{O}}; \widetilde{HC}_q(\mathcal{A}^{\natural}_{(\mathcal{O})})) \Longrightarrow HC_{p+q}(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}},$$

$$E^2_{p,q} = HC_p(H_q(\mathcal{N}_{\mathcal{O}}; \mathcal{A}^{\natural}_{(\mathcal{O})})) \Longrightarrow HC_{p+q}(\mathcal{A} \rtimes \mathcal{G})_{\mathcal{O}},$$

and the analogue for HH_*.

The localizations at units (i.e. at $\mathcal{O} = \mathcal{G}^{(0)}$) are usually denoted by the subscript [1] instead of $\mathcal{O}$. We get they are equal to $HH_*(\mathcal{G};\mathcal{A}^\natural)$, $HC_*(\mathcal{G};\mathcal{A}^\natural)$, $HP_*(\mathcal{G};\mathcal{A}^\natural)$ and the corresponding spectral sequences have at the second level:

$$H_p(\mathcal{G};\widetilde{HH}_q(\mathcal{A}^\natural)) \Longrightarrow HH_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{(1)} \quad , \quad HH_p(H_q(\mathcal{G};\mathcal{A}^\natural) \Longrightarrow HH_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{[1]} \ ,$$

$$H_p(\mathcal{G};\widetilde{HC}_q(\mathcal{A}^\natural)) \Longrightarrow HC_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{(1)} \quad , \quad HC_p(H_q(\mathcal{G};\mathcal{A}^\natural) \Longrightarrow HC_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{[1]} \ .$$

In the case of crossed products by groups, this is 2.6 in [83]. We also know (from 2.2.13) the form of the boundaries $d^2_{p,q}$. This generalizes a similar result for crossed products by groups (see Prop. 3.2 in [9]).

4.5.12 Hyperbolic case: We call $\mathcal{O} \subset B^{(0)}$ hyperbolic if it is $\mathcal{G}$-invariant and $ord(\gamma) = \infty$, for all $\gamma \in \mathcal{O}$. Denote by $e_{\mathcal{O}} \in H^2(\mathcal{N}_{\mathcal{O}};\mathbb{C})$ the Euler class of the (hyperbolic) cyclic groupoid $\mathcal{Z}_{\mathcal{O}}$. From Proposition 4.5.7, and Proposition 4.4.8 we get (compare to A6.1 in [45], 1.8 in [22], section 3 in [83]):

Theorem *(hyperbolic case): If $\mathcal{O}$ is hyperbolic, then the $HC_*(\mathcal{A}\rtimes\mathcal{G})_{\mathcal{O}}$ are modules over the ring $H^*(\mathcal{N}_{\mathcal{O}};\mathbb{C})$ and S in the SBI-sequence is the (cap-) product by the Euler class $e_{\mathcal{O}} \in H^2(\mathcal{N}_{\mathcal{O}};\mathbb{C})$. Moreover, the $\widetilde{HH}_q(\mathcal{A}^\natural_{\mathcal{O}}) \in Sh(\mathcal{Z}_{\mathcal{O}})$ are $\mathcal{N}_{\mathcal{O}}$-sheaves and there are spectral sequences:*

$$E^2_{p,q} = H_p(\mathcal{Z}_{\mathcal{O}};\widetilde{HH}_q(\mathcal{A}^\natural_{\mathcal{O}})) \Longrightarrow HH_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{\mathcal{O}},$$

$$E^2_{p,q} = H_p(\mathcal{N}_{\mathcal{O}};\widetilde{HH}_q(\mathcal{A}^\natural_{\mathcal{O}})) \Longrightarrow HC_{p+q}(\mathcal{A}\rtimes\mathcal{G})_{\mathcal{O}}.$$

4.6 Cyclic Homology of Smooth Étale Groupoids

In this subsection, using the results of the previous section, we compute the cyclic homology of the convolution algebra of a smooth étale groupoid.

Let $\mathcal{G}$ be a smooth étale groupoid. Recall that if $\mathcal{G}$ is Hausdorff then its convolution algebra is defined as $\mathcal{C}^\infty_c(\mathcal{G}) = \{a : \mathcal{G}^{(1)} \longrightarrow \mathbb{C} : a$ is compactly supported and smooth $\}$ with the convolution product: $(ab)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} a(\gamma_1)b(\gamma_2)$. This is a locally convex algebra, which is non-unital in general. Its Hochschild and cyclic homologies are computed (cf. 1.5.6) by using the (b,b')-complex of $\mathcal{C}^\infty_c(\mathcal{G})^\natural$ and certain topological tensor product. As remarked in [20], the relevant definition is:

$$\mathcal{C}^\infty_c(\mathcal{G})^\natural(n) = \mathcal{C}^\infty_c(\mathcal{G}^{n+1}),$$

with the structure maps:

$$d_i(a_0,\ldots,a_n) = \begin{cases} (a_0,\ldots,a_ia_{i+1},\ldots,a_n) & \text{if } 0 \le i \le n-1 \\ (a_na_0,a_1,\ldots,a_{n-1}) & \text{if } i = n \end{cases} ,$$

$$t(a_0,\ldots,a_n) = (a_n,a_0,a_1,\ldots,a_{n-1}) \ .$$

In the general (i.e. non-Hausdorff) case, $\mathcal{C}^\infty_c(\mathcal{G})$ still makes sense as an algebra (see 4.5.3.2). Rather than going into details concerning the topology, following [20], we take the previous equalities as definition of the cyclic vector space $\mathcal{C}_c^\infty(\mathcal{G})^\natural$ (this is the relevant object for defining the Chern-character). It is important to emphasize here that, in contrast with [20] where Connes' definition of $\mathcal{C}_c^\infty(M)$ for a non-Hausdorff manifold M is used, we work with our definition as given in Section 2.5 (and this is essential). The homologies of the cyclic vector space $\mathcal{C}_c^\infty(\mathcal{G})^\natural$ (see 1.5.5) are denoted by $HH_*(\mathcal{C}_c^\infty(\mathcal{G})), HC_*(\mathcal{C}_c^\infty(\mathcal{G})), HP_*(\mathcal{C}_c^\infty(\mathcal{G}))$.

Denote $\mathcal{C}_{\mathcal{G}^{(0)}}^\infty = \mathcal{A}$; it is a $\mathcal{G}$-sheaf cf. 4.2.2. In the definitions of $\mathcal{A}^{\otimes n}$, $\mathcal{A}^{\boxtimes n}$ we take into account the topology, as explained in 1.5.11 and 4.2.2. The following is an extension of 3.2 in [20] to the non-Hausdorff case:

Proposition 4.6.1 *: For any smooth étale groupoid $\mathcal{G}$, $HH_*(\mathcal{C}_c^\infty(\mathcal{G}))$, $HC_*(\mathcal{C}_c^\infty(\mathcal{G}))$, $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))$ are isomorphic to $HH_*(\mathcal{A}\rtimes\mathcal{G}), HC_*(\mathcal{A}\rtimes\mathcal{G}), HP_*(\mathcal{A}\rtimes\mathcal{G})$.*

Since $\mathcal{G}$ is étale, the elements $u \in \mathcal{C}_c^\infty(\mathcal{G}) = \Gamma_c(\mathcal{G}; s^*\mathcal{A})$ can be viewed as functions $\mathcal{G} \ni \gamma \mapsto u(\gamma) \in (s^*\mathcal{A})_\gamma = \mathcal{A}_{s(\gamma)}$ and the convolution product becomes $(u * v)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2}(u(\gamma_1)\gamma_2)v(\gamma_2)$ (in other words, $\mathcal{C}_c^\infty(\mathcal{G}) = \mathcal{A}\rtimes_{alg}\mathcal{G}$). We will simply denote $u * v$ by uv. In the same way, the elements:

$$u \in \mathcal{C}_c^\infty(\mathcal{G})^\natural(n) = \mathcal{C}_c^\infty(\mathcal{G}^{n+1}) = \Gamma_c(\mathcal{G}^{n+1}; s_{n+1}^*\mathcal{A}^{\boxtimes(n+1)})$$

(here $s_n = s \times \ldots \times s$ for n times and we use the notations from 1.5.11) can be viewed as functions:

$$\mathcal{G}^{n+1} \ni (\gamma_0, \ldots, \gamma_n) \mapsto u(\gamma_0, \ldots, \gamma_n) \in (\mathcal{A}\boxtimes \ldots \boxtimes \mathcal{A})_{(s(\gamma_0),\ldots,s(\gamma_n))}.$$

This is the only way we are going to look at elements in $\mathcal{C}_c^\infty(\mathcal{G}^{n+1})$. With this it is straightforward to write the formulas for the t's and the d_i's in the general case.

proof of 4.6.1: From 4.5.5 we get a projection (restriction to the $B^{(n)}$'s in fact) $\pi : \mathcal{C}_c^\infty(\mathcal{G})^\natural \longrightarrow (\mathcal{A}\rtimes\mathcal{G})^\natural$ which is compatible with the t's and the d_i's. It is enough to prove that π is a quasi-isomorphism between both the b and the b' complexes. Denote $C_* = \mathcal{C}_c^\infty(\mathcal{G})^\natural, A_* = Ker(\pi)$; so $(\mathcal{A}\rtimes\mathcal{G})^\natural = C_*/A_*$. We prove that $((\mathcal{A}\rtimes\mathcal{G})^\natural, b'), (C_*, b')$ and (A_*, b) are acyclic. For the two b' complexes this is standard. Indeed, for the first complex we use the degeneracy s_n to get a contraction. For the second complex we use local units to get local contractions; more precisely, if $u \in C_n$ has $b'u = 0$, then $u = b'v$ where $v = u \otimes \varphi \in C_{n+1}$, $\varphi \in \mathcal{C}_c^\infty(\mathcal{G}^{(0)})$ such that $\varphi = 1$ around the compact $K = \{s(\gamma_n) : u(\gamma_0, \ldots, \gamma_n) \neq 0\}$ in $\mathcal{G}^{(0)}$.

We are left with (A_*, b). We will construct two double complexes $\{C_{p,q}; p \geq -1, q \geq 0\}, \{A_{p,q}; p \geq -1, q \geq 0\}$ with $C_{-1,*} = C_*, A_{-1,*} = A_*$ and such that all the rows $A_{*,q}, q \geq 0$ and all the columns $A_{p,*}, p \geq 0$ are acyclic. Of course, it is enough to construct $A_{*,*}$ with these properties; the only role of $C_{*,*}$ is to facilitate the definition of $A_{*,*}$. For this we put for $p, q \geq 0$:

$$C_{p,q} = \mathcal{C}_r^\infty(\mathcal{G}^{q+2} \times \mathcal{G}^p),\ \ A_{p,q} = \{u \in C_{p,q} : u|_{\mathcal{G}^{(q+2)}\times\mathcal{G}^p} = 0\},$$

and $C_{-1,*} = C_*, A_{-1,*} = A_*$. The boundaries are defined as follows:

The boundaries for the column $p = constant$ of $C_{*,*}$ are defined by analogy with $(C_*, b')\widehat{\otimes}C_r^\infty(\mathcal{G}^p)$ i. e. :

$$\ldots \xrightarrow{d^c} C_{p,1} \xrightarrow{d^c} C_{p,0}, \quad d^c = \sum(-1)^i d_i^c,$$

$$d_i^c(u_0, ..., u_{q+1}; v_1, ..., v_p) = (u_0, ..., u_i u_{i+1}, ..., u_{q+1}; v_1, ..., v_p),\ 0 \le i \le q.$$

We keep the same formulas to define the column $p = constant$ of $A_{*,*}$. To see that $A_{p,*}$ is acyclic for $p \ge 0$ remark that $(C_{p,*}/A_{p,*}, b')$ and $(C_{p,*}, b')$ are acyclic; this can be viewed in the same way as for $(C_*/A_*, b')$ and (C_*, b').

The boundaries for the row $q = constant$ of $C_{*,*}$ are defined by analogy with the complex computing $HH_*(\mathcal{C}_c^\infty(\mathcal{G}), C_c^\infty(\mathcal{G}^{q+2}))$ together with an augmentation; more precisely, define them by:

$$\dots \xrightarrow{d^r} C_{1,q} \xrightarrow{d^r} C_{0,q} \xrightarrow{d^r_{(-1)}} C_{-1,q} = C_q, \quad d^r = \sum_{i=0}^{p} (-1)^i d_i^r,$$

$$d_i^r(u_0, ..., u_{q+1}; v_1, ..., v_p) = \begin{cases} (u_0, ..., u_q, u_{q+1}v_1, v_2, ..., v_p) & \text{if } i = 0 \\ (u_0, ..., u_{q+1}; v_1, .., v_i v_{i+1}, ..., v_p) & \text{if } 0 \le i \le p-1 \\ (v_p u_0, ..., u_{q+1}; v_1, ..., v_{p-1}) & \text{if } i = p \end{cases},$$

$$d^r_{(-1)}(u_0, ..., u_{q+1}) = (u_{q+1}u_0, ..., u_q).$$

The same formulas define the row $q = constant$ of $A_{*,*}$. To prove that $A_{*,q}$ is acyclic, assume for simplicity of notation that $q = 0$. For any $\varphi \in C_c^\infty(\mathcal{G}^{(0)})$, define a degree 1 linear map of $C_{*,0}$ by:

$$h_\varphi : C_{p,0} \longrightarrow C_{p+1,0} \qquad h_\varphi(u_0, u_1; v_1, ..., v_p) = (u_0, \varphi; \varphi u_1, v_1, ..., v_p), \quad p \ge 0$$

$$h_\varphi : C_{-1,0} \longrightarrow C_{0,0} \qquad h_\varphi(u_0) = (\varphi u_0, \varphi).$$

For any $u \in C_{p,0} = C_c^\infty(\mathcal{G}^{p+2})$ we have the naive formulas (correct in the Hausdorff case):

$$(d^r h_\varphi u)(\gamma_0, ..., \gamma_{p+1}) = \varphi^2(t(\gamma_1))u(\gamma_0, ..., \gamma_{p+1}) + \varphi(\gamma_1)\varphi(t(\gamma_2))(d^r u)(\gamma_0, \gamma_2, ..., \gamma_{p+1}),$$

which can be written in general:

$$(d^r h_\varphi u)(\gamma_0, ..., \gamma_{p+1}) = (1 \otimes germ_{t(\gamma_1)}(\varphi^2) \otimes 1 \otimes ... \otimes 1) * u(\gamma_0, ..., \gamma_{p+1}) +$$
$$+ i^{(2)}_{germ_{\gamma_1}(\varphi)}((1 \otimes germ_{t(\gamma_2)}(\varphi) \otimes 1 \otimes ... \otimes 1) * (d^r u)(\gamma_0, \gamma_2, ..., \gamma_{p+1})).$$

Here $i_v^{(2)}$ denotes "inserting v on the second place" and "$*$" is the stalkwise product. In general we do not have: $h_\varphi(A_{p,0}) \subset A_{p+1,0}$. Fix a metric ρ defining the topology of $\mathcal{G}^{(0)}$. Take $u \in A_{p,0}$ to be a cycle. From 2.5.8 we can choose $\epsilon > 0$ such that:

$$\rho(s(\gamma_0), t(\gamma_1)) < \epsilon \Rightarrow u(\gamma_0, \gamma_1, ..., \gamma_p) = 0.$$

For any $\varphi \in C_c^\infty(\mathcal{G}^{(0)})$ with $diam(supp\varphi) < \epsilon/6$ we see that $h_\varphi(u) \in A_{p+1,0}$. Now $u = d^r v$ is the boundary of $v = \sum h_{\varphi_i} u$ where $\varphi_i \in C_c^\infty(\mathcal{G}^{(0)})$ is a finite set of functions as before such that $\sum \varphi_i^2 = 1$ around the compact $K = \{t(\gamma_1) : u(\gamma_0, ..., \gamma_n) \neq 0\} \subseteq \mathcal{G}^{(0)}$. $\square$

The following four theorems are generalizations of the computations performed by Burghelea for groups (Theorem I in [22]). In the case of smooth Hausdorff étale groupoids, the computation of the elliptic components was done by Brylinski and Nistor: Theorem 5.6 in [20] computes these components in terms of homology of some double complexes. Our Theorem 4.6.4 is an extension of that result, in a slightly more precise form. Emphasize also that our proof of Theorem 4.6.4, in contrast to the one given in [20], will make use of the quasi-isomorphism ensured by 1.5.14 just for $\varphi = id$.

From 4.5.9 we get (see also Proposition 3.3 in [20]):

Theorem 4.6.2 *(localization): For $\mathcal{O} \subset B^{(0)}$ a $\mathcal{G}$-invariant subset, the localized homologies at $\mathcal{O}$ are defined. We have linear maps $HH_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} \longrightarrow HH_*(\mathcal{C}_c^\infty(\mathcal{G}))$ if $\mathcal{O}$ is open in $B^{(0)}$, and $HH_*(\mathcal{C}_c^\infty(\mathcal{G})) \longrightarrow HH_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ if $\mathcal{O}$ is closed in $B^{(0)}$, and the same applies to HC_*, HP_*.*

If $B^{(0)} = \bigcup \mathcal{O}$ is a $\mathcal{G}$-invariant disjoint open covering of $B^{(0)}$, then:

$$HH_*(\mathcal{C}_c^\infty(\mathcal{G})) = \bigoplus_{\mathcal{O}} HH_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}},\ HC_*(\mathcal{C}_c^\infty(\mathcal{G})) = \bigoplus_{\mathcal{O}} HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}},$$

and the analogue for HP_*; moreover, everything is compatible with the SBI-sequences.

Theorem 4.6.3 *(localization at units): For any smooth étale groupoid $\mathcal{G}$:*

$$HP_0(\mathcal{C}_c^\infty(\mathcal{G}))_{(1)} = \prod_{k-even} H_k(\mathcal{G};\mathbb{C}),$$

$$HP_1(\mathcal{C}_c^\infty(\mathcal{G}))_{(1)} = \prod_{k-odd} H_k(\mathcal{G};\mathbb{C}).$$

proof: First of all, using Lemma 1.5.12 we get a quasi-isomorphism of sheaves on $\mathcal{G}^{(0)}$:

$$(\mathcal{A}^\natural, b) \xrightarrow{\sim} (\Omega^*_{\mathcal{G}^{(0)}}, 0).$$

It is well known [28] that it makes the B-boundary compatible with de Rham boundary. This, Proposition 4.6.1, Theorem 4.5.11 for $\mathcal{O} = \mathcal{G}^{(0)}$ and Lemma 2.2.2 give the proof. □

Theorem 4.6.4 *(elliptic case): For any smooth étale groupoid $\mathcal{G}$, and any elliptic $\mathcal{O} \subset B^{(0)}$:*

$$HP_0(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \prod_{k-even} H_k(\mathcal{N}_{\mathcal{O}};\mathbb{C}) = \prod_{k-even} H_k(\mathcal{Z}_{\mathcal{O}};\mathbb{C}),$$

$$HP_1(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \prod_{k-odd} H_k(\mathcal{N}_{\mathcal{O}};\mathbb{C}) = \prod_{k-odd} H_k(\mathcal{Z}_{\mathcal{O}};\mathbb{C}).$$

(for the precise definition of the centralizer $\mathcal{Z}_{\mathcal{O}}$ and the normalizer $\mathcal{N}_{\mathcal{O}}$ of $\mathcal{O}$, see 4.5.8).

proof: From Proposition 4.6.1 and Theorem 4.5.11 the groups $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ are isomorphic to $HP_*(\mathcal{N}_{\mathcal{O}}; \mathcal{A}^\natural_{(\mathcal{O})})$. From the description of $\mathcal{A}^\natural_{(\mathcal{O})}$ (see 4.5.8) it is not difficult to see that $\mathcal{A}^\natural_{(\mathcal{O})} = (\mathcal{C}_V^\infty)^\natural_{|\mathcal{O}}$ where V is an open neighborhood of $\mathcal{O}$ in $B^{(0)}$ which is a submanifold of $\mathcal{G}$ (the existence of V is ensured by the fact that $\mathcal{O}$ is elliptic). Using 1.5.12, and the fact that $0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0_M|_A \longrightarrow \Omega^0_M|_A \longrightarrow \ldots$ is a c-soft resolution of $\mathbb{C} \in Sh(A)$ for any manifold M and any $A \subset M$ [63] we get, as in the previous proof, the relations expressing HP_* in terms of the homology of $\mathcal{N}_{\mathcal{O}}$. The passage to $\mathcal{Z}_{\mathcal{O}}$ is ensured by the Hochschild-Serre spectral sequence 2.3.4 applied to the projection map $\varphi : \mathcal{Z}_{\mathcal{O}} \longrightarrow \mathcal{N}_{\mathcal{O}}$ (it degenerates since $\gamma/\varphi \simeq <\gamma>$ is a finite cyclic group for all $\gamma \in \mathcal{O}$). □

Remark that this theorem, together with Theorem 4.6.2, ensures us that for any elliptic $\mathcal{O}$, such that $\mathcal{O}$ is closed in $B^{(0)}$, there is a localized Connes-Chern-Karoubi character:

$$Ch_{\mathcal{O}} : K_*^{alg}(\mathcal{C}_c^\infty(\mathcal{G})) \longrightarrow H_*(\mathcal{Z}_{\mathcal{O}};\mathbb{C}).$$

Given a loop $\gamma \in B^{(0)}$, we call it stable if its germ $\tilde{\gamma}; (\mathcal{G}^{(0)}, s(\gamma)) \longrightarrow (\mathcal{G}^{(0)}, s(\gamma))$ (as defined in 1.1.2) is stable (in the sense of 1.5.13). Given $\mathcal{O} \subset B^{(0)}$ we say that $\mathcal{O}$ *is stable* if it is $\mathcal{G}$-invariant, and every $\gamma \in \mathcal{O}$ is stable.

Theorem 4.6.5 *(hyperbolic case): For any smooth étale groupoid $\mathcal{G}$, and any hyperbolic $\mathcal{O} \subset B^{(0)}$, $HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ is a module over $H^*(\mathcal{N}_{\mathcal{O}}; \mathbb{C})$ and the S in the SBI-sequence is the (cap-) product by the Euler class $e_{\mathcal{O}} \in H^2(\mathcal{N}_{\mathcal{O}})$.*

Moreover, if $\mathcal{O}$ is stable:

$$HH_n(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \bigoplus_{p+q=n} H_p(\mathcal{Z}_{\mathcal{O}}; \Omega^q_{B^{(0)}}|_{\mathcal{O}}),$$

$$HC_n(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \bigoplus_{p+q=n} H_p(\mathcal{N}_{\mathcal{O}}; \Omega^q_{B^{(0)}}|_{\mathcal{O}}),$$

and the SBI-sequence is the Gysin sequence (of Proposition 4.3.4) for $\mathcal{Z}_{\mathcal{O}} \longrightarrow \mathcal{N}_{\mathcal{O}}$.

Remark that if $\mathcal{O}$ is stable, there is a neighborhood V of $\mathcal{O}$ in $B^{(0)}$ which is a submanifold of $\mathcal{G}$. In particular $\Omega_V^* \in Sh(V)$ makes sense. The notation $\Omega^*_{B^{(0)}}|_{\mathcal{O}}$ should be understood as the restriction of Ω_V^* to $\mathcal{O}$; when $\mathcal{O}$ is open in $B^{(0)}$, this is simply $\Omega^*_{\mathcal{O}}$.

proof: The first part is a consequence of Proposition 4.6.1 and Theorem 4.5.12. We are left with the last two equalities. We prove the second one (the first can be proved in the same way or as a consequence of 1.5.14, 4.6.1 , 4.5.7). The proof is an improvement of the second spectral sequence of Proposition 4.4.8 for the cyclic category $(\mathcal{Z}_{\mathcal{O}}, \theta)$. There we used the Hochschild-Serre spectral sequence of $\pi_2' : \Lambda_\infty \wedge \mathcal{Z}_{\mathcal{O}} \longrightarrow \mathcal{N}_{\mathcal{O}}$ and the strong deformation retract $\gamma/\pi_2' \xleftarrow{\sim} \Lambda_\infty \wedge <\gamma>= \Lambda_\infty$ for all $\gamma \in \mathcal{O}$. Assume for simplicity that $\mathcal{O}$ is open in $B^{(0)}$. We have from 4.5.9:

$$HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = H_*(\Lambda_\infty \wedge \mathcal{Z}_{\mathcal{O}}; \mathcal{A}^\natural_{\mathcal{O}}).$$

Due to the SBI trick (see 4.4.3) and 1.5.14, we can replace $\mathcal{A}^\natural_{\mathcal{O}}$ by the θ-cyclic $\mathcal{Z}_{\mathcal{O}}$-sheaf $\mathcal{B}^\natural = (\mathcal{C}^\infty_{\mathcal{O}})^\natural$. This is in fact a cyclic $\mathcal{N}_{\mathcal{O}}$-sheaf. We use (cf the proof of Theorem 2.3.4) that the spectral sequence for π_2' comes from an equality:

$$H_*(\Lambda_\infty \wedge \mathcal{Z}_{\mathcal{O}}; \mathcal{B}^\natural) = \mathbb{H}_*(\mathcal{N}_{\mathcal{O}}; (\mathcal{L}\pi_2')_! \mathcal{B}^\natural),$$

where $\mathbb{H}_q((\mathcal{L}\pi_2')_! \mathcal{B}^\natural) \in Sh(\mathcal{N}_{\mathcal{O}})$ has the stalk at $\gamma \in \mathcal{O}$ (cf. Proposition 2.3.3 and Lemma 2.2.15):

$$H_q(\gamma/\pi_2'; \mathcal{B}^\natural) = H_q(\Lambda_\infty; \mathcal{B}^\natural_\gamma)$$

We get in this way a quasi-isomorphism of complexes of sheaves on $\mathcal{O}$: $(\mathcal{L}\pi_2')_! \mathcal{B}^\natural \simeq (\mathcal{B}^\natural, b)$, b =the Hochschild boundary. This is in $Sh(\mathcal{N}_{\mathcal{O}})$ so:

$$HC_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \mathbb{H}_*(\mathcal{N}_{\mathcal{O}}; (\mathcal{B}^\natural, b)).$$

Using Lemma 1.5.12 once again, $(\mathcal{B}^\natural, b) \simeq (\Omega^*_{\mathcal{O}}, 0)$ and the second equality follows. $\square$

4.6.6 Corollary *(compare to [83]): If $\mathcal{O}$ is hyperbolic and $\mathcal{N}_{\mathcal{O}}$ has finite cohomological dimension, then $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = 0$.*

4.7 The case of cohomology

In this section we indicate how the methods of the previous sections can be dualized, giving the computation of the cyclic cohomology associated to the convolution algebra.

4.7.1 Dual sheaves: Let M be a space, $\mathcal{A} \in Sh(M)$ c-soft. The correspondence

$$U \mapsto \Gamma_c(U; \mathcal{A})^\vee$$

(recall that "$\vee$" stands for the algebraic dual) defines a sheaf on M, denoted $\mathcal{A}^\vee$; it is flabby (even injective; see [63]). This is well known in the Hausdorff case and carries over to the general case ([35]). The construction has the property that for any $f : M \longrightarrow N$ continuous, and for any c-soft sheaf $\mathcal{A} \in Sh(M)$:

$$f_!(\mathcal{A})^\vee = f_*(\mathcal{A}^\vee).$$

When applied to sheaves on an étale category, it gives a correspondence:

$$Sh(\mathcal{G}) \ni \mathcal{A}\ ,\ c-\text{soft}\ \mapsto \mathcal{A}^\vee \in Sh(\mathcal{G}^{op})\ ,\ \mathcal{G}-\text{acyclic},$$

and $B_*(\mathcal{G}; \mathcal{A})^\vee = B^*(\mathcal{G}; \mathcal{A}^\vee)$.

4.7.2 Currents: Let M be a manifold (not necessarily Hausdorff) of dimension n. There is an obvious notion of q-currents: $\Omega_q(M) = \{u : \Omega^q_c(M) \longrightarrow \mathbb{C} : u\,|_U \in \Omega_q(U)$ for all local coordinates charts $U\}$ ($\Omega_q(U)$ has the usual meaning if U is Hausdorff). We get the sheaf of q-currents on M which is a kind of topological dual of the sheaf Ω^q_M; we denote it by $(\Omega^q_M)'$. In general we have two different resolutions of the (complex) orientation sheaf [15, 63]:

$$or_M \longrightarrow (\Omega^n_M)' \longrightarrow (\Omega^{n-1}_M)' \longrightarrow \ldots$$

and

$$or_M \longrightarrow (\Omega^n_M)^\vee \longrightarrow (\Omega^{n-1}_M)^\vee \longrightarrow \ldots\ ,$$

and an obvious "forgetting continuity" map of complexes in $Sh(M)$:

$$((\Omega^*_M)', d'_{dRh}) \longrightarrow ((\Omega^*_M)^\vee, d^\vee_{dRh}).$$

Applying to this the global sections functor, we get two cohomologies: de Rham cohomology with coefficients in or_M (or, equivalently, closed de Rham homology) and the sheaf cohomology of or_M, together with a linear map between them:

$$H^*_{dRh}(M; or_M) \longrightarrow H^*(M; or_M).$$

In the Hausdorff case, all sheaves $(\Omega_M)'$ and $(\Omega_M)^\vee$ are soft and the map above is an isomorphism; in the non-Hausdorff case, $(\Omega_M)'$ may fail to be acyclic for cohomology and the two cohomologies are not isomorphic in general. The same discussion carries over to the case of étale groupoids (see [55]).

4.7.3 Definition: *For a smooth étale groupoid $\mathcal{G}$, define $HH^*(\mathcal{C}^\infty_c(\mathcal{G}))$, $HC^*(\mathcal{C}^\infty_c(\mathcal{G}))$, $HP^*(\mathcal{C}^\infty_c(\mathcal{G}))$ as the cohomologies of $(\mathcal{C}^\infty_c(\mathcal{G})^\natural)^\vee$.*

Emphasize that these are not the algebraic cohomologies of $\mathcal{C}_c^\infty(\mathcal{G})$ since in the definition of $\mathcal{C}_c^\infty(\mathcal{G})^\natural$ we used the topology. It is possible also to use $(\mathcal{C}_c^\infty(\mathcal{G})^\natural)'$ for defining the cyclic cohomologies. Because of the facts explained in 4.7.2 they give the same *periodic* cyclic cohomologies in the case of Hausdorff groupoids. In the general case the difference between them is the same as the difference between the two cohomologies described in 4.7.2.

Remark that we chose the maximal definition such that we keep the pairing with the cyclic homology and such that it is a receptacle for "Chern character" maps ([29, 85]). And, as we shall see, it is "computable".

Theorem 4.7.4 : *For any smooth étale groupoid $\mathcal{G}$, and any elliptic open $\mathcal{O} \subset B^{(0)}$ which is a topological manifold (not necessarily Hausdorff) of dimension q:*

$$HP^0(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \bigoplus_{k-even} H^{k+q}(\mathcal{N}_{\mathcal{O}}; or_{\mathcal{O}}),$$

$$HP^1(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = \bigoplus_{k-odd} H^{k+q}(\mathcal{N}_{\mathcal{O}}; or_{\mathcal{O}}),$$

and the pairing with $HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}$ is in fact the Poincaré-duality pairing (see 2.4.13).

In particular, the Connes pairing $HP^* \times HP_* \longrightarrow \mathbb{C}$ induces an inclusion:

$$HP^*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} \hookrightarrow HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}^\vee.$$

proof: We explain how to make the analogy with the case of homology (and get more of the lemmas we need as consequences of that case). The analogue of 4.6.1 we get for free (as a consequence). The analogy with the sub-paragraph 2.4 is ensured by 1.2.13 and 4.7.1. To see this, keep the notations from 4.5.4, 4.5.6. Since our $\mathcal{A} = \mathcal{C}^\infty_{\mathcal{G}^{(0)}}$ is c-soft and since $(\mathcal{A} \rtimes \mathcal{G})^\natural(n)$ is isomorphic to $\Gamma_c(\Omega(\mathcal{G}); \mathcal{A}^\natural_{tw})$ (see the isomorphism in the proof of 4.5.7) we get

$$((\mathcal{A} \rtimes \mathcal{G})^\natural(n))^\vee \cong \Gamma_c(\Omega(\mathcal{G}); (\mathcal{A}^\natural_{tw})^\vee)$$

and each $(\mathcal{A}^\natural_{tw})^\vee$ is injective (cf. 4.7.1). From this point on, the analogy is ensured by 1.2.13 and the isomorphism in 4.7.1. The analogue of 1.5.14 we also get for free (just by dualizing) and we get down to the resolutions of the orientation sheaf given by the duals of the sheaves of forms. This also introduces the shifting in the result. The identification with the Poincaré pairing is straightforward. $\Box$

We leave the statement for the hyperbolic case to the reader. As an obvious (but interesting) consequence we have:

4.7.5 Corollary: *Let $\mathcal{O} \subset B^{(0)}$ be $\mathcal{G}$-invariant such that it is a disjoint union of manifolds. Assume that $\mathcal{N}_{\mathcal{O}}$ has finite cohomological dimension.Then:*

1. *if $\mathcal{O}$ is elliptic: $HP^*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} \simeq HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}}^\vee$ (induced by the Connes pairing),*
2. *if $\mathcal{O}$ is hyperbolic: $HP^*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = HP_*(\mathcal{C}_c^\infty(\mathcal{G}))_{\mathcal{O}} = 0$.*

4.8 The case of foliations

Let $(M, \mathcal{F})$ be a foliated manifold. The holonomy groupoid (see 1.1.3 and 1.3.3), $Hol(M, \mathcal{F})$ is a smooth groupoid; when restricted to a complete transversal T it becomes an étale groupoid $Hol_T(M, \mathcal{F})$. As proved in [60], the choice of T is not important when we talk about the associated non-commutative spaces (i. e. the associated C^*-algebras). The case of their smooth convolution algebras seems to be slightly more difficult. However we have:

Theorem 4.8.1 : *For any foliated manifold $(M, \mathcal{F})$ the correspondences:*

$$T \mapsto HH_*, HC_*, HP_*, HH^*, HC^*, HP^*(\mathcal{C}_c^\infty(Hol_T(M, \mathcal{F})))$$

do not depend on the choice of the complete transversal T; so they give well defined invariants of the ("leaf space" of the) foliation.

proof: Let T, T' be two complete transversals; $\mathcal{G} = Hol_T(M, \mathcal{F}), \mathcal{N}$ and $\mathcal{B} = \mathcal{A}^\natural_{B^{(0)}}$ be the constructions from 4.5.6 for $\mathcal{G}$; the similar constructions for T' are denoted by $\mathcal{G}', \mathcal{N}', \mathcal{B}'$. Replacing T by $T \coprod T'$ if needed, we can assume $T \subset T'$. We have continuous functors $\mathcal{G} \longrightarrow \mathcal{G}', \mathcal{N} \longrightarrow \mathcal{N}'$ which are Morita equivalences. Since $HH_*(\mathcal{C}_c^\infty(\mathcal{G})) = \mathbb{H}_*(\mathcal{N}; (\mathcal{B}, b))$ (cf 4.6.1, 4.5.7, 4.4.3) and the analogue for $\mathcal{G}'$, it is enough to use Morita invariance for homology and an SBI argument. $\square$

Since in practice one (sometimes) has to deal with the cyclic homology of the (full) holonomy groupoid $Hol(M, \mathcal{F})$, the following isomorphism (and the explicit formulas) is relevant:

4.8.2 Theorem: *For any foliation $(M, \mathcal{F})$, and any complete transversal T, there is a "canonical" isomorphism:*

$$HC_*(C_c^\infty(Hol(M, \mathcal{F}))) \cong HC_*(C_c^\infty(Hol_T(M, \mathcal{F})))$$

(and similarly for Hochschild and periodic cyclic).

We emphasize that "canonical" is part of the theorem and refers to the fact that the isomorphism does not depend on the additional choices we make (e.g. partitions of unity).

4.8.3 At units: For the localization at units we get (with $\mathcal{G} = Hol(M, \mathcal{F})$):

$$HP^*(C_c^\infty(\mathcal{G}))_{(1)} \cong \bigoplus_{k \equiv q + * mod 2} H^k(M/\mathcal{F}; or).$$

This is a common point between two different approaches to model the leaf-space $M/\mathcal{F}$ as a generalized space: one in the spirit of non-commutative geometry and one in the spirit of Grothendieck, by looking at all the sheaves on $Hol(M; \mathcal{F})$ (see [72]). In particular, the right side is the cohomology of the orientation sheaf of M inside the category of sheaves of complex vector spaces on $Hol(M, \mathcal{F})$ (hence, by 1.2.7, we get the connection with classifying space).

The dual result is:

$$HP_*(C_c^\infty(\mathcal{G}))_{(1)} \cong H_c^*(M/\mathcal{F})$$

(where the grading on the right is the even/odd one). This is relevant in the longitudinal index theorem for foliations, in the process of describing the Chern character for convolutions algebras. Using the general Chern character in cyclic homology (see 1.5.2 or 4.9.1 below), the localization result (Theorem 4.6.2), and the computation at units (Theorem 4.6.3), one gets, if $\mathcal{G}$ is Hausdorff (so that one can restrict compact supports on $\mathcal{G}$ to compact supports on M), a Chern character localized at units:

$$Ch^1 : K_*(C_c^\infty(\mathcal{G})) \longrightarrow H_c^*(M/\mathcal{F}) \ .$$

Passing to basic forms (via the map j_b in (2.6) or of 4.9.3 below), we get a (weaker) basic Chern character that [59] asks for.

The the next section will be devoted to the study of the basic Chern character (as well as its extensions to loops other then units), and to describe explicit formulas for it. The remaining of this section is devoted to the proof of Theorem 4.8.2 above. In order to explain/prove the theorem, as well as to construct an explicit isomorphism, we briefly recall the definition of the convolution algebra associated to the full holonomy groupoid, and establish two preliminary lemmas. We restrict here to the Hausdorff case (since our arguments are local, one can extend them to the non-Hausdorff setting using the same ideas we have already used).

4.8.4 Convolution algebras: To define the (smooth) convolution algebra of a Lie groupoid $\mathcal{G}$, one uses the convolution product:

$$(\phi * \psi)(g) = \int_{g_1 g_2 = g} \phi(g_1)\psi(g_2), \quad x \xleftarrow{g} y \ . \tag{4.1}$$

If $\mathcal{G}$ is étale, then the integration is simply summation, but, in general, one has to give a precise meaning to the integration in the previous formula. For this, one has some choices to make. If one wants to work with functions $\phi, \psi \in C_c^\infty(\mathcal{G})$, then one has to fix a smooth Haar system for $\mathcal{G}$ (sometimes called a tangential measure) (we refer to [92] for more precise definitions). Instead, one can use a line bundle $\mathcal{L}$ of "densities" which is isomorphic to the trivial bundle (in a non-canonical way), and to work with compactly supported smooth sections of $\mathcal{L}$, $\phi, \psi \in C_c^\infty(\mathcal{G}; \mathcal{L})$. Fixing a trivialisation of $\mathcal{L}$ induces a Haar system on $\mathcal{G}$, and gives an isomorphism $C_c^\infty(\mathcal{G}; \mathcal{L}) \cong C_c^\infty(\mathcal{G})$.

Let me recall Connes' choice of $\mathcal{L}$ for the case where $\mathcal{G}$ is the holonomy groupoid of a foliation $(M, \mathcal{F})$ (in general, one has to replace the bundle $\mathcal{F}$ by the Lie algebroid of $\mathcal{G}$). Denote by $\mathcal{D}^{1/2}$ the line bundle on M consisting on half-densities along the leaves (i.e. the fiber at $x \in M$ consists of maps ρ from the exterior power $\Lambda^p \mathcal{F}_x$, $p = dim\mathcal{F}$, to $\mathbf{C}$ such that $\rho(\lambda v) = |\lambda|^{1/2}\rho(v)$ for all $\lambda \in \mathbb{R}$, $v \in \Lambda^p \mathcal{F}_x$), and similarly $\mathcal{D}^r$ for any r; the bundle of densities ($r = 1$) is usually denoted by $\mathcal{D}$. We put $\mathcal{L} = s^*\mathcal{D}^{1/2} \otimes t^*\mathcal{D}^{1/2}$. Then (4.1) makes sense for $\phi, \psi \in C_c^\infty(\mathcal{G}; \mathcal{L})$. Indeed, looking at the variable $g_2 = h$, one has to integrate $\phi(gh^{-1})\psi(h) \in \mathcal{D}_x^{1/2} \otimes \mathcal{D}_z \otimes \mathcal{D}_y^{1/2}$ with respect to $z \xleftarrow{h} y$ varying in $\mathcal{G}_y = s^{-1}(y)$. But $\mathcal{D}_z$ is (canonically isomorphic) to the fiber at h of the bundle of densities on the manifold $\mathcal{G}_y$, hence the integration makes sense and gives an element

$(\phi * \psi)(g) \in \mathcal{D}_x^{1/2} \otimes \mathcal{D}_y^{1/2} = \mathcal{L}_g$. Later we will come back at the choice of $\mathcal{L}$.

In the sequel we will omit $\mathcal{L}$ from the notation $C_c^{\infty}(\mathcal{G}; \mathcal{L})$. One may think that we have choosen a trivialisation of $\mathcal{D}^{1/2}$ (hence a Haar system for $\mathcal{G}$, hence a product on the true $C_c^{\infty}(\mathcal{G})$, and an algebra isomorphism $C_c^{\infty}(\mathcal{G}; \mathcal{L}) \cong C_c^{\infty}(\mathcal{G})$).

The following is a version of the preliminary lemma of [60].

4.8.5 A lemma on foliations: *Let $(M, \mathcal{F})$ be a codimension q foliation, and let p be its dimension. There exists an open locally finite covering $\mathcal{U} = \{U_i : 1 \leq i \leq \infty\}$ of M, together with foliation charts $\varphi_i : I_2^p \times I_2^q \cong U_i$, such that, denoting $\varphi_i(I_1^p \times I_1^q)$ by V_i, one has:*

(1) $\bar{V}_i \cap \bar{V}_j = \emptyset$, $\forall i \neq j$,

(2) each leaf of $\mathcal{F}$ intersects at least one V_i.

Moreover, we can choose $\lambda_i \in C_c^{\infty}(U_i)$ such that $\{\lambda_i^2\}$ is a partition of unity, and $\lambda_i|_{V_i} = 1$ for all i's.
(recall that $I_r^n = \{x \in \mathbb{R}^n : ||x|| < r\}$).

4.8.6 A lemma on cyclic homology: The usual Morita invariance for cyclic homology says that the inclusion $i : A \longrightarrow M_{\infty}(A)$ of a unital algebra A as the upper-left corner of its matrix algebra induces isomorphism i_* in Hochschild and cyclic homology/cohomology. We will have to deal with homomorphisms of type:

$$i^{\lambda} : A \longrightarrow M_{\infty}(A), \ i^{\lambda}(a) = (\lambda_i a \lambda_j)_{1 \leq i,j \leq \infty}, \tag{4.2}$$

and the induced maps in cyclic homology. In practice, $\lambda = \{\lambda_i\}_{1 \leq i \leq \infty}$ is a family of elements of an algebra A_0 which acts on A both from the left and from the right. We have to require that

$$\{i : \lambda_i x \neq 0\}, \quad \{i : x\lambda_i \neq 0\}$$

are finite for any element x of A_0 or of A. In particular i^{λ} is well defined, and, in order to have an algebra homomorphism, we also require $\sum_{i=1}^{\infty} \lambda_i^2 x = \sum_{i=1}^{\infty} x\lambda_i^2 = x$ for any x as above. If these conditions are satisfied, we say that $\{\lambda_i^2\}_{1 \leq i \leq \infty}$ is a partition of unity of A_0 relative to A. For instance, if A_0 is unital, and λ is finite (i.e. $\lambda_i = 0$ for i greater then a fixed n), then there is just one condition: $\sum_1^n \lambda_i^2 = 1$.

4.8.7 Lemma: *Let A be an algebra. If A_0 is an algebra which acts on A both from the left and from the right, and if $\lambda = \{\lambda_i\}_{1 \leq i \leq \infty}$ is a family of elements of A_0 with the property that $\{\lambda_i^2\}_{1 \leq i \leq \infty}$ is a partition of unity of A_0 relative to A, then the algebra homomorphism (4.2) induces an isomorphism in periodic cyclic homology/cohomology:*

$$i_*^{\lambda} : HP_*(A) \xrightarrow{\sim} HP_*(M_{\infty}(A)) \ ,$$

which is independent of A_0 or λ (hence coincides with i_*).

The same is true also for cyclic and Hochschild homology/cohomology, provided A is H-unital.

proof: We first assume that A_0 is a subalgebra of A, and the left/right actions of A_0 are the canonical ones. In this case the conditions imply that A is H-unital (the elements $\sum_1^n \lambda_i^2$, $n \geq 0$ are local units), hence it suffices to work on the usual Hochschild complex $C_*(A) = A^{\otimes(*+1)}$ with:

$$b(a^0, a^1, \ldots, a^n) = \sum_{i=0}^{n-1}(-1)^i(a^0, \ldots, a^i a^{i+1}, \ldots, a^n) + (-1)^n(a^n a^0, a^1, \ldots, a^{n-1}) .$$

Considering the trace-map $Tr_* : C_*(M_\infty(A)) \longrightarrow C_*(A)$,

$$Tr_*(a^0, a^1, \ldots, a^n) = \sum_{i_0, \ldots, i_n} (a^0_{i_0 i_1}, a^1_{i_1 i_2}, \ldots, a^n_{i_n i_0}), \ a^i \in M_\infty(A),$$

it suffices to show that $Tr_* i^\lambda_* : C_*(A) \longrightarrow C_*(A)$ is homotopic to the identity (it is well known that, at the level of homology, Tr_* is an isomorphism whose inverse is i_*). For this, we construct the homotopy:

$$\begin{aligned} h(a^0, a^1, \ldots, a^n) &= \sum_{i_0, \ldots, i_n} (a^0 \lambda_{i_0}, \lambda_{i_0} a^1 \lambda_{i_1}, \ldots, \lambda_{i_{n-1}} a^n \lambda_{i_n}, \lambda_{i_n}) - \\ &- \sum_{i_0, \ldots, i_{n-1}} (a^0 \lambda_{i_0}, \lambda_{i_0} a^1 \lambda_{i_1}, \ldots, \lambda_{i_{n-2}} a^{n-1} \lambda_{i_{n-1}}, \lambda_{i_{n-1}}, a^n) + \ldots + \\ &+ (-1)^{n-1} \sum_{i_0, i_1} (a^0 \lambda_{i_0}, \lambda_{i_0} a^1 \lambda_{i_1}, \lambda_{i_1}, a^2, \ldots, a^n) + \\ &+ (-1)^n \sum_{i_0} (a^0 \lambda_{i_0}, \lambda_{i_0}, a^1, \ldots, a^n). \end{aligned}$$

In the general case, we use the new algebra $A_0 \rtimes A$ which is $A_0 \oplus A$ with the product:

$$(\lambda, a)(\eta, b) = (\lambda\eta, \lambda b + a\eta + ab), \ \lambda, \eta \in A_0, a, b \in A,$$

Remark that $A_0 \rtimes A$ contains A_0 as a subalgebra (with the inclusion $\rho(\lambda) = (\lambda, 0)$), and the map i^λ lifts to a map between short-exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "i^\lambda"] & A_0 \rtimes A \arrow[r, "\pi"] \arrow[d, "i^\lambda"] & A_0 \arrow[r] \arrow[d, "i^\lambda"] & 0 \\ 0 \arrow[r] & M_\infty(A) \arrow[r, "\tilde{i}"] & M_\infty(A_0 \rtimes A) \arrow[r, "\tilde{\pi}"] & M_\infty(A_0) \arrow[r] & 0 \end{tikzcd}$$

($i(a) = (0, a)$, $\pi(\lambda, a) = \lambda$, and $\tilde{i}, \tilde{\pi}$ are induced by i, π).

Note that ρ is an algebra splitting of π. By the previous discussion, the statement is true for $A_0 \rtimes A$ and A_0; to deduce it for A, it suffices to use Cuntz-Quillen's excision theorem for periodic cyclic homology, or, in the H-unital case, Wodzicki's excision for Hochschild/cyclic homology. □

Proof of Theorem 4.8.2: With the notations from 4.8.5, we :

(i) consider the transversals $T_i = \phi_i(\{0\} \times I_2^q) \subset U_i$, $S_i = \phi_i(\{0\} \times I_1^q) \subset V_i$, and the complete transversal $T = \coprod_i T_i$, $S = \coprod_i S_i$;

(ii) consider the groupoids $\mathcal{G}_\mathcal{U}$, $\mathcal{G}_\mathcal{V}$ (see 1.1.3), and their convolution algebras $C_c^\infty(\mathcal{G}_\mathcal{U})$, $C_c^\infty(\mathcal{G}_\mathcal{V})$ (defined in analogy with $C_c^\infty(\mathcal{G})$). Hence

$$C_c^\infty(\mathcal{G}_\mathcal{U}) = \oplus_{i,j} C_c^\infty(\mathcal{G}_{U_i}^{U_j})\sigma_{j,i}$$

where $\mathcal{G}_{U_i}^{U_j} \subset \mathcal{G}^{(1)}$ stands for the space of arrows from points of U_i to points in U_j, and $\sigma_{i,j}$ are just formal symbols used to emphasize that $C_c^\infty(\mathcal{G}_{U_i}^{U_j}) \subset C_c^\infty(\mathcal{G})$ is considered as a subspace of $C_c^\infty(\mathcal{G}_\mathcal{U})$. The product in $C_c^\infty(\mathcal{G}_\mathcal{U})$ is simply

$$(\phi\sigma_{i,j})(\psi\sigma_{j,k}) = (\phi\psi\sigma_{i,k}) \ ,$$

and $(\phi\sigma_{i,j})(\psi\sigma_{j',k}) = 0$ for $j \neq j'$.

Moreover, due to the triviality of $\mathcal{F}$ on each U_i, one has $\mathcal{G}_\mathcal{U} \cong \mathcal{G}_T^T \times R$ (and $\mathcal{G}_\mathcal{V} \cong \mathcal{G}_S^S \times R'$) where R is the groupoid $(I_2^p \times I_2^p)$ over I_2^p (and similarly R' with I_1^p instead of I_2^p). The convolution algebra $\mathcal{R}$ of R consists of compactly supported smooth functions on $k(x,y)$ on $I_2^p \times I_2^p$, with the product:

$$(k_1k_2)(x,z) = \int k_1(x,y)k_2(y,z)dy \ .$$

A similar discussion applies to the convolution algebra $\mathcal{R}' \subset \mathcal{R}$ of R'.

(iii) choose u such that:

$$u \in C_c^\infty(I_1^p), \quad \int u(x)^2 dx = 1 \ ,$$

and we put $\alpha := u \otimes u \in \mathcal{R}' \subset \mathcal{R}$. Denoting by $j_\alpha : C_c^\infty(\mathcal{G}_T^T) \longrightarrow C_c^\infty(\mathcal{G}_T^T) \otimes \mathcal{R}$ the algebra homomorphism $\phi \mapsto \phi \otimes \alpha$, and by $j : C_c^\infty(\mathcal{G}_\mathcal{V}) \hookrightarrow C_c^\infty(\mathcal{G}_\mathcal{U})$, and $j_0 : C_c^\infty(\mathcal{G}_S^S) \hookrightarrow C_c^\infty(\mathcal{G}_T^T)$ the obvious inclusions, we have a commutative diagram:

$$\begin{tikzcd} C_c^\infty(\mathcal{G}_\mathcal{V}) \cong C_c^\infty(\mathcal{G}_S^S) \otimes \mathcal{R}' \arrow[d, "j"'] & C_c^\infty(\mathcal{G}_S^S) \arrow[l, "j_\alpha"'] \arrow[d, "j_0"'] \\ C_c^\infty(\mathcal{G}_\mathcal{U}) \cong C_c^\infty(\mathcal{G}_T^T) \otimes \mathcal{R} & C_c^\infty(\mathcal{G}_T^T) \arrow[l, "j_\alpha"'] \end{tikzcd}$$

(iv) construct the algebra homomorphism:

$$\lambda : C_c^\infty(\mathcal{G}) \longrightarrow C_c^\infty(\mathcal{G}_\mathcal{U}), \ \lambda(\phi) = \sum_{i,j} (\lambda_i * \phi * \lambda_j)\sigma_{i,j},$$

where we are using the left/right action of $A_0 = C_c^\infty(\mathcal{G}^{(0)})$ on $C_c^\infty(\mathcal{G})$: for $f \in A_0$, $\phi \in C_c^\infty(\mathcal{G})$, and for any arrow $x \xleftarrow{\gamma} y$ in $\mathcal{G}$,

$$f * \phi\,(\gamma) = f(x)\phi(\gamma), \quad \phi * f\,(\gamma) = \phi(\gamma)f(y) \ .$$

The obvious extension by zero $e : C_c^\infty(\mathcal{G}_\mathcal{V}) \longrightarrow C_c^\infty(\mathcal{G})$, $\sum_{i,j} \phi_{i,j}\sigma_{i,j} \overset{e}{\mapsto} \sum_{i,j} \phi_{i,j}$, is an algebra homomorphism (by condition (1) of 4.8.5), and, since $\lambda_i|_{V_i} = 1$, one has a commutative diagram:

$$\begin{tikzcd} & C_c^\infty(\mathcal{G}_\mathcal{V}) \arrow[dl, "e"'] \arrow[d, "j"] \\ C_c^\infty(\mathcal{G}) \arrow[r, "\lambda"] & C_c^\infty(\mathcal{G}_\mathcal{U}) \end{tikzcd}$$

We will show that all the maps in the previous two diagrams induce isomorphism in periodic cyclic homology. For j_0 it is a result of [34] (since $\mathcal{G}_T^T$ and $\mathcal{G}_S^S$ are étale groupoids). Let's now prove that, for any étale groupoid $\mathcal{T}$,

$$j_\alpha : C_c^\infty(\mathcal{T}) \longrightarrow C_c^\infty(\mathcal{T}) \otimes \mathcal{R}, \ j_\alpha(a) = a \otimes \alpha$$

induces an isomorphism in the cohomology with respect to b and b' boundaries. For this we construct the chain map:

$$\tau_* : C_*(C_c^\infty(\mathcal{T}) \otimes \mathcal{R}) \longrightarrow C_*(C_c^\infty(\mathcal{T})),$$

$$\tau_*(a^0 \otimes k^0, \dots, a^n \otimes k^n) := \tau(k^0 \dots k^n)(a^0, \dots, a^n),$$

where τ is the usual trace:

$$\tau : \mathcal{R} \longrightarrow \mathbf{C}, \ \tau(k) = \int k(x,x)dx.$$

Since $\tau_* j_\alpha = Id$, it suffices to show that

$$j_\alpha \tau_* : C_*(C_c^\infty(\mathcal{T}) \otimes \mathcal{R}) \longrightarrow C_*(C_c^\infty(\mathcal{T}) \otimes \mathcal{R}),$$

$$(a^0 \otimes k^0, \dots, a^n \otimes k^n) \mapsto \tau(k^0 \dots k^n)(a^0 \otimes \alpha, \dots a^n \otimes \alpha)$$

induces Id in our homologies. Let us first assume that $\mathcal{T}^{(0)}$ is compact. We then have the following homotopy:

$$h(a^0 \otimes x^0 \otimes y^0, a^1 \otimes x^1 \otimes y^1, \ \dots, a^n \otimes x^n \otimes y^n) = \sum_{k=0}^{n} (-1)^k \tau(y^0 \otimes x^1)\tau(y^1 \otimes x^2) \dots \tau(y^{k-1} \otimes x^k)$$

$$(a^0 \otimes x^0 \otimes u, a^1 \otimes u \otimes u, \ \dots, a^k \otimes u \otimes u, 1 \otimes u \otimes y^k, a^{k+1} \otimes x^{k+1} \otimes y^{k+1}, \ \dots, a^n \otimes x^n \otimes y^n)$$

for all $a^i \in C_c^\infty(\mathcal{T})$, $x^i \otimes y^i \in \mathcal{R}$ (it is straightforward to write the corresponding formula for the general elements in $C_c^\infty(\mathcal{T} \times I_2^p \times I_2^p)$). When $\mathcal{T}^{(0)}$ is not compact, we have to replace the unit $1 \in C_c^\infty(\mathcal{T}^{(0)}) \subset C_c^\infty(\mathcal{T})$ by local units (compactly supported smooth functions on $\mathcal{T}^{(0)}$, which are constantly 1 inside compacts which exhaust $\mathcal{T}^{(0)}$).

This shows that the maps j, j_0, j_α (on the diagram of (iii)) induce isomorphism in our cyclic homologies. Hence, using also the commutative diagram in (iv), it follows that the maps induced by λ in homologies are surjective. On the other hand, one has an obvious inclusion $i : C_c^\infty(\mathcal{G}_\mathcal{U}) \hookrightarrow M_\infty(C_c^\infty(\mathcal{G}))$ which sends $\phi\sigma_{i,j}$ to the matrix $\phi e_{i,j}$ (ϕ on the position (i,j)). By Lemma 4.8.7 (we refer to [37] for the proof that $C_c^\infty(\mathcal{G})$ is H-unital), the composition $i\lambda : C_c^\infty(\mathcal{G}) \longrightarrow \mathcal{M}_\infty(C_c^\infty(\mathcal{G}))$ induces an isomorphisms. In particular, the maps induced by λ are also injective, and this shows that all the maps in our diagrams induce isomorphisms. □

4.8.8 Remark: Note that the isomorphism does not really depend on the choices we made. For instance it does not depend on the partition $\{\lambda_i\}$ (since the map e is defined without the use of this partition), and it does not depend on the choice of the covering $\mathcal{U}$. As before, a map $C_*(C_c^\infty(\mathcal{G})) \longrightarrow C_*(C_c^\infty(\mathcal{G}_T^T))$ can be defined for any covering $\mathcal{U}$ by foliation charts and any subordinate partition; the map induced in cohomology is an isomorphism which does not depend on these choices, since any such cover has a refinement as in Lemma 4.8.5.

4.8.9 More general convolution algebras: Connected to the choice of the line bundle $\mathcal{L}$ in 4.8.4 so that the integration (4.1) makes sense for $\phi, \psi \in C_c^\infty(\mathcal{G}; \mathcal{L})$, let us remark that there are various other choices. For instance one could take $\mathcal{L} =$

$s^*\mathcal{D}^r \otimes t^*\mathcal{D}^{1-r}$ for any r. A very general type of convolution algebras is suggested in [88] and they inevitably appear when considering longitudinal operators acting on vector bundles (e.g. as in [59]). For any bundle E over $\mathcal{G}^{(0)} = M$, the convolution product obviously makes sense on:

$$C_c^\infty(\mathcal{G}; \mathcal{L} \otimes End_{\mathcal{G}}(E)), \quad \mathcal{L} = s^*\mathcal{D}^{1/2} \otimes t^*\mathcal{D}^{1/2},$$

where $End_{\mathcal{G}}(E) = End(s^*E, t^*E)$. For instance, $s^*\mathcal{D}^r \otimes t^*\mathcal{D}^{1-r} = \mathcal{L} \otimes End_{\mathcal{G}}(\mathcal{D}^{r-1/2})$. Remark that our proof easily extends (one just has to assume that $\mathcal{U}$ trivializes also E, and to replace $\mathcal{R} = C_c^\infty(\mathbb{R}^p \times \mathbb{R}^p)$ by $C_c^\infty(\mathbb{R}^p \times \mathbb{R}^p, End(\mathbb{R}^k)) = M_k(\mathcal{R})$, $k = dim(E)$). Hence:

4.8.10 Corollary: *For any vector bundle E on M:*

$$HC_*(C_c^\infty(\mathcal{G}; \mathcal{L} \otimes End_{\mathcal{G}}(E))) \cong HC_*(C_c^\infty(\mathcal{G}_T^T)) \ .$$

4.8.11 More general Lie groupoids: Of course, since the cyclic homology for étale groupoids is now quite well understood, one would like to apply our results to all Lie groupoids which are equivalent to an étale one, and to extend Theorem 4.8.2. This is possible (the proof of 4.8.2 will apply with minor changes) , provided we make use of the description of this type of groupoids given in the recent [37]. Let me mention here that [37] proves that, for a Lie groupoid $\mathcal{G}$, the following assertions are equivalent:

(i) $\mathcal{G}$ is Morita equivalent to an étale groupoid;

(ii) the anchor map of $\mathcal{G}$ is injective (i.e. $\mathcal{G}$ integrates a foliation $\mathcal{F}$ on $M = \mathcal{G}^{(0)}$);

(iii) the isotropy groups $\mathcal{G}_x^x$ are discrete.

Moreover, any $x_0 \in M$ has a neighborhood U in M so that $\mathcal{F}|_U$ is defined by a submersion $\pi : U \longrightarrow T$ and, around x_0, $\mathcal{G}$ is diffeomorphic to the groupoid $U \times_T U$ (i.e. there is a diffeomorphism ϕ from $U \times_T U$ into an open neighborhood $\mathcal{U}$ of id_{x_0} in $\mathcal{G}^{(1)}$, so that $\phi(x, y)$ is an arrow from y to x, $\phi(x, y)^{-1} = \phi(y, x)$, $\phi(x, x) = id_x$, $\phi(x, y)\phi(y, z) = \phi(x, z)$).

4.9 The basic Chern character for leaf spaces

Using the localization at units of the Chern character (see 4.8.3), and the canonical map $j_b : H_c^*(M/\mathcal{F}) \longrightarrow H_{c,bas}^*(M/\mathcal{F})$ (see also 4.9.3 below), we get the basic Chern character at units:

$$Ch_{bas}^1 : K_*(C_c^\infty(\mathcal{G})) \longrightarrow H_{c,bas}^*(M/\mathcal{F}) \ , \quad \mathcal{G} = Hol(M/\mathcal{F}). \tag{4.3}$$

This provides the link between two different approaches to longitudinal index theorems for foliations: the non-commutative approach [29], and the extension of Bismut's methods to leaf spaces [59]. We mention that the the basic Chern character of the analytical index [29] coincides with the Bismut-type Chern character of the index bundle (defined in [59]). This follows formally from the topological interpretations (given by the index theorems) of these analytical invariants; of course, one expects a direct proof of this connection. Motivated by this, the aim of this section is to give explicit descriptions of Ch_{bas}^1 (as a partial Chern character associated to a geometric cycle), and to extend it to more general loops. We first briefly review the construction of partial Chern characters, their connections with the Chern character in cyclic homology, and the construction of compactly supported cohomologies associated to groupoids. We mention that for defining Ch^1_{bas} the Hausdorffness assumption on the holonomy groupoid is essential because we need to restrict compact supports from $\mathcal{G}$ to M (for this we need M to be closed on $\mathcal{G}$, and this is equivalent to the Hausdorffness of $\mathcal{G}$). Note however that the basic Chern characters can be defined also in the non-Hausdorff case, provided one restricts to other *closed* spaces of loops in $\mathcal{G}$.

4.9.1 Cyclic homology, and the Chern-character: Let us recall that, given an algebra A, its cyclic homology is computed by the double complex $\mathcal{B}(A)$:

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] & {} \arrow[d, "b"'] \\
\Omega^3(A) \arrow[d, "b"'] & \Omega^2(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^2(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] \\
\Omega^2(A) \arrow[d, "b"'] & \Omega^1(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] & \\
\Omega^1(A) \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] & & \\
\Omega^0(A) & & &
\end{tikzcd}$$

Here, $(\Omega^*(A), d)$ is the algebra of non-commutative differential forms on A. Recall that it consists of (finite) sums of elements of type $\tilde{a}_0 da_1 \ldots da_n$, with $\tilde{a}_0 \in \tilde{A}$, $a_1, \ldots, a_n \in A$, and $d(\omega\eta) = d(\omega)\eta + (-1)^{deg(\omega)}\omega d(\eta)$ for all $\omega, \eta \in \Omega^*(A)$. The operators b, B are given by $b(\omega da) = (-1)^{deg(\omega)}[\omega, a]$, $B = (1 + k + \ldots + k^n)d$ on $\Omega^n(A)$, where k is the Karoubi operator $k(\omega da) = (-1)^{deg(\omega)} da\omega$. Extending the double complex $\mathcal{B}(A)$ to the left by periodicity, and forming the total complex using products along the diagonals (rather than sums), one gets a periodic complex which we denote $Tot^{per}(\mathcal{B}(A))$, and which defines the periodic cyclic homology $HP_*(A)$. The Chern character on K_0:

$$Ch : K_0(A) \longrightarrow HP_0(A) \tag{4.4}$$

is defined as follows: if $E \in M_\infty(\tilde{A})$ is an idempotent, we form $Ch(E)_0 = Tr(E) \in \tilde{A}$ and

$$Ch(E)_{2n} = (-1)^n \frac{(2n)!}{n!} Tr(E - \frac{1}{2})(dE)^{2n} \in \Omega^{2n}(A),\ n \geq 1$$

If $E, F \in M_\infty(\tilde{A})$ are two idempotents congruent modulo $M_\infty(A)$, then the Chern character of $[E] - [F] \in K_0(A)$ is represented by the cyclic cycle $Ch(E) - Ch(F) \in Tot^{per}(\mathcal{B}(A))$.

4.9.2 Partial Chern-characters: Recall [28, 29] that an *n-dimensional cycle* over A is a triple (Ω, d, τ) together with an algebra morphism $\rho : A \longrightarrow \Omega^0$, where (Ω, d) is a differential graded algebra, and $\tau : \Omega^n \longrightarrow \mathbf{C}$ is a trace (i.e. vanishes on the graded commutators $[\omega, \eta] = \omega\eta - (-1)^{deg(\omega)\deg(\eta)}\eta\omega)$ which is closed (i.e. $\tau(d\omega) = 0$ for all $\omega \in \Omega^{n-1}$).

More generally, given a cochain complex $\mathcal{C}$, *a $\mathcal{C}$-valued cycle on A* is a triple (Ω, d, τ) and a morphism ρ as before, except that τ is this time a $\mathcal{C}$-valued closed trace on Ω, i.e. $\tau : \Omega \longrightarrow \mathcal{C}$ is a chain map vanishing on all graded commutators. Using the even/odd grading on $H^*(\mathcal{C})$, any such $\mathcal{C}$-valued cycle induces a partial Chern character:

$$Ch_\tau : K_0(A) \longrightarrow H^*(\mathcal{C})$$

that we briefly describe. Let me assume A to be unital for the moment (in general, one adjoins a unit). If $\mathcal{E}$ is a finitely generated projective module over A, one can always choose a Ω-connection on $\mathcal{E}$, i.e. $\nabla : \mathcal{E} \longrightarrow \Omega^1 \otimes_A \mathcal{E}$ satisfying $\nabla(ax) = d(a)x + a\nabla(x)$ for all $a \in A, x \in \mathcal{E}$ (here we use the notation $d(a) = d(\rho(a)) \in \Omega^1$ for $a \in A$); we can extend such a connection to $\nabla : \Omega^* \otimes_A \mathcal{E} \longrightarrow \Omega^{*+1} \otimes_A \mathcal{E}$ satisfying the usual identity $\nabla(\omega\eta) = d(\omega)\eta + (-1)^{\deg(\omega)}\omega\nabla(\eta)$ for all $\omega \in \Omega, \eta \in \Omega \otimes_A \mathcal{E}$. From this it follows that the curvature ∇^2 is Ω-linear:

$$\nabla^2 \in End_\Omega(\Omega \otimes_A \mathcal{E}) \ .$$

Since E is finitely generated projective, it follows that so is the Ω-module $\Omega \otimes_A \mathcal{E}$, hence there is a well defined (graded) trace $Tr : End_\Omega(\Omega \otimes_A \mathcal{E}) \longrightarrow \Omega_\sharp$ where $\Omega_\sharp = \Omega/[\Omega, \Omega]$. Composing with τ, one gets a new trace:

$$Tr_\tau : End_\Omega(\Omega \otimes_A \mathcal{E}) \longrightarrow \mathcal{C} \ .$$

In general:

$$Tr_\tau(e^{-\nabla^2}) \in \mathcal{C}^*$$

is a cocycle, whose cohomology class does not depend on the choice of ∇, and represents the Chern character Ch_τ of $\mathcal{E}$.

This type of partial Chern characters are just an instance of the full Chern character in cyclic homology. Let me describe this relation, which can be taken as a definition of Ch_τ (the advantage of the previous construction is the freedom of representing cocycles representing $Ch_\tau(\mathcal{E})$). Any $\mathcal{C}$-valued cycle as above induces a map:

$$j_\tau(a_0 da_1 \ldots da_n) = \frac{1}{n!}\tau(\rho(a_0)d\rho(a_1)\ldots d\rho(a_n))$$

from the bicomplex $\mathcal{B}(A)$ to the bicomplex

$$\begin{tikzcd}
{} \arrow[d, "0"'] & {} \arrow[d, "0"'] & {} \arrow[d, "0"'] & {} \arrow[d, "0"'] \\
\mathcal{C}^3 & \mathcal{C}^2 \arrow[l, "d"'] \arrow[d, "0"'] & \mathcal{C}^2 \arrow[l, "d"'] \arrow[d, "0"'] & \mathcal{C}^0 \arrow[l, "d"'] \\
\mathcal{C}^2 \arrow[from=u, "0"'] & \mathcal{C}^1 \arrow[l, "d"'] \arrow[d, "0"'] & \mathcal{C}^0 \arrow[l, "d"'] & \\
\mathcal{C}^1 \arrow[from=u, "0"'] \arrow[d, "0"'] & \mathcal{C}^0 \arrow[l, "d"'] & & \\
\mathcal{C}^0 & & &
\end{tikzcd}$$

The partial Chern character Ch_τ is then the composition of the Chern character (4.4) with the map induced by j_τ,

$$j_\tau : HP_*(A) \longrightarrow H^*(\mathcal{C}) \ . \tag{4.5}$$

To see this, we may assume that A is unital; if $\mathcal{E} = A^n E$ is defined by an idempotent E, there is a preferred connection ∇_E on $\mathcal{E}$ coming form the decomposition $A^n = A^n E \oplus A^n(1-E)$, and $Ch_\tau(E) = Tr_\tau(e^{-\nabla_E^2})$ is easily seen to be equal (as cocycles of $\mathcal{C}$) with $j_\tau(Ch(E))$.

Sometimes, when constructing cycles over A, one first constructs triples (Ω, d, τ) with all the properties above except for $d^2 = 0$. Instead, one has $d^2 = ad_\theta$ for some $\theta \in \Omega^2$ with $d(\theta) = 0$. Even more, in the non-unital case $\theta = (l, r)$ is rather a multiplier of Ω as explained in [87] (i.e. a pair of linear maps $l, r : \Omega \longrightarrow \Omega$ satisfying $l(ab) = l(a)b$, $r(ab) = ar(b)$, and $al(b) = r(a)b$ for any $a, b \in \Omega$), and the previous conditions on d and θ read $d^2 = l - r$, $[d, l] = [d, r] = 0$. Such a triple (Ω, d, τ) is called a generalized $\mathcal{C}$-valued cycle over A. As explained by Connes, starting with a generalized cycle one can produce an ordinary cycle (see [29], page 229), hence one has an induced map j_τ as in (4.5) and an induced (partial) Chern character. One can also give explicit formulas for the map (4.5) associated to a generalized $\mathcal{C}$-valued cycle (dualize the formulas in [51]).

4.9.3 Cohomology theories for étale groupoids (overview): As we have already seen, one of the important properties that distinguishes étale groupoids from general ones is that any arrow $g : x \longrightarrow y$ induces a (canonical) germ $\tilde{g} : (U, x) \xrightarrow{\sim} (V, y)$ from a neighborhood U of x in $\mathcal{G}^{(0)}$ to a neighborhood V of y (see 1.1.2). Using the differential of these germs, we get a (right) action of $\mathcal{G}$ on the sheaf of p-forms $\Omega^p \in Sh(\mathcal{G}^{(0)})$: for $x \xleftarrow{g} y$ in $\mathcal{G}$, the differential of $\tilde{g}$ induces the action between the stalks, denoted:

$$\Omega^p_x \longrightarrow \Omega^p_y, \quad a \mapsto a^g \tag{4.6}$$

Moreover, the stalk at $x \xleftarrow{g} y$ of the sheaf of p-forms on $\mathcal{G}^{(1)}$ is isomorphic to Ω^p_x (via the differential of the target map), hence the forms $\omega \in \Omega^p(\mathcal{G}^{(1)})$ can be viewed as functions:

$$\mathcal{G}^{(1)} \ni g \mapsto \omega(g) \in \Omega^p_x, \quad x = t(g) \tag{4.7}$$

(which are continuous with respect to the sheaf topology). This way of representing the forms on $\mathcal{G}^{(1)}$ is useful for writing down explicit formulas; we call it *the germ representation* of forms.

One of the classical constructions which extends from manifolds to étale groupoids (and is of interest in this section) is the compactly supported cohomology (introduced in Chapter 2 as a homology theory), and its basic version. The complex $\Omega^*_{bas}(\mathcal{G}^{(0)})$ of basic (or invariant) differential forms on $\mathcal{G}^{(0)}$ is the subcomplex of $\Omega^*(\mathcal{G}^{(0)})$ consisting on forms ω with the property that $s^*(\omega) = t^*(\omega) \in \Omega^*(\mathcal{G}^{(1)})$. Hence, denoting $\delta^* = t^* - s^*$, $\Omega^*_{bas}(\mathcal{G}^{(0)})$ is defined by the exact sequence $0 \longrightarrow \Omega^*_{bas}(\mathcal{G}^{(0)}) \longrightarrow \Omega^*(\mathcal{G}^{(0)}) \xrightarrow{\delta^*} \Omega^*(\mathcal{G}^{(1)})$; using the germ representation (4.7) of forms on $\mathcal{G}^{(1)}$, one has $\delta^*(\omega)(g) = \omega(x) - \omega(y)^{g^{-1}} \in \Omega^*_x$ for $x \xleftarrow{g} y$. One defines the basic cohomology $H^*_{bas}(\mathcal{G})$ as the cohomology of this complex.

Dual to $H^*_{bas}(\mathcal{G})$ is the compactly supported basic cohomology $H^*_{c,bas}(\mathcal{G})$; the corresponding complex $\Omega^*_{c,bas}(\mathcal{G}^{(0)})$ is defined this time as a quotient of $\Omega^*_c(\mathcal{G}^{(0)})$, by means of the exact sequence:

$$\Omega^*_c(\mathcal{G}^{(1)}) \xrightarrow{\delta} \Omega^*_c(\mathcal{G}^{(0)}) \longrightarrow \Omega^*_{c,bas}(\mathcal{G}^{(0)}) \longrightarrow 0. \tag{4.8}$$

Here $\delta = \int_t - \int_s$, where $\int_t$ stands for integration along the fibers of t. Using again the germ representation (4.7), the formula is:

$$\delta(\omega)(x) = \sum_{t(g)=x} \omega(g) - \sum_{s(g)=x} \omega(g)^g \in \Omega^*_x.$$

To define the full compactly supported cohomology $H^*_c(\mathcal{G})$, one has to use the double complex $\{\Omega^p_c(\mathcal{G}^{(q)}) : p, q \geq 0\}$ so that $\Omega^*_c(\mathcal{G}^{(q)})$ gives a continuation to the left of the sequence of complexes (4.8). Hence it is endowed with the DeRham differential along p, and the simplicial boundary δ along q: $\delta = \sum (-1)^i \delta_i$ where $\delta_i = \int_{d_i} : \Omega^p_c(\mathcal{G}^{(q)}) \longrightarrow \Omega^p_c(\mathcal{G}^{(q-1)})$ is the summation along the fibers of the simplicial maps d_i (see 1.1.13). To get a cochain complex, we consider the indexing $B^{p,-q}(\mathcal{G}) = \Omega^p_c(\mathcal{G}^{(q)})$, and denote by $H^*_c(\mathcal{G})$ its cohomology. There is an obvious map:

$$j_b : H^*_c(\mathcal{G}) \longrightarrow H^*_{c,bas}(\mathcal{G}) \tag{4.9}$$

which is not an isomorphism in general. These new groups $H^*_c(\mathcal{G})$ are wilder then $H^*_{c,bas}(\mathcal{G})$, but they have much nicer properties (which we have already used in this chapter): apart from Morita invariance, they also satisfy a Leray spectral sequence, they can be defined for arbitrary sheaves on $\mathcal{G}$, and they extend Poincaré duality to étale groupoids (and, as we have seen, they are the localization at units of $HP_*(C^\infty_c(\mathcal{G}))$).

Let me recall (cf. 2.2.8) that the connecting homomorphism (4.9) is an isomorphism in the case of étale groupoids representing orbifolds (i.e. proper groupoids, cf. our preliminaries).

4.9.4 Back to foliations: According to the general strategy, when looking at leaf spaces, $H^*_c(M/\mathcal{F})$ and $H^*_{c,bas}(M/\mathcal{F})$ are defined as the cohomologies of the reduced holonomy groupoid. By the Morita invariance of these cohomologies, the definition does not depend on the choice of a transversal. When representing the leaf space by the reduced holonomy groupoid $Hol_T(M, \mathcal{F})$ associated to a transversal T, we denote by $\Omega^*_{c,bas}(T/\mathcal{F})$ the corresponding basic complex. Point out that the basic groups $H^*_{c,bas}(M/\mathcal{F})$ are well known in foliation theory (and have been introduced by Haefliger in his study of foliations by minimal leaves), and the basic complex $\Omega^*_{c,bas}(T/\mathcal{F})$ represented on a complete transversal T coincides with the complex $\Omega^*_c(Tr(\mathcal{F}))$ of [56].

When $\mathcal{F}$ is oriented, there are integration along the leaves maps (see 3.2.6, and 2.2.7). The basic version of this integration comes from a chain map defined by Haefliger [56] $\int_{\mathcal{F}} : \Omega^*_c(M) \longrightarrow \Omega^{*-p}_{c,bas}(M/\mathcal{F})$. For reasons which will become clear, we need to consider integration of densities, hence a version of Haefliger's integration over leaves:

$$\int_{\mathcal{F}} : C^\infty_c(M; \mathcal{D} \otimes \Lambda^k \nu^*) \longrightarrow \Omega^k_{c,bas}(T/\mathcal{F}), \quad k \geq 0 \tag{4.10}$$

which makes sense in general (recall that $\mathcal{D}$ denotes the line bundle of 1-densities along the leaves). If $T = \coprod_i T_i$ is the complete transversal associated to a cover $\mathcal{U} = \{U_i\}$ as in the previous section (or just a good cover in the sense of [56]), and $\eta_i \in C_c^\infty(U_i)$ is a partition of unity (with our choice of $\mathcal{U}$ as in Lemma 4.8.5, $\eta_i = \lambda_i^2$), then $\int_{\mathcal{F}}(\omega) := \sum_i \int_{\pi_i} \eta_i \omega$ for all $\omega \in \Omega_c^*(M)$, where $\int_{\pi_i} : \Omega_c^*(U_i) \longrightarrow \Omega_c^*(T_i)$ is the integration along the fibers of $\pi_i : U_i \longrightarrow T_i$ (the definition does not depend on our choices).

4.9.5 The basic Chern character for leaf spaces: Using the Chern character in cyclic cohomology, the reduction to the étale setting (Theorem 4.8.2), the restriction at units res_1, the computation of the localization at units (Theorem 4.6.3), and the map into basic cohomology (i.e. (4.9) applied to the reduced holonomy groupoid), we get a basic Chern character at units defined as the composition:

$$K_*(C_c^\infty(\mathcal{G})) \xrightarrow{Ch} HP_*(C_c^\infty(\mathcal{G})) \cong HP_*(C_c^\infty(\mathcal{G}_T^T)) \xrightarrow{res_1}$$
$$\xrightarrow{res_1} HP_*(C_c^\infty(\mathcal{G}_T^T))_{(1)} \cong H_c^*(M/\mathcal{F}) \xrightarrow{j_b} H_{c,bas}^*(M/\mathcal{F}) .$$

We know give a more explicit description of Ch_{bas}^1 as a partial Chern character associated to a cycle. Although the description is more effective by reducing first to a complete transversal, let me first indicate a global description, at the level of the full holonomy groupoid $\mathcal{G}$. As explained by Connes (see $III.7.\alpha$ in [29]), one can define such cycles by making use of the transversal derivation d_H on $C_c^\infty(\mathcal{L} \oplus t^*\Lambda^*\nu^*)$ (recall that $\mathcal{L} = s^*\mathcal{D}^{1/2} \oplus t^*\mathcal{D}^{1/2}$ as in 4.8.4). For this we need to choose a horizontal distribution H, i.e. a vector bundle $H \subset TM$ such that $TM = \mathcal{F} \oplus H$. Of course, one can get rid of the density bundles and have a canonical H. The typical situation is (under the assumption that $\mathcal{F}$ is orientable) when one fixes an orientation on $\mathcal{F}$, a Riemannian metric on M , and $H = \mathcal{F}^\perp$. Even in this situation (when we have a specified isomorphism $\mathcal{D} \cong \Lambda^p\mathcal{F}^* \cong \mathbf{C}$), it is still convenient to work with densities; otherwise one has to twist the "usual formulas" by the volume form $v_{\mathcal{F}} \in \mathcal{C}_c^\infty(M, \Lambda^p\mathcal{F}^*)$. For instance, Haefliger's integration $(C_c^\infty(M, \Lambda^*\nu^*), d_H) \longrightarrow (\Omega_{c,bas}^*(T/\mathcal{F}), d)$, $\omega \longrightarrow \int_{\mathcal{F}} \omega \wedge v_{\mathcal{F}}$ is not compatible with the differentials, and to correct this, one should replace d_H by $\tilde{d}_H(\omega) = d_H(\omega) + \frac{d_H(v_{\mathcal{F}})}{v_{\mathcal{F}}}\omega$ (see also below).

Hence, let H be a horizontal distribution. One has $\Lambda^k(T^*M) = \oplus_{r+s=k}\Lambda^r(\mathcal{F}^*) \otimes \Lambda^s(\nu^*)$. For a form ω on M of type (r,s) (i.e. $\omega \in C^\infty(M, \Lambda^r(\mathcal{F}^*) \otimes \Lambda^s(\nu^*))$, its De Rham differential is of type $d_{DRh}(\omega) = d_L(\omega) + d_H(\omega) + d_{0,2}(\omega)$, where $d_L(\omega)$ is of type $(r+1,s)$, $d_H(\omega)$ is of type $(r, s+1)$, and $d_{0,2}(\omega)$ is of type $(r, s+2)$. Unlike d_L (which is just the leafwise differential),

$$d_H : C^\infty(M, E \otimes \Lambda^s\nu^*) \longrightarrow C^\infty(M, E \otimes \Lambda^{s+1}\nu^*), \quad E = \Lambda^r\mathcal{F}^*$$

depends on H, and d_H^2 is not zero unless H is integrable. One extends d_H to density bundles (i.e. $E = \mathcal{D}^r$ above) by the Leibniz rule and the requirement

$$d_H(\rho) = r\ \rho\ \frac{d_H\omega}{\omega}, \quad \text{for } \rho = |\omega|^r,\ \omega \in C^\infty(M, \Lambda^p\mathcal{F}) \tag{4.11}$$

(any r-density can be written at least locally as $f|\omega|^r$). Moreover, d_H lifts at the holonomy groupoid $\mathcal{G}$. To explain this lifting, the first step is to describe $d_H : C^\infty(\mathcal{G}) \longrightarrow C^\infty(\mathcal{G}, t^*\nu^*)$. It comes from DeRham differential on $\mathcal{G}$ and the restriction $T^*\mathcal{G} \longrightarrow t^*\nu^*$ induced by H. The inclusion $\tilde{i}_H : t^*\nu \longrightarrow T(\mathcal{G})$ inducing this restriction can be described fiberwise, using the inclusion $i_H : \nu \longrightarrow TM$; at $\gamma : x \longrightarrow y$, $\tilde{i}_H$ is defined by the commutative diagram:

$$\begin{tikzcd}
(t^*\nu)_\gamma = \nu_x \arrow[r, "\tilde{i}_H"] \arrow[d, "(\gamma,id)"'] & T_\gamma(\mathcal{G}) \arrow[d, "(ds,dt)"] \\
\nu_x \times \nu_y \arrow[r, "i_H\times i_H"] & T_xM \times T_yM
\end{tikzcd}$$

Again, keeping the Leibniz rule and the formula (4.11) for densities, there is a unique extension of d_H to a derivation:

$$d_H : C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*) \longrightarrow C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*) \ .$$

The algebra structure on $C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*)$ is given by the convolution product (one extends the definition in 4.8.4). Note that, although it is not an algebra homomorphism, the restriction $C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*) \longrightarrow C_c^\infty(M, \mathcal{D}\otimes\Lambda\nu^*)$ is compatible with the differentials. Hence one gets in particular a chain map:

$$\tau_{bas} : C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*) \longrightarrow \Omega^*_{c,bas}(T/\mathcal{F}) \ .$$

4.9.6 Theorem: *The triple* $(C_c^\infty(\mathcal{G}; \mathcal{L}\otimes t^*\Lambda\nu^*), d_H, \tau_{bas})$ *is a* $\Omega^*_{c,bas}(T/\mathcal{F})$*-valued generalized cycle over the convolution algebra* $C_c^\infty(\mathcal{G}; \mathcal{L})$*, whose partial Chern character coincides with the basic Chern character* Ch^1_{bas}*.*

4.9.7 The basic Chern character for étale groupoids: Before proving the theorem, let me first restrict to complete transversals. There the transversal differentiation becomes canonical, and provides us with a (non-generalized) cycles. It also gives a better understanding of Ch^1_{bas}, helps to prove the theorem above, and gives more freedom in choosing cocycles representing Ch^1_{bas}. So let Γ be an étale groupoid. As above, using the computation of $HP_*(C_c^\infty(\Gamma))$ at units, and the map (4.9) one gets a $H^*_{c,bas}(\Gamma)$-valued basic Chern character for Γ. Associated to the groupoid Γ there is a DG algebra:

$$\Omega^*_c(\Gamma): \quad \Omega^0_c(\Gamma^{(1)}) \xrightarrow{d} \Omega^1_c(\Gamma^{(1)}) \xrightarrow{d} \Omega^0_c(\Gamma^{(2)}) \xrightarrow{d} \ \ldots \ ,$$

where d is DeRham differential, and the product on $\Omega^*_c(\Gamma)$ is the convolution product; hence, using the germ representation (4.7) for forms on $\Gamma^{(1)}$,

$$(\omega\eta)(g) = \sum_{g_1g_2=g} \omega(g_1)\omega(g_2)^{g_1^{-1}} \ .$$

4.9.8 Theorem: *Let* Γ *be a (Hausdorff) étale groupoid, and let* $\tau_{bas} : \Omega^*_c(\Gamma) \longrightarrow \Omega^*_{c,bas}(\Gamma^{(0)})$ *be the map induced by the restriction of forms from* $\Gamma^{(1)}$ *to* $\Gamma^{(0)}$*. Then:*

(i) The triple $(\Omega^*_c(\Gamma), d, \tau_{bas})$ *is a* $\Omega^*_{c,bas}(\Gamma^{(0)})$*-valued cycle over* $C_c^\infty(\Gamma)$*;*

(ii) The associated partial Chern character coincides with the basic Chern character Ch^1_{bas}*.*

Proofs: Let me sketch the proofs of theorems 4.9.6 and 4.9.8. We start with 4.9.8. To prove that τ_{bas} is a (graded) trace, it suffices to show the existence of $\tilde{r}$ making the following diagram commutative:

$$\begin{tikzcd}
\Omega^*_c(\Gamma)\otimes\Omega^*_c(\Gamma) \arrow[r, "\tilde{r}"] \arrow[d, "b"'] & \Omega^*_c(\Gamma^{(1)}) \arrow[d, "\delta"] \\
\Omega^*_c(\Gamma) \arrow[r, "r"] & \Omega^*_c(\Gamma^{(0)})
\end{tikzcd}$$

where r is the restriction, δ is the map used in (4.8) to define $\Omega^*_{c,bas}(\Gamma^{(0)})$, and $b(\omega,\eta) = [\omega,\eta]$. But, computing:

$$rb(\omega,\eta)(x) = \sum_{t(g)=x} \omega(g)\eta(g^{-1})^{g^{-1}} - (-1)^{deg(\omega)deg(\eta)} \sum_{t(g)=x} \eta(g)\omega(g^{-1})^{g^{-1}} =$$

$$= \sum_{t(g)=x} \omega(g)\eta(g^{-1})^{g^{-1}} - \sum_{s(g)=x} \omega(g)^{g}\eta(g^{-1}) \ ,$$

we see that there is an obvious choice: $\tilde{r}(\omega,\eta)(g) := \omega(g)\eta(g^{-1})^{g^{-1}}$.

One still has to prove that the map (4.5) (applied to $(\Omega^*_c(\Gamma), \tau_{bas})$) coincides with the composition of the localization at units $res_1 : HP_*(C^\infty_c(\Gamma)) \longrightarrow HP_*(C^\infty_c(\Gamma))_{[1]} \cong H^*_c(\Gamma)$ with (4.9). But this follows from the construction of res_1 in the previous sections or in [34, 20] (the complicated formulas there, at the level of the bicomplex $\Omega^*_c(\mathcal{G}^{(*)})$, simplify a lot since we have to take care just of the components in $\Omega^*(\Gamma^{(0)})$). For theorem 4.9.6 one still has to prove that τ_{bas} is a trace, and the induced partial Chern character coincides with (4.3) (the remaining part about d_H^2 is explained in [29]). We use the notations of the previous section (note that we have a preferred choice of H: one has the obvious horizontal distributions on each U_i, and we patch them using the partition λ_i^2; by the properties of λ_i's, H restricted to each V_i coincides with the canonical horizontal distribution on $I_1^q \otimes I_1^p$). We already know that (4.3) is induced by the algebra homomorphism $\lambda : C^\infty_c(\mathcal{G}) \longrightarrow C^\infty_c(\mathcal{G}^T_T) \otimes \mathcal{R}$, and the cycle $(\Omega^*_c(\mathcal{G}^T_T) \otimes \mathcal{R}, d \otimes id_{\mathcal{R}}, \tau_{bas} \otimes \tau_{\mathcal{R}})$. Moreover, using the partition λ_i^2 in describing the integration over leaves (4.10), the there is a commutative diagram:

$$\begin{tikzcd}
C^\infty_c(\mathcal{G}; \mathcal{L} \otimes t^*\Lambda^*\nu^*) \arrow[r, "res"] \arrow[d, "\lambda\otimes\tau_{\mathcal{R}}"'] \arrow[dr, "\tau bas"] & C^\infty_c(M; \mathcal{D} \otimes \Lambda^*\nu^*) \arrow[d, "\int_{\mathcal{F}}"] \\
\Omega^*_c(\mathcal{G}^T_T) \arrow[r, "\tau bas"] & \Omega^*_{c,bas}(T/\mathcal{F})
\end{tikzcd}$$

(here $\lambda : C^\infty_c(\mathcal{G}; \mathcal{L} \otimes t^*\Lambda^*\nu^*) \longrightarrow C^\infty_c(\mathcal{G}_{\mathcal{U}}; \mathcal{L} \otimes t^*\Lambda^*\nu^*) \cong \Omega^*_c(\mathcal{G}^T_T) \otimes \mathcal{R}$ stands for $\omega \mapsto \sum \lambda_i\omega\lambda_j\sigma_{i,j}$). $\Box$

4.10 Other particular cases

In this section we look at some simple examples, and at some particular cases as discrete group actions on manifolds, Lie group actions with discrete stabilizers, and orbifolds.

4.10.1 Examples:

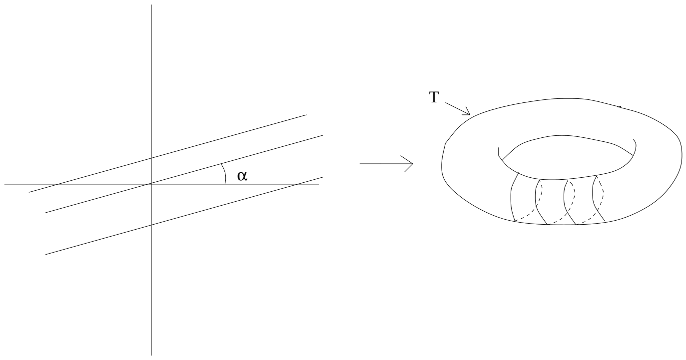
(i) Consider the étale groupoid $\mathcal{G}^{(0)} = (-1,1), \mathcal{G}^{(1)} = (-1,1) \cup_D (-1,1)$ where $D = (-1,0) \cup_* (0,1)$. It is not Hausdorff and has just elliptic loops. Applying 4.6.4 we get:

$$HP_0(C^\infty_c(\mathcal{G})) = 0 \ , \ HP_1(C^\infty_c(\mathcal{G})) = \mathbb{C} \ .$$

(ii) Consider the Kronecker foliation on the torus (see I.4.β in [29]) which comes from the foliation of the plane by lines of slope $\alpha = 2\pi\theta$, $\theta \in \mathbb{R} - \mathbb{Q}$. Choosing T as in the picture, the reduced groupoid becomes $S^1 \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on S^1 by rotations by α. The only elliptic loops are the units. From 4.6.4 we get $HP_*(M/\mathcal{F}) = HP_*(M/\mathcal{F})_{(1)}$ and from 4.6.3 this is computable in terms of $H_*(S^1 \rtimes \mathbb{Z})$. An easy algebraic computation shows that the last group is $\mathbb{C}$ if $* \in \{-1, 1\}$, $\mathbb{C} \oplus \mathbb{C}$ if $* = 0$ and 0 otherwise. So:

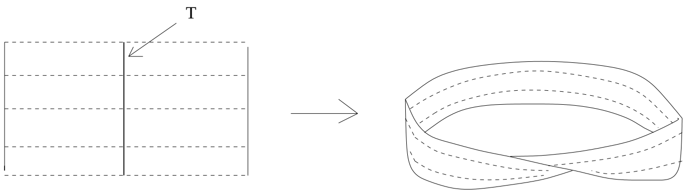
$$HP_0(C_c^\infty(Hol(M, \mathcal{F}))) = HP_1(C_c^\infty(Hol(M, \mathcal{F}))) = \mathbb{C} \oplus \mathbb{C}.$$

A similar result is obtained for cohomology. Compare to theorem 53 in [28].



(iii) Consider the foliation of the open Moebius band and the complete transversal T as in the picture. Then $\mathcal{G} = Hol_T(M, \mathcal{F})$ has Applying 4.6.4 we get:

$$HP_0(C_c^\infty(Hol(M, \mathcal{F}))) = \mathbb{C}\ ,\ \ HP_1(C_c^\infty(Hol(M, \mathcal{F}))) = \mathbb{C}\ .$$



(iv) For Haefliger's groupoid Γ^q (which is non-Hausdorff), the localization at units gives the homology and cohomology of Γ^q. The last one gives universal cocycles which induce characteristic classes for foliations of codimension q. This is the case for the Godbillon-Vey invariant or, more generally, for classes coming from Gelfand-Fuchs cohomology. It is this non-Hausdorff groupoid Γ^q the one which provides the link between these (topological) characteristic classes and the characteristic classes coming from non-commutative geometry (living in cyclic cohomology). See [55] and [27]. So our results describe precisely this connection.

4.10.2 Group actions on manifolds: We use the previous results to describe the homologies of the cross-product (locally convex) algebra $C_c^\infty(M) \rtimes G$. Here G is a discrete group acting smoothly on the Hausdorff manifold M. This algebra coincides with the convolution algebra of the smooth étale groupoid $\mathcal{G} = M \rtimes G$ (see 1.1.3). For any $g \in G$, denote by $Z_g = \{h \in G : hg = gh\}$ and $N_g = Z_g / < g >$ the centralizer and the normalizer of g, and by M^g the points fixed by g. Denote by "$\sim$" the conjugacy relation on G and put $< G >= G/\sim$. The loop space of $\mathcal{G}$ is $B^{(0)} = \{(x, g) \in M \times G : xg = x\}$ and is usually denoted by $\widehat{M}$ ([8]). Any $g \in G$ defines an invariant open $\mathcal{O}_g = \{(x, h) \in \widehat{M} : h \sim g\}$ and $\widehat{M} = \coprod_{g \in <G>} \mathcal{O}_g$. In particular we have the well-known decomposition (see [19], [45], [83]):

$$HH_*(C_c^\infty(M) \rtimes G) = \bigoplus_{g \in <G>} HH_*(C_c^\infty(M) \rtimes G)_{(g)},$$

and the analogue for HC_*, HP_*. For any g we have obvious Morita equivalences $\mathcal{Z}_{\mathcal{O}_g} \simeq M^g \rtimes Z_g$, $\mathcal{N}_{\mathcal{O}_g} \simeq M^g \rtimes N_g$.

In the elliptic case we get:

4.10.3 Corollary: *If $g \in G$ is elliptic, then*

$$HP_*(C_c^\infty(M) \rtimes G)_{(g)} = \prod_{k \equiv *(mod 2)} H_k(M^g \rtimes N_g; \mathbb{C}).$$

Remark that, with 2.4.13 and 1.2.7 in mind, this is similar to the description given in [20]. Remark also that if G acts properly on M we get (using 2.3.4 for $M^g \rtimes N_g \longrightarrow M^g/N_g$ and 2.2.5) the following computation (which, as expected, coincides with the target of the geometric Chern character defined in [8]):

4.10.4 Corollary: *If G is a discrete group acting properly on a manifold M, then*

$$HP_*(\mathcal{C}_c^\infty(M) \rtimes G) = \bigoplus_{g \in <G>} \prod_{k = * mod 2} H_c^k(M^g/Z_g)$$

Assume now that $g \in G$ is hyperbolic. We have a projection $(\mathcal{Z}_{\mathcal{O}_g}, \theta) \longrightarrow (Z_g, g)$ of hyperbolic cyclic groupoids (see example 3 in 4.3.2) which induces a map:

$$H^*_{N_g}(pt; \mathbb{C}) = H^*(N_g; \mathbb{C}) \longrightarrow H^*_{N_g}(M^g; \mathbb{C}) = H^*(\mathcal{N}_{\mathcal{O}_g}; \mathbb{C}).$$

From 4.3.4, $e_{\mathcal{O}_g} \in H^2_{N_g}(M^g; \mathbb{C})$ is the image by this map of the Euler class of (Z_g, g). This Euler class is denoted by $e_g \in H^2(N_g)$; by the previous work of Nistor ([83]) we know it is represented by the extension $< g > \longrightarrow Z_g \longrightarrow N_g$. We get the following :

4.10.5 Corollary: *For $g \in G$ hyperbolic, $HC_*(\mathcal{C}_c^\infty(M) \rtimes G)_{(g)}$ is a module over the equivariant cohomology ring $H^*_{N_g}(M^g; \mathbb{C})$ and S in the SBI-sequence is the product by the image of the Euler class $e_g \in H^2(N_g; \mathbb{C})$ in $H^2_{N_g}(M^g; \mathbb{C})$.*

Moreover, if g acts stably (for instance if it preserves a metric) then:

$$HH_n(\mathcal{C}_c^\infty(M) \rtimes G)_{(g)} = \bigoplus_{p+q=n} H_p(Z_g; \Omega_c^q(M^g)),$$

$$HC_n(\mathcal{C}_c^\infty(M) \rtimes G)_{(g)} = \bigoplus_{p+q=n} H_p(N_g; \Omega_c^q(M^g)),$$

and the SBI-sequence is the sum of the Gysin-sequences for $< g > \longrightarrow Z_g \longrightarrow N_g$.

Compare to [22, 83]. Also we have the dual results for cohomology. In particular:

4.10.6 Corollary: *If $g \in G$ is hyperbolic and the image of $e_g \in H^2(N_g; \mathbb{C})$ in the equivariant cohomology ring $H^2_{N_g}(M^g; \mathbb{C})$ is nilpotent, then $HP_*(C_c^\infty(M) \rtimes G)_{(g)}$ and $HP^*(C_c^\infty(M) \rtimes G)_{(g)}$ vanish.*

4.10.7 Lie group actions: Let G be a Lie group acting on a manifold M. By the remark in 4.8.11, one can use our computations for the translation groupoid $M \rtimes G$, provided all the stabilizers $G_x = \{g \in G : xg = x\}$ are discrete. Note that $M \rtimes G$ coincides with the holonomy groupoid of the induced foliation on M provided G is connected and it acts transversally faithful (i.e. the action of the stabilizers G_x on slices are faithful). To state our computations in general, we should describe the cohomology "$H_c^*(M/G)$" as in the foliation case; in the case of the cyclic cohomology, one can use the Borel cohomology groups (which are Morita invariant). Hence an instance of this discussion is:

4.10.8 Corollary: *Let G be a Lie group acting on the manifold M, so that all the stabilizers G_x are discrete. Then $HP^*(C_c^\infty(M) \rtimes G)$ contains $H_G^{*-q}(M;\mathcal{O})$ as a direct summand (here H_G^* denotes the equivariant cohomology, with the even/odd grading, $q = dim(M) - dim(G)$ is the transversal dimension, and $\mathcal{O}$ is the transversal orientation sheaf on M).*

4.10.9 Orbifolds: Let $\mathcal{M} = (M, \mathcal{U})$ be an orbifold (M is the underlying topological space, $\mathcal{U}$ an orbifold atlas). Associated to $\mathcal{M}$ there is another orbifold $\Omega(\mathcal{M})$ ("of loops"). This space has been defined also by Kawasaki [69] in order to define a (geometric) Chern character for orbifolds. Realizing $\mathcal{M}$ as a quotient N/L [75], where L is a compact Lie group acting on M, with finite stabilizers, then:

$$\Omega(\mathcal{M}) = \widehat{N}/L,$$

where $\widehat{N} = \{\langle x, \gamma \rangle \in M \times L : x\gamma = x\}$ is Brylinski's space, with the L-action $\langle x, \gamma \rangle g = \langle xg, g^{-1}\gamma g \rangle$. Alternatively, taking any representation of $\mathcal{M}$ by a proper étale groupoid $\mathcal{G}$, the loop groupoid $\Omega(\mathcal{G})$ is again a proper étale groupoid, whose quotient is precisely $\Omega(\mathcal{M})$ (and $\Omega(\mathcal{G})$ represents this orbifold of loops). Using (2.16) (applied to $\Omega(\mathcal{G})$), (2.17), and our remark in 4.10.7:

4.10.10 Corollary: *Let $\mathcal{M}$ be an orbifold. For any convolution algebra $\mathcal{A}$ associated to $\mathcal{M}$ (i.e. the convolution algebra of a smooth étale groupoid representing $\mathcal{M}$, or the crossed product $C_c^\infty(N) \rtimes L$ if $M = N/L$ is the quotient by the action of a compact Lie group with finite stabilizers):*

$$HP_i(\mathcal{A}) = \prod_k H_c^{i+2k}(\Omega(\mathcal{M})) \quad , i \in \{0, 1\} \ .$$

Moreover, since in the proper case the basic cohomology coincides with the full one (see 2.3.9), one can describe the full Chern character in cyclic homology as a partial Chern character associated to an explicit cycle. Its target is $H_c^*(\Omega(\mathcal{M}))$, and we get explicit formulas. This also makes the connection with Kawasaki's definition of the Chern character for orbifolds [69].

4.10.11 Corollary: *The Chern character associated to an orbifold $\mathcal{M}$ becomes:*

$$Ch : K_*(\mathcal{A}) \longrightarrow H_c^*(\Omega(\mathcal{M})),$$

and it coincides with the partial Chern character of an explicit cycle.

Chapter 5

Cyclic cohomology of Hopf algebras

5.1 Introduction

In the transversal index theorem for foliations (cohomological form), the characteristic classes involved are a priori cyclic cocycles on an algebra $\mathcal{A}$ associated to the foliation. In their computation, A. Connes and H. Moscovici [31] have discovered that the action of the operators appearing from the non-commutative index formula can be organized in an action of a Hopf algebra $\mathcal{H}_T$ on $\mathcal{A}$, and that the cyclic cocycles are made out just by combining the action with a certain invariant trace $\tau : \mathcal{A} \longrightarrow \mathbf{C}$. In other words, they define a cyclic cohomology $HC^*_\delta(\mathcal{H}_T)$, in such a way that the cyclic cocycles involved are in the target of a characteristic map $k_\tau : HC^*_\delta(\mathcal{H}_T) \longrightarrow HC^*(\mathcal{A})$, canonically associated to the pair $(\mathcal{A}, \tau)$. When computed, $HC^*_\delta(\mathcal{H}_T)$ gives the Gelfand-Fuchs cohomology and the characteristic map k_τ is a non-commutative version of the classical geometric characteristic map $k^{geom}_{\mathcal{F}}(see 1.4.4) : H^*(WO_q) \longrightarrow H^*(M/\mathcal{F})$ for codimension q foliations $(M, \mathcal{F})$ (for the relation between $HC^*(\mathcal{A})$ and $H^*(M/\mathcal{F})$, see [20, 34], or our Chapter 4). The definition of $HC^*_\delta(\mathcal{H})$ given in [31] applies to any Hopf algebra $\mathcal{H}$ endowed with a character δ, satisfying certain conditions (see the end of 5.3.4). In the context of $\mathcal{H}_T$ this provides a new beautiful relation of cyclic cohomology with Gelfand-Fuchs cohomology, while, in general, it can be viewed as a non-commutative extension of the Lie algebra cohomology (see Theorem 5.6.6).

Our first goal is to show that Connes-Moscovici formulas can be used under the minimal requirement $S^2_\delta = Id$ (which answers a first question raised in [31]), and to give a new definition/interpretation of $HC^*_\delta(\mathcal{H})$ in the spirit of Cuntz-Quillen's formalism.

Independently, in the work of Gelfand and Smirnov [46] on universal Chern-Simons classes, there are implicit relations with cyclic cohomology. Our second goal is to make these connections explicit. This leads us to a noncommutative Weil complex $W(\mathcal{H})$ associated to a coalgebra, which extends the constructions from [46, 89].

Our third goal is to show that $W(\mathcal{H})$ is intimately related to the cyclic cohomology $HC^*_\delta(\mathcal{H})$ and to the construction (see 5.8.4, 5.8.6) of characteristic homomorphisms k_τ associated to invariant higher traces τ (which is a second problem raised in [31]). The construction of k_τ is inspired by the construction of the usual Chern-Weil homomorphism (see e.g [43] or Section 1.4), and of the secondary characteristic classes for foliations (see [13] or our preliminary section 1.4).

This chapter is strongly influenced by the Cuntz-Quillen approach to cyclic cohomology ([40, 41, 90] etc).

Here is an outline of the paper. In Section 5.2 we bring together some basic results about characters δ and the associated twisted antipodes S_δ on Hopf algebras. In section 5.3 we present some basic terminology, describe the problem (see 5.3.4) of defining $HC^*_\delta(\mathcal{H})$, and explain why the case $S_\delta^2 = Id$ is better behaved (see Proposition 5.3.5). Under this requirement, we define a cyclic cohomology $HC^*_{\delta-inv}(R)$ for any $\mathcal{H}$-algebra R, and we indicate how Cuntz-Quillen machinery can be adapted to this situation (Theorem 5.3.7). The relevant information which is needed for the cyclic cohomology of Hopf algebras, only requires a small part of this machinery. This is captured by a localized X-complex (denoted by $X_\delta(R)$); in Section 5.4, after recalling the X-complex interpretation of S-operations (see 5.4.1), we introduce $X_\delta(R)$ (see 5.4.2) and compute it in the case where $R = T(V)$ is the tensor algebra of an $\mathcal{H}$-module V (see Proposition 5.4.4).

In Section 5.5 we prove that $HC^*_\delta(\mathcal{H})$ can be defined under the minimal requirement $S_\delta^2 = Id$ (see Proposition 5.5.4); also, starting with the question "which is the target of characteristic maps $k_\tau : HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A)$, associated to pairs (A, τ) consisting of a $\mathcal{H}$-algebra A and a δ-invariant trace τ (see 5.3.4)?", we explain/interpret the definition of $HC^*_\delta(\mathcal{H})$ in terms of localized X-complexes (see 5.5.6, and Theorem 5.5.7). This interpretation is the starting point in constructing the characteristic maps associated to higher traces (Section 5.8). We also recall Connes-Moscovici's recent proposal to extend the definition to the non-unimodular case, and we show how our approach extends to this setting.

In Section 5.6 we give some examples, including the quantum enveloping algebra $\mathcal{H} = U_q(sl_2)$, and a detailed computation of the fundamental example where $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$ (see Theorem 5.6.6).

In section 5.7 we introduce the non-commutative Weil complex (by collecting together 'forms and curvatures' in a non-commutative way). We show that there are two relevant types of cocycles involved (which, in the case considered by Gelfand and Smirnov, correspond to Chern classes, and Chern-Simons classes, respectively), we describe the Chern-Simons transgression, and prove that it is an isomorphism between these two types of cohomologies (Theorem 5.7.7). In connection with cyclic cohomology, we show that the non-commutativity of the Weil complex naturally gives rise to an S-operator, and to cyclic bicomplexes computing our cohomologies (see 5.7.8-5.7.10).

In section 5.8 we come back to Hopf algebra actions, and higher traces, and we show how the non-commutative Weil complex can be used to construct the characteristic map k_τ associated to higher traces (see 5.8.4, 5.8.6). To prove the compatibility with the S-operator (Theorems 5.8.5 and 5.8.7), we also show that the truncations of the Weil complex can be expressed in terms of relative X-complexes (Theorem 5.8.9). When $\mathcal{H} = \mathbf{C}$, we re-obtain the cyclic cocycles (and their properties) described by Quillen [90]. In general, the truncated Weil complexes still compute $HC^*_\delta(\mathcal{H})$, as explained by Theorem 5.8.3. Section 5.9 is devoted to the proof of this theorem, and the construction of characteristic maps associated to equivariant cycles.

A well known principle is that quantum groups correspond to "non-commutative Lie groups", and they can be described either by their Hopf algebras of operators (quantum enveloping algebras), or by their dual algebras of functions (as in [100]). While the theory presented in the first sections seems more relevant for the first type of algebras (and is sensible to the presence of traces), in the last section we dualize the construction so that it becomes relevant to the second type of Hopf algebras (and to *co*traces rather than traces). As expected, for $\mathcal{H} = C^{\infty}(G)$ for some compact Lie group G we get the Lie algebra cohomology, while for $\mathcal{H} = \mathbf{C}\Gamma$ for some discrete group Γ we get the group cohomology.

5.2 Preliminaries on Hopf Algebras

In this section we review some basic properties of Hopf algebras (see [95]) and prove some useful formulas on twisted antipodes.

Let $\mathcal{H}$ be a Hopf algebra. As usual, denote by S the antipode, by ϵ the counit, and by $\Delta(h) = \sum h_0 \otimes h_1$ the coproduct. Recall some of the basic relations they satisfy:

$$\sum \epsilon(h_0)h_1 = \sum \epsilon(h_1)h_0 = h, \tag{5.1}$$

$$\sum S(h_0)h_1 = \sum h_0 S(h_1) = \epsilon(h) \cdot 1, \tag{5.2}$$

$$S(1) = 1, \epsilon(S(h)) = \epsilon(h), \tag{5.3}$$

$$S(gh) = S(h)S(g), \tag{5.4}$$

$$\Delta S(h) = \sum S(h_1) \otimes S(h_0). \tag{5.5}$$

Throughout this chapter, the notions of $\mathcal{H}$-module and $\mathcal{H}$-algebra have the usual meaning, with $\mathcal{H}$ viewed as an algebra. The tensor product $V \otimes W$ of two $\mathcal{H}$-modules is an $\mathcal{H}$-module with the diagonal action:

$$h(v \otimes w) = \sum h_0(v) \otimes h_1(w). \tag{5.6}$$

A *character* on $\mathcal{H}$ is any non-zero algebra map $\delta : \mathcal{H} \longrightarrow \mathbb{C}$. Characters will be used for 'localizing' modules: for an $\mathcal{H}$-module V, define V_δ as the quotient of V by *the space of co-invariants* (linear span of elements of type $h(v) - \delta(h)v$, with $h \in \mathcal{H}, v \in V$). In other words,

$$V_\delta = \mathbb{C}_\delta \otimes_{\mathcal{H}} V,$$

where $\mathbb{C}_\delta = \mathbb{C}$ is viewed as an $\mathcal{H}$-module via δ. Before looking at very simple localizations (see 5.2.3), we need to discuss the *twisted antipode*:

$$S_\delta : = \delta * S$$

associated to a character δ (recall that $*$ denotes the natural product on the space of linear maps from the coalgebra $\mathcal{H}$ to the algebra $\mathcal{H}$, [95]). Explicitly,

$$S_\delta(h) = \sum \delta(h_0)S(h_1), \quad \forall h \in \mathcal{H}.$$

Lemma 5.2.1 *The following identities hold:*

$$\sum S_\delta(h_0)h_1 = \delta(h) \cdot 1, \tag{5.7}$$

$$S_\delta(1) = 1, \epsilon(S_\delta(h)) = \delta(h), \tag{5.8}$$

$$\Delta S_\delta(h) = \sum S(h_1) \otimes S_\delta(h_0), \tag{5.9}$$

$$S_\delta(gh) = S_\delta(h)S_\delta(g), \tag{5.10}$$

$$\sum S^2(h_1)S_\delta(h_0) = \delta(h) \cdot 1. \tag{5.11}$$

proof: These follow easily from the previous relations. For instance, the first relation follows from the definition of S_δ, (5.2), and (5.1), respectively:

$$\begin{aligned}\sum S_\delta(h_0)h_1 &= \sum \delta(h_0)S(h_1)h_2 = \\ &= \sum \delta(h_0)\epsilon(h_1)\cdot 1 = \delta(\sum h_0\epsilon(h_1))\cdot 1 = \delta(h)\cdot 1 \ .\end{aligned}$$

The other relations are proved in a similar way. □

Lemma 5.2.2 . *For any two $\mathcal{H}$-modules V, W:*

$$h(v)\otimes w \equiv v \otimes S_\delta(h)(w) \quad \textit{mod co-invariants}$$

proof: From the definition of S_δ,

$$v\otimes S_\delta(h)w = \sum \delta(h_0)v\otimes S(h_1)w \ ,$$

so, modulo coinvariants, it is

$$\sum h_0(v)\otimes h_1S(h_2)(w) = \sum \epsilon(h_1)h_0(v)\otimes w = h(v)\otimes w \ ,$$

where for the last two equalities we have used (5.2) and (5.1), respectively. □

It follows easily that:

Corollary 5.2.3 . *For any $\mathcal{H}$-module V, there is an isomorphism:*

$$\begin{aligned}(\mathcal{H}\otimes V)_\delta &\cong V \ , \\ (h, v) &\mapsto S_\delta(h)v \ .\end{aligned}$$

There is a well known way to recognize Hopf algebras with $S^2 = Id$ (see [95], pp. 74). We extend this result to twisted antipodes:

Lemma 5.2.4 . *For a character δ, the following are equivalent:*

(i) $S_\delta^2 = Id$,

(ii) $\sum S_\delta(h_1)h_0 = \delta(h)\cdot 1, \ \forall h\in\mathcal{H}$

proof: The first implication follows by applying S_δ to (5.7), using (5.10), and *(i)*. Now, assume *(ii)* holds. First, remark that $S{\circ}S_\delta = \delta$. Indeed,

$$\begin{aligned}(S*(S{\circ}S_\delta))(h) &= \sum S(h_0)S(S_\delta(h_1)) = \\ &= \sum S(S_\delta(h_1)h_0) = \delta(h)\cdot 1\end{aligned}$$

(where we have used the definition of $*$, (5.5), and (ii), respectively.) Multiplying this relation by Id on the left, we get

$$S{\circ}S_\delta = Id * \delta \ .$$

Using the definition of S_δ, (5.9), and the previous relation, respectively,

$$\begin{aligned} S_\delta^2(h) &= \sum \delta(S_\delta(h)_0) S(S_\delta(h)_1) = \\ &= \sum \delta(S(h_1) S(S_\delta(h_0)) = \sum \delta(S(h_2)) h_0 \delta(h_1), \end{aligned}$$

which is (use that δ is a character, and the basic relations again):

$$\delta(\sum h_1 S(h_2)) h_0 = \sum \delta(\epsilon(h_1)) h_0 = \sum \epsilon(h_1) h_0 = h. \ \square$$

5.3 Invariant traces

In this section we present some basic terminology like invariant traces, $\mathcal{H}$-algebras. For such an algebra R, the non-commutative differential forms on R can be localized, under the hypothesis $S_\delta^2 = Id$, and a cyclic cohomology $HC^*_{\delta-inv}(R)$ shows up. For completeness, we indicate how the Cuntz-Quillen machinery [40] can be adapted to this context (see Theorem 5.3.7); this extends, in particular, the usual correspondence ([40]) between δ-invariant cyclic cocycles and δ-invariant higher traces (with equivariant linear splitting).

5.3.1 Flat algebras: Let A be an algebra, not necessarily unital. An action $\mathcal{H} \otimes A \longrightarrow A$ of $\mathcal{H}$ (viewed as an algebra) on A is called *flat* (and say that A is a $\mathcal{H}$*-algebra*) if:

$$h(ab) = \sum h_0(a) h_1(b), \quad \forall h \in \mathcal{H}, \ a, b \in A \tag{5.12}$$

The motivation for the terminology is that, in our interpretations (see 5.5.6), it plays a role similar to the usual flat connections in geometry.

5.3.2 Invariant traces: Let $\mathcal{H}$ be a Hopf algebra endowed with a character δ, and A a $\mathcal{H}$-algebra. A δ*-invariant trace* is any trace $\tau : A \longrightarrow \mathbb{C}$ with the property:

$$\tau(ha) = \delta(h)\tau(a), \quad \forall h \in \mathcal{H}, \ a \in A.$$

If $\delta = \epsilon$ is the counit, we simply call τ invariant.

Recall [90] that an even (n dimensional) higher trace on an algebra R is given by an extension $0 \longrightarrow I \longrightarrow L \longrightarrow R \longrightarrow 0$ and a trace on L/I^{n+1}, while an odd higher trace is given by an extension as before, and an I- adic trace, i.e. a linear functional on I^{n+1} vanishing on $[I^n, I]$. Starting with an extension of $\mathcal{H}$-algebras, and a δ-invariant trace τ, we talk about equivariant (or δ-invariant) higher traces.

5.3.3 Examples: If $\mathcal{H} = \mathbf{C}[\Gamma]$ is the group algebra of a discrete group Γ (recall that $\Delta(\gamma) = \gamma \otimes \gamma, \epsilon(\gamma) = 1$ if $\gamma = 1$, and 0 otherwise), $\mathcal{H}$-algebras are precisely Γ-algebras.

If G is a connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra, then $\mathcal{H}$-algebras are precisely infinitesimal G-algebras; that is, algebras A endowed with linear maps (Lie derivatives) $L_v : A \longrightarrow A$, linear on $v \in \mathfrak{g}$, such that $L_{[v,w]} = L_v L_w - L_w L_v, L_v(ab) = L_v(a)b + aL_v(b)$. If $\Delta : G \longrightarrow \mathbf{C}$ is a character, it induces an infinitesimal character δ on $U(\mathfrak{g})$: $\delta(v) = (\frac{d}{dt})_{t=0}\Delta(exp(tv))$, $v \in \mathfrak{g}$. If A is a topological (locally convex) G-algebra, δ-invariance of traces on A is equivalent to $\tau(ga) = \Delta(g)\tau(a)$, $\forall g \in G, a \in A$.

Remark that, in general, the action of a Hopf algebra on itself is not flat. A basic example of flat action is the diagonal action of $\mathcal{H}$ on its tensor algebra $T\mathcal{H} = \oplus_{n\geq 1}\mathcal{H}^{\otimes n}$ (see (5.6)). Another basic example is the algebra $\Omega^*(R)$ of noncommutative differential forms on a $\mathcal{H}$-algebra R. Recall that:

$$\Omega^n(R) = \tilde{R} \otimes R^{\otimes n},$$

where $\tilde{R}$ is R with a unit adjoined. Extending the action of $\mathcal{H}$ to $\tilde{R}$ by $h \cdot 1 := \epsilon(h)1$, we have an action of $\mathcal{H}$ on $\Omega^*(R)$ (the diagonal action). To check the flatness condition: $h(\omega\eta) = \sum h_0(\omega)h_1(\eta)$, $\forall \omega, \eta \in \Omega(R)$, remark that one can formally reduce to the case where ω and η are degree 1 forms, in which case the computation is easy.

Recall also the usual operators d, b, B, k acting on $\Omega^*(R)$ (see [39], paragraph 3 of [40]): $d(a_0da_1 \ldots da_n) = da_0da_1 \ldots da_n$, $b(\omega da) = (-1)^{deg(\omega)}[\omega, a]$, $k = 1 - (bd + db)$, $B = (1 + k + \ldots + k^n)d$ on $\Omega^n(R)$.

5.3.4 The Problem: Let δ be a character on a Hopf algebra $\mathcal{H}$. The problem of defining a cyclic cohomology $'HC^*_\delta(\mathcal{H})'$, should answer the question: which are the nontrivial cyclic cocycles on a $\mathcal{H}$-algebra A, arising from a δ-invariant trace τ, and the action of $\mathcal{H}$ on A. In particular, for any pair (A, τ) one should have an associated characteristic map:

$$k_\tau : HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A) \ ,$$

compatible with the S-operation on cyclic cohomology. There is a similar problem for invariant *higher* traces.

In [31], Connes and Moscovici have introduced $HC^*_\delta(\mathcal{H})$ under the hypothesis that there is an algebra $\mathcal{A}$, endowed with an action of $\mathcal{H}$, and with a δ-invariant *faithful* trace $\tau : \mathcal{A} \to \mathbb{C}$. As pointed out in [31], this requirement is quite strong; a more natural hypothesis would be the weaker condition $S^2_\delta = Id$.

Proposition 5.3.5 . *If $S^2_\delta = Id$, then for any $\mathcal{H}$-algebra R, the operators d, b, k, B, acting on $\Omega^*(R)$ (see 5.3.3), descend to $\Omega^*(R)_\delta$.*

proof: Since d commutes with the action of $\mathcal{H}$, and k, B (and all the other operators appearing in paragraph 3 of [40]) are made out of d, b, it suffices to prove that, modulo co-invariants,

$$b(h \cdot \eta) \equiv b(\delta(h)\eta), \quad \forall h \in \mathcal{H}, \ \eta \in \Omega^*(R).$$

For $\eta = \omega da$, one has$(-1)^{|\omega|}b(h \cdot \omega da) = \sum h_0(\omega)h_1(a) - \sum h_1(a)h_0(\omega)$.
Using Lemma 5.2.2 and (5.7), $\sum h_0(\omega)h_1(a) \equiv \sum \omega \cdot S_\delta(h_0)h_1 a = \delta(h)\omega a$.
Using Lemma 5.2.2, and (ii) of Lemma 5.2.4, $\sum h_1(a)h_0(\omega) \equiv \sum a \cdot S_\delta(h_1)h_0\omega = \delta(h)a\omega$, which ends the proof. $\square$

Definition 5.3.6 *($S^2_\delta = Id$) Define the localized cyclic cohomology $HC^*_{\delta-inv}(R)$ of R as the cyclic cohomology of the mixed complex $\Omega^*(R)_\delta$. Similarly for Hochschild and periodic cyclic cohomologies, and also for homology.*

This cohomology is not used in the next sections, but it fits very well in our discussion of higher traces. Recall that, via a certain notion of homotopy, higher traces correspond exactly to cyclic cocycles on R (for the precise relations, see pp. 417- 419 in [40]). Using $HC^*_{\delta-inv}(R)$ instead of $HC^*(R)$, this relation extends to the equivariant setting (provided one restricts to higher traces which admit an equivariant linear splitting). The main ingredient is the following theorem which we include for completeness. It is analogous to one of the main results in [40] (Theorem 6.2). The notation $T(R)$ stands for the (non-unital) tensor algebra of R, and $I(R)$ is the kernel of the multiplication map $T(R) \to R$. Recall also that if M is a mixed complex [67], θM denotes the associated Hodge tower of M, which represents the cyclic homology type of the mixed complex (for more details on the notations and terminology see [40]).

Theorem 5.3.7 *There is a homotopy equivalence of towers of super-complexes:*

$$\mathcal{X}_\delta(TR, IR) \simeq \theta(\Omega^*(R)_\delta).$$

proof: The proof from [40] can be adapted. For this, one uses the fact that the projection $\Omega^*(R) \longrightarrow \Omega^*(R)_\delta$ is compatible with all the structures (with the operators, with the mixed complex structure). All the formulas we get for free, from [40]. The only thing we have to do is to take care of the action. For instance, in the computation of $\Omega^1(TR)_\natural$ (pp. $399-401$ in [40]), the isomorphism $\Omega^1(TR)_\natural \cong \Omega^-(R)$ is not compatible with the action of $\mathcal{H}$, but, using the same technique as in 5.2.2, it descends to localizations (which means that we can use the natural (diagonal) action we have on $\Omega^-(R)$). With this in mind, the analogous of Lemma 5.4 in [40] holds, that is, $\mathcal{X}_\delta(TR, IR)$ can be identified (without regarding the differentials) with the tower $\theta(\Omega^*(R)_\delta)$. Denote by k_δ the localization of k. The spectral decomposition with respect to k_δ is again a consequence of the corresponding property of k ([40], pp $389-391$ and pp. $402-403$), and the two towers are homotopically concentrated on the nillspaces of k_δ, corresponding to the eigenvalue 1. Lemma 6.1 of [40] identifies the two boundaries corresponding to this eigenvalues, which concludes the theorem. $\Box$

5.4 S-operations and X-complexes

In this section we recall Quillen's interpretation of a certain degree two cohomology operation ('S-operators') in terms of X-complexes, and describe a localized version (to be used in sections 5.5 and 5.7). As before, $\mathcal{H}$ is a Hopf algebra endowed with a character δ such that $S_\delta^2 = Id$.

5.4.1 S-operations: If R is a DG algebra, denote by $R_\natural = R/[R, R]$ the complex obtained dividing out by the linear span of graded commutators. In examples like tensor algebras, the algebras considered by Gelfand, Smirnov etc (see [46] and references therein), the noncommutative Weil complex of Section 5.7, and, in general when R is 'free', one encounters a very interesting degree two operation in the cohomology of $R_\natural$, $S : H^*(R_\natural) \longrightarrow H^{*+2}(R_\natural)$. This phenomenon, due to the non-commutativity of R, has been very nicely explained by Quillen ([90, 89]). In general, for any algebra R, there is a sequence:

$$0 \longrightarrow R_\natural \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \xrightarrow{\natural} R_\natural \longrightarrow 0, \tag{5.13}$$

Here $\Omega^1(R)_\natural = \Omega^1(R)/[\Omega^1(R), R]$, $b(xdy) = [x,y]$, and $\natural$ is the projection. In our graded setting, one uses graded commutators, and (5.13) is a sequence of complexes. In general, it is exact in the right. When it is exact (and this happens in our examples), it can be viewed as an Ext^2 class, and induces a degree 2 operator $S : H^*(R_\natural) \longrightarrow H^{*+2}(R_\natural)$, explicitly described by the following diagram chasing ([89], pp. 120). Given $\alpha \in H^k(R_\natural)$, we represent it by a cocycle c, and use the exactness to solve successively the equations:

$$c = \natural(u),\ \partial(u) = b(v),\ \partial(v) = d(w).$$

where ∂ stands for the vertical boundary. Then $S(\alpha) = [\natural(w)] \in H^{k+2}(R_\natural)$.

Equivalently, pasting together (5.13), we get a resolution, usually denoted by $X^+(R)$:

$$0 \longrightarrow R_\natural \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \longrightarrow \dots$$

Emphasize that, when working with bicomplexes with anti-commuting differentials, one has to introduce a '$-$' sign for the even vertical boundaries (i.e. for those of R). So, one can use the cyclic bicomplex $X^+(R)$ to compute the cohomology of $R_\natural$, and then S is simply the shift operator.

The X-complex of R is simply the full version of $X^+(R)$, that is, the super-complex:

$$X(R): \quad R \underset{d}{\overset{b}{\leftrightarrows}} \Omega^1(R)_\natural \quad , \tag{5.14}$$

where $b(xdy) = [x,y], d(x) = dx$. It is defined in general, for any algebra, and it can be viewed as the degree one level of the Hodge tower associated to $\Omega^*(R)$. In our graded setting, it is a cyclic bicomplex.

5.4.2 The localized X-complex: When $R = T\mathcal{H}$ is the tensor DG algebra of $\mathcal{H}$, then $T\mathcal{H}_\natural$ computes the cyclic cohomology of $\mathcal{H}$, viewed as a coalgebra (cf. Theorem 5.5.2), and our previous discussion describes the usual S-operation in cyclic cohomology. We need a similar construction for $T\mathcal{H}_{\natural,\delta}$. Here, if R is a DG algebra endowed with a flat action of $\mathcal{H}$ compatible with the differentials (a $\mathcal{H}$-DG algebra on short), $R_{\natural,\delta} := R/[R,R] + (\text{coinvariants})$ denotes the complex obtained dividing out R by the linear span of graded commutators and coinvariants (i.e. elements of type $h(x) - \delta(h)x$, with $h \in \mathcal{H}, x \in R$).

Since $S_\delta^2 = Id$, we know (cf. Proposition 5.3.5) that b, d descend, and we define *the localized X-complex* of R as the degree one level of the Hodge tower associated to $\Omega^*(R)_\delta$. In other words, this is simply the super-complex (a cyclic bicomplex in our graded setting):

$$X_\delta(R): \quad R_\delta \underset{d}{\overset{b}{\leftrightarrows}} \Omega^1(R)_{\natural,\delta} \quad ,$$

where:

$$\Omega^1(R)_{\natural,\delta} := \Omega^1(R)_\delta / b\Omega^2(R)_\delta = \Omega^1(R)/[\Omega^1(R), R] + (\text{coinvariants}),$$

and the formulas for b, d are similar to the ones for $X(R)$. There is one remark about the notation: $\Omega^1(R)_{\natural,\delta}$ is not the localization of $\Omega^1(R)_\natural$; in general, there is no natural action of $\mathcal{H}$ on it.

5.4.3 Example. Before proceeding, let us look at a very important example: the (non-unital) tensor algebra $R = T(V)$ of an $\mathcal{H}$-module V. Adjoining a unit, one gets the unital tensor algebra $\tilde{R} = \widetilde{T}(V) = \oplus_{n\geq 0} V^{\otimes n}$. The computation of $X(R)$ was carried out in [90], Example 3.10. One knows that ([40], pp. 395) $R = T(V)$, $\Omega^1(R)_\natural = V \oplus \widetilde{T}(V) = T(V)$, and also the description of the boundaries: $d = \sum_{i=0}^{i=n} t^i, b = (t-1)$ on $V^{\otimes(n+1)}$, where t is the backward-shift cyclic permutation. The second isomorphism is essentially due to the fact that, since V generates $T(V)$, any element in $\Omega^1(T(V))$ can be written in the form $xd(v)y$, with $x, y \in \widetilde{T(V)}, v \in V$ (see also the proof of the next proposition). To compute $X_\delta(R)$, one still has to compute its odd part. The final result is:

Proposition 5.4.4 . *For $R = T(V)$:*

$$X^0_\delta(R) = T(V)_\delta, \quad X^1_\delta(R) = T(V)_\delta,$$

where the action of $\mathcal{H}$ on $T(V)$ is the usual (diagonal), and the boundaries have the same description as the boundaries of $X(R)$: they are $(t-1)$, N (which descend to the localization). The same holds when V is a graded $\mathcal{H}$-module, provided we replace the the backward-shift cyclic permutation t by its graded version.

proof: One knows ([40], pp. 395):

$$\tilde{R} \otimes V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R), \quad x \otimes v \otimes y \mapsto x(dv)y,$$

which, passing to commutators, gives (compare to [40], pp. 395):

$$R = V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R)_\natural, \quad v \otimes y \longrightarrow \natural(dvy),$$

and the projection map $\natural : \Omega^1(R) \longrightarrow \Omega^1(R)_\natural$ identifies with:

$$\natural : \tilde{R} \otimes V \otimes \tilde{R} \longrightarrow V \otimes \tilde{R}, \quad x \otimes v \otimes y \longrightarrow v \otimes yx.$$

So $X^1_\delta(R)$ is obtained from $T(V)$, dividing out by the linear subspace generated by elements of type:

$$\natural(h \cdot x \otimes v \otimes y - \delta(h)x \otimes v \otimes y) = \sum h^1(v) \otimes h^2(y)h^0(x) - \delta(h)v \otimes yx \in T(V).$$

Now, for $y = 1$, this means exactly that we have to divide out by coinvariants (of the diagonal action of $\mathcal{H}$ on $T(V)$). But this is all, because modulo these coinvariants we have (from Lemma 5.2.2):

$$\sum h_1(v) \otimes h_2(y)h_0(x) \equiv \sum v \otimes S_\delta(h_1) \cdot (h_2(y)h_0(x)),$$

while, from (5.9), (5.2) and (ii) of Lemma 5.2.2, (5.1):

$$\sum S_\delta(h_1) \cdot (h_2(y)h_0(x)) = \sum S(h_2)h_3(y)S_\delta(h_1)h_0(x) = \sum \epsilon(h_1)\delta(h_0)yx = \delta(h)yx. \quad \Box$$

5.5 Cyclic Cohomology of Hopf Algebras

In this section we introduce the cyclic cohomology of Hopf algebras (endowed with a character δ as before). First we prove that Connes-Moscovici's formulas can be used under the minimal condition $S_\delta^2 = Id$ (see 5.5.4). Next (see 5.5.6) we present a second approach to defining $HC_\delta^*(\mathcal{H})$ as the natural solution to our problem 5.3.4. The two approaches coincide, which leads us to a X-complex interpretation of our cohomology (see Theorem 5.5.7). This interpretation is also the starting point in dealing with higher traces (section 5.8).

Let $\mathcal{H}$ be a Hopf algebra endowed with a character δ.

5.5.1 Cyclic cohomology of coalgebras: Looking first just at the coalgebra structure of $\mathcal{H}$, one defines the cyclic cohomology of $\mathcal{H}$ by duality with the case of algebras. As in [31], we define the Λ-module (see 1.5.5), denoted $\mathcal{H}^\natural$, which is $\mathcal{H}^{\otimes(n+1)}$ in degree n, whose co-degeneracies are:

$$d^i(h^0, \ldots, h^n) = \begin{cases} (h^0, \ldots, h^{i-1}, \Delta h^i, h^{i+1}, \ldots, h^n) & \text{if } 0 \leq i \leq n \\ \sum(h^0_{(1)}, h^1, \ldots, h^n, h^0_{(0)}) & \text{if } i = n+1 \end{cases} .$$

and whose cyclic action is:

$$t(h^0, \ldots, h^n) = (h^1, h^2, \ldots, h^n, h^0).$$

Emphasize that, dual to the case of algebras, the terminology is here: Λ-module = cyclic module := *covariant* functor on Λ. The definitions in 1.5.5 obviously dualize, and, for such covariant functors one has associated Hochschild, cyclic, and periodic cyclic *co*homology (as well as the associated complexes). Denote by $HH^*(\mathcal{H})$, $HC^*(\mathcal{H})$, $HP^*(\mathcal{H})$ the cohomologies associated to $\mathcal{H}^\natural$, by $C_\lambda^*(\mathcal{H})$ the associated cyclic complex, and by $CC^*(\mathcal{H})$ the associated cyclic (upper plane) bicomplex ((b, b')-bicomplex).

Recall that the DG tensor algebra of $\mathcal{H}$, denoted $T(\mathcal{H})$, is $\mathcal{H}^{\otimes n}$ in degrees $n \geq 1$ and 0 otherwise, and has the differential $b' = \sum_0^n (-1)^i d^i$. The following proposition shows that the S-operator acting on $HC^*(\mathcal{H})$ (a priori described by the shift on $CC^*(\mathcal{H})$), is the S-operator described by an X-complex:

Proposition 5.5.2 *Up to a shift in degrees, the cyclic bicomplex of $\mathcal{H}$, $CC^*(\mathcal{H})$ coincides with the X-complex of the DG algebra $T(\mathcal{H})$, and the cyclic complex $C_\lambda^*(\mathcal{H})$ is isomorphic to $T(\mathcal{H})_\natural$. This is true for any coalgebra.*

proof: It follows from the computation in the proof of Lemma 5.4.4, or by dualizing the analogous result for algebras (Theorem 4 and Lemma 2.1 of [90]).

□

Let us be more precise about the shifts. In a precise way, the proposition identifies $CC^*(\mathcal{H})$ with the super-complex of complexes:

$$\ldots \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow X^0(T\mathcal{H})[-1] \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow \ldots \quad ,$$

and gives an isomorphism: $C_\lambda^*(\mathcal{H}) \cong T(\mathcal{H})_\natural[-1]$.

5.5.3 Cyclic cohomology of Hopf algebras: Localizing the cyclic module $\mathcal{H}^\sharp$, we obtain a new object, denoted $\mathcal{H}^\sharp_\delta$. By Lemma 5.2.3, it is $\mathcal{H}^{\otimes n}$ in degree n, and the projection becomes:

$$\pi : \mathcal{H}^\sharp \longrightarrow \mathcal{H}^\sharp_\natural, \ \pi(h^0, h^1, \ldots, h^n) = S_\delta(h^0) \cdot (h^1, \ldots, h^n),$$

where '$\cdot$' stands for the diagonal action of $\mathcal{H}$ (cf. Section 2, (5.6)).

It is not true in general that the structure maps of $\mathcal{H}^\sharp$ descend to maps $d^i_\delta, s^i_\delta, t_\delta$ on $\mathcal{H}^\sharp_\delta$, but the compatibility with π forces the following formulas, which make sense in general (compare to [31], formulas (37) – (40)):

$$d^i_\delta(h^1, \ldots, h^n) = \begin{cases} (1, h^1, \ldots, h^n) & \text{if } i = 0 \\ (h^1, \ldots, h^{i-1}, \Delta h^i, h^{i+1}, \ldots, h^n) & \text{if } 1 \leq i \leq n \\ (h^1, \ldots, h^n, 1) & \text{if } i = n+1 \end{cases} .$$

$$s^i_\delta(h^1, \ldots, h^n) = (h^1, ..., \epsilon(h^{i+1}), \ldots, h^n), \quad 0 \leq i \leq n-1,$$
$$t_\delta(h^1, \ldots, h^n) = S_\delta(h_1) \cdot (h^2, \ldots, h^n, 1)$$

Apriori $\mathcal{H}^\sharp_\delta$ is just an ∞-cyclic [45] module (in the sense that the cyclic relation $t^{n+1}_\delta = 1$ is not necessarily satisfied). As pointed out by Connes and Moscovici, checking directly the cyclic relation $t^{n+1}_\delta = Id$ (which forces $S^2_\delta = Id$) is not completely trivial. They have proved it in [31] under the assumption mentioned in 5.3.4.

Proposition 5.5.4 *Given a Hopf algebra $\mathcal{H}$ and a character δ, the previous formulas make $\mathcal{H}^\sharp_\delta$ into a cyclic module if and only if $S^2_\delta = Id$. More precisely:*

$$t^{n+1}_\delta(h^1, h^2, \ldots, h^n) = (S^2_\delta(h^1), \ldots, S^2_\delta(h^n)) \ .$$

Proof: Dualizing the construction for algebras (see [34, 45, 83]), to any coalgebra homomorphism $\theta : \mathcal{H} \longrightarrow \mathcal{H}$ one associates a ∞-cyclic module $\mathcal{H}^\sharp(\theta)$. It is a slight modification of $\mathcal{H}^\sharp$ of 5.5.1, obtained by replacing d^{n+1}, t in 5.5.1 by:

$$d^{n+1}(h^0, \ldots, h^n) = \sum (h^0_{(1)}, h^1, \ldots, h^n, \theta(h^0_{(0)})),$$

$$t(h^0, \ldots, h^n) = (h^1, h^2, \ldots, h^n, \theta(h^0)).$$

Choosing $\theta := S^2_\delta$, since $\pi : \mathcal{H}^\sharp(\theta) \longrightarrow \mathcal{H}^\sharp_\delta$ is surjective, it suffices to show that π is compatible with the structure maps. The non-trivial formulas are $\pi d^{n+1}_\theta = d^{n+1}_\theta \pi$, $\pi t_\theta = t_\theta \pi$. We prove the last one. We need the following two relations which follow easily from (5.5), (5.9):

$$\Delta^{n-1} S_\delta(h) = \sum S(h_{(n)}) \otimes \ldots \otimes S(h_{(2)}) \otimes S_\delta(h_{(1)}), \tag{5.15}$$

$$\Delta^{n-1} S_\delta S(h) = \sum S^2(h_{(1)}) \otimes \ldots \otimes S^2(h_{(n-1)}) \otimes S_\delta S(h_{(n)}). \tag{5.16}$$

(where the sums are over $\Delta^{n-1} h = \sum h_{(0)} \otimes ... \otimes h_{(n)}$.) We have :

$$\begin{aligned} & t_\delta \pi(h^0, h^1, \ldots, h^n) = \\ & \quad = \sum t_\delta((S(h^0_{(n)}), \ldots, S(h^0_{(2)}), S_\delta(h^0_{(1)})) \star (h^1, \ldots, h^n)) \\ & \quad = \sum S_\delta(h^1) S_\delta S(h^0_{(n)}) \cdot (S(h^0_{(n-1)}), \ldots, S(h^0_{(2)}), S_\delta(h^0_{(1)}), 1) \star (h^2, \ldots, h^n, 1) \ . \end{aligned}$$

where $\star$ stands for the componentwise 'product' on $\mathcal{H}^{\otimes n}$. We want to prove it equals to $\pi t_\theta(h^0, h^1, ..., h^n) = S_\delta(h^1)\cdot(h^2, ..., h^n, S_\delta^2(h^0)) = S_\delta(h^1)\cdot(1, ..., 1, S_\delta^2(h^0))\star(h^2, ..., h^n, 1)$, so it suffices to show that for any $h^0 = h \in \mathcal{H}$:

$$\sum S_\delta S(h_{(n)})\cdot(S(h_{(n-1)}), \ldots, S(h_{(2)}), S_\delta(h_{(1)}), 1) = (1, \ldots, S_\delta^2(h)). \tag{5.17}$$

Using (5.16), the left hand side is:

$$\sum(S^2(h_{(n)})S(h_{(n-1)}), \ldots, S^2(h_{(2n-3)})S(h_{(2)}), S^2(h_{(2n-2)})S(h_{(1)}), S_\delta S(h_{(2n-1)})) \tag{5.18}$$

Using successively (5.11) for $\delta = \epsilon$, (5.1), and the coassociativity of Δ, this is:

$$\sum(1, ..., 1, \epsilon(h_{(2)}), S^2(h_{(3)})S_\delta(h_{(1)}), S_\delta S(h_{(4)})) =$$
$$= \sum(1, ..., 1, S^2(h_{(2)})S_\delta(h_{(1)}), S_\delta S(h_{(3)})) = \sum(1, ..., 1, \delta(h_{(1)}), S_\delta S(h_{(2)}))\ ,$$

by (5.11). Since $\sum \delta(h_{(1)})S_\delta S(h_{(2)}) = S_\delta(\sum \delta(h_{(1)})S(h_{(2)})) = S_\delta^2(h)$, we obtain the right hand side of (5.17). $\square$

Definition 5.5.5 *If $S_\delta^2 = Id$, define $HC_\delta^*(\mathcal{H})$ as the cohomology defined by the cyclic module $\mathcal{H}_\delta^\natural$; denote by $C_{\lambda,\delta}^*(\mathcal{H})$ the associated cyclic complex, and by $CC_\delta^*(\mathcal{H})$ the associated cyclic bicomplex.*

In connection to our problem 5.3.4, to any δ-invariant trace τ on a $\mathcal{H}$-algebra A one associates a characteristic map $k^\tau : HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A)$,

$$k^\tau(h_1, \ldots, h_n)(a_0, a_1, \ldots, a_n) = \tau(a_0 h_1(a_1)\ldots h_n(a_n)), \tag{5.19}$$

which is compatible with the S-operator (since it exists at the level of cyclic modules). Next we interpret/motivate this characteristic map, as well as the cohomology under discussion.

5.5.6 The (localized) characteristic map: Let A be a $\mathcal{H}$-algebra, and let $\tau : A \longrightarrow \mathbb{C}$ be a trace on A. There is an obvious map induced in cyclic cohomology (which uses just the coalgebra structure of $\mathcal{H}$):

$$\gamma^\tau : HC^*(\mathcal{H}) \longrightarrow HC^*(A),\ (h^0, \ldots, h^n) \longrightarrow \gamma(h^0, \ldots, h^n), \tag{5.20}$$
$$\gamma(h^0, \ldots, h^n)(a_0, \ldots, a_n) = \tau(h^0(a_0)\cdot \ldots \cdot h^n(a_n)).$$

In order to find the relevant complexes in the case of invariant traces, we give a different interpretation of this simple map. We can view the action of $\mathcal{H}$ on A, as a linear map:

$$\gamma_0 : \mathcal{H} \longrightarrow Hom(B(A), A)^1 = Hom_{lin}(A, A)$$

where $B(A)$ is the (DG) bar coalgebra of A. Recall that $B(A)$ is $A^{\otimes n}$ in degrees $n \geq 1$ and 0 otherwise, with the coproduct:

$$\Delta(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1}(a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n),$$

and with the usual b' boundary as differential. Then $Hom(B(A), A)$ is naturally a DG algebra (see [90]), with the product: $\phi * \psi : = m \circ (\phi \otimes \psi) \circ \Delta$ (m stands for the multiplication on A). Explicitly, for $\phi, \psi \in Hom(B(A), A)$ of degrees p and q, respectively,

$$(\phi * \psi)(a_1, \ldots, a_{p+q}) = (-1)^{pq} \phi(a_1, \ldots, a_p) \psi(a_{p+1}, \ldots, a_{p+q}) \ ,$$

The map γ_0 uniquely extends to a DG algebra map:

$$\tilde{\gamma} : T(\mathcal{H}) \longrightarrow Hom(B(A), A). \tag{5.21}$$

This can be viewed as a characteristic map for the flat action (see Proposition 5.7.2). Recall also ([90]) that the norm map N can be viewed as a closed cotrace $N : C_*^{\lambda}(A)[1] \to B(A)$ on the DG coalgebra $B(A)$, that is, N is a chain map with the property that $\Delta \circ N = \sigma \circ \Delta \circ N$, where σ is the graded twist $x \otimes y \mapsto (-1)^{deg(x) deg(y)} y \otimes x$. A formal property of this is that, composing with N and τ, we have an induced trace:

$$\tau_{\natural} : Hom(B(A), A) \longrightarrow C_{\lambda}^*(A)[1], \quad \tau_{\natural}(\phi) = \tau \circ \phi \circ N. \tag{5.22}$$

Composing with $\tilde{\gamma}$, we get a trace on the tensor algebra:

$$\tilde{\gamma}^{\tau} : T(\mathcal{H}) \longrightarrow C_{\lambda}^*(A)[1], \tag{5.23}$$

and then a chain map:

$$\gamma^{\tau} : T(\mathcal{H})_{\natural} \longrightarrow C_{\lambda}^*(A)[1]. \tag{5.24}$$

Via Proposition 5.5.2, it induces (5.20) in cohomology.

Let us now start to use the Hopf algebra structure of $\mathcal{H}$, and the character δ. First of all remark that the map $\tilde{\gamma}$ is $\mathcal{H}$-invariant, where the action of $\mathcal{H}$ on the right hand side of (5.21) comes from the action on A: $(h \cdot \phi)(a) = h\phi(a)$, $\forall\ a \in B(A)$. To check the invariance condition: $\tilde{\gamma}(hx) = h\tilde{\gamma}(x)$, $\forall\ x \in T(\mathcal{H})$, remark that the flatness of the action reduces the checking to the case where $x \in \mathcal{H} = T(\mathcal{H})^1$, and that is obvious. Secondly, remark that if the trace τ is δ-invariant, then so is (5.22). In conclusion, $\tilde{\gamma}^{\tau}$ in (5.23) is an invariant trace on the tensor algebra, so our map (5.24) descends to a chain map:

$$\gamma_{\delta}^{\tau} : T(\mathcal{H})_{\natural, \delta} \longrightarrow C_{\lambda}^*(A)[1],$$

So $H^*(TH_{\natural, \delta})$ naturally appears as the solution of our problem 5.3.4; also, using the localized X-complex $X_{\delta}(T\mathcal{H})$ (see 5.4.2), we have a short exact sequence:

$$0 \longrightarrow T\mathcal{H}_{\natural, \delta} \xrightarrow{N} T\mathcal{H}_{(\delta)} \xrightarrow{1-t} T\mathcal{H}_{(\delta)} \longrightarrow T\mathcal{H}_{\natural, \delta} \longrightarrow 0 \tag{5.25}$$

describing an S-operation (cf 5.4.1) in our cohomology $H^*(TH_{\natural, \delta})$. These new objects are related to 5.5.5 by the following (compare to Proposition 5.5.2):

Theorem 5.5.7 *Given a Hopf algebra $\mathcal{H}$ and a character δ such that $S_{\delta}^2 = Id$, one has isomorphisms:*

$$C_{\lambda, \delta}^*(\mathcal{H}) \cong TH_{\natural, \delta}, \quad CC_{\delta}^*(\mathcal{H}) \cong X_{\delta}(T\mathcal{H}),$$

up to the same degree shift as in Proposition 5.5.2.

proof: We have seen in Proposition 5.4.4:

$$\Omega^1(T\mathcal{H})_\natural \cong T\mathcal{H}, \quad \Omega^1(T\mathcal{H})_{\natural,\delta} \cong (T\mathcal{H})_\delta.$$

The first isomorphism is the one which gives the identification $X(T\mathcal{H}) \cong CC^*(\mathcal{H})$ of Proposition 5.5.2. The second isomorphism, combined with the isomorphism (cf. Lemma 5.2.3):

$$(T\mathcal{H})_\delta^{n+1} \cong \mathcal{H}^{\otimes n}, \ [h_0 \otimes h_1 \otimes \ldots \otimes h_n] \mapsto S_\delta(h_0)\cdot(h_1 \otimes \ldots \otimes h_n),$$

(with the inverse $h_1 \otimes \ldots \otimes h_n \mapsto [1 \otimes h_1 \otimes \ldots \otimes h_n]$), gives the identification $X_\delta(T\mathcal{H}) \cong CC_\delta^*(\mathcal{H})$. $\square$

5.5.8 The uni-modular case: Motivated by examples like quantum groups, compact matrix groups [100] and their duals, Connes and Moscovici have recently proposed [32] an extension of $HC_\delta^*(\mathcal{H})$ to the more general case where S_δ is not necessarily involutive, but there exists an invertible group-like element $\sigma \in \mathcal{H}$ such that:

$$S_\delta^2(h) = \sigma h \sigma^{-1} \ \ \forall\, h \in \mathcal{H}, \quad \delta(\sigma) = 1. \tag{5.26}$$

In the terminology of [32], one says that (δ, σ) is a *modular pair*. For any such pair (δ,σ), one defines a cyclic module $\mathcal{H}^\natural_{\delta,\sigma}$ by the same formulas as in 5.5.3 except for:

$$d_{\delta,\sigma}^{n+1}(h^1, \ldots, h^n) = (h^1, \ldots, h^n, \sigma)\ ,$$

$$t_{\delta,\sigma}(h^1, \ldots, h^n) = S_\delta(h_1)(h^2, \ldots, h^n, \sigma)\ .$$

Let $C^*_{\lambda,\delta,\sigma}(\mathcal{H})$, $CC^*_{\delta,\sigma}(\mathcal{H})$ be the associated cyclic complex, and cyclic bicomplex, respectively. The resulting cohomology is denoted by $HC^*_{\delta,\sigma}(\mathcal{H})$, and appears as the target of characteristic maps associated to pairs (A, τ) with τ a δ-invariant σ-trace (i.e. $\tau(ab) = \tau(b\sigma(a))$).

Our interpretations extend to this setting. For any $\mathcal{H}$-algebra R, we define the following localized complex:

$$X_{\delta,\sigma}(R): \quad R_\delta \underset{d}{\overset{b_\sigma}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}\ \ ,$$

where, this time, $b_\sigma(dxy) = -[x,y]_\sigma$, where $[x,y]_\sigma$ is the twisted commutator $xy - y\sigma(x)$, and $\Omega^1(R)_{\natural,\delta}$ is the quotient of $\Omega^1(R)$ by the subspace linearly spanned by coinvariants and twisted commutators $[x,\omega]_\sigma$ ($x \in R$, $\omega \in \Omega^1(R)$). Similarly one defines $R_{\natural,\delta}$, which fits into a sequence (exact on the right):

$$0 \longrightarrow R_{\natural,\delta} \xrightarrow{d} \Omega^1(R)_{\natural,\delta} \xrightarrow{b_\sigma} R_\delta \xrightarrow{\natural} R_{\natural,\delta} \longrightarrow 0.$$

The construction applies also to the graded case.

Theorem 5.5.9 *Let (δ,σ) be as before. Then, for any $\mathcal{H}$ (DG) algebra R, $X_{\delta,\sigma}(R)$ is a well defined complex. For $R = T\mathcal{H}$:*

$$T\mathcal{H}_{\natural,\delta} \cong C^*_{\lambda,\delta,\sigma}(\mathcal{H}), \quad , X_{\delta,\sigma}(T\mathcal{H}) \cong CC^*_{\delta,\sigma}(\mathcal{H})\ ,$$

up to the same degree shift as in Proposition 5.5.2.

proof: The first part follows from the fact that $d : R \longrightarrow \Omega^1(R)$, and $b_\sigma : \Omega^1(R)$ map coinvariants into coinvariants (with the same proof as for 5.3.5), b_σ kills the twisted commutators, $b_\sigma d = 0$ modulo coinvariants (straightforward), and $db_\sigma = 0$ in $\Omega^1(R)_{\natural,\delta}$. The last assertion follows from $\delta(\sigma) = 1$, and the relation:

$$db_\sigma(dxy) = [x, dy]_\sigma - [\sigma^{-1}(y), dx]_\sigma + (\sigma^{-1}(\omega) - \omega),$$

where $\omega = yd(\sigma(x))$. The second part is a straightforward extension of 5.4.4, 5.5.7. $\square$

One can also extend our interpretations 5.5.6 of the characteristic map.

5.6 Some Examples

In this section we compute the cohomology under discussion in several examples. Unless specified, (δ, σ) is a pair consisting of a character, and an invertible group-like element as in 5.5.8 (i.e. satisfying $S_\delta^2(h) = \sigma h \sigma^{-1}$). In most of our examples, $\sigma = 1$.
As a technical tool, let us remark that the complex computing $HH^*_{\delta,\sigma}(\mathcal{H})$ depends just on the coalgebra structure of $\mathcal{H}$, and the group like elements $1, \sigma \in \mathcal{H}$. More precisely, denoting by $\mathbf{C}_\sigma$ the (left/right) one-dimensional $\mathcal{H}$ comodule induced by the group-like element σ, and by $\mathbf{C}$ the one corresponding to $\sigma = 1$, we have:

Lemma 5.6.1 *There are isomorphisms:*

$$HH^*_{\delta,\sigma}(\mathcal{H}) \cong Cotor^*_{\mathcal{H}}(\mathbb{C}, \mathbb{C}_\sigma)\ ,$$

proof: For any group-like element σ one has a standard resolution $\mathbf{C}_\sigma \xrightarrow{\sigma} B(\mathcal{H}, \mathbb{C}_\sigma)$ of $\mathbf{C}_\sigma$ by (free) left $\mathcal{H}$ comodules. Here $B(\mathcal{H}, \mathbb{C}_\eta)$ is $\mathcal{H}^{\otimes(n+1)}$ in degree n, and has the boundary:

$$d_\sigma{}'(h^0, \ldots, h^n) = \sum_{i=0}^{n}(-1)^i(h^0, \ldots, \Delta(h^i), \ldots, h^n) + (-1)^{n+1}(h^0, \ldots, h^n, \sigma). \quad (5.27)$$

Hence $Cotor_{\mathcal{H}}(\mathbb{C}, \mathbb{C}_\sigma)$ is computed by the chain complex $\mathbb{C}\square_{\mathcal{H}} B(\mathcal{H}, \mathbb{C}_\sigma)$, that is, by the Hochschild complex of $\mathcal{H}^\sharp_\delta$. $\square$

5.6.2 Example (group-algebras): If $\mathcal{H} = \mathbf{C}[\Gamma]$ is the group algebra of a discrete group Γ (see 5.3.3), we have:

$$HP^0_\epsilon(\mathbf{C}[\Gamma]) \cong \mathbf{C}, \quad HP^1_\epsilon(\mathbf{C}[\Gamma]) \cong 0$$

($\epsilon =$ the counit, $\sigma = 1$).

proof: We have the following periodic resolution I^* of $\mathbf{C}$ by free $\mathbf{C}[\Gamma]$-comodules:

$$0 \longrightarrow \mathbf{C} \xrightarrow{\eta} \mathbf{C}[\Gamma] \xrightarrow{\alpha} \mathbf{C}[\Gamma] \xrightarrow{\beta} \mathbf{C}[\Gamma] \xrightarrow{\alpha} \mathbf{C}[\Gamma] \longrightarrow \ldots$$

where $\eta(1) = 1$, $\alpha(g) = g$ for $g \neq 1$ and $\alpha(1) = 0$, $\beta(g) = 0$ for $g \neq 1$ and $\beta(1) = 1$. Hence $HH_\epsilon(\mathcal{H}) = Cotor_{\mathcal{H}}(\mathbf{C}, \mathbf{C})$ is computed by $\mathbf{C}\square_{\mathcal{H}} I^*$, that is, by $0 \longrightarrow \mathbf{C} \xrightarrow{0} \mathbf{C} \xrightarrow{id} \mathbf{C} \xrightarrow{0} \mathbf{C} \xrightarrow{id} \ldots$. So $HH^*_\epsilon(\mathcal{H}) = \mathbf{C}$ if $n = 0$ and 0 otherwise, and the statement follows from the SBI sequence. $\square$

5.6.3 Example (algebras with Haar measures): Recall that a left Haar measure for the Hopf algebra $\mathcal{H}$ is a linear map $\tau : \mathcal{H} \longrightarrow \mathbf{C}$ with the property $\tau(1) = 1$, $\sum \tau(h_0)h_1 = \tau(h)\cdot 1$ for all $h \in \mathcal{H}$. Basic Hopf algebras which admit Haar measure are: finite dimensional Hopf algebras (by 5.1.6 of [95]), group-algebras, algebras of smooth functions on a compact quantum group G (by the fundamental Theorem 4.2 of [100]). We recall that in the case of compact matrix groups there is a preferred choice of the character δ, namely the modular character f_{-1} of Theorem 5.6 [100]. One has:

Proposition 5.6.4 *If $\mathcal{H}$ admits a left Haar measure then:*

$$HP^0_\delta(\mathcal{H}) \cong \mathbf{C}, \quad HP^1_\delta(\mathcal{H}) \cong 0$$

proof: Use the SBI sequence and the fact that the left measure τ induces a contraction $(h^0, \ldots, h^n) \mapsto \tau(h^0)(h^1, \ldots, h^n)$ of the Hochschild complex. $\square$

5.6.5 Example (enveloping algebras): Following [31] (Theorem 6.(i)), We present now a detailed computation for the case where $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$. Let δ be a character of $\mathfrak{g}$ (i. e. $\delta : \mathfrak{g} \to \mathbb{C}$ linear, with $\delta|_{[\mathfrak{g},\mathfrak{g}]} = 0$), and extend it to $U(\mathfrak{g})$. Denote by $\mathbb{C}_\delta$ the $\mathfrak{g}$ module $\mathbb{C}$ with the action induced by δ. Since $S_\delta(v) = -v + \delta(v)$ for all $v \in \mathfrak{g}$, we are in the uni-modular case ($\sigma = 1$). The final result of our computation is:

Theorem 5.6.6 *For any Lie algebra $\mathfrak{g}$, and any $\delta \in \mathfrak{g}^*$:*

$$HP^*_\delta(U(\mathfrak{g})) \cong \bigoplus_{i \equiv * \bmod 2} H_i(\mathfrak{g}; \mathbb{C}_\delta).$$

As a first step in the proof of 5.6.6, let us look at the symmetric (Hopf) algebra $S(V)$ on a vector space V. Recall that the coproduct is defined on generators by $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\forall v \in V$.

Lemma 5.6.7 *For any vector space V, the maps $A : \Lambda^n(V) \longrightarrow V^{\otimes n} \subset S(V)^{\otimes n}$, $v_1 \wedge \ldots \wedge v_n \mapsto (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$ induce isomorphisms:*

$$HH^*_\delta(S(V)) \cong \Lambda^*(V).$$

proof: We will use a Koszul type resolution for the left $S(V)$ comodule $\mathbb{C}_\delta$. Let $e_1, \ldots, e_k$ be a basis of V, and $\pi^i \in V^*$ the dual basis. The linear maps π^i extend uniquely to derivations $\pi^i : S(V) \longrightarrow S(V)$. Remark that each of the π^i's are maps of left $S(V)$ comodules. Indeed, to check that $(1 \otimes \pi^i)\circ\Delta = \Delta\circ\pi^i$, since both sides satisfy the Leibniz rule, it is enough to check it on the generators $e_i \in S(V)$, and that is easy. Consider now the co-augmented complex of left $S(V)$ comodules:

$$0 \longrightarrow \mathbb{C}_\eta \xrightarrow{\eta} S(V) \otimes \Lambda^0(V) \xrightarrow{d} S(V) \otimes \Lambda^1(V) \xrightarrow{d} \ldots,$$

with the boundary $d = \sum \pi^i \otimes e_i$, that is:

$$d(x \otimes v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{k} \pi^i(x) \otimes e_i \wedge v_1 \wedge \ldots \wedge v_n.$$

Point out that the definition does not depend on the choice of the basis, and it is dual to the Cartan boundary on the Weil complex of V, viewed as a commutative Lie algebra. This also explains the exactness of the sequence. Alternatively, one can use a standard 'Koszul argument', or, even simpler, remark that $(S(V)\otimes\Lambda^*(V))\otimes(S(W)\otimes\Lambda^*(W))\cong(S(V\oplus W)\otimes\Lambda^*(V\oplus W))$ as chain complexes (for any two vector spaces V and W), which reduces the assertion to the case where $dim(V)=1$. So we get a resolution $\mathbb{C}_\eta\longrightarrow S(V)\otimes\Lambda^*(V)$ by free (hence injective) left $S(V)$ comodules. Then 5.6.1 implies that $HH^*_\delta(S(V))$ is computed by $\mathbb{C}_\eta\Box_{S(V)}(S(V)\otimes\Lambda^*(V))$, that is, by $\Lambda^*(V)$ with the zero differential. This proves the second part of the theorem.

To show that the isomorphism is induced by A, we have to compare the previous resolution with the standard bar resolution $B(S(V),\mathbb{C}_\eta)$ (see the proof of 5.6.1). We define a chain map of left $S(V)$ comodules:

$$P: B(S(V),\mathbb{C}_\eta)\longrightarrow S(V)\otimes\Lambda^*(V)\ ,$$

$$P(x_0\otimes x_1\otimes\ldots\otimes x_n)=x_0\otimes pr(x_1)\wedge\ldots\wedge pr(x_n),$$

where $pr: S(V)\longrightarrow V$ is the obvious projection map. We check now that it is a chain map, i.e.:

$$dP(x_0\otimes x_1\otimes ...\otimes x_n)=Pd(x_0\otimes x_1\otimes\ldots\otimes x_n).$$

First of all, we may assume $x_1,\ldots,x_n\in V$ (otherwise, both terms are zero). The left hand side is then:

$$\sum_{i=1}^{k}\pi^i(x_0)\otimes e_i\wedge x_1\wedge\ldots\wedge x_n,$$

while the right hand side is:

$$P(\Delta(x_0)\otimes x_1\otimes\ldots\otimes x_n)=(id\otimes pr)(\Delta(x_0))\wedge x_1\wedge\ldots\wedge x_n\ .$$

So we are left with proving that:

$$(id\otimes pr)\Delta(x)=\sum_{i=1}^{k}\pi^i(x)\otimes e_i,\quad\forall\ x\in S(V),$$

and this can be checked directly on the linear basis $x=e_{i_1}\ldots e_{i_n}\in S(V)$. In conclusion, P is a chain map between our free resolutions of $\mathbb{C}_\eta$ (in the category of left $S(V)$ comodules). By the usual homological algebra, the induced map $\bar{P}$ obtained after applying the functor $\mathbb{C}_\eta\Box_{S(V)}$- , induces isomorphism in cohomology. From the explicit formula:

$$\bar{P}(x_1\otimes\ldots\otimes x_n)=pr(x_1)\wedge\ldots\wedge pr(x_n),$$

we see that $\bar{P}\circ A=Id$, so our isomorphism is induced by both $\bar{P}$ and A. $\Box$

proof of 5.6.6: Consider the mixed complex [67]:

$$\Lambda:\ \Lambda^0(\mathfrak{g})\underset{0}{\overset{d_{Lie}}{\rightleftarrows}}\Lambda^1(\mathfrak{g})\underset{0}{\overset{d_{Lie}}{\rightleftarrows}}\Lambda^2(\mathfrak{g})\underset{0}{\overset{d_{Lie}}{\rightleftarrows}}\ \cdots\ ,$$

where d_{Lie} stands for the usual boundary in the Chevalley-Eilenberg complex computing $H_*(\mathfrak{g})$. Denote by $\mathcal{B}$ the mixed complex associated to the cyclic module $\mathcal{H}^\sharp$, and by $\mathcal{B}_\delta$ its localization, i.e. the mixed complex associated to the cyclic module $\mathcal{H}^\sharp_\delta$ (so they are the mixed complexes computing $HC^*(\mathcal{H})$, and $HC^*_\delta(\mathcal{H})$, respectively). Here $\mathcal{H} = U(\mathfrak{g})$. Let $\pi : \mathcal{B} \to \mathcal{B}_\delta$ be the projection map, which, after our identifications (see 5.5.3), is degree-wise given by:

$$\pi : \mathcal{H}^{\otimes(n+1)} \longrightarrow \mathcal{H}^{\otimes n},\ \pi(h_0 \otimes \ldots \otimes h_n) = S_\delta(h_0) \cdot (h_1 \otimes \ldots \otimes h_n).$$

Denote by B and B_δ the usual (degree (-1)) 'B- boundaries' of the two mixed complexes $\mathcal{B}, \mathcal{B}_\delta$. Recall that $B = N\sigma_{-1}\tau$, where:

$$\sigma_{-1}(h_0, \ldots, h_n) = \epsilon(h_0)(h_1, \ldots, h_n),\ \tau(h_0, \ldots, h_n) = (-1)^n(h_1, \ldots, h_n, h_0)\ ,$$

and $N = 1 + \tau + \ldots + \tau^n$ on $\mathcal{H}^{\otimes(n+1)}$.

We will show that Λ and $\mathcal{B}_\delta$ are quasi-isomorphic mixed complexes (which easily implies the theorem), but for the computation we have to use the mixed complex $\mathcal{B}$, where explicit formulas are easier to write. We define the map:

$$A : \Lambda^n(\mathfrak{g}) \longrightarrow \mathcal{H}^{\otimes n},\ A(v_1 \wedge \ldots \wedge v_n) = (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!.$$

The fact that the (localized) Hochschild boundary depends just on the coalgebra structure of $U(\mathfrak{g})$ and on the unit, which are preserved by the Poincaré-Birkhoff-Witt Theorem (see e.g. [97]), together with Lemma 5.6.7, shows that A is a quasi-isomorphism of mixed complexes, provided we prove its compatibility with the degree (-1) boundaries, that is:

$$B_\delta(A(x)) = A(d_{Lie}(x)),\quad \forall\ x = v_1 \wedge \ \ldots \wedge v_n \in \Lambda^n(\mathfrak{g}). \tag{5.28}$$

Using that $A(x) = \pi(y)$, where $y = (\sum sign(\sigma) 1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$, we have

$$\begin{aligned}
B_\delta A(x) &= \pi(B(y)) = \\
&= \pi N\sigma_{-1}(\sum sign(\sigma)\,(1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} - (-1)^n v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \otimes 1))/n! \\
&= \pi(N(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}))/n! \\
&= \pi(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/(n-1)!\ .
\end{aligned}$$

But

$$\pi(v \otimes v_1 \otimes \ldots \otimes v_n) = \delta(v) v_1 \otimes \ldots \otimes v_n - \sum_{i=1}^n v_1 \otimes \ldots \otimes vv_i \otimes \ldots v_n\ ,$$

and, with these, it is straightforward to see that $B_\delta A(x)$ equals to:

$$A(\sum_{i=1}^n (-1)^{i+1}\delta(v_1) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_n + \sum_{i<j}(-1)^{i+j}[v_i, v_j] \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_n),$$

i.e. with $A(d_{Lie}(x))$. $\square$

5.6.8 Example (the quantum enveloping algebra of sl_2): We look now at the simplest example of a quantized envelopping algebra, namely $U_q(sl_2)$. As an algebra, it is generated by the symbols E, F, K, K^{-1}, subject to the relations: $KE = q^2 EK, KF =$

$q^{-2}FK, KK^{-1} = K^{-1}K = 1, [E, F] = (K - K^{-1})/(q - q^{-1})$. The co-algebra structure is given by:

$$\Delta(K) = K \otimes K,\ \Delta(E) = 1 \otimes E + E \otimes K,\ \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\epsilon(K) = 1,\ \epsilon(E) = \epsilon(F) = 0,$$

while for the antipode: $S(K) = K^{-1}$, $S(E) = -EK^{-1}$, $S(F) = -KF$. One has $S^2(h) = KhK^{-1}$ for all $h \in U_q(sl_2)$, hence this is a first example with $\sigma \neq 1$.

Proposition 5.6.9 *$HP^0(U_q(sl_2)) = 0$, and $HP^1(U_q(sl_2)) \cong \mathbf{C}^2$ with the generators represented by E and KF.*
(we have omitted the indices $\delta = \epsilon$, $\sigma = K$ from the notation)

proof: Denote $E = x$, $K = \sigma$, $KF = y$. Clearly x, and y define cyclic cocycles. Using the SBI sequence, it suffices to prove the similar statement for Hochschild cohomology. We first prove that, for any n,

$$Cotor_{\mathcal{H}}(\mathbf{C}, \mathbf{C}_{\sigma^a}) = 0, \quad \forall\ a\ < 0\ . \tag{5.29}$$

We use induction over n. It is obvious for $n = 0$; let us assume it is true for any $k < n$. Remark that, by the proof of 5.6.1, $Cotor(\mathbf{C}_\alpha, \mathbf{C}_\beta)$ (α, β- group-like elements) is computed by the complex $B(\mathcal{H}; \mathbf{C}_\alpha, \mathbf{C}_\beta)$, which is $\mathcal{H}^{\otimes n}$ in degree n and has the boundary $u \mapsto (\alpha, u) - d_\beta{}'(u)$ (see (5.27)). Denote $B_a = B(\mathcal{H}; \mathbf{C}, \mathbf{C}_{\sigma^a})$, and d_a its boundary.
One has the following basis of $\mathcal{H}$:

$$\{x^m y^k \sigma^p : m, k, p\ -\text{integers}\ \ , m, k \geq 0\}\ . \tag{5.30}$$

Let '$\leq$' be the order $(m, k) \leq (m', k')$ iff $m \leq m'$, or $m = m'$ and $k \leq k'$. For any pair (m, k) of positive integers, denote by $L_{m,k}$, and $L^{<}_{m,k}$ the subcomplexes of B_a linearly spanned by elements of type $(x^i y^j, \dots)$, with $(i, j) \leq (m, k)$, and $(i, j) < (m, k)$, respectively.
For the proof of (5.29), let $[z] \in Cotor^n(\mathbf{C}, \mathbf{C}_{\sigma^a})$, represented by a cocycle $z \in B_a$. We claim that:

$$\exists m, k \geq 0,\ \exists u \in L_{m,k}\ :\ \ [z] = [u]\ . \tag{5.31}$$

Indeed, defining $\tau : \mathcal{H} \longrightarrow \mathbf{C}$ on the basis (5.30) by $\tau(1) = 1$ and 0 otherwise, and $\theta = \tau \otimes Id_{\mathcal{H}} \otimes \dots \otimes Id_{\mathcal{H}} : B_a \longrightarrow B_a$, we have $d_a\theta + \theta d_a = Id - \phi$ where ϕ is identity on elements of type $(x^m y^k, \dots)$, $(m, k) \neq (0, 0)$, and 0 otherwise. Choose (m, k) minimal such that (5.31) holds. Hence we find $v \in B_a^{n-1}$ such that:

$$u \equiv (x^m y^k, v)\ \ \text{mod}\ \ \ L^{<}_{m,k}\ .$$

Assume first $(m, k) \neq (0, 0)$. Since $d_a(u) \equiv -(x^m y^k, \sigma^{m+k}, v) + (x^m y^k, d_a{}'(v))$ mod $L^{<}_{m,k}$, we must have $(\sigma^{m+k}, v) = d_a{}'(v)$, i.e. v represents a $(n-1)$-cocycle in the standard complex computing $Cotor(\mathbf{C}_{\sigma^{m+k}}, \mathbf{C}_{\sigma^a})$. This complex is isomorphic (by the multiplication by σ^{-m-k}) to the standard complex computing $Cotor(\mathbf{C}, \mathbf{C}_{\sigma^{a-m-k}})$, hence, by the induction hypothesis, $v = (\sigma^{m+k}, w) - d_a{}'(w)$ for some w. Choosing $u' = u + d_a(x^m y^k, w)$, we then have $[z] = [u']$, and $u' \in L^{<}_{m,k}$, which contradicts the minimality of (m, k). We

are left with the case $(m,k)=(0,0)$, when, since $\phi(u)=0$, one gets $[u]=0$.
A completely similar argument shows that:

$$Cotor^n(\mathbf{C},\mathbf{C})=0 \;\; \forall\, n\geq 1\;, \quad Cotor^n(\mathbf{C},\mathbf{C}_\sigma)=0 \;\; \forall\, n\geq 2\;,$$

while clearly $Cotor^0(\mathbf{C},\mathbf{C}_\sigma)=0$. Let $[z]\in Cotor^1(\mathbf{C},\mathbf{C}_\sigma)$. As above, we find a minimal (m,k) such that (5.31) holds, and let $\eta\in\mathbf{C}$ such that $u\equiv \eta x^m y^k \bmod L^{<}_{m,k}$. Again, since $d_1(u)\equiv \eta((x^m y^k,\sigma^{m+k})-(x^m y^k,\sigma)) \bmod L^{<}_{m,k}$, we must have $\eta\sigma^{m+k}=\eta\sigma$, hence $\eta=0$, or $m+k=1$. In other words, $Cotor^1(\mathbf{C},\mathbf{C}_\sigma)\cong \mathbf{C}^2$, with the generators $[x],[y]$.
$\square$

5.7 A non-commutative Weil complex

In this section we introduce/describe a non-commutative Weil complex associated to a coalgebra, which extends/explains some results in [46, 89], and will naturally appear in the construction of characteristic maps associated to higher traces (section 5.8). We describe the relevant cohomologies (analogues of Chern, Chern/Simons classes), and (using section 5.4) the S-operators acting on them.

Let $\mathcal{H}$ be a coalgebra. Define its *non-commutative Weil algebra* $W(\mathcal{H})$ as the (non-commutative, non-unital) DG algebra freely generated by the symbols h of degree 1, $\omega(h)$ of degree 2, linear on $h\in\mathcal{H}$. The differential of $W(\mathcal{H})$ is similar to the b' differential of $T(\mathcal{H})$ (see 5.5.1). It is denoted by ∂, and is the unique derivation which acts on generators by:

$$\partial(h)=\omega_h-\sum h_0h_1,$$
$$\partial(\omega_h)=\sum \omega_{h_0}h_1-\sum h_0\omega_{h_1}.$$

Example 5.7.1 This algebra is intended to be a non-commutative analogue of the usual Weil complex of a Lie algebra (see the preliminary section 1.4, formulas (1.8), or [23, 43]). Particular cases have been used in the study of universal Chern-Simons forms. When $\mathcal{H}=\mathbf{C}\rho$ (i.e. $\mathbf{C}$, with 1 denoted by ρ), with $\Delta(\rho)=\rho\otimes\rho$, it is the algebra introduced in [89]; for $\mathcal{H}=\mathbf{C}\rho_1\bigoplus\ldots\bigoplus\mathbf{C}\rho_n$ with $\Delta(\rho_i)=\rho_i\otimes\rho_i$, we obtain one of the algebras studied on [46].

We discuss now its 'universal property'. Given a DG algebra Ω^*, and a linear map:

$$\phi:\mathcal{H}\longrightarrow\Omega^1,$$

define its *curvature*:

$$\omega_\phi:\mathcal{H}\longrightarrow\Omega^2,\;\; \omega_\phi(h)=d\phi(h)+\sum\phi(h_0)\phi(h_1).$$

Alternatively, using the natural DG algebra structure of $Hom(\mathcal{H},\Omega^*)$,

$$\omega_\phi:=d(\phi)-1/2[\phi,\phi]\in Hom(\mathcal{H},\Omega)^2.$$

There is a unique algebra homomorphism (the characteristic map of ϕ):

$$k(\phi):W(\mathcal{H})\longrightarrow\Omega^*,$$

sending h to $\phi(h)$ and ω_h to $\omega_\phi(h)$.

One can easily see that (compare with the usual Weil complex of a Lie algebra):

Proposition 5.7.2 . *The previous construction induces a* $1-1$ *correspondence between linear maps* $\phi : \mathcal{H} \longrightarrow \Omega^1$ *and DG algebra maps* $k : W(\mathcal{H}) \longrightarrow \Omega^*$. *In particular, there is a* $1-1$ *correspondence between flat linear maps* $\phi : \mathcal{H} \longrightarrow \Omega^1$ *(i. e. with the property that* $\omega_\phi = 0$*), and DG algebra maps* $k : T(\mathcal{H}) \longrightarrow \Omega^*$.

An immediate consequence is that $W(\mathcal{H})$ does not depend on the co-algebra structure of $\mathcal{H}$. Actually one can see directly that $(W(\mathcal{H}), \partial) \cong (W(\mathcal{H}), d)$, where d is the derivation on $W(\mathcal{H})$ defined on generators by $d(h) = \omega_h$, $d(\omega_h) = 0$ (i.e. the differential corresponding to $\mathcal{H}$ with the trivial co-product). An explicit isomorphism sends h to h and ω_h to $\omega_h + \sum h_0 h_1$.

Corollary 5.7.3 . *The Weil algebra* $W(\mathcal{H})$*, and the complex* $W(\mathcal{H})_\natural$ *are acyclic.*

5.7.4 Extra-structure on $W(\mathcal{H})$**:** Now we look at the extra-structure of $W(\mathcal{H})$. First of all, denote by $I(\mathcal{H})$ the ideal generated by the curvatures ω_h. The powers of $I(\mathcal{H})$, and the induced *truncations* are denoted by:

$$I_n(\mathcal{H}) := I(\mathcal{H})^{n+1}, \quad W_n(\mathcal{H}) := W(\mathcal{H})/I(\mathcal{H})^{n+1}.$$

These are analogous to the complexes which appear in the construction of the geometric characteristic classes for foliations (see (1.22) and (1.23) in our Preliminary section 1.4). Remark that $W_0(\mathcal{H}) = T(\mathcal{H})$ is the tensor (DG) algebra of $\mathcal{H}$ (up to a minus sign in the boundary, which is irrelevant, and will be ignored). Dual to even higher traces, we introduce the complex:

$$W_n(\mathcal{H})_\natural := W_n(\mathcal{H})/[W_n(\mathcal{H}), W_n(\mathcal{H})]$$

obtained dividing out the (graded commutators). In the terminology of [46] (pp. 103), it is the space of 'cyclic words'. Dual to odd higher traces:

$$I_n(\mathcal{H})_\natural := I_n(\mathcal{H})/[I(\mathcal{H}), I_{n-1}(\mathcal{H})].$$

It is interesting that all these complexes compute the same cohomology (independent of n !), namely the cyclic cohomology of $\mathcal{H}$ viewed as a coalgebra. This is the content of Theorem 5.7.7, Proposition 5.5.2, and Section 5.9.

Secondly, we point out a bi-grading on $W(\mathcal{H})$: defining ∂_0 such that $\partial = \partial_0 + d$, then $W(\mathcal{H})$ has a structure of bigraded differential algebra, with $deg(h) = (1, 0), deg(\omega_h) = (1, 1)$. Actually $W(\mathcal{H})$ can be viewed as the tensor algebra of $\mathcal{H}^{(1,0)} \bigoplus \mathcal{H}^{(1,1)}$ (two copies of $\mathcal{H}$ on the indicated bi-degrees). With this bi-grading, q in $W^{p,q}$ counts the number of curvatures. The boundary d increases q, while ∂_0 increases p.

5.7.5 Example: Let us have a closer look at $\mathcal{H} = \mathbf{C}\rho$ with $\Delta(\rho) = h\rho \otimes \rho$, for which the computations were carried out by D. Quillen [89], recalling the main features of our complexes:

(1) ω^n are cocycles of $W(\mathcal{H})_\natural$ (where $\omega = \omega_{h\rho}$). They are trivial in cohomology (cf. Corollary 5.7.3).

(2) the place where ω^n give non-trivial cohomology classes is $I_{m,\natural}$, with m sufficiently large.

(3) the cocycles ω^n (trivial in $W(\mathcal{H})_\natural$) transgress to certain (Chern-Simons) classes.

The natural complex in which these classes are non-trivial (in cohomology) is $W_m(\mathcal{H})_\natural$.

(4) there are striking 'suspensions' (by degree 2 up) in the cohomology of all the complexes $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$, $\tilde{I}_n(\mathcal{H})_\natural$.

Our intention is also to explain these phenomena (in our general setting).

5.7.6 'Chern-Simons contractions'. Starting with two linear maps:

$$\rho_0, \rho_1 : \mathcal{H} \longrightarrow \Omega^1,$$

we form:

$$t\rho_0 + (1-t)\rho_1 := \rho_0 \otimes t + \rho_1 \otimes (1-t) : \mathcal{H} \longrightarrow (\Omega^* \otimes \Omega(1))^1,$$

where $\Omega(1)$ is the algebraic DeRham complex of the line: $\mathbb{C}[t]$ in degree 0, and $\mathbb{C}[t]dt$ in degree 1, with the usual differential. Composing its characteristic map $W(\mathcal{H}) \longrightarrow \Omega \otimes \Omega(1)$, with the degree -1 map $\Omega \otimes \Omega(1) \longrightarrow \Omega$ coming from the integration map $\int_0^1 : \Omega(1) \longrightarrow \mathbb{C}$ (emphasize that we use the graded tensor product, and the integration map has degree -1), we get a degree -1 chain map:

$$k(\rho_0, \rho_1) : W(\mathcal{H}) \longrightarrow \Omega.$$

As usual, $[k(\rho_0, \rho_1), \partial] = k(\rho_1) - k(\rho_0)$.
The particular case where $\Omega = W(\mathcal{H}), \rho_0 = 0, \rho_1 = Id_{\mathcal{H}}$ gives a contraction of $W(\mathcal{H})$:

$$H := k(Id_{\mathcal{H}}, 0) : W(\mathcal{H}) \longrightarrow W(\mathcal{H}).$$

We point out that H preserves commutators, and induces a chain map

$$CS : I_n(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural[1], \quad [x] \mapsto [H(x)], \tag{5.32}$$

to which we will refer as the Chern Simons map. The formulas for the contraction H resemble the usual ones ([46, 90, 89]). For instance, at the level of $W(\mathcal{H})_\natural$, one has:

$$H(\frac{\omega_h^{n+1}}{(n+1)!}) = \int_0^1 \frac{1}{n!} h(t\omega_h + (t^2 - t)\sum_{(h)} h_0 h_1)^n dt \tag{5.33}$$

Theorem 5.7.7 *The Chern-Simons map (5.32) induces an isomorphism* $H^*(I_n(\mathcal{H})_\natural) \xrightarrow{\sim} H^{*-1}(W_n(\mathcal{H})_\natural)$ *(compatible with the* S*-operator described below).*

proof: We consider the following slight modification of $I_n(\mathcal{H})$:

$$\tilde{I}_n(\mathcal{H})_\natural := I_n(\mathcal{H}) / I_n(\mathcal{H}) \cap [W(\mathcal{H}), W(\mathcal{H})].$$

One has a short exact sequence:

$$0 \longrightarrow \tilde{I}_n(\mathcal{H})_\natural \longrightarrow W(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0,$$

and, using Corollary 5.7.3, the boundary of the long exact sequence induced in cohomology gives an isomorphism $\tilde{\partial} : H^{*-1}(W_n(\mathcal{H})_\natural) \xrightarrow{\sim} H^*(\tilde{I}_n(\mathcal{H})_\natural)$. The same formula (5.32) defines a chain map $\tilde{CS} : \tilde{I}_n(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural[1]$, and one can easily check that $\tilde{CS} \circ \tilde{\partial} = Id$. Now, since CS is the composition of $\tilde{CS}$ with the canonical projection $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$, it suffices to show that the last map induces isomorphism in cohomology. We prove this after describing the S- operator. $\square$

5.7.8 The S-operator: The discussion in 5.4.1 applies to the Weil complex $W(\mathcal{H})$, explaining the 'suspensions' (by degree 2 up) in the various cohomologies we deal with. It provides cyclic bicomplexes computing our cohomologies, in which S can be described as a shift. As in cyclic cohomology, one can introduce these bicomplexes directly, and prove all the formulas in a straightforward manner. Here we prefer to apply the formal constructions of 5.4.1 to $W(\mathcal{H})$ and to compute its X-complex. This computation can be carried out exactly as in the case of the tensor algebra (see Example 5.4.3), and this is done in the proof of Theorem 5.8.9. We end up with the following exact sequence of complexes (which can be taken as a definition):

$$\ldots \longrightarrow W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \xrightarrow{N} W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \longrightarrow \ldots,$$

where we have to explain the new objects. First of all, $W^b(\mathcal{H})$ is the same as $W(\mathcal{H})$ but with a new boundary $b = \partial + b_t$ with b_t described below. The t operator is the backward cyclic permutation:

$$t(ax) = (-1)^{|a||x|} xa,$$

for $a \in H$ or of type ω_h. This operator has finite order in each degree of $W(\mathcal{H})$: we have $t^p = 1$ on elements of bi-degree (p, q), so $t^{k!} = 1$ on elements of total degree k. The norm operator N is $N := 1 + t + t^2 + \ldots + t^{p-1}$ on elements of bi-degree (p, q). The boundary b of $W^b(\mathcal{H})$ is $b = \partial + b_t$,

$$b_t(ax) = t(\partial_0(a)x),$$

for $a \in \mathcal{H}$ or of type ω_h. For all the operators involved, see also section 5.9. Obviously, the powers $I(\mathcal{H})^{n+1}$ are invariant by $b, t-1, N$, so we get similar sequences for $I_n(\mathcal{H})$, $W_n(\mathcal{H})$.
For reference, we conclude:

Corollary 5.7.9 *There are exact sequences of complexes:*

$$CC(W_n(\mathcal{H})) : \ \ldots \longrightarrow W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \ \ldots \tag{5.34}$$

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \ \ldots \tag{5.35}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow \ \ldots \tag{5.36}$$

Corollary 5.7.10 *There are short exact sequences of complexes:*

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0 \tag{5.37}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow I_n(\mathcal{H})_\natural \longrightarrow 0 \tag{5.38}$$

In particular, (5.35), (5.36), give bicomplexes which compute the cohomologies of $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$. They are similar to the (first quadrant) cyclic bicomplexes appearing in cyclic cohomology, and come equipped with an obvious shift operator, which defines our S- operation:

$$S : H^*(W_n(\mathcal{H})_\natural) \longrightarrow H^{*+2}(W_n(\mathcal{H})_\natural),$$

(and similarly for $I_n(\mathcal{H})_\natural$). Alternatively, one can obtain S as cup-product by the Ext^2 classes arising from Corollary 5.7.10.

End of proof of theorem 5.7.7: Denote for simplicity by $\underline{CC}^*(I_n), \underline{CC}^*(W), \underline{CC}^*(W_n)$ the (first quadrant) cyclic bicomplexes (or their total complexes) of I_n, W, and W_n, respectively. We have a map of short exact sequences of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{I}_n(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & W(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & W_n(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & 0 \\
0 \arrow[r] & \underline{CC}^*(I_n) \arrow[r] & \underline{CC}^*(W) \arrow[r] & \underline{CC}^*(W_n) \arrow[r] & 0
\end{tikzcd}$$

where we have used the fact that $N : I_n(\mathcal{H})_\natural \longrightarrow I_n(\mathcal{H})$ factors through the projection $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$ (being defined on the entire $W(\mathcal{H})_\natural$). Applying the five lemma to the exact sequences induced in cohomology by the previous two short exact sequences, the statement follows. $\square$

5.7.11 Example: There are canonical Chern and Chern-Simons classes induced by any group-like element $\rho \in \mathcal{H}$ (i.e. with the property $\Delta(\rho) = \rho \otimes \rho$). Denote by ω its curvature. Since $\partial(\omega^n) = [\omega^n, \rho]$ is a commutator, ω^n define cohomology classes:

$$ch_{2n}(\rho) := [\natural(\frac{1}{n!}\omega^n)] \in H^{2n}(I_m(\mathcal{H})_\natural)\ , \tag{5.39}$$

for any $n \geq m$. The associated Chern-Simons class $cs_{2n-1}(\rho) := CS(ch_{2n}(\rho))$ is given by the formula (see (5.33)):

$$cs_{2n-1}(\rho) = [\natural\{\frac{1}{(n-1)!}\int_0^1 \rho(t\partial(\rho) + t^2\rho^2)^{n-1}dt\}] \in H^{2n-1}(W_m(\mathcal{H})_\natural).$$

To compute $S(ch_{2n}(\rho))$, we have to solve successively the equations:

$$\begin{cases} \partial(\frac{1}{n!}\omega^n) & = & (t-1)(v) \\ b(v) & = & N(w) \end{cases}$$

and then $S(ch_{2n}(\rho)) = [\natural(w)]$. The first equation has the obvious solution $v = \frac{1}{n!}\rho\omega^n$, whose $b(v) = \frac{1}{n!}\omega^{n+1}$, so the second equation has the solution $w = \frac{1}{(n+1)!}\omega^{n+1}$. In conclusion,

$$S(ch_{2n}(\rho)) = ch_{2(n+1)}(\rho), \quad S(cs_{2n-1}(\rho)) = cs_{2n+1}(\rho). \tag{5.40}$$

(where the second relation follows from the first one and Theorem 5.7.7.)

5.8 The Weil complex and higher traces

We explain now how the Weil complex introduced in the previous section appears naturally in the construction of characteristic maps associated to higher traces and Hopf algebra actions, as a non-commutative analogue of the construction of the geometric characteristic map for foliations ((1.23) in the preliminary Section 1.4). The main reason that $HC^*_\delta(\mathcal{H})$ is still the target of these characteristic maps is that it can be computed by the truncation of the Weil complex (see Theorem 5.8.3, whose proof is postponed until the next section). To prove the compatibility with the S-operator, we first have interpret the complexes introduced in the previous section in terms of Cuntz-Quillen's (tower of) relative X-complexes. We will obtain in particular the case of usual traces discussed in Section 5.5. Also, for $\mathcal{H} = \mathbf{C}\rho$ (example 5.7.1), we re-obtain the results , and interpretations of some of the computations of [90] (see Example 5.8.11 below).

In this section $\mathcal{H}$ is a Hopf algebra, δ is a character on $\mathcal{H}$, and A is a $\mathcal{H}$-algebra. We assume for simplicity that $S_\delta^2 = Id$.

5.8.1 Localizing $W(\mathcal{H})$: First of all remark that the Weil complex $W(\mathcal{H})$ is naturally an $\mathcal{H}$ DG algebra. By this we mean a DG algebra, endowed with a (flat) action, compatible with the grading and with the differentials. The action is defined on generators by:

$$g \cdot i(h) := i(gh), \quad g \cdot \omega_h := \omega_{gh}, \quad \forall\, g, h \in \mathcal{H}.$$

and extended by $h(xy) = \sum h_0(x)h_1(y)$. Here, to avoid confusions, we have denoted by $i : \mathcal{H} \longrightarrow W(\mathcal{H})$ the inclusion. Remark that the action preserves the bi-degree (see 5.7.4), so $W(\mathcal{H})_\delta$ has an induced bi-grading. We briefly explain how to get the localized version for the constructions and the properties of the previous section. First of all one can localize with respect to δ as in Section 3, and (with the same proof as of Proposition 5.3.5), all the operators descend to the localized spaces. The notation $I_n(\mathcal{H})_{\natural,\delta}$ stands for $I_n(\mathcal{H})$ divided out by commutators and co-invariants. For Theorem 5.7.7, remark that the contraction used there is compatible with the action. To get the exact sequences from Corollary 5.7.9 and 5.7.10, we may look at them as a property for the cohomology of finite cyclic groups, acting (on each fixed bi-degree) in our spaces. Or we can use the explicit map $\alpha : W(\mathcal{H}) \to W(\mathcal{H})$ defined by $\alpha := (t+2t^2+\ldots+(p-1)t^{p-1})$ on elements of bi-degree (p, q), which has the properties: $(t-1)\alpha + N = pId$, $\alpha(I(\mathcal{H})^{n+1}) \subseteq I(\mathcal{H})^{n+1}$, and α descends (because t does). So, also the analogue of Theorem 5.7.7 follows. In particular $H_\delta^*(W_n(\mathcal{H})_\natural)$ is computed either by the complex $W_n(\mathcal{H})_{\natural,\delta}$, or by the (localized) cyclic bicomplex $CC_\delta^*(W_n(\mathcal{H}))$ (analogous to (5.34)). Similarly, we consider the S operator, and the periodic versions of these cohomologies. Due to the shift in the degree already existent in the case of the tensor algebra (see 5.5.2), we re-index these cohomologies:

Definition 5.8.2 *Define $HC_\delta^*(\mathcal{H}, n) := H^{*+1}(W_n(\mathcal{H})_{\natural,\delta})$, and denote by $CC_\delta^*(\mathcal{H}, n)$ the cyclic bicomplex computing it, that is, $CC_\delta^*(W_n(\mathcal{H}))$ shifted by one in the vertical direction.*

Remark that for $n = 0$ we obtain Connes-Moscovici's cyclic cohomology and:

$$CC^*(\mathcal{H}, 0) = CC^*(\mathcal{H})\ , \quad CC_\delta^*(\mathcal{H}, 0) = CC_\delta^*(\mathcal{H}),$$

while, in general, there are obvious maps:

$$\ldots \xrightarrow{\pi_3} HC_\delta^*(\mathcal{H}, 2) \xrightarrow{\pi_2} HC_\delta^*(\mathcal{H}, 1) \xrightarrow{\pi_1} HC_\delta^*(\mathcal{H}, 0) \cong HC_\delta^*(\mathcal{H}). \tag{5.41}$$

In the next section we will prove:

Theorem 5.8.3 *$HC^*_\delta(\mathcal{H}, n) \cong HC^{*-2n}_\delta(\mathcal{H})$, and the tower (5.41) is the S operation tower for $HC^*_\delta(\mathcal{H})$. More precisely, there are isomorphisms*

$$\beta : HC^*_\delta(\mathcal{H}, n) \xrightarrow{\sim} HC^{*-2}_\delta(\mathcal{H}, n-1)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] \arrow[dr, "\pi"] & HC^*_\delta(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+2}_\delta(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+4}_\delta(\mathcal{H}, 2) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] \arrow[dr, "\pi"] & HC^{*-2}_\delta(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^*_\delta(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & HC^{*+2}_\delta(\mathcal{H}, 1) \arrow[r, "S"] \arrow[d, "\beta"] \arrow[dr, "\pi"] & \cdots \\
\cdots \arrow[r] & HC^{*-4}_\delta(\mathcal{H}, 0) \arrow[r, "S"] & HC^{*-2}_\delta(\mathcal{H}, 0) \arrow[r, "S"] & HC^*_\delta(\mathcal{H}, 0) \arrow[r, "S"] & \cdots
\end{tikzcd}$$

5.8.4 The case of even equivariant traces: Consider now an equivariant even trace over A, i.e. an extension:

$$0 \longrightarrow I \longrightarrow R \stackrel{u}{\longrightarrow} A \longrightarrow 0 \tag{5.42}$$

and a δ-invariant trace $\tau : R \longrightarrow \mathbb{C}$ vanishing on I^{n+1}. To describe the induced characteristic map, we choose a linear splitting $\rho : A \longrightarrow R$ of (5.42). As in the case of the usual Weil complex, there is a unique equivariant map of DG algebras:

$$\tilde{k} : W(\mathcal{H}) \longrightarrow Hom(B(A), R),$$

sending $1 \in \mathcal{H}$ to ρ. This follows from Proposition 5.7.2 and from the equivariance condition (with the same arguments as in 5.5.6). Here, the action of $\mathcal{H}$ on $Hom(B(A), R)$ is induced by the action on R. Since ρ is a homomorphism modulo I, $\tilde{k}$ sends $I(\mathcal{H})$ to $Hom(B(A), I)$, so induces a map $W_n(\mathcal{H}) \longrightarrow Hom(B(A), R/I^{n+1})$. As in 5.5.6, composing with the δ-invariant trace:

$$\tau_\natural : Hom(B(A), R/I^{n+1}) \longrightarrow C^*_\lambda(A)[1], \quad \phi \mapsto \tau \circ \phi \circ N \ ,$$

we get a δ-invariant trace on $W_n(\mathcal{H})$, so also a chain map:

$$k^{\tau,\rho} : W_n(\mathcal{H})_{\natural,\delta} \longrightarrow C^*_\lambda(A)[1] \ . \tag{5.43}$$

Denote by the same letter the map induced in cohomology:

$$k^{\tau,\rho} : HC^*_\delta(\mathcal{H}, n) \longrightarrow HC^*(A), \tag{5.44}$$

or, using the isomorphism of Theorem 5.8.3:

$$k^{\tau,\rho} : HC^{*-2n}_\delta(\mathcal{H}) \longrightarrow HC^*(A) \tag{5.45}$$

(compare with the construction of the geometrical characteristic map for foliations, (1.20), (1.23) in Section 1.4).

Theorem 5.8.5 *The characteristic map (5.45) of the even higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

proof:(compare to [90]) We use 5.7.6. If ρ_0, ρ_1 are two liftings, form $\rho = t\rho_0 + (1-t)\rho_1 \in Hom(A, R[t])$, viewed in the degree one part of the DG algebra $Hom(B(A), R \otimes \Omega(1))$. It induces a unique map of $\mathcal{H}$DG algebras $\tilde{k}_\rho : W(\mathcal{H}) \longrightarrow Hom(B(A), R \otimes \Omega(1))$, sending 1 to ρ, which maps $I(\mathcal{H})$ to the DG ideal $Hom(B(A), I \otimes \Omega(1))$ (since ω_ρ belongs to the former). Using the trace $\tau \otimes \int : R/I^{n+1} \otimes \Omega(1) \longrightarrow \mathbb{C}$, and the universal cotrace on $B(A)$, it induces a chain map:

$$k_{\rho_0,\rho_1} : W_n(\mathcal{H}) \longrightarrow C^*_\lambda(A)[1],$$

which kills the coinvariants and the commutators. The induced map on $W_n(\mathcal{H})_{\natural,\delta}$ is a homotopy between k^{τ,ρ_0} and k^{τ,ρ_1}. The compatibility with S follows from the fact that the characteristic map (5.43) can be extended to a map between the cyclic bicomplexes $CC^*_\delta(\mathcal{H}, n)$ and $CC^*(A)$. We will prove this after shortly discussing the case of odd higher traces. $\square$

5.8.6 The case of odd equivariant traces: A similar discussion applies to the case of odd equivariant traces on A, i.e. extensions (5.42) endowed with a δ-invariant linear map $\tau : I^{n+1} \longrightarrow \mathbb{C}$, vanishing on $[I^n, I]$. The resulting map $H^{*+1}(I_n(\mathcal{H})_{\natural,\delta}) \longrightarrow HC^{*-1}(A)$, combined with Corollary 5.7.7, the comments in 5.8.1, and Theorem 5.8.3, give the characteristic map:

$$k^{\tau,\rho} : HC^{*-2n-1}_\delta(\mathcal{H}) \cong HC^{*-1}_\delta(\mathcal{H}, n) \longrightarrow HC^*(A), \tag{5.46}$$

which has the same properties as in the even case:

Theorem 5.8.7 *The characteristic map (5.46) of the odd higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

5.8.8 The localized tower $\mathcal{X}_\delta(R, I)$: Recall that given an ideal I in the algebra R, one has a tower of super-complexes $\mathcal{X}_\delta(R, I)$ given by ([40], pp. 396):

$$\mathcal{X}^{2n+1}(R, I) : R/I^{n+1} \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural/\natural(I^{n+1}dR + I^n dI) \ ,$$

$$\mathcal{X}^{2n}(R, I) : R/(I^{n+1} + [I^n, R]) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural/\natural(I^n dR) \ ,$$

where $\natural : \Omega^1(R) \to \Omega^1(R)_\natural$ is the projection. The structure maps $\mathcal{X}^n(R, I) \to \mathcal{X}^{n+1}(R, I)$ of the tower are the obvious projections. We have a localized version of this, denoted by $\mathcal{X}_\delta(R, I)$, and which is defined by:

$$\mathcal{X}^{2n+1}_\delta(R, I) : R/(I^{n+1} + \text{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^{n+1}dR + I^n dI) \ ,$$

$$\mathcal{X}^{2n}_\delta(R, I) : R/(I^{n+1} + [I^n, R] + \text{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^n dR) \ ,$$

where, this time, $\natural$ denotes the projection $\Omega^1(R) \to \Omega^1(R)_{\natural,\delta}$.

Remark that the construction extends to the graded case, and each $\mathcal{X}^n(R, I)$ is a super-complex of complexes.

Theorem 5.8.9 *The cyclic bicomplex $CC^*_\delta(\mathcal{H},n)$ is isomorphic to the bicomplex $\mathcal{X}^{2n+1}_\delta(W(\mathcal{H}),I(\mathcal{H}))$.*

proof: The computation is similar to the one of $X(T\mathcal{H})$ (see Example 5.4.3 and Proposition 5.4.4). Denote $W=W(\mathcal{H})$, $I=I(\mathcal{H})$, and let $V\subset W(\mathcal{H})$ be the linear subspace spanned by h's and ω_h's. Remark that W, as a graded algebra, is freely generated by V. This allows us to use exactly the same arguments as in 5.4.3, 5.4.4 to conclude that $\Omega^1W\cong\tilde{W}\otimes V\otimes\tilde{W}$, $\Omega^1(W)_\natural\cong V\otimes\tilde{W}=W$, $\Omega^1(W)_{\natural,\delta}\cong W_\delta$; the projection $\natural:\Omega^1(W)\to\Omega^1(W)_\natural$ identifies with:

$$\natural:\Omega^1(W)\longrightarrow V\otimes\tilde{W}=W,\quad x\partial_u(v)y\mapsto(-1)^\mu vyx\ , \tag{5.47}$$

for $x,y\in\tilde{W}, v\in V$. Here $\mu=deg(x)(deg(v)+deg(y))$ introduces a sign, due to our graded setting, and $\partial_u:W\to\Omega^1(W)$ stands for the universal derivation of W. Using this, we can compute the new boundary of W, coming from the isomorphism $W\cong\Omega^1(W)_\natural$, and we end up we the b-boundary of W, defined in Section 5. For instance, if $x=hx_0\in W$ with $h\in\mathcal{H}$, since $\natural(\partial_u(h)x_0)=x$ by (5.47), its boundary is:

$$\begin{aligned}
&\natural(\partial_u(\partial(h)x_0)-\partial_u(h)\partial(x_0))=\\
&\quad=\natural(\partial_u(\omega_h-\sum h_0h_1)x_0-\partial_u(h)\partial(x_0))=\\
&\quad=\natural(\partial_u(\omega_h)x_0-\sum\partial_u(h_0)h_1x_0-\sum h_0\partial_u(h_1)x_0-\partial_u(h)\partial(x_0))=\\
&\quad=\omega_hx_0-\sum h_0h_1x_0-\sum(-1)^{deg(x)}h_1x_0h_0-h\partial(x_0)=\\
&\quad=\partial(h)x_0-\sum t(h_0h_1x_0)-h\partial(x_0)=\\
&\quad=\partial(hx_0)+t(\partial_0(h)x_0)=b(hx_0)
\end{aligned}$$

Remark also that our map (5.47) has the property:

$$\natural(I^n\partial_uI+I^{n+1}\partial_uW)=I^{n+1}. \tag{5.48}$$

These give the identification $\mathcal{X}^{2n+1}(W,I)\cong CC^*(\mathcal{H},n)$. The localized version of this is just a matter of checking that the isomorphism $\Omega^1(W)_{\natural,\delta}\cong W_\delta$ already mentioned, induces $\Omega^1(W)_{\natural,\delta}/\natural(I^n\partial_uI+I^{n+1}\partial_uW)\cong(W/I^{n+1})_\delta$, which follows from (5.48). $\square$

5.8.10 Proof of the S-relation: We freely use the dual constructions for (DG) coalgebras B, such as the universal coderivation $\Omega_1(B)\to B$, the space of co-commutators $B^\natural=Ker(\Delta-\sigma\circ\Delta:B\to B\otimes B)$, and the X-complex $X(B)$ (see [90]). Denote $B=B(A)$, $L=Hom(B,R)$, $J=Hom(B,I)$. Our goal is to prove that the characteristic map (5.43) can be defined at the level of the cyclic bicomplexes. Consider first the case of even traces τ. Since the $\mathcal{H}$ DG algebra map $\tilde{k}:W(\mathcal{H})\to L$ maps $I(\mathcal{H})$ inside J, there is an induced map $\mathcal{X}^{2n+1}_\delta(W(\mathcal{H}),I(\mathcal{H}))\to\mathcal{X}^{2n+1}_\delta(L,J)$, extending $W_n(\mathcal{H})_{\natural,\delta}\to(L/J^{n+1})_{\natural,\delta}$. So, it suffices to show that the map $(L/J^{n+1})_{\natural,\delta}\to Hom(B^\natural,(R/I^{n+1})_{\natural,\delta})$ (constructed as (5.43)), lifts to a map of super-complexes (of complexes)

$$\mathcal{X}^{2n+1}_\delta(L,J)\to Hom(X(B),(R/I^{n+1})_{\natural,\delta}) \tag{5.49}$$

Indeed, using Theorem 5.8.9, the (similar) computation of $X(B)$ (as the cyclic bicomplex of A), the interpretation of the norm map N as the universal cotrace of B (see

[90]), and the fact that any τ as above factors through $(R/I^{n+1})_{\natural,\delta} \to \mathbf{C}$, the map (5.49) is 'universal' for our problem. The construction of (5.49) is quite simple. The composition with the universal coderivation of B is a derivation $L \to Hom(\Omega_1(B), R)$ on L, so it induces a map $\chi : \Omega^1(L) \to Hom(\Omega_1(B), R)$. Since χ is a L-bimodule map, and it is compatible with the action of $\mathcal{H}$, it induces a map $\Omega^1(L)_\natural \to Hom(\Omega_1(B)^\natural, (R/I^{n+1})_{\natural,\delta})$, which kills $\natural(J^n dJ + J^{n+1}dL + coinv)$. This, together with with the obvious $(L/J^{n+1})_\delta \to Hom(B, (R/I^{n+1})_{\natural,\delta})$, give (5.49). For the case of odd higher traces we proceed similarly, and use the remark that (5.49) was a priori defined at the level of L, and $\Omega^1(L)$, and one can restrict to the ideals (instead of dividing out by them). $\square$

5.8.11 Examples: Choosing $\rho = 1 \in \mathcal{H}$ (the unit of $\mathcal{H}$) in Example 5.7.11, and applying the characteristic map to the resulting classes, we get the Chern/Chern-Simons classes (in the cyclic cohomology of A), described in [90]. Remark that our proof of the compatibility with the S operator consists of two steps: the first one proves the universal formulas (5.40) at the level of the Weil complex, while the second one shows, in a formal way, that the characteristic map can be defined at the level of the cyclic bicomplexes. This allows us to avoid the explicit cochain computations.
Another interesting example is when $\mathcal{H} = U(\mathfrak{g})$ as in Example 5.3.3, $\delta =$ the counit. Via the computation of Theorem 5.6.6, our construction associates to any G-algebra A, and any G-invariant higher trace τ on A, of parity i, a $\mathbf{Z}/2\mathbf{Z}$ graded characteristic maps:

$$k_\tau : H_*(\mathfrak{g}) \longrightarrow HP^{*+i}(A). \tag{5.50}$$

When τ is a usual invariant trace $\tau : A \longrightarrow \mathbf{C}$, we have the following formula (use (5.19) and the map A used in the proof of Theorem 5.6.6):

$$\begin{aligned} &k_\tau(v_1 \wedge \ldots \wedge v_n)(a_0, a_1, \ldots, a_n) = \\ &\quad = \frac{1}{n!} \sum_\sigma sign(\sigma) \tau(a_0 v_{\sigma(1)}(a_1) \ldots v_{\sigma(n)}(a_n)) \ , \end{aligned}$$

(where $v(a) := L_v(a)$ is the Lie derivative). This coincides with the characteristic map described in [24].

5.9 Proof of Theorem 5.8.3, and equivariant cycles

This section is devoted to the proof of Theorem 5.8.3. At the end we illustrate how the new complexes computing $HC^*_\delta(\mathcal{H})$ which arise during the proof can be used to construct characteristic maps associated to equivariant cycles.

We first concentrate on the non-localized version, whose proof uses explicit formulas which can be easily localized. So, we construct isomorphisms

$$\beta : H^*(W_n(\mathcal{H})_\natural) \xrightarrow{\sim} H^{*-2}(W_{n-1}(\mathcal{H})_\natural)$$

(and explicit inverses) such that the following diagram is commutative:

$$\begin{tikzcd} H^*(W_{n,\natural}) \arrow[r, "S"] \arrow[d, "\beta"'] \arrow[dr, "\pi"] & H^{*+2}(W_{n,\natural}) \arrow[d, "\beta"] \\ H^{*-2}(W_{n-1,\natural}) \arrow[r, "S"] & H^*(W_{n-1,\natural}) \end{tikzcd}$$

Let us start by fixing some notations. Denote $W_n(\mathcal{H}) = W_n$, $I(\mathcal{H}) = I$, $I^{(n)} = I^n/I^{n+1}$ viewed as the subspace of W spanned by elements having exactly n curvatures. The only grading we consider is the total grading (with $deg(h) = 1, deg(\omega_h) = 2$); notations like $(W_n, \partial), (W_n, b)$ are used to specify the complexes we are working with. In general, if the (signed) cyclic permutation acts on a vector space X, denote $X_\natural = X/Im(1-t)$.
We review now the various operators we have. First of all,

$$\partial = \partial_0 + d,$$

where ∂_0, d are the degree 1 derivations given on generators by:

$$\partial_0(h) = -\sum h_0 h_1, \quad \partial_0(\omega_h) = -\sum \omega_{h_0} h_1 - h_0 \omega_{h_1},$$

$$d(h) = \omega_h, \quad d(\omega_h) = 0.$$

Secondly, the operator $b = \partial + b_t$, where:

$$b_t(hx) = \sum (-1)^{degx} h_1 x h_0, \quad b_t(\omega_h x) = \sum h_1 x \omega_{h_0} - (-1)^{degx} \omega_{h_1} x h_0.$$

Define also $b_0 = \partial_0 + b_t$. It is straightforward to check:

$$b = b_0 + d, \quad b_0^2 = d^2 = [b_0, d] = 0.$$

Point out that d commutes with t.
For the construction of β we need the following degree -1 operator:

$$\theta : W \longrightarrow W, \quad \theta(hx) = 0, \quad \theta(\omega_h x) = hx.$$

For constructing the inverse of β, we will use the degree 0 operators $\phi_i : I \longrightarrow W$, $i = 0, 1$. On $I^{(n)}$,

$$\phi_1(hx) = 0, \quad n\phi_1(\omega_h x) = x\omega_h.$$

For $y = x_0 x_1 \ldots x_p \in I^{(n)}$, where each of the x_i's are of type h or ω_h, we put $\lambda_i(y) = \#\{j \leq i : x_j \text{ is of type } \omega_h\}$, and define $n\phi_0(y) = \sum_1^{n-1} \lambda_i(y) t^i(y)$. For a conceptual motivation, see the next proof. We can actually forget about these formulas, and just keep their relevant properties:

Lemma 5.9.1 *(i)* $[\theta, b_0] = 0,\ [\theta, \partial_0] = 0,\ [\theta, d] = 1,\ \theta^2 = 0,$
(ii) $\phi_1 N - (1-t)\phi_0 = 1, N\phi_1 - \phi_0(1-t) = 1,$
(iii) $\phi_1 b_0 = \partial_0 \phi_1$ *modulo* $Im(1-t)$, $\phi_1 \theta = 0,$
(iv) $\phi_0 \partial_0 = b_0 \phi_0$ *modulo* $Im\theta$.

Proof: (i) and (iii) follow by direct computation. For instance, for the first part of (iii) one has $\phi_1 b_0 = 0 = \partial_0 \phi_1$ on elements of type hx, while on elements of type $\omega_h x \in I^{(n)}$:

$$\begin{aligned} \phi_1 b_0(\omega_h x) = & \ \phi_1((\omega_{h_0} h_1 - h_0 \omega_{h_1})x + \omega_h \partial_0(x) + h_1 x \omega_{h_0} - (-1)^{deg(x)} \omega_{h_1} x h_0) = \\ & = n(h_1 x \omega_{h_0} + \partial_0(x)\omega_h - (-1)^{deg(x)} x h_0 \omega_{h_1}) \end{aligned}$$

$$\partial_0 \phi_1(\omega_h x) = n\partial_0(x\omega_h) = n(\partial_0(x)\omega_h + (-1)^{deg(x)} x \omega_{h_0} h_1 - (-1)^{deg(x)} x h_0 \omega_{h_1}),$$

and the two expressions are clearly the same modulo $Im(1-t)$.
One can check directly also (ii). Instead, let us explain that ϕ_0, ϕ_1 have been constructed in such a way that (ii) holds. On the graded algebra $W = \oplus I^{(n)}$, we have a Goodwillie [50] type derivation: multiplication by the number of curvatures. Since W is a tensor algebra, it comes equipped with a canonical connection (see the end of section 3 in [41]). We know that the X- complex of W is the cyclic bicomplex, and the general Cartan homotopy formula of [40] for our derivation D, gives precisely the homotopy $(n\phi_0, n\phi_1)$ on $I^{(n)}$. For (iv), remark first that $Im\theta = Ker\theta$, so it suffices to show that $A := (\theta\phi_0)\partial_0 - b_0(\theta\phi_0)$ is zero. From the first formula of (iii), the second of (ii), and (i), it follows that $A(1-t) = 0$. So, it is enough to check $A = 0$ on homogeneous monomials having a curvature as first element. Such an element can be written as $X = \omega(h^1)X^1\ldots\omega(h^n)X^n \in I^{(n)}$, where $X^i \in \widetilde{T}(\mathcal{H})$, $\omega(h) = \omega_h$. On X, $\theta\phi_0(X) = \sum_1^n \epsilon_i(i-1)h^iX^i\omega(h^{i+1})X^{i+1}\ldots\omega(h^n)X^n\omega(h^1)X^1\ldots\omega(h^{i-1})X^{i-1}$ (ϵ_i are corresponding signs), and the proof becomes a lengthy straightforward computation. $\square$

To define β, we need the right complexes computing $H^*(W_{n,\natural})$. One of them is given by the following:

Lemma 5.9.2 *(i) There are isomorphisms $p : H^*(W_{n,\natural}) \xrightarrow{\sim} H^*(I_\natural^{(n)}/Imd, \partial_0)$, compatible with the S-operations.*

(ii) One has short exact sequences:

$$0 \longrightarrow (I_\natural^{(n-1)}/Imd, \partial_0) \xrightarrow{d} (I_\natural^{(n)}, \partial_0) \longrightarrow (I_\natural^{(n)}/Imd, \partial_0)[1] \longrightarrow 0, \tag{5.51}$$

whose induced boundary in cohomology identifies, via p, with π:

$$\begin{array}{ccc} H^*(W_{n,\natural}) & \xrightarrow{\quad\pi\quad} & H^*(W_{n-1,\natural}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ H^*(I_\natural^{(n)}/Imd, \partial_0) & \xrightarrow{\quad\delta\quad} & H^*(I_\natural^{(n-1)}/Imd, \partial_0) \end{array}$$

Here, we view $(W_{n,\natural}, \partial)$ as the total complex of the double complex:

$$0 \longrightarrow (I_\natural^{(0)}, \partial_0) \xrightarrow{d} (I_\natural^{(1)}, \partial_0) \xrightarrow{d} \ldots \xrightarrow{d} (I_\natural^{(n)}, \partial_0) \longrightarrow 0 \longrightarrow \ldots$$

and p is induced by the obvious augmentation sending $[\sum_0^n x_i]$ into $[x_n]$ ($x_i \in I^{(i)}$). The S operation on $H^*(I_\natural^{(n)}/Imd, \partial_0)$ of (i) is defined by the cyclic bicomplex which is augmentation of (5.35), or, similar to (5.37), by the Ext^2 class defined by the extension:

$$0 \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \xrightarrow{N} (I^{(n)}/Imd, b_0) \xrightarrow{t-1} (I^{(n)}/Imd, \partial_0) \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \longrightarrow 0 \tag{5.52}$$

Using that W is contractible along d (cf. 5.9.1 (i)), (i) is clear. Using that $d\circ t = t\circ d$, and that taking invariants under the action of a finite group does not affect exactness, also the first part of (ii) follows, while the last part is a routine spelling out of the boundary of long exact sequences.
There is a slight modification of (5.52) which can be used to compute $H^*(W_{n,\natural})$, obtained as follows: (5.52) splits into two short exact sequence:

$$0 \longrightarrow (I_\natural^{(n)}/Imd, \partial_0) \xrightarrow{N} (I^{(n)}/Imd, b_0) \longrightarrow (I^{(n)}/Imd + ImN, b_0) \longrightarrow 0, \tag{5.53}$$

$$0 \longrightarrow (I^{(n)}/Imd + ImN, b_0) \xrightarrow{t-1} (I^{(n)}/Imd, \partial_0) \longrightarrow (I_{\natural}^{(n)}/Imd, \partial_0) \longrightarrow 0 \tag{5.54}$$

Since the middle complex of (5.54) is acyclic, e.g. by using the contraction $s_{-1}(hx) = \epsilon(h)x$, $s_{-1}(\omega_h x) = 0$ (which commutes with d), we get a quasi-isomorphism (which, in cohomology, is independent of the contraction):

$$s_{-1}(1-t) : (I^{(n)}/Imd + ImN, b_0) \xrightarrow{q.i.} (I_{\natural}^{(n)}/Imd, \partial_0)[1] \tag{5.55}$$

Via this, the S operator is simply the boundary of the long exact sequence induced by (5.53).
Now, our map is defined as the chain map:

$$\beta : (I_{\natural}^{(n)}/Imd, \partial_0) \longrightarrow (I^{(n-1)}/Imd + ImN, b_0))[1] \tag{5.56}$$

induced by $-\theta{\circ}N$. To understand our choice of complexes, let us just mention that (5.56) is an isomorphism when $n = 1$. Note also that β, as well as the chain map α bellow describing its homotopical inverse, do not depend on the structure of $\mathcal{H}$, other then the vector space structure.
Now, to see that β is compatible with the S operation, and to construct its inverse (in cohomology), we make use of the following diagram with exact rows and columns:

$$\begin{tikzcd}
(I_{\natural}^{(n-1)}/Imd, \partial_0)[1] \arrow[r, "d"] \arrow[d, "N"'] & (I_{\natural}^{(n)}, \partial_0) \arrow[r, "p_5"] \arrow[d, shift right, "\tilde{N}"'] \arrow[dl, "-\tilde{\beta}"'] & (I_{\natural}^{(n)}/Imd, \partial_0) \arrow[d, "N"] \\
(I^{(n-1)}/Imd, b_0)[1] \arrow[r, shift left, "\tilde{d}"] \arrow[d] & (I^{(n)}, b_0) \arrow[l, shift left, "\tilde{\theta}"] \arrow[r, shift left, "p_1"] \arrow[u, shift right, "\tilde{\phi}_1"'] \arrow[d, shift right, "p_2"'] & (I^{(n)}/Imd, b_0) \arrow[l, shift left, "r"] \arrow[d, "p_3"] \arrow[dl, "p_2 \circ r"] \\
(I^{(n-1)}/Imd + ImN, b_0)[1] \arrow[r, "d"'] & (I^{(n)}/ImN, b_0) \arrow[r, "p_4"'] \arrow[u, shift right, "s"'] & (I^{(n)}/Imd + ImN, b_0)
\end{tikzcd}$$

Here $\tilde{N}, \tilde{\phi}_1, \tilde{d}, \tilde{\theta}$ are induced by N, ϕ_1, d, θ, respectively, p_1, p_2 are the obvious projections, r is the map induced by θd, s is the one induced by $\phi_0(1-t)$, and $\tilde{\beta}$ is the one induced by $-\theta N$. From 5.9.1 (ii), (iii), $\tilde{\phi}_1, s$ are chain maps with:

$$\tilde{\phi}_1 N = Id,\ p_2 s = Id,\ N\tilde{\phi}_1 + sp_2 = Id. \tag{5.57}$$

Also, from (i) of the same Lemma, $\tilde{\theta}, r$ are chain maps with:

$$\tilde{\theta}\tilde{d} = Id,\ p_1 r = Id,\ rp_1 + \tilde{d}\tilde{\theta} = Id. \tag{5.58}$$

Since $-\tilde{\beta}d = \tilde{\theta}Nd = \tilde{\theta}\tilde{d}N = N$, $\tilde{\beta}$ induces a map between the Cokernels of d and N, and this is precisely our map (5.56). Moreover, $\tilde{\beta}$ induces a map between the left vertical short exact sequence, and the upper horizontal one. The boundaries of the long exact sequences induced in cohomology are, by the previous remarks and by (ii) of Lemma 5.9.2, the $-S$ operator, and π, respectively (the '$-$' sign in front of S is due to the fact that, given a short exact sequence, and shifting by one, the boundary induced in cohomology is '$-$ the initial boundary'; it also explains the '$-$' sign in our definition of β). Hence, by naturality, $\pi = (-S)(-\beta) = S\beta$. Similar arguments show that $p_2 r$ induces a chain map β' between the kernels of p_3 and p_4. By a diagram chasing and (5.58), we have $d{\circ}\beta'{\circ}p_5 = d{\circ}\beta{\circ}p_5$, hence $\beta' = \beta$. Using the naturality of the long exact

sequences induced in cohomology by the right vertical and the bottom horizontal, we find $\pi = \beta S$.
Hence we are left with proving that β induces isomorphism in cohomology. We define now a new map on our diagram:

$$\tilde{\alpha} := -\tilde{\phi}_1\tilde{d} : (I^{(n-1)}/Imd, b_0)[1] \longrightarrow (I_\natural^{(n)}, \partial_0).$$

First of all, $-\tilde{\alpha}N = \tilde{\phi}_1\tilde{d}N = \tilde{\phi}_1 Nd = d$ by (5.57), so $\tilde{\alpha}$ induces a map between the Cokernels of N and d:

$$\alpha : (I^{(n-1)}/Imd + ImN, b_0)[1] \longrightarrow (I_\natural^{(n)}/Imd, \partial_0).$$

Since $\phi_1 d\theta N = \phi_1(1-\theta d)N$, $\phi_1 N \equiv 1$ modulo $Im(1-t)$, and $\phi_1\theta = 0$ (cf. 5.9.1), we have:

$$\alpha \circ \beta = 1 .$$

Now we show that $\beta\alpha = 1$ in cohomology. Since $\theta N \phi_1 d = \theta(1 + \phi_0(1-t))d \equiv 1 + \theta\phi_0(1-t)d$ modulo $Im(d)$, it suffices to show that:

$$\theta\phi_0 d(1-t) : (I^{(n-1)}/Imd + ImN, b_0) \longrightarrow (I^{(n-1)}/Imd + ImN, b_0)$$

is trivial in cohomology. For this we remark that our map factors as:

$$(I^{(n-1)}/Imd + ImN, b_0) \stackrel{1-t}{\longrightarrow} (I^{(n-1)}/Imd, \partial_0) \stackrel{\theta\phi_0 d}{\longrightarrow} (I^{(n-1)}/Imd + ImN, b_0),$$

where the second map is a chain map by the non-trivial 5.9.1 (iv), and the middle complex is contractible (by the usual s_{-1}).
Now, using Lemma 5.2.2, and Lemma 5.2.4, it is easy to see that all the formulas and arguments localize without any problem; this concludes the proof of Theorem 5.8.3.

5.9.3 Example (Equivariant cycles): We point out that the new complexes computing $HC^*_\delta(\mathcal{H})$, arising from Lemma 5.9.2, appear naturally in the construction of characteristic maps associated to equivariant cycles. Recall [29] that a *chain* of dimension n is a triple $(\Omega, d, \int)$ where $\Omega = \oplus_{j=0}^n \Omega^n$ is a DG algebra, and $\int : \Omega^n \longrightarrow \mathbf{C}$ is a graded trace on Ω. It is a *cycle* if $\int$ is closed. If (Ω, d) is a $\mathcal{H}$ DG algebra, and $\int$ is δ-invariant, we say that $(\Omega, d, \int)$ is a *$\mathcal{H}$-chain.* For instance, if a Lie group G acts smoothly on the Ω^j's, $d(g\omega) = gd(\omega)$, $\int g\omega = \int \omega$, for $g \in G$, $\omega \in \Omega$ (i. e. $(\Omega, d, \int)$ is a G- equivariant chain), then, with the induced infinitesimal action , $(\Omega, d, \int)$ is an $U(\mathfrak{g})$-chain. A *$\mathcal{H}$-cycle* over an algebra A is given by such a cycle, together with an algebra homomorphism $\rho : A \longrightarrow \Omega^0$. We view ρ as an element of (total) degree 1 on the bigraded differential algebra $L = Hom(BA, \Omega)$. The structure on L is the one induced by the graded structures on $B(A)$, and Ω, respectively. That is, the bigrading: $L^{p,q} = Hom(A^{\otimes p}, \Omega^q)$, the differentials $d(f) = d\circ f$, $\partial_0(f) = -(-1)^{deg(f)} f \circ b'$, the product: $(\phi * \psi)(a_1, \ldots, a_{p+q}) = (-1)^{pdeg(\psi)}\phi(a_1, \ldots, a_p)\ \psi(a_{p+1}, \ldots, a_{p+q})$. As in 5.5.6, there is a unique $\mathcal{H}$ DG algebra map $k : W(\mathcal{H}) \longrightarrow Hom(BA, \Omega)$, sending 1 to ρ; it is compatible with the bigraded differential structure. On the other hand, using $\int : \Omega^n \longrightarrow \mathbf{C}[n]$, we have a graded trace $\int^\natural : Hom(BA, \Omega) \longrightarrow C_\lambda(A)[n]$, similar to (5.22), $\int^\natural(f) = \int \circ f \circ N$, whose composition with k is still denoted by $k : W(\mathcal{H}) \longrightarrow C_\lambda(A)[n]$.

Now, since $\int$ is closed, δ-invariant, and supported on degree n, the relevant target of

k is the complex $(I^{(n)}_{\natural,\delta}/Imd, \partial_0)$, hence it induces a characteristic map $k : HC^{*-2n-1}_{\delta}(\mathcal{H}) \cong H^*(I^{(n)}_{\natural,\delta}/Imd, \partial_0) \longrightarrow H^*(C_\lambda(A)[n+1]) \cong HC^{*-n-1}(A)$. As in the proof of 5.8.5, it is compatible with the S-operator.
Let us assume now that $(\Omega, d, \int)$ is cobordant to the trivial cycle, that is, there is an n-dimensional chain $(\tilde{\Omega}, \tilde{d}, \tilde{\int})$, $\tilde{\rho} : A \longrightarrow \tilde{\Omega}^0$, and an equivariant chain map $r : \tilde{\Omega} \longrightarrow \Omega$ such that $\tilde{\int} \circ \tilde{d} = \int \circ r$, $r \circ \tilde{\rho} = \rho$. As before, we have an induced map $\tilde{k} : (I^{(n+1)}_{\natural,\delta}/Imd, \partial_0)[-1] \longrightarrow C_\lambda(A)[n+1]$. Since $\tilde{\int} \circ \tilde{d} = \int \circ r$, we have $\tilde{k} \circ d = k$, and we are on the localized version of the short exact sequence (5.51). Since its boundary identifies with π, hence with the S operator cf. 5.8.3, we deduce that, after stabilizing by S, k is trivial (in cohomology). We summarize our discussion:

Corollary 5.9.4 *Let A be an algebra. The previous construction associates to any n-dimensional equivariant cycle $(\Omega, d, \int)$ over A a characteristic map compatible with the S-operator:*

$$k : HC^*_\delta(\mathcal{H}) \longrightarrow HC^{*+n}(A) .$$

If k_0, k_1 are associated to cobordant cycles, then $S \circ k_0 = S \circ k_1$.

For instance, for $\mathcal{H} = U(\mathfrak{g})$ previously mentioned, using Theorem 5.6.6 we get:

Corollary 5.9.5 *Given a Lie group G, and a smooth n-dimensional G-cycle over an algebra A, we have induced (**Z**/**2Z** graded) characteristic maps:*

$$k : H_*(\mathfrak{g}) \longrightarrow HP^{*+n}(A) .$$

Cobordant G-cycles induce the same map.

5.10 A dual cyclic cohomology for quantum groups?

The cyclic cohomology of Hopf algebras that we considered so far in this chapter extends Lie algebra cohomology from the commutative case (Lie groups) to the non-commutative one (quantum groups), provided one works with the Hopf algebra of operators (enveloping algebra). While we have seen it is trivial when the Hopf algebra admits a Haar integral (Proposition 5.6.4), such integrals exist as part of the axioms or part of crucial theorems regarding the representation theory of the group (as for instance in Woronowicz's approach to quantum groups [100]) whenever one works with Hopf algebras representing the algebra of functions on a quantum group. Here we adopt the dual point of view, looking at quantum groups through their algebras of functions, and we dualize the previous constructions. We show that this new cohomology (which we denote by $HC^*_{inv}(\mathcal{H})$) represent the relevant cohomologies in the classical examples: for group-algebras they give the (discrete) group-cohomology, and for the algebra of smooth functions on a compact Lie group they give the Lie algebra cohomology. It would be interesting to proceed with more computations, and the compact quantum groups of Woronowicz [100] are certainly the right objects to look at (actually that would be our main interest).

The given data is quite general: a Hopf algebra $\mathcal{H}$, and a character δ such that $S^2 = \delta * Id * \bar{\delta}$, but our mind is especially at the (smooth) Hopf algebra associated to a compact quantum group G, and at the modular character f_1 (see [100]). In particular, when $S^2 = Id$, we automatically assume that δ is trivial, case which we refer to as the modular case.

As general terminology: we use S, and ϵ to denote the antipode and the co-unit, and we use Sweedler's notation $\Delta(h) = \sum h_{(0)} \otimes h_{(1)}$ for the co-product. We denote by "$*$" the convolution product $\phi * \psi(h) = \sum \phi(h_{(0)})\psi(h_{(1)})$. A (left) coaction of $\mathcal{H}$ on a vector space V is a linear map $\Phi : V \longrightarrow \mathcal{H} \otimes V$ satisfying $(\Delta \otimes Id_V)\Phi = (Id_{\mathcal{H}} \otimes \Phi)\Phi$, $(\epsilon \otimes Id_V)\Phi = Id_V$; the invariant elements are $Inv(V) = \{v \in V : \Phi(v) = 1 \otimes V\}$. If $V = A$ is an algebra, a coaction $\Phi : A \longrightarrow \mathcal{H} \otimes A$ which is an algebra homomorphism is referred to as an algebra coaction. In this situation, a trace $\tau : A \longrightarrow \mathbb{C}$ is said to be invariant if $(Id_{\mathcal{H}} \otimes \tau)\Phi = \tau$.

5.10.1 Definition of $HC^*_{inv}(\mathcal{H})$: We define the cyclic module $\mathcal{H}^{\natural}_{inv} := (\{\mathcal{H}^{\otimes n}\}_{n\geq 0}, d_i, s_i, t)$ by the formulas (dual to the ones in [31]):

$$d_i(h^1, \ldots, h^n) = \begin{cases} \epsilon(h^1)(h^2, \ldots, h^n) & \text{if } i = 0 \\ (h^1, \ldots, h^{i-1}, h^i h^{i+1}, \ldots, h^n) & \text{if } 1 \leq i \leq n-1 \\ \delta(h^n)(h^1, \ldots, h^{n-1}) & \text{if } i = n \end{cases} . \tag{5.59}$$

$$s_i(h^1, \ldots, h^n) = (h^1, ..., h^i, 1, h^{i+1}, \ldots, h^n), \quad 0 \leq i \leq n, \tag{5.60}$$

$$t(h^1, \ldots, h^n) = \sum \delta(h^n_{(1)})(S(h^1_{(0)}h^2_{(0)}...h^n_{(0)}), h^1_{(1)}, ..., h^{n-1}_{(1)}). \tag{5.61}$$

5.10.2 Lemma/Definition: *The previous formulas make $\mathcal{H}^{\natural}$ into a cyclic module. The resulting cyclic cohomology is denoted by $HC^*_{inv}(\mathcal{H})$.*

There are various ways to see this. For instance one can deduce it from the corresponding property of $HC^*_{\delta}(\mathcal{H})$, or by direct computation. Here it will become obvious from our (explicit) interpretations.

We will first concentrate on the uni-modular case, explaining at the end the minor modifications needed in general. The previous cyclic module is dictated by the following property of $HC^*_{inv}(\mathcal{H})$ (dual to the one in [31, 36]): for any algebra A endowed with an algebra coaction $\Phi : A \longrightarrow \mathcal{H} \otimes A$ and an invariant trace τ, there is an induced map:

$$\Phi_\tau : HC^*_{inv}(\mathcal{H}) \longrightarrow HC^*(A),$$

$$(\Phi_\tau \phi)(a^0, a^1, ..., a^n) = \sum \tau(a^0 a^1_{(1)}...a^n_{(1)})\phi(h^1_{(0)}, ..., h^n_{(0)}), \tag{5.62}$$

where we have used the notation $\Phi(a^i) = \sum h^i_{(0)} \otimes a^i_{(1)}$.

5.10.3 Interpretation via non-commutative forms ($S^2 = Id$): We now explain that the homology/cohomology of 5.10.2 is computed by the (mixed) complex $\Omega^*_{inv}(\mathcal{H})$ of invariant non-commutative differential forms. Recall (see [39, 40]) that the algebra $\Omega^*(\mathcal{H})$ comes equipped with operators d, b, k, B_0 defined by $d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$, $b(\omega da) = (-1)^{deg(\omega)}[\omega, a]$, $k(\omega da) = (-1)^{deg(\omega)} da\omega$, $B^0 = (1 + k + \ldots + k^n)d$ on $\Omega^n(\mathcal{H})$. As vector spaces, $\Omega^0(\mathcal{H}) \cong \mathcal{H}$, and, for $n \geq 1$, $\Omega^n(\mathcal{H}) \cong \mathcal{H} \otimes \bar{\mathcal{H}}^{\otimes n}$, $(a_0 da_1 \ldots da_n) \leftrightarrow (a_0, a_1 \ldots a_n)$, where $\bar{\mathcal{H}} = \mathcal{H}/\mathbb{C}$. The cyclic cohomology of $\mathcal{H}$ viewed as an algebra is computed by the mixed complex [67] $(\Omega^*(\mathcal{H}), b, B^0)$. Alternatively, it is computed by the cyclic module $\mathcal{H}^\natural = (\{\mathcal{H}^{\otimes(n+1)}\}_{n\geq 0}, d_i, s_i, t)$ with:

$$d_i(h^0, \ldots, h^n) = \begin{cases} (h^0, \ldots, h^{i-1}, h^i h^{i+1}, \ldots, h^n) & \text{if } 0 \leq i \leq n-1 \\ (h^n h^0, h^1, \ldots, h^{n-1}) & \text{if } i = n \end{cases}, \tag{5.63}$$

$$s_i(h^0, \ldots, h^n) = (h^0, ..., h^i, 1, h^{i+1}, \ldots, h^n), \quad 0 \leq i \leq n, \tag{5.64}$$

$$t(h^0, \ldots, h^n) = (h^n, h^0, \ldots, h^{n-1}), \tag{5.65}$$

and $(\Omega^*(\mathcal{H}), b, B^0)$ is the usual normalized mixed complex associated to the cyclic module $\mathcal{H}^\natural$.

In our setting, any of the spaces $\mathcal{H}^{\otimes(n+1)}$ is endowed with a coaction of $\mathcal{H}$:

$$\Phi_n : \mathcal{H}^{\otimes(n+1)} \longrightarrow \mathcal{H} \otimes \mathcal{H}^{\otimes(n+1)},$$

$$\Phi_n(h^0, h^1, \ldots, h^n) = \sum (h^0_{(0)} h^1_{(0)} ... h^n_{(0)}, h^0_{(1)}, ... h^n_{(1)}).$$

Similar formulas endow $\Omega^*(\mathcal{H})$ with an algebra coaction. Our remark is that one has isomorphisms:

$$\mathcal{H}^{\otimes n} \cong Inv(\mathcal{H}^{\otimes(n+1)}) \subset \mathcal{H}^{\otimes(n+1)}, \tag{5.66}$$

$$(h^1, \ldots, h^n) \mapsto \sum (S(h^1_{(0)} ... h^n_{(0)}), h^1_{(1)}, \ldots, h^n_{(1)}),$$

compatible with the operators d_i, s_i, t, hence $\mathcal{H}^\natural_{inv} \cong Inv(\mathcal{H}^\natural)$. This simple remark shows that $\mathcal{H}^\natural_{inv}$ is a cyclic module, $\Omega^*_{inv}(\mathcal{H})$ is preserved by the operators d, b, k, B^0, and $HC^*_{inv}(\mathcal{H})$ is computed by the mixed complex $(\Omega^*_{inv}(\mathcal{H}), b, B^0)$. In particular, there is an obvious restriction map (which in interesting cases is surjective, see 5.10.10):

$$\pi : HC^*(\mathcal{H}) \longrightarrow HC^*_{inv}(\mathcal{H}) \tag{5.67}$$

Using this interpretation, the computation of $HP^*(C^\infty(M))$ for a compact manifold M, and the fact that the Chevalley Eilenberg complex identifies with the invariant part of DeRham complex, we easily deduce:

5.10.4 Corollary: *If $\mathcal{H} = C^\infty(G)$, G = a compact Lie group, then $HP^*_{inv}(\mathcal{H}) \cong H_*(\mathfrak{g})$, Lie algebra cohomology indexed modulo 2.*

Also the formula (5.62) can be easily explained. For an algebra co-action $\Phi : A \longrightarrow \mathcal{H} \otimes A$, there is an obvious induced map $A^{\otimes(n+1)} \longrightarrow \mathcal{H}^{\otimes(n+1)} \otimes A^{\otimes(n+1)}$. Using a trace τ on A, and the product of A, one gets a map of cyclic modules $\Phi_\tau : A^\natural \longrightarrow \mathcal{H}^\natural$, $(a^0, \ldots, a^n) \mapsto \sum \tau(a^0_{(0)} ... a^0_{(0)})\ h^0_{(1)} \otimes \ldots \otimes h^n_{(1)}$. A formal argument shows that, if τ is invariant, then $\Phi_\tau(A^\natural) \subset Inv(\mathcal{H}^\natural)$, hence:

5.10.5 Corollary: *For any pair (A, τ) consisting on an algebra A endowed with an algebra co-action $\Phi : A \longrightarrow \mathcal{H} \otimes A$ and an invariant trace $\tau : A \longrightarrow \mathbb{C}$, the formula (5.62) induces a map $\Phi_\tau : HC^*_{inv}(\mathcal{H}) \longrightarrow HC^*(A)$, compatible with the S-operator.*

5.10.6 Interpretation via restriction at units ($S^2 = Id$): We now interpret $HC^*_{inv}(\mathcal{H})$ as the restriction at units of the cyclic cohomology of the *algebra* $\mathcal{H}$, in analogy with the case of group-algebras: $HC^*_{inv}(\mathcal{H}) \cong HC^*(\mathcal{H})_{(1)}$.

We consider the right $\mathcal{H}$ module B which is $\mathcal{H}$ as a vector space, with the adjoint action $B \otimes \mathcal{H} \longrightarrow B$, $(\gamma, h) \mapsto ad_h(\gamma) := \sum h_{(1)}\gamma S(h_{(0)})$. Let us use the Hopf algebra structure on $\mathcal{H}$ to re-write the cyclic module $\mathcal{H}^\sharp$ in a more convenient way. For any n there is an isomorphisms:

$$j : \mathcal{H}^{\otimes(n+1)} \xrightarrow{\sim} \mathcal{H}^{\otimes n} \otimes B, \tag{5.68}$$

$$j(h^0, h^1, \ldots, h^n) = \sum(h^1_{(1)}, \ldots, h^n_{(1)}, h^0 h^1_{(0)} ... h^n_{(0)}),$$

with the explicit inverse:

$$j^{-1}(h^1, \ldots, h^n, \gamma) = \sum(\gamma S(h^1_{(0)} ... h^n_{(0)}), h^1_{(1)}, \ldots, h^n_{(1)}).$$

We obtain a cyclic module $B^\sharp(\mathcal{H}) = (\{\mathcal{H}^{\otimes n} \otimes B\}_{n\geq 0}, \bar{d}_i, \bar{s}_i, \bar{t})$ isomorphic to $\mathcal{H}^\sharp$, with the formulas:

$$\bar{d}_i(h^1, \ldots, h^n, \gamma) = \begin{cases} \epsilon(h^1)(h^2, \ldots, h^n, \gamma) & \text{if } i = 0 \\ (h^1, \ldots, h^{i-1}, h^i h^{i+1}, \ldots, h^n, \gamma) & \text{if } 1 \leq i \leq n-1 \\ (h^1, \ldots, h^{n-1}, ad_{h^n}(\gamma)) & \text{if } i = n \end{cases}, \tag{5.69}$$

$$\bar{s}_i(h^1, \ldots, h^n, \gamma) = (h^1, ..., h^i, 1, h^{i+1}, \ldots, h^n, \gamma), \quad 0 \leq i \leq n, \tag{5.70}$$

$$\bar{t}(h^1, \ldots, h^n) = \sum(\gamma_{(1)} S(h^1_{(0)} h^2_{(0)} ... h^n_{(0)}), h^1_{(1)}, ..., h^{n-1}_{(1)}, ad_{h^n_{(1)}}(\gamma_{(0)}). \tag{5.71}$$

It is clear now that for any ad-invariant subspace $\mathcal{O} \subset \mathcal{H}$ (i.e. sub-module of B) one can restrict to $\gamma \in \mathcal{O}$; the resulting cyclic module and cyclic cohomology are denoted $B^\sharp_{\mathcal{O}}(\mathcal{H})$, and $HC^*(\mathcal{H})_{\mathcal{O}}$, respectively. Under our assumption ($S^2 = Id$), the unit spans an ad-invariant subspace $(1) := \mathbb{C} \subset \mathcal{H}$, and $HC^*_{inv}(\mathcal{H}) \cong HC^*(\mathcal{H})_{(1)}$ is obvious.

The previous formulas are inspired by the case of group algebras [83], and we deduce:

5.10.7 Corollary: *If $\mathcal{H} = \mathbb{C}[\Gamma]$, $\Gamma =$ a discrete group, then $HP^*_{inv}(\mathcal{H}) \cong H^*(\Gamma)$, group cohomology indexed modulo 2.*

5.10.8 Using the Haar measure$(S^2 = Id)$ Let us now assume that $\mathcal{H}$ admits a left Haar measure $\int : \mathcal{H} \longrightarrow \mathbb{C}$ which is a trace (the last condition is automatic in the framework of [100], since we are still in the uni-modular case). By Haar measure we mean here that $\sum \int(h_{(0)})h_{(1)} = \int(h)1$, $\forall\, h \in \mathcal{H}$, $\int(1) = 1$. A straightforward computation shows that:

$$\int^\sharp : \mathcal{H}^\sharp \longrightarrow \mathcal{H}^\sharp_{inv},$$

$$(h^0, h^1, \ldots, h^n) \mapsto \sum \int (h^0 h^1_{(0)} ... h^n_{(0)})(h^1_{(1)}, \ldots, h^n_{(1)})$$

is a map of cyclic modules which is a right inverse of the inclusion, hence:

5.10.9 Corollary: *If $\mathcal{H}$ admits a Haar measure which is a trace, then the natural (restriction at units) map $\pi : HC^*(\mathcal{H}) \longrightarrow HC^*_{inv}(\mathcal{H})$ is surjective, and has a right inverse (compatible with the SBI-sequence).*

The previous conditions are automatically satisfied if $\mathcal{H}$ is finite dimensional. In this case, we can find an element $\sigma \in \mathcal{H}$ which is dual to the integral, that is, $\sigma h = \epsilon(h)\sigma$, $\forall\ h \in \mathcal{H}$, $\epsilon(\sigma) = 1$. In general, the existence of such an element implies that our cohomology is quite trivial (this is true even in the modular case). Indeed, the Hochschild complex of 5.10.2 admits the following contraction (in positive degrees): $(h^1, \ldots, h^n) \mapsto (\sigma, h^1, \ldots, h^n)$. Combining with the SBI-sequence, we deduce:

5.10.10 Corollary: *If $\mathcal{H}$ is finite dimensional, then $HC^*_{inv}(\mathcal{H}) = \mathbb{C}$ if $*$ is even, and 0 if $*$ is odd.*

5.10.11 The modular case: Let us briefly explain how the previous interpretations extend to the general case, when δ is not necessarily trivial. We denote by θ the automorphism $\delta * Id : \mathcal{H} \longrightarrow \mathcal{H}$. To make the isomorphism (5.66) compatible with the operators d^i, s^i, t, we have to replace $\mathcal{H}^\natural$ by its twisted version $\mathcal{H}^\natural(\theta)$. The difference with (5.63)-(5.65) is that we replace d_n, t by:

$$d_{\theta,n}(h^0, \ldots, h^n) = (\theta(h^n)h^0, h^1, \ldots, h^{n-1}),$$

$$t_\theta(h^0, \ldots, h^n) = (\theta(h^n), h^0, \ldots, h^{n-1}).$$

The new object $\mathcal{H}^\natural(\theta)$ is not cyclic any more (t_θ^{n+1} = the action of θ), but it is easy to see that its invariant elements are fixed by θ. Hence $\mathcal{H}^\sharp_{inv} \cong Inv(\mathcal{H}^\sharp(\theta))$ is a cyclic module. Similarly for forms: this time we have a twisted version of the operators: $b_\theta(\omega da) = (-1)^{deg(\omega)}[\omega, a]_\theta = (-1)^{deg(\omega)}\omega a - \theta(a)\omega$, $k_\theta(\omega da) = (-1)^{deg(\omega)} d\theta(a)\omega$, $B^0_\theta = (1 + k_\theta + \ldots + k^n_\theta)d$. The basic relations [40] become $b_\theta d + db_\theta = 1 - k_\theta$, $k^n_\theta = \theta - k^n_\theta bd$, $(k^{n+1}_\theta - \theta)(k^n_\theta - \theta) = 0$ on $\Omega^n(\mathcal{H})$, and $\theta = 1$ on $\Omega_{inv}(\mathcal{H})$ (in particular, one has the same type of spectral decomposition with respect to k_θ on $\Omega_{inv}(\mathcal{H})$ as in [40]). The normalized mixed complex associated to $\mathcal{H}^\sharp$ will be $(\Omega_{inv}(\mathcal{H}), b_\theta, B^0_\theta)$.

The isomorphism (5.68) induces now an isomorphism $\mathcal{H}^\natural(\theta) \cong B^\sharp(\mathcal{H}, \theta)$, where the formulas for the last module are the same as those of $B^\sharp(\mathcal{H})$ (see (5.69)-(5.71)), with the only difference that one has to replace ad by ad^θ, $ad^\theta_h(\gamma) = \theta(h_{(1)})\gamma S(h_{(0)})$. Straightforward computations show that the ad^θ_h's commute with θ, and $\bar{t}^{n+1}_{tw}$ = the action of θ on the last component γ. These show that the maximal submodule of $B^\sharp(\mathcal{H}, \theta)$ which is cyclic is obtained by restricting to $\mathcal{O}_{max} = \{\gamma \in \mathcal{H} : \theta(\gamma) = \gamma\}$. Restricting to the ad^θ-invariant induced by the unit $(1) \subset \mathcal{O}_{max}$, one gets $HC^*_{inv}(\mathcal{H})$.

What is not clear is how to use a Haar measure on $\mathcal{H}$. Recall that in the case of compact quantum groups of [100], the Haar measure is not a trace, but it does satisfy the relation $\int ab = \int b\sigma(a)$, $\forall\ a, b \in \mathcal{H}$, where $\sigma = \delta * Id * \delta$. It would be interesting to give a good interpretation of this relation in our context. Probably this would be a hint to the computation of $HC^*_{inv}(\mathcal{H})$ in more interesting cases ($S_\nu U(2)$ for instance [100]).

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Samenvatting

Dit Proefschrift behandelt cohomologietheorieën en karakteristieke klassen voor de bladenruimtes (*leaf spaces*) van foliaties. Daarnaast wordt ingegaan op de interactie tussen de klassieke (Grothendieck, Bott - Haefliger) en de niet-commutatieve benadering (Connes - Moscovici) van deze theorieën.

Bladenruimtes zijn in veel gevallen voorbeelden van singuliere ruimtes waarop de standaard theorie niet direct van toepassing is. Grothendieck, Bott en Haefliger vinden hiervoor een oplossing in het bestuderen van de categorie van étale groepoïden in plaats van de beperktere categorie van (niet singuliere) ruimtes. Hoofdstukken 2 en 3 volgen deze benadering.

We introduceren in hoofdstuk 2 een homologietheorie, waarvoor er een Poincaré-dualiteitsrelatie is met Haefligers cohomologie. In hoofdstuk 3 geven we een meer meetkundig model (Čhech - De Rham) voor Haefligers cohomologie. Dit stelt ons in staat om karakteristieke klassen voor étale groepoïden, en dus van bladenruimtes, op een meetkundige wijze te construeren, en om formules van Bott te verklaren en te generaliseren. Deze hoofdstukken zijn tot stand gekomen in samenwerking met Ieke Moerdijk.

Ook in de niet-commutatieve meetkunde geven bladenruimtes vele interessante voorbeelden. Daar worden ze gemodelleerd door hun associatieve convolutie algebras. In deze context is cyclische cohomologie de relevante cohomologietheorie.

In hoofdstuk 4 berekenen we de cyclische cohomologie van degelijke convolutiealgebras van étale groepoïden. Hier vinden we een relatie met de klassieke benadering (Grothendieck, Bott - Haefliger) gebaseerd op onze homologie theorie uit hoofdstuk 1. Onze berekeningen generaliseren die van Brylinski, Burghelea, Connes, Karoubi en Nistor. In de laatste paragrafen concentreren we ons, gemotiveerd door de relatie met longitudinale indextheorie, op de holonomie groepoïden van foliaties en de daarmee geassocieerde Chern-karakters.

Connes en Moscovici hebben recentelijk een diep verband weten te leggen tussen meetkundige karakteristieke klassen voor foliaties en niet-commutatieve karakteristieke klassen, die voortkomen uit het Chern-karakter in de cyclische cohomologiegroepen uit hoofdstuk 4. Dit verband is gebaseerd op de cyclische cohomologie van een Hopf algebra van meetkundige operatoren.

In hoofdstuk 5 bestuderen we deze cohomologietheorie. Ten eerste tonen we aan dat deze theorie voor willekeurige Hopf algebras de klassieke Lie-algebracohomologie generaliseert. Ten tweede leggen we een verband met de benadering van cyclische cohomologie in termen van X-complexen door Cuntz en Quillen. Daarnaast geven we, geïnspireerd door de (vrij klassieke) constructie van van karakteristieke klassen voor foliaties in termen van het afgeknotte (*truncated*) Weil-complex (zie hoofdstuk 1), een niet commutatieve versie van het standaard Weil-complex. Dit complex blijkt sterk gerelateerd aan het Cuntz-Quillen X-complex. Het kan gebruikt worden om karakteristieke afbeeldingen geassocieerd met hogere sporen (*higher traces*) te construeren.