Generalized complex structures and Lie brackets

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Abstract

We remark that the equations underlying the notion of generalized complex structure have simple geometric meaning when passing to Lie algebroids/groupoids.

Contents

1 Introduction

Generalized complex structures have recently been introduced by N. Hitchin [9] and further studied by M. Gualtieri [8], as a bridge between symplectic and complex geometry. When making the conceptual definition more explicit, one arises at puzzling equations (see Section 2.1). In this paper we remark that such complicated explicit formulas have miraculously simple interpretations after allowing algebroids and groupoids into the picture. Since underlying a generalized complex structure there is a Poisson bivector, we use either as a guide, or explicitly, some basic principles from Poisson geometry. The first one we have in mind is that, when passing to the global objects behind Poisson structures, one arises at symplectic structures (and similarly after restricting to leaves). Here, instead of "symplectic", please read "*non-degenerate* Poisson".

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Using the groupoid associated to the underlying Poisson structure, we show that the puzzling equations mentioned above translate into simple structures on the groupoid, giving rise to what we call here "Hitchin groupoids". In particular, we will see that, at the level of the groupoid, one has an induced *non-degenerate* generalized complex structure (but, in comparison with Poisson geometry, the situation is even better since the structure is more then non-degenerate). This is done in the last section, which starts with a short presentation of algebroids/groupoids, as well as of the main ingredient that we use, namely the multiplicative 2-forms discussed in [2] (however, the presentation here is self-contained). The first section is "groupoid-free"; there we look at the definition of generalized complex manifolds (in terms of the tensors involved), we introduce the generalized holomorphic maps (pointing out "reduction" in this context), and we have a closer look at generalized complex structures which are non-degenerate. We end the paper with a few remarks inspired from Poisson and Dirac geometry.

Our results can be viewed as "integration results", i.e. as describing the global structure behind generalized complex structures. Since underlying our discussion is the symplectic groupoid of the Poisson manifold, or the phase-space of the Poisson sigma model [4], it would be interesting to find the relationship between our work and the "Hitchin sigma model" [10, 13]. Another interesting (and probably related) question is to find the relationship between our integration results and the integration of complex Lie algebroids proposed by A. Weinstein [14]. Finally, this work is similar to [1], in the sense that we study generalized complex structures from the point of view of Poisson (but also Dirac) geometry.

2 Generalized complex structures

2.1 Generalized complex structures

Let M be a (real) manifold, and let

$$\mathcal{T}M := TM \oplus T^*M,$$

which we will call the generalized tangent bundle of M. It relates to the usual tangent bundle via the obvious projection, denoted:

$$\pi : \mathcal{T}M \longrightarrow TM,$$

The space of sections of $\mathcal{T}M$ is endowed with a bracket, known as **the Courant bracket** [5], which extends the Lie bracket on vector fields. Explicitly,

$$[(X,\xi),(Y,\eta)] = ([X,Y], L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)).$$

Note that the Courant bracket is antisymmetric but it does not satisfy the Jacobi identity.

The generalized tangent bundle $\mathcal{T}M$ carries various other interesting structures, such as the canonical (fiberwise) symplectic form, or the non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$:

$$\langle (X,\xi),(Y,\eta)\rangle = \xi(Y) + \eta(X).$$

Although these will not play much role in this paper, they should be viewed as part of the picture. For more details on the relation between $\langle\cdot,\cdot\rangle$ and the Courant bracket, we refer to [5, 11].

By replacing the usual tangent bundle with $\mathcal{T}M$ and the Lie bracket with the Courant bracket, many aspects of tensorial geometry can be carried out in this "generalized" setting. The famous Newlander-Nirenberg theorem enables us to apply this principle to complex structures. Let us first recall that, for a bundle map $a : TM \longrightarrow TM$ covering the identity, its torsion $\mathcal{N}_a$ is the $(1,2)$-tensor:

$$\mathcal{N}_a(X,Y) = [aX, aY] + a^2[X,Y] - a([aX,Y] + [X,aY]),$$

and one says that a is torsion-free if $\mathcal{N}_a = 0$. With this, the Newlander-Nirenberg theorem asserts that there is a 1-1 correspondence between complex structures on M and almost complex structures J on M (i.e. vector bundle maps $J : TM \longrightarrow TM$ satisfying $J^2 = -Id$) which are torsion-free. Hence one arises [8, 9] at

Definition 2.1 *A* ***generalized complex structure on*** *M (g.c. structure on short) is a complex structure $\mathcal{J}$ on the bundle $\mathcal{T}M$ (i.e. a bundle map $\mathcal{J} : \mathcal{T}M \longrightarrow \mathcal{T}M$ with $\mathcal{J}^2 = -Id$), satisfying:*

$$[\mathcal{J}\alpha, \mathcal{J}\beta] - [\alpha, \beta] - \mathcal{J}([\mathcal{J}\alpha, \beta] + [\alpha, \mathcal{J}\beta]) = 0, \tag{2.1}$$

for all sections α, β of $\mathcal{T}M$.

To make the definition of generalized complex structures more explicit, we need to fix some notations: for a two-form σ on M we denote by $\sigma_\sharp : TM \longrightarrow T^*M$ the map $X \mapsto i_X\sigma$, while for a bivector π on M we denote by $\pi^\sharp : T^*M \longrightarrow TM$ the contraction with π, i.e. defined by

$$\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta).$$

Also, we denote by $[\cdot,\cdot]_\pi$ the bracket on the space of 1-forms on M defined by (see also the last section):

$$[\xi, \eta]_\pi = L_{\pi^\sharp\xi}\eta - L_{\pi^\sharp\eta}\xi - d\pi(\xi, \eta). \tag{2.2}$$

Next, after spelling out (2.1), reducing the (apparently longer) list of equations to a minimal one, and then writing and organizing them in a convenient way, we arrive at the following (in local coordinates, the equations bellow also appear in [10]).

Proposition 2.2 *A generalized complex structure $\mathcal{J}$ on M is necessarily of type*

$$\mathcal{J} = \begin{pmatrix} a & \pi^\sharp \\ \sigma_\sharp & -a^* \end{pmatrix} \tag{2.3}$$

where π is a bivector on M, σ is a two-form on M, and $a : TM \longrightarrow TM$ is a bundle map. Moreover, $\mathcal{J}$ is a generalized complex structure if and only if the following conditions are satisfied

(C1) π satisfies the equation:

$$\pi^\sharp([\xi, \eta]_\pi) = [\pi^\sharp(\xi), \pi^\sharp(\eta)].$$

(C2) π and a are related by the following two formulas:

$$a\pi^\sharp = \pi^\sharp a^* \tag{2.4}$$

$$a^*([\xi, \eta]_\pi) = L_{\pi^\sharp\xi}(a^*\eta) - L_{\pi^\sharp\eta}(a^*\xi) - d\pi(a^*\xi, \eta) \tag{2.5}$$

(C3) π, a and σ are related by the following two formulas:

$$a^2 + \pi^\sharp\sigma_\sharp = -Id \tag{2.6}$$

$$\mathcal{N}_a(X,Y) = \pi^\sharp i_{X\wedge Y}(d\sigma) \tag{2.7}$$

(C4) σ and a are related by the following two formulas:

$$a^*\sigma_\sharp = \sigma_\sharp a \tag{2.8}$$

$$d\sigma_a(X,Y,Z) = d\sigma(aX,Y,Z) + d\sigma(X,aY,Z) + d\sigma(X,Y,aZ), \tag{2.9}$$

where $\sigma_a(X,Y) = \sigma(aX,Y)$.

(the previous equations must hold for all 1-forms ξ and η, and all vector fields X, Y and Z).

Note that condition (C1) is equivalent to that fact that π is a **Poisson bivector** (i.e. the bracket $\{f,g\} = \pi(df,dg)$ is a Lie algebra bracket on the $C^\infty(M)$). In particular, the distribution $\pi^\sharp(T^*M) \subset TM$ is integrable, and its leaves (called the symplectic leaves) carry natural symplectic structures; also, the bracket $[\cdot,\cdot]$ restricts to a Lie bracket on $\mathfrak{g}_x = Ker(\pi^\sharp_x)$, making it into a Lie algebra called the **isotropy Lie algebra** at $x \in M$. Of course, the first equations in (C2) and (C3) imply that $\mathfrak{g}_x$ is a complex Lie algebra (with the restriction of a as the complex structure).

For latter use note also that if (2.3) is a generalized complex structure, then so is

$$\overline{\mathcal{J}} = \begin{pmatrix} a & -\pi^\sharp \\ -\sigma_\sharp & -a^* \end{pmatrix}, \tag{2.10}$$

which will be called **the opposite** of $\mathcal{J}$.

Proof: First of all, note that $\mathcal{J}$ must be orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$:

$$\langle \mathcal{J}(\alpha), \mathcal{J}(\beta)\rangle = \langle \alpha, \beta\rangle$$

for all sections α, β of $\mathcal{T}M$. This follows from the $C^\infty(M)$-linearity of the equation (2.1) with respect to β and the formula:

$$[\alpha, f\beta] = f[\alpha,\beta] + L_{\pi(\alpha)}(f)\beta - \langle\alpha,\beta\rangle df,$$

for $f \in C^\infty(M)$. From this orthogonality property it immediately follows that $\mathcal{J}$ must be of type (2.3), while the condition $\mathcal{J}^2 = -Id$ immediately implies that π, a and σ satisfy the linear equations appearing in (C2), (C3) and (C4). We claim that the remaining (differential) equations are equivalent to the integrability condition (2.1). Let us look closer at (2.1). The equation has two components hence it gives us two equations. And this happens in each of the three cases we have to consider. The first one is when α and β are 1-forms, call them ξ and η, when the resulting equations are easily seen to be (C1) (for the first component) and the second one in (C2) (for the second component). The next case is when α is a vector field X and β is a 1-form ξ. The first component of the resulting equation translates into:

$$[a(X), \pi^\sharp(\xi)] - a([X, \pi^\sharp(\xi)]) = \pi^\sharp(L_{aX}(\xi) - L_X(a^*(\xi))). \tag{2.11}$$

while the second component gives us:

$$\begin{aligned} \sigma_\sharp([\pi^\sharp(\xi), X]) - L_{\pi^\sharp(\xi)}(\sigma_\sharp(X)) &= L_X(\xi) + d\xi(\pi^\sharp\sigma_\sharp(X)) \\ &+ L_{a(X)}(a^*(\xi)) - a^*(L_{a(X)}(\xi) - L_X(a^*(\xi))), \end{aligned} \tag{2.12}$$

The third case is when α and β are vector fields, call them X and Y. The first component of the resulting equation is

$$\begin{aligned} &[X,Y] - [a(X), a(Y)] + a([a(X),Y] + [X, a(Y)]) = \\ &-\pi^\sharp(L_X(\sigma_\sharp(Y)) - L_Y(\sigma_\sharp(X)) + d\sigma(X,Y)), \end{aligned} \tag{2.13}$$

while the second component gives us:

$$-L_{a(X)}(\sigma_\sharp(Y)) + L_{a(Y)}(\sigma_\sharp(X)) - d\sigma_a(X,Y) + \sigma_\sharp([a(X),Y] + [X,a(Y)]) = \\ a^*(L_X(\sigma_\sharp(Y)) - L_Y(\sigma_\sharp(X)) + d\sigma(X,Y)). \tag{2.14}$$

Now, the equation (2.11) is easily seen to be equivalent to the the second equation in (C2) (again). Also, a straightforward computation shows that (2.12) and (2.13) are equivalent. Making use of the formula:

$$i_{X\wedge Y}(d\sigma) = L_X(i_Y(\sigma)) - L_Y(i_X(\sigma)) + d(i_{X\wedge Y}\sigma) - i_{[X,Y]}\sigma, \tag{2.15}$$

(2.13) is easily seen to be equivalent to the second equation of (C3). Finally, making use of the Koszul-type formula

$$\begin{aligned}(d\sigma)(X,Y,Z) \quad &= L_X\sigma(Y,Z) + L_Y\sigma(Z,X) + L_Z\sigma(X,Y) \\ &-\sigma([X,Y],Z) - \sigma([Z,X],Y) - \sigma([Y,Z],X),\end{aligned} \tag{2.16}$$

a straightforward computation shows that (2.14) is equivalent to the second equation in (C4). □

Finally, let us point out the relation with Dirac geometry. Recall that a Dirac structure on M is a sub-bundle $L \subset \mathcal{T}M$ of rank equal to the dimension of M, with the property that the canonical pairing $\langle\cdot,\cdot\rangle$ vanishes when restricted to L, and which satisfies the involutivity condition $[\Gamma(L),\Gamma(L)] \subset \Gamma(L)$ (with respect to the Courant bracket). The standard examples of Dirac structures are graphs of closed 2-forms or of Poisson bivectors on M. In our context, condition (C2) is equivalent to saying that

$$L_{\pi,a} := \{(\pi^\sharp(\xi), a^*(\xi)) : \xi \in T^*M\} \tag{2.17}$$

is a Dirac structure on M.

Complex Dirac structures are defined similarly, by replacing $\mathcal{T}M$ by its complexification

$$\mathcal{T}_{\mathbb{C}}M = \mathcal{T}M \otimes \mathbb{C} = T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M.$$

The main point of this discussion is that generalized complex structures $\mathcal{J}$ on M can be identified with complex Dirac structures L on M with the property that $L \oplus \overline{L} = \mathcal{T}_C M$ [8]. In one direction, given $\mathcal{J}$, L is the $+i$-eigenspace of $\mathcal{J}$ on $\mathcal{T}M \otimes \mathbb{C}$. We see that, if $\mathcal{J}$ is represented by (2.3), then the associated L is:

$$L := \{(X - iX', \xi - i\xi') : X' = aX + \pi^\sharp\xi, \xi' = \sigma_\sharp X - a^*\xi\}.$$

2.2 The non-degenerate case

In this section we discuss the generalized complex structure which are non-degenerate in the sense that the associated bivector π is non-degenerate (i.e. $\pi^\sharp$ is an isomorphism). Such structures arise after restricting to the symplectic leaves of π or after passing to the global objects (see the next section). Also, they are particularly simple since some of the structure data (tensors and equations) are consequences of the non-degeneracy condition. To make this explicit, we first introduce some notations.

Definition 2.3 *Given a 2-form ω on M, and a bundle map $a : TM \longrightarrow TM$, we say that ω and a commute if $\omega_\sharp a = a^* \omega_\sharp$, i.e.:*

$$\omega(aX, Y) = \omega(X, aY),$$

for all X, Y-vector fields on M. In this case, we denote by ω_a the 2-form:

$$\omega_a(X, Y) = \omega(aX, Y).$$

An ***symplectic+complex structure*** *on M consists of a pair (ω, J) with ω-symplectic, and J-complex structure on M, which commute.*

For instance, in the (real) cotangent bundle of any complex manifold, the canonical symplectic form together with the lifted complex structure is a symplectic+complex structure structure. Also, the co-adjoint orbits of a complex Lie algebra carry such structures. In our context, the symplectic+complex structures correspond to those non-degenerate generalized complex structures with $\sigma = 0$ (see below).

Lemma 2.4 *Let (M, ω) be a symplectic manifold. Then*

$$a \mapsto \omega_a$$

defines a 1-1 correspondence between $(1,1)$-tensors $a : TM \longrightarrow TM$ commuting with ω and 2-forms on M. Moreover, (ω, a) is a symplectic+complex structure if and only if

$$d\omega_a = 0, \ \omega + a^*\omega = 0.$$

Definition 2.5 *A* ***Hitchin pair*** *on M is a pair (ω, a) consisting of a symplectic form ω and a $(1,1)$-tensor a with the property that ω and a commute and*

$$d\omega_a = 0.$$

The twist of (ω, a) is the 2-form

$$\sigma := -(\omega + a^*\omega).$$

Note that in this case a is neither an almost complex structure nor torsion free. Instead, as it will follow from the computation below, one has:

$$a^2 = -Id - \omega_\sharp^{-1}\sigma_\sharp, \ \mathcal{N}_a(X, Y) = \omega_\sharp^{-1} i_{X\wedge Y}(d\sigma).$$

Proposition 2.6 *There is a 1-1 correspondence between generalized complex structures*

$$\mathcal{J} = \begin{pmatrix} a & \pi \\ \sigma_\sharp & -a^* \end{pmatrix} \tag{2.18}$$

with π non-degenerate, and Hitchin pairs (ω, a). In this correspondence, π is the inverse of ω, and σ is the twist of the Hitchin pair (ω, a).

The remaining of this section is devoted to the proof of Lemma 2.4 and of Proposition 2.6. We start with the proposition. At the level of the tensors (bivectors, 2-forms, $(1,1)$-tensors), the correspondence is completely described in the statement, and we only have to take care of the equations they satisfy. The following is the well-known correspondence between non-degenerate Poisson bivectors and symplectic forms, which we include for completeness.

Lemma 2.7 *If π is a non-degenerate bivector on M, and ω is the inverse 2-form (defined by $\omega_{\sharp} = (\pi^{\sharp})^{-1}$), then π satisfies (C1) if and only if ω is closed.*

PROOF: Apply (C1) to $\xi = i_X(\omega)$, $\eta = i_Y(\omega)$, where X and Y are arbitrary vector fields, then apply $\omega_{\sharp}$ to the resulting formula, and use (2.15). $\square$

Next, we take care of (C2).

Lemma 2.8 *Given a symplectic form ω, π the associated non-degenerate bivector (i.e. $\pi^{\sharp} = \omega_{\sharp}^{-1}$), and $a : TM \longrightarrow TM$ a bundle map, then π and a satisfy (C2) if and only if ω and a commute and ω_a is closed.*

PROOF: Clearly, (2.4) is equivalent to ω and a commuting. Next, to take care of (2.5), we apply it to $\xi = i_X\omega$, $\eta = i_Y\omega$, where X and Y are arbitrary vector fields. Using the $a^*(i_X(\omega)) = i_X(\omega_a)$, and applying formula (2.15) to our closed ω, we get:

$$i_{[X,Y]}(\omega_a) = L_X(i_Y(\omega_a) - L_Y(i_X(\omega_a)) + d\omega_a(X, Y).$$

Hence, using again (2.15), we see that (2.5) is equivalent to ω_a being closed. $\square$

Next, the equation (2.6), under the hypothesis that π is non-degenerate, clearly forces $\sigma = -\omega - a^*\omega$, where ω is the inverse of π. Hence we are left with proving that if (ω, a) is as in the statement of the proposition, then J given by (2.3) with π the inverse of ω and $\sigma = -\omega - a^*\omega$ automatically satisfies equation (2.7), (2.8), (2.9). The first one is taken care by the following general computation:

Lemma 2.9 *For any 2-form ω and any $a : TM \longrightarrow TM$ commuting with ω, one has:*

$$i_{\mathcal{N}_a(X,Y)}(\omega) = i_{aX\wedge Y + X\wedge aY}(d\omega_a) - i_{aX\wedge aY}(d\omega) - i_{X\wedge Y}(d(a^*\omega)).$$

PROOF: Using the Koszul-type formula (2.16) applied to each of the four terms appearing in the right hand side of the equation, and the fact that ω and a commute, we easily arrive to the left hand side. $\square$

Hence, under our hypothesis (ω and ω_a-closed), the equation in the lemma implies that

$$i_{X\wedge Y}(d(a^*\omega)) = -i_{\mathcal{N}_a(X,Y)}(\omega). \tag{2.19}$$

Hence $\sigma = -\omega - a^*\omega$ has:

$$i_{X\wedge Y}(d\sigma) = i_{\mathcal{N}_a(X,Y)}(\omega),$$

and then, applying $\pi^{\sharp}$, we arrive at (2.7). Next, equation (2.8) is clear. Finally, we look at (2.9). Writing the equation as

$$i_{X\wedge Y}(d\sigma_a) = i_{aX\wedge Y + X\wedge aY}(d\sigma) + a^*(i_{X\wedge Y}(d\sigma)),$$

since $\sigma = -\omega - a^*\omega$ and since ω and ω_a are closed, the desired equation is equivalent to:

$$i_{X\wedge Y}d(a^*\omega_a) = i_{aX\wedge Y + X\wedge aY}d(a^*\omega) + a^* i_{X\wedge Y}d(a^*\omega). \tag{2.20}$$

To compute the left hand side, we use Lemma 2.9 applied to ω_a and we obtain:

$$i_{X\wedge Y} d(a^*\omega_a) = i_{aX\wedge Y + X\wedge aY} d(a^*\omega) - i_{\mathcal{N}_a(X,Y)}(\omega_a).$$

Using $i_{\mathcal{N}}(\omega_a) = a^* i_{\mathcal{N}}(\omega)$ and (2.19), we arrive at the desired equation (2.20). This concludes the proof of Proposition 2.6.

We now prove Lemma 2.4. The only part which needs some explanation is to show that if a is almost complex then $\mathcal{N}_a = 0$ if and only if $d\omega_a = 0$. If ω_a is closed, then Lemma 2.9 together with the fact that ω is closed, $a^*\omega = -\omega$ and ω is non-degenerate, implies that $\mathcal{N}_a = 0$. The converse follows again from a more general equation which is stated in the following lemma.

Lemma 2.10 *If J is a complex structure on M then, for any 2-form ω on M commuting with J one has:*

$$2d\omega_J(X,Y,Z) =$$
$$d\omega(JX,Y,Z) + d\omega(X,JY,Z) + d\omega(X,Y,JZ) + d\omega(JX,JY,JZ).$$

PROOF: This follows from the general properties of complex forms, or can be checked directly, e.g. by using again the Koszul formula (2.16). □

2.3 Generalized holomorphic maps

Next, we discuss the notion of morphism between generalized complex manifolds. At first one may expect that, based on the interpretation of generalized complex structures as complex Dirac structures (see the end of section 2.1), one could use the notion of morphism in Dirac geometry. However, there are two such notions: "forward" morphisms which are well-behaved with respect to bivectors but not 2-forms, and "backward" morphisms which are well behaved with respect to 2-forms but not bivectors; see [2, 3]. However, since generalized complex structures are made up of both 2-forms and bivectors, both of these two notions would be quite restrictive. However, with the mind at the matrix representation (2.3), there is a natural notion of morphism which sits in between the two notions of forward and backward Dirac maps.

Definition 2.11 *Let $(M_i, \mathcal{J}_i)$, $i \in \{1,2\}$, be two generalized complex manifolds, and let a_i, π_i, σ_i be the components of $\mathcal{J}_i$ in the matrix representation (2.3). A map $f : M_1 \longrightarrow M_2$ is called* ***generalized holomorphic*** *if f maps π_1 into π_2, $f^*\sigma_2 = \sigma_1$ and $(df)\circ a_1 = a_2 \circ (df)$.*

Lemma 2.12 *If $f : M_1 \longrightarrow M_2$ is generalized holomorphic and $x \in M_2$ is regular for f, then the fiber $\mathcal{C} = f^{-1}(x)$ is a complex manifold (with complex structure $a_1|_{\mathcal{C}}$).*

PROOF: Let $T\mathcal{C} = Ker(df)$ and since (df) commutes with the a_i's, a_1 restricts to an $(1,1)$-tensor on $\mathcal{C}$. Using the two equations in (C3) applied to J_1 and the fact that $f^*\sigma_2 = \sigma_1$, we deduce that $a_1^2(X) = -X$ and $\mathcal{N}_{a_1}(X,Y) = 0$ if $X \in Ker(df)$. □

With the mind at Poisson manifolds, it is very tempting to discuss reduction in our setting.

Definition 2.13 *A Hitchin realization of a generalized complex manifold $(M, \mathcal{J})$ consists of a manifold S endowed with a Hitchin pair (ω, a) and a smooth map $\mu : S \longrightarrow M$ with the property that μ is generalized holomorphic.*

Recall from the previous subsection that Hitchin pairs correspond to non-degenerate generalized complex structures. A basic example of Hitchin realization is the inclusion of the symplectic leaves into M.

We now discuss reduction. Let $\mu : S \longrightarrow M$ be a Hitchin realization, and let π, a, σ be the components of $\mathcal{J}$. Assume that $x \in M$ is regular for μ, and we look at the level set

$$\mathcal{C} = \mu^{-1}(x).$$

There is an induced infinitesimal action of the isotropy Lie algebra $\mathfrak{g}_x = Ker(\pi^{\sharp}_x)$ on $\mathcal{C}$, encoded into a Lie algebra map $\rho : \mathfrak{g}_x \longrightarrow \mathcal{X}(\mathcal{C})$: for $\alpha \in \mathfrak{g}_x$, $\rho(\alpha)$ is the unique solution of the moment map equation

$$i_{\rho(\alpha)}(\omega) = \mu^*(\alpha).$$

We now assume that the quotient space $\mathcal{C}/\mathfrak{g}_x$ is smooth. The reduction from Poisson geometry says that $\mathcal{C}/\mathfrak{g}_x$ carries a canonical symplectic structure $\omega_{\mathcal{C}}$ (uniquely determined by the condition that its pull-back to $\mathcal{C}$ coincides with the restriction of ω to $\mathcal{C}$). The following shows that $\mathcal{C}/\mathfrak{g}_x$ carries a canonical symplectic+complex structure.

Proposition 2.14 *$\mathcal{C}/\mathfrak{g}_x$ carries a complex structure $J_{\mathcal{C}}$ unique with the property that the projection $\mathcal{C} \longrightarrow \mathcal{C}/\mathfrak{g}_x$ is a holomorphic map. Moreover, $J_{\mathcal{C}}$ commutes with the reduced symplectic structure $\omega_{\mathcal{C}}$.*

PROOF: One can check directly that J vanishes on the image of ρ, and then prove that J can be pushed forward to a $(1,1)$-tensor on $\mathcal{C}/\mathfrak{g}_x$ with the desired properties. Alternatively, the same argument that shows that ω can be reduced also applies to ω_J and we obtain a second symplectic structure $\underline{\omega}_J$ on $\mathcal{C}/\mathfrak{g}_x$ (this can also be viewed as reduction in the Dirac setting [2] since (S, ω_J) is a presymplectic realization [2] of M endowed with the Dirac structure (2.17)). Then one considers $\underline{J} = \underline{\omega}^{-1}\underline{\omega}_J$, so that J automatically commutes with $\underline{\omega}$ and $\underline{\omega_J}$ is closed. Since $\omega + J^*\omega$ is a pull-back by μ, it is zero when restricted to $\mathcal{C}$, and then we deduce that $\underline{J}^2 = -Id$. $\square$

Remark 2.15 Similar to the Poisson case (see e.g. [7]), one can perform reduction starting with any generalized holomorphic map $\mu : N \longrightarrow M$ between two generalized complex manifolds. Then, for $x \in M$, we consider the isotropy Lie algebra $\mathfrak{g}_x$ at x, and we make the same assumptions as above (x is a regular point and $\mathcal{C}/\mathfrak{g}_x$ is smooth). This time, the action of $\mathfrak{g}_x$ on $\mathcal{C}$ is $\alpha \mapsto \pi^{\sharp}\mu^*(\alpha)$, where π is the bivector on N. Then $\mathcal{C}/\mathfrak{g}_x$ will carry a natural Hitchin pair.

3 Bringing Lie algebroids/groupoids into the picture

In this section we show that conditions (C1)-(C3) (see Proposition 2.2) have simple interpretations after passing to groupoids. We start by recalling some basic facts on Lie groupoids and multiplicative forms.

3.1 Lie algebroids/groupoids

Recall that a Lie algebroid over M is a vector bundle $A \longrightarrow M$ together with a Lie algebra bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$ and a bundle map $\rho : A \longrightarrow TM$, called the anchor of A, satisfying the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + L_{\rho(\alpha)}(f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$, $f \in C^\infty(M)$. In what follows, we will write $L_{\rho(\alpha)}$ simply as L_α. The basic examples are Lie algebras (when M is a point) and tangent bundles (when ρ is the identity).

Lie groupoids are the global counterparts of Lie algebroids. In order to fix our notation, we recall that a Lie groupoid over a manifold M consists of a manifold $\mathcal{G}$ together with surjective submersions $\mathsf{t}, \mathsf{s} : \mathcal{G} \to M$, called **target** and **source**, a partially defined multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} \,|\, \mathsf{s}(g) = \mathsf{t}(h)\}$, a **unit section** $\varepsilon : M \to \mathcal{G}$ and an **inversion** $i : \mathcal{G} \to \mathcal{G}$, all related by the appropriate axioms (see e.g. [6] and the references therein). To simplify our notation, we will often identify an element $x \in M$ with its image $\varepsilon(x) \in \mathcal{G}$. We say that $\mathcal{G}$ is **source-simply-connected** if all the fibers s^{-1} are connected and simply connected.

For a Lie groupoid $\mathcal{G}$, the associated Lie algebroid $A(\mathcal{G})$ consists of the vector bundle

$$\ker(d\mathsf{s})|_M \to M,$$

with anchor $\rho = d\mathsf{t} : \ker(d\mathsf{s})|_M \to TM$ and bracket induced from the Lie bracket on $\mathcal{X}(\mathcal{G})$ via the identification of sections $\Gamma(\ker(d\mathsf{s})|_M)$ with right-invariant vector fields on $\mathcal{G}$ tangent to the s-fibers.

An **integration** of a Lie algebroid A is a Lie groupoid $\mathcal{G}$ together with an isomorphism $A \cong A(\mathcal{G})$. Unlike Lie algebras, not every Lie algebroid admits an integration, see [6] for a description of the obstructions. On the other hand, if a Lie algebroid is integrable, then there exists a canonical source-simply-connected integration $\mathcal{G}(A)$, and any other source-simply-connected integration is smoothly isomorphic to $\mathcal{G}(A)$; see [6]. To avoid endless repetitions, *from now on we assume that all Lie groupoids are source-simply-connected.*

Next, we recall a few facts about multiplicative forms. A form ω on a Lie groupoid $\mathcal{G}$ is called **multiplicative** if

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega, \tag{3.1}$$

where $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the canonical projections. Note that, for any form ϕ on M, $s^*\phi - t^*\phi$ is multiplicative; we say that a multiplicative form **vanishes cohomologically** if it is of this type. Cohomological vanishing is often as good as honest vanishing. According to this principle, given a 3-form ϕ on M, one says that a 2-form ω on $\mathcal{G}$ is ϕ-closed if $d\omega = s^*\phi - t^*\phi$.

On the infinitesimal side, for an arbitrary Lie algebroid A, one has the notion of **IM form on** A (read infinitesimal multiplicative form): am IM form on A is a bundle map

$$u : A \longrightarrow T^*M$$

satisfying the following properties:

$$\begin{aligned} \langle u(\alpha), \rho(\beta)\rangle &= -\langle u(\beta), \rho(\alpha)\rangle; & (3.2)\\ u([\alpha,\beta]) &= L_\alpha(u(\beta)) - L_\beta(u(\alpha)) + d\big\langle u(\alpha), \rho(\beta')\big\rangle, & (3.3) \end{aligned}$$

for $\alpha, \beta \in \Gamma(A)$ (here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between a vector space and its dual).

In general, if A is the Lie algebroid of a Lie groupoid $\mathcal{G}$, then a closed multiplicative 2-form ω on $\mathcal{G}$ induces a IM form u_ω of A by:

$$\langle u_\omega(\alpha), X\rangle = \omega(\alpha, X).$$

This has been explained in [2] (Proposition 3.5), to which we refer for more details on multiplicative 2-forms. In particular, we will need the following reconstruction result (Theorem 5.1 in [2]):

Theorem 3.1 *If A is an integrable Lie algebroid and if $\mathcal{G}$ is its (source-simply-connected) integration, then $\omega \mapsto u_\omega$ is a 1-1 correspondence between closed multiplicative 2-forms on $\mathcal{G}$ and IM forms of A.*

A few more words on multiplicative structures on groupoids. Similar to 2-forms (and symplectic groupoids), one can talk about the multiplicativity of various other structures. For instance, the multiplicativity of bivectors (and then Poisson groupoids) appears in [12]. Of relevance here are the $(1,1)$-tensors: given a Lie groupoid $\mathcal{G}$, we say that a $(1,1)$-tensor $J : T\mathcal{G} \longrightarrow \mathcal{G}$ is multiplicative if for any $(g,h) \in \mathcal{G}^{(2)}$ and any $v_g \in T_g\mathcal{G}$, $w_h \in T_h\mathcal{G}$ such that (v_g, w_h) is tangent to $\mathcal{G}^{(2)}$ at (g,h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}(Jv_g, Jw_h) = J((dm)_{g,h}(v_g, w_h)).$$

With these, the various structures that we have discussed have a groupoid-version by requiring multiplicativity. For instance, a **symplectic groupoid** is a Lie groupoid endowed with a symplectic form which is multiplicative. Also, a **symplectic+complex groupoid** is a Lie groupoid $\mathcal{G}$ together with a symplectic+complex structure (ω, J) such that ω and J are multiplicative.

3.2 Taking care of (C1)

Important for us here is the relation of Poisson geometry with algebroids and symplectic groupoids that we shortly recall. To a Poisson bivector π on M, one associates a Lie algebroid structure on T^*M: the anchor is $\pi^\sharp$, while the bracket is precisely the bracket $[\cdot,\cdot]_\pi$ defined in (2.2). Note that this bracket is uniquely determined by the condition that $[df, dg] = d\{f,g\}$ and the Leibniz identity.

Now, (C1) says that $\pi^\sharp$ preserves the bracket, and this is easily seen to be equivalent to the fact that π is Poisson (actually, this is also equivalent to the fact that $[\cdot,\cdot]_\pi$ satisfies the Jacobi identity). One says that (M,π) is an **integrable Poisson manifold** (or that π is integrable) if the induced algebroid is, and in this case we denote by $\Sigma(M) = \Sigma(M,\pi)$ the associated source-simply connected Lie groupoid. Also, we say that a generalized complex structure is integrable if the underlying Poisson manifold is. We will be interested in the integrable case only.

What is also important (but maybe too obvious to be noticed) is that the identity map Id_{T^*M} is an IM form on T^*M, hence, under the integrability assumption, it induces a multiplicative closed 2-form on $\Sigma(M)$ (cf. Theorem 3.1), that we denote by ω. This is easily seen to be non-degenerate, hence $(\Sigma(M), \omega)$ is a symplectic groupoid. Conversely, if Σ is a symplectic groupoid over M, then there is an induced Poisson structure π on M uniquely determined by the condition that the source-map is Poisson; moreover, the Lie algebroid structure on T^*M induced by π is integrable (integrated by Σ). These constructions describe the well-known correspondence (see [7] and the references therein):

Theorem 3.2 *There is a natural 1-1 correspondence between:*

(i) integrable Poisson structures on M (i.e. π's satisfying (C1), integrable).

(ii) symplectic groupoids (Σ, ω) over M.

3.3 Taking care of (C2)

Theorem 3.3 *Let π be an integrable Poisson structure on M, and let (Σ, ω) be the associated symplectic groupoid over M. Then there is a natural 1-1 correspondence between* *(i)* $(1,1)$*-tensors* a *on* M *satisfying (C2).*

(ii) multiplicative $(1,1)$*-tensors* J *on* Σ *with the property that* (J,ω) *is a Hitchin pair.*

In particular, there is a 1-1 correspondence between pairs (π, a) *on* M *satisfying (C1) and (C2), with* π *integrable, and groupoids* Σ *over* M *together with a Hitchin pair* (ω, J) *on* Σ *with* ω *and* J *multiplicative.*

PROOF: With the bracket $[\cdot,\cdot]_\pi$ and the IM conditions (equations (3.2), (3.3)) in mind, the meaning of (C2) is now clear: (C2) is equivalent to the fact that a^* is an IM form on the algebroid T^*M associated to the Poisson structure π. Hence we obtain a 1-1 correspondence with closed multiplicative forms on Σ, i.e., cf. Lemma 2.4 (see also the comment before Proposition 2.6), with $(1,1)$-tensors J on Σ with the property that (ω, J) is a twisted s.c. structure and ω_J is multiplicative. Finally, remark that Lemma 2.4 applied to symplectic groupoids (Σ, ω) has a bonus: ω_J is multiplicative if and only if J is. $\square$

3.4 Taking care of (C3)

We have seen that (C1) and (C2) alone insure us that Σ has an induced Hitchin pair (ω, J). Note that the twist (i.e. the 2-form $-(\omega + J^*\omega)$) is a multiplicative 2-form, but it may be non-zero. We will see below that what (C3) forces is that the twist vanishes cohomologically. This can be interpreted also using generalized holomorphic maps. To motivate the statement recall that one of the main properties of symplectic groupoids Σ over M is that the target map t is Poisson while s is anti-Poisson. Equivalently, $(t,s): \Sigma \longrightarrow M \times \overline{M}$ is a Poisson map, where $\overline{M}$ is M together with the opposite Poisson tensor $-\pi$. There are various other properties of groupoids that can be expressed in terms of the map (t,s). For instance, a groupoid Σ is proper if (t,s) is a proper map, or the presymplectic groupoids of [2] have the main property that (t,s) defines a Dirac realization of the base manifold. In our context, if $M = (M, \mathcal{J})$ is a generalized complex manifold, then $\overline{M}$ is the opposite generalized complex manifold, i.e. M together with $\overline{\mathcal{J}}$) (see (2.10)).

Theorem 3.4 *Assume that* (π, a) *satisfy (C1), (C2), with* π *integrable, and let* (Σ, ω, J) *be the induced symplectic groupoid over* M *together with the induced multiplicative* $(1,1)$*-tensor. Then, for a 2-form* σ *on* M*, the following are equivalent:*

(i) (C3) is satisfied.

(ii) $\omega + J^*\omega = t^*\sigma - s^*\sigma$*.*

(iii) $(t,s): \Sigma \longrightarrow M \times \overline{M}$ *is generalized holomorphic.*

PROOF: We first prove the equivalence between (i) and (ii). We consider the 2-form

$$\phi = \tilde{\sigma} - t^*\sigma + s^*\sigma,$$

where $\tilde{\sigma} = -\omega - J^*\omega$ is the twist of (ω, J) ($J = J_a$), and let $A = Ker(ds)|_M$. We already know from Theorem 3.1 that a closed multiplicative 2-form θ on Σ is zero if and only if $u_\theta = 0$, i.e.

$$\theta(X_x, \alpha_x) = 0,$$

for all $X_x \in T_xM$, $\alpha_x \in A_x$. The proof of this fact is actually just a very simple step in the proof of the theorem, and it appears as Corollary 3.4 in [2]. We remark that exactly the same argument applies to higher degree forms. In particular, a 3-form θ is zero if and only if

$$\theta_x(X_x, Y_x, \alpha_x) = 0 \tag{3.4}$$

for all $X_x, Y_x \in T_xM$, $\alpha_x \in T_x\Sigma$. We can apply this to $d\phi$ (so that we can check when ϕ is closed) and then the original criteria to the 2-form ϕ. Recalling also that $i_{X\wedge Y}(d\tilde{\sigma}) = i_{\mathcal{N}_J(X,Y)}(\omega)$ (cf. the previous section on the non-degenerate case), we deduce that ϕ vanishes if and only if:

$$\omega(\mathcal{N}_J(X_x, Y_x), \alpha_x) = (d\sigma)(X_x, Y_x, \rho(\alpha_x), \tag{3.5}$$

$$-\omega(X_x, \alpha_x) - \omega(J(X_x), J(\alpha_x)) = \sigma(X_x, \rho(\alpha_x)), \tag{3.6}$$

for all $X_x, Y_x \in T_xM$, $\alpha_x \in A_x$. To make this more explicit, we need some remarks. First of all, after identifying A with T^*M we know (from the definition of ω and ω_J) that:

$$\omega(\alpha_x, X_x) = \alpha_x(X_x), \omega_J(\alpha_x, X_x) = \alpha(a(X)).$$

Note that the second equation shows that J restricted to T_xM equals to a, while restricted to $A_x = T^*_xM$ it equals to a^*. In particular, J preserves the space of vector fields on Σ tangent to M; since this space is closed under the Lie bracket, we deduce that $\mathcal{N}_J(X_x, Y_x)$ is tangent to M, and equals to $\mathcal{N}_J(X_x, Y_x)$, for all $X_x, Y_x \in T_xM$. With these, we see that (3.5) and (3.6) say that, for all $X_x \in T_xM$ and $\alpha_x \in T^*_xM$,

$$-\alpha_x(\mathcal{N}_a(X_x, Y_x)) = i_{X_x\wedge Y_x}(d\sigma)(\pi^\sharp(\alpha_x))(= -\alpha_x(\pi^\sharp i_{X_x\wedge Y_x}(d\sigma)))$$
$$\alpha_x(X_x + a^2X_x) = -\alpha_x(\pi^\sharp\sigma_\sharp(X_x)),$$

and this is precisely (C3).

We now prove the equivalence of (ii) and (iii). Clearly, (ii) says that (t, s) is compatible with the 2-forms. Also, (t, s) is compatible with the bivectors since Σ is a symplectic groupoid. We claim that also the compatibility of (s, t) with the $(1, 1)$-tensors is independent of (C3) (i.e. it follows from the previous conditions (C1) and (C2)). Explicitly, this compatibility translates into the equations

$$(dt) \circ J = a \circ (dt), \quad (ds) \circ J = -a \circ (ds).$$

We prove the first one (the second one being proven similarly, or deduced from the first one using $s = t \circ i$, $i(J(V)) = -J(V)$). The main remark is that, for any multiplicative 2-form ω on any groupoid Σ,

$$\omega(\alpha, V) = \omega(\alpha, dt(V)) = \langle u_\omega(\alpha), dt(V)\rangle,$$

for all $\alpha \in \Gamma(A)$ and all $V \in \mathcal{X}(\Sigma)$ (where α is extended to a vector field on Σ using right translations). This is one of the first consequences of the multiplicativity of ω (see equation (3.4) in [2]). We apply this equation to our symplectic form ω (for which $u_\omega = Id$) and to the multiplicative form ω_J (for which $u_{\omega_J} = a^*$):

$$\langle \alpha, adt(V)\rangle = \langle a^*\alpha, dt(V)\rangle = \omega_J(\alpha, V) = \omega(\alpha, J(V)) = \langle \alpha, dt(J(V))\rangle,$$

for all $\alpha \in T^*M$, hence the desired equation. $\square$

Definition 3.5 *A **Hitchin groupoid** is a Lie groupoid Σ together with a Hitchin pair (ω, J) on Σ and a 2-form σ on the the base such that $\omega + J^*\omega = t^*\sigma - s^*\sigma$.*

With these, we conclude that there is a 1-1 correspondence between triples (π, a, σ) satisfying (C1), (C2) and (C3), with π integrable, and Hitchin groupoids (Σ, ω, J) over M. Note that an immediate consequence of the definition is that the isotropy groups $\Sigma_x = s^{-1}(x) \cap t^{-1}(x)$ are complex Lie groups with the complex structure coming from the restriction of J (see also Corollary 2.12).

3.5 A few more remarks

As we have seen throughout this paper, the Poisson and Dirac geometry underlying generalized complex structures can serve as a guide for various constructions and results in generalized complex geometry. We end with a few remarks along this line.

Remark 3.6 As it follows from the Poisson case, given a generalized complex manifold M, the associated Hitchin groupoid Σ acts on any Hitchin realization $\mu : S \longrightarrow M$ with S compact (see e.g. [7, 2]). The action $\Sigma \times_M S \longrightarrow S$ satisfies certain compatibility relations with the generalized complex structure (of course, one of them ensures that the action is Hamiltonian). In particular, the complex Lie group Σ_x acts on the fiber $\mu^{-1}(x)$, and the reduced space $\mu^{-1}(x)/\Sigma_x$ is precisely the reduced space $\mathcal{C}/\mathfrak{g}_x$ discussed at the end of section 2.3.

Remark 3.7 One can talk about dual pairs and Morita equivalence in the setting of generalized complex structures. Given two generalized complex manifolds M_1 and M_2, a Morita equivalence will be given by a manifold M together with a Hitchin structure on it, and two submersions $\pi_i : M \longrightarrow M_i$ $(i \in \{1, 2\})$ so that M defines a Morita equivalence between the underlying Poisson manifolds and $(\pi_1, \pi_2) : M \longrightarrow M_1 \times \overline{M}_2$ is generalized holomorphic. With these, the Hitchin groupoid of a generalized complex manifold $(M, \mathcal{J})$ defines a self Morita equivalence.

Remark 3.8 Recall [8] that, given a generalized complex structures $\mathcal{J}$ on M, and a closed 2-form B, the gauge transformation of $\mathcal{J}$ with respect to B is the new generalized complex structure $\mathcal{J}_B$ given by the matrix

$$\mathcal{J}_B = \begin{pmatrix} 1 & 0 \\ -B & 0 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ B & 0 \end{pmatrix} \tag{3.7}$$

One says that $\mathcal{J}$ and $\mathcal{J}'$ are gauge equivalent if $\mathcal{J}' = \mathcal{J}_B$ for some closed 2-form B. As in [3] (see Theorem 5.1 there), if $(M, \mathcal{J})$ is integrable and (Σ, ω, J) is the associated Hitchin groupoid, then one can recover the Hitchin groupoid of $(M, \mathcal{J}_B)$ and conclude that $(M, \mathcal{J})$ and $(M, \mathcal{J}_B)$ are Morita equivalent. More precisely, one defines the gauge transformation of J as the new $(1, 1)$-tensor on Σ given by

$$J_B = J + \omega^{-1}(s^*B - t^*B),$$

and then (Σ, ω, J_B) is the Hitchin groupoid of $(M, \mathcal{J}_B)$. Moreover, $(\Sigma, \omega, J - \omega^{-1}t^*(B))$ defines a Morita equivalence between $(M, \mathcal{J})$ and $(M, \mathcal{J}_B)$.

Remark 3.9 Some of the remarks above also follow from the corresponding results for Dirac structures and the fact that, if (Σ, ω, J) is the Hitchin groupoid of a generalized complex structure $\mathcal{J}$, then (Σ, ω_J) is the presymplectic groupoid [2] of the Dirac structure (2.17) underlying $\mathcal{J}$.

References

[1] Abouzaid, M., Boyarchenko, M.: *Local structure of generalized complex manifolds*, work in progress

[2] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C.: *Integration of twisted Dirac brackets.* Duke Math. J. **123** (2004), 549-607.

[3] Bursztyn, H., Radko, O. : *Gauge equivalence of Dirac structures and symplectic groupoids.* Ann. Inst. Fourier (Grenoble), **53** (2003), 309-337.

[4] Cattaneo, A., Felder, G.: *Poisson sigma models and symplectic groupoids.* In: *Quantization of singular symplectic quotients*, Progr. Math. 198, 61–93, Birkhäuser, Basel, 2001.

[5] T. Courant : *Dirac manifolds.* Trans. Amer. Math. Soc. **319** (1990), 631-661.

[6] M. Crainic and R. Fernandes: *Integrability of Lie brackets.* Ann. of Math. **157** (2003), 575–620.

[7] M. Crainic and R. Fernandes: *Integrability of Poisson brackets.* J. of Differential Geometry **66** (2004) 71-137.

[8] M. Gualtieri: *Generalized complex geometry.* Preprint math.DG/0404451.

[9] N. Hitchin: *Generalized Calabi-Yau manifolds.* Q. J. Math. **54** (2003) 281–308.

[10] U. Lindstrom, R. Minasian, A. Tomasiello and M. Zabzine: *Generalized complex manifolds and supersymmetry.* to appear in Comm.Math.Phys, e-Print Archive: hep-th/0405085.

[11] Z.J. Liu, A. Weinstein and P. Xu: *Manin triples for Lie bialgebroids.* J. Differential Geom. **45** (1997), 547–574.

[12] K. Mackenzie and P. Xu: *Lie bialgebroids and Poisson groupoids.* Duke Math. J. **73** (1994), 415–452.

[13] Zucchini, R. : *A sigma model field theoretic realization of Hitchin's generalized complex geometry.* Preprint hep-th/0409181.

[14] Weinstein, A. : *Algébroides de Lie complexes sur les variétés réelles* . mini-course (and private discussions) at l'Ecole Polytechnique, November 2004.