INTEGRABILITY OF JACOBI STRUCTURES

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Abstract. We discuss the integrability of Jacobi manifolds by contact groupoids, and then look at what the Jacobi point of view brings new into Poisson geometry. In particular, using contact groupoids, we prove a Kostant-type theorem on the prequantization of symplectic groupoids, which answers a question posed by Weinstein and Xu [20]. The methods used are those of Crainic-Fernandes on A-paths and monodromy group(oid)s of algebroids. In particular, most of the results we obtain are valid also in the non-integrable case.

Contents

1. Introduction

The "integrability" in the title refers to the global geometric structures behind infinitesimal data. Examples of "integrations" are given by Lie's third theorem (which integrates finite dimensional Lie algebras), by Palais' work on integrability of infinitesimal Lie algebra actions [18], by Weinstein's symplectic groupoids which integrate Poisson structures [19] (and variations, e.g. Dirac structures [2]) or by the integrability of general "Lie brackets of geometric type", i.e. Lie algebroids [6].

The structures that we want to integrate here are Lichnerowicz's *Jacobi manifolds* [15], known also as Kirillov's "local Lie algebra structures" on $C^\infty(M)$ [14].

Definition 1.1. *A Jacobi manifold (M, Λ, R) is a manifold M together with a bivector field Λ and a vector field R satisfying*

$$[\Lambda, \Lambda] = 2R \wedge \Lambda, \quad [\Lambda, R] = 0, \tag{1}$$

The vector field R is called the Reeb vector field of M.

In the equation above, $[\cdot, \cdot]$ stands for the Schouten-Nijenhuis bracket on multivector fields. Using Kirillov's terminology, $C^\infty(M)$ together with the bracket $\{f, g\} = \Lambda(df, dg) + R(f)g - fR(g)$ is a local Lie algebra (and Λ and R can be recovered from this bracket).

There are three types of "extreme examples":

1. *Contact manifolds:* To give a contact form θ on an $(2n+1)$-dimensional manifold M, i.e. a 1-form with the property that $\theta \wedge (d\theta)^n \neq 0$, is equivalent to

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giving a Jacobi structure with the property that $\Lambda^n \wedge R \neq 0$; the defining formula is $i_\theta(\Lambda) = 0$, $i_R(\theta) = 1$.

2. *Vector fields:* Clearly, vector fields on M can be seen as Jacobi structures with vanishing bivector.

3. *Poisson manifolds:* Also, a Poisson structure on M is the same thing as a Jacobi structure with vanishing Reeb vector field; then Λ is called a Poisson bivector. Note that multiplying a Poisson structure Λ on M by a smooth function f, the new structure $\Lambda_f = f\Lambda$ will no longer be Poisson unless the Hamiltonian vector field X_f (obtained by contracting *df* and Λ) is zero; instead, Λ_f together with X_f always defines a Jacobi structure.

There is yet another connection between Jacobi and Poisson manifolds: to any Jacobi manifold (M, Λ, R) one can associate a Poisson manifold [15].

Definition 1.2. *The poissonization of a Jacobi manifold (M, Λ, R) is the Poisson manifold $M \times \mathbb{R}$ with the bivector:*

$$\tilde{\Lambda} = e^{-s}\left(\Lambda + \frac{\partial}{\partial s} \wedge R\right), \tag{2}$$

where s is the coordinate on $\mathbb{R}$. When M is contact, $M \times \mathbb{R}$ will be called the symplectification of M (since $\tilde{\Lambda}$ is non-degenerate, it defines a symplectic form).

To understand the global picture behind Jacobi structures it is useful to first recall what happens in the Poisson case, when one discovers Weinstein's symplectic groupoids [19]. First of all, given a Poisson manifold P, one has an associated topological groupoid $\Sigma_s(P)$ which shows up as the phase space of the Poisson-sigma model [4], or as the "cotangent monodromy groupoid" of the Poisson manifold [6, 7]. We will call it *the symplectic monodromy groupoid of P*. The terminology, and the subscript "s" come from the fact that, when $\Sigma_s(P)$ is smooth, then it is naturally a *symplectic groupoid* of P, i.e. it comes endowed with a symplectic form which is multiplicative[1] and it induces the Poisson structure on P.

This correspondence between Poisson manifolds and symplectic groupoids, which is "almost one-to-one", is best explained through the infinitesimal version of Lie groupoids, i.e. Lie algebroids. At this point, let us fix some notations and basic definitions (for more details and proper references, please see [6]). For a groupoid Σ over M (hence Σ is the space of arrows and M is the space of objects, also identified with the subspace of Σ consisting of the identity arrows 1_x at points $x \in M$), we denote by α the source map, by β the target map, and by $m(g, h) = gh$ the multiplication (defined when $\alpha(g) = \beta(h)$). Hence $\alpha, \beta : \Sigma \longrightarrow M$, $m : \Sigma_2 \longrightarrow \Sigma$, where Σ_2 is the space of composable arrows. Lie groupoids will have smooth structure maps, α and β will be submersions (so that Σ_2 is a manifold), and, although the base manifold M and the α-fibers $\alpha^{-1}(x)$ are assumed to be Hausdorff, Σ may be a non-Hausdorff manifold (important examples come from bundles of Lie algebras and foliations). Recall also that a Lie algebroid A over M is a vector bundle together with a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$ and a bundle map $\rho : A \longrightarrow TM$ so that the Leibniz identity holds:

$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + \mathcal{L}_{\rho(\xi_1)}(f)\xi_2$$

for all sections ξ_1 and ξ_2 and all functions f; here $\mathcal{L}$ stands for Lie derivatives along vector fields. ρ is called *the anchor* of A. As in the case of Lie groups and Lie algebras, any Lie groupoid has an associated Lie algebroid (obtained by taking the tangent spaces along the α-fibers at each identity element 1_x). However, not all

[1] recall that a form ω on a Lie groupoid Σ is called *multiplicative* if $m^*\omega = pr_1^*\omega + pr_2^*\omega$, where pr_1, pr_2 are the projections, and m is the multiplication, all defined on the space Σ_2 of pairs of composable arrows of Σ.

Lie algebroids arise in this way; one says that A is *integrable* if it comes from a Lie groupoid. Nevertheless, any Lie algebroid A has an associated *monodromy groupoid* $G(A)$ (called also the Weinstein groupoid of A), made out of homotopy classes of paths in the algebroid world, and it is a topological groupoid which is the universal candidate for integrating A [6] (this will be recalled in section 3).

Going back to Poisson manifolds, the cotangent bundle T^*P of any Poisson manifold P is naturally a Lie algebroid: the anchor is the contraction by the Poisson tensor:

$$\Lambda^\sharp : T^*M \longrightarrow TM,\ \Lambda(\omega,\eta) = \eta(\Lambda^\sharp(\omega)), \tag{3}$$

and the bracket is

$$[\omega,\eta]_\Lambda = \mathcal{L}_{\Lambda^\sharp\omega}\eta - \mathcal{L}_{\Lambda^\sharp\eta}\omega - d\Lambda(\omega,\eta). \tag{4}$$

Note that the bracket is uniquely determined by the Leibniz identity and $[df, dg] = d\{f, g\}$. With these, the symplectic monodromy groupoid of P is just

$$\Sigma_s(P) = G(T^*P),$$

the monodromy groupoid associated to T^*P.

For Jacobi manifolds, there are partial results which are similar to those in the Poisson case, and the necessary objects are already known [9, 13].

Definition 1.3. *A contact groupoid over a manifold M is a Lie groupoid Σ over M together with a contact form θ, and a function r on Σ, with the property that θ is r-multiplicative in the sense that*

$$m^*\theta = pr_2^*(e^{-r}) \cdot pr_1^*\theta + pr_2^*\theta. \tag{5}$$

(where, as in the previous footnote, m is the multiplication and pr_i are the projections). The function r is called the Reeb function, or the Reeb cocycle of Σ.

For a discussion on the non-symmetry of the previous equation, and versions which use the point of view of contact hyperplanes instead of contact forms, please see Example 6 in the last section.

As we shall explain, Reeb cocycles come from integrating Reeb vector fields of Jacobi manifolds. The term "cocycle" comes from the fact that the definition above forces the cocycle condition $r(gh) = r(g)+r(h)$, whenever gh is defined. This implies that the base M of a contact groupoid has an induced Jacobi structure [13] (and it will also follow from the next section).

As in the Poisson case, any Jacobi manifold has an associated Lie algebroid [13], hence a monodromy groupoid.

Definition 1.4. *The Lie algebroid of the Jacobi manifold (M, Λ, R) is $T^*M \oplus \mathbb{R}$, with the anchor $\rho : T^*M \oplus \mathbb{R} \longrightarrow TM$ given by*

$$\rho(\omega, \lambda) = \Lambda^\sharp(\omega) + \lambda R,$$

and the bracket

$$[(\omega, 0), (\eta, 0)] = ([\omega, \eta]_\Lambda, 0) - (i_R(\omega \wedge \eta), \Lambda(\omega, \eta))$$
$$[(0, 1), (\omega, 0)] = (L_R(\omega), 0),$$

(where $\Lambda^\sharp$ and $[\cdot,\cdot]_\Lambda$ are given by (3) and (4) above). The associated groupoid

$$\Sigma_c(M) = G(T^*M \oplus \mathbb{R}),$$

is called the contact monodromy groupoid of the Jacobi manifold M. We say that M is integrable as a Jacobi manifold if the associated algebroid $T^*M \oplus \mathbb{R}$ is integrable (or, equivalently, if $\Sigma_c(M)$ is smooth, cf. [6]).

The fact that R can be integrated to define a multiplicative function on $\Sigma_c(M)$ (which will be explained in detail in the main body), together with the local result of Dazord [9], suggests that $\Sigma_c(M)$ is a contact groupoid whenever it is smooth. Our first main result proves that this is indeed the case, and also describes the relation between the integrability of M and of its Poissonization. To describe this relation at the groupoid level, one remarks that any multiplicative function r on a groupoid Σ over M can be used to define an "r-twisted multiplication by $\mathbb{R}$", which is a groupoid $\Sigma \times_r \mathbb{R}$ over $M \times \mathbb{R}$ (cf. Definition 2.3). We then have:

Main theorem 1. *For any Jacobi manifold M,*

(i) *there is an isomorphism of topological groupoids*

$$\Sigma_s(M \times \mathbb{R}) \cong \Sigma_c(M) \times_r \mathbb{R},$$

and M is integrable if and only if the Poisson manifold $M \times \mathbb{R}$ is integrable.

(ii) *M is integrable if and only if $\Sigma_c(M)$ is smooth. Moreover, in this case $\Sigma_c(M)$ has a natural structure of contact groupoid.*

Next, we concentrate on Poisson geometry by viewing Poisson manifolds as Jacobi ones with vanishing Reeb vector fields. In other words, we look at what the "Jacobi point of view" (rather than Jacobi structures) brings new into Poisson geometry. First of all, it shows that a Poisson manifold P comes not only with the symplectic monodromy groupoid $\Sigma_s(P)$, but also with a contact monodromy groupoid $\Sigma_c(P)$. Of course, it is not a surprise that the relation between the two heavily depends on the Poisson geometry of P. However, this relation also provides new insights, e.g. into the geometric prequantization of Poisson manifolds. First of all, we concentrate on describing the relation between the two groupoids. Here we restrict to the case where P is integrable as a Poisson manifold (the general case is treated in Section 4). We have a bundle of groups over P, $\mathcal{P}_\Lambda$, whose fiber at $x \in P$ is the period group of the restriction of Ω to the α-fiber $\alpha^{-1}(x)$, where Ω is the symplectic form of $\Sigma_s(P)$. We also define $\mathcal{G}_\Lambda$ as the quotient of the trivial bundle with fiber $\mathbb{R}$ by $\mathcal{P}_\Lambda$.

Main theorem 2. *For any Poisson manifold P, there is a short exact sequence of topological groupoids*

$$1 \longrightarrow \mathcal{G}_\Lambda \longrightarrow \Sigma_c(P) \longrightarrow \Sigma_s(P) \longrightarrow 1.$$

Moreover, if P is integrable as a Poisson manifold, the following are equivalent

(i) *P is integrable as a Jacobi manifold.*
(ii) *$\mathcal{P}_\Lambda$ is smooth.*
(iii) *$\mathcal{G}_\Lambda$ is smooth.*

Next, we will use the contact groupoid $\Sigma_c(P)$ to clarify the prequantization of the symplectic groupoid $\Sigma_s(P)$. In particular, we will see that any prequantization is a contact groupoid. As a simplified theorem that we can state in this introduction, we mention here:

Main theorem 3. *For any integrable Poisson manifold P, the following are equivalent:*

(i) *$\Sigma_s(P)$ is prequantizable.*
(ii) *$\mathcal{P}_\Lambda \subset P \times \mathbb{Z}$.*

Moreover, if $\Sigma_s(P)$ is Hausdorff, the conditions above are also equivalent to

(iii) *The symplectic form of $\Sigma_s(P)$ is integral.*

In this case, the prequantization $\tilde{\Sigma}$ is Hausdorff.

And, finally, we will describe the connection with the Van Est map. More precisely, the Poisson tensor can be viewed as an algebroid 2-cocycle on T^*M, and it makes sense to ask when it is integrable (i.e. when it comes from a 2-cocycle on the symplectic groupoid, via the Van Est map). We will show (compare with the previous theorem):

Main theorem 4. *Let (P, Λ) be an integrable Poisson manifold, and $\Sigma_s(P)$ is Hausdorff. Then the following are equivalent:*

(i) *Λ is integrable as an algebroid cocycle.*
(ii) *The symplectic form of $\Sigma_s(P)$ is exact.*
(iii) *$\mathcal{P}_\Lambda = 0$.*

Moreover, in this case $\Sigma_c(P)$ is isomorphic to the product $\Sigma_s(P)$ with $\mathbb{R}$, with the multiplication twisted by a cocycle on $\Sigma_s(P)$ integrating Λ.

The paper is organized as follows. In the first section we give more details on the poissonization process, including a groupoid version. Next, there is one section devoted to each of the main theorems, which provides the details on the definitions, more precise statements, and the proofs. In the last section we look at several particular cases and examples.

Late comment: Finally, we would like to mention that, at the time the first version of this paper was written, we were not aware of several related works. Most notably, the non-trivial remark that the integrability of Jacobi structures is intimately related to prequantization already appears in [10]. Also, the idea of viewing the Reeb vector field of a Jacobi manifold as an algebroid cocycle already appears in [12] (where it plays a central role). However, regarding the main results, there is hardly any overlap, and our methods (and point of view) is very different from the existing ones. One of the reasons is due to the fact that we use the A-path approach of [6] (and this has been proven to be more powerful also in the Poisson case [7]).

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2. Poissonization and homogeneity

2.1. The manifold case [15, 11]. A *homogeneous Poisson manifold* is a Poisson manifold $(P, \tilde{\Lambda})$ together with a vector field Z, called the *homogeneous vector field* of P, with the property that

$$\tilde{\Lambda} = -\mathcal{L}_Z(\tilde{\Lambda}).$$

If $\tilde{\Lambda}$ comes from a symplectic form ω, then the equation above becomes $\omega = \mathcal{L}_Z(\omega)$, and one calls (P, ω, Z) a *homogeneous symplectic manifold.* Recall also that the *poissonization* of a Jacobi manifold (M, Λ, R) is the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$, where $\tilde{\Lambda}$ is given by the formula (2) in the introduction. We have (cf. subsection 11 in [15], or the theorem on pp. 443 of [11]):

Proposition 2.1. *The poissonization construction defines a one-to-one correspondence between Jacobi structures on M and homogeneous Poisson structures on $M \times \mathbb{R}$ with homogeneous vector field $\frac{\partial}{\partial s}$.*

Moreover, when restricted to contact manifolds, this induces a one-to-one correspondence between contact structures on M and homogeneous symplectic structures on $M \times \mathbb{R}$ with homogeneous vector field $\frac{\partial}{\partial s}$.

Proof. The proof is based on some simple remarks which are interesting on their own. First, for the inverse of the poissonization construction we remark that, given a homogeneous Poisson structures $\tilde{\Lambda}$ on $M \times \mathbb{R}$ with homogeneous vector field $\frac{\partial}{\partial s}$, one has an induced Jacobi structure on M with

$$\Lambda = (pr_1)_*(e^s\tilde{\Lambda}), \qquad R = e^s\tilde{\Lambda}^\sharp(ds),$$

where $pr_1 : M \times \mathbb{R} \longrightarrow M$ is the projection. Next, when the Jacobi manifold M is actually contact with contact form θ, then its poissonization is actually symplectic, with the symplectic form

$$\omega = d(e^s pr_1^*\theta).$$

And, finally, if $(M \times \mathbb{R}, \omega)$ is symplectic and $\mathcal{L}_{\frac{\partial}{\partial s}}\omega = \omega$, then the induced Jacobi structure on M comes from the contact form $\theta = \iota(\frac{\partial}{\partial s})\omega$. □

2.2. The groupoid case. We now discuss the groupoid version of the proposition above. Corresponding to Poisson manifolds are symplectic groupoids, i.e. Lie groupoids $\Sigma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} P$ equipped with a symplectic form ω on Σ which is multiplicative, i.e. which satisfies

$$m^*\omega = pr_1^*\omega + pr_2^*\omega, \tag{6}$$

where the equation is on the space Σ_2 of pairs of composable elements, pr_j is the projection into the j-th component, and m is the multiplication. Recall also that a multiplicative vector field on Σ consists of a vector field Z on Σ and a vector field Z_0 on the base manifold M, with the property that the flow ϕ^t_Z is a local Lie groupoid morphism over the flow $\phi^t_{Z_0}$. That means that for any arrow g from x to y so that $g' = \phi^t_Z(g)$ is defined, $x' = \phi^t_{Z_0}(x)$ and $y' = \phi^t_{Z_0}(y)$ are defined too and g' is an arrow from x' to y', and the multiplicativity condition $\phi^t_Z(gh) = \phi^t_Z(g)\phi^t_Z(h)$ holds whenever the right hand side is defined. These conditions can be reformulating by saying that $Z : \Sigma \longrightarrow T\Sigma$ is a groupoid morphism with base map $Z_0 : M \longrightarrow TM$, where $T\Sigma$ is with the induced structure of groupoid over TM (for details, see [17]. Since Z_0 can be recovered from Z (push down along the source map), we simply say that Z is a multiplicative vector field.

Finally,

Definition 2.2. *A homogeneous symplectic groupoid (Σ, ω, Z) is a symplectic groupoid together with a multiplicative vector field Z, with the property that $\mathcal{L}_Z(\omega) = \omega$.*

Next, we need the groupoid version of "multiplying with the reals" that appears in the poissonization procedure above.

Definition 2.3. *Given a groupoid Σ (Lie or not), and a multiplicative function r on Σ, we define the groupoid $\Sigma \times_r \mathbb{R} = \Sigma \times \mathbb{R}$ over $M \times \mathbb{R}$, with source α, target β and multiplication given by*

$$\alpha(g,s) = (\alpha(g), s), \beta(g,s) = (\beta(g), s - r(g)), (g_1, s_1)(g_2, s_2) = (g_1g_2, s_2). \tag{7}$$

Actually, it is easy to check that $\Sigma \times_r \mathbb{R}$ is a groupoid if and only if $r(g_1g_2) = r(g_1) + r(g_2)$, i.e. r is multiplicative,

Proposition 2.4. *Let Σ be a Lie groupoid endowed with a smooth multiplicative function r. Then there is a one-to-one correspondence between contact groupoid structures on the Lie groupoid Σ with Reeb function r, and homogeneous symplectic groupoid structures on the Lie groupoid $\Sigma \times_r \mathbb{R}$ with homogeneous vector field $\frac{\partial}{\partial s}$.*

Proof. By the second part of the previous proposition we are left with showing that, if the groupoid Σ comes equipped with the multiplicative function r and a contact form θ, and if ω is the induced symplectic structure on $\Sigma \times \mathbb{R}$, then the multiplicativity of ω (i.e. equation (6)) is equivalent to the r-multiplicativity of θ

(i.e. equation (5)). For this one recalls that $\omega = d(e^s\theta)$, and one remarks that the space of composable pairs of arrows in $\Sigma \times_r \mathbb{R}$ can be identified with $\Sigma_2 \times \mathbb{R}$ by

$$((g, s - r(g')), (g', s)) \mapsto (g, g', s).$$

Taking the component of (6) containing ds, we obtain (5) immediately. The other direction follows by multiplying by e^s and taking derivatives. $\square$

Definition 2.5. *Given a contact groupoid Σ with Reeb function r, the associated symplectic groupoid $\Sigma \times_r \mathbb{R}$ is called the symplectification of Σ.*

2.3. Compatibility. We now point out the compatibility between the correspondences described by the previous two propositions. Recall that, given a symplectic groupoid Σ over P, there is an induced Poisson structure on P, uniquely determined by the property that the source map is a Poisson map[2] (i.e. preserves the Poisson bivector). A similar result holds for contact groupoids and Jacobi manifolds:

Lemma 2.6. *Given a contact groupoid Σ over M, there exists an unique Jacobi structure with the property that the source map $\alpha : \Sigma \longrightarrow M$ is a Jacobi map. In this case, we call Σ a contact groupoid of the Jacobi manifold M.*

The proof of this lemma will also show the following which, although straightforward, proves that the correspondence of Proposition 2.4 implies that of Proposition 2.1.

Proposition 2.7. *If (Σ, θ, r) is a contact groupoid over M, and $(\Sigma \times_r \mathbb{R}, \omega)$ is the associated homogeneous symplectic groupoid, then the poissonization of the Jacobi structure induced on M (by the contact groupoid Σ) coincides with the Poisson structure induced on $M \times \mathbb{R}$ (by the symplectic groupoid $\Sigma \times_r \mathbb{R}$).*

Proof. (of Lemma 2.6 and of Proposition 2.7) The uniqueness in the lemma is clear since α is a submersion. We prove the rest. Remark that, in general, a Poisson tensor $\tilde{\Lambda}$ is homogeneous with respect to a vector field Z if and only if ϕ_Z^t maps $\tilde{\Lambda}$ into $e^t\tilde{\Lambda}$. On the other hand, if Γ is a homogeneous symplectic groupoid over P with homogeneous vector field Z, we know that the flow of Z is a (local) groupoid homomorphism over the flow of Z_0. We deduce that the induced Poisson structure on P is a homogeneous one, with homogeneous vector field Z_0. Now, given a contact groupoid Σ over M, we form the homogeneous symplectic groupoid $\Sigma \times_r \mathbb{R}$, and it follows that the induced Poisson structure on $M \times \mathbb{R}$ is homogeneous with vector field $\frac{d}{ds}$. Hence it comes from a Jacobi structure on M. One still needs to remark that, by the correspondence of Proposition 2.1, Jacobi maps correspond to Poisson maps (and we apply this to the source map). $\square$

3. Symplectic and contact monodromy groupoids

3.1. The main theorem of the section. In this section we investigate the effect that the poissonization process has on the monodromy groupoids. We first recall the construction of the monodromy groupoid $G(A)$ associated to a Lie algebroid A. As mentioned in the introduction, when applied to the algebroid T^*P of a Poisson manifold P and to the algebroid $T^*M \oplus \mathbb{R}$ of a Jacobi manifold M, one defines

$$\Sigma_s(P) = G(T^*P), \quad \Sigma_c(M) = G(T^*M \oplus \mathbb{R}),$$

called the symplectic monodromy groupoid of P, and the contact monodromy groupoid of M, respectively. The purpose of this section will then be to prove the following stronger version of the Main Theorem 1 stated in the introduction.

[2]This construction actually gives one-to-one correspondences between symplectic groupoids over P and integrable Poisson structures on P.

Theorem 1. *Let M be a Jacobi manifold with Reeb vector field R, let $\Sigma_c(M)$ be the contact monodromy groupoid of M and let $\Sigma_s(M\times\mathbb{R})$ be the symplectic monodromy groupoid of the Poissonization of M. Then:*

(i) *By integration, the Reeb vector field R induces a multiplicative function r on $\Sigma_c(M)$.*

(ii) *One has an isomorphism of topological groupoids*

$$\Sigma_s(M\times\mathbb{R})\cong\Sigma_c(M)\times_r\mathbb{R}, \tag{8}$$

(where r is the one from (i), and "$\times_r$" was introduced in Definition 2.3).

(iii) *M is integrable as a Jacobi manifold if and only if $M\times\mathbb{R}$ is integrable as a Poisson manifold.*

(iv) *In the integrable case, $\Sigma_c(M)$ is the source-simply connected contact groupoid of M with Reeb function r, and its symplectification (cf. Definition 2.5) is isomorphic to the symplectic groupoid $\Sigma_s(M\times\mathbb{R})$ of Poisson manifold $M\times\mathbb{R}$.*

3.2. Monodromy groupoids. We now recall from [6] the construction of the monodromy groupoid $G(A)$ associated to a Lie algebroid A.

Definition 3.1. *Given a Lie algebroid $A\xrightarrow{\pi}M$ with anchor $\rho:A\longrightarrow TM$, an* A-path *of A is a C^1 map a: $I=[0,1]\longrightarrow A$ with the property that*

$$\rho(a(t))=\frac{d\gamma}{dt}(t),$$

where $\gamma(t)=\pi\circ a(t)$ is called the base path of a. We denote by $P_a(A)$ the set of all A-paths of A.

The choice of the order of smoothness is not very important, and we choose it finite in order to work with Banach manifolds and not with Frechet ones. In particular, the space $P(A)$ of all paths in A will be a Banach manifold modelled by the Banach space $P(\mathbb{R}^n)=C^1(I,\mathbb{R}^n)$, and $P_a(A)$ will be a Banach submanifold. For more details, see [6].

Definition 3.2. [6] *Let ∇ be a connection on A with torsion T_∇ defined as*

$$T_\nabla(\alpha,\beta)=\nabla_{\rho(\beta)}\alpha-\nabla_{\rho(\alpha)}\beta+[\alpha,\beta],$$

and let ∂_t be the induced derivative operator (which associates to a path $a=a(t)$ in A, the path in A $\partial_t a$ which is the ∇-horizontal component of $\frac{da}{dt}$). An A-homotopy is a family $a_\epsilon(t)=a(\epsilon,t)$ of A-paths of class C^2 in ϵ with the property that their base paths $\gamma_\epsilon(t)=\gamma(\epsilon,t)$ have fixed end points, and the solution $b=b(\epsilon,t)$ of the equation

$$\partial_t b-\partial_\epsilon a=T_\nabla(a,b),\qquad b(\epsilon,0)=0 \tag{9}$$

satisfies $b(\epsilon,1)=0$ for all ϵ. We say that two A-paths a_0 and a_1 are homotopic, and write $a_0\sim a_1$, if there exists an A-homotopy $a(\epsilon,t)$ with the property that $a_i(t)=a(i,t)$, $i=0,1$.

Recall [6] that the solution b of the previous equation (hence also the homotopy relation) does not depend on the choice of the connection ∇. Intuitively, A-homotopies are "algebroid homotopies with fixed end-points", and the equation (9) above is just the algebroid version of the equation (in $\mathbb{R}^n$) $\frac{d}{dt}\frac{d}{d\epsilon}=\frac{d}{d\epsilon}\frac{d}{dt}$ which can be used to compute $b=\frac{d\gamma}{d\epsilon}$ from $a=\frac{d\gamma}{dt}$.

Finally, the monodromy groupoid of A is defined as

$$G(A):=\big(P_a(A)/\sim\big)\overset{\alpha}{\underset{\beta}{\rightrightarrows}}M.$$

The source and target maps α and β are given by

$$\alpha([a])=\gamma(0),\qquad\beta([a])=\gamma(1),$$

where γ is the base path of a. The multiplication is basically the concatenation of paths:

$$a \odot b(t) \equiv \begin{cases} 2b(2t), & 0 \leq t \leq \frac{1}{2} \\ 2a(2t-1), & \frac{1}{2} < t \leq 1 \end{cases}$$

Strictly speaking, this forces us to consider the pathwise smooth A-paths. Instead, since reparametrization does not affect the homotopy class [6], we can first reparametrize A-paths by a cut-off function $\tau \in C^{\infty}(I, I)$ (the reparametrization of a is $a^{\tau}(t) = \tau'(t)a(\tau(t))$, and then define $[a] \cdot [b] = [a^{\tau} \odot b^{\tau}]$.

Since $G(A)$ is the quotient of the Banach manifold $P_a(A)$, it follows that $G(A)$ is a topological groupoid, and we can talk unambiguously about its smoothness: we are only interested on smooth structures for which the projection from $P_a(A)$ onto $G(A)$ is a submersion, and there is at most one such structure. It then follows that A is integrable if and only if $G(A)$ is smooth, in which case $G(A)$ will be the unique Lie groupoid integrating A which has simply connected α-fibers [6].

3.3. Passing to 1-cocycles. To prove (i), (ii) and (iii) of Theorem 1, it is useful to concentrate on the algebroid

$$A = T^*M \oplus \mathbb{R},$$

to identify the Reeb vector field with the section $R \in \Gamma(A^*)$ which vanishes in the $\mathbb{R}$-direction:

$$R(\omega, \lambda) = \omega(R),$$

and to remark that R becomes an algebroid 1-cocycle, i.e.

$$R([\alpha, \beta]) = \mathcal{L}_{\rho\alpha}(R(\beta)) - \mathcal{L}_{\rho\beta}(R(\alpha))$$

for all $\alpha, \beta \in \Gamma(A)$.

Definition 3.3. *Given a Lie algebroid A and a 1-cocycle $R \in \Gamma(A^*)$, define $A \times_R \mathbb{R}$ as the algebroid over $M \times \mathbb{R}$, which, as a vector bundle, is the pull-back of A to $M \times \mathbb{R}$ along the projection, has the anchor $\rho_R : A \times_R \mathbb{R} \longrightarrow T^*(M \times \mathbb{R})$,*

$$\rho_R(\alpha) = \rho(\alpha) - R(\alpha)\frac{\partial}{\partial s},$$

and the Lie bracket:

$$[\alpha, \beta]_R = [\alpha, \beta] - R(\alpha)\frac{\partial \beta}{\partial s} + R(\beta)\frac{\partial \alpha}{\partial s},$$

where ρ and $[\cdot, \cdot]$ are anchor and Lie bracket on A.

Via the remark that 1-cocycles $R \in \Gamma(A^*)$ are the same thing as Lie algebroid actions of A on $M \times \mathbb{R}$, the algebroid $A \times_R \mathbb{R}$ is the standard action algebroid (or pull-back algebroid) associated to the action (see e.g. [16, 17]). The point of this definition is that it allows us to include the algebroid $T^*(M \times \mathbb{R})$ into the general picture:

Lemma 3.4. *For the Lie algebroid $A = T^*M \oplus \mathbb{R}$, and the Reeb vector field R viewed as a 1-cocycle of A, one has an isomorphism of algebroids*

$$T^*(M \times \mathbb{R}) \cong A \times_R \mathbb{R}.$$

Proof. One uses the bundle isomorphism $\psi(v, t) = (e^{-t}v, t)$. □

We now have the following general result:

Proposition 3.5. *Let A be a Lie algebroid, and let $R \in \Gamma(A^*)$ be an 1-cocycle. Then*

(i) *The integral*

$$r(a) = \int_a R := \int_0^1 \langle R(\gamma(t)), a(t)\rangle dt$$

only depends on the homotopy class of the A-path a, and induces a multiplicative function r on $G(A)$.

(ii) *There is an isomorphism of topological groupoids*

$$G(A \times_R \mathbb{R}) \cong G(A) \times_r \mathbb{R}.$$

(iii) *A is integrable if and only if $A \times_R \mathbb{R}$ is. In this case, the previous isomorphism is a Lie groupoid isomorphism.*

Proof. That $\int_a R$ only depends on the homotopy class of a has been proven for Lie algebroids coming from Poisson manifolds in [7], and exactly the same proof applies in general. That r is multiplicative is clear from the additivity of integration. Next, it is easy to see that an A-path of $A \times_R \mathbb{R}$ is the same thing as an A-path a of A, together with a path γ_0 in $\mathbb{R}$, satisfying

$$\frac{d\gamma_0}{dt} = -R(a(t)). \tag{10}$$

In turn, this determines γ_0 from the initial point $s = \gamma_0(0)$. This defines a bijection $P_a(A \times_R \mathbb{R}) \cong P_a(A) \times \mathbb{R}$ ((a, γ_0) corresponds to (a, s)), which is clearly smooth. Moreover, choosing a connection ∇ on A and the pull-back $\tilde{\nabla}$ on $A \times_R \mathbb{R}$ in order to write the homotopy equations (9) for the two algebroids, it is straightforward to see that this correspondence preserves the homotopy, hence it descends to the isomorphism of topological spaces:

$$G(A \times_R \mathbb{R}) \cong G(A) \times \mathbb{R}.$$

It is straightforward to identify the groupoid structure on the right hand side with $G(A) \times_r \mathbb{R}$. For instance, the source and the target of $([a], s)$ will be $(\gamma_1(0), \gamma_0(0)) = (\alpha([a]), s)$, and $(\gamma_1(1), \gamma_0(1)) = (\beta([a]), s - r(a))$, respectively, where we have integrated (10) to compute $\gamma_0(1)$.

For (iii) we use again ∇. We can talk about geodesic A-paths, and define the exponential map

$$exp_{\nabla} : A \longrightarrow G(A)$$

which associates to $v \in A$ the homotopy class of the geodesic A-path that starts at v. As in the classical case, exp_{∇} is defined only for small enough v's, but we make an abuse of notation and write as if it was defined globally. The point is that A is integrable if and only if, locally around the zero points, exp_{∇} is injective. This has been explained in [6], to which we refer also for more details on the exponential map. Then, for the first part of (iii), it suffices to remark that, after the identification $G(A \times_R \mathbb{R}) \cong G(A) \times_r \mathbb{R}$, $exp_{\tilde{\nabla}}$ is identified with $exp_{\nabla} \times id$. Also, since the smooth structure on $G(A)$ is constructed with the help of the exponential map (and that is why we need it to be injective), the last part of (iii) follows. □

3.4. Proof of Theorem 1. By Lemma 3.4, (i), (ii) and (iii) in Theorem 1 follow from Proposition 3.5. We are now left with part (iv). Since $\Sigma_s(M \times \mathbb{R})$ is a symplectic groupoid, the isomorphism (8) makes $\Sigma_c(M) \times_r \mathbb{R}$ into a symplectic groupoid with symplectic form denoted by ω. Using Proposition 2.4, it suffices to show that $\frac{d}{ds}$ is multiplicative and $(\Sigma_c(M) \times_r \mathbb{R}, \omega)$ is homogeneous with respect to the vector field $\frac{d}{ds}$. We only have to show that $\mathcal{L}_{\frac{d}{ds}}\omega = \omega$. We will make some general remarks. First of all, if (P, Λ) is a Poisson manifold, and λ is a non-zero real, it is immediate from the construction of the symplectic form ω_Λ on $\Sigma_s(P,\Lambda)$ (see [7]) that

$$\Sigma_s(P,\lambda\Lambda) = \Sigma_s(P,\Lambda), \quad \omega_{\lambda\Lambda} = \lambda^{-1}\omega_\Lambda.$$

Next, if $\phi : (M_1,\Lambda_1) \longrightarrow (M_2,\Lambda_2)$ is an isomorphism of two Poisson manifolds, then it induces an isomorphism of algebroids $\phi_* : T^*M_1 \longrightarrow T^*M_2$ which, on each fiber, is given by the inverse $(d\phi)^{-1}$ of the differential of ϕ. In the integrable case, it induces a map $\phi_* : \Sigma_s(M_1) \longrightarrow \Sigma_s(M_2)$ of symplectic groupoids. Applying this to $\phi_u : M\times\mathbb{R} \longrightarrow M\times\mathbb{R}$, $\phi_u(x,s) = (x,s+u)$, $\Lambda_1 = \tilde{\Lambda}$, $\Lambda_2 = e^{-u}\tilde{\Lambda}$, and using also the previous remark, we obtain

$$(\phi_u)_* : (\Sigma_s(M\times\mathbb{R}),\omega) \longrightarrow (\Sigma_s(M\times\mathbb{R}), e^u\omega).$$

After the identification from (iii), we see that $(\phi_u)_*(g,s) = (g,s+u)$ is the flow of $\frac{d}{ds}$. Hence $\frac{d}{ds}$ is multiplicative. Taking derivatives in $(\phi_u)_*(\omega) = e^u\omega$ (with respect to u, at the origin), we obtain the desired equation $\mathcal{L}_{\frac{d}{ds}}\omega = \omega$.

Remark 1. Let us go back to Theorem 1 and conclude with the explicit formulas. First of all, the Reeb function on $\Sigma_c(M)$ is given by

$$r([a]) = \int_a R := \int_0^1 \langle R(\gamma(t)), a_1(t)\rangle dt,$$

for any A-path $a = (a_1,a_0)$ of $T^*M\oplus\mathbb{R}$ with base path γ. Also, the isomorphism in (iii) comes from a diffeomorphism at the level of A-paths,

$$\begin{array}{ccc} P_a(T^*(M\times\mathbb{R})) & \longrightarrow & P_a(T^*M\oplus_M\mathbb{R})\times\mathbb{R} \\ \tilde{a}_1+\tilde{a}_0ds & \leftrightarrow & ([a_1,a_0],s). \end{array} \tag{11}$$

On the left hand side, $\tilde{a} = \tilde{a}_1+\tilde{a}_0ds$ is an A-path of $T^*(M\times\mathbb{R})$ over the base path $\gamma = (\gamma_0,\gamma_1)$ in $M\times\mathbb{R}$, while on the right hand side we have a pair consisting of an A-path $a = (a_1,a_0)$ of $T^*M\oplus_M\mathbb{R}$ over the base path γ_1 in M, together with a real number s. The explicit formulas are:

$$a_i(t) = e^{-\gamma_0(t)}\tilde{a}_i(t), \quad (i=0,1), \quad s = \gamma_0(0)$$

for computing the right hand side from the left one, while for the other direction:

$$\begin{aligned} \tilde{a}_i(t) &= e^{\gamma_0(t)}a_i(t), \\ \gamma_0(t) &= -\int_0^t i(E)a_1(t)dt + s. \end{aligned}$$

4. The Poisson case I: general discussion

4.1. The main theorem of the section. In this section (as well as in the next two) we look at what the Jacobi point of view brings new into Poisson geometry. In other words, we start with a Poisson manifold (P,Λ), and we view it both as a Poisson manifold, as well as a Jacobi manifold with trivial Reeb vector field. Then (P,Λ) will have two associated groupoids: the symplectic monodromy groupoid $\Sigma_s(P)$ (with the role of integrating the Poisson structure), and the contact monodromy groupoid $\Sigma_c(P)$ (with the role of integrating the Jacobi structure). The aim of this section is to describe the relation between the two. Emphasize here that the smoothness of the one does not imply the smoothness of the other, i.e. P can be integrable as Poisson manifold without being integrable as Jacobi, or the other way around (see the examples).

Given (P,Λ), we will define a bundle of groups $\mathcal{P}_\Lambda$ over P, where each fiber $\mathcal{P}_{\Lambda,x}$ is an additive subgroup of $\mathbb{R}$. When P is integrable as a Poisson manifold, i.e. when

$\Sigma_s(P)$ is symplectic Lie groupoid, with symplectic form denoted by Ω, then $\mathcal{P}_{\Lambda,x}$ can be described as the group of periods of $\Omega|_{\alpha^{-1}(x)}$:

$$Per(\Omega|_{\alpha^{-1}(x)}) = \{\int_g \Omega : [g] \in \pi_2(\alpha^{-1}(x))\}. \tag{12}$$

It is remarkable that these groups can be defined without any integrability assumption. We also consider the quotient $\mathcal{G}_\Lambda$ of $P \times \mathbb{R}$ by $\mathcal{P}_\Lambda$, i.e. the bundle of groups over P whose fiber above $x \in P$ is $\mathbb{R}/\mathcal{P}_{\Lambda,x}$. The main result to be discussed in this section is the following improvement of Main Theorem 2 stated in the introduction.

Theorem 2. *For any Poisson manifold (P, Λ), there is a short exact sequence of topological groupoids*

$$1 \longrightarrow \mathcal{G}_\Lambda \longrightarrow \Sigma_c(P) \xrightarrow{\pi} \Sigma_s(P) \longrightarrow 1.$$

If P is integrable as a Poisson manifold, then the following are equivalent:

(i) *P is integrable as a Jacobi manifold.*
(ii) *$\mathcal{P}_\Lambda$ is locally uniformly discrete.*
(iii) *$\mathcal{P}_\Lambda$ is smooth.*
(iv) *$\mathcal{G}_\Lambda$ is smooth.*

Moreover, in this case the symplectic form Ω of $\Sigma_s(P)$ and the contact form θ of $\Sigma_c(P)$ are related by

$$\pi^*\Omega = d\theta,$$

and the Reeb vector field of $\Sigma_c(P)$ is

$$R = \frac{d}{ds},$$

the infinitesimal generator of the action of $\mathcal{G}_\Lambda$ on $\Sigma_c(P)$ (or, equivalently, of the induced action of $\mathbb{R}$ via the projection $\mathbb{R} \longrightarrow \mathcal{G}_\Lambda$).

We mention here that, as for the $G(A)$'s, when talking about the smoothness of $\mathcal{P}_\Lambda$ or $\mathcal{G}_\Lambda$ we refer to the natural smooth structures, i.e. $\mathcal{P}_\Lambda$ as a subspace of $P \times \mathbb{R}$, and $\mathcal{G}_\Lambda$ as a quotient of it. In particular, there is at most one such smooth structure. Also, the condition that $\mathcal{P}_\Lambda$ is locally uniformly discrete means that, for $x \in P$, the distance between the zero element and $\mathcal{P}_{\Lambda,y} - \{0_y\}$ is bounded from below by a positive number, for y in a neighborhood of x.

4.2. Monodromy maps. To define the groups $\mathcal{P}_\Lambda$ in the nonintegrable case (and to proceed with the proof of the theorem), we need to recall the construction of the monodromy map of an algebroid (at a first reading, readers can restrict themselves to the integrable case, and skip the general definition of $\mathcal{P}_\Lambda$).

So, let A be a Lie algebroid over P. For $x \in P$ we denote by $\mathfrak{g}_x(A)$ the kernel of the anchor at x, and we call it the isotropy Lie algebra of A at x. The Lie bracket can be restricted to this kernel, and this shows that $\mathfrak{g}_x(A)$ is indeed a Lie algebra. As for any Lie algebroid, we can form the associated groupoid $G(\mathfrak{g}_x(A))$, which is a group since the base is a point. This is nothing but the unique simply connected Lie group integrating $\mathfrak{g}_x(A)$, viewed as $\mathfrak{g}_x(A)$-homotopy classes of paths $a_1 : I \longrightarrow \mathfrak{g}_x(A)$. Also, the image of ρ defines a singular foliation on P, whose leaves are the orbits of A.

Definition 4.1. *Let A be a Lie algebroid over P, let $x \in P$, and we denote by L_x the orbit through x. The monodromy map at x,*

$$\partial_A : \pi_2(L, x) \longrightarrow G(\mathfrak{g}_x(A)),$$

associates to the homotopy class of $\gamma : I \times I \longrightarrow L$ *(representing an element in* $\pi_2(L,x)$*) the class of a (any)* $\mathfrak{g}_x$*-path* $a_1 : I \longrightarrow \mathfrak{g}_x$ *with the property that there exists an* A*-homotopy* $a(\epsilon,t)$ *(cf. Definition 3.2), sitting over* $\gamma(\epsilon,t)$*, and which connects the zero path (i.e.* $a(0,t)=0$*) to* a_1*.*

The image of ∂_A is called the (extended) monodromy group of A at x, and these groups are central for understanding the integrability of A. In particular, A is integrable if and only if these groups are discrete, locally uniformly with respect to x. For this, and more details (e.g. to see that a above can be chosen, and that $[a_1]$ does not depend on the choice of a, but only on the homotopy class of γ), we refer to [6]. We describe here what happens in the integrable case, when these constructions become more transparent. Then $G(A)$ is smooth, and the isotropy group $G_x(A)$ (i.e. arrows of $G(A)$ that start and end at x) will be a Lie group that integrates the Lie algebra $\mathfrak{g}_x(A)$. It may be different from $G(\mathfrak{g}_x(A))$, and the main reason is that it may fail to be simply connected. Applying the homotopy long exact sequence associated the fibration $\beta : \alpha^{-1}(x) \longrightarrow L$, we get a surjective boundary map

$$\partial : \pi_2(L) \longrightarrow \pi_1(G_x(A)),$$

and this is basically ∂_A after one views $\pi_1(G_x(A))$ as a subgroup of $G(\mathfrak{g}_x(A))$ (which is naturally the case since $G_x(A)$ integrates $\mathfrak{g}_x(A)$).

4.3. The period groups in the general case. When (P,Λ) is a Poisson manifold, we denote by

$$\partial_s : \pi_2(L,x) \longrightarrow G(\mathfrak{g}_x(T^*P)),$$

the monodromy map associated to T^*P, and, similarly, by ∂_c the one associated to $T^*P \oplus \mathbb{R}$ (the algebroid associated to P viewed as a Jacobi manifold). Note that L is the symplectic leaf through x, whose tangent space is spanned by the Hamiltonian vectors $X_f = \Lambda^\sharp(df) = \{f,\cdot\}$, and with the symplectic form

$$\omega_L(X_f, X_g) = \{f,g\}.$$

Definition 4.2. *Given a Poisson manifold* (P,Λ)*,* $x \in P$*, we define the period group of* Λ *at* x*,*

$$\mathcal{P}_{\Lambda,x} = \{\int_\gamma \omega_L : [\gamma] \in \pi_2(L,x) \text{ and } \partial_s\gamma = 1_x\} \subset \mathbb{R},$$

and we define the period bundle $\mathcal{P}_\Lambda$ *of* Λ *whose fiber at* x *is* $\mathcal{P}_{\Lambda,x}$*, and the structural group bundle* $\mathcal{G}_\Lambda$ *of* Λ *whose fiber at* x *is* $\mathbb{R}/\mathcal{P}_{\Lambda,x}$*.*

As promised, in the integrable case we have

Lemma 4.3. *If* P *is integrable as a Poisson manifold, then* $\mathcal{P}_{\Lambda,x}$ *coincides with the group of periods (12) of the restriction of the symplectic form of* $\Sigma_s(P)$ *to the* α*-fiber at* x*.*

Proof. Since T^*P is integrable, we can use the description of ∂_s as the boundary of the homotopy long exact sequence (see the end of subsection 4.2). We deduce that

$$\mathcal{P}_{\Lambda,x} = \{\int_{\beta_*(u)} \omega : u \in \pi_2(\alpha^{-1}(x))\}.$$

On the other hand, since $\beta^*\omega = -\Omega$ on $\alpha^{-1}(x)$ (β is anti-Poisson), we have

$$\int_{\beta_*(u)} \omega = \int_u \beta^*\omega = -\int_u \Omega,$$

and the lemma follows. □

Next, the two monodromy maps ∂_s and ∂_c are related as follows:

Lemma 4.4. *Let P be a Poisson manifold and $x \in P$. Denote by $\mathfrak{g}_x^s$ the isotropy Lie algebra at x of T^*P and by $\mathfrak{g}_x^c$ the isotropy Lie algebra at x of $T^*P \oplus \mathbb{R}$. Then*

$$G(\mathfrak{g}_x^c) \cong G(\mathfrak{g}_x^s) \times \mathbb{R},$$

and, after this identification, the monodromy maps ∂_s and ∂_c of T^*P, and $T^*P \oplus \mathbb{R}$, respectively, are related by:

$$\partial_c \gamma = (\partial_s \gamma, -\int_\gamma \omega_L),$$

for every $[\gamma] \in \pi_2(L, x)$.

Proof. The first part follows from the remark that $\mathfrak{g}_x^c = \mathfrak{g}_x^s \times \mathbb{R}$, which is clear at the level of vector spaces, and, at the level of Lie algebras, it follows immediately from the formulas defining the Lie brackets of T^*P and $T^*P \oplus \mathbb{R}$. Consider now $[\gamma] \in \pi_2(L)$, let $\tilde{a}(\epsilon, t)$ be an A-homotopy over γ connecting the trivial path with $\tilde{a}_1 = (a_1, u_1)$ (so that $\partial_c([\gamma]) = [\tilde{a}_1]$), and let b be the solution of the equation (9). Write $a = (a, u)$ and, similarly, $b = (b, v)$. The first component of the equation (9) tells us that a is a homotopy between the zero path and a_1, hence $\partial_c(\gamma) = (\partial_s(\gamma), [u_1])$. On the other hand, $G(\mathbb{R}) \cong \mathbb{R}$, where the homotopy class of an $\mathbb{R}$-path u_1 is identified (homotopic) to the number $\int_0^1 u_1$. We now look at the $\mathbb{R}$-component of the equation (9), which gives us

$$\partial_t v - \partial_\epsilon u = \Lambda(a, b).$$

Since

$$\Lambda^\sharp(a) = \frac{d\gamma}{dt}, \qquad \Lambda^\sharp(b) = \frac{d\gamma}{d\epsilon},$$

and γ stays entirely in the leaf L, we have

$$\partial_t v - \partial_\epsilon u = \omega_L(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}).$$

So

$$\begin{aligned}\int_0^1 \int_0^1 \omega_L(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}) d\epsilon dt &= \int_0^1 \int_0^1 \partial_t v d\epsilon dt - \int_0^1 \int_0^1 \partial_\epsilon u d\epsilon dt \\ &= \int_0^1 (v(\epsilon, 1) - v(\epsilon, 0)) d\epsilon - \int_0^1 (u(1, t) - u(0, t)) dt \\ &= -\int_0^1 u(1, t) dt,\end{aligned}$$

i.e. $\int_\gamma \omega_L = -\int_0^1 u(1, t)dt$. This proves the lemma. □

4.4. <u>Proof of Theorem 2.</u> We now proceed with the proof of the theorem. The projection $T^*P \oplus \mathbb{R} \longrightarrow T^*P$ is an algebroid morphism, hence it induces a groupoid morphism

$$\pi : \Sigma_c(P) \longrightarrow \Sigma_s(P)$$

which sends an A-path (a, u) of $T^*P \oplus \mathbb{R}$ into the A-path a of T^*P. This is clearly surjective, and we denote by $\mathcal{G}$ its kernel. We will show that $\mathcal{G} = \mathcal{G}_\Lambda$. Recall [6] that, for any algebroid A over P, the isotropy group at x of the monodromy groupoid of A (denoted by $G_x(A)$) has $\pi_0(G_x(A))$ isomorphic to $\pi_1(L)$, and the connected component of the identity in $G_x(A)$ is

$$(G_x(A))^0 = G(\mathfrak{g}_x(A))/Im(\partial_A),$$

independently of the integrability of A. Applying this to our algebroids, we see that

$$\mathcal{G}_x = Ker(\pi : G(\mathfrak{g}_x^c)/Im(\partial_c) \longrightarrow G(\mathfrak{g}_x^s)/Im(\partial_s)),$$

and, using the previous lemma, this is precisely $\mathbb{R}/\mathcal{P}_{\Lambda,x}$. This proves the exact sequence in the theorem. Note that the inclusion of $\mathcal{G}_\Lambda$ into $\Sigma_c(P)$ sends the class of the real number λ into the A-homotopy class of the path $(0, \lambda)$ of $T^*P \oplus \mathbb{R}$.

We now prove the equivalence of (i)-(iv) in the Theorem. We first show that (i) is equivalent to (ii). By the general result of [6], (i) is equivalent to the groups $Im(\partial_{c,x})$ being locally uniformly discrete. This means that, if (x_i) is a sequence in P converging to x, and $[\gamma_i] \in \pi_2(L, x_i)$ satisfies

$$\lim_{n\to+\infty} \text{distance}((\partial_c(\gamma_i), 0)) = 0, \tag{13}$$

then $\partial_c(\gamma_i) = 0$ for i large enough. On the other hand, the similar condition for ∂_s is satisfied since T^*P is integrable. Hence, using Lemma 4.4, the condition becomes: if (x_i) is a sequence in P converging to x, and $[\gamma_i] \in \pi_2(L, x_i)$ satisfies

$$\int_\gamma \omega_{L_{x_i}} \longrightarrow 0,$$

then these integrals must vanish for i large enough. I.e., $\mathcal{P}_\Lambda$ must be locally uniformly discrete. Next, (i) implies (iv) because π will be a submersion and $\mathcal{G}_\Lambda = \pi^{-1}(P)$ (P is embedded in $\Sigma_s(P)$ as the space of identity elements). Similarly, (iv) implies (iii) because $\mathcal{P}_\Lambda$ is $\pi_0^{-1}(P)$ where the projection $\pi_0 : P \times \mathbb{R} \longrightarrow \mathcal{G}_\Lambda$ is a submersion. Assume now (iii), and prove (ii). The condition (iii) implies that $\mathcal{P}_{\Lambda_x}$ is a smooth submanifold of $\mathbb{R}$. But $\mathcal{P}_{\Lambda_x}$ is at most countable, since it is a quotient of the second homotopy group $\pi_2(L)$ of the leaf through x. It follows that $\mathcal{P}_{\Lambda_x}$ is discrete and the projection from $\mathcal{P}_\Lambda$ into P is a local diffeomorphism. This implies (ii).

We are left with proving the last part of the theorem. Using the correspondence we have established in (11) and the formula of the symplectic form in the path space [4], we have

$$\mathcal{L}_{\frac{d}{ds}}\theta = 0, \qquad i(\frac{d}{ds})\theta = 1, \tag{14}$$

i.e. $\frac{d}{ds}$ is the Reeb vector field of θ. From these formulas it also follows that $d\theta$ is a basic form, i.e. $d\theta = \pi^*\omega$ for some 2-form ω on $\Gamma_s(P)$. Since θ is a contact form and it is multiplicative, it follows that ω is a symplectic form on $\Sigma_s(P)$ which is multiplicative. Since the source map of $\Sigma_c(P)$ is Jacobi, it follows that the source map $\alpha : (\Sigma_s(P), \omega) \longrightarrow P$ is Poisson. By uniqueness of the symplectic groupoid integrating the Poisson manifold P, we must have $\omega = \Omega$.

5. The Poisson case II: Application to prequantization

5.1. <u>The main theorem of the section.</u> Recall that a prequantization of a symplectic manifold (S, ω) is a complex line bundle L together with a connection ∇ so that ω represents the first Chern class $c_1(L, \nabla)$. Equivalently, this is the same as a principal S^1-bundle $\pi : \tilde{S} \longrightarrow S$ together with a connection 1-form $\theta \in \Omega^1(\tilde{S})$ with the property that $\pi^*\omega = d\theta$. Kostant's theorem (sometimes also attributed to Kobayashi [1] or to Souriau) says that this is possible if and only if ω is integral. Similarly, Weinstein and Xu have introduced the notion of prequantization of symplectic groupoids, with the aim of quantizing Poisson manifolds "all at once" [20]. More precisely:

Definition 5.1. *One calls prequantization of the symplectic groupoid* (Σ, Ω) *any Lie groupoid extension of* Σ *by the trivial bundle of Lie groups* S^1*,*

$$1 \longrightarrow S^1 \longrightarrow \tilde{\Sigma} \xrightarrow{\pi} \Sigma \longrightarrow 1,$$

(and this makes $\tilde{\Sigma}$ *into a principal* S^1*-bundle over* Σ*), together with a connection 1-form* $\theta \in \Omega^1(\tilde{\Sigma})$ *which is multiplicative and satisfies* $\pi^*\Omega = d\theta$*.*

When saying "the trivial bundle of S^1's", we really mean triviality in the sense of extensions, i.e., besides the triviality as a bundle, the action of Σ on the bundle must be trivial too. That simply means that $gz = zg$ for all $g \in \tilde{\Sigma}$ and $z \in S^1$. In particular, there is no ambiguity when talking about $\tilde{\Sigma}$ as a principal S^1-bundle over Σ. This corresponds to the "no-holonomy above identity elements" condition that appears in [20] (where uniqueness is proven).

In this section we show that, for a Poisson manifold P, $\Sigma_c(P)$ is intimately related to prequantizing the symplectic groupoid $\Sigma_s(P)$. More precisely, we will prove the following result which is an extension of Kostant's theorem to symplectic groupoids, and an improvement of Main Theorem 3 stated in the introduction.

Theorem 3. *Let P be an integrable Poisson manifold, with associated symplectic groupoid $\Sigma_s(P)$. The following are equivalent:*

(i) *$\Sigma_s(P)$ is prequantizable.*

(ii) *$\mathcal{P}_\Lambda \subset P \times \mathbb{Z}$.*

Moreover, if $\Sigma_s(P)$ is Hausdroff, the conditions above are also equivalent to

(iii) *Ω is integral.*

Finally, in the prequantizable case, the prequantization $\tilde{\Sigma}$ together with the connection 1-form becomes a contact groupoid which is a quotient of $(\Sigma_c(P), \theta)$, and $\tilde{\Sigma}$ is Hausdorff if $\Sigma_c(P)$ is.

5.2. <u>Proof of Theorem 3</u> The implications (i)$\Longrightarrow$ (ii)$\Longrightarrow$ (iii) are clear (independent of the Hausdorfness assumption). Assume now (ii). First of all, it follows that $\mathcal{P}_\Lambda$ is uniformly discrete, hence, by Theorem 2, $\Sigma_c(P)$ is a Lie groupoid.

We put $\tilde{\Sigma} := \Sigma_c(P)/\mathbb{Z}$, where the action of $\mathbb{Z}$ is the one induced by the action of $\mathbb{R}$ (see Theorem 2). From the hypothesis and Theorem 2 it also follows that $P \times \mathbb{Z}/\mathcal{P}_\Lambda$ is a smooth (étale) sub-bundle of $\mathcal{G}_\lambda$. Since $\Sigma_c(P)$ is a principal $\mathcal{G}_\Lambda$-bundle over $\Sigma_s(P)$, it follows that $\tilde{\Sigma}$ is smooth, and it is a principal $\mathcal{G}_\Lambda/\mathbb{Z} = \mathbb{R}/\mathbb{Z} = S^1$-bundle over $\Sigma_s(P)$. Denote by $\tilde{\pi} : \tilde{\Sigma} \longrightarrow \Sigma_s(P)$ the projection. This will be a morphism of Lie groupoids, whose kernel is the trivial bundle of groups with fiber S^1.

By (14), θ is $\mathbb{R}$-invariant, so it descends to a 1-form $\tilde{\theta} \in \Omega^1(\tilde{\Sigma})$ such that $\tilde{\pi}^*\tilde{\theta} = \theta$. The same equations (14) imply the similar equations for $\tilde{\theta}$, where the Reeb vector field will be the generator of the action of S^1 on $\tilde{\Sigma}$. This shows that $\tilde{\theta}$ is a connection 1-form on our principal bundle. Moreover, $\pi^*\Omega = d\theta$ implies that $\tilde{\pi}^*\Omega = d\tilde{\theta}$. Hence $(\tilde{\Sigma}, \tilde{\theta})$ is a prequantization of our symplectic groupoid $\Sigma_s(P)$. By construction, it is a quotient of $(\Sigma_c(P), \theta)$ and it inherits the contact groupoid structure from $\Sigma_c(P)$. When $\Sigma_s(P)$ is Hausdorff, so is $\tilde{\Sigma}$ as a S^1 principal bundle over $\Sigma_s(P)$.

6. The Poisson case III: Relation to the Van Est map

6.1. <u>The main theorem of the section.</u> In this section we discuss the Poisson bivector from the point of view of 2-cocycles. The main remark is that the algebroid $T^*P \oplus \mathbb{R}$ (provided by the Jacobi point of view) is made out of the algebroid T^*P (provided by the Poisson point of view) together with an extra-piece of data, namely a 2-cocycle. This will show that most of our results belong to a more general class of results, depending on closed 2-cocycles on algebroids. As a consequence of our discussions, labelled here as the main theorem of the section, we have the following result which gives the precise conditions for when the relation between $\Sigma_c(P)$ and $\Sigma_s(P)$ is the simplest possible. More precisely, we will prove the following:

Theorem 4. *Let (P, Λ) be a Poisson manifold, and assume that the symplectic monodromy groupoid $\Sigma_s(P)$ is smooth and Hausdorff. Denote by Ω the symplectic form on $\Sigma_s(P)$. Then the following are equivalent:*

(i) *Λ is integrable as an algebroid cocycle.*

(ii) *Ω is exact.*
(iii) $\mathcal{P}_\Lambda = 0$.
Moreover, if c integrates Λ, then $\Sigma_c(P) \cong \Sigma_s(P) \ltimes_c \mathbb{R}$.

This section is organized as a discussion around this theorem, which will provide the precise definitions and the proof.

6.2. 2-cocycles and the Van Est map.[3] To see that 2-cocycles are at the heart of $T^*P \oplus \mathbb{R}$ we remark that, if $[\cdot,\cdot]$ and ρ are the Lie bracket and the anchor of T^*P, then the bracket of $T^*P \oplus \mathbb{R}$ is

$$\begin{aligned}[(\omega_1,\omega_0),(\eta_1,\eta_0)] =&([\omega_1,\eta_1],\\ &\mathcal{L}_{\rho(\omega_1)}(\eta_0) - \mathcal{L}_{\rho(\eta_1)}(\omega_0) + \Lambda(\eta_1,\omega_1))\end{aligned}$$

(and the anchor is $(\omega_1,\omega_0) \mapsto \rho(\omega_1)$). More abstractly, given any algebroid A and any section $\Lambda \in \Gamma(\Lambda^2 A^*)$, the previous formula defines a bracket on $A \oplus \mathbb{R}$. On the other hand, the spaces $\Gamma(\Lambda^p A^*)$ together define an "A-De Rham complex", with the differential

$$\begin{aligned} d_A(\Lambda)(X_1,\ldots,X_{p+1}) &= \sum_{i<j}(-1)^{i+j-1}\Lambda([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{p+1})) \\ &+ \sum_{i=1}^{p+1}(-1)^i\mathcal{L}_{X_i}(\Lambda(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})) \qquad (15)\end{aligned}$$

(where $\Lambda \in \Gamma(\Lambda^p A^*)$ is arbitrary). The resulting complex is denoted by $\Omega^*(A)$, and the cohomology is denoted by $H^*(A)$. For instance, when $A = TP$ one recovers the usual De Rham cohomology, and when $A = \mathfrak{g}$ is a Lie algebra one recovers the Lie algebra cohomology.

For $\Lambda \in \Gamma(\Lambda^2 A^*)$, the resulting structure on $A \oplus \mathbb{R}$ is a Lie algebroid structure if and only if Λ is a 2-cocycle, i.e. $d_A\Lambda = 0$. The resulting algebroid is denoted by $A \ltimes_\Lambda \mathbb{R}$. It is not difficult to see that the isomorphism class of $A \ltimes_\Lambda \mathbb{R}$ only depends on the cohomology class of Λ. Coming back to our Poisson manifold, we conclude that

Lemma 6.1. *Given a Poisson manifold (P,Λ), the Poisson tensor is a closed algebroid 2-cocycle for T^*P, and*

$$T^*P \oplus \mathbb{R} \cong T^*P \ltimes_\Lambda \mathbb{R}.$$

We now turn to the global picture, i.e. to cocycles on groupoids. Assume that G is a Lie groupoid over P. A differentiable p-cocycle on G is a smooth function c defined on the space of p-tuples $(g_1,\ldots,g_p)$ of composable elements of G (i.e. such that $g_1 \ldots g_p$ is defined). The differential of c is the $(p+1)$-cocycle

$$(dc)(g_1,\ldots,g_p,g_{p+1}) = c(g_2,\ldots,g_{p+1}) + \qquad (16)$$

$$\sum_{i=1}^{p}(-1)^i c(g_1,\ldots,g_ig_{i+1},\ldots,g_{p+1}) + \qquad (17)$$

$$(-1)^{p+1}c(g_1,\ldots,g_p). \qquad (18)$$

We will only work with normalized cocycles, i.e. with the property that $c(g_1,\ldots,g_p) = 0$ whenever one of the entries is an identity arrow. The resulting complex is denoted by $C^*_{diff}(G)$, and the cohomology is denoted by $H^*_{diff}(G)$.

As in the algebroid case, any (normalized) 2-cocycle c induces a Lie groupoid structure on $G \times \mathbb{R}$, with $(g_1,\lambda_1)(g_2,\lambda_2) = (g_1g_2, \lambda_1+\lambda_2+c(g_1,g_2))$. We denote

[3] here we only give a brief outline on algebroid cocycles, groupoid cocycles, and the Van Est map. More details can be found in [5].

this groupoid by $G \ltimes_c \mathbb{R}$. Again, the isomorphism class of $G \ltimes_c \mathbb{R}$ only depends on the cohomology class of c.

Finally, by differentiation, groupoid cocycles on G induce algebroid cocycles on the Lie algebroid A of G (see [5]). This construction induces a map of complexes $VE : C^*_{diff}(G) \longrightarrow \Omega^*(A)$, hence also a map in cohomology

$$VE : H^*_{diff}(G) \longrightarrow H^*(A),$$

known as the Van Est map. Elements in the image of the Van Est map are called integrable algebroid cocycles (note that this property only depends on the cohomology class of a cocycle).

Definition 6.2. *Given a Poisson manifold (P, Λ), we say that Λ is integrable if the Poisson manifold P is integrable, and Λ comes from a differentiable groupoid 2-cocycle c on $\Sigma_s(P)$. In this case we also say that c integrates Λ.*

In general, the algebroid associated to an integrable algebroid 2-cocycle is integrable. More precisely, if c is a groupoid 2-cocycle on a Lie groupoid G and Λ is the induced algebroid 2-cocycle, it follows that the Lie algebroid of $G \ltimes_c \mathbb{R}$ is precisely $A \ltimes_\Lambda \mathbb{R}$ [5].

6.3. Proof of Theorem 4. Clearly (ii) implies (iii), and we now prove the converse. We use Theorem 2 and the fact that $\mathcal{G}_x = \mathbb{R}$ (the hypothesis). It follows that $\pi : \Sigma_c(P) \longrightarrow \Gamma_s(P)$ is a principal $\mathbb{R}$-bundle, with connection form θ and curvature Ω. Since $\mathbb{R}$ is contractible, π^* induces an isomorphism from $H^2(\Gamma_s(P))$ to $H^2(\Sigma_c(P))$. So $[\pi^*\Omega] = [d\theta]$ implies that $[\Omega] = 0$, i.e. Ω is exact.

To prove that (i) is equivalent to (iii), we use the following characterization of integrable algebroid cocycles from [5]. Assume that ν is an algebroid 2-cocycle on A, and that G is an α-simply connected Hausdorff Lie groupoid integrating A. Since each fiber A_x is identified with the tangent space of $\alpha^{-1}(x)$ at the identity element at x, using right translations, ν will induce a 2-form on $\alpha^{-1}(x)$, call it $\omega_{\nu,x}$. Then, ν is in the image of the Van Est map if and only if $Per(\omega_{\nu,x}) = 0$ for all $x \in P$. In our case, note that $\omega_{\Lambda,x} = \Omega|_{\alpha^{-1}(x)}$, hence the equivalence of (i) and (iii) follows.

Corollary 6.3. *For a compact s-simply connected contact groupoid of an integrable Poisson manifold, the Reeb vector field always has at least one closed orbit.*

Proof. If the Reeb vector field has no closed orbit, then according to Theorem 2, $\mathcal{G}_\Lambda$ is the trivial $\mathbb{R}$ bundle. From Theorem 4, one has $\Sigma_c(P) \cong \Sigma_s(P) \times \mathbb{R}$, which contradicts with the compactness. □

Corollary 6.4. *If every symplectic leaf in an integrable Poisson manifold P has exact symplectic form, then the symplectic form Ω of $\Sigma_s(P)$ is also exact.*

Proof. It is a direct conclusion from Theorem 4 and Lemma 4.3. □

7. Special cases and examples

Example 1 (Symplectic manifolds). Let (S, ω) be a symplectic manifold, and assume for simplicity that S is simply connected (in general, one has to replace the pair groupoid below with the homotopy groupoid). Then the symplectic groupoid of S (viewed as a Poisson manifold) is

$$\Gamma_s(S) = (S \times S, (\omega, -\omega)),$$

the pair groupoid (source and target are the projections, and the multiplication is $(x, y)(y, z) = (x, z)$), endowed with the symplectic form $pr_1^*\omega - pr_2^*\omega$. In this case $\mathcal{P}_\Lambda$ is a trivial group bundle over S with fiber $Per(\omega)$, hence, by Theorem 2, S is integrable as a Jacobi manifold if and only if $Per(\omega)$ is a discrete group, i.e.

$Per(\omega) = a\mathbb{Z}$ for some real number a. The interesting case is when $a \neq 0$. In this case $\omega_a = \frac{1}{a}\omega$ is integral, hence we find a principal S^1-bundle $\pi : \tilde{S} \longrightarrow S$ and a connection 1-form $\theta_a \in \Omega^1(\tilde{S})$ so that $\pi^*\omega_a = d\theta_a$. The gauge groupoid of $\tilde{S}$ is $\tilde{S} \otimes_{S^1} \tilde{S}$, the quotient of the pair groupoid $\tilde{S} \times \tilde{S}$ by the diagonal action of S^1 (hence the base manifold is $\tilde{S}/S^1 = S$). Moreover, the 1-form $(\theta_a, -\theta_a)$ is basic, hence induces a 1-form $\bar{\theta}_a$ on $\tilde{S} \otimes_{S^1} \tilde{S}$, and this makes the gauge groupoid into a contact groupoid. It is not difficult to see that

$$\Sigma_c(S) = (\tilde{S} \otimes_{S^1} \tilde{S}, a\bar{\theta}_a, 1).$$

Example 2 (Contact manifolds). Let (M, θ) be a contact manifold, and assume for simplicity that M is simply connected (as in the previous example, in the general case one has to replace the pair groupoid by the homotopy groupoid). Then the contact groupoid of M (viewed as a Jacobi manifold) can be described as follows. Consider the product $M \times M \times \mathbb{R}$ of the pair groupoid with $\mathbb{R}$. The 1-form

$$\theta = -(\exp \circ p_3)p_1^*\theta_0 + p_2^*\theta_0$$

where p_i, $1 \leq i \leq 3$, is the projection on the i-th component, will be a contact form, and it will be multiplicative with respect to the Reeb cocycle $r = \exp \circ p_3$. With these,

$$\Sigma_c(M) = (M \times M \times \mathbb{R}, \theta, r).$$

Example 3 (Vector fields). Given a vector field X on a manifold M, one can view (M, X) as a Jacobi manifold with vanishing bivector. Note that the orbits of the associated Lie algebroid $T^*M \oplus \mathbb{R}$ are precisely the orbits of X hence, since they are 1-dimensional, it follows that M is integrable as a Jacobi manifold. Hence the associated contact groupoid, denoted here by $\Sigma_c(M, X)$, is smooth. Let us describe $\Sigma_c(M, X)$ in more detail. First, let us mention two other simpler groupoids which are associated to X.

1. The flow of X, $\mathcal{D}(X)$, is probably the best known example of Lie groupoid. One has $\mathcal{D}(X) \subset M \times \mathbb{R}$ as the domain of the local flow ϕ_X^t of X, consisting of pairs (x, t) with the property that $\phi_X^t(x)$ is defined. The elements $(x, t) \in \mathcal{D}(X)$ are viewed as arrows from $\phi_X^t(x)$ into x, and the composition is given by the rule $\phi_X^t \phi_X^s = \phi_X^{t+s}$.

2. In general, for any finite dimensional vector space V and any vector $v \in V$, one has an associated Lie algebra $\mathfrak{g}(v)$, which is V^* endowed with the bracket

$$[\alpha, \beta] = -\alpha(v)\beta + \beta(v)\alpha.$$

The associated simply connected Lie group, denoted by $\mathcal{G}(v)$, can be described as follows:

$$\mathcal{G}(v) = \{\lambda \in V^* : \phi_\lambda := Id_V + \lambda v \in Aut^+(V)\}$$

(where $Aut^+(V)$ is the group of orientation preserving automorphisms of V). The product $\lambda\eta$ is defined by

$$u \mapsto \lambda(u) + \eta(u) + \lambda(v)\eta(u),$$

i.e. so that $\phi_{\lambda\eta} = \phi_\lambda\phi_\eta$. Applying this to each $X_x \in T_xM$, we obtain a bundle of Lie groups over M, denoted by $\mathcal{G}(X)$.

Note that $\mathcal{D}(X)$ acts on $\mathcal{G}(X)$: for each $(x, t) \in \mathcal{D}(X)$ viewed as an arrow from $y = \phi_X^t(x)$ into x, $(d\phi_X^t)_x : T_xM \longrightarrow T_yM$ preserves X, hence it induces a Lie group map from $\mathcal{G}(X_y)$ into $\mathcal{G}(X_x)$, denoted by $\phi_{x,t}$. One then forms the semi-direct product $\mathcal{G}(X) \rtimes \mathcal{D}(X)$, which consists of triples

$$(\lambda, x, t) \ \text{ with } \ (x, t) \in \mathcal{D}(X), \lambda \in \mathcal{G}(X_x),$$

such a triple is viewed as an arrow from $\phi_X^t(x)$ into x, and the composition is given by

$$(\lambda, x, t)(\lambda', x', t') = (\lambda + \phi_{x,t}(\lambda'), x, t + t').$$

With these, the contact groupoid is

$$\Sigma_c(M, X) = \mathcal{G}(X) \rtimes \mathcal{D}(X). \tag{19}$$

In particular, this tells us that the period group of X at x,

$$Per_x(X) = \{t : \phi_X^t(x) = x\},$$

acts on the Lie group $\mathcal{G}(X_x)$, and the isotropy group of $\Sigma_c(M, X)$ (i.e. arrows that start and end at x) is

$$\Sigma_c(M, X)_x = \mathcal{G}(X_x) \rtimes Per_x(X).$$

To see (19), one looks at what happens at the infinitesimal level. The Lie algebroid of $\mathcal{D}(X)$, denoted by $\mathbb{L}_X$, is the trivial line bundle together with the bracket $[f, g]_X := -X(f)g + X(g)f$ for $f, g \in \Gamma(\mathbb{L}_X) = C^\infty(M)$, and the anchor is given by multiplication by X. Moreover, the Lie algebras $\mathfrak{g}(X_x)$ fit into a bundle of Lie algebras, denoted by $\mathfrak{g}(X)$; this is T^*M, together with the Lie bracket on 1-forms $[\omega, \theta]_X := -i_X(\omega \wedge \theta)$. There is an obvious exact sequence of algebroids

$$0 \longrightarrow \mathfrak{g}(X) \longrightarrow T^*M \oplus_M \mathbb{R} \longrightarrow \mathbb{L}_X \longrightarrow 0,$$

and, from the explicit formulas for the bracket on $T^*M \oplus \mathbb{R}$, it is not difficult to see that $T^*M \oplus_M \mathbb{R}$ is a semi-direct product of Lie algebroids, where the action of $\mathbb{L}_X$ on $\Gamma(\mathfrak{g}(X)) = \Omega^1(M)$ is the Lie derivative with respect to X. Passing to the global picture, we find (19).

Example 4 (Homogeneous Poisson manifolds)**.** Let $(P, \tilde{\Lambda})$ be a homogeneous Poisson manifold (see subsection 2.1). Note that the homogeneity equation $\mathcal{L}_Z\tilde{\Lambda} = -\tilde{\Lambda}$ can be reformulated in terms of the cohomology complex of the algebroid $A = T^*P$ (see subsection 6.2) as $\Lambda = -d_{T^*M}(Z)$. In particular, Theorem 4 tells us that $\mathcal{P}_\Lambda = 0$. We deduce the following:

Corollary 7.1. *Any integrable homogenous Poisson manifold is also Jacobi integrable, and* $\Sigma_c(P) \cong \Sigma_s(P) \times \mathbb{R}$.

Example 5 (Conformal versions)**.** Given a Jacobi manifold (M, Λ, E), and a smooth nowhere vanishing function τ on M, one defines the conformal transformation of (Λ, E) by τ as the new Jacobi structure given by

$$\Lambda_\tau = \tau\Lambda, \qquad E_\tau = \tau E + \Lambda^\sharp(d\tau).$$

One says that two Jacobi structures are conformal equivalent if they are related by such a transformation; such an equivalence class of Jacobi structures is called *conformal Jacobi structure*. Of course, when restricted to contact manifolds, this becomes the usual notion of conformal equivalence of contact forms (see e.g. [1]): θ and θ' are equivalent if $\theta' = \tau\theta$ for some non-vanishing function τ. Equivalently, this corresponds to the fact that the contact distribution of θ, $\mathcal{H}_\theta = Ker(\theta)$, coincides with the one of θ'.

Similarly, given a contact groupoid Σ over M with contact form θ and Reeb function r, and a smooth nowhere vanishing function τ on M,

$$\theta_\tau = \alpha^*(\tau)\theta, \quad r_\tau = r + \lg(\frac{\alpha^*\tau}{\beta^*\tau})$$

define a new contact form and Reeb function so that $(\Sigma, \theta_\tau, r_\tau)$ is a contact groupoid. Exactly as before, Σ, together with an equivalence class of a pair (θ, r), will be called a *conformal contact groupoid.*

We then have:

Corollary 7.2. *There is a 1-1 correspondence between conformal contact groupoids over M which are α-simply connected, and integrable conformal Jacobi structures on M.*

This follows immediately from our results and the following two simple remarks:

(i) The integrability of Jacobi structures is stable under conformal equivalences, hence one can talk about the integrability of conformal Jacobi structure.

(ii) If $f : (N, \Lambda^{'}, E^{'}) \longrightarrow (M, \Lambda, E)$ is a Jacobi map, and τ is a non-vanishing function on M, then f is Jacobi also as a map from $(N, \Lambda^{'}_{\tau \circ f}, E^{'}_{\tau \circ f})$ to $(M, \Lambda_{\tau}, E_{\tau})$.

Example 6 (Locally conformal versions)**.** Apparently, the contact groupoid equation

$$m^{*}\theta = pr_2^{*}(e^{-r}) \cdot pr_1^{*}\theta + pr_2^{*}\theta$$

is not symmetric. The "mirror symmetry" of the previous equation is

$$m^{*}\theta' = pr_1^{*}\theta' + pr_1^{*}(e^{-r'}) \cdot pr_2^{*}\theta',$$

and this is obtained by the transformation $\theta' = e^{r}\theta$ and $r' = -r$. A more symmetric version of the equation is obtained by choosing $\theta_0 = e^{-\frac{r}{2}}\theta$, $r_0 = \frac{r}{2}$, for which we have

$$m^{*}\theta_0 = pr_2^{*}(e^{-r_0}) \cdot pr_1^{*}\theta_0 + pr_1^{*}(e^{r_0}) pr_2^{*}\theta_0.$$

Of course, all these describe essentially the same contact groupoid, and this is the point of view adapted in [8, 9]. The relation between all these descriptions comes from the fact that $Ker(\theta) = Ker(\theta') = Ker(\theta_0)$. This is only one of the several motivations for considering "locally conformal versions" of the structures we have been discussing.

To formulate our conclusions, we recall here that a locally conformal Jacobi structure on M is described by

(LCJ1) an open cover $\{U_i\}$ of M together with Jacobi structures (Λ_i, R_i) on each open U_i, so that, on the overlaps $U_i \cap U_j$, the restrictions of the two Jacobi structures are conformal equivalent via $\tau_{i,j}$, and so that the $\tau_{i,j}$'s satisfy the cocycle condition $\tau_{i,j}\tau_{j,k} = \tau_{i,k}$ on $U_i \cap U_j \cap U_k$.

Of course, different covers and local Jacobi structures can lead to the same locally conformal Jacobi structure, i.e. one has to consider a certain equivalence relation. This is completely similar to the description of vector bundles in terms (of equivalence classes!) of transition functions, and, as there, there is an alternative global description:

(LCJ2) A (isomorphism class of a) line bundle L over M together with a Lie algebra structure $[\cdot, \cdot]$ on the space $\Gamma(L)$ of sections, so that $[\cdot, \cdot]$ is local. (The isomorphisms are realized by bundle maps covering the identity and inducing Lie algebra isomorphisms.)

The global picture (LCJ2) is obtained by gluing the local data of (LCJ1): one glues the trivial line bundles over U_i using the transition functions $\tau_{i,j}$, and then the local Lie brackets defined on each U_i by the Jacobi structures (see the introduction) will fit together into a local Lie bracket on $\Gamma(L)$.

Restricting to Jacobi structures coming from contact ones, we obtain what we will call here locally conformal contact structure, and which are well known in the literature (often under various other names). Similar to the discussion above (and well known), such a structure on M is described by either

(LCC1) an open cover $\{U_i\}$ of M together with contact forms θ_i on each U_i and nowhere vanishing functions $\tau_{i,j}$ defined on the overlaps $U_i \cap U_j$, such that $\theta_j = \tau_{i,j}\theta_i$.

(LCC2) a contact hyperplane $\mathcal{H}$, i.e. a smooth field of tangent hyperplanes $\mathcal{H} \subset TM$ so that, locally, $\mathcal{H}$ is of type $Ker(\theta)$ for some (locally defined) contact 1-form θ.

Note that a conformal Jacobi structure is the same thing as a locally conformal Jacobi structure with orientable line bundle, and a conformal contact structure is the same thing as a locally conformal contact structure whose contact hyperplane is induced by a globally defined contact form. In particular, if M is simply connected, then "locally conformal= conformal".

Now, with our terminology, a locally conformal contact groupoid will be *a groupoid Σ over M together with a contact hyperplane $\mathcal{H}$ with the property that $\mathcal{H}$ is compatible with the groupoid structure in the sense that*

(i) *The inversion $i : \Sigma \to \Sigma$ leaves $\mathcal{H}$ invariant.*
(ii) *For all $X, Y \in \mathcal{H}$ for which $X \cdot Y = (dm)(X, Y)$ is defined, we have: $X \cdot Y \in \mathcal{H}$.*

These have been introduced in [8, 9] under the name of contact groupoids (see also [21]). With this, our main theorem on contact groupoids and the correspondence with Jacobi structures has a locally conformal version (and this completes the study of [8, 9]). There are various ways to see this. For instance, Dazord shows that, if $(\Sigma, \mathcal{H})$ is a locally conformal contact groupoid so that $\mathcal{H} = Ker(d\theta)$ for some globally defined contact form θ, then θ can be choosen so that (Σ, θ, r) is a contact groupoid (for some uniquely defined multiplicative function r), and two choices θ_1 and θ_2 define the same locally conformal contact groupoid if and only if the associated contact groupoids are conformal equivalent (see Proposition 4.1 in [9] or Appendix I in [21]). In particular, if M is simply connected and Σ is α-simply connected, then locally conformal structures on Σ are the same thing as conformal structures on Σ. One can then pull-back Σ to the universal cover $\tilde{M}$, use our main result there, and show that it decends down to M (this requires some care; in particular, the trivial line bundle over $\tilde{M}$ will descend to a possibly non-trivial line bundle over M). Alternatively, one can check that all our arguments, after suitable modifications, make sense in the locally conformal setting as well. For instance, if M is locally conformal with associated bundle L_M, and if $(\Sigma, \mathcal{H})$ is a locally conformal groupoid, then:

(i) the Poissonization of (M, L_M) will be L_M viewed as a manifold.
(ii) the Lie algebroid of (M, L_M) will be the jet bundle $J^1(L_M)$, with the bracket of two 1-jets given by the jet of the local Lie bracket on $\Gamma(L_M)$.
(iii) the symplectification of $(\Sigma, \mathcal{H})$ will be $L_\Sigma = T\Sigma/\mathcal{H}$, a symplectic groupoid over L_M.

All these have been already explained in [9]. A bit more care is needed when working with A-paths. Nevertheless, one can use a connection on L_M to write the jet algebroid as $L_M \oplus \mathbb{R}$, so that the discussion from section 3 (where the Lie bracket, the A-paths, and the corresponding equations are all written componentwise) can be carried out in this setting. In particular, we have:

Corollary 7.3. *There is a 1-1 correspondence between locally conformal contact groupoids over M which are α-simply connected, and integrable locally conformal Jacobi structures on M.*

Example 7 (De-poissonization). Inverse to the "poissonization process" (Section 2), one can obtain Jacobi-type structures out of homogeneous Poisson manifolds. The result is also a very good illustration of the different Jacobi-type structures described in the previous two examples. More precisely, given a homogeneous Poisson manifold $(P, \tilde{\Lambda})$ with homogeneous vector field Z, assuming that Z is nowhere zero and that the set $\bar{P}$ of all trajectories of Z admits a manifold structure such that the projection $\pi : P \longrightarrow \bar{P}$ is a submersion, then

(i) $\bar{P}$ has an induced locally conformal Jacobi structure.
(ii) if Z is the infinitesimal generator of a free action of $\mathbb{R}_+$ on P, then $\bar{P}$ has a canonically induced conformal Jacobi structure.
(iii) under the condition of (ii), if π has a distinguished section, then $\bar{P}$ has a distinguished Jacobi structure.

Part (i) is explained in [10] (Corollary 2.5), while (ii) and (iii) are just simpler versions that we explain here independently of (i). For (iii), using the given section we identify P with $\bar{P} \times \mathbb{R}_+$ with the action on the second component. Then, completely similar to Section 2, one gets an induced Jacobi structure on $\bar{P}$. For (ii), since the fibers of π are contractible, we can allways find globally defined sections, and consider the induced Jacobi structures induced from (iii); different choices of sections produce conformal equivalent Jacobi structures, i.e. it is only the conformal Jacobi structure that is independent of the choice of the section.

Of course, starting with a symplectic manifold, the quotient will be a (locally conformal) contact manifold. When looking at a homogeneous symplectic groupoid Σ over P, one can show that the induced quotient $\bar{\Sigma}$ inherits a groupoid structure, and it is a (locally conformal) contact groupoid over $\bar{P}$. Of course, the quotient of $\Sigma_s(P)$ will coincide with $\Sigma_c(\bar{P})$.

Example 8 (Sphere bundles). A particular case of the previous example comes from duals of Lie algebroids. Relevant here is that, given a vector bundle A over M, there is a 1-1 correspondence between Poisson structures on A^* and Lie algebroid structures on A (see e.g. [3]). In particular, given a Lie algebroid A, the Poisson manifold A^* will be homogeneous with respect to the generator of the (fiberwise linear!) action of $\mathbb{R}_+$: $t \cdot a = ta$. It follows that the sphere bundle

$$S(A^*) = (A^* - \{0\})/\mathbb{R}_+$$

has a conformal Jacobi structure. When A comes equipped with a metric, $S(A^*)$ takes the more familiar form consisting of vectors of norm 1, and we will have an induced Jacobi structure on $S(A^*)$ (in (iii) of the previous example, one uses the obvious section induced by the metric). This applies in particular to $A = TM$, when one obtains $S(T^*M)$ with its canonical contact structure (or conformal contact if one does not fix a metric).

On the other hand, if A comes from a Lie groupoid G that we assume to be α-simply connected, one knows that T^*G is naturally a groupoid over A^* which, together with the canonical symplectic form on the cotangent bundle, becomes a symplectic groupoid (cf. e.g. [9]). Of course, this is the symplectic monodromy groupoid of A^*. Passing to sphere bundles, $S(T^*G)$ together with the canonical (conformal) contact structure and the induced groupoid structure, becomes the conformal contact groupoid associated to $S(A^*)$. Interesting particular cases are obtained when A is a Lie algebra, or a tangent bundle.

Example 9 (2-dimensional Poisson case). It is known that every 2-dimensional Poisson manifold P is integrable [7]. The main reason is that each symplectic leaf of P is either a point or 2 dimensional, and in the former case the isotropy Lie algebra of T^*P at x is zero. These force that all monodromy maps ∂_s (see subsection 4.3) are zero, hence, by the main result of [6], T^*P must be integrable. Note that the previous discussion also shows that $\mathcal{P}_{\Lambda,x}$ coincides with the period group $Per(\omega_L)$, where L is the symplectic leaf through x and ω_L is its symplectic form. Hence, Theorem 2 becomes

Corollary 7.4. *A 2-dimensional Poisson manifold is integrable as a Jacobi manifold if and only if $Per(\omega_L)$ are locally uniformly discrete.*

Example 10 (Non-integrable examples)**.** Using the previous corollary or Example 1, it is easy to produce Poisson manifolds which are integrable as Poisson manifolds but which are not integrable as Jacobi manifolds: it suffices to consider a symplectic form whose period group is dense. We now show that there are also Poisson manifolds which are Jacobi integrable, but which are not Poisson integrable. This exploits one of the examples of [7] (see Example 3.8 there), which we now recall. Let $M_a = \mathbb{R}^3$, endowed with the Poisson structure described by the bracket of the coordinate functions x^i as follows:

$$\{x^2, x^3\} = ax^1, \quad \{x^3, x^1\} = ax^2, \quad \{x^1, x^2\} = ax^3,$$

where $a = a(r)$ is a function depending only on the radius r, with the property that $a(r) > 0$ for $r > 0$. The symplectic leaves of M_a are the spheres S_r centered at the origin (including the degenerated sphere: the origin itself), and the leafwise symplectic forms are

$$\omega_r = \frac{r^2}{a}(x^1 dx^2 dx^3 + x^2 dx^3 dx^1 + x^3 dx^1 dx^2).$$

Central to the conclusion is the symplectic area function,

$$A_a(r) = \int_{S_r} \omega_r = \frac{4\pi r}{a(r)}.$$

With these, [7] shows that M_a is Poisson integrable if and only if either $A_a(r)$ is constant, or A_a has no critical points and $\lim_{r\to 0} A_a'(r) \neq 0$. One can do exactly the same type of computations as in [7], but this time for the algebroid $T^*M \oplus \mathbb{R}$. For those familiar with [7] (to which we refer also for notations), here are the main modifications: the splitting to be used is $(\sigma, 0)$, the curvature becomes (Ω, ω_r), the monodromy group is $(A_a^{'}(r)\mathbb{Z}n, A_a(r)\mathbb{Z})$, and the function $r_{\mathcal{N}}$ will be $A_a{}' + A_a$ away from the origin, and $+\infty$ at the origin. The conclusion is that M_a is integrable as a Jacobi manifold if and only if $\lim_{r\to 0} A_a'(r) + A_a(r) \neq 0$. It is now easy to find various non-integrable examples:

(i) For $a(r) = re^r$, M_a is Poisson integrable but it is not Jacobi integrable.
(ii) For $a(r) = 1/(\sin r + 2)$, M_a is Jacobi integrable but it is not Poisson integrable.
(iii) For a such that $a(r) = re^{\phi(r)r}$ where ϕ is a smooth function that equals to 1 near the origin and equals to 0 for r large enough, M_a is neither Poisson nor Jacobi integrable.

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