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Rigidity and Flexibility in Poisson Geometry

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Abstract

We study rigidity and flexibility phenomena in the context of Poisson geometry.

1 Introduction

Rigidity and flexibility phenomena are well-known in symplectic geometry. Since a Poisson manifold is a generalization of a symplectic manifold, it is natural to look for rigidity and flexibility phenomena in the context of Poisson geometry. In the present work we will survey recent results in this direction, give a few new results, and pose some conjectures. It should came as no surprise that these kind of questions are even harder to investigate in this more general context.

In symplectic geometry there are two basic results, going in opposite directions:

- For *open manifolds*, a celebrated theorem by Gromov ([17]) states that every non-degenerate two form is homotopic to a symplectic form (*flexibility*).
- For *compact manifolds*, Moser's stability theorem (see, e.g., [20] Theorem 3.17) shows that two symplectic forms that are homotopic in the same cohomology class must be symplectomorphic (*rigidity*).

These two results have important consequences. For example, the first result reduces the problem of existence of a symplectic form on an open manifold to the problem of existence of a non-degenerate 2-form, which is a relatively easy problem in obstruction theory. In contrast, for compact 4-manifolds there are highly non-trivial obstructions to the existence of symplectic forms. In Poisson geometry, one would like to have generalizations of these results, and in this paper we discuss some recent progress in this direction.

In Section 2, we start by recalling Bertelson's theorem ([1]), which gives a partial analogue of Gromov's result in Poisson geometry. It states that on an *open foliated manifold* every leafwise non-degenerate 2-form is homotopic, in the class of non-degenerate 2-forms, to a leafwise symplectic form, i.e., a Poisson structure. Therefore, for an *open foliation* $\mathcal{F}$ the problem of existence of a Poisson structure whose underlying foliation is $\mathcal{F}$, can be reduced to a problem in obstruction theory.

We point out that Bertelson's results can be better understood in the setting of groupoids. Unfortunately, these results are only valid for regular foliations, while in Poisson geometry, some of the most interesting Poisson manifolds have singular symplectic foliations. It would be very interesting to find generalizations of these results to singular foliations, but no such result is available at present.

In Section 3, we turn to rigidity. Given a Poisson structure π, one may ask if every nearby Poisson structure, satisfying some cohomological condition, must be diffeomorphic to π. We don't know of such analogue of Moser's stability theorem in Poisson geometry. In fact, it is not clear how such a general result should be stated since there is no obvious way of comparing Poisson cohomology classes attached to distinct Poisson structures. Even the question of what is the right compactness assumption is an interesting problem. We will see that one is led naturally to the study of *proper* Poisson manifolds (Poisson manifolds which integrate to proper symplectic groupoids) and of Poisson manifolds of *compact type* (Poisson manifolds which integrate to compact symplectic groupoids). Properness is enough to guarantee rigidity around leafs, as illustrated by recent results of Weinstein and Zung on linearization around leaves ([25, 27]). Some kind of global rigidity should also hold for Poisson manifolds of compact type. However this is an intriguing class of which very little seems to be known. We will give some properties of this class, and we will state a rigidity result in the regular case.

In face of such difficulties, one is led naturally to the study of weaker properties of Poisson manifolds, but still keeping a flavor of rigidity/flexibility. In this direction, one very natural question which does not seem to have been discussed before, is the problem of *stability* of symplectic leaves: a leaf of a Poisson structure is called stable if every nearby Poisson structure has a nearby diffeomorphic leaf. In Section 4, we discuss the following result which gives a sufficient condition for stability for fixed points, i.e., zero dimensional symplectic leafs:

Theorem 1.1. *Let $x_0 \in M$ be a fixed point of a Poisson structure π on M. If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ has vanishing second Chevalley-Eilenberg cohomology, i.e., $H^2(\mathfrak{g}_{x_0}) = 0$, then x_0 is a stable fixed point.*

We will also argue that this is the best possible general result in this direction, and we will formulate a conjecture concerning stability of symplectic leafs of higher dimension, which uses the relative Poisson cohomology of the leaf due to Ginzburg and Lu [15].

2 Flexibility

In this section we start by recalling the results due to Melanie Bertelson [1], which extend Gromov's results to the foliated case.

Recall that a manifold M is open if and only if it carries a positive, proper Morse function, without any local maximum. The existence of such a Morse function is precisely what is need in order to proof Gromov's h-principle for open invariant relations (see, e.g., [16], Theorem 3.3). Loosely speaking, the h-principle states that for every open, invariant relation $\mathcal{R}$ on a jet space J^kE, the space of sections $\Gamma(\mathcal{R})$ is weak homotopy equivalent to the space of holonomic sections $\Gamma_{\mathrm{hol}}(\mathcal{R})$. The h-principle, in turn, when applied to the relation

$$\mathcal{R} = \left\{ j^1\alpha(x) \in J^1T^*M : \mathrm{d}\alpha(x) \text{ is non-degenerate} \right\},$$

yields the following important result:

Theorem 2.1 (Gromov [17]). *Every non-degenerate two form is homotopic in the set of non-degenerate two forms to a symplectic form.*

Let now $\mathcal{F}$ be a foliation of a manifold M, and denote by $\Omega^2(\mathcal{F})$ the space of foliated 2-forms on $\mathcal{F}$. Note that a leafwise symplectic 2-form $\omega \in \Omega^2(\mathcal{F})$ is the same thing as a Poisson structure on M with symplectic foliation equal to $\mathcal{F}$. In order to obtain an analogue of Gromov's result, one needs the concept of an open foliation:

Definition 2.1 (Bertelson [1]). A foliated manifold $(M, \mathcal{F})$ is said to be **open** if there exists a smooth function $f : M \to [0, \infty)$with the following properties:

(a) f is proper;

(b) f has no leafwise local maxima;

(c) f is $\mathcal{F}$-generic.

The condition that f is $\mathcal{F}$-generic is a transversality condition on its jet which plays a role analogous to the Morse condition. The main result of [1] then states that on an open foliated manifold, any open, foliated, invariant differential relation satisfies the parametric h-principle. Applying this result to the relation

$$\mathcal{R} = \left\{ j^1\alpha(x) \in J^1T^*\mathcal{F} : \mathrm{d}_{\mathcal{F}}\alpha(x) \text{ is non-degenerate} \right\},$$

one obtains a foliated version of Gromov's theorem:

Theorem 2.2 (Bertelson [1]). *Let $(M, \mathcal{F})$ be an open foliated manifold. Any leafwise nondegenerate 2-form is homotopic, in the class of leafwise nondegenerate 2-forms, to a leafwise symplectic form.*

The leaves of an open foliation are necessarily open manifolds. However, this condition is not sufficient for the conclusions of the theorem to hold. The following example is presented in [2].

Example 2.1. Let M be a compact m-manifold and $\mathcal{F}$ a dimension $2k$ foliation of M. If $\mathcal{F}$ is transversely orientable then $\mathcal{F}$ does not carry an exact leafwise symplectic form. In fact, let μ be a transverse volume form, i.e., a nowhere vanishing, closed, $(m-2k)$-form such that $i_X\mu = 0$, for any tangential vector field $X \in \mathfrak{X}(\mathcal{F})$. If $\omega = \mathrm{d}_{\mathcal{F}}\alpha$ is an exact leafwise symplectic form, we choose extensions $\widetilde{\omega}$, $\widetilde{\alpha}$ of ω and α to forms on M satisfying $\widetilde{\omega} = \mathrm{d}\widetilde{\alpha}$, and we observe that:

$$\mu \wedge \widetilde{\omega}^k = \mathrm{d}(\mu \wedge \widetilde{\omega}^{k-1} \wedge \widetilde{\alpha})$$

is an exact volume form on M. This contradicts the assumption that M was compact.

For example, take $M = \mathbb{T}^2 \times \mathbb{S}^3$ with the foliation $\mathcal{F} = \mathcal{F}_a \times \mathbb{S}^3$, where $\mathcal{F}_a$ is the irrational foliation of the torus of slope a. By this we mean the foliation of $\mathbb{T}^2$ generated by the vector field $X_a = \frac{\partial}{\partial\theta_1} + a\frac{\partial}{\partial\theta_2}$, where (θ_1, θ_2) are coordinates on $\mathbb{T}^2$ and $a \in \mathbb{R} - \mathbb{Q}$. Now observe that:

(a) $\mathcal{F}$ carries a leafwise non-degenerate 2-form: let α be the 1-form on $\mathbb{T}^2$ such that $\alpha(X) = 1$, and let β be the contact form on $\mathbb{S}^3$. Then $\alpha \wedge \beta + \mathrm{d}\beta$ gives a leafwise non-degenerate 2-form on $\mathcal{F}$.

(b) $\mathcal{F}$ is transversely orientable: a transverse volume form is given by the 1-form $\mu = \pi_1^*(a\mathrm{d}\theta_1 - \mathrm{d}\theta_2)$, where $\pi_1 : \mathbb{T}^2 \times \mathbb{S}^3 \to \mathbb{T}^2$ denotes projection to the first factor.

Therefore, $\mathcal{F}$ has open leafs and admits a leafwise non-degenerate 2-form. However, it does not carry any exact leafwise symplectic form.

Several other examples are given by M. Bertelson in [2], including examples with non-compact M. It should be observed that in all such examples one has $H^2(\mathcal{F}) = 0$. For instance, in the example above this follows from:

$$H^2(\mathcal{F}) = \bigoplus_{i+j=2} H^i(\mathcal{F}_a) \otimes H^j(\mathbb{S}^3) = 0.$$

The condition $H^2(\mathcal{F}) = 0$ can be interpreted as an obstruction to the existence of a leafwise symplectic form on a foliation which exhibits compactness. In order to explain this, let us recall that a Poisson tensor $\pi \in \mathfrak{X}^2(M)$ is said to be *exact* if its Poisson cohomology class $[\pi] \in H^2_\pi(M)$ is trivial. Note that this means that:

$$\mathcal{L}_X\pi = \pi,$$

for some vector field $X \in \mathfrak{X}(M)$. Now we have:

Proposition 2.1. *Let $\omega \in \Omega^2(\mathcal{F})$ be a leafwise symplectic form. If the class $[\omega] \in H^2(\mathcal{F})$ vanishes then the associated Poisson tensor π is exact.*

Proof. Let $\alpha \in \Omega^1(M)$ be a 1-form and denote by $\alpha^\sharp \in \mathfrak{X}(M)$ the vector field obtained by contraction with the Poisson tensor π. A simple computation shows that:

$$\mathcal{L}_{\alpha^\sharp}\pi(\mathrm{d}f, \mathrm{d}g) = \mathrm{d}\alpha(\mathrm{d}f^\sharp, \mathrm{d}g^\sharp).$$

Therefore, if $\omega = \mathrm{d}_\mathcal{F}\alpha$, we conclude that:

$$\mathcal{L}_{\alpha^\sharp}\pi(\mathrm{d}f, \mathrm{d}g) = \omega(\mathrm{d}f^\sharp, \mathrm{d}g^\sharp) = \pi(\mathrm{d}f, \mathrm{d}g).$$

Hence, the vector field $X = \alpha^\sharp$ satisfies $\mathcal{L}_X\pi = \pi$, and π is exact. □

Recall that an *integrable Poisson manifold* (M, π) is a Poisson manifold which arises as the space of units of a symplectic groupoid $\mathcal{G} \rightrightarrows M$ (see [25]). The results in [9, 10] show that the Poisson structures in this class are the Poisson structures which have non-singular variations of symplectic areas in directions transverse to the leaves. Simple conditions on the foliation $\mathcal{F}$ will guarantee that *any* leafwise symplectic form on $\mathcal{F}$ determines an *integrable* Poisson tensor. For example, if $\mathcal{F}$ is a foliation such that:

(i) $\mathcal{F}$ has no vanishing cycles;

(ii) $\pi_2(L)$ is finite for every leaf L of $\mathcal{F}$;

then any Poisson structure with symplectic foliation $\mathcal{F}$ will be integrable by an Hausdorff symplectic groupoid (see [10]).

For an integrable Poisson manifold (M, π), we will denote by $\mathcal{G}(M) \rightrightarrows M$ its source 1-connected symplectic groupoid, and we let $\Omega \in \Omega^2(\mathcal{G}(M))$ denote its symplectic form. The following proposition relates exactness of π and of Ω:

Proposition 2.2. *An integrable Poisson structure π on M is exact if and only if the symplectic form Ω on the associated symplectic groupoid is exact.*

Proof. Assume that π is exact, so there exists a vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X\pi = \pi$. Then one checks easily that the flow ϕ^t of X satisfies:

$$(\phi^t)_*\pi = e^t\pi.$$

This means that, for each t, the map

$$\phi^t : (M, \pi) \to (M, e^t\pi)$$

is a Poisson diffeomorphism. By Lie's second theorem for Lie algebroids (see [9]), this integrates to a Lie groupoid isomorphism:

$$\begin{array}{ccc} \mathcal{G}(M,\pi) & \xrightarrow{\Phi^t} & \mathcal{G}(M, e^t\pi) \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \xrightarrow[\phi^t]{} & M \end{array}$$

From the description of $\mathcal{G}$ given in [9, 10], one sees that the Lie groupoids $\mathcal{G}(M,\pi)$ and $\mathcal{G}(M,e^t\pi)$ can be identified as Lie groupoids and they only differ on their symplectic forms Ω and Ω_t, which satisfy:

$$\Omega_t = e^t\Omega.$$

In other words, we have that $\Phi^t : \mathcal{G}(M) \to \mathcal{G}(M)$ satisfies:

$$(\Phi^{-t})^*\Omega = e^t\Omega.$$

If $\tilde{X}$ denotes the vector field on $\mathcal{G}(M)$ with flux Φ^{-t}, we conclude that:

$$\Omega = \mathcal{L}_{\tilde{X}}\Omega = \mathrm{d}i_{\tilde{X}}\Omega,$$

so Ω is exact. The converse can be proved by reversing the argument. □

We have not used the fact that the symplectic form Ω on a symplectic groupoid $\mathcal{G}$ is also *multiplicative*: if $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is the multiplication in $\mathcal{G}$, and $s,t : \mathcal{G} \to M$ are the source and target maps, then:

$$m^*\Omega = s^*\Omega - t^*\Omega.$$

Using this condition one can show that (see [8], Corollary 5.3):

Proposition 2.3. *The symplectic form Ω is exact if and only if the restriction of Ω to each s-fiber is exact.*

Therefore, we see that that the condition $H^2(\mathcal{F}) = 0$ places strong restrictions on the topology of the groupoid integrating any leafwise symplectic form on $\mathcal{F}$.

Remark 2.1. A different setting where one can discuss all these questions, including Bertelson's notion of an open foliation, is the theory of twisted Dirac brackets (see [3]). In fact, given a foliation $\mathcal{F}$ of a manifold M, a leafwise 2-form $\omega \in \Omega^2(\mathcal{F})$ defines a ϕ-twisted Dirac bracket on M, where $\phi = \mathrm{d}\omega$. Under some nice assumptions, this twisted Dirac structure can be integrated to a presymplectic groupoid $(\mathcal{G},\Omega)$, where the presymplectic form Ω is ϕ-twisted:

$$\mathrm{d}\Omega = s^*\phi - t^*\phi.$$

When one varies the leafwise 2-form, the multiplication in the groupoid and the presymplectic form changes but the total space and the source and target maps stay the same. Hence, one can try to find a homotopy, preserving the fibers, and which changes the presymplectic form Ω to a symplectic form, and this should depend on topological properties of $\mathcal{G}$.

All what we have said above applies only to *regular* Poisson structures. If one fixes a manifold M and a *singular* foliation $\mathcal{F}$, one can look at the family of bivector fields $\theta \in \mathfrak{X}^2(M)$ which generate this foliation: each bivector θ determines a bundle map $T^*M \to TM$ and we are interested in those θ for which the image is $T\mathcal{F}$. Then one asks if, given such an $\mathcal{F}$-compatible θ, is it possible to find a homotopy θ_t of $\mathcal{F}$-compatible bivector fields with $\theta_0 = \theta$ and θ_1 a Poisson bivector field. Nothing seems to be known about this more general problem.

3 Rigidity

Let us now turn to *rigidity* in Poisson geometry. Our main motivation is the well-known stability theorem in symplectic geometry due to Moser:

Theorem 3.1. *Let M be a compact manifold, and let $\omega_t \in \Omega^2(M)$ be a homotopy of symplectic structures in M such that its cohomology class $[\omega_t] \in H^2(M)$ is constant. Then ω_0 and ω_1 are symplectomorphic.*

A proof can be found in any standard text in symplectic geometry , such as the monograph of MacDuff and Salamon ([20]). We would like to have analogues of this result in Poisson geometry. The first difficulty we face is to find the appropriate compactness assumption.

Recall that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *proper* if the map $(s,t) : \mathcal{G} \to M \times M$ is a proper map. We propose the following definitions:

Definition 3.1. A Poisson manifold (M,π) is called **proper** if it is integrable and its symplectic groupoid $\mathcal{G}(M) \rightrightarrows M$ is proper. A Poisson manifold (M,π) is of **compact type** if it is compact and proper.

Note that (M,π) is of compact type if and only it is integrable and its source 1-connected groupoid $\mathcal{G}(M)$ is compact.

Example 3.1. A compact Poisson manifold may not be of compact type. For example, take any compact Poisson manifold with the zero Poisson bracket: its symplectic groupoid is $T^*M \to M$ (source and target coincide, and the product is addition on the fibers), which is never proper.

Example 3.2. An example of a proper Poisson manifold is given by the linear Poisson manifold $\mathfrak{g}^*$, where $\mathfrak{g}$ is a compact semi-simple Lie algebra: its symplectic groupoid is $T^*G \rightrightarrows \mathfrak{g}^*$, where G is the (compact) simply connected Lie group integrating $\mathfrak{g}$. Here the source/target maps are given by right/left translations to the identity. This Poisson manifold is not of compact type, since $\mathfrak{g}^*$ is not a compact manifold.

Example 3.3. An example of a Poisson manifold of compact type is provided by a compact symplectic manifold with finite fundamental group: its symplectic groupoid is the fundamental groupoid $\pi_1(M) \rightrightarrows M$, with source/target maps given by the initial/end points of the homotopy class of a path.

For a proper Poisson manifold we have the following properties:

Proposition 3.1. *Let (M,π) be a proper Poisson manifold. Then:*

(i) All isotropy Lie algebras are of compact type.

(ii) All symplectic leafs are closed submanifolds.

(iii) *Every Poisson vector field is Hamiltonian:* $H^1_\pi(M) = 0$.

(iv) *There is a measure* μ *on* M *which is invariant under all Hamiltonian diffeomorphisms.*

Proof. Assume that $\mathcal{G}(M) \rightrightarrows M$ is proper so that the map $(s,t) : \mathcal{G}(M) \to M \times M$ is a proper map. If $x \in M$ is a fixed point, its isotropy Lie algebra $\mathfrak{g}_x$ integrates to the Lie group $G_x = (s,t)^{-1}(x,x)$ which is 1-connected and compact, so (i) holds.

Let $x_0 \in M$ and let L be the leaf through x_0. If $x_n \in L$ is sequence converging to some $x \in M$, then there are elements $g_n \in \mathcal{G}$ such that $(s,t)(g_n) = (x_0, x_n)$. The set

$$K = \{(x_0, x_n) : n \in \mathbb{N}\} \cup \{(x_0, x)\} \subset M \times M$$

is compact and $g_n \in (s,t)^{-1}(K)$. Therefore, the sequence g_n has a convergent subsequence: $g_{n_i} \to g$. Observe that:

$$s(g) = \lim s(g_{n_i}) = x_0,$$
$$(x_0, x) = \lim (s,t)(g_{n_i}) = (s,t)(g).$$

Hence $x \in L$ and L is a closed leaf, so (ii) holds.

To prove (iii), we use the fact that ([7]):

$$H^1_\pi(M) = H^1_d(\mathcal{G}),$$

where $H^k_d(\mathcal{G})$ denotes the groupoid cohomology with differentiable cochains. Since $\mathcal{G}$ is proper, we have $H^1_d(\mathcal{G}) = 0$, and (iii) follows. In particular, the modular class must vanish and so (iv) also holds. □

Proper Poisson manifolds already exhibit a lot of rigidity, as it is shown by the following result which is a consequence of general results on linearization of proper groupoids due to Weinstein [26] and Zung [27]:

Theorem 3.2. *Let* (M, π) *be a proper Poisson manifold, and let* L *be a compact symplectic leaf. Then* M *is linearizable around* L.

We refer to [26, 27] for exact statements and proofs.

In order to obtain *global* rigidity results one should require some global compactness, and it is natural to consider Poisson manifolds of compact type. At present, the only examples of Poisson manifolds of compact type which we are aware of are symplectic. Nevertheless, let us list some properties that any such manifold must satisfy.

We start with the following basic property, which shows that Poisson manifolds of compact type extend compact symplectic manifolds:

Proposition 3.2. *If* (M, π) *is a Poisson manifold of compact type then the Poisson cohomology class* $[\pi] \in H^2_\pi(M)$ *is non-trivial.*

Proof. If π was exact then, by Proposition 2.2, the symplectic form Ω in $\mathcal{G}(M)$ would be exact, which contradicts the fact that $\mathcal{G}(M)$ is compact. $\square$

Remark 3.1. Note that there are proper Poisson manifolds with $H^2_\pi(M) = 0$. For example, this happens for the Lie-Poisson structure on $M = \mathfrak{g}^*$, when $\mathfrak{g}$ is semi-simple of compact type.

The following somewhat surprising property shows that the class of Poisson manifolds of compact type is quite rigid:

Proposition 3.3. *Let (M, π) be a Poisson manifold of compact type. Then π does not have any fixed points.*

Proof. Assume that π has a fixed point $x_0 \in M$. Then $s^{-1}(x_0) = t^{-1}(x_0)$ is a compact, 1-connected Lie group. Hence, it must be homologically 2-connected. By stability, all s-fibers are homologically 2-connected.

It was shown in [7] that the classical Van Est homomorphism can be extended to Lie groupoids: it relates the Lie algebroid cohomology, the groupoid cohomology, and the ordinary cohomology of the source fibers; when applied to Poisson manifolds it gives a map:

$$\Phi : H^k_d(\mathcal{G}) \to H^k(T^*M) = H^k_\pi(M).$$

This map is an isomorphism up to degrees $k \leq n$, and injective in degree $n+1$, provided the source fibers are homologically n-connected.

In the present situation, we conclude that:

$$H^2_\pi(M) = H^2_d(\mathcal{G}).$$

Another basic result of [7], states that the differentiable groupoid cohomology of a proper Lie groupoid vanishes (this is a generalization of a well-known result in Lie theory stating that the group cohomology of a compact Lie group vanishes). We conclude that the class $[\pi] = 0$, which contradicts Proposition 3.2. $\square$

To construct examples of Poisson manifolds of compact type one could try to start with a proper Poisson manifold and restrict to a compact Poisson submanifold. However, one must be careful since a Poisson submanifold of an integrable Poisson manifold may be non-integrable, as shown by the following example:

Example 3.4. Let $\mathfrak{g}$ be a semi-simple Lie algebra of compact type of dimension $d+1$. We take $M = \mathfrak{g}^*$ with the Lie-Poisson bracket. The (negative of the) Killing form defines an inner product on $\mathfrak{g}^*$ which is Ad^*-invariant. It follows that the unit sphere $\mathbb{S}^d = \{\xi \in \mathfrak{g}^* : ||\xi|| = 1\}$ is a Poisson submanifold of $\mathfrak{g}^*$.

The Weinstein groupoid (see [10]) of $\mathbb{S}^d$, which we still denote by $\mathcal{G}(\mathbb{S}^d)$, can be obtained as follows: consider the symplectic groupoid of $\mathfrak{g}^*$, namely $T^*G \rightrightarrows \mathfrak{g}^*$, where G is the (compact) 1-connected Lie groupoid integrating $\mathfrak{g}$. Restricting to $\mathbb{S}^d$, we obtain the presymplectic groupoid $S(T^*G) \rightrightarrows \mathbb{S}^d$, where $S(T^*G)$ is the sphere bundle. This groupoid is still smooth, but to obtain the Weinstein groupoid of $\mathbb{S}^d$ we have to factor out the kernel of the presymplectic form. This may lead to a non-smooth groupoid. For example, one can check that this happens if $\mathfrak{g} = \mathfrak{su}(n)$, for $n \geq 3$.

Another possible method to produce Poisson manifolds of compact type is suggested by the following remark: Let (S, ω) be a symplectic manifold and let $G \times S \to S$ be a proper and free action of a Lie group G by symplectomorphisms. Then $M = S/G$ is an integrable Poisson manifold. In fact, its symplectic groupoid $\mathcal{G}(M)$ is obtained as the symplectic quotient of the symplectic groupoid of S:

$$\mathcal{G}(S/G) = \mathcal{G}(S)//G.$$

For a proof of this fact and generalizations to singular cases, we refer to the upcoming paper [14]. If one starts with a compact symplectic manifold (S, ω) with finite fundamental group, this could lead to a Poisson manifold of compact type. However, any canonical action on a compact symplectic manifold with finite fundamental group cannot be free. So one is forced to consider actions with a fixed isotropy type, and this makes this method much harder to work.

In the regular case, we can state the following rigidity result:

Proposition 3.4. *Let $\mathcal{F}$ be a regular foliation of a manifold M. Let $\omega_t \in \Omega^2(\mathcal{F})$ be a homotopy of foliated symplectic structures such that its cohomology class $[\omega_t] \in H^2(\mathcal{F})$ is constant and the associated Poisson structures π_t are of compact type. Then π_0 and π_1 are Poisson diffeomorphic.*

Proof. We sketch a proof of this result.

Recall that there is a homomorphism $\mathrm{d}_\nu : H^2(\mathcal{F}) \to H^2(\mathcal{F}; \nu^*)$, which can defined as follows: given a class $[\omega] \in H^2(\mathcal{F})$ represented by a foliated 2-form $\omega \in \Omega^2(\mathcal{F})$, pick an extension $\widetilde{\omega} \in \Omega^2(M)$ of ω. Since $\mathrm{d}\widetilde{\omega}|_{T\mathcal{F}} = 0$, we obtain a well-defined map $\Gamma(\wedge^2 T\mathcal{F}) \to \Gamma(\nu^*)$ given by:

$$(X, Y) \mapsto \mathrm{d}\widetilde{\omega}(X, Y, \cdot).$$

One checks easily that this is a closed foliated 2-form with values in ν^*, whose cohomology class $\mathrm{d}_\nu[\omega] \in H^2(\mathcal{F}; \nu^*)$ does not depend on any choices.

Now, if $\omega_t \in \Omega^2(\mathcal{F})$ is a homotopy of foliated symplectic structures such that the cohomology class $[\omega_t] \in H^2(\mathcal{F})$ is constant, the class $\mathrm{d}_\nu[\omega_t] \in H^2(\mathcal{F}, \nu^*)$ is independent of t. The results of [10] show that the groupoid $\mathcal{G}(M, \pi_t)$ only depends on the class $\mathrm{d}_\nu[\omega_t]$, so we see that the Poisson structures π_t integrate to the same Lie groupoid $\mathcal{G}(M)$, but with varying symplectic structures Ω_t. Since $\mathcal{G}(M)$ is compact, one can apply Moser's trick to produce an isotopy of Lie groupoid automorphisms Φ_t such that $(\Phi_t)^*\Omega_t = \Omega$. This isotopy covers the desired isotopy of Poisson diffeomorphisms. □

Again in the non-regular case, just like in the previous section, nothing seems to be known. In fact, it is not clear how such a general result should be stated since there is no obvious way of comparing Poisson cohomology classes attached to distinct Poisson structures. In face of these difficulties, we now turn to milder rigidity properties of Poisson structures.

4 Stability of leaves

The space Poiss(M) of Poisson structures on a manifold M is a subset of the space of smooth bivector fields $\mathfrak{X}^2(M)$:

$$\mathrm{Poiss}(M) = \left\{\pi \in \mathfrak{X}^2(M) : [\pi, \pi] = 0\right\}.$$

Since $\mathfrak{X}^2(M)$ is the space of sections of the vector bundle $\wedge^2 TM$, we can consider on it the C^k $(0 \le k \le +\infty)$ compact-open topology. This induces topologies on Poiss(M) which, in general, are extremely complicated.

At the formal level, around a Poisson structure $\pi \in \mathrm{Poiss}(M)$, one knows that the second Poisson cohomology group $H^2_\pi(M)$ is the space of infinitesimal deformations of Poisson structures. In other words, let $\mathcal{M}$ denote the moduli space of Poisson structures on M, obtained by factoring out diffeomorphic Poisson structures:

$$\mathcal{M} = \mathrm{Poiss}(M)/\mathrm{Diff}\,(M).$$

Then, formally, one has (see, e.g., [4]):

$$T_\pi\mathcal{M} = H^2_\pi(M).$$

Hence, it is natural to study points π where this tangent space vanishes, where one hopes to have rigidity. In other words, one is led to the question:

- If $H^2_\pi(M) = 0$ is every nearby Poisson structure diffeomorphic to π?

Of course, this question should be taken with care since $H^2_\pi(M)$ only parametrizes the formal deformations of π.

Let us take, for example, $M = \mathfrak{g}^*$ and assume that $\mathfrak{g}$ is of compact type, so that $H^2_\pi(\mathfrak{g}^*) = 0$. As a first step towards understanding stability, we recall Conn's Linearization Theorem [6]:

Theorem 4.1. *Let $\pi \in$ Poiss(M) be a Poisson structure, $x_0 \in M$ such that $\pi_{x_0} = 0$, and assume that the isotropy Lie algebra $\mathfrak{g}_{x_0}$ has compact type. Then there exists a neighborhood V of x_0 which is Poisson diffeomorphic to a neighborhood of zero in $\mathfrak{g}^*_{x_0}$.*

Hence, if one could prove that every Poisson structure π nearby $\mathfrak{g}^*$ must have a zero x_0 with isotropy Lie algebra $\mathfrak{g}_{x_0} \simeq \mathfrak{g}$, then, around x_0, π would look like its linearization $\mathfrak{g}^*$. Therefore, one is led naturally to the study of the *stability* of zeros of Poisson structures:

Definition 4.1. A fixed point x_0 of a Poisson structure $\pi \in \mathrm{Poiss}(M)$ is said to be **stable** if, given any neighborhood V of x_0 in M, there is a neighborhood U of π in $\mathrm{Poiss}(M)$, such that each Poisson structure $\theta \in U$ has a fixed point in V.

In other words, a fixed point of π is stable if all nearby Poisson structures have nearby fixed points. We have the following criterion for stability of fixed points:

Theorem 4.2. *Let π be a Poisson structure with a fixed point $x_0 \in M$, and denote by $\mathfrak{g}_{x_0}$ the isotropy Lie algebra at x_0. If $H^2(\mathfrak{g}_{x_0}) = 0$ then x_0 is a stable fixed point.*

For a proof of this result we refer to [12]. In fact, there we give a more precise version of the theorem showing that, under the assumption $H^2(\mathfrak{g}_{x_0}) = 0$, the set of fixed points of π in a neighborhood of x_0 is a submanifold tangent to the fixed point set of the linear approximation to π at x_0, which is stable under perturbations of the Poisson structure.

The condition in the theorem is, in some sense, not too far from being a necessary condition. This can be seen by considering the case of *linear* Poisson structures. For these we can prove:

Proposition 4.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and take $M = \mathfrak{g}^*$ with the linear Lie-Poisson bracket. If the origin is a stable fixed point then $H^2(\mathfrak{g}) = 0$.*

Proof. Let us denote by π_0 the Lie-Poisson bracket on $\mathfrak{g}^*$, so that:

$$\pi_0(\mathrm{d}f, \mathrm{d}g)(\xi) = -\langle \xi, [\mathrm{d}_\xi f, \mathrm{d}_\xi g] \rangle,$$

for any $\xi \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$. A cohomology class $[c] \in H^2(\mathfrak{g})$ is represented by a skew-symmetric map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Consider the associated (constant) bivector field on $\mathfrak{g}^*$ defined by:

$$\pi_1(\mathrm{d}f, \mathrm{d}g)(\xi) = c(\mathrm{d}_\xi f, \mathrm{d}_\xi g),$$

for any $\xi \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$. Since π_1 is constant, it defines a Poisson bivector field, i.e., the Schouten bracket with itself vanishes: $[\pi_1, \pi_1] = 0$. Denote by $\delta : \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1} \mathfrak{g}^*$ the Chevalley-Eilenberg differential. Then:

$$\delta c = 0 \quad \Longleftrightarrow \quad [\pi_0, \pi_1] = 0,$$

and we conclude that the 1-parameter family of bivector fields $\pi_t = \pi_0 + t\pi_1$ is a pencil of Poisson structures.

Assume now that the origin is a stable fixed point point for π_0. Then for $t > 0$, π_t must also have a fixed point $\xi \in \mathfrak{g}^*$, and we find:

$$\pi_t(\xi) = \pi_0(\xi) + t\pi_1(\xi) = 0 \quad \Longleftrightarrow \quad tc = \delta\xi,$$

so c must be a coboundary. Hence, if the origin is a stable fixed point, we must have $H^2(\mathfrak{g}) = 0$. □

As a corollary of Proposition 4.1 and Theorem 4.2 we obtain:

Corollary 4.1. *For a linear Poisson structure $M = \mathfrak{g}^*$ the origin is a stable fixed point if and only if $H^2(\mathfrak{g}) = 0$.*

Example 4.1. Let π be a Poisson structure on a manifold M, and let x_0 be a fixed point of π for which the isotropy Lie algebra $\mathfrak{g}_{x_0}$ is semisimple. By the Second Whitehead Lemma, we have $H^2(\mathfrak{g}_{x_0}) = 0$, so x_0 is a stable fixed point.

Remark 4.1. It was pointed out to us by Jean-Paul Dufour, that our criterion for fixed points can be generalized to higher degree singularities (see [13]).

Fixed points of a Poisson structure are symplectic leaves of dimension zero. The problem of stability of fixed points naturally generalizes to higher dimensional symplectic leaves. This is made precise by the following definition:

Definition 4.2. A symplectic leaf $S \subset M$ of a Poisson structure π is said to be a **stable leaf** if, for every tubular neighborhood V of S in M, there is a neighborhood U of π in $\mathrm{Poiss}(M)$, such that each Poisson structure $\theta \in U$ has a leaf contained in V which is mapped diffeomorphically onto S by the tubular neighborhood projection.

In the case where $\dim S = 0$, i.e., $S = \{x_0\}$ is a fixed point, this definition reduces to the definition of stability of a fixed point given before.

We conjecture that our criterion for stability of fixed points (Theorem 4.2) can be generalized to leaves. For that recall the *relative Poisson cohomology* of a symplectic leaf S of (M, π) due to Ginzburg and Lu (see [15]): one considers the complex $(\mathfrak{X}^k(M,S), \delta)$ where:

$$\mathfrak{X}^k(M,S) = \Gamma(\wedge^k T^*_S M),$$

is the space of k-multivector fields along the symplectic leaf S, and $\delta : \mathfrak{X}^\bullet(M,S) \to \mathfrak{X}^{\bullet+1}(M,S)$ is given by:

$$\delta\theta = [\theta, \pi].$$

Since π is tangent to S, one checks easily that δ is well-defined. On the other hand, the vanishing of the Schouten bracket $[\pi,\pi] = 0$ implies that $\delta^2 = 0$, so δ is a differential. One calls the cohomology of the resulting complex the **relative Poisson cohomology** of the leaf S, and denotes it by $H^\bullet_\pi(M,S)$. In terms of Lie algebroids, $H^\bullet_\pi(M,S)$ coincides with the Lie algebroid cohomology of the restriction $T^*_S M$.

Example 4.2. If $\dim S = 0$, so that $S = \{x_0\}$ is a fixed point, it follows from the definitions that the relative Poisson cohomology is just the Lie algebra cohomology of the isotropy Lie algebra at x_0: $H^\bullet_\pi(M,S) = H^\bullet_S(\mathfrak{g}_{x_0})$.

In this example, $H^\bullet_\pi(M,S)$ is finite dimensional. In fact, one can show that the complex $(\mathfrak{X}^k(M,S),\delta)$ is elliptic, hence the relative Poisson cohomology is finite dimensional provided S is of finite type (e.g., if S is compact). This is in sharp contrast with the absolute Poisson cohomology, which is often infinite dimensional.

Now we pose the following natural conjecture:

Conjecture 4.1. *Let π be a Poisson structure with a compact symplectic leaf $S\subset M$. If the 2nd Poisson cohomology relative to S vanishes, i.e., $H^2_\pi(M,S)=0$, then S is a stable symplectic leaf.*

Further evidence for this conjecture is given in [12].

One may consider stronger notions of stability. For example, we may call a symplectic leaf S **strongly stable** if all nearby Poisson structures have nearby symplectic leaves which are symplectomorphic (rather than just diffeomorphic) to L, or one can even require the restricted Lie algebroids T^*_SM to be isomorphic. In the case of fixed points, this would mean that nearby fixed points should have isomorphic isotropy Lie algebras. These notions of stability seem to be much harder to study.

Remark 4.2. For group (or Lie algebra) actions on a manifold M the situation is similar, as one may consider, also, two notions of stability: an orbit $O\subset M$ of an action is called *stable* (respectively, *strongly stable*) if every nearby action has a nearby leaf which is diffeomorphic to (respectively, has the same orbit type as) S. While the stability of orbits has been studied extensively (cf. [22]), little seems to be known about strong stability.

Similarly, for foliations of a manifold M of codimension k, one also has two notions of stability: a leaf $L\subset M$ of a foliation $\mathcal{F}$ is called *stable* (respectively, *strongly stable*) if every nearby foliation has a nearby leaf which is diffeomorphic to (respectively, has the same holonomy as) L. While the stability of leafs has been studied extensively (cf. the classical stability results of Reeb [21], Thurston [23], Langevin and Rosenberg [19]), little seems to be known about strong stability.

To finish, motivated by Conn's linearization result, we would like to formulate a conjecture on linearization around leafs in the sense of Vorobjev ([24]):

Conjecture 4.2. *Let π be a Poisson structure with a compact symplectic leaf $S\subset M$. If the restricted Lie algebroid T^*_SM integrates to a compact, source 1-connected, Lie algebroid, then π is linearizable around S.*

One should be able to prove this conjecture using the methods of [11]. One can also look into milder properties of a leaf which are shared by (some) nearby leaves. For example, given a leaf L, one may ask if all nearby Poisson structures have a a nearby leaf of the same dimension, etc. We refer the reader to the upcoming paper [12] for a detailed discussion of the problem of stability of symplectic leaves.

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