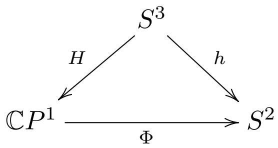
EXERCISE GIVEN ON OCTOBER 4

Consider $H : S^3 \to \mathbb{C}P^1$ and $h : S^3 \to S^2$ as in the previous homework and

$$\Phi : \mathbb{C}P^1 \to S^2, \quad \Phi([z_0 : z_1]) = h(z_0, z_1) \quad (\text{for } (z_0, z_1) \in S^3)$$



For simple formulas, h was using complex numbers. Using real coordinates, it is

$$h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

On S^3 we define the following vector fields:

$$V^1 := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}.$$

$$V^2 := -z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t}.$$

$$V^3 := -t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}.$$

More precisely: V^1 associates to each point $p = (x, y, z, t) \in S^3$ the tangent vector

$$V^1_p := -y\left(\frac{\partial}{\partial x}\right)_p + x\left(\frac{\partial}{\partial y}\right)_p + t\left(\frac{\partial}{\partial z}\right)_p - z\left(\frac{\partial}{\partial t}\right)_p \quad (\text{and similarly for } V^2 \text{ and } V^3).$$

Remark 1. Let $F : M \to N$ be a map between two manifolds, where M is a submanifold of some $\mathbb{R}^k$. If F is the restriction of a smooth map $\tilde{F} : \mathbb{R}^k \to N$, then F is itself smooth. Moreover, if N is a submanifold of some $\mathbb{R}^l$, then F (or $\tilde{F}$) is smooth if and only if it is smooth as a map with values in $\mathbb{R}^l$.

Exercise 1. Show that:

(1) using the previous remark and without using any other charts than the two describing the smooth structure on $\mathbb{C}P^1$, show that h, H and Φ are smooth.
(2) show that, for any $p \in S^3$, $\{V^1_p, V^2_p, V^3_p\}$ is a basis of T_pS^3.
(3) show that each V^i is h-projectable, i.e. there exist three vector fields E^1, E^2 and E^3 on S^2 (which you have to find explicitly) such that:

$$(dh)_p(V^i_p) = E^i_{h(p)} \quad \text{for all } p \in S^3.$$

(4) using the formulas you found for E^1, E^2, E^3, deduce that h is a submersion.
(5) without any further computations deduce that Φ is a local diffeomorphism (hint1: the chain rule+ the inverse fct thm; hint2: a dimension argument).
(6) assuming you already know that Φ is bijective, show it is a diffeomorphism.
(7) if F is a fiber of h and $p = (z_0, z_1) \in F$, using the previous diagram (and no further computations), show that F is the image of the map:

$$\sigma : S^1 \to S^3, \quad \sigma(\lambda) = (\lambda \cdot z_0, \lambda \cdot z_1).$$

(8) then show that the curve $\gamma : \mathbb{R} \to S^3$, $\gamma(s) = \sigma(e^{is})$ satisfies $\frac{d\gamma}{ds}(s) = V_{\gamma(s)}$, where $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$ (a vector field on S^3).
(9) using that $(d\gamma)_s(\frac{d}{ds}) = \frac{d\gamma}{ds}(s)$ deduce that γ is an immersion. Then deduce that σ is a diffeomorphism (hint for the last part: chain rule + dimension).