EXERCISE GIVEN ON OCTOBER 11

Here we make use of the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in $\mathbb{H}$ is uniquely determined by the fact that it is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. We will identify S^3 with the subspace of $\mathbb{H}$ consisting of elements of norm one:

$$S^3 = \{u \in \mathbb{H} : |u| = 1\},$$

where recall that the norm of a quaternion

$$u = x + iy + jz + kt \in \mathbb{H}$$

is

$$|u| = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

The basic property $|u \cdot v| = |u| \cdot |v|$ implies that the product in $\mathbb{H}$ restricts to a product

$$S^3 \times S^3 \to S^3, \quad (u, v) \mapsto u \cdot v$$

or, more explicitly,

$$(x + iy + jz + kt) \cdot (x' + iy' + jz' + kt') = xx' - yy' - zz' - tt' + i(\ldots) + \ldots .$$

This makes S^3 into a Lie group (take that for granted), with the identity $1 \in \mathbb{H}$.

Exercise 1. *Remark first that*

$$e_1 = \left(\frac{\partial}{\partial y}\right)_1, \quad e_2 = \left(\frac{\partial}{\partial z}\right)_1, e_3 = \left(\frac{\partial}{\partial t}\right)_1$$

form a basis of the Lie algebra $\mathfrak{g} = T_1 S^3$ of S^3. Then:

(1) *Show that the resulting vector fields $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ on S^3 coincide with the vector fields on S^3 from the previous homework:*

$$V^1 := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t},$$
$$V^2 := -z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t},$$
$$V^3 := -t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}.$$

(2) *Compute the Lie brackets $[e_1, e_2]$, $[e_2, e_3]$ and $[e_3, e_1]$ in the Lie algebra $\mathfrak{g}$.*

(3) *Compute the flow $\phi^t_{V^1}$ of V^1 explicitly.*
(for nice final formulas, use complex numbers at the end).

Remark: Keep in mind that, if $f : M \to N$ is a smooth map which is the restriction of a smooth map $\tilde{f} : \tilde{M} \to \tilde{N}$, with $M \subset \tilde{M}$ and $N \subset \tilde{N}$ submanifolds, then, for $p \in M$, $(df)_p : T_pM \to T_{f(p)}N$ is the restriction of $(d\tilde{f})_p : T_p\tilde{M} \to T_{f(p)}\tilde{N}$ to the subspace $T_pM \subset T_p\tilde{M}$.