EXERCISE GIVEN ON SEPTEMBER 26 (TO BE HANDED IN ON OCTOBER 4TH)

Recall that the m-dimensional complex projective space $\mathbb{C}P^m$ is the collection of all complex lines in $\mathbb{C}^{m+1}$ passing through the origin (i.e. complex vector subspaces of $\mathbb{C}^{m+1}$ of complex dimension 1). To work with it, we write

$$\mathbb{C}P^m = \{[z_0 : z_1 : \ldots : z_m] : (z_0, z_1, \ldots, z_m) \in \mathbb{C}^m \setminus \{0\}$$

where $[z_0 : z_1 : \ldots : z_m]$ is just a notation for the line through the origin and the point $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Hence

$$[z_0 : z_1 : \ldots : z_m] = [\lambda \cdot z_0 : \lambda \cdot z_1 : \ldots : \lambda \cdot z_m] \quad \text{for } \lambda \in \mathbb{C}^*.$$

Recall that $\mathbb{C}P^m$ is a manifold with an atlas given by

$$U_i = \{[z_0 : z_1 : \ldots : z_m] \in \mathbb{C}P^m : z_i \neq 0\},$$

$$\chi^i : U_i \to \mathbb{C}^m = \mathbb{R}^{2m}, \quad \chi^i([z_0 : z_1 : \ldots : z_m]) = (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}).$$

Using $\mathbb{R}^{2m} = \mathbb{C}^m$ we also represent the $(2m+1)$-dimensional sphere as:

$$S^{2m+1} = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} : |z_0|^2 + \ldots + |z_m|^2 = 1\};$$

there is an obvious map

$$H : S^{2m+1} \to \mathbb{C}P^m, \quad H(z_0, \ldots, z_m) = [z_0 : \ldots : z_m].$$

We now look at the particular case when $m = 1$:

$$\boxed{H : S^3 \to \mathbb{C}P^1, \quad H(z_0, z_1) = [z_0 : z_1].}$$

And we also consider

$$\boxed{h : S^3 \to S^2, \quad h(z_0, z_1) := (|z_0|^2 - |z_1|^2, 2i \cdot \overline{z_0} \cdot z_1).}$$

(btw: both H and h are related to the name Hopf (... fibration)).

Exercise 1. Show that:

(1) H is a smooth submersion.

(2) h is well defined and it is a smooth submersion.

(3) $\mathbb{C}P^1$ is diffeomorphic to S^2.

(4) each fiber of h is an embedded submanifold of S^3 which is diffeomorphic to a circle.