LAST HOMEWORK (TO BE HANDED IN RIGHT AT THE START OF THE EXAM)

Remember the map $h : S^3 \to S^2$ given by

$$h(z_0, z_1) := (|z_0|^2 - |z_1|^2, 2i \cdot \overline{z_0} \cdot z_1)$$

in complex coordinates or, using real coordinates,

$$h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

We will denote by $h_1, h_2, h_3 \in \mathcal{C}^\infty(S^3)$ these three components of h.

And remember the vector fields on S^3 given by

$$V^1 := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t},$$

$$V^2 := -z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t},$$

$$V^3 := -t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}.$$

as well as a slightly different one:

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

The last one was considered in the homework from October 8, part (8); actually that part tells us what the flow of V,

$$\phi_V^s : S^3 \to S^3$$

is (here we use the notation s for the time parameter, since we already use t for the last coordinate in $\mathbb{R}^4$!). Using complex coordinates for $(z_0, z_1) \in S^3$ we have (according to that exercise)

$$\phi_V^s(z_0, z_1) = (e^{is} \cdot z_0, e^{is} \cdot z_1).$$

Or, using real coordinates $(x, y, z, t) \in \mathbb{R}^4$,

$$\phi_V^s(x, y, z, t) = (x\cos s - y\sin s, x\sin s + y\cos s, z\cos s - t\sin s, z\sin s + t\cos s).$$

(I am pointing out the flow of V since below, although it is not absolutely necessary, you may want to use it).

While we have seen that V^1, V^2, V^3 form, at each point in S^3, a basis of the tangent space, we pass to the dual basis and we consider the resulting 1-forms

$$\theta_1, \theta_2, \theta_3 \in \Omega^1(S^3).$$

Hence they are uniquely determined by the condition that $\theta_i(V^j) = \delta_j^i$ (1 if $i = j$ and 0 otherwise).

Exercise 1. Do the following:

(1) Show that the θ_is are given by formulas similar to V^is:
$$\theta_1 := -y\,dx + x\,dy + t\,dz - z\,dt,$$
$$\theta_2 := -z\,dx - t\,dy + x\,dz + y\,dt,$$
$$\theta_3 := -t\,dx + z\,dy - y\,dz + x\,dt$$
(where, according to our conventions, dx and the other forms on the right hand side are the restrictions from $\mathbb{R}^4$ to S^3- just that we do not write $dx|_{S^3}$ etc all the time).

(2) Show that:
$$d\theta_1 = -2\theta_2 \wedge \theta_3, \quad d\theta_2 = -2\theta_3 \wedge \theta_1, \quad d\theta_3 = -2\theta_1 \wedge \theta_2.$$
(here, for a simpler argument, you may want to use what you should have computed before: that $[V^1, V^2] = 2V^3, [V^2, V^3] = 2V^1, [V^3, V^1] = 2V_2$).

(3) Show that
$$L_V(\theta_1) = L_V(\theta_2) = L_V(\theta_3) = 0.$$

(4) Remark that the components of h are precisely
$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3)$$
and then prove that
$$dh_1 = 2h_2 \cdot \theta_3 - 2h_3 \cdot \theta_2,$$
$$dh_2 = 2h_3 \cdot \theta_1 - 2h_1 \cdot \theta_3,$$
$$dh_3 = 2h_1 \cdot \theta_2 - 2h_2 \cdot \theta_1$$

(5) Show that $dh_1 \wedge dh_2 \wedge dh_3 = 0$.

(6) Now prove the previous point directly (no computation at all!) by remarking that $dh_1 \wedge dh_2 \wedge dh_3 = h^*(dx \wedge dy \wedge dz)$ (the proof is a one line remark!).

(7) Consider the volume form on S^3:
$$\mu := \theta_1 \wedge \theta_2 \wedge \theta_3 \in \Omega^3(S^3).$$
Since this is a volume form (by construction), any other 3-form on S^3 is of type $\phi \cdot \mu$ for some $\phi \in \mathcal{C}^\infty(S^3)$. Find ϕ such that
$$dx \wedge dy \wedge dz = \phi \cdot \mu$$
(again, $dx \wedge dy \wedge dz$ is the restriction to S^3). (Hint: $\mu(V^1, V^2, V^3) = 1$).

(8) Consider the chart
$$\chi : S^3 \setminus \{p_N\} \to \mathbb{R}^3$$
given by the stereographic projection w.r.t. the north pole- hence
$$\chi_1(x, y, z, t) = \frac{x}{1-t}, \quad \chi_2(x, y, z, t) = \frac{y}{1-t}, \quad \chi_3(x, y, z, t) = \frac{z}{1-t}.$$
Compute the representing function of μ w.r.t. this chart, i.e. the function $f \in \mathcal{C}^\infty(\mathbb{R}^3)$ with the property that
$$\mu|_{S^3\setminus\{p_N\}} = f \circ \chi \cdot d\chi_1 \wedge d\chi_2 \wedge d\chi_3.$$
(here you may want to remember that $t = \frac{||\chi||^2-1}{||\chi||^2+1}$).

(9) Compute $\int_{S^3} \mu$.
(Hint: either use the previous point or try to use Stokes).