7. Exercises given on September 12:

First solve exercise 2.1. Then the following:

Exercise 2.13. Consider two smooth functions

$$U \xrightarrow{f} U' \xrightarrow{g} \mathbb{R}^p,$$

defined on opens $U \subset \mathbb{R}^m$, $U' \subset \mathbb{R}^n$. Using the interpretation of linear maps as matrices (as made precise on page 23) show that the chain rule becomes:

$$\frac{\partial g \circ f}{\partial x_i}(x) = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k}(f(p))\frac{\partial f_k}{\partial x_i}(x)$$

for all $x \in U$ and $1 \leq i \leq m$.

Exercise 2.14. Show that for any function $g : R \to \mathbb{R}$, the function

$$\tilde{g} : \mathbb{R}^m \to \mathbb{R}, \quad \tilde{g}(x_1, \ldots, x_m) = g((x_1)^2 + \ldots + (x_m)^2)$$

is not a submersion at $x = 0$.

Exercise 2.15. Assume that $f : U_0 \to \mathbb{R}^n$ is a smooth map, $U \subset \mathbb{R}^m$ open, $p \in U$. Let

$$\chi : U \to \Omega \subset \mathbb{R}^m, \quad \chi' : U' \to \Omega' \subset \mathbb{R}^n$$

be charts, of $\mathbb{R}^m$ around p and of $\mathbb{R}^n$ around $f(p)$, respectively. What is the (maximal) domain of definition of $f_{\chi}^{\chi'}$?

Exercise 2.16. Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{x^2 + 2xy + 2y^2}$$

and look around $p = (1, 0)$. Find a chart χ of $\mathbb{R}^2$ around p and a chart χ of $\mathbb{R}$ around $g(p) = 1$ such that, w.r.t. these charts,

$$f_{\chi}^{\chi'}(u, v) = u^2 + v^2.$$

Exercise 2.17. Show that

$$g : \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (\cos(t), \sin(t))$$

is an immersion at each point. Then, looking around $t = 0$, find a chart χ' of $\mathbb{R}^2$ around $f(0) = (1, 0)$ such that, w.r.t. this chart,

$$f_{\chi'}(t) = (t, 0).$$

Exercise 2.18. Show that if $f : U \to \mathbb{R}^n$, $p \in U$ satisfy the conclusion of the submersion theorem, then f must be a submersion at p. Similarly for the immersion theorem.
(Hint: try it! If it really doesn't work, then look at the next exercise).

Exercise 2.19. Assume that $f : U \to \mathbb{R}^n$ is a smooth map, $U \subset \mathbb{R}^m$ open, $p \in U$. Let χ be a chart of $\mathbb{R}^m$ around p and let χ' be a chart of $\mathbb{R}^n$ around $f(p)$. Show that f is a submersion/immersion at p if and only if $f_{\chi}^{\chi'}$ is a submersion/immersion at $\chi(p)$.

Exercise 2.20. Consider the stereographic projection w.r.t. the north pole p_N, denoted

$$\chi_N : S^2 \setminus \{p_N\} \to \mathbb{R}^2$$

and similarly the one w.r.t. the south pole, denoted χ_S. Show that

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

is a diffeomorphism.

Exercise 2.21. Don't forget to read the proof of theorem 2.12 (short) and do Exercise 2.3.

Exercise 2.22. For any $\epsilon > 0$ describe a smooth function

$$f : \mathbb{R}^n \to [0, 1]$$

with the property that $f(0) > 0$ and whose support (in $\mathbb{R}^m$) is contained in the ball $\{x \in \mathbb{R}^n : ||x|| < \epsilon\}$.
(Hint: page 24- but give the precise formulas/details).

Exercise 2.23 (HOMEWORK). *Assume that $f : U \to U'$ and $g : U' \to U''$ are two smooth functions, with $U \subset \mathbb{R}^m$, $U' \subset \mathbb{R}^n$ and $U'' \subset \mathbb{R}^p$ opens. Show that if $g \circ f$ is a local diffeomorphism around a given point $x \in U$, then:*

1. *f is an immersion at x and g is a submersion at $f(x)$.*
2. *however, it may happen that f is not a submersion at x and g is not an immersion at $g(x)$ (describe an example!).*
3. *if, furthermore, f is a submersion at x or g is an immersion at $f(x)$, then both f and g are local diffeomorphisms (around x and $f(x)$, respectively).*