11. Homework for September 27

In general, for any two manifolds M and N of dimensions M and n, their product has a canonical structure of $(m+n)$-dimensional manifold: if $\chi : U \to \Omega \subset \mathbb{R}^m$ is a smooth chart of M and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ one of N, then

$$\chi \times \chi' : U \times U' \to \Omega \times \Omega' \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}, \quad (p,q) \mapsto (\chi(p), \chi'(q))$$

will define a smooth chart of $M \times N$ (we do not ask you to prove anything here).

Take now $M = N = S^1$ the unit circle with the standard smooth structure, and we consider the resulting manifold $S^1 \times S^1$. While $S^1 \subset \mathbb{R}^2$, $S^1 \times S^1$ sits naturally inside $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. We now want to embed it in $\mathbb{R}^3$, i.e. find a submanifold T of $\mathbb{R}^3$ that is diffeomorphic to $S^1 \times S^1$.

EXERCISE 3.19. *We define $T \subset \mathbb{R}^3$ as the image of the map:*

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(\alpha, \beta) = ((2 + \cos\alpha)\cos\beta, (2 + \cos\alpha)\sin\beta, \sin\alpha) .$$

Do the following:

(1) *write T as $g^{-1}(0)$ (the zero-set of g) for some smooth map $g : \mathbb{R}^3 \to \mathbb{R}$ which is a submersion at each point in T. This ensures that T is indeed a submanifold of $\mathbb{R}^3$.*

(2) *show that f is an immersion.*

(3) *using f find a homeomorphism $f_0 : S^1 \times S^1 \to T$ and prove that it is an immersion.*

(4) *then show that $f_0 : S^1 \times S^1 \to T$ is a diffeomorphism.*

(5) *if you haven't done that already: draw a picture of T inside $\mathbb{R}^3$.*

(Hint: try to avoid ugly computations and use what we have already discussed in the class).