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Chapter 1
Reminders on Topology and Analysis

1.1 Reminder 1: TOPOLOGY

Here is a very brief reminder on the basic notions from Topology. For those which are not so familiar with these basics, one may skip the later parts of this section (most notably the part on partitions of unity) and return to it later on, when necessary.

1.1.1 Topological spaces and continuous maps

A **The objects of Topology**

B **The maps and isomorphisms of Topology**

C **Metric topologies; bases**

D **Topological manifolds**

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A. The objects of Topology

First of all, the main objects of Topology: **a topological space** is a set X endowed with a **topology**, i.e. a collection $\mathscr{T}$ of subsets of X (called **the opens of the topological space**, or simply **opens in** X) such that $\emptyset$ and X are open in X, arbitrary unions of opens are open and finite intersections of opens are open. We usually omit $\mathscr{T}$ from the notations, and we simply say that X is a topological space; hence that means that X is a set and we can talk about the subsets of X that are open (in X).

A topology on X allows us to make sense of the central phenomena of Topology: "two points being close to each other". First of all we can make sense of neighborhoods in a topological space X: given $x \in X$, a **neighborhood** (in X) of x is any subset $V \subset X$ that contains at least an open neighborhood of x, i.e. an open U with $x \in U$. In turn, this allows us to talk about convergence: a sequence $(x_n)_{n \geq 1}$ of elements of X **converges** (in the topological space X) to $x \in X$ if for any neighborhood V of x there exists an integer n_V such that $x_n \in V$ for all $n \geq n_V$.

The notion of neighborhoods also allows to talk also about an important property one requires on topological spaces in order to exclude pathological examples- Hausdorffness: a topological space X called **Hausdorff** if for any $x, y \in X$ distinct, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$. Hence, intuitively, this means that "if two are distinct, then they cannot be too close to each other" (yes, not having this sounds pathological but, since this condition is not automatic, it is often imposed precisely to avoid "strange/pathological spaces).

B. The maps and isomorphisms of Topology

The relevant maps (the only ones that really matter) in Topology are the continuous ones: a map $F : X \to Y$ between topological spaces is called **continuous** if for any U-open in Y, its pre-image $F^{-1}(U)$ is open in X. "Isomorphism" between topological spaces are known under the name of **homeomorphisms**: they are the bijections $F : X \to Y$ with the property that both F as well as F^{-1} are continuous.

In the language of "Category Theory", Topology is the category whose objects are topological spaces, and whose morphisms (between objects) are the continuous maps.

Remark 1.1. Note that, while proving that two topological spaces are homeomorphic (i.e there exists a homeomorphism between them) is relatively easy in principle (one just has to produce ONE single homeomorphism between them- and for that it is often enough to follow ones intuition), proving that two spaces are not homeomorphic is much harder. One way to proceed is by understanding the specific "topological properties" of the spaces under discussion (such as Hausdorffness, compactness, etc); if one of them has such a topological property and the other one does not, then they cannot be homeomorphic. A more advanced approach consists of constructing topological invariants of algebraic nature (such as numbers, groups, etc)- and that is what Algebraic Topology is about.

C. Metric topologies; bases

One of the largest class of topological spaces are metric spaces (X,d): any metric $d : X \times X \to \mathbb{R}$ induces a topology $\mathscr{T}_d$ on X: a subset $U \subset X$ is open iff for any $x \in U$ there exists $r > 0$ such that U contains the d-ball

$$B_d(x,r) := \{y \in X : d(x,y) < r\}. \tag{1.1.1}$$

In general, a topological space X is called **metrizable** if there is a metric d on X such that the original topology on X coincides with $\mathscr{T}_d$ (note also that, if such a d exists, in general it is far from being unique; e.g. already $2d$, $3d$, $\frac{d}{d+1}$ would do the same job). One of the basic questions about topological spaces is how to recognize which are metrizable; **metrizability theorems** aim at finding simple topological conditions that imply metrizability.

Considering the Euclidean metric d on $\mathbb{R}^m$, or on any subset $A \subset \mathbb{R}^m$, we see that A is endowed with a canonical topology- called **the Euclidean topology** on the subset $A \subset \mathbb{R}^m$ (exercise: show that also the square metric induces the same topology). Note that, in this case, the resulting notion of convergence (and continuity) coincides with the one from Analysis.

Remark 1.2. Continuing the previous remark, let us point out that showing that two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of different dimensions $m \neq n$ are not homeomorphic is non-trivial. When $m = 1$, this can be done using the notion of connectedness but, for $m,n \geq 2$, one has to appeal to tools from Algebraic Topology.

We now return to general metric spaces. A metric d on X allows us to talk about the open balls $B_d(x,r)$ for $x \in X$, $r \in \mathbb{R}_+$ (see (1.1.1)), giving rise to the collection of open balls induced by d:

$$\mathscr{B}_d = \{B_d(x,r) : x \in X, r \in \mathbb{R}_+\}.$$

This is not a topology on X, but $\mathscr{T}_d$ is the smallest topology containing $\mathscr{B}_d$.

Recall also (simple exercise) that any family $\mathscr{B}$ of subsets of X gives rise to a topology $\mathscr{T}(\mathscr{B})$ on X, defined as the smallest one containing $\mathscr{B}$. It is called **the topology generated by** $\mathscr{B}$. In general, the members of $\mathscr{T}(\mathscr{B})$ are arbitrary unions of finite intersections of members of $\mathscr{T}$.

Depending on the properties of $\mathscr{B}$, the members of $\mathscr{T}(\mathscr{B})$ may have simpler descriptions. The most common case is when $\mathscr{B}$ is a **topology basis**, i.e. satisfies the following axioms: any $x \in X$ is contained in at least one member B of $\mathscr{B}$ and, for any $B_1, B_2 \in \mathscr{B}$ and any $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ containing x with $B \subset B_1 \cap B_2$. In this case, for $U \subset X$, the following are equivalent:

(0) U belongs to $\mathscr{T}(\mathscr{B})$.
(1) for any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) U is a union of members of $\mathscr{B}$.

For instance, the collection $\mathscr{B}_d$ associated to a metric d is a topology basis, and (1) is the original definition of $\mathscr{T}_d$.

One can change a bit the point of view and, starting with a topology $\mathscr{T}$ on X, look for collections $\mathscr{B}$ generating $\mathscr{T}$, i.e. such that $\mathscr{T} = \mathscr{T}(\mathscr{B})$. Of course, one possibility is to take $\mathscr{B} = \mathscr{T}$, but that is the least interesting one. The more interesting choices are the ones for which $\mathscr{B}$ is smaller- e.g. countable. And here is the precise terminology: given a topological space X, a **basis for the topological space** X is any collection $\mathscr{B}$ of subsets of X such that it is topology basis and $\mathscr{T} = \mathscr{T}(\mathscr{B})$. As above, for a collection $\mathscr{B}$ of subsets of X, the following are equivalent:

(0) $\mathscr{B}$ is a basis for the space X.
(1) for any open U in X and any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) any open in X is a union of members of $\mathscr{B}$.

(in particular, each of the conditions (1) and (2) imply that $\mathscr{B}$ is a topology basis).

Repeating what we said before, but with a slightly different wording, we have that the metric topology associated to a metric d admits $\mathscr{B}_d$ as basis. Another possible basis for the space $(X, \mathscr{T}_d)$, slightly smaller, is

$$\mathscr{B}_d^{\mathbb{N}} = \left\{ B_d\left(x, \frac{1}{n}\right) : x \in X, n \in \mathbb{N} \right\}.$$

For the Euclidean metric d_{Eucl} on $\mathbb{R}^m$ we can do even better:

$$\mathscr{B}_d^{\mathbb{Q}} := \left\{ B_{d_{\text{Eucl}}}\left(q, \frac{1}{n}\right) : q \in \mathbb{Q}^m, n \in \mathbb{N} \right\}$$

is still a basis for the Euclidean topology on $\mathbb{R}^m$, but it it "much smaller": it is countable.

In general, one says that a topological space X is **2nd countable** if it admits a basis $\mathscr{B}$ which is countable.

D. Topological manifolds

The second countability condition is a very subtle one and turns out to be of capital importance in establishing some central results in Topology and Geometry- such as metrizability and embedding theorems. In particular, it is part of the basic axioms for the notion of manifolds. For now:

Definition 1.3. A **topological *m*-dimensional manifold** is a topological space X satisfying:

(TM0): any point $x \in X$ admits a neighborhood X which is homeomorphic to an open subset of $\mathbb{R}^m$.
(TM1): it is Hausdorff.
(TM2): it is 2nd countable.

A homeomorphism
$$\chi : U \to \Omega \subset \mathbb{R}^m$$
from an open subset U of X to an open subset Ω in $\mathbb{R}^m$ is called a ***m*-dimensional topological chart for** X, and U is called the domain of the chart- so that axiom (TM0) can also be read as:

(TM0): X can be covered by (domains) of m-dimensional topological charts.

You should convince yourself (or remember) why some of the usual examples of topological spaces such as spheres, tori, Moebius band, etc are topological manifolds. Also, in all these examples, one should concentrate first on the condition (TM0) (... as the labelling indicates). Note however that, while the notion of dimension is intuitively clear (at least in all examples), handling it theoretically is not such a piece of cake; see Remark 1.2. This is due to the fact that there is no obvious topological characterization of the (intuitive notion) of dimension. This will be much less of a problem as soon as we move to (differentiable) manifolds.

Remark 1.4. Since the notion of "topological space" is built on the notion of "open", so are most of the basic definitions in Topology- such as continuity, Hausdorffness, compactness, etc etc. However, under rather mild assumptions, such definitions can be rephrased more intuitively, using sequences. The main "mild assumption" that we have in mind here is that of "first countability"; please see the basic course on Topology. This condition is weaker even than the second countability condition. For instance, metric topologies are always first countable but may fail to be 2nd countable. For our purpose, it is enough to know that either of the conditions (TM0) or (TM2) implies 1st countability (and, if you look at the definitions, you will see that this statement is completely trivial).

What is interesting to know here is that, when restricting to spaces X which are first countable, many of the basic notions can be reformulated in terms of sequences. E.g.:

- X is Hausdorff iff any convergent sequence in X has at most one limit.
- $F : X \to Y$ is continuous iff it is sequential continuous i.e.: if $(x_n)_{n\geq 1}$ is a sequence converging in X to $x \in X$, then $(f(x_n))_{n\geq 1}$ converges in Y to $F(x)$.

E. Inside a topological space

Recall that, given a space X, a subset $A \subset X$ is said to be **closed in** X if its complement $X \setminus A$ is open. Of course, knowing the closed subsets of X is equivalent to knowing the open ones- hence one could have introduced the notion of topology completely in terms of closed subsets (which would then be the axioms?). Opens are preferred because some of the the most important properties can be described more directly in terms of opens (and perhaps also because they are closer in spirit to the notion of "ball" in a metric space). However, closed subsets often have some very nice properties- e.g. when talking about compactness.

Given the axioms of a topology (the fact that arbitrary unions of opens is open or, equivalently, that arbitrary intersections of closed is closed), it follows that for any subset A of a topological space X one can talk about:

- the largest open contained in A- and this is called **the interior of** A (in the space X), and denoted $\text{Int}(A)$.
- the smallest closed containing A- and this is called **the closure of** A (in the space X), and denoted $\text{Cl}(A)$,

Recall also that, under the first countability axiom (in particular, for topological manifolds), the closure has a particularly nice description in terms of sequences:

$$\text{Cl}(A) = \{x \in X : \exists \text{ a sequence in } A \text{ converging to } x\}.$$

1.1.2 Constructions and topological properties

A **Construction of topological spaces**

B **Topological properties: connected, compact, normal, locally compact, Baire**

C **Partitions of unity, paracompactness and 2nd countability**

A. Construction of topological spaces

We have already seen two (related) ways of constructing topologies on a set X: the metric topology $\mathscr{T}_d$ induced by any metric d on X, and the topology $\mathscr{T}(\mathscr{B})$ generated by any family $\mathscr{B}$ of subsets of X (with the particularly nice situation when $\mathscr{B}$ is a topology basis).

There are various other important constructions of topologies out of the old ones. For instance, given any two topological spaces X and Y, the Cartesian product

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

carries a canonical topology, called **the product topology**. There is a slight complication: while we would like that the products of opens is open,

$$\mathscr{B}_{X\times Y} := \{U\times V : U - \text{open in } X, V - \text{open in } Y\}$$

is not a topology on $X\times Y$; instead, it is a topology basis, and the product topology is defined as the topology generated by $\mathscr{B}_{X\times Y}$. Equivalently, and more conceptually, it is the smallest topology on $X\times Y$ with the property that the projections

$$\mathrm{pr}_X : X\times Y\to X, \mathrm{pr}_Y : X\times Y\to Y$$

are continuous.

The last description is more conceptual because it follows a general philosophy that one should apply when looking for topologies: require that the most interesting maps that you have around to be continuous, and look for "the best (least boring)" topology that does that (usually "the best" means "the largest" or "the smallest").

Another example of this philosophy is **the induced topology**: given a topological space X, any subset $A\subset X$ carries a canonical, induced, topology: it is the smallest topology with the property that the canonical inclusion

$$i : A\to X, \quad i(a) = a$$

is continuous (why would looking for the largest topology with this property be "boring"?). Explicitly, the opens in A (endowed with the topology induced from X) are the intersections $A\cap U$ of A with opens U of X.

Yet another example is that of **quotient topology**. In some sense, it is the other extreme compared to the previous example. While before we started with an inclusion $I : A\to X$, we now start with a surjection

$$\pi : X\to Y,$$

where X is a topological space and Y is just a set (on which we would like to induce a topology). This time, looking for "the most interesting" topology on Y, we are lead to looking at the largest topology on Y with the property that π is continuous (why?). We obtain the quotient topology on Y: a subset $U\subset Y$ is an open of this topology if an donly if $\pi^{-1}(U)$ is open in X (check that this is, indeed, a topology on Y).

The terminology "quotient" comes from the fact that, typically, the situation of having a surjection $\pi : X\to Y$ arises when starting with X and an equivalence relation R on X. Then, with the intuition that we want to glue the points of X that are equivalent (w.r.t. the equivalence relation R), we obtain the quotient space

$$Y = X/R$$

consisting of R-equivalence classes $[x]_R = \{y\in X : (x,y)\in R\}$ of points $x\in X$, together with the canonical projection

$$\pi_R : X\to X/R, \quad \pi_R(x) = [x]_R.$$

Therefore, starting with an equivalence relation R on a topological space X, we see that the resulting quotient X/R carries a canonical (quotient) topology.

One of the most interesting examples of quotient topologies is the canonical topology on the projective space

$$\mathbb{P}^m := \{l : l - \text{line through the origin in } \mathbb{R}^{m+1}\}$$

(all 1-dimensional vector subspaces of $\mathbb{R}^{m+1}$). We can place ourselves in the previous situation by considering

$$\pi : \mathbb{R}^{m+1}\setminus\{0\}\to\mathbb{P}^m, x\mapsto l_x,$$

where l_x is the line through the origin and x (i.e. the vector subspace $\mathbb{R}\cdot x$ spanned by x). In terms of equivalence relations, we deal with the equivalence relation on $\mathbb{R}^{m+1}\setminus\{0\}$ given by:

$$x\sim y \Longleftrightarrow l_x = l_y \Longleftrightarrow y = \lambda\cdot x \text{ for some } \lambda\in\mathbb{R}.$$

Using the Euclidean topology on $\mathbb{R}^{m+1} \setminus \{0\}$ we obtain a natural topology on $\mathbb{P}^m$; endowed with this topology, $\mathbb{P}^m$ is called **the projective space** (of dimension m). You should convince yourself that convergence in this topology corresponds to the intuitive idea of "lines getting close to each other".

Finally, given the notion of induced topology, one can make use of that of embeddings: an (topological) **embedding** of a topological space X into a topological space Y is any map $i : X \to Y$ that is continuous, injective and, when interpreted as a continuous map $i : X \to i(X)$ and we endow $i(X) \subset Y$ with the topology induced from Y, it is a homemorphism (note that the last map is automatically continuous and bijective, but that does not imply that its inverse is continuous as well!). Next to metrizability theorems (see above), one of the most interesting problems in Topology/Geometry is that of deciding whether a space X can be embedded in a Euclidean space; results in this direction are usually labelled as **embedding theorems**. Looking at the notion of topological manifold, it is worth pointing out that, due also to the axioms (TM1) and (TM2), it follows that any topological manifold is metrizable and can be embedded in some Euclidean space!

B. Topological properties: connected, compact, locally compact, Baire

As we have pointed out in Remark 1.1, to distinguish topological spaces from each other (or to understand better each specific one), it is useful to isolate the various topological properties that spaces may have. By a topological property we mean any property that can be described by only using the notion of opens or, equivalently, any property that is preserved via homeomorphisms. We have already mentioned several such properties: Hausdorffness and second countability. Here we recall a few more.

The first one is that of connectedness: a space X is called **connected** if it cannot be written as $X = U \cup V$ with U, V-disjoint non-empty opens in X. Or, equivalently, if the only subsets of X that are both open and closed are $\emptyset$ and X. In general, if X is not connected, it can be "broken" into connected pieces; more precisely, recall that a **connected component** of a space X is any connected subset $C \subset X$ which (when endowed with the induced topology) is connected, and which is maximal (w.r.t. the inclusion) with this property. Then the set of connected components defines a partition of X by closed subspace. The easiest way to prove that a space is connected is to prove that it is **path connected** (well, if it is). That means that any two points can be connected by a continuous curve. The spaces that we will be interested in will all be locally path connected (in the sense that any point admits a path connected neighborhood). In that case each connected component of X is not only closed but also open in X. In examples, the partition into connected components is usually easy to guess intuitively; here is a simple exercise that can be used as a recipe to confirm such guesses: assume that we manage to write X as

$$X = X_1 \cup \ldots \cup X_k, \quad \text{with } X_i \cap X_j = \emptyset \text{ for } i \neq j.$$

Assume also that all the X_is are open or, equivalently (why?), that all the X_is are closed. Then $\{X_1, \ldots, X_k\}$ must coincide with the partition into connected components.

Remark 1.5. Of course, the number of connected components may sometimes be infinite (even non-countable). However, for topological manifolds M, due to the second countability axiom, the family of connected components is at most countable (and finite if M is compact). Actually, one often restricts the attention to connected manifolds.

Another important topological property is that of compactness. While this is a property that one usually encounters in the first courses in Analysis (compacts in $\mathbb{R}^m$ being the subsets $A \subset \mathbb{R}^m$ that are closed and bounded), the fact that this is a topological property (i.e. can be described by appealing only to the notion of opens in A, without any reference to the Euclidean metric or to the way that A sits inside $\mathbb{R}^m$) is not at all obvious. That makes the resulting general definition less intuitive and a bit hard to digest at first: a topological space X is said to be **compact** if for any open cover

$$\mathscr{U} = \{U_i : i \in I\}$$

of X (i.e. each U_i is open in X, their union is X, and I is an indexing set), one can extract a finite subcover, i.e. there exists $i_1, \ldots, i_k \in I$ such that $\{U_{i_1}, \ldots, U_{i_k}\}$ is still a cover of X- i.e.

$$X = U_{i_1} \cup \ldots \cup U_{i_k}.$$

Here is the list of the most important properties of compactness:

1. Compact inside Hausdorff is closed: if X is a topological space, $A \subset X$ is endowed with the induced topology:

$$A - \text{compact}, X - \text{Hausdorff} \Longrightarrow A - \text{is closed in} X.$$

2. Closed inside compact is compact: if X is a topological space, $A \subset X$ is endowed with the induced topology:

$$A - \text{is closed in } X, X - \text{compact} \Longrightarrow A - \text{is compact}$$

3. Any compact Hausdorff space is automatically normal:

$$X - \text{compact} \Longrightarrow X - \text{normal}$$

   Recall here that a topological space X is said to be **normal** if for any $A, B \subset X$ closed disjoint subsets, one can find opens in X, U containing A and V containing B, such that $U \cap V = \emptyset$.
4. Product of compacts is compact: if X and Y are spaces, $X \times Y$, endowed with the product topology then

$$X, Y - \text{compact} \Longrightarrow X \times Y - \text{compact}$$

5. Continuous applied to compact is compact: if $F : X \to Y$ is continuous and $A \subset X$ (with the induced topology):

$$F : X \to Y \text{ continuous}, A - \text{compact inside } X \Longrightarrow f(A) - \text{compact}$$

6. In particular: quotients of compacts are compacts.
7. A continuous bijection from a compact space to a Hausdorff one is automatically a homeomorphism:

$$(\text{continuous } F) : (\text{compact space } X) \to \ (\text{Hausdorff space } Y) \Longrightarrow F \text{ is a homeomorphism}$$

   More generally: a continuous injection from compact to Hausdorff is automatically an embedding (see above).

Exercise 1.6. Assuming that you know that the unit interval $[0,1]$ (with the Euclidean topology) is compact, use the properties listed above to deduce that: for subsets of $\mathbb{R}^m$ endowed with the Euclidean topology:

$$A \subset \mathbb{R}^m \text{ is compact} \Longleftrightarrow A \text{ is closed and bounded in } \mathbb{R}^m$$

A related topological property is the local version of compactness: one says that a space X is **locally compact** if any point $x \in X$ admits a compact neighborhood. If X is also Hausdorff, it follows that any point in X admits ”arbitrarily small compact neighborhoods”: for any neighborhood U of x in X there exists a compact neighbhborhood of x, contained in U. In general, Hausdorff locally compact spaces can be compactified by adding one extra-point. More on the 1-point compactification can be found in the lecture note on Topology. For topological manifolds, axiom (MT0) ensures that they are automatically locally compact. But also axioms (MT1), (MT2) interact nicely with local compactness: they ensure the existence of ”exhaustions” (cf. Theorem 4.48 in the 2024 Topology notes:

Theorem 1.7. *Any locally compact, Hausdorff, 2nd countable space X admits an* ***exhaustion****, i.e. a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all n.*

Proof. Let $\mathscr{B}$ be a countable basis and consider $\mathscr{V} = \{B \in \mathscr{B} : \overline{B} - \text{compact}\}$. Then $\mathscr{V}$ is a basis: for any open U and $x \in X$ we choose a compact neighborhood N inside U; since $\mathscr{B}$ is a basis, we find $B \in \mathscr{B}$ s.t. $x \in B \subset N$; this implies $\overline{B} \subset N$ and then $\overline{B}$ must be compact; hence we found $B \in \mathscr{V}$ s.t. $x \in B \subset U$. In conclusion, we may assume that we have a basis $\mathscr{V} = \{V_n : n \in \mathbb{Z}_+\}$ where $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We put $K_1 = \overline{V}_1$. Since $\mathscr{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we put

$$K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}.$$

This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \mathring{K}_2$; since $K_1 \subset D_1$, we have $K_1 \subset \mathring{K}_2$. Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we put

$$K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}.$$

As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing this process, we construct the family K_n, which clearly covers X.

Finally, recall that a topological space X is called a **Baire space** if the following condition holds: *for any countable family $\{U_n : n \in \mathbb{N}\}$ consisting of dense open subsets of X, $\cap_n U_n$ is dense in X.* Again, in the Topology notes there are two main results discussed on Baire spaces (Theorem 4.55 and 5.3 in those notes); together:

Theorem 1.8. *All locally compact Hausdorff spaces, as well as all complete metric spaces, are Baire spaces.*

1.1.2.1 Real-valued functions

A **The algebra of continuous functions**

B **Partitions of unity**

A. The algebra of continuous functions

Given a topological space X, an "observable on X" has a precise meaning: it is a continuous function

$$f : X \to \mathbb{R}.$$

The set of all such continuous functions is denoted by

$$\mathscr{C}(X).$$

One of the simplest but most fundamental ideas in various parts of Geometry is that of understanding a space X via the associated "object" $\mathscr{C}(X)$. This will allow one to consider "more relevant observables": e.g. for subspaces $X \subset \mathbb{R}^m$, one can consider only fs that are smooth, or polynomials. Or even to handle "spaces" which, although are quite intuitive, are not topological spaces in the strict sense of the word. All together, this point of view gives rise to several directions in Geometry: Differential Geometry (where the key-word is "smooth" instead of "continuous"), Algebraic Geometry (where the key-word is "polynomial", or "complex analytic"), Noncommutative Geometry (where X does not even make sense, but $\mathscr{C}(X)$ does).

Of course, what makes these work is the rich structure that $\mathscr{C}(X)$ posses- making the "object" $\mathscr{C}(X)$ (a priory just a set) into a more interesting mathematical object. We recall here the most important part of the algebraic structure present on $\mathscr{C}(X)$: it is an algebra. Recall here:

Definition 1.9. A (real) **algebra** is a vector space A over $\mathbb{R}$ together with an operation

$$A \times A \to A, \ \ (a,b) \mapsto a \cdot b$$

which is unital in the sense that there exists an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in A,$$

and which is $\mathbb{R}$-bilinear and associative, i.e., for all $a, a', b, b', c \in A$, $\lambda \in \mathbb{R}$,

$$(a + a') \cdot b = a \cdot b + a' \cdot b, \ a \cdot (b + b') = a \cdot b + a \cdot b',$$

$$(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b),$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

We say that A is commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$. Similarly one talks about complex algebras: then A is a vector space over $\mathbb{C}$ and $\lambda \in \mathbb{C}$.

For a topological space X, the algebra structure on $\mathscr{C}(X)$ is defined simply by pointwise addition and multiplication: for $f, g \in \mathscr{C}(X)$ and $\lambda \in \mathbb{R}$, $f + g, f \cdot g, \lambda \cdot f \in \mathscr{C}(X)$ are given by:

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x), \quad (\lambda \cdot f)(x) = \lambda f(x).$$

And, considering the space $\mathscr{C}(X, \mathbb{C})$ of $\mathbb{C}$-valued continuous functions on X, one obtains a complex algebra.

The fact that, under certain assumptions, a topological space X can be recovered from the algebra $\mathscr{C}(X)$, is the content of the Gelfand-Naimark theorem. While we refer to the basic course on Topology for the full statement and details, here is the very brief summary:

Theorem 1.10 (informative version of Gelfand Naimark theorem). *There is a way to associate to any algebra A a topological space $X(A)$ (the spectrum of A) so that, when applied to $A = \mathscr{C}(X)$- the algebra of continuous functions on a compact Hausdorff space X, one recovers X: $X(\mathscr{C}(X))$ is homeomorphic to X.*

Remark 1.11 (Some details). The spectrum $X(A)$ of an algebra A is defined as the set of characters on A, i.e. maps

$$\chi : A \to \mathbb{R}$$

which preserve the algebra structure, i.e. which are linear, multiplicative ($\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in A$) and send the unit of A to $1 \in \mathbb{R}$. The topology on $X(A)$ is "the best" for which all the evaluation maps

$$\mathrm{ev}_a : X(A) \to \mathbb{R}, \quad \chi \mapsto \chi(a) \quad \text{(one for each } a \in A\text{)}$$

are continuous. For instance, when $A = \mathscr{C}(X)$ for a compact Hausdorff space X, then any point $x \in M$ gives rise to a character χ_x on $\mathscr{C}(X)$, namely the evaluation at x, and the resulting map

$$X \to X(A), \quad x \mapsto \chi_x$$

is the one the realises the desired homeomorphism (of course, there are things to prove along the way). Let us give here a direct argument showing that, if X is a compact space, then any character on $\mathscr{C}(X)$,

$$\chi : \mathscr{C}(X) \to \mathbb{R},$$

is necessarily of type χ_x for some $x \in M$ (proving that the previous map is surjective). $\square$

And, with the mind at the fact that we may want to consider more restrictive conditions than continuity (e.g. smoothness), here is the resulting relevant abstract notion:

Definition 1.12. Given an algebra A (over the base field $\mathbb{R}$ or $\mathbb{C}$), a **subalgebra** of A is any vector subspace $B \subset A$, containing the unit 1 of A and such that

$$b \cdot b' \in B \quad \forall\, b, b' \in B.$$

When we want to be more specific about the base field, we talk about real or complex subalgebras.

For instance, for $X \subset \mathbb{R}^m$, smooth or polynomial functions gives a sequence of sub-algebras:

$$\mathscr{C}^{\text{polyn}}(X) \subset \mathscr{C}^{\infty}(X) \subset \mathscr{C}(X).$$

Finally, when looking at a subset

$$\mathscr{A} \subset \mathscr{C}(X),$$

(subalgebra or not), there are several interesting properties that turn out to be interesting- and we say that:

(1) $\mathscr{A}$ is **point separating** for any $x, y \in X$ distinct there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$ or, equivalently, if there exists $f \in \mathscr{A}$ such that $f(x) = 0$ and $f(y) = 1$.
(2) $\mathscr{A}$ is **normal** if for any two disjoint closed subset $A, B \subset X$, there exists $f \in \mathscr{A}$ such that $f|_A = 0$, $f|_B = 1$.
(3) $\mathscr{A}$ is **closed under sums** if $f + g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$.
(3) $\mathscr{A}$ is **closed under quotients** if $f/g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$ and g is nowhere vanishing.

For instance, the Stone-Weierstrass theorem (which will not be used in this course) says that: for compact Hausdorff spaces X any point-separating sub-algebra $\mathscr{A} \subset \mathscr{C}(X)$ is dense in $\mathscr{C}(X)$; with particular cases of the type: real valued continuous functions on $[0,1]$ (or other similar spaces) can be approximated by polynomial functions.

Note that, for a general topological space X, even the entire $\mathscr{A} = \mathscr{C}(X)$ need not be point separating or normal. Actually, it is a rather simple exercise to check that the point separation of $\mathscr{C}(X)$ implies that X must be Hausdorff, while the normality of $\mathscr{C}(X)$ implies that the topological space X must be normal (i.e., as recalled above: any two disjoint closed subsets $A, B \subset X$ can be separated topologically: there exist opens $U, V \subset X$ containing A and B, respectively, with $U \cap V = \emptyset$). What is far less obvious (actually one of the most non-trivial basic results in Topology) is the converse, known as the Urysohn lemma: if a topological space X is Hausdorff and normal then $\mathscr{C}(X)$ is normal; more precisely, for any two disjoint closed subsets $A, B \subset X$ there exists

$$f : X \to [0,1] \text{ continuous and such that } f|_A = 0, f|_B = 1.$$

This will not be used later in the course; we mention it here just for completeness.

B. Partitions of unity

Finally, one more basic topic from Topology- but this time one that is difficult to appreciate (and perhaps even to digest) without entering the realm of Differential Geometry and/or Analysis: partitions of unity. To be able to talk about partitions of unity that are not just continuous (as we will be interested only on smooth functions), we can place ourselves in the following setting: X is a topological space and

$$\mathscr{A} \subset \mathscr{C}(X)$$

is a given vector subspace; we will be looking at partitions of unity that belong to $\mathscr{A}$. For the main definition, we first need to recall the notion of support: given $\eta : X \to \mathbb{R}$ continuous, **the support of** η **in** X, denoted $\mathrm{supp}_X(\eta)$ or simply $\mathrm{supp}(\eta)$ is the closure in X of the set $\eta \neq 0$ of points of X on which η does not vanish:

$$\mathrm{supp}_X(\eta) := \overline{\{\eta \neq 0\}} = \overline{\{x \in X : \eta(x) \neq 0\}}^X.$$

Given an open $U \subset X$, we say that η **is supported in** U if $\mathrm{supp}(\eta) \subset U$. This condition allows one to promote functions that are defined only on U, $f : U \to \mathbb{R}$, to functions on X, at least after multiplying by η; namely, $\eta \cdot f$, a priory defined only on U, if extended to X by declaring it to be zero outside U, the resulting function

$$\eta \cdot f : X \to \mathbb{R}$$

will be continuous (check this and, by looking at examples, convince yourselves that this does not work if the condition $\mathrm{supp}(\eta) \subset U$ is replaced by the weaker one that $\{\eta \neq 0\} \subset U$).

We now move to partitions of unity; we start with the finite ones.

Definition 1.13. Let X be a topological space, $\mathscr{U} = \{U_1, \ldots, U_n\}$ a finite open cover of X. A continuous partition of unity subordinated to $\mathscr{U}$ is a family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\eta_1 + \ldots + \eta_k = 1, \ \ \mathrm{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

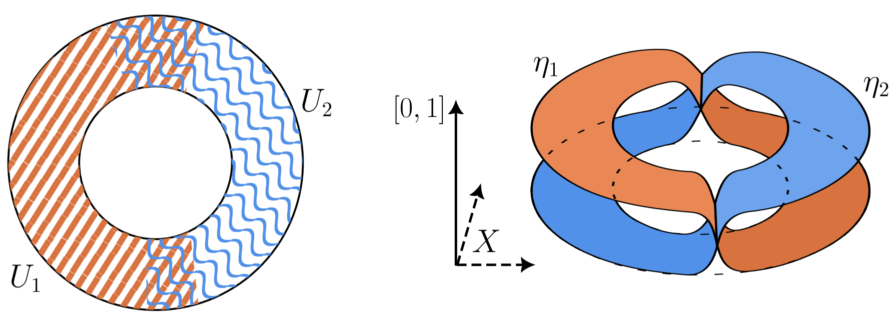


Fig. 1.1 On the left, an annulus X is covered by two open sets U_1 and U_2. The graph on the right shows two functions $\eta_i : X \to [0,1]$ that form a partition of unity subordinate to this cover.

Theorem 1.14. *Let X be a topological space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal and is closed under sums and quotients. Then, for any finite open cover $\mathscr{U}$, $\exists$ an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.*

Addendum (just for completeness, again): in particular if X is Hausdorff and normal, by the Uryshon Lemma, any finite open cover $\mathscr{U}$ admits a continuous partition of unity subordinated to $\mathscr{U}$.

Proof (sketch; for more details, see the lecture notes on Topology). The main ingredients are:

(St1) the remark made above that the normality of $\mathscr{A}$ implies that X is a normal space (simple exercise).

(St2) the fact that, in a normal space X, whenever we have $A \subset U$ with A-closed in X and U-open in X, one can find a smaller open V such that

$$A \subset V \subset \overline{V} \subset U$$

(short proof, but a bit tricky).

(St3) the shrinking lemma: for any finite open cover $\mathscr{U} = \{U_1, \dots, U_k\}$ of a normal space X one can find another cover $\mathscr{V} = \{V_1, \dots, V_n\}$ such that

$$\overline{V}_i \subset U_i \quad \forall\, i \in \{1, \dots, k\}.$$

(this follows by applying the previous step inductively, starting with $U = U_1$ $A = X \setminus (U_2 \cup \ldots \cup U_k)$.

Now the proof of the theorem. Apply the shrinking lemma twice and choose open covers $\mathscr{V} = \{V_i\}$, $\mathscr{W} = \{W_i\}$, with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. For each i, we use the separation property of $\mathscr{A}$ for the disjoint closed sets $(\overline{W}_i, X - V_i)$. We find $f_i : X \to [0,1]$ that belongs to $\mathscr{A}$, with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f := f_1 + \ldots + f_k$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathscr{W}$ is a cover of X. From the properties of $\mathscr{A}$, each

$$\eta_i := \frac{f_i}{f_1 + \ldots + f_k} : X \to [0,1]$$

is continuous. Clearly, their sum is 1. Finally, $\mathrm{supp}(\eta_i) \subset U_i$ because $\overline{V}_i \subset U_i$ and $\mathrm{supp}(\eta_i) = \mathrm{supp}(f_i)$. 😀

And here is a nice application of the existence of (finite) partitions of unity:

Theorem 1.15. *Any compact topological manifold M can be embedded in some Euclidean space $\mathbb{R}^m$.*

Proof. Cover M by opens that are homeomorphic to $\mathbb{R}^d$, where d is the dimension of M. Using that M is compact, we find an open cover $\mathscr{U} = \{U_1, \dots, U_n\}$ together with homeomorphisms $\chi_i : U_i \to \mathbb{R}^d$. Since M is compact it is also normal hence we find a partition of unity $\{\eta_1, \dots, \eta_n\}$ subordinated to $\mathscr{U}$. Each of the functions $\eta_i \cdot \chi_i : U_i \to \mathbb{R}^d$ is extended to M by declaring it to be zero outside U_i; by the previous comments, the resulting functions $\tilde{\chi}_i : M \to \mathbb{R}^d$ are continuous. Consider now

$$i = (\eta_1, \dots, \eta_k, \tilde{\chi}_1, \dots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

One check directly that i is injective; since M is compact and $\mathbb{R}^{k(d+1)}$ is Hausdorff, by the properties recalled on compactness, i will be an embedding. 😀

Finite partitions of unity are useful mainly when working over compacts (so that one can ensure finite open covers). For the more general case one first has to make precise sense of "infinite sums $\sum_i \eta_i$". For that first recall that, given a topological space X, a family $\mathscr{S}$ of subsets of X is said to be **locally finite** (in X) if for any $x \in X$ there exists a neighborhood V of x which intersect only a finite number of members of $\mathscr{S}$. Given a family $\{\eta_i\}_{i \in I}$

(I some indexing set) of continuous functions $\eta_i : X \to \mathbb{R}$, we say that $\{\eta_i\}_{i\in I}$ **is locally finite** in X if their supports (in X) $\text{supp}(\eta_i)$ form a locally finite family of subsets of X. Note that in this case the sum

$$\sum_{i\in I} \eta_i : X \to \mathbb{R}$$

can be defined pointwise (at any $x \in X$ only a finite number of terms do not vanish), and the resulting function is continuous. For a subset $\mathscr{A} \subset \mathscr{C}(X)$, we say that is **closed under locally finite sums** if for any locally finite family $\{\eta_i\}$ with $\eta_i \in \mathscr{A}$, $\sum_i \eta_i$ is again in $\mathscr{A}$. With these, we can now talk about infinite partitions of unity:

Definition 1.16. Let X be a topological space, $\mathscr{U} = \{U_i : i \in I\}$ an open cover of X. A partition of unity subordinated to $\mathscr{U}$ is a locally finite family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\sum_{i\in I} \eta_i = 1, \;\; \text{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

For the pragmatic reader that cares only about what is used later on in this course, the main result is:

Theorem 1.17. *Let X be a Hausdorff, locally compact, 2nd countable space, $\mathscr{A} \subset \mathscr{C}(X)$ and assume that:*

- *$\mathscr{A}$ is closed under locally finite sums and under quotients and*
- *$\mathscr{A}$ satisfies: for any $x \in M$ and any open neighborhood U of x, $\exists\, f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

C. Partitions of unity, paracompactness and 2nd countability

For the curious student, here is the more detailed discussion to which the previous theorem belongs (with the explanation of how the proof goes). The existence of partitions of unity subordinated to arbitrary open covers forces a topological property of X called paracompactness: we say that a topological space X is **paracompact** if for any open cover $\mathscr{U}$ of X, there exists a locally finite open cover $\mathscr{V}$ that is a refinement of $\mathscr{U}$ in the sense that any $V \in \mathscr{V}$ is included inside some $U \in \mathscr{U}$. The existence of arbitrary partitions of unity is ensured by the following:

Theorem 1.18. *Let X be a paracompact Hausdorff space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal, closed under locally finite sums and closed under quotients. Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.*

Paracompact spaces are automatically normal and Uryshon's lemma can can be used. Hence, in paracompact Hausdorff spaces, a for any open cover there exists a continuous partition of unity subordinated to the cover.

The proof of the previous theorem is almost identical with the one from the finite case- just that one now has to establish an infinite version of the shrinking lemma (and that is where paracompactness enters),

To apply the previous theorem, there are two points that may be difficult to check: the paracompactness of X and, when working with arbitrary $\mathscr{A}$, that $\mathscr{A}$ is normal. For the first one, the following comes in handy:

Theorem 1.19. *Any Hausdorff, locally compact and 2nd countable space is paracompact.*

In particular, topological manifolds are automatically paracompact. However, one can show even more: assuming that axioms (TM0) and (TM1) from Definition 1.3 are satisfied and, for simplicity, also that M is connected, then the following are equivalent:

- axiom (TM2): M is 2nd countable
- M paracompact
- M admits an exhaustion (as in Theorem 1.7)

This should make clear the role of the axiom (TM2) as well as the importance of the perhaps not so cool looking Theorem 1.7. If one gives up on the condition that M is connected, then the second item should be replaced by "M paracompact and the collection of connected components is at most countable".

Proof. We use an exhaustion $\{K_n\}$ of X (Theorem 1.7). Let $\mathscr{U}$ be an open cover of X. For each $n \in \mathbb{Z}_+$ there is a finite family $\mathscr{V}_n$ which covers $K_n - \mathrm{Int}(K_{n-1})$, consisting of opens V with the properties: $V \subset \mathrm{Int}(K_{n+1}) - K_{n-1}$, $V \subset U$ for some $U \in \mathscr{U}$. Indeed, for any $x \in K_n - \mathrm{Int}(K_{n-1})$ let V_x be the intersection of $\mathrm{Int}(K_{n+1}) - K_{n-1}$ with any member of $\mathscr{U}$ containing x; since $K_n - \mathrm{Int}(K_{n-1})$ is compact, just take a finite subcollection $\mathscr{V}_n$ of $\{V_x\}$, covering $K_n - \mathrm{Int}(K_{n-1})$. Set $\mathscr{V} = \cup_n \mathscr{V}_n$; it covers X since each $K_n - K_{n-1} \subset K_n - \mathrm{Int}(K_{n-1})$ is covered by $\mathscr{V}_n$. Finally, it is locally finite: if $x \in X$, choosing n and V such that $V \in \mathscr{V}_n$, $x \in V$, we have $V \subset \mathrm{Int}(K_{n+1}) - K_{n-1}$, hence V can only intersect members of $\mathscr{V}_m$ with $m \leq n+1$ (a finite number of them!).

Finally, to check the normality of $\mathscr{A}$ needed in Theorem 1.18, the following comes in handy:

Theorem 1.20. *Let X be a Hausdorff paracompact space and $\mathscr{A} \subset \mathscr{C}(X)$ closed under locally finite sums and under quotients. If X is also locally compact, then the following are equivalent:*

1. *$\mathscr{A}$ is normal.*
2. *for any $x \in M$ and any open neighborhood U of x, $\exists\, f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

In particular, for a topological manifold M, checking that a subset $\mathscr{A} \subset \mathscr{C}(M)$ is normal is a local matter- and that is very useful since, locally, topological manifolds look just like Euclidean spaces.

Proof. That 1 implies 2 is clear: apply the separation property to $\{x\}$ and $X - V$. Assume 2. We claim that for any $C \subset X$ compact and any open U such that $C \subset U$, there exists $f \in \mathscr{A}$ supported in U, such that $f|_C > 0$. Indeed, by hypothesis, for any $c \in C$ we can find an open neighborhood V_c of c and $f_c \in \mathscr{A}$ positive such that $f_c(c) > 0$; then $\{f_c \neq 0\}_{c \in C}$ is an open cover of C in X, hence we can find a finite subcollection (corresponding to some points $c_1, \ldots, c_k \in C$) which still covers C; finally, set $f = f_{c_1} + \ldots + f_{c_k}$.

To prove 1, let $A, B \subset X$ be two closed disjoint subsets. As terminology, $D \subset X$ is called relatively compact if $\overline{D}$ is compact. Since X is locally compact, any point has arbitrarily small relatively compact open neighborhoods. For each $y \in X - A$, we choose such a neighborhood $D_y \subset X - A$. For each $a \in A$, since $a \in X - B$, applying step (St2) from the proof of Theorem 1.14, we find an open D_a such that $a \in D_a \subset X - B$. Again, we may assume that $\overline{D}_a$ is relatively compact. Then $\{D_x : x \in X\}$ is an open cover of X; let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite refinement. We split the set of indices as $I = I_1 \cup I_2$, where I_1 contains those i for which $U_i \cap A \neq \emptyset$, while I_2 those for which $U_i \subset X - A$. Using the shrinking lemma (the infinite version of the one described in (St3) of he proof of Theorem 1.14) we can also choose an open cover of X, $\mathscr{V} = \{V_i : i \in I\}$, with $\overline{V}_i \subset U_i$. Note that, by construction, each U_i (hence also each V_i) is relatively compact. Hence, by the claim above, we can find $\eta_i \in \mathscr{A}$ such that

$$\eta_i|_{\overline{V}_i} > 0, \;\; \mathrm{supp}(\eta_i) \subset U_i.$$

Finally, we define

$$f(x) = \frac{\sum_{i \in I_1} \eta_i(x)}{\sum_{i \in I} \eta_i(x)}$$

From the properties of $\mathscr{A}$, $f \in \mathscr{A}$. Also, $f|_A = 1$. Indeed, for $a \in A$, a cannot belong to the U_i's with $i \in I_2$ (i.e. those $\subset X - A$); hence $\eta_i(a) = 0$ for all $i \in I_2$, hence $f(a) = 1$. Finally, $f|_B = 0$. To see this, we show that $\eta_i(b) = 0$ for all $i \in I_1$, $b \in B$. Assume the contrary. We find $i \in I_1$ and $b \in B \cap U_i$. Now, from the construction of $\mathscr{U}$, $U_i \subset D_x$ for some $x \in X$. There are two cases. If $x = a \in A$, then the defining property for D_a, namely $D_a \cap B = \emptyset$, is in contradiction with our assumption ($b \in B \cap U_i$). If $x = y \in X - A$, then the defining property for D_y, i.e. $D_y \subset X - A$, is in contradiction with the fact that $i \in I_1$ (i.e. $U_i \cap A \neq \emptyset$).

1.2 Reminder 2: ANALISYS

The relationship between Analysis and Differential Geometry is subtle. On one hand, Differential Geometry relies on the very basics of Analysis. On the other hand, various notions/results from Analysis become much more transparent/intuitive once the geometric perspective/intuition is brought into picture. In some sense, in many cases, the geometric point of view indicates the (expected) results while analysis provides the tools to prove them.

1.2.1 Smoothness in $\mathbb{R}^n$

A **The spaces** $\mathbb{R}^n$

B **The differential and the inverse function theorem**

C **Directional/partial derivatives**

D **The implicit function theorem**

E **Moving to other subspaces** $M \subset \mathbb{R}^m$

A. The spaces $\mathbb{R}^n$

The basic playground for multivariate analysis is the standard Euclidean space

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}.$$

Despite its simplicity, this "space" has many different (but related) structures- and often the problem with handling $\mathbb{R}^n$ comes from the fact that it may not be completely clear which of the structures present on $\mathbb{R}^n$ is relevant for the specific discussions. Here are some of the many interesting structures present on $\mathbb{R}^n$:

- it is a vector space. When we want to emphasize this structure, we will denoted by

  $$v = (v_1, \ldots, v_n) \in \mathbb{R}^n$$

  its elements and we will think of them as "vectors"/"directions". Intrinsic in this notation is the presence of yet another piece of structure: it is not just a vector space- it comes with a preferred (standard) basis:

  $$e_1, \ldots, e_n \in \mathbb{R}^n;$$

  in coordinates, e_i has 1 on the i-th position and 0 everywhere else.
- it is a vector space endowed with an inner product:

  $$\langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot w_i,$$

  hence it is also a normed vector space, with the norm:

  $$||v|| = \sqrt{\langle v, v \rangle}.$$

- it is a topological space- endowed with the standard Euclidean topology. When we want to emphasize this structure, we will denote by

  $$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

its elements and we will think of them as "points". For instance, when looking at a circle in $\mathbb{R}^2$, the vector space space structure on $\mathbb{R}^2$ is not so relevant, and we think of the circle as made by points rather then vectors. Also, when talking about the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$, the vector space structure of $\mathbb{R}^n$ is not relevant (though it may be useful).
Recall also that the topology on $\mathbb{R}^n$ is a shadow of yet another structure: $\mathbb{R}^m$ is also a metric space, with the standard Euclidean metric:

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

We say a "shadow" because uses part of what the metric allows us top talks about: points being "close to each other" (or, more precisely: convergence and continuity). In particular, there are several other natural metrics on $\mathbb{R}^n$ that induce the same Euclidean topology- e.g. the so called square metric

$$d'(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

- even when thinking about $\mathbb{R}^n$ as a topological space, so of its elements as points, each point $x \in \mathbb{R}^n$ can be represented using "canonical coordinates"- used already above. Again, the coordinates are not so relevant/important: they are useful and can be used, but they are not intrinsic to the structure. For instance, a circle in $\mathbb{R}^2$ can be described using coordinates by the equation $x^2 + y^2 = 1$, but the circle itself can be drawn without any coordinate axes at our disposal.
  Note that the standard coordinates we mentioned are the simplest illustration of the notion of "chart"- to be discussed in a bit more detail below, and essential in defining the notion of manifold.
- it is a topological space "on which analysis can be performed" (... i.e. a manifold).
- etc.

Of course, all these are inter-related but, in each situation, it is important to realize which of these structures really matter. In particular, whenever one encounters a definition or result, it is instructive to figure out whether the elements in $\mathbb{R}^m$ that show up play the role of points and which ones of vectors, and how much the definition/result depends on the coordinates. This is the first step towards a geometric understanding of Analysis.

B. The differential and the inverse function theorem

One can talk about various notions of derivatives of a function F at a point

$$x \in \mathbb{R}^n$$

whenever we have a function F defined on a neighborhood of x- so that the expressions $F(y)$ used below makes sense for all y near x or, equivalently, $F(x+v)$ is defined for small vectors v. Typically one assumes that F is defined on an open subset $\Omega \subset \mathbb{R}^n$ and takes values in some other Euclidean space $\mathbb{R}^k$,

$$F : \Omega \to \mathbb{R}^k,$$

so that it makes sense to talk about derivatives of F at any point in its domain, $x \in \Omega$.

The most intrinsic notion of derivative is that of "**total derivative**", also called **the differential of** F (at the given point $x \in \Omega$). This notion arises when trying to approximate F, near x, by simpler (linear-like) functions. Understanding F near x is about understanding

$$v \mapsto F(x+v)$$

for $v \in \mathbb{R}^n$ near 0. It sends $v = 0$ to $F(x)$, hence the best one can hope for is to approximate

$$v \mapsto F(x+v) - F(x)$$

by functions that are linear in v. Think that we try to write the last expression as a function linear in v, plus one that is qudratic in v, etc (plus eventually an "error term"), but we are interested only in the linear term A. We see we are looking for a linear map

$$A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$$

with the property that

$$\lim_{v \to 0} \frac{F(x+v) - F(x) - A(v)}{||v||} = 0. \tag{1.2.1}$$

It is easy to see that, if such a map A exists, then it is unique. And that is what the differential of F at x is.

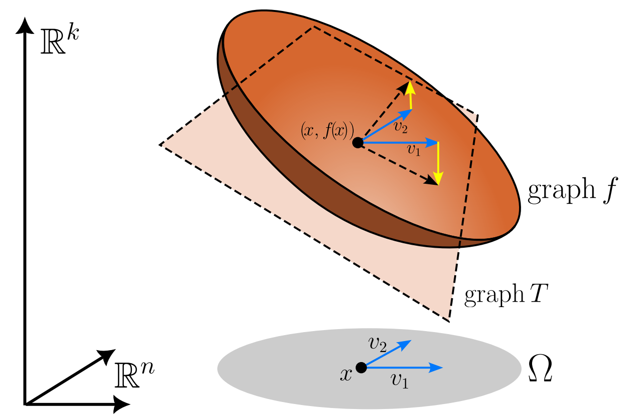


Fig. 1.2 Visualization of the total derivative for a function $F : \Omega \to \mathbb{R}^k$ defined over an open $\Omega \subseteq \mathbb{R}^n$, in this case with $n = 2$ and $k = 1$. Infinitesimal movements through a point $x \in \Omega$ are represented by the blue horizontal vectors v_i and the resulting infinitesimal movement in the codomain $\mathbb{R}^k$ is represented by the yellow vertical vectors $D_xF(v_i)$. They sum up to dashed vectors of the form $(v_i, D_xF(v_i)) \in \mathbb{R}^n \times \mathbb{R}^k$ that are tangent to the graph of F in the point $(x, F(x))$. The graph of the best affine approximation T of F in x (which sends any $\tilde{x} \in \mathbb{R}^n$ to $T(\tilde{x}) = F(x) + D_xF(\tilde{x} - x)$) is an affine plane spanned by the dashed vectors.

Definition 1.21. We say that $F : \Omega \to \mathbb{R}^k$ is **differentiable at** x if there exists a linear map $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ satisfying (1.2.1). The linear map A (necessarily unique) is called **the total derivative of** F **at the point** x (or the **differential!total derivative** of F at x) and is denoted

$$D_xF = (DF)_x : \mathbb{R}^n \to \mathbb{R}^k.$$

We say that F is **differentiable** if it differentiable at all points x in its domain Ω. We say that F is **of class** C^1 if it is differentiable and the resulting map

$$DF : \Omega \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k), \quad x \mapsto (DF)_x$$

is continuous (recall here that, via the matrix representation of linear maps, $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ can be interpreted as the Euclidean space $\mathbb{R}^{n \cdot k}$).

We say that F is of class C^2 if it differentiable and dF is of class C^1; proceeding inductively, we can talk about F being of class C^l for any $l \in \mathbb{N}$. We say that F is **smooth** if it is of class C^l for all l.

Recall here also the chain rule that allows one to compute the differential of a composition of two functions:

Proposition 1.22 (the chain rule). *Given opens $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^k$ and functions*

$$\Omega \xrightarrow{F} \Omega' \xrightarrow{G} \mathbb{R}^l,$$

if F is differentiable at $x \in \Omega$ and G is differentiable at $F(x) \in \Omega'$, then $G \circ F$ is differentiable at x and

$$(D(G \circ F))_x = (DG)_{F(x)} \circ (DF)_x.$$

Despite the fact that the differential $(DF)_x$ arises as "the linear approximation" of F near x, it contains a great deal of information of F near x- and that makes it extremely useful. Probably the best and most fundamental illustration is the inverse function theorem. Recall here that

Definition 1.23. A map $F : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is said to be a **diffeomorphism** if it is bijective and both F and F^{-1} are smooth.

We say that F is a **local diffeomorphism** around $x \in \Omega$ if there exist opens $\Omega_x \subset \Omega$ and $\Omega'_{F(x)} \subset \Omega'$ with $x \in \Omega_x$, such that $F|_{\Omega_x} : \Omega_x \to \Omega'_{F(x)}$ is a diffeomorphism.

It is interesting to draw an analogy with Topology, where the main objects are topological spaces, the relevant maps are the continuous ones and two spaces are "isomorphic in Topology" (homeomorphic) if there exists a bijection F between them such that both F as well as F^{-1} are continuous. However, in topology is it usually very hard to prove that two given spaces are not homeomorphic (and one often has to appeal to methods from Algebraic Topology); for instance, just the simple the fact that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homeomorphic only when $n = k$ is very hard to prove. In contrast, the similar statements for diffeomorphisms are much easier to prove thanks to the notion of differential. Indeed, using the chain rule, the following should be a rather easy exercise:

Exercise 1.24. Show that if a map $F : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is a local diffeomorphism around $x \in \Omega$, then

$$(DF)_x : \mathbb{R}^n \to \mathbb{R}^k$$

is a linear isomorphism. Deduce that if two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ are diffeomorphic, then $n = k$.

Although this clearly shows the usefulness of the differential, its great power is due to the inverse function theorem (and its immediate consequences, such as the implicit function theorem -see below). Indeed, we see that a condition on the differential of F at a single (given) point x tells us information about F around x:

Theorem 1.25 (The inverse function theorem). *Given a smooth map $F : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$, if F is differentiable at a point $x \in \Omega$ and $(DF)_x$ is an isomorphism, then F is a local diffeomorphism around x.*

C. Directional/partial derivatives

Note that, when talking about the differential $(DF)_x(v)$ (hence $F : \Omega \to \mathbb{R}^k$, with $\Omega \subset \mathbb{R}^n$ open), $x \in \Omega$ should be thought of as a point, while $v \in \mathbb{R}^n$ as a direction (vector). This becomes more apparent if we reformulate the total derivative as a directional derivative.

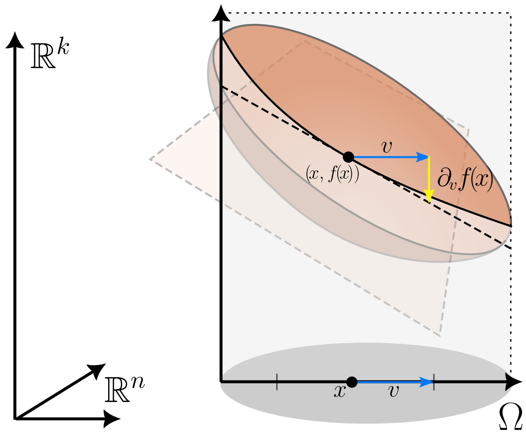


Fig. 1.3 The directional derivative $\partial_v F(x)$ of the function F from Fig. 1.2 can be seen to arise geometrically in the following way: The map $t \mapsto x + vt$ traces a line through Ω. The slice of the graph of F over this line is exactly the graph of the function $t \mapsto F(x+tv)$ if we label the horizontal axis by integer multiples of v. The directional derivative can now be defined as the normal derivative of this function since we have a one-dimensional domain.

Definition 1.26. With $F : \Omega \to \mathbb{R}^k$ and $x \in \Omega$ as above, and an arbitrary vector $v \in \mathbb{R}^n$, **the derivative of** F **at** x **in the direction** v is defined as the vector

$$\partial_v(F)(x) = \frac{\partial F}{\partial v}(x) := \left.\frac{d}{dt}\right|_{t=0} F(x+tv) = \lim_{t \to 0} \frac{F(x+tv) - F(x)}{t} \in \mathbb{R}^k.$$

When this derivative exists, we say that F is differentiable at x in the v-direction.

The relationship with the total differential is immediate: just replace in (1.2.1) v (small enough) by tv with $v \in \mathbb{R}^n$ fixed (but arbitrary) and $t \in \mathbb{R}$ approaching 0; using that A is linear, we find that:

$$(DF)_x(v) = \frac{\partial F}{\partial v}(x).$$

This relationship is visualized in Fig. 1.3. In particular, if F is differentiable at x then it is differentiable in all directions. The converse is not true; however, one can show that if F is of class C^1 if and only if all the directional derivatives $\frac{\partial F}{\partial v}$ exist and are continuous (see also the discussion below on partial derivatives).

Applying the definition to $v \in \{e_1, \ldots, e_n\}$, a vector in the standard basis of $\mathbb{R}^n$, we obtain the partial derivatives

$$\frac{\partial F}{\partial x_i}(x) := \frac{\partial F}{\partial e_i}(x) = \left.\frac{d}{dy}\right|_{y=x_i} F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in \mathbb{R}^k.$$

Its components are the partial derivatives of the components F_i of F:

$$\frac{\partial F}{\partial x_i}(x) = \left(\frac{\partial F_1}{\partial x_j}(x), \ldots, \frac{\partial F_k}{\partial x_j}(x)\right) \quad (\text{where } F = (F_1, \ldots, F_k)).$$

These partial derivatives contain the same information as $(DF)_x$, just in a less intrinsic way; however, they allow one to handle $(DF)_x$ more concretely, via matrices. For that recall that, due to the fact that the standard Euclidean spaces come with a preferred basis, linear maps

$$A : \mathbb{R}^n \to \mathbb{R}^k$$

can be represented as matrices

$$A = (A^i_j)_{1 \le i \le k, 1 \le j \le n} = \begin{pmatrix} A^1_1 & \dots & A^1_n \\ \dots & \dots & \dots \\ A^k_1 & \dots & A^k_n \end{pmatrix}$$

To make a distinction between the matrix A and the linear map A, one may want to denote by $\hat{A}$ the linear map, at least for a while. Then the relationship between the two is (by definition):

$$\hat{A}(e_j) = \sum_{i=1}^{k} A^i_j e_i$$

or, on a general vector $v = v^1 e_1 + \ldots + v^n e_n \in \mathbb{R}^n$, one has

$$\hat{A}(v) = \sum_{i=1}^{k} (\sum_{j=1}^{n} A^i_j v^j) e_i.$$

To write also this formula in terms of matrix multiplication, we interpret any $v \in \mathbb{R}^n$ as a row matrix and we denote by v^T its transpose (column matrix):

$$v^T = \begin{pmatrix} v^1 \\ \dots \\ v^n \end{pmatrix}$$

With this, the previous formula becomes

$$\hat{A}(v)^T = A \cdot v^T$$

It follows immediately that the standard multiplication of matrices,

$$(A \cdot B)^i_j = \sum_k A^i_k B^k_j,$$

corresponds to the composition of linear maps:

$$\widehat{AB} = \hat{A} \circ \hat{B}.$$

All together, there should be no confusion in identifying A with $\hat{A}$ even notationally.

In the case of the differential $(DF)_x$, to see the matrix representing it we write

$$(DF)_x(e_j) = \frac{\partial F}{\partial x_j}(x) = \left(\frac{\partial F_1}{\partial x_j}(x), \ldots, \frac{\partial F_k}{\partial x_j}(x)\right) = \sum_{i=1}^{k} \frac{\partial F_i}{\partial x_j}(x) e_j$$

i.e., in the matrix notation,

$$(DF)_x = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \dots & \frac{\partial F_1}{\partial x_n}(x) \\ \dots & \dots & \dots \\ \frac{\partial F_k}{\partial x_1}(x) & \dots & \frac{\partial F_k}{\partial x_n}(x) \end{pmatrix}. \tag{1.2.2}$$

Note that, with this, the fact that $(DF)_x$ is an isomorphism is equivalent to the fact that the matrix above is invertible. More generally, the rank of $(DF)_x$ as a linear map coincides with the rank as a matrix. With these:

Proposition 1.27. *A function $F : \Omega \to \mathbb{R}^k$ is of class C^1 if and only if all the partial derivatives $\frac{\partial F}{\partial x_i}$ exist and are continuous functions on Ω.*

In this case we can further look at partials derivatives of order two etc. Hence the higher, order l, partial derivatives are defined inductively:

$$\frac{\partial^l F}{\partial x_{i_1} \dots \partial x_{i_l}} = \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial}{\partial x_{i_2}} \left(\dots \left(\frac{\partial F}{\partial x_{i_l}} \right) \right) \right).$$

The previous proposition that extends to a characterization of F being of class C^l; in particular, F is smooth if and only if all its higher partial derivatives exist. We will denote by

$$\mathscr{C}^\infty(\mathbb{R}^n)$$

the space (algebra!) of smooth functions on $\mathbb{R}^n$. A **smooth partitions of unity** on $\mathbb{R}^n$ (subordinated to an open cover) is any partition of unity whose members η_i are smooth- i.e. Definition 1.13 (finite case) and Definition 1.16 (general case) applied at $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n) \subset \mathscr{C}(\mathbb{R}^n)$.

Theorem 1.28. *Any open cover of* $\mathbb{R}^n$ *admits a smooth partition subordinated to it.*

Proof. We want to use Theorem 1.18 for $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n)$. This is clearly closed under quotients and, for the same reason that locally finite sums of continuous functions are continuous, it is closed under locally finite sums. We still have to check the last condition on $\mathscr{A}$ or, equivalently: for any $x \in \mathbb{R}^n$ and any ball centered at x, $B(x, \varepsilon)$, there exists a smooth functions $f : \mathbb{R}^n \to [0,1]$ such that $f(x) > 0$ and f is supported in the ball. It is clear that we may assume that $x = 0$. Also, by rescaling the argument of f (i.e. multiply it by a constant) we may assume that $\varepsilon = 1$. Then set $f(x) = g(x_1^2 + \dots + x_n^2)$ where $g : \mathbb{R} \to [0,1]$ is any smooth function with $g(0) > 0$ and $g = 0$ outside $[-\frac{1}{2}, \frac{1}{2}]$. That such a function exists should be clear by thinking of its graph. The following exercise provides and explicit formula. 😀

Exercise 1.29. Show that

$$g_0 : \mathbb{R} \to \mathbb{R}, \quad g_0(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is a smooth function. Then show that

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = g_0\left(x + \frac{1}{2}\right) g_0\left(\frac{1}{2} - x\right).$$

is a smooth function with the properties required at the end of the previous proof.

D. The implicit function theorem

We now recall the implicit function theorem- which is one the important and rather immediate consequences of the inverse function theorems. The importance is rather geometric, as it arises when looking at curves, surfaces (or higher dimensional ... submanifolds) in $\mathbb{R}^n$. While such subspaces are usually given by equations of type $F(x_1, \dots, x_n) = 0$ (think e.g. of $x^2 + y^2 = 1$, defining the unit circle in the plane), one would like to express some of the coordinates x_i in terms of the others (or, equivalently, describe our subspace as a graph).

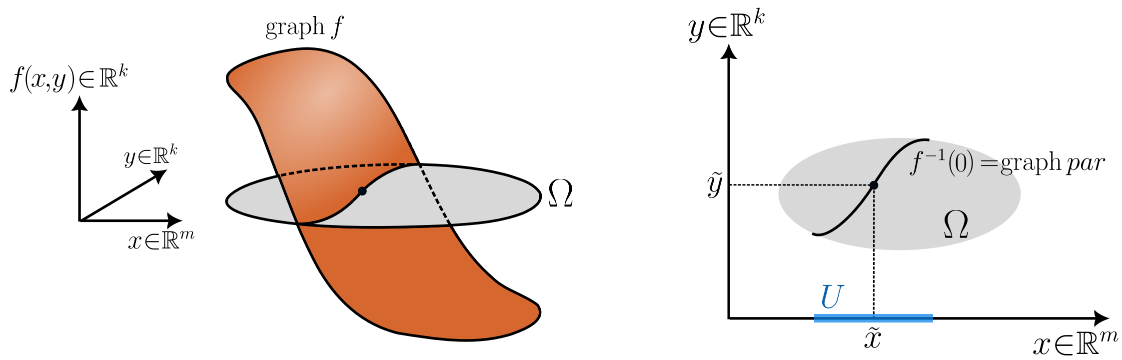


Fig. 1.4 The implicit function Theorem 1.30 illustrated for a concrete choice of $F:\Omega\to\mathbb{R}^k$. The theorem establishes that the preimage $F^{-1}(0)$ (under appropriate assumptions) can locally be written as a graph of a function $par:U\to\mathbb{R}^k$ over a subset of the variables. In other words, the condition that F vanishes *implicitly* defines the function par.

Theorem 1.30. *Let $F:\Omega\to R^k$ be a smooth map defined on an open $\Omega\subset\mathbb{R}^m\times\mathbb{R}^k$ whose elements we label as $(x,y)=(x_1,\dots,x_m,y_1,\dots,y_k)$. Furthermore let $(\tilde{x},\tilde{y})\in\Omega$ be a point where $F(\tilde{x},\tilde{y})=0$ and the matrix*

$$\left(\frac{\partial F_i}{\partial y_j}(\tilde{x},\tilde{y})\right)_{1\le i,j\le k}$$

is non-singular. Then there exists a function $par:U\to\mathbb{R}^k$ defined in a neighborhood U of $\tilde{x}$ such that for all (x,y) near $(\tilde{x},\tilde{y})$, one has

$$F(x,y)=0\Longleftrightarrow y=par(x).$$

The matrix appearing in this theorem is exactly the Jacobian matrix of the map $y\mapsto F(\tilde{x},y)$ at $y=\tilde{y}$. You can convince yourself using Fig. 1.4 that its non-singularity is a neccessary condition to find a smooth function par: In the depicted situation, it corresponds exactly to a tangency of $F^{-1}(0)$ in the y-direction at $(\tilde{x},\tilde{y})$.

Remark 1.31. Note that this theorem is not completely canonical: It gives preference to the last k components of the arguments of F, i.e. depends on how we split the components of elements of Ω into x and y-components. For example, there is an obvious modification in which the starting assumption is that the Jacobian with respect to the *first* k components is regular, instead of the last ones. Such modifications are necessary even when looking at the simplest examples: E.g., for the unit circle where $F:\mathbb{R}^2\to\mathbb{R}$, $F(x,y)=x^2+y^2-1$, the condition

$$\frac{\partial F}{\partial y}(x,y)\neq 0$$

(necessary for the theorem) is valid at *almost* all the points (x,y) in the circle (and, indeed, we can always solve $y=\pm\sqrt{1-x^2}$), except for the points $(1,0)$ and $(-1,0)$ (and, indeed, there is a problem there: we would need a function two take two values simultaneously close to $y=0$ in order for it's graph to match $F^{-1}(0)$). However, at those points one can switch the roles of x and y- and, indeed, around those points which are problematic for $\pm\sqrt{1-x^2}$, one can write $x=\pm\sqrt{1-y^2}$.

A more intrinsic version of the theorem (and with exactly the same proof) can be obtained by requiring that $(DF)_x$ has maximal rank k, without specifying which minor is non-singular. the conclusion will be that there exists a permutation $\sigma\in S_{m+k}$ such that $F(p)=0$ near $\tilde{p}\in\mathbb{R}^{m+k}$ is equivalent to

$$\pi_y(\sigma\cdot p)=par(\pi_x(\sigma\cdot p)),$$

where π_x and π_y are the projections of $\mathbb{R}^{m+k}$ onto the first m and last k components, respectively, and we write $\sigma \cdot p$ for the result of permuting p by σ. But perhaps the most geometric formulation is what is know as the submersion theorem- see below.

Proof. Consider the map

$$\mathcal{F} : \Omega \to \mathbb{R}^{m+k}, \quad \mathcal{F}(x,y) := (x, F(x,y)).$$

Then the non-singularity condition in the statement precisely means that $(D\mathcal{F})_{(\tilde{x},\tilde{y})}$ is non-singular. Hence, by the inverse function theorem, we find a smooth inverse $\mathcal{G}$ of $\mathcal{F}$, defined near $\mathcal{F}(\tilde{x},\tilde{y})$. Given the form of $\mathcal{F}$, it follows that $\mathcal{G}$ is of a similar form:

$$\mathcal{G}(x,z) = (x, G(x,z)).$$

That $\mathcal{G} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{G}$ are the identity maps (near $(\tilde{x},\tilde{y})$, and $\mathcal{F}(\tilde{x},\tilde{y})$, respectively) translates into

$$G(x, F(x,y)) = y \qquad \text{and} \qquad F(x, G(x,z)) = z. \tag{1.2.3}$$

The first equation shows that

$$F(x,y) = 0 \Longrightarrow G(x,0) = y,$$

hence we have an obvious candidate $par(x) := G(x,0)$. Note that the assumption $F(\tilde{x},\tilde{y}) = 0$ guarantees that G is defined for (x,z) near $(\tilde{x},0)$. The fact that, indeed, $F(x, par(x)) = 0$ for x close to $\tilde{x}$ is just the second equation in (1.2.3) applied when $z = 0$. 😀

E. Moving to other subspaces $M \subset \mathbb{R}^n$

Finally, it is useful to extend some of the previous concepts to other subsets of $\mathbb{R}^n$, more general than just opens. First of all, for the notion of smoothness we keep the local character and make use of the ambient space $\mathbb{R}^n$:

Definition 1.32. Let $F : M \to M'$ be a function between subsets $M \subset \mathbb{R}^n$, $M' \subset \mathbb{R}^{n'}$. We say that:

(a) F is **smooth around** $p \in M$ if the restriction $F|_U$ to some neighborhood U of p in M admits a smooth extension $\widetilde{F} : \widetilde{U} \to \mathbb{R}^{n'}$ to an open inside $\mathbb{R}^L$ containing U.
(b) F is **smooth** if it is smooth around any $p \in M$.
(c) F is a **diffeomorphism** if it is a bijection and both F and F^{-1} are smooth.

Here is a nice application of the existence of smooth partitions of unity (see Theorem 1.28 below).

Exercise 1.33. Show that if $M \subset \mathbb{R}^n$ is a closed subset that any smooth function $F : M \to \mathbb{R}^k$ admits a smooth extension $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^k$. (Hint: $\mathbb{R}^n \setminus M$ is open; get an open cover of $\mathbb{R}^n$ out of one of M.)

Next, for the notion of total derivative in this more general context, the point of view provided by the directional derivatives becomes more useful. First of all, it brings us closer to the intrinsic nature of (p,v) when talking about $(DF)_p(v)$: that of tangent vector. The key point is that $(DF)_p(v)$ depends only on the behaviour of F near p, in "the direction of v"- and how we realise that "direction" is less important. This is best seen by looking at arbitrary paths through p with the original speed v,i.e. any smooth map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ (with $\varepsilon > 0$) satisfying

$$\gamma(0) = p, \quad \frac{d\gamma}{dt}(0) = v. \tag{1.2.4}$$

For instance, one could take $\gamma(t) = p + tv$, but the point is that the variation of $F(p+tv)$ at $t = 0$ does not depend on this specific choice of γ.

Lemma 1.34. *If $\Omega \subset \mathbb{R}^n$ open and $F : \Omega \to \mathbb{R}^k$ is differentiable at p then, for any path γ satisfying (1.2.4), one has*

$$(DF)_p(v) = \frac{\partial F}{\partial v}(p) = \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)). \tag{1.2.5}$$

In particular, if F is constant along such a path γ, then $(DF)_p(v) = 0$.

With this in mind, when moving to functions between more general subspaces of the Euclidean spaces, we first discover tangent spaces.

Definition 1.35. Given $M \subset \mathbb{R}^n$ and a point $p \in M$:

(a) a **smooth curve in** M is any smooth map $\gamma : I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$, which takes values in M. We say that it starts at p of $0 \in I$ and $\gamma(0) = p$.
(b) A **vector tangent to** M **at** p is any vector $v \in \mathbb{R}^n$. which can be written as

$$v = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)$$

for some smooth curve in M starting at p.
(c) The collection of such vectors is called the tangent space of M at p and is denoted

$$T_p^{\text{class}} M \subset \mathbb{R}^n.$$

We can now return to differentials of smooth maps

$$F : M \to M'$$

with $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ and conclude that they are now defined as maps

$$(DF)_p : T_p^{\text{class}} M \to T_{F(p)}^{\text{class}} M'.$$

Explicitly, for $v \in T_p^{\text{class}} M$, choose as smooth curve γ in M such that (1.2.4) and then define $(DF)_p$ by the formula (1.2.5). Notice that the derivative still makes sense because $F \circ \gamma$ is smooth (smoothness in the sense of Definition 1.32 is still preserved under composition) and lands in $T_{F(p)}^{\text{class}} M'$ because $F \circ \gamma$ is a smooth path in N starting at $F(p)$. Furthermore, it is well-defined: it does not depend on the choice of the curve realising v as in (1.2.4). To see this we choose a smooth extension $\widetilde{F}$ near p as in Definition 1.32, and then apply Lemma 1.34 to $\widetilde{F}$ (incidentally, this also shows that $(D\widetilde{F})_p(v)$ for $v \in T_p^{\text{class}} M$ only depends on $\tilde{F}|_M$), which would give a slightly different way to describe the differential of F).

1.2.2 Embedded submanifolds of $\mathbb{R}^m$

A **Some explanations**

B **Coordinates (charts)**

C **Implicit equations (submersions)**

D **Parametrizations (immersions)**

E **All together now**

F **More exercises**

A. Some explanations

We now move to the notion of (smooth) embedded submanifold of $\mathbb{R}^n$. A good example to have in mind are the m-dimensional spheres naturally sitting inside $\mathbb{R}^{m+1}$ (hence $n = m+1$)

$$S^m = \{(x_0,\ldots,x_m) \in \mathbb{R}^{m+1} : (x_0)^2 + (x_1)^2 + \ldots + (x_m)^2 = 1\},$$

or a torus inside $\mathbb{R}^3$. In principle, an m-dimensional embedded submanifold of $\mathbb{R}^n$ is any subset which locally "looks like $\mathbb{R}^m$ sitting inside $\mathbb{R}^n$" as

$$\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}.$$

There are several ways to look at this, such as:

(1) locally, the points of M can be parametrised by m coordinates (appropriately chosen).
(2) locally, M can be describe by equations of type $x_{m+1} = \ldots = x_n = 0$ (for appropriately chosen coordinates).

Notice that the dimension on M is the number of coordinates you need in (1) to describe M, while in (2) it is the dimension of the ambient space minus the number of equations. For instance, for the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

which is $2 = 3-1$-dimensional, if we look in the first quadrant $U = \{(x,y,z) \in S^2 : x,y,z > 0\}$,

1. inside that quadrant S^2 can be parametrised by just the coordinates (x,y), since z is forced to be $\sqrt{1-x^2-y^2}$.
2. changing the coordinates of $\mathbb{R}^3$ (inside the first quadrant) to

$$x' = x, \quad y' = y, \quad z' = \sqrt{1-x^2-y^2}$$

then $U \subset S^2$ will be described by $z' = 0$.

On the other hand, if you ask someone to describe S^2 parametrically or by equations, the most likely answer is:

1. "the usual" spherical coordinates (θ,ϕ) describing the points of S^2 as

$$x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta$$

2. "obviously" the defining equation

$$x^2 + y^2 + z^2 = 1$$

Similarly, for general submanifolds M, there are many possible choices of parametrizations or equations describing M locally, some which are perhaps more natural, and some more useful for certain purposes (e.g. computing an integral or solving a differential equation). Anyway, to lay down the foundations, the main concepts that are emerging from this discussion are:

coordinates, parametrizations and implicit equations

The "smoothness" in "smooth submanifolds" translates into regularity conditions on the functions that enter the parametrization and the implicit equations so that, indeed, M will look like a copy of $\mathbb{R}^m$ inside $\mathbb{R}^n$.

B. Coordinates (charts)

The standard coordinates in $\mathbb{R}^n$, despite being "obvious", are often not the best ones to use in specific problems. E.g.: often when dealing with (algebraic or differential) equations or computing integrals, one proceeds to a change of variables (i.e. passing to more convenient coordinates). Baby example: looking at the curve in $\mathbb{R}^2$ defined by

$$5x^2+2xy+2y^2=1,$$

a change of coordinates of type

$$x=\frac{u+v}{3}, \quad y=\frac{u-2v}{3} \tag{1.2.6}$$

brings us to the simpler looking equation $u^2+v^2=1$. To formalise such changes of coordinates, one talks about charts:

Definition 1.36. A **smooth chart of** $\mathbb{R}^n$ is a pair (U,ϕ) consisting of an open $U\subset\mathbb{R}^n$, called the **domain of the chart**, and a diffeomorphism

$$\chi=(\chi_1,\ldots,\chi_n):U\to\Omega\subset\mathbb{R}^n$$

between U and another open $\Omega\subset\mathbb{R}^n$. For $p\in U$,

$$(\chi_1(p),\ldots,\chi_n(p))$$

are called the **coordinates of** p **w.r.t. the chart**; we also say that (U,χ) is a smooth chart around p.

For instance, a very common change of coordinates in $\mathbb{R}^2$ is the passing to polar coordinates:

$$x=r\cos(\theta), \quad y=r\sin(\theta). \tag{1.2.7}$$

Computing the inverse of $(r,\theta)\mapsto(r\cos(\theta),r\sin(\theta))$, one finds the chart

$$\chi(x,y)=\left(\sqrt{x^2+y^2}, arctg(\frac{y}{x})\right).$$

See also the baby Example 1.37 below.

The purpose of choosing a new chart is to change the coordinates in order to make certain functions look simpler. The typical situation is when we have a function

$$F:\mathbb{R}^n\to\mathbb{R}^{n'}$$

and we change coordinates in $\mathbb{R}^n$ using some chart (U,χ), and in $\mathbb{R}^{n'}$ using some chart (U',χ'),

$$\chi:U\to\Omega\subset\mathbb{R}^n, \quad \chi':U'\to\Omega'\subset\mathbb{R}^{n'}.$$

For simplicity, assume that $F(U)\subset U'$. **The representation of** F **w.r.t. the charts** χ **and** χ' is defined as

$$F_\chi^{\chi'}=\chi'\circ F\circ\chi^{-1}:\Omega\to\Omega'.$$

This is the function how the χ'-coordinates of $F(p)$ can be described in terms of the χ-coordinates of p:

$$\Big(\underbrace{\chi_1'(F(p)),\ldots,\chi_m'(F(p))}_{\text{the }\chi'\text{ coordinates of }F(p)}\Big)=F_\chi^{\chi'}\Big(\underbrace{\chi_1(p),\ldots,\chi_n(p)}_{\text{the }\chi\text{ coordinates of }p}\Big)$$

Of course, for the standard charts (the identity maps) one obtains back F. If just χ', or just χ, is the standard chart then we use the notations F_χ and $F^{\chi'}$, respectively:

$$F_\chi=F_\chi^{\mathrm{id}}=F\circ\chi^{-1}, \quad F^{\chi'}=F_{\mathrm{id}}^{\chi'}=\chi\circ F.$$

Example 1.37. Returning to the baby example above, the change of coordinates (1.2.6) is about the chart

$$\chi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \chi(x,y) = (2x+y, x-y) \tag{1.2.8}$$

so that, in the new coordinates, a point $p = (x,y)$ will have the coordinates (w.r.t. χ)

$$u(x,y) = 2x+y, \quad v(x,y) = x-y.$$

The relevant function in that discussion is

$$F : \mathbb{R}^3 \to \mathbb{R}, \quad F(x,y) = 5x^2 + 2xy + 2y^2$$

and, with respect to the new chart, it becomes

$$F_\chi(u,v) = u^2 + v^2.$$

In general, it is interesting to try to write smooth functions in the simplest possible way, modulo change of coordinates. The following is rather a "cheap exercise" (rather tautological) but still interesting since it marks a shift on the way of looking at diffeomorphisms.

Exercise 1.38. Show that for a smooth map $F : \mathbb{R}^n \to \mathbb{R}^{n'}$ as above, and $p \in U$, the following are equivalent:

1. F is a local diffeomorphism around p.
2. there exists a chart χ around p such that $F_\chi = \mathrm{Id}$.
3. there exists a chart χ' around $F(p)$ such that $F^{\chi'} = \mathrm{Id}$.
4. there exists charts χ and χ' around p sand $F(p)$, respectively, such that $F_\chi^{\chi'} = \mathrm{Id}$.

Returning to the previous discussion on subspaces $M \subset \mathbb{R}^n$, we can now make sense of: it locally looks like $\mathbb{R}^m = \mathbb{R}^m \times \{0\}$ sitting inside $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$'.

Definition 1.39. Given $M \subset \mathbb{R}^n$ and $p \in M$, we say that M satisfies the (m-dimensional) **embedded submanifold condition** around p if there exists a chart

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$$

for $\mathbb{R}^n$ around p such that $\widetilde{\chi}(M \cap \widetilde{U}) = \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$ or, equivalently,

$$M \cap \widetilde{U} = \left\{ p \in \widetilde{U} : \widetilde{\chi}_{m+1}(p) = \ldots = \widetilde{\chi}_n(p) = 0 \right\} \tag{1.2.9}$$

A chart $(\widetilde{U}, \widetilde{\chi})$ of $\mathbb{R}^n$ with this property is call **chart adapted to** M.

Notice that any such adapted chart $(\widetilde{U}, \widetilde{\chi})$ induces a topological chart for M (cf. Definition 1.3):

$$U := M \cap \widetilde{U}, \quad \chi|_U : U \to \Omega,$$

where $\Omega = \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$ can be seen as an open in $\mathbb{R}^m$. This allows us to encode the points of M by m coordinates. It is a smooth chart in the following sense.

Definition 1.40. Given $M \subset \mathbb{R}^n$ and $p \in M$, a **smooth (m-dimensional) chart** for M is any topological chart of M, i.e., a homeomorphism

$$\chi : U \to \Omega$$

from an open U in M to an open $\Omega \subset \mathbb{R}^m$, such that χ is also a diffeomorphism (cf. Definition 1.32).

Not every smooth chart χ of M is induced by an adapted smooth chart $\widetilde{\chi}$ of $\mathbb{R}^n$. However:

Proposition 1.41. *For $M \subset \mathbb{R}^n$ and $p \in M$, the manifold condition for M at p is equivalent to the existence of a smooth chart of $\mathbb{R}^n$ around p, that is adapted to M.*

The proof will be discussed together with the proof of Theorem 1.50 below.

C. Implicit equations (submersions)

We now make precise the idea of describing submanifolds implicitly, i.e., by equations. One may say that that main problem here is to make sense of the "regularity conditions" one has to impose on the equation so that the resulting manifold is smooth. For instance, the equation $xy = 0$ described two intersecting lines which, around the intersection points it doesn't look very smooth (and certainly doesn't look like an interval).

The power of the inverse function theorem and its various consequences (like the implicit function theorem, as well as the submersion and immersion theorems mentioned below) is that the regularity conditions can be controlled entirely infinitesimally, i.e., by looking at the total derivatives of our functions. If $(DF)_p$ is invertible, then F was a diffeomorphism around p and then, modulo change of coordinates (cf. Exercise 1.38) F looks like the identity. What we need here is basically just the first variation of the inverse function theorem, i.e. the implicit function theorem, but reformulated more canonically as follows.

Theorem 1.42 (the submersion theorem). *Let $F : U \to \mathbb{R}^k$ be a smooth functions defined on an open $U \subset \mathbb{R}^n$ and let $p \in U$. Then the following are equivalent:*

- *$(DF)_p : \mathbb{R}^n \to \mathbb{R}^k$ is surjective or, equivalently, $n \geq k$ and (1.2.2) has rank k.*
- *there exists a smooth chart χ of $\mathbb{R}^n$ around p such that, around $\chi(p)$,*

$$F_\chi(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k).$$

Definition 1.43. Given $F : U \to \mathbb{R}^k$ as above, we say that F is a **submersion at** p if the equivalent properties from the last theorem are satisfied. When that holds for all $p \in U$, we say that F is a submersion.

Exercise 1.44. Show that F is a submersion at p if and only if, in a neighborhood of p, it admits a right inverse.

Proof (of Theorem 1.42:). Since the matrix representing $(DF)_p$ is of maximal rank, one of its maximal minors (an $k \times k$ matrix) is invertible; we may assume that the invertible minor is precisely the one made of the last k rows (why?)- which is also the hypothesis of the implicit function theorem (Theorem 1.30). Looking at the proof of the theorem, one remarks that the desired chart is $\chi = \tilde{F}$. 😀

Definition 1.45. Given $M \subset \mathbb{R}^n$, by a **implicit equation** for M around $p \in M$ we mean a submersion

$$eq : \widetilde{U} \to \mathbb{R}^k \quad (\text{where } k = n - m)$$

defined on an open neighborhood $\widetilde{U} \subset \mathbb{R}^n$ of p such that

$$M \cap \widetilde{U} = \{q \in \widetilde{U} : eq(q) = 0\}.$$

D. Parametrizations (immersions)

Theorem 1.46 (the immersion theorem). *Let $F : U \to \mathbb{R}^n$ be a smooth functions defined on an open $U \subset \mathbb{R}^m$ and let $p \in U$. Then the following are equivalent:*

- *$(DF)_p : \mathbb{R}^m \to \mathbb{R}^n$ is injective or, equivalently, $m \leq n$ and (1.2.2) has rank m.*
- *there exists a smooth chart χ' of $\mathbb{R}^n$ around $F(p)$ such that, around p,*

$$F^{\chi'}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m \text{ zeros}}). \tag{1.2.10}$$

Addendum: *furthermore, the chart (U', χ') around $F(p)$ and the neighborhood $U_p \subset U$ of p where (1.2.10) holds can be chosen such that*

$$F(U_p) = \{u \in U'_{F(p)} : \chi'_{m+1}(u) = \ldots = \chi'_n(u) = 0\}.$$

Definition 1.47. Given $F : U \to \mathbb{R}^n$ as above, we say that F is an **immersion at** p if the equivalent from the last theorem are satisfied. When that holds for all $p \in U$, we say that F is an immersion.

Exercise 1.48. Show that F is an immersion at p if and only if, in a neighborhood of p, it admits a left inverse.

Proof (of Theorem 1.46:). Let us give a proof that makes reference to $(DF)_p$ as a linear map and not as a matrix. Since $(DF)_p : \mathbb{R}^m \to \mathbb{R}^n$ is injective, we find a second linear map $B : \mathbb{R}^{n-m} \to \mathbb{R}^n$ such that

$$((DF)_p, B) : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^k$$

is an isomorphism. Consider then

$$h : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \quad h(x_1, x_2) = F(x_1) + B(x_2).$$

We see that h satisfies the hypothesis of the inverse function theorem at the point $(x, 0)$. Hence it is a diffeomorphism around a neighborhood of $(x, 0)$. We denote by

$$\chi' : U' \to \Omega'$$

its inverse. Note that:

1. U' is an open neighborhood of $F(p)$ in $\mathbb{R}^n$.

2. Ω' is an open neighborhood of $(p,0)$ in $\mathbb{R}^n$, contained in $U \times \mathbb{R}^{n-m}$.
3. the intersection of $\Omega' \subset \mathbb{R}^n$ with $\mathbb{R}^m \times \{0\}$,

$$\Omega := \{u \in \mathbb{R}^m : (u,0) \in \Omega'\},$$

is an open neighborhood of p included in the domain of F.

Note that, since $\chi'(h(x_1,x_2) = 0$ for all $(x_1,x_2) \in \Omega'$ and $h(x,0) = F(x)$, we have $\chi'(F(x)) = (x,0)$ for all $x \in \Omega$. This proves the main part of the theorem; for the last part, note that we have, by the first part, that $F(\Omega)$ is inside the zero set of $\chi_2' : U' \to \mathbb{R}^{n-m}$ (the second component of the chart χ' w.r.t. the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$). For the reverse inclusion, let $x \in U'$ with $\chi_2'(x) = 0$; since $h \circ \chi' = \mathrm{id}$ on U', we obtain

$$x = h(\chi'(x)) = h(\chi_1'(x),0) = F(\chi_1'(x))$$

where, for the last equality, we used the explicit formula for h. Moreover, since $\chi'(x) \in \Omega'$ and since $\chi'(x) = (\chi_1'(x),0)$, by the definition of Ω, we have $\chi_1'(x) \in \Omega$. With the previous equality in mind, we obtain $x \in F(\Omega)$. 😀

Definition 1.49. Given $M \subset \mathbb{R}^n$ and $p \in M$, by a **parametrization** of M around p we mean an immersion

$$par : \Omega \to \mathbb{R}^n$$

defined on an open $\Omega \subset \mathbb{R}^m$, such that its image (denoted $par(\Omega)$) is a neighborhood of p in M

$$U := par(\Omega) \subset M$$

and, as a map $\Omega \to U$, par is a homeomorphism.

E. All together now

Theorem 1.50. *Given a subset $M \subset \mathbb{R}^n$, $p \in M$, the following are equivalent:*

1. *M satisfies the m-dimensional embedded submanifold condition around p (as in Definition 1.39).*
2. *M admits an m-dimensional smooth chart around p (as in Definition 1.40).*
3. *M admits an m-dimensional parametrization around p (as in Definition 1.49).*
4. *M can be described by an m-dimensional implicit equation around p (as in Definition 1.45).*

Proof. For keeping track of notations note that, throughout the proof, we look around the given point $p \in M \subset \mathbb{R}^n$ and around the corresponding point

$$x = \chi(p) = par^{-1}(p) \in \Omega \subset \mathbb{R}^m.$$

Therefore, we will deal with

- neighborhoods U_p of p in M, and $\widetilde{U}_p$ of p in $\mathbb{R}^n$.
- neighborhoods Ω_x of x in $\mathbb{R}^m$.

The points in the neighborhoods of p will be denoted by q, while the ones in the neighborhoods of x by y; for them, we may be looking at similar neighborhoods U_q, $\widetilde{U}_q$ and Ω_y.

We first prove that (2) implies (3). We start with the the chart $\chi : U \to \Omega \subset \mathbb{R}^m$ defined in a neighborhood U of p in M. Setting $par = \chi^{-1}$ we have to check that par is a homeomorphism -which is clear by construction (it has the continuous χ as inverse)- and that, as a map $\Omega \to \mathbb{R}^n$, it is an immersion. For the last part use that the composition

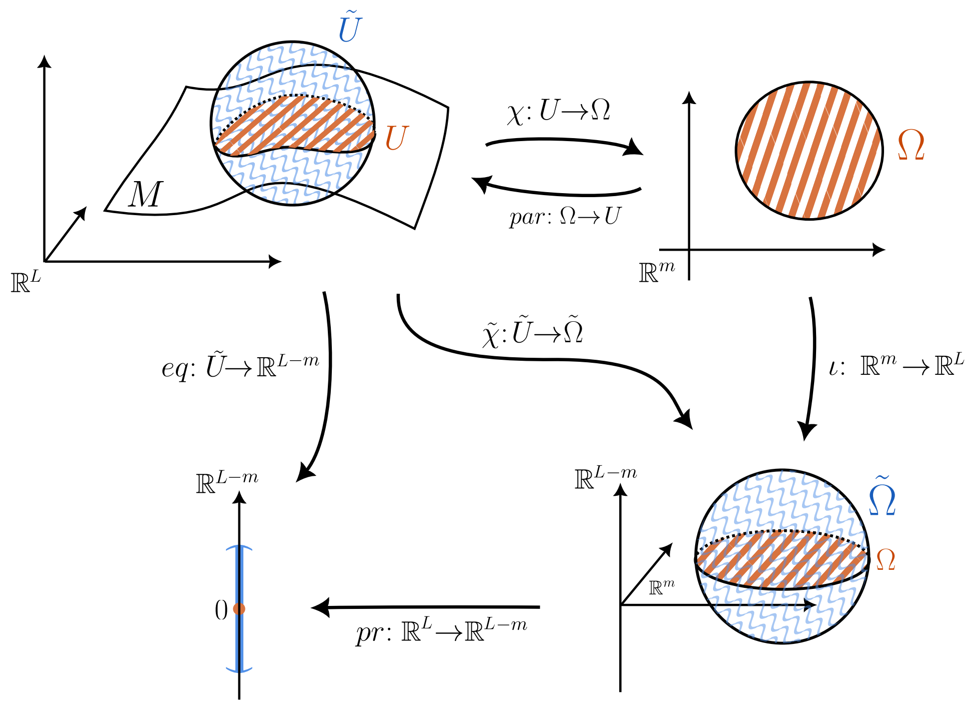


Fig. 1.5 The equivalent ways of phrasing the manifold condition in Theorem 1.50 involve (adapted) charts, parametrizations or implicit equations that are related as shown in this figure. Note that $U = \tilde{U} \cap M$ and $\Omega = \tilde{\Omega} \cap \mathbb{R}^m$. ι and pr are canonical inclusions and projections of the product space $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ in the lower right. All these maps commute where they can be evaluated. Note: L in the picture is n from the text.

$$\Omega \xrightarrow{par} U \xrightarrow{\chi} \Omega$$

is the identity on Ω and then apply the chain rule to deduce that, for each point $y \in \Omega$, $(D\chi)_{par(y)} \circ (D\,par)_y$ is the identity- hence, in particular, $(D\,par)_y$ will be injective.

We now prove that (3) implies both (2) as well as (4). Hence we start with a parametrization $par : \Omega \to U \subset M$; as above, we set $\chi = par^{-1} : U \to \Omega$. To get (2), we still have to check that χ is smooth in the sense of Definition 1.32: i.e., around any point $q \in U$, it is obtained by restricting a smooth map defined on an open $\widetilde{U}_q \subset \mathbb{R}^n$. For that we use the immersion theorem (Theorem 1.46) applied to $par : \Omega \to \mathbb{R}^n$ around

$$y = \chi(q) \in \Omega.$$

We find:

- an open neighborhood Ω_y of y in $\Omega \subset \mathbb{R}^m$
- a diffeomorphism $\widetilde{\chi} : \widetilde{U}_q \to \widetilde{\Omega}_q$ from an open neighborhood $\widetilde{U}_q \subset \mathbb{R}^n$ of p' to an open $\widetilde{\Omega}_q \subset \mathbb{R}^n$,

so that, on Ω_y, $\widetilde{\chi} \circ par$ becomes the inclusion on the first factors. We now write $\widetilde{\chi} = (\widetilde{\chi}_1, \widetilde{\chi}_2)$ where we use again the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. We deduce that $\widetilde{\chi}_1(par(z)) = z$ for all $z \in \Omega_y$. Since $z = \chi(par(z))$ for all $z \in \Omega$, we deduce that $\widetilde{\chi}_1(r) = \chi(r')$ for all $r \in par(\Omega_q)$. In this way, on the neighborhood $par(\Omega_q)$ of q in U, χ is now the restriction of a smooth function defined on an open neighborhood of q in $\mathbb{R}^n$- namely $\widetilde{\chi}_1 : \widetilde{U}_q \to \mathbb{R}^m$.

To prove (4) (still assuming (3)), we use $q = p$ in the previous reasoning and the resulting diffeomorphism $\widetilde{\chi} : \widetilde{U}_p \to \widetilde{\Omega}_p$; in principle, the desired function eq (as in Definition 1.45) will be $\widetilde{\chi}_2$, but we have to choose the domain of definition carefully. For that we use the last part of Theorem 1.46 which says that we may assume that

$$par(\Omega_x) = \{y \in \widetilde{U}_p : \widetilde{\chi}_{m+1}(y) = 0, \ldots, \widetilde{\chi}_n(y) = 0\}.$$

Since this is open in M, we can write it as $M \cap W_p$ for some open $W_p \subset \mathbb{R}^n$. Considering now

$$\widetilde{U} := \widetilde{U}_p \cap W_p, \quad eq = \widetilde{\chi}_2|_{\widetilde{U}} : \widetilde{U} \to \mathbb{R}^{n-m}$$

one checks right away that $M \cap \widetilde{U}$ is the zero set of eq (why is eq a submersion?). Notice that this argument actually shows that (4) implies (1), once you do the following exercise:

Exercise 1.51. Conclude now that $\widetilde{\chi}$ is actually an adapted chart.

Since (1) clearly implies (2), we are now left with proving that (4) implies (2). Let $eq : \widetilde{U} \to \mathbb{R}^{n-m}$ satisfying the conditions from the hypothesis. Note that if we replace $\widetilde{U}$ by a smaller open neighborhood of p in $\mathbb{R}^n$ (and eq by its restriction), those conditions will still be satisfied. Therefore, using the submersion theorem applied to eq, we may assume that we also find a diffeomorphism $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega}$ into an open subset of $\mathbb{R}^n$, such that $eq = \widetilde{\chi}_2$. This chart will then take the zero set of eq into the zero set of the second projection $\mathrm{pr}_2 : \widetilde{\Omega} \to \mathbb{R}^{n-m}$, i.e. into

$$\Omega := \{u \in \mathbb{R}^m : (u, 0) \in \widetilde{\Omega}\}.$$

We deduce that the restriction of $\widetilde{\chi}$ to $U = M \cap \widetilde{U}$,

$$\chi := \widetilde{\chi}|_U : U \to \Omega \subset \mathbb{R}^m,$$

is a smooth chart of M (around p). 😄

F. More exercises

Exercise 1.52. Consider two smooth functions $U \xrightarrow{F} U' \xrightarrow{G} \mathbb{R}^p$ defined on opens $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$. Using the interpretation of linear maps as matrices (as made precise on page 23) show that the chain rule becomes:

$$\frac{\partial G \circ F}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}(F(p)) \frac{\partial F_j}{\partial x_i}(x)$$

for all $x \in U$ and $1 \leq i \leq n$.

Exercise 1.53. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}, \quad F(x,y) = 3 \cdot \sqrt[3]{x^2 + 2xy + 2y^2}$$

Find a chart χ of $\mathbb{R}^2$ around $p = (1,0)$ and a chart χ' of $\mathbb{R}$ around $F(p) = 3$ such that, w.r.t. these charts,

$$F_{\chi}^{\chi'}(u,v) = u^2 + v^2.$$

Exercise 1.54. Assume that $F : U \to U'$ and $G : U' \to U''$ are two smooth functions, with $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$ and $U'' \subset \mathbb{R}^p$ opens. Show that if $G \circ F$ is a local diffeomorphism around a given point $x \in U$, then:

1. F is an immersion at x and G is a submersion at $F(x)$.
2. however, it may happen that F is not a submersion at x and G is not an immersion at $G(x)$ (describe an example!).

3. if, furthermore, F is a submersion at x or G is an immersion at $F(x)$, then both F and G are local diffeomorphisms (around x and $F(x)$, respectively).

Exercise 1.55. For any $\varepsilon > 0$ describe a smooth function

$$f : \mathbb{R}^n \to [0,1]$$

with the property that $f(0) > 0$ and whose support (in $\mathbb{R}^n$) is contained in the ball $\{x \in \mathbb{R}^n : ||x|| < \varepsilon\}$.

Exercise 1.56. Consider the stereographic projection w.r.t. the north pole p_N, denoted

$$\chi_N : S^2 \setminus \{p_N\} \to \mathbb{R}^2$$

and similarly the one w.r.t. the south pole, denoted χ_S. Show that

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

is a diffeomorphism.

Exercise 1.57. Pove the chain rule still holds for differentials of smooth functions between arbitrary subsets of Euclidean spaces. Deduce that if $F : M \to N$ is a diffeomorphism, then $(DF)_p$ is a bijection between $T_p^{\text{class}}M$ and $T_{F(p)}^{\text{class}}N$ for all $p \in M$.

Although we use the name "tangent *space*", in general (for completely random Ms inside $\mathbb{R}^n$), $T_p^{\text{class}}M$ is just a subset of $\mathbb{R}^n$ (... but it is a vector subspace if M is "nice").

Exercise 1.58. Compute $T_p^{\text{class}}M$ when:

1. $M \subset \mathbb{R}^2$ is the unit circle and $p = (1,0)$.
2. $M \subset \mathbb{R}^2$ is the union of the coordinate axes and $p = (0,0)$.

Exercise 1.59. Assume that $M \subset \mathbb{R}^n$ is defined by an equation $F(x) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function. For $p \in \mathbb{R}^n$ we denote by $\text{Ker}_p(DF)$ the kernel (= the zero set) of the differential $(DF)_p : \mathbb{R}^n \to \mathbb{R}^k$. Show that, in general,

$$T_p^{\text{class}}(M_F) \subset \text{Ker}_p(DF),$$

but the inclusion may be strict.

Exercise 1.60. With the notations from the previous exercise show that for all $p \in M_F$ at which F is a submersion

$$T_p^{\text{class}}(M_F) = \text{Ker}_p(DF).$$

Chapter 2
Smooth manifolds

2.1 Manifolds and smooth maps

2.1.1 Charts, smooth atlases, smooth structures

The difference between topological manifolds (see Definition 1.3) and smooth manifolds is, as the terminology suggests, that we assume smoothness for all the objects one considers (so that, on smooth manifolds, unlike on topological ones, we will be able to talk about speeds of curves, tangent vectors, differentials, etc). For subspaces $M \subset \mathbb{R}^n$, making use of the ambient space $\mathbb{R}^n$ we managed to make sense of smoothness of charts- giving rise to the notion of smooth submanifold of $\mathbb{R}^n$ (Definition 1.40). However, in the general setting, there is no intrinsic way to make sense of smoothness of a bare topological space M (not necessarily embedded into $\mathbb{R}^n$)- instead, we need extra-data on M that serves precisely that purpose. And that is the notion of smooth atlas that we start with here.

Recall from Definition 1.3 that, given a topological space M, an m-dimensional chart is a homeomorphism χ between an open U in M and an open subset $\chi(U)$ of $\mathbb{R}^m$,

$$\chi : U \to \chi(U) \subset \mathbb{R}^m. \quad \text{(coordinate charts)}$$

We also say that (U, χ) is a chart for M, and we call U the domain of the chart. Given such a chart, each point $p \in U$ is determined/parametrized by its coordinates w.r.t. χ:

$$(\chi_1(p), \ldots, \chi_m(p)) \in \mathbb{R}^m$$

(a more intuitive notation would be: $(x^1_\chi(p), \ldots, x^m_\chi(p))$). Given a second chart $\chi' : U' \to \chi'(U') \subset \mathbb{R}^m$, the map

$$c^{\chi'}_{\chi} := \chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U') \quad \text{(coordinate changes)}$$

is a homeomorphism between two opens in $\mathbb{R}^m$. It will be called **the change of coordinates map** from the chart χ to the chart χ'. The terminology is motivated by the fact that, denoting

$$c^{\chi'}_{\chi} = (c_1, \ldots, c_m),$$

the coordinates of a point $p \in U \cap U'$ w.r.t. χ' can be expressed in terms of those w.r.t. χ by:

$$\chi'_i(p) = c_i(\chi_1(p), \ldots, \chi_m(p)).$$

Definition 2.1. We say that two charts (U,χ) and (U',χ') are **smoothly compatible** if the change of coordinates map $c_{\chi}^{\chi'}$, which is a map between two opens in $\mathbb{R}^m$, is a diffeomorphism.

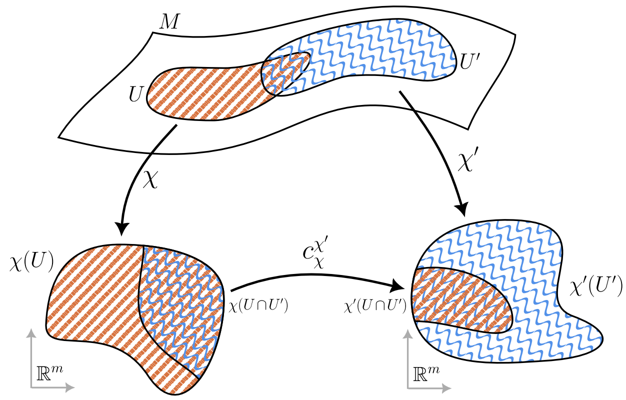


Fig. 2.1 Two charts $\chi : U \to \chi(U)$ and $\chi' : U' \to \chi'(U') \subseteq \mathbb{R}^m$ together with the change of coordinates map $c_{\chi}^{\chi'} : \chi(U \cap U') \to \chi'(U \cap U')$. These maps commute wherever they can be evaluated sensibly, i.e. $c_{\chi}^{\chi'} \circ \chi = \chi'$ holds on $U \cap U'$.

Definition 2.2. A (m-dimensional) **smooth atlas** on a topological space M is a collection $\mathscr{A}$ of (m-dimensional) charts of M with the following properties:

1. the domains of the charts that belong to $\mathscr{A}$ cover M entirely.
2. each two charts from $\mathscr{A}$ are smoothly compatible.

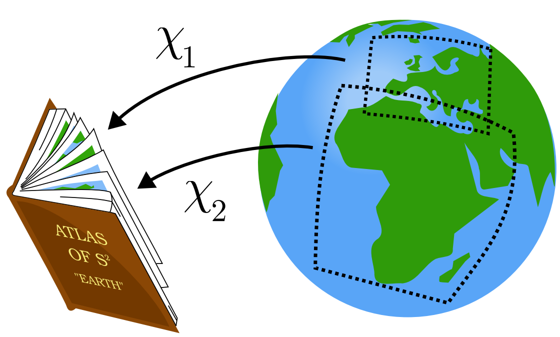


Fig. 2.2 An atlas of Earth is exactly a collection of charts that covers the surface of the globe. The same region might be included in several charts of a smooth atlas in a way that measured distances, angles and areas do not match - e.g. one chart might use the *Mercator projection* whereas another might faithfully depict areas. However, the smooth compatibility between charts does guarantee at least that a smooth path in one chart cannot develop any kinks in another.

Example 2.3. (vector spaces) You have encountered and used charts already in linear algebra, although you did not use quite that name. More precisely, when dealing with a general vector space V, you have probably used the notion of "coordinates of a vector $v \in V$ w.r.t. a basis of V". To clarify the relationship, assume that $M = V$ is a real vector spaces of finite dimension m. To work with V one often fixes a basis $\mathfrak{e} = \{e_1, \ldots, e_m\}$ of V That allows us to represent each vector $v \in V$ uniquely as a linear combination $v = \sum_{i=1}^{m} \lambda_i \cdot e_i$ for some numbers $\lambda_i \in \mathbb{R}$- and those are the coordinates of v with respect to $\mathfrak{e}$ mentioned above. As the name indicates, these correspond to a chart. More precisely, denoting them by $\lambda_i =: x_i^{\mathfrak{e}}(v)$, one obtains a chart with domain the entire V (!):

$$\chi^{\mathfrak{e}} = (x_1^{\mathfrak{e}}, \ldots, x_m^{\mathfrak{e}}) : V \to \mathbb{R}^m.$$

Hence each basis $\mathfrak{e}$ gives rise to a chart. When choosing another basis $\mathfrak{e}'$, then the coordinate change $\chi_{\mathfrak{e}'} \circ \chi_{\mathfrak{e}}$ is precisely the linear map that encodes the change from one basis to the other. In particular, one has a smooth atlas:

$$\mathscr{A}_V^{\mathrm{lin}} := \{\chi^{\mathfrak{e}} : \mathfrak{e} - \text{basis of } V\}.$$

Example 2.4. (Euclidean spaces) On $M = \mathbb{R}^m$ there are several interesting atlases. At the extreme ends one has: $\mathscr{A}_{\mathbb{R}^m}$ consisting of only one chart, namely the identity chart $\mathrm{Id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$, and $\mathscr{A}_{\mathbb{R}^m}^{\max}$ consisting of all smooth charts of $\mathbb{R}^m$ in the sense of Definition 1.36. In between there is the one coming from the previous example:

$$\mathscr{A}_{\mathbb{R}^m} \subset \mathscr{A}_{\mathbb{R}^m}^{\mathrm{lin}} \subset \mathscr{A}_{\mathbb{R}^m}^{\max}.$$

Example 2.5. (polar coordinates) While $M = \mathbb{R}^2$ carries already interesting charts defined on the entire $\mathbb{R}^2$, sometimes there are coordinates that one would like to use but are well-defined (as a chart) only on parts of $\mathbb{R}^2$. The best example is probably provided by polar coordinates:

$$x = r\cos\theta, \; y = r\sin\theta$$

These formulas are, in principle, used to parametrise points $(x,y) \in \mathbb{R}^2$. The radius and the angle are viewed as functions of x and y and, together, they define a chart

$$\chi(x,y) = (r(x,y), \theta(x,y)).$$

In other words, χ is the inverse of the function $\Phi(r,s) = (r\cos\theta, r\sin\theta)$. The only problem is that, as described,

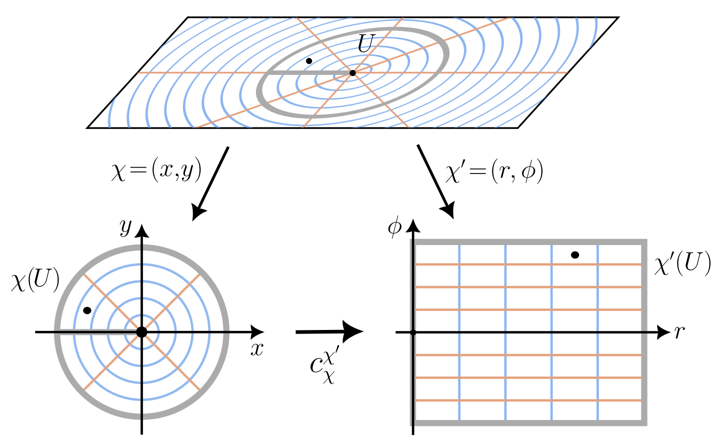


Fig. 2.3 Example of two smoothly compatible charts over the same open subset $U = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 5\} \setminus (-\infty, 0] \times \{0\}$ of the plane: The identity chart $\chi = (x,y)$ as well as polar coordinates $\chi' = (r,\phi)$. The change of coordinate map going from right to left, for example, can be visualized by first collapsing the left boundary of the rectangle into the origin and then glueing the two boundary components that touch the origin together on the negative x-axis.

χ is not smooth and Φ is not even bijective. That is the reason that polar coordinates have to be used with care: they provide charts only when restricted to certain opens $U \subset \mathbb{R}^2$. For instance, Φ is a diffeomorphism between $\Omega_0 = (-\pi, \pi) \times (0, \infty)$ and

$$U_0 := \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\},$$

and, this gives rise to a chart (U_0, χ_0) of $\mathbb{R}^2$; explicitly,

$$\chi_0 = (\Phi|_{\Omega_0})^{-1} : U_0 \to \Omega_0.$$

To handle points outside U_0, still using by polar coordinates, one can use $\theta \in (0, 2\pi)$, i.e. replace Ω_0 by $\Omega_1 = (0, \infty) \times (0, 2\pi)$ and then one obtains a chart (U_1, χ_1) with $U_1 = \mathbb{R}^2 \setminus [0, \infty) \times \{0\}$. One should check that the new chart is smoothly compatible with (U_0, χ_0). Note however that 0 is still left out, since $U_0 \cup U_1 = \mathbb{R}^2 \setminus \{0\}$. To include also the origin in the domain of a "polar-coordinate chart", one can for instance translate Φ by a vector $v \in \mathbb{R}^2$. A single non-zero vector would be enough but, for symmetry, it would be more natural to consider all. One gets new charts (U_0^v, χ_0^v), (U_1^v, χ_1^v) and, all together, a "polar atlas" for the plane $\mathbb{R}^2$

$$\mathscr{A}_{\mathbb{R}^2}^{\text{polar}} = \left\{ (U_i^v, \chi_i^v) : i \in \mathbb{R}^2, v \in \mathbb{R}^2 \right\}.$$

You should check that this is, indeed, a smooth atlas!

Example 2.6. (inside Euclidean spaces) For embedded m-dimensional submanifolds $M \subset \mathbb{R}^n$ (cf. Definition 1.40), there are two interesting atlases on M: the atlas $\mathscr{A}_M^{\max}$ consisting of all smooth m-dimensional charts of M in the sense of Definition 1.40, and the atlas $\mathscr{A}_M^{\text{adapt}}$ consisting of all charts that arise from smooth charts of $\mathbb{R}^n$ that are adapted to M (see the discussion following Definition 1.40).

The charts of an atlas are used to transfer notions and properties that involve smoothness from the Euclidean spaces (and opens inside) $\mathbb{R}^m$ to M; the compatibility of the charts ensures that the resulting notions (now on M) do not depend on the choice of the charts from the atlas. Hence one may say that an atlas on M allows us to put a "smooth structure" on M. To test this philosophy, let us introduce temporarily the following concept/notation: *given a smooth m-dimensional atlas $\mathscr{A}$ on the space M, a function $f : M \to \mathbb{R}$ is called $\mathscr{A}$***-smooth** *if for any chart (U, χ) that belongs to $\mathscr{A}$,*

$$f_\chi := f \circ \chi^{-1} : \chi(U) \to \mathbb{R}$$

is smooth in the usual sense ($\chi(U)$ is an open in $\mathbb{R}^m$!). The collection of such $\mathscr{A}$-smooth-maps will be denoted:

$$\mathscr{C}^\infty(M, \mathscr{A}). \tag{2.1.1}$$

The first remark one can make is: two rather different atlases may give rise to the same notion of smoothness.

Example 2.7. Returning to Example 2.4 and Example 2.5 one has:

$$\mathscr{C}^\infty\left(\mathbb{R}^2, \mathscr{A}_{\mathbb{R}^2}\right) = \mathscr{C}^\infty\left(\mathbb{R}^2, \mathscr{A}_{\mathbb{R}^2}^{\text{lin}}\right) = \mathscr{C}^\infty\left(\mathbb{R}^2, \mathscr{A}_{\mathbb{R}^2}^{\text{polar}}\right) = \mathscr{C}^\infty\left(\mathbb{R}^2, \mathscr{A}_{\mathbb{R}^2}^{\max}\right)$$

Proving these equalities may serve as an exercise; for a more conceptual argument, one should return at this exercise at the end of this section.

This brings us to the notion of equivalence of atlases:

Definition 2.8. We say that two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are **smoothly equivalent**, and we we write $\mathscr{A}_1 \sim \mathscr{A}_2$, if any chart in $\mathscr{A}_1$ is smoothly compatible with any chart in $\mathscr{A}_2$ (or, shorter: if $\mathscr{A}_1 \cup \mathscr{A}_2$ is again a smooth atlas).

In the proposition below, by a **maximal** smooth atlas we mean a smooth atlas that is maximal w.r.t. inclusions, i.e., with the property that there is no strictly larger smooth atlas containing it.

Proposition 2.9. Given a space M,

(i) for any smooth atlas $\mathscr{A}$ on M the collection $\mathscr{A}^{\max}$ consisting of the charts of M that are smoothly compatible with all the charts from $\mathscr{A}$ has the following properties:

(i1) $\mathscr{A}^{\max}$ is a maximal smooth atlas;
(i2) $\mathscr{A}^{\max}$ is smoothly equivalent to $\mathscr{A}$;
(i3) $\mathscr{A}^{\max}$ is the unique maximal smooth atlas containing $\mathscr{A}$.

(ii) for any two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ on M one has:

$$\mathscr{A}_1 \sim \mathscr{A}_2 \iff \mathscr{A}_1^{\max} = \mathscr{A}_2^{\max}.$$

In particular, $\sim$ is an equivalence relation and $\mathscr{A} \mapsto \mathscr{A}^{\max}$ induces a 1-1 correspondence

$$\left\{ \begin{matrix} \text{equivalence classes of} \\ \text{smooth atlases} \\ \text{on } M \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{maximal} \\ \text{atlases} \\ \text{on } M \end{matrix} \right\} \tag{2.1.2}$$

Proof. We first prove that $\mathscr{A}^{\max}$ is a smooth atlas. Since $\mathscr{A} \subset \mathscr{A}^{\max}$, we only have to show that each two $\chi_1, \chi_2 \in \mathscr{A}^{\max}$ are smoothly compatible. We look at $\chi_2 \circ \chi_1^{-1}$, which is a continuous map between two opens in $\mathbb{R}^m$, say

$$\chi_2 \circ \chi_1^{-1} : \Omega_{1,2} \to \Omega_{2,1}.$$

We claim that this is smooth. Since smoothness is a local property, let us look around an arbitrary point $x \in \Omega_{1,2}$. Consider the corresponding point $p = \chi_1^{-1}(x) \in M$. Since $\mathscr{A}$ is an atlas, we find a chart $\chi \in \mathscr{A}$ whose domain contains p. By hypothesis, χ is smoothly compatible with both χ_1 and χ_2. Therefore,

$$\chi_2 \circ \chi_1^{-1} = (\chi_2 \circ \chi^{-1}) \circ (\chi \circ \chi_2^{-1})$$

is smooth on the open in $\mathbb{R}^m$ on which the compositions from the right hand side are defined. That open contains x and, therefore, $\chi_2 \circ \chi_1^{-1}$ is smooth around x. By symmetry, also $\chi_1 \circ \chi_2^{-1}$ will be smooth, hence χ_1 and χ_2 are smoothly compatible.

Since $\mathscr{A}$ is a smooth atlas, one has $\mathscr{A} \subset \mathscr{A}^{\max}$. Next, remark that if $\widetilde{\mathscr{A}}$ is any other atlas containing $\mathscr{A}$, then $\widetilde{\mathscr{A}} \subset \mathscr{A}^{\max}$ (indeed, if $\widetilde{\chi} \in \widetilde{\mathscr{A}}$, since the charts of $\widetilde{\mathscr{A}}$ are compatible with each other and $\mathscr{A} \subset \widetilde{\mathscr{A}}$, $\widetilde{\chi}$ will be smoothly compatible with all charts from $\mathscr{A}$, hence $\widetilde{\chi} \in \mathscr{A}^{\max}$). The last remark implies both the maximality of $\mathscr{A}^{\max}$ as well as item (i3). Item (i2) holds by construction.

We now move to (ii). The reverse implication is clear and we do focus on the converse. Hence start from $\mathscr{A}_1 \sim \mathscr{A}_2$. Due to symmetry, it suffices to show that any $\widetilde{\chi} \in \mathscr{A}_1^{\max}$ is also in $\mathscr{A}_2^{\max}$ Hence, for arbitrary $\widetilde{\chi} \in \mathscr{A}_1^{\max}$ and $\chi_2 \in \mathscr{A}_2$, we have to show that $\widetilde{\chi} \circ \chi_2^{-1}$ and $\chi_2^{-1} \circ \widetilde{\chi}$ are smooth. Again, one proceeds locally. E.g., for the first composition, for x in its domain, one chooses $\chi_1 \in \mathscr{A}_1$ whose domain contains $\chi_2^{-1}(x)$, and then we can write $\widetilde{\chi} \circ \chi_2^{-1} = (\widetilde{\chi} \circ \chi_1^{-1}) \circ (\chi_2^{\circ} \chi_2^{-1})$, and this will be smooth. 😀

Example 2.10. (various atlases on Euclidean spaces) On $\mathbb{R}^m$, next to the identity atlas

$$\mathscr{A}_{\mathbb{R}^m} := \{\mathrm{Id}_{\mathbb{R}^m}\}, \tag{2.1.3}$$

one can also consider two small variations"

$$\mathscr{A}_1 := \{\mathrm{Id}_U : U - \text{open in } \mathbb{R}^m\}, \quad \mathscr{A}_2 := \{\mathrm{Id}_B : B - \text{open ball in } \mathbb{R}^m\}.$$

These all induce the smooth structure on $\mathbb{R}^m$ (the standard one) but they are not all comparable:

$$A_1^{\max} = \mathscr{A}_2^{\max} = \mathscr{A}_{\mathbb{R}^m}^{\max}, \quad \mathscr{A}_{\mathbb{R}^m} \subset \mathscr{A}_1 \supset \mathscr{A}_2, \quad \mathscr{A}_{\mathbb{R}^m} \cap \mathscr{A}_2 = \{\emptyset\}..$$

Exercise 2.11. Next, return to Exercise 2.7 and do it slightly differently. First prove that

$$(\mathscr{A}_{\mathbb{R}^2})^{\max} = \left(\mathscr{A}^{\mathrm{lin}}_{\mathbb{R}^2}\right)^{\max} = \left(\mathscr{A}^{\mathrm{polar}}_{\mathbb{R}^2}\right)^{\max} = \mathscr{A}^{\max}_{\mathbb{R}^2}.$$

Then prove that, for any smooth atlas $\mathscr{A}$ on any space M, one has

$$\mathscr{C}^\infty(M,\mathscr{A}) = \mathscr{C}^\infty(M,\mathscr{A}^{\max}),$$

As we shall see a bit later (see Proposition 2.63 below), if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two smooth atlases on M, then

$$\mathscr{C}^\infty(M,\mathscr{A}_1) = \mathscr{C}^\infty(M,\mathscr{A}_2) \iff \mathscr{A}_1 \sim \mathscr{A}_2. \tag{2.1.4}$$

The discussion above and the support provided by Proposition 2.9, it is becoming clear what is the structure on a space M that encodes smoothness.

Definition 2.12. An m-dimensional **smooth structure** $\mathscr{S}$ on a topological space M is an equivalence class of m-dimensional smooth atlases on M. By 2.1.2, one may also interpret $\mathscr{S}$ as single (m-dimensional) *maximal* smooth atlas on M.

For an arbitrary atlas $\mathscr{A}$ on M, the corresponding equivalence class or, equivalently, the maximal atlas $\mathscr{A}^{\max}$, is called **the smooth structure induced by the atlas $\mathscr{A}$**.

Remark 2.13. (Why care about atlases which are not maximal?) Since smooth structures can be described using maximal atlases, why would one care about non-maximal atlases? The simple answer is that there are many natural atlases that are not maximal- e.g. the various atlases that we mentioned on the Euclidean space including the polar atlas on $\mathbb{R}^2$. However, equally important is the fact that nothing is lost by considering such natural but possibly non-maximal atlases; the key points are:

- after all, any atlas $\mathscr{A}$ induces a maximal one $\mathscr{A}^{\max}$, hence a smooth structure.
- for many notions that involve conditions to be satisfied for all charts in a maximal atlas, it suffices to check those condition on a/any smaller atlas included in the original one (e.g., for the notion of smoothness of $\mathbb{R}$-valued functions, this is the subject of the previous exercise!).

One may even say that it is desirable to describe a smooth structure by an atlas that is "as small as possible". Furthermore, for various parts of the theory, it is important to be able to find atlases that are at most countable- and that is one of the main reasons for which our spaces will be required to be 2nd countable. More precisely, here is an interesting exercise on Topology:

Exercise 2.14. Given a smooth structure on a space M, the smooth structure can be defined by an atlas that is at most countable if and only if M is 2nd countable.

2.1.2 Manifolds

We now come to the main objects of study of this course.

Definition 2.15. A **smooth** m**-dimensional manifold** is a Hausdorff, 2nd countable topological space M together with am m-dimensional smooth structure on M.

Given a smooth m-dimensional manifold M, when saying that (U,χ) is a **chart of the smooth manifold** M we mean that (U,χ) belongs to the maximal atlas $\mathscr{A}$ defining the smooth structure on M.

Example 2.16 (Euclidean spaces and opens inside them). The standard smooth structure on $\mathbb{R}^m$ is, by definition, the one whose smooth charts are precisely the standard smooth charts on $\mathbb{R}^m$, i.e., the ones from Definition 1.36. We have already mentioned several atlases inducing the smooth structure, starting from the simplest one consisting of the identity chart only (2.1.3). Similarly, any open

$$\Omega \subset \mathbb{R}^m$$

comes with a standard smooth structure making it into an m-dimensional manifold. The maximal atlas consists of the standard smooth charts of $\mathbb{R}^m$ with domain inside Ω. At the other extreme, we have again the identity atlas

$$\mathscr{A}_\Omega := \{\mathrm{Id}_\Omega\};$$

Actually, these are all the possible manifolds for which the smooth structure can be induced by an atlas consisting of one chart only- see Exercise 2.52.

Example 2.17 (Disguised Euclidean spaces: vector spaces, matrices). First of all, any finite dimensional vector space has a very natural atlas $\mathscr{A}_V^{\mathrm{lin}}$- the one whose coordinate charts are precisely the ones that correspond to bases $\mathfrak{e}$ of V- see Example 2.3. The resulting smooth structure will be called the standard smooth structure on V. On the other hand, if we do fix a basis of V, then one obtains an "identification" of V with $\mathbb{R}^n$. Notice that, by this identification, the standard smooth structures of V and $\mathbb{R}^n$ correspond to each other. Hence, in some sense, V is just a disguised version of $\mathbb{R}^n$.

One such example of spaces that is an Euclidean space without really thinking about it that way is the space of all $m \times n$ matrics, say with real coefficients:

$$\mathscr{M}_{m,n}(\mathbb{R}).$$

This is an mn dimensional vector spaces (which even comes with a canonical basis!), hence can already be identified with $\mathbb{R}^{mn}$. As manifold, one can think of the smooth structure on $\mathscr{M}_{m,n}(\mathbb{R})$ as induced by a single chart:

$$\mathscr{M}_{m,n}(\mathbb{R}) \ni A = (A^i_j)_{ij} \quad \stackrel{\chi}{\longmapsto} \quad (A^1_1,\ldots,A^1_n,A^2_1,\ldots,A^2_n,\ldots,A^m_1,\ldots,A^m_n) \in \mathbb{R}^{mn}.$$

There are various interesting examples of manifolds that arise as spaces of matrices. Let us mention here the simplest one, as it is an open in our Euclidean n^2-dimensional space $\mathscr{M}_n(\mathbb{R})$:

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathscr{M}_n(\mathbb{R}) : A - \text{is invertible}\}.$$

It is open because $\det : \mathscr{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

A similar discussion applies to matrices with complex entries. Since $\mathbb{C}$ can be identified with $\mathbb{R}^2$, $\mathscr{M}_{m,n}(\mathbb{C})$ will be a $4mn$-dimensional smooth manifold, while $\mathrm{GL}_n(\mathbb{C})$ will be a $4n^2$ one.

Example 2.18 (General opens). Given an m-dimensional manifold M, any non-empty open $U \subset M$ carries a natural (induced) smooth structure that makes U itself into an m-dimensional manifold: the charts of U are, by definition, the charts of M whose domain are contained in U.

Example 2.19 (Connected components). Given an m-dimensional manifold M it follows from Topology that each connected component of M is both open and closed in M (this is true because M is locally path connected, since it is locally Euclidean). In particular, M is partitioned into open submanifolds. Of course, one can also go the other way around: given m-dimensional manifolds M_i parametrised by $i \in I$, their disjoint union

$$\bigsqcup M_i$$

carry a smooth structure as well, obtained by putting together all the charts of all the M_is. To ensure that the second countability axiom is still satisfied, one has to assume that I is at most countable.

In conclusion, any m-dimensional manifold M is a countable disjoint union of connected m-dimensional manifolds. For that reason, most of the time one works with connected manifolds (sometimes even forgetting to mention that, unfortunately).

Example 2.20 (The circle). The unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{(\cos t, \sin t) : t \in \mathbb{R}\}$$

(with the topology induced from $\mathbb{R}^2$) has a natural atlas in with four charts, with domains the four half planes

$$U_\pm = \{(x,y) \in S^2 : \pm y > 0\}, \quad V_\pm = \{(x,y) \in S^2 : \pm x > 0\},$$

and coordinate functions the restrictions of the two projections

$$\mathrm{pr}_1|_{U_\pm} : U_\pm \to (-1,1), \quad \mathrm{pr}_2|_{V_\pm} : V_\pm \to (-1,1),$$

This atlas allows us to turn S^1 into a smooth, 1-dimensional manifold. From now on, when talking about S^1 as a manifold, we always endow it with this smooth structure, that we will call canonical. One should be aware however that there are a lot more atlases that one can choose to describe this smooth structure.

Example 2.21 (product of manifolds). One of the simplest way to get new manifolds is by using products. In general, the product $M \times N$ of two manifolds M and N can be made into a manifold, with $\dim M \times N = \dim M + \dim N$. The key remark is that, starting with smooth charts $\chi : U \to \Omega \subset \mathbb{R}^m$ of M and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ of N, their product

$$\chi \times \chi' : U \times U' \to \Omega \times \Omega' \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}, \quad (p,q) \mapsto (\chi(p), \chi'(q))$$

is a chart on the topological space $M \times N$. Of course, any two such product charts are smoothly compatible, and all together define the desired smooth structure on $M \times N$. This is called **the product smooth structure**.

Notice that, by the previous construction, any smooth atlases $\mathscr{A}_M$ and $\mathscr{A}_N$ inducing the smooth structures of M and N, respectively, gives rise to a similar atlas $\mathscr{A}_M \times \mathscr{A}_N$; however, the latter is almost never maximal even if the former ones are.

Example 2.22 (the torus). Let us consider the familiar torus or, to be more concrete, let us consider a model of the torus obtained by rotation a circle around the OZ axis, as shown in the picture. One has the explicit parametric and implicit descriptions

$$T^2 = T^2_{R,r} = \left\{ \Big((R + r \cdot \cos a) cos\, b,\ (R + r \cdot \cos a) sin\, b,\ r \cdot \sin a \Big) :\ a, b \in [0, 2\pi] \right\} \tag{2.1.5}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}. \tag{2.1.6}$$

A more abstract description of the torus would be as a product

$$S^1 \times S^1;$$

the way to identify it with the model in $\mathbb{R}^3$ is indicated in Figure 2.5, where the blue and the red circles indicate the parallels and the meridians on the torus, both parametrised by S^1. In formulas, the identification between the two is given by the homeomorphism

$$F : S^1 \times S^1 \to T^2_{R,r}, \quad (e^{i \cdot a}, e^{i \cdot b}) \mapsto ((R + r \cdot \cos a) cos\, b,\ (R + r \cdot \cos a) sin\, b,\ r \cdot \sin a). \tag{2.1.7}$$

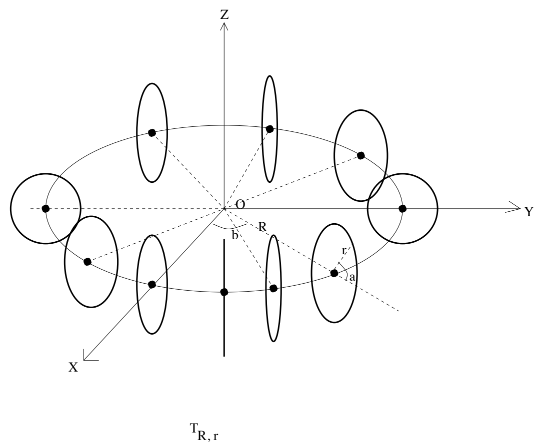


Fig. 2.4

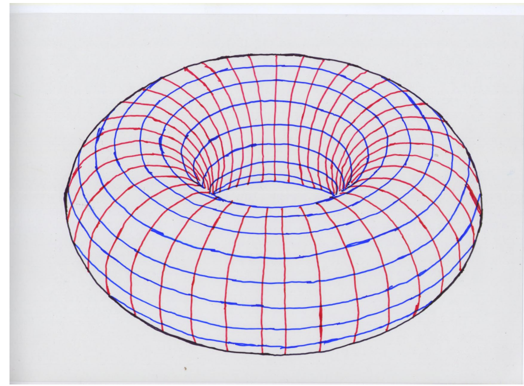


Fig. 2.5

The smooth structure on the S^1 combined with the general construction of taking products of manifolds provide the torus $S^1 \times S^1$ with a smooth structure, called again "the standard one".

Example 2.23 (Embedded submanifolds of Euclidean spaces). Any embedded submanifold $M \subset \mathbb{R}^n$ carries a natural smooth structure, whose charts are the standard smooth charts, i.e., the ones in the sense of Definition 1.40. This will be called **the standard smooth structure on the embedded submanifold**. Another atlas that describes the same smooth structure is the atlas $\mathscr{A}_M^{\text{adapt}}$ consisting of all charts that arise restrictions to M of smooth charts of $\mathbb{R}^n$ that are adapted to M (see the discussion following Definition 1.40).

Often in examples there are smaller and/or nicer atlases inducing the standard smooth structure on M. For instance, one can now check that for the circle $S^1 \subset \mathbb{R}^2$ and the torus $T_{R,r} \subset \mathbb{R}^3$ (embedded submanifolds!) the smooth structures that we described above coincide with the standard smooth structures. Another nice example is the atlas on the sphere provided by the stereographic projections - see below.

On the other hand, one may be aware that any manifold M can be "embedded" in some Euclidean space. That is a very interesting result, but the conclusion may be a bit misleading. The "general" theory frees the embedded submanifolds from the ambient spaces shading light even on the notions that are described, originally, using those embeddings. Think e.g. of what happens in Topology, when compactness was originally described for subsets of $\mathbb{R}^n$ requiring them to be closed and bounded, and then turned out to be a completely topological property (with important consequences). Moreover, some manifolds just do not come with "natural embeddings" into some Euclidean spaces- see e.g. the projective spaces! Finally, at some point we want to talk about various geometric structures on manifolds, and such structures usually clash with ambient Euclidean spaces.

One of the main tools for proving that a subset $M \subset \mathbb{R}^n$ is an embedded submanifold is to describe it via equations and then use the following version of the regular value theorem".

Theorem 2.24 (the regular value theorem in $\mathbb{R}^n$). *Assume that $M \subset \mathbb{R}^n$ can be written as the zero-set of a smooth map $F : \Omega \to \mathbb{R}^k$ defined on an open subset $\Omega \subset \mathbb{R}^n$ and assume that F is a submersion at each point $p \in M$. Then M is an $m = n - k$ dimensional embedded submanifold of $\mathbb{R}^n$.*

Example 2.25 (the unit circle again). The unit circle S^1 can be described as in the theorem: just take $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = x^2 + y^2 - 1$. The total derivative $(Df)_p$ at a point $p = (a,b)$ is the row matrix $(a\ b)$, which is clearly non-zero when $a^2 + b^2 = 1$. Hence the theorem can be applied. A similar argument works also for the higher dimensional spheres.

Exercise 2.26. Return to the torus (Example 2.22) and use the last theorem to deduce that $T_{R,r}^2$ is an embedded submanifold of $\mathbb{R}^3$.

Example 2.27 (the Klein bottle). A good friend of the torus is the Klein bottle. It was introduced in Topology as the space obtained from the square by gluing its edges as indicated in Figure 2.6 and you have probably seen also

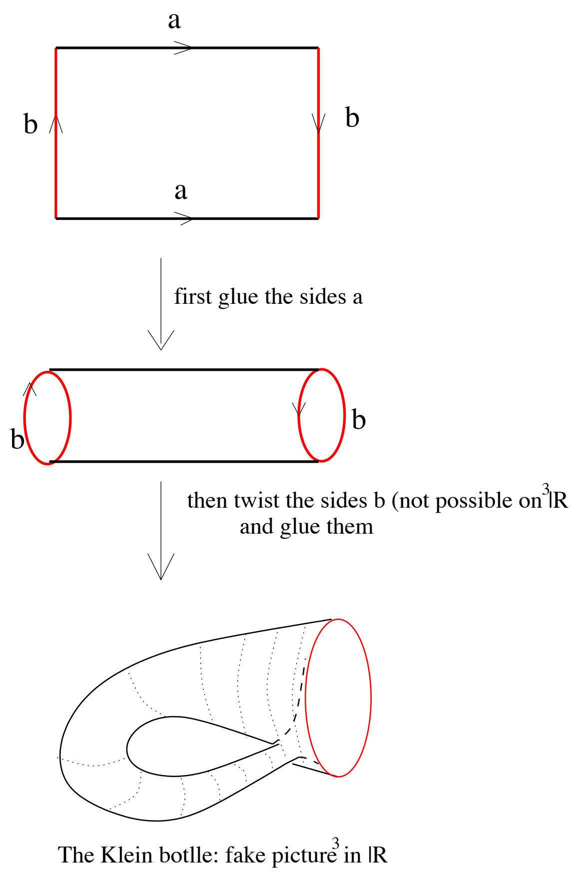


Fig. 2.6

the following concrete (topological) embedding into $\mathbb{R}^4$

$$K = \{((2+cos(2\pi s))cos(2\pi t),(2+cos(2\pi s))sin(2\pi t),sin(2\pi s)cos(\pi t),sin(2\pi s)sin(\pi t)) : s,t \in [0,2\pi]\}. \tag{2.1.8}$$

Exercise 2.28. Show that (2.1.8) makes K into an embedded submanifold of $\mathbb{R}^4$.

Example 2.29 (Embedded submanifolds). More generally, one can look at embedded submanifolds inside any other manifold. To make this precise, it is useful to start from the description of embedded submanifolds of Euclidean spaces in terms of adapted charts (Proposition 1.41 in subsection 1.2.2). The key remark is that the notion of adapted chart makes sense for $\mathbb{R}^n$ replaced by any other manifold N.

Definition 2.30. Given an n-dimensional manifold N, an (m-dimensional) **embedded submanifold** of N is any subset $M \subset N$ with the property that for any $p \in M$ there exists a chart

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$$

for N around p such that $\widetilde{\chi}(\widetilde{U} \cap M) = \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$ or, equivalently,

$$\widetilde{U} \cap M = \{p \in \widetilde{U} : \widetilde{\chi}_{m+1}(p) = \ldots = \widetilde{\chi}_n(p) = 0\} \tag{2.1.9}$$

A chart $(\widetilde{U}, \widetilde{\chi})$ of N satisfying this equality is called **chart of N adapted to M**.

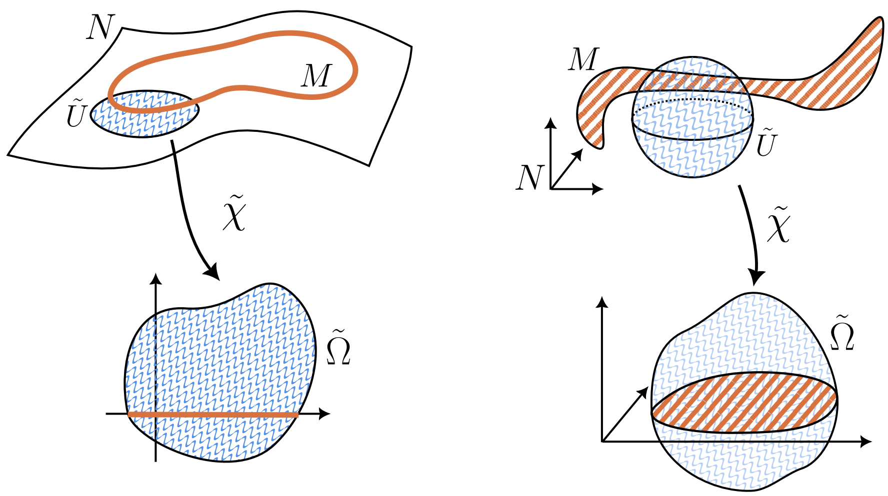


Fig. 2.7 Two examples of charts $\tilde{\chi} : \tilde{U} \to \tilde{\Omega}$ of an ambient manifold N adapted to an embedded submanifold M. The dimensions of the manifolds M, N are $(1,2)$ on the left hand side, and $(2,3)$ on the right hand side. Can you draw a picture for dimensions $(1,3)$?

For any such chart one can talk about **the restriction of the adapted chart to M**:

$$\widetilde{\chi}|_M : U \to \Omega, \quad \text{where } U := \widetilde{U} \cap M,\ \Omega := \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\}),$$

with Ω interpreted as an (open) subset of $\mathbb{R}^m$. With these we obtain an **induced smooth structure** on M.

Exercise 2.31. Check that, indeed, the collection of all charts of type $\widetilde{\chi}|_M$ obtained from charts $\widetilde{\chi}$ of N that are adapted to M, define a smooth structure on M.

Exercise 2.32. Look at the end of the reminder on Topology and at Exercise 2.180 and deduce that for any Hausdorff space M endowed with a smooth structure, the following are equivalent:

- M is 2nd countable (hence a manifold).
- M admits a countable atlas.
- M admits an exhaustion (as in Theorem 1.7).
- M is paracompact and the collection of connected components is at most countable.

Remark 2.33. **(Manifolds without Topology?)** A small but non-zero percentage of the students that follow the Manifolds course did not follow Topology. This subsection is especially for them and the question: can we talk about manifolds without topology? Strictly speaking that is possible, as it follows from the following simple remark: given a manifold M, $O \subset M$ is open if and only if for any smooth chart $\chi : U \to \Omega$ of M, one has that $\chi(U \cap O)$ is open in Ω.

However, being possible does not mean it is also a good idea, or that it should be done. One problem with such an approach is that its "directness" (avoiding topology) is only apparent, and the various topological notions that inevitably show up over and over again would look even more complicated and also rather mysterious (for instance, to hide the second countability axiom we would have to require countable atlases- as indicated by Exercise 2.14). Another problem with such an approach (even more important than the previous one) is the fact that the various manifolds that we encounter are, before anything, topological spaces in a very natural way. Not making any reference to such natural topologies would be artificial, less intuitive and things would become harder. Therefore, my strong advise for those with no background on Topology: spend first a couple of days (say a full week) studying the summary from the first chapter, to get a grasp on the main concepts and results. However, here is an exercise that indicates how one could (but should not) proceed.

Exercise 2.34. Let M be a set and let $\mathscr{A}$ be a collection of bijections

$$\chi : U \to \chi(U) \subset \mathbb{R}^m$$

between subsets $U \subset M$ and opens $\chi(U) \subset \mathbb{R}^m$, such that the domains of all the functions $\chi \in \mathscr{A}$ cover M entirely. We look for topologies on M with the property that each $\chi \in \mathscr{A}$ becomes a homeomorphism. We call them topologies compatible with $\mathscr{A}$.

(i) Show that M admits at most one topology compatible with $\mathscr{A}$.
(ii) Assume now that, for any $(U,\chi),(U',\chi') \in \mathscr{A}$, $\chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U')$ is a homeomorphism between opens in $\mathbb{R}^m$. Show that M admits a topology compatible with $\mathscr{A}$. (Hint: topology basis).
(iii) However, please be aware that the resulting topology may fail to be Hausdorff. Actually, even better: please find an example when that happens.

On the other hand, it is interesting how to see how one can put a manifold structure on a set M on which we do have a topology but we did not put yet a smooth structure. If you have solved the previous exercise, you are almost done proving also the following:

Lemma 2.35. Let M be a set and let $\mathscr{A}$ be a countable collection of bijections

$$\chi : U \to \chi(U) \subset \mathbb{R}^m$$

between subsets $U \subset M$ and opens $\chi(U) \subset \mathbb{R}^m$, such that their domains U cover M entirely. We assume that:

(i) For any $(U,\chi),(U',\chi') \in \mathscr{A}$,

$$\chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U')$$

is a smooth map between opens in $\mathbb{R}^m$.

(ii) for all $p, p' \in M$ distinct, there exists (U,χ), (U',χ') in $\mathscr{A}$ and opens $\Omega, \Omega' \subset \mathbb{R}^m$ such that $\chi^{-1}(\Omega)$ contains p, $\chi'^{-1}(\Omega')$ contains p' and the two are disjoint.

Then M can be made into a manifold for which $\mathscr{A}$ is a smooth atlas in one, and only one way.

2.1.3 *Smooth maps; diffeomorphisms*

Having introduced the main objects (manifolds), we now move to the maps between them. The idea will always be the same: use charts to move to Euclidean spaces.

Definition 2.36. Consider an m-dimensional manifold M, an n-dimensional one N and a map between them

$$F : M \to N$$

Given charts (U,χ) of M and (U',χ') of N, the **representation of F with respect to χ and χ'** is defined as

$$F_\chi^{\chi'} := \chi' \circ F \circ \chi^{-1}.$$

We say that F is **smooth** if it is continuous and, for any chart χ of M and χ' of N, $F_\chi^{\chi'}$ is smooth (in the usual sense of Analysis).

The map $F_\chi^{\chi'}$ makes sense when applied to a point of type $\chi(p) \in \mathbb{R}^d$ with $p \in U$ with the property that $F(p) \in U'$, i.e. $p \in U$ and $p \in F^{-1}(U')$. Therefore, it is a map

$$F_\chi^{\chi'} : \mathrm{Domain}(F_\chi^{\chi'}) \to \mathbb{R}^n.$$

whose domain is the following open subset of $\mathbb{R}^m$:

$$\mathrm{Domain}(F_\chi^{\chi'}) = \chi(U \cap F^{-1}(U')) \subset \mathbb{R}^m.$$

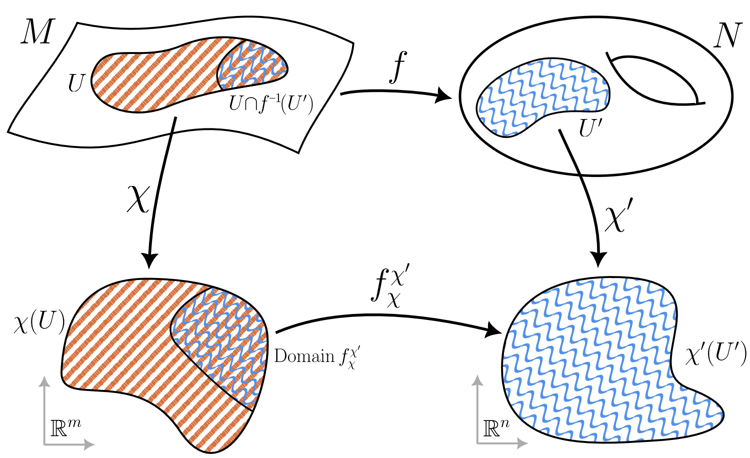


Fig. 2.8 A map $F : M \to N$ between an m- and n-dimensional manifold equipped with local charts χ and χ' defined on opens $U \subseteq M$ and $U' \subseteq N$, respectively, can locally be characterized by the representative $F_\chi^{\chi'}$: Domain $F_\chi^{\chi'} \to \mathbb{R}^n$. These maps commute wherever they can be evaluated sensibly, i.e. $\chi' \circ F = F_\chi^{\chi'} \circ \chi$ holds on $U \cap F^{-1}(U')$.

The next exercise supports Remark 2.13 on the usefulness of having non-maximal atlases available.

Exercise 2.37. Let $\mathscr{A}_M$ and $\mathscr{A}_N$ be two atlases (not necessarily maximal) inducing the smooth structure on M and N, respectively. Show that is $F_{\chi}^{\chi'}$ is smooth for all $\chi \in \mathscr{A}_M$ and $\chi' \in \mathscr{A}_N$ then F is smooth.

Then show that smoothness can also be checked locally, i.e., it suffices to show that: for any $p \in M$ there exists a smooth chart χ for M around p and a smooth chart χ' for N around $F(p)$ such that $F_{\chi}^{\chi'}$ is smooth.

Exercise 2.38. Show that composition of smooth maps is again smooth.

Exercise 2.39. We consider the circle endowed with the standard smooth structure (cf. Example 2.20). Prove, using the definition, that the map $F: S^1 \to S^1$, $F(\cos t, \sin t) = (\cos 2t, \sin 2t)$ is smooth.

Example 2.40 (charts). Any chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of an m-dimensional manifold M is itself smooth as a map between U with the smooth structure induced from the one of M (Example 2.18) and the similar one on $\Omega \subset \mathbb{R}^m$.

Example 2.41 (products). For the product $M \times N$ of smooth manifolds (Example 2.21), the canonical projections $\mathrm{pr}_M : M \times N \to M$, $\mathrm{pr}_N : M \times N \to N$ are smooth.

Exercise 2.42. Given manifolds M, N and P, and a map

$$H = (F, G) : P \to M \times N,$$

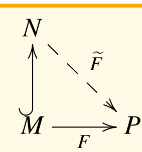
show that H is a smooth if and only if F and G are. Use Exercise 2.38 to prove one of the directions.

Example 2.43 (inclusions of embedded submanifolds). For any embedded submanifold $M \subset N$, the inclusion

$$i : M \hookrightarrow N$$

is smooth, where M is endowed with the smooth structure. Combined with the Exercise 2.38 we see that, if we are given map $F : M \to P$ into another smooth manifold P, then

$$\left.\begin{array}{r} F : M \to P \\ \exists \text{ smooth extension } \widetilde{F} : N \to P \end{array}\right\} \Longrightarrow F : M \to P \text{ is smooth} \tag{2.1.10}$$

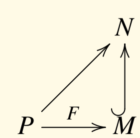


This is handy already for embedded submanifolds of $\mathbb{R}^m$ when one can use $N = \mathbb{R}^m$ or opens inside it. E.g., $f : S^{m-1} \to \mathbb{R}$, $f(x) = \sin(x_1)x_2/||x||^2$ is smooth simply because it make sense as a function on the open $N = \mathbb{R}^m \setminus \{0\}$.

Example 2.44. Still assuming that $M \subset N$ is an embedded submanifold, but using them as codomain, we have that:

$$\forall \text{ smooth manifold } P \text{ and } \forall F : P \to M \text{ one has:} \tag{2.1.11}$$

$$F \text{ is smooth} \Longleftrightarrow F \text{ is smooth as an } N\text{-valued function}$$



Proof. Being smooth as an N-valued function is the same as saying that the composition of F with the inclusion $i : M \to N$, $i \circ F : P \to N$, is smooth. Hence the direct implication follows from the last exercise. For the reverse implication, to see that $F : P \to M$ is smooth, one checks the definition where for M one uses only charts induced from N (which is allowed by Exercise 2.37).

Exercise 2.45. Using (2.1.10) and (2.1.11):

a) prove again that the map from Exercise 2.39 is smooth.
b) prove that the map $F : S^1 \times S^1 \to T_{R,r}$ from (2.1.7) is smooth.
c) go on and deduce a general criteria for the smoothness of maps $F : M \to M'$ between embedded submanifolds $M \subset P$ and $M' \subset P'$.

Example 2.46 (maps between embedded submanifolds). The outcome of Example 2.43 and of the last exercise does not have a converse in full generality. However, it does if one proceeds locally. That explains the rather ad-hoc Definition 1.32 for the smoothness of maps between embedded submanifolds of Euclidean spaces. It is now time to reconciliate between that definition and the one introduced in this section. Therefore, assume that $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ are embedded submanifolds endowed with their standard smooth structure (Example 2.23).
Claim: the smoothness of a map $F : M \to M'$ in the sense discussed here is equivalent to the smoothness of F as a function between subsets of Euclidean spaces as discussed in Section 1.2, Definition 1.32.

Proof. Let us call "A-smoothness" the notion from Section 1.2 of Chapter 1, and smoothness the one from this chapter. It is clear that the two coincide when M and N are open in their ambient Euclidean spaces.

Assume first that F is A-smooth. Since smoothness can be checked locally (see Exercise 2.37), we may focus on neighborhoods of an arbitrary point $p \in M$. Choosing neighborhoods of type $U = \widetilde{U} \cap M$ with $\widetilde{U}$ chosen so that $F|_U$ admits a smooth extension to $\widetilde{U}$, the discussion above implies that F is indeed smooth when restricted to U. However, let us spell it out. Since F is A-smooth, we may assume that $F|_U = \tilde{F}|_U$ with $\tilde{F} : \tilde{U} \to \tilde{U}'$ is smooth; in turn, $\tilde{F}$ induces $\Phi := \tilde{F}_{\tilde{\chi}}^{\tilde{\chi}'} : \Omega \to \Omega'$, whose restriction to $\Omega \cap (\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{\tilde{m}}$ is precisely $F_{\chi}^{\chi'}$ (and takes values in $\Omega' \cap (\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{\tilde{n}})$. Hence we are in the situation of having a smooth map $\Phi : \Omega \to \Omega'$ between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, taking $\Omega_0 = \Omega \cap (\mathbb{R}^m \times \{0\}$ to $\Omega_0' = \Omega' \cap (\mathbb{R}^n \times \{0\}$ and we want to show that $\Phi|_{\Omega_0} : \Omega_0 \to \Omega_0'$ is smooth as a map between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$; that should be clear.

Assume now that F is smooth. Tho check that it is A-smooth, we fix $p \in M$ and consider charts $\tilde{\chi}$ and $\tilde{\chi}'$ and proceed as above and with the same notation. This time but we do not have $\tilde{F}$, but having it is equivalent to having Φ extending $\Phi_0 = F_{\chi}^{\chi'}$. Then we are in a similar general situation as above, when we have a smooth map $\Phi_0 : \Omega_0 \to \Omega_0'$ between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$ and we want to extend it to a smooth map Φ between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, defined in a neighborhood of $\chi(p)$. Using the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and similarly for $\mathbb{R}^{n'}$, we just set $\Phi(x,y) = (\Phi_0(x), 0)$. 😀

Example 2.47 (smooth curves). An extreme is when $M = I \subset \mathbb{R}$ is an open interval (which, according to our conventions, is always endowed with the standard smooth structure- see Example 2.16) and N is an arbitrary manifold. Then smooth maps

$$\gamma : I \to N$$

are called **(smooth) curves in** N. On I one can use the atlas consisting of the identity chart only, hence the smoothness of γ is checked using charts (U, χ) for M; the resulting representation of γ in the chart χ,

$$\gamma^{\chi} := \chi \circ \gamma : I_{\chi} \subset \mathbb{R}^m$$

(with domain $I_{\chi} = \gamma^{-1}(U) \subset I$) will then have to be smooth in the usual sense.

Finally, we can now look at the correct notion of "isomorphisms" in Differential Geometry (analogous to linear isomorphisms in Linear Algebra, isomorphisms of groups in Group Theory, homeomorphisms in Topology, etc).

Definition 2.48. A **diffeomorphism** between two manifolds M and N is a bijective map $F : M \to N$ with the property that both F and F^{-1} are smooth. Two manifolds M and N are said to be **diffeomorphic** if such a diffeomorphism exists.

A diffeomorphism allows one to pass from whatever differential geometric object/property on M to N and backwards; for that reason, two manifolds that are diffeomorphic are usually thought of as being "(basically) the same as manifolds".

Exercise 2.49. Show that composition of diffeomorphisms is again a diffeomorphism.

Exercise 2.50. Recall that any finite dimensional vector space V can be seen as a smooth manifold (see Example 2.17). Show that any linear map $F : V \to W$ is smooth, when seen as a map between manifolds. Deduce that any m-dimensional vector space is diffeomorphic to $\mathbb{R}^m$.

Example 2.51 (charts). Any chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of an m-dimensional manifold M is a diffeomorphism between the domain U and its image Ω, both endowed with the induced smooth structures as in Example 2.18.

Exercise 2.52. Show that the only smooth manifolds with the property that their smooth structure can be induced by an atlas consisting of only one chart are those that are diffeomorphic to open subsets $\Omega \subset \mathbb{R}^m$.

Example 2.53. The homeomorphism $F : S^1 \times S^1 \to T_{R,r}$ from (2.1.7) is actually a diffeomorphism. However, proving this in detail at this point is not such a pleasant task; we will return to it a bit later.

Example 2.54 (equal versus diffeomorphic). In general, $\mathbb{R}^m$ has many smooth structures, though many of them (but not all!) are actually diffeomorphic. For the easy part of this assertion, here is a construction that you may find puzzling at first:

- any homeomorphism $\chi : \mathbb{R}^m \to \mathbb{R}^m$ gives rise to a smooth atlas on $\mathbb{R}^m$, namely $\mathscr{A}_\chi := \{\chi\}$ (a single chart!).
- if χ is not a diffeomorphism, the smooth structure induced by $\mathscr{A}_\chi$ on $\mathbb{R}^m$ is different from the standard one.
- however, for any χ, $\mathbb{R}^m$ endowed with the smooth structure induced by $\mathscr{A}_\chi$ is diffeomorphic to $\mathbb{R}^m$ endowed with the standard smooth structure.

Exercise 2.55. Prove the last claim. Then stare at the previous example until it no longer seems weird.

Remark 2.56 (For the interested student: exotic smooth structures). As the previous example shows, the interesting question for $\mathbb{R}^m$, and as a matter of fact for any topological space M, is:

how many non-diffeomorphic smooth structures does a topological space M admit?

Even for $M = \mathbb{R}^m$ this is a highly non-trivial question, despite the fact that the answer is deceivingly simple: each of the spaces $\mathbb{R}^m$ with $m \neq 4$ admits only one such smooth structure, while $\mathbb{R}^4$ admits an infinite number (called "exotic" smooth structures on $\mathbb{R}^4$)!

There are some small variations of the notion of diffeomorphism. First of all, one can require that condition locally: a smooth map $F : M \to N$ is a **local diffeomorphism** around a point $p \in M$ if there are open neighborhoods U of p in M, and V of $F(p)$ in N, such that F restricts to a diffeomorphism $F|_U : U \to V$ (where U and V are endowed with the induced smooth structure, cf. Example 2.18).

Exercise 2.57. First, convince yourself that the map

$$\exp : \mathbb{R} \to S^1, \quad t \mapsto e^{it} = (\cos t, \sin t)$$

is a local homeomorphism around any point. Then prove that that the standard smooth structure on S^1 (cf. Example 2.20) is the only one which makes exp into a local diffeomorphism.

The following shows how to reduce checking that a map is a diffeomorphism to the easier task of checking that it is a local diffeomorphism.

Lemma 2.58. A smooth function $F : M \to N$ is a diffeomorphism if and only if it is a local diffeomorphism around any point, as well as a bijection.

Proof. The direct implication is obvious. For the reverse one, the bijectivity assumption allows us to talk about the inverse $G : N \to M$ of F. The question is whether F and G are smooth. But smoothness is a local condition, that can be checked around points. Since F is a local diffeomorphism, it follows that both F as well as G are smooth around any point in their domain. Hence they are both smooth and, therefore, F is a diffeomorphism. 😀

Finally, there is yet another rather small variation of the notion of diffeomorphisms: embeddings. This time, we are allowed to shrink the codomain: we require the map to be a diffeomorphism into its image. Let us define this more carefully. First of all recall that a map

$$F : M \to N$$

between two topological spaces is a **topological embedding** if it is injective and, as a map from M to $F(M)$ (where the second space is now endowed with the topology induced from N), is a homeomorphism. For the smooth version of this notion one encounters a small problem: while any subset of a topological space inherits a natural induced topology, for smooth structures that is not the case- hence it has to be imposed.

Definition 2.59. A map $F : M \to N$ between manifolds M and N is called a **smooth embedding** if:

1. $F(M)$ is an embedded submanifold of N.
2. as a map from M to $F(M)$, F is a diffeomorphism.

Remark 2.60. In some sense, this is just a slight variation on the notion of embedded submanifold: an embedding of a manifold M in another manifold N is just a diffeomorphism between M and an embedded submanifold of N.

Theorem 2.61. *Any manifold can be embedded in an Euclidean space.*

2.1.4 $\mathbb{R}$-valued functions

Similar to the space (algebra) of continuous functions $\mathscr{C}(M)$ on a topological space M (see the reminder on Topology, Section 1.1.2.1), one has the space of real-valued smooth functions on M:

$$\mathscr{C}^{\infty}(M) := \{f : M \to \mathbb{R} : f - \text{smooth}\},$$

(temporarily denoted $\mathscr{C}^{\infty}(M, \mathscr{A})$ in the previous sections (see (2.1.1).) The main point is that, similar to the fact that $\mathscr{C}(M)$ reflects the topology on M, $\mathscr{C}^{\infty}(M$ reflects the smooth structure on M. This was indicated already in the claim made in (2.1.4), which we are able to prove now. We start with a preparatory lemma, which is very important in various other places. Conceptually, the lemma reveals that smooth functions are rather flexible- much more flexible than analytic, holomorphic, or polynomial functions.

Lemma 2.62 (the fundamental property of smooth functions). For any manifold M, for any $p \in M$ and any open neighborhood U of p, there exists $f \in \mathscr{C}^{\infty}(M)$ that is supported in U and such that $f(p) \neq 0$. Actually, one may arrange f so that $f = 1$ in a (small enough) neighborhood of p.

Proof. Given our discussion from $\mathbb{R}^m$ (namely the proof of Theorem 1.28), there is very little that we have to do: we may assume that U is the domain of a coordinate chart $\chi : U \xrightarrow{\sim} \mathbb{R}^m$ (why?) sending p to $0 \in \mathbb{R}^m$, then choose a smooth function $f : \mathbb{R}^m \to [0,1]$ supported in the ball $B(0,1)$ and such that $f(0) \neq 0$ (as in the proof of Theorem 1.28), then move it to U via χ, i.e. consider $f \circ \chi : U \to [0,1]$; given the support property of f, it follows that if we extend f to M by declaring it to be zero outside U, we get a smooth functions $\tilde{f} : M \to [0,1]$ satisfying the desired property.

For the last part, we would need a function f on $\mathbb{R}^m$ as above, with the extra-property that $f = 1$ in a neighborhood of the origin. For that, one has to slightly improve the choice of g in the proof of Theorem 1.28: choose $g : \mathbb{R} \to \mathbb{R}$ is any smooth function that is 1 when $|t| < \frac{1}{3}$ and is 0 when $t \geq \frac{1}{2}$ and set $f(x) = g(||x||^2)$. 😀

Here is the first illustration of the fact that $\mathscr{C}^{\infty}(M)$ reflects the smooth structure on M.

Proposition 2.63. Let M be a set. Assume that we have two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on M, a smooth structure $\mathscr{A}_1$ on the space $(M, \mathscr{T}_1)$ and, similarly, a smooth structure $\mathscr{A}_2$ on the space $(M, \mathscr{T}_2)$. Assume that the two smooth structures give rise to the same notions of smoothness of $\mathbb{R}$-valued functions, i.e., for arbitrary functions $f : M \to \mathbb{R}$, one has:

$$f \text{ is smooth w.r.t. } \mathscr{A}_1 \iff f \text{ is smooth w.r.t. } \mathscr{A}_2.$$

Then $\mathscr{T}_1 = \mathscr{T}_2$ and $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two maximal smooth atlases in M such that $\mathscr{C}^{\infty}(M, \mathscr{A}_1) = \mathscr{C}^{\infty}(M, \mathscr{A}_2)$. First we show that the topologies $\mathscr{T}_i$ underlying $\mathscr{A}_i$, $i \in \{1,2\}$, must coincide. Let $U \in \mathscr{T}_1$. The previous lemma shows that, for any $p \in U$ we find $f_p \in \mathbb{C}^{\infty}(M, \mathscr{A}_1)$ such that

$$V_p := \{q \in M : f_p(q) \neq 0\} \subset U.$$

and q belongs to V_p. Since $\mathbb{C}^{\infty}(M, \mathscr{A}_1) = \mathbb{C}^{\infty}(M, \mathscr{A}_2)$, f_p is smooth also w.r.t. $\mathscr{A}_2$ and, therefore, $V_p \in \mathscr{T}_2$. Since $U = \cup_{p \in U} V_p$, it follows that $U \in \mathscr{T}_2$ as well. Hence $\mathscr{T}_1 \subset \mathscr{T}_2$ and, similarly, the other inclusion holds as well. Hence $\mathscr{T}_1 = \mathscr{T}_2$.

We now show that $\mathscr{A}_1 \subset \mathscr{A}_2$. Let $\chi \in \mathscr{A}_1$, $\chi : U \to \Omega \subset \mathbb{R}^m$, with $U \in \mathscr{T}_1 = \mathscr{T}_2$. Then each component χ_k of χ is smooth w.r.t. χ; however, we cannot say that it belongs to $\mathscr{C}^{\infty}(M, \mathscr{A}_1)$ because it is defined only on U. Therefore, for any point $p \in U$ we apply the previous lemma to find $\eta_p \in \mathscr{C}^{\infty}(M, \mathscr{A}_1)$ such that $\eta = 1$ in a neighborhood $V_p \subset U$ of p, and is 0 outside the closure in M of a larger neighborhood $\widetilde{V}_p \subset U$. It follows then that each $\eta_p \cdot \chi_k$, extended to the entire M by declaring to be 0 outside U, belongs to $\mathscr{C}^{\infty}(M, \mathscr{A}_1)$, hence also to $\mathscr{C}^{\infty}(M, \mathscr{A}_2)$. In particular, $\chi_k|_{V_p} = (\eta_p \cdot \chi_k)|_{V_p}$ is smooth w.r.t. $\mathscr{A}_2$. Since each $p \in U$ admits a neighborhood $V_p \subset U$ on which χ_k is smooth (w.r.t. $\mathscr{A}_2$ now), χ_k is smooth as a function on U, and so is χ (again, w.r.t. $\mathscr{A}_2$). Therefore, for any $\chi' \in \mathscr{A}_2$, $\chi_{\chi'} = (\chi \circ \chi')^{-1}$ will be smooth; the fact that $\mathscr{A}_2$ is maximal implies now that, indeed, $\chi \in \mathscr{A}_2$. This proves the inclusion $\mathscr{A}_1 \subset \mathscr{A}_2$, end then it suffices to invoke the maximality of $\mathscr{A}_2$.

On the other hand, $\mathscr{C}^\infty(M)$ not only "reflects the smooth structure on M" but actually contain

On the other hand, one may remember that for large classes of spaces M, $\mathscr{C}(M)$ not only "reflects the topology on M" but actually encodes it completely. There one subtlety here: when saying "$\mathscr{C}(M)$" we do not mean just the set itself, but the set together with algebraic structure that is present. The algebraic operations present on $\mathscr{C}(M)$ are addition, multiplication by scalars and multiplication- together making $\mathscr{C}(M)$ into an algebra (cf. Definition 1.9). Therefore, when saying "$\mathscr{C}(M)$" we now mean "the algebra $\mathscr{C}(M)$". This also gives rise to some expectations from $\mathscr{C}^\infty(M)$ (supported so far by the previous proposition) as well as some guidelines. First of all, we should not forget about the algebraic structure. Therefore, while the following is rather simple to check, it is conceptually important.

Exercise 2.64. Show that $\mathscr{C}^\infty(M)$ is a sub-algebra of $\mathscr{C}(M)$, i.e. the sum of two smooth functions, the multiplication by a scalar of a smooth function, as well as products of smooth functions, they are all smooth.

Then show also that $\mathscr{C}^\infty(M)$ is closed under locally finite sums (as discussed in subsection 1.1.2.1), i.e. for any locally finite family $\{\eta_i\}_{i\in I}$ of real-valued smooth functions on M, $\sum_i \eta_i$ is again smooth.

Here is another interesting consequence of Lemma 2.62.

Exercise 2.65. Show that, for any manifold M, $\mathscr{C}^\infty(M)$ is point separating, i.e. for any $p,q \in M$ distinct, there exists $f \in \mathscr{C}^\infty(M)$ such that $f(p) = 0$, $f(q) = 1$.

However, the best way to understand the importance of Lemma 2.62 comes from the general discussion of subspaces $A \subset \mathscr{C}(M)$ and the existence of $\mathscr{A}$-partitions of unity- as discussed in the reminder on Topology, subsection 1.1.2.1. To apply Theorem 1.17 to $A = \mathscr{C}^\infty(M)$, one first has to check some simple algebraic properties and then there was a second, more subtle, condition. But that is precisely what Exercise 2.64 and the first part of Lemma 2.62 are taking care of! Therefore one obtains:

Theorem 2.66. *On any manifold M, for any open cover $\mathscr{U}$ of M, there exists a smooth partition of unity on M subordinated to $\mathscr{U}$.*

Diving a bit more into the details of subsection 1.1.2.1 one see that, from the various properties discussed there, the one that is most subtle is "paracompactness". But Theorem 1.19 immediately implies that any manifold is automatically paracompact. One the other hand, when it comes to properties of subspaces $\mathscr{A} \subset C^\infty(M)$ discussed in subsection 1.1.2.1, the one that is more subtle is "normality"; from this perspective, Lemma 2.62 and Exercise 2.64 check the criteria for normality provided by Theorem 1.20. Therefore one also obtains:

Corollary 2.67. *For any manifold M, $\mathscr{C}^\infty(M)$ is normal. In other words, for any two disjoint closed subset $A, B \subset M$, there exists a smooth function $f : M \to [0,1]$ with the property that $f|_A = 0$, $f|_B = 1$.*

Remark 2.68 (For the curious students: a smooth Gelfand-Naimark ...). The "Gelfand-Naimark message" from Topology is that, for reasonable topological space X, the topological information on X can be completely recovered from the $\mathscr{C}(X)$ and its algebraic structure (the sums and products that make it into an algebra). As recalled in Remark 1.11, the way one "reconstructs" the space X from the algebra $\mathscr{C}(X)$ is by associating to any algebra A a topological space $X(A)$, called the spectrum of A, and defined as the set of all characters $\chi : A \to \mathbb{R}$ (see Remark 1.11).

Since the notion of character makes sense for any algebra, we can apply it to

$$A := \mathscr{C}^\infty(M),$$

the algebra of smooth functions on a manifold M. As before, any point $p \in M$ gives rise to a character

$$\chi_p : A \to \mathbb{R}, \quad \chi_p(f) := f(p);$$

and this gives rise to a map

$$\mathrm{GN} : M \to X(A), \quad p \mapsto \chi_p.$$

Note that the Exercise 2.65 says precisely that this map is injective! What about surjectivity? I.e., is is true that any character

$$\chi : \mathscr{C}^\infty(M) \to \mathbb{R}$$

is of type χ_p for some $p \in M$?

Theorem 2.69. *If M is a compact manifold then the map GN is 1-1.*

Proof. We are left with proving surjectivity. Hence we start with a character χ and we look for $p \in M$ such that $\chi = \chi_p$. First remark that, for the last equality to hold (given p), it suffices to require something apparently weaker condition

$$\text{if } f \in \mathscr{C}^\infty(M) \text{ is killed by } \chi \text{ (i.e. } \chi(f) = 0) \Longrightarrow f(p) = 0.$$

Indeed, for an arbitrary $f \in \mathscr{C}^\infty(M)$, since $f - \chi(f) \cdot 1$ is anyway killed by χ, the previous implication would imply $\chi(f) = f(p)$ for all f, i.e. $\chi = \chi_p$.

Therefore, it suffices to show that there exists $p \in M$ for which the previous implication holds. We proceed by contradiction. If no such p exists then we find, for each p, a function f_p with

$$\chi(f_p) = 0, \quad f_p(p) \neq 0.$$

We may actually assume that $f_p(p) > 0$ (otherwise replace f_p by its square). Since f_p is smooth (hence also continuous), for each p we find a neighborhood U_p of p s.t. $f_p > 0$ on the entire U_p. Then $\{U_p : p \in M\}$ forms an open cover of M and, using the compactness of M, we cab extract a finite subcover- corresponding to a finite number of points $p_1, \ldots, p_k$. Then the sum f of the corresponding functions f_{p_i} will have the property that

$$f \in \mathscr{C}^\infty(M), \quad \chi(f) = 0, \quad f > 0 \text{ everywhere on } M.$$

But then we can write the constant function $1 = f \cdot \frac{1}{f}$, with both f and $\frac{1}{f}$, so that

$$\chi(1) = \chi(f) \cdot \chi(\frac{1}{f}) = 0,$$

which provides us with a contradiction we were looking for. 😀

Exercise 2.70. Adapt the previous proof to show that, if M is not necessarily compact then, for any character χ on $\mathscr{C}^\infty(M)$, there exists $p \in M$ such that $\chi(f) = f(p)$ for all $f \in \mathscr{C}^\infty(M)$ that are compactly supported (i.e. vanish outside some compact).

Exercise 2.71. Given two manifolds M and N, any smooth $F : M \to N$ induces

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M),$$

which is a morphism of algebras (i.e. is linear, multiplicative and sends the unit to the unit). Prove that conversely, any morphism from the algebra $\mathscr{C}^\infty(N)$ to $\mathscr{C}^\infty(M)$ arises in this way.

Then deduce that, for any two manifolds M and N,

$$M \text{ and } N \text{ are diffeomorphic} \quad \Longleftrightarrow \quad \mathscr{C}^\infty(M) \text{ and } \mathscr{C}^\infty(N) \text{ are isomorphic as algebras}$$

2.2 First interesting examples

2.2.1 The spheres S^m

The first example in our list are the m-dimensional spheres

$$S^m := \{(x_0, \ldots, x_m) \in \mathbb{R}^{m+1} : (x_0)^2 + (x_1)^2 + \ldots + (x_m)^2 = 1\}.$$

Of course, this example fits into the general discussion of embedded submanifolds of Euclidean spaces mentioned in Example 2.23. However, it is one of the examples which are instructive to consider separately, even as abstract manifolds, and notice its rather special properties.

First of all, as with any subspace of a Euclidean space, we endow it with the Euclidean topology: opens are intersections of S^m with opens in $\mathbb{R}^{m+1}$. In this way, S^m is a Hausdorff, 2nd countable space (... even compact).

Intuitively it should be clear that, locally, S^m looks like (opens inside) $\mathbb{R}^m$- and that is something that we use everyday (we do live on some sort of sphere, remember?). For instance, drawing small disks on the sphere and projecting them on planes through the origin, one can easily build charts. For instance, the upper hemisphere

$$U_0^+ = \{(x_0, \ldots, x_m) \in S^m : x_0 > 0\}$$

(not even so small!) and the projection into the horizontal plane $\{0\} \times \mathbb{R}^m \subset \mathbb{R}^{m+1}$ gives rise to

$$\chi_0^+ : U_0^+ \to \mathbb{R}^m, \quad \chi(x_0, \ldots, x_m) = (x_1, \ldots, x_m).$$

Considering the similar charts with $x_0 < 0$, as well as the ones that use the other coordinates x_i instead of x_0, all together, we get a smooth atlas

$$(U_0^+, \chi_0^+), \ (U_0^-, \chi_0^-), \ \ldots, \ (U_m^+, \chi_m^+), \ (U_m^-, \chi_m^-) \tag{2.2.1}$$

defining a ”natural” smooth structure on S^m, called **the standard smooth structure on S^m**.

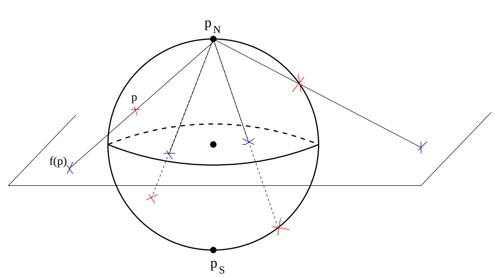
One can actually use the intuition to build similar charts and atlases; however, if you really follow your intuition, you will obtain the same smooth structure (and therefore the name ”natural”). Here is an example of another smooth atlas on the sphere (... describing the same smooth structure). It is probably the most elegant one; at least it uses the least amount of charts: two. These are the so-called stereographic projections w.r.t. the north and the south poles,

$$p_N = (0, \ldots, 0, 1), \quad p_S = (0, \ldots, 0, -1) \in S^m,$$

respectively. The one w.r.t. to p_N is the map

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^m$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 2.2.1).



The stereographic projection (sending the red points to the blue ones)

Fig. 2.9

Computing the intersections, we find the precise formula:

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^n, \quad \chi_N(x_0, x_1, \ldots, x_m) = \left(\frac{x_0}{1 - x_m}, \ldots, \frac{x_{m-1}}{1 - x_m} \right)$$

(while for χ_S we find a similar formula, but with $+$s instead of the $-$s). Reversing the process, we find

$$\chi_N^{-1} : \mathbb{R}^m \to S^m \setminus \{p_N\}, \quad \chi_N^{-1}(u_1, \ldots, u_m) = \left(\frac{2u_1}{|u|^2 + 1}, \ldots, \frac{2u_m}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right)$$

hence χ_N is a homeomorphism. Similarly for χ_S and, computing the coordinate change between the two we find

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \chi_S \circ \chi_N^{-1}(u) = \frac{u}{|u|^2}.$$

This is clearly smooth, composed with itself is the identity, therefore it is a diffeomorphism (... therefore also solving Exercise 1.56 from Chapter 1). We deduce that the two charts

$$(S^m \setminus \{p_N\}, \chi_N), \quad (S^m \setminus \{p_S\}, \chi_S) \tag{2.2.2}$$

define a smooth m-dimensional atlas on S^m.

Exercise 2.72. Show that the stereographic projections give rise to the standard smooth structure on S^m. Or, more precisely, show that the smooth atlas (2.2.2) induces the same smooth structure as (2.2.1).

Remark 2.73 (For the curious students: exotic spheres). As for the Euclidean spaces, it becomes very interesting (and exciting!) when we ask about the existence of other smooth structures on S^m (endowed with the Euclidian topology). Of course, for the reasons we explained, we only look at non-diffeomorphic smooth structures. Again, the answers are deceivingly simple (and the proofs highly non-trivial):

- except for the exceptional case $m = 4$ (see below), for $m \leq 6$ the standard smooth structure on S^m is the only one we can find (up to diffeomorphisms).
- S^7 admits precisely 28 (!?!) non-diffeomorphic smooth structures.
- S^8 admits precisely 2.
- ...
- and S^{11} admits 992, while S^{12} only one!
- and S^{31} more than 16 million, while S^{61} only one (and actually, next to the case S^m with $m \leq 6$, S^{61} is the only odd-dimensional sphere that admits only one smooth structure).
- ...

Moreover, some of the exotic (i.e. non-diffeomorphic to the standard) spheres can be described rather simply. For instance, fixing $\varepsilon > 0$ small enough, inside the small sphere of radius ε, $S^9_\varepsilon \subset \mathbb{C}^5$,

$$W_k := \{(z_1, z_2, z_3, z_4, z_5) \in S^9_\varepsilon : z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\} \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$$

(one for each integer $k \geq 1$) are all homeomorphic to S^7, they are all endowed with smooth structures induced from the standard (!) one on $\mathbb{R}^{10}$, but the first 28 of them are each two non-diffeomorphic (and after that they start repeating). In this way one obtains a rather explicit description of all the 28 smooth structures on S^7.

While the number of smooth structures on S^m is well understood for $m \neq 4$, the case of S^4 remains a mystery: is there just one smooth structure? Is there a finite number of them (and how many?)? Or there is actually an infinite number of them? The smooth Poincare conjecture says that there is only one; however, nowadays it is believed that the conjecture is false.

2.2.2 The projective spaces $\mathbb{P}^m$

Probably the simplest example of a smooth manifold that does not sit *naturally* inside a Euclidean space (therefore for which the abstract notion of manifold is even more appropriate) is the m-dimensional projective space $\mathbb{P}^m$. Recall that it consists of all lines through the origin in $\mathbb{R}^{m+1}$:

$$\mathbb{P}^m = \{l \subset \mathbb{R}^{m+1} : l - \text{one dimensional vector subspace}\}.$$

Each point $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to the line l_x through the origin and x, hence $l_x := \mathbb{R} \cdot x \subset \mathbb{R}$, so that we can write

$$\mathbb{P}^m = \left\{l_x : x \in \mathbb{R}^{m+1} \setminus \{0\}\right\}, \quad \text{where } (l_x = l_y) \Longleftrightarrow (y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}^*) \tag{2.2.3}$$

$\mathbb{P}^m$ is best understood by relating it back to $\mathbb{R}^{m+1} \setminus \{0\}$, via the map

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{P}^m, \quad x \mapsto l_x.$$

For instance, this gives rise to a natural topology on $\mathbb{P}^m$: the quotient topology w.r.t. π i.e., the largest one that makes π continuous, explicitly described as follows:

$$U \subset \mathbb{P}^m \text{ is open} \quad \Longleftrightarrow \quad \pi^{-1}(U) \text{ is open in } \mathbb{R}^{m+1}.$$

All these define $\mathbb{P}^n$ as a set, and as a topological space. For the smooth structure, it is handy to change the notation for the lines l_x to

$$[x_0 : x_1 : \ldots : x_m] := l_x = \mathbb{R} \cdot x, \quad \text{for } x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}.$$

The use of the symbol : in the notation should suggest ”division” and is motivated by

$$[x_0 : x_1 : \ldots : x_m] = [y_0 : y_1 : \ldots : y_m] \Longleftrightarrow y_k = \lambda \cdot x_k \text{ for some } \lambda \in \mathbb{R}^* \text{ and all } k\text{s}. \tag{2.2.4}$$

The coordinate notation is also more appropriate when one is searching for coordinate charts. The natural smooth structure on $\mathbb{P}^m$ is obtained starting from a simple observation: the last equality above allows us in principle to make the first coordinate equal to 1:

$$[x_0 : x_1 : \ldots : x_m] = \left[1 : \frac{x_1}{x_0} : \ldots : \frac{x_m}{x_0}\right],$$

i.e. to use just coordinates from $\mathbb{R}^m$; with one little problem- when $x_0 = 0$ (this ”little problem” is what forces us to use more than one chart). We arrive at a very natural chart for $\mathbb{P}^m$: with domain

$$U_0 = \{[x_0 : x_1 : \ldots : x_m] \in \mathbb{P}^m : x_0 \neq 0\}$$

and defined as

$$\chi^0 : U_0 \to \mathbb{R}^m,\ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_1}{x_0}, \ldots, \frac{x_m}{x_0}\right).$$

And similarly when trying to make the other coordinates equal to 1:

$$\chi^i : U_i \to \mathbb{R}^m,\ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i}\right), \tag{2.2.5}$$

defined on U_i defined by $x_i \neq 0$.

Exercise 2.74. Show that, indeed,

$$(U_0, \chi^0), (U_1, \chi^1), \ldots, (U_m, \chi^m),$$

is, indeed, a smooth atlas (hence it gives rise to a smooth structure on $\mathbb{P}^m$).

Remark 2.75 (gluing the antipodal points of the sphere). By rescaling elements $x \in \mathbb{R}^{m+1} \setminus \{0\}$ to force them to be of norm one or, more geometrically, by intersecting the lines through the origin with the unit sphere, one can replace $\mathbb{R}^{m+1} \setminus \{0\}$ by S^m in the discussion above. Hence

$$\mathbb{P}^m = \{l_x : x \in S^m\} \quad \text{where, for } x, y \in S^m\text{: } \ (l_x = l_y) \Longleftrightarrow (y = x \text{ or } y = -x) \tag{2.2.6}$$

and it is not very difficult to show that the quotient topology induced by π coincides with the one induced by its restriction

$$H = \pi|_{S^m} : S^m \to \mathbb{P}^m, \quad x \mapsto l_x. \tag{2.2.7}$$

These allow us to think of $\mathbb{P}^m$ as being obtained from S^m by gluing any point $x \in S^m$ to its antipodal $-x$. More elegantly, one may say that we deal with an action of $\mathbb{Z}_2$ on S^m, and

$$\mathbb{P}^m = S^m/\mathbb{Z}_2,$$

at least as topological spaces. Notice that one of the gains of using S^m is that it follows right away that $\mathbb{P}^m$ is compact (as the quotient of a compact space). This discussion can also be used to provide another description of the smooth structure on $\mathbb{P}^m$, but we have to wait a bit (until section 2.4.2) for doing that.

Exercise 2.76. Show that the map $\pi_0 : S^m \to \mathbb{P}^m$, $x \mapsto l_x$ is a local diffeomorphism. Then show that the smooth structure on $\mathbb{P}^m$ that we introduced here is the only one with this property.

Remark 2.77 (For the interested students: minimal number of charts). Note that this smooth structure on $\mathbb{P}^m$ is also a first example in which, at least intuitively, we need many more than just two charts to form an atlas. The question, valid for any manifold M, asking what is the minimal number of charts one needs to obtain an atlas of M, is a very interesting one. In general one can show that one can always find an atlas consisting of no more than $m+1$ charts where m is the dimension of M (not very deep, but not trivial either!). Therefore, denoting by $N_0(M)$ the minimal number of charts that we can find, one always has $N_0(M) \leq \dim(M) + 1$. The atlas described above confirms this inequality for $\mathbb{P}^m$. However, in concrete examples, we can always do better. E.g. we have seen that $N_0(S^m) = 2$ (well, why can't it be 1?). The same holds for all compact orientable surfaces. Thinking a bit (but not too long) about $\mathbb{P}^m$ one may expect that $N_0(\mathbb{P}^m) = m+1$; however, that is not the case. The precise computation was carried out by M. Hopkins, except for the cases $n = 31$ and $n = 47$. For those cases we know that $N_0(\mathbb{P}^{31})$ is either 3 or 4, while $N_0(\mathbb{P}^{46})$ is either 5 or 6. For all the other cases, writing $m = 2^k a - 1$ with a odd, one has

$$N_0(\mathbb{P}^m) = \begin{cases} \max\{2, a\} & \text{if } k \in \{1,2,3\} \\ \text{the least integer} \geq \frac{m+1}{2(k+1)} & \text{otherwise} \end{cases} \quad \square$$

2.2.3 The complex projective spaces $\mathbb{CP}^m$

Returning to the basics, note that there is a complex analogue of $\mathbb{P}^m$; for that reason $\mathbb{P}^m$ is sometimes denoted $\mathbb{RP}^m$ and called **the real projective space**. The m-dimensional **complex projective space** $\mathbb{CP}^m$ is defined completely analogously but using complex lines $l_z \subset \mathbb{C}^{m+1}$, i.e. 1-dimensional complex subspaces of $\mathbb{C}^{m+1}$ (and 1-dimensional is in the complex sense). Again, one can write

$$\mathbb{CP}^m = \left\{[z_0 : z_1 : \ldots : z_m] : (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1} \setminus \{0\}\right\}$$

where $[z_0 : z_1 : \ldots : z_m]$ is just a notation for the line through the origin and the point $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Hence

$$[z_0 : z_1 : \ldots : z_m] = [\lambda \cdot z_0 : \lambda \cdot z_1 : \ldots : \lambda \cdot z_m] \quad \text{for } \lambda \in \mathbb{C}^*.$$

Analogously to the real case, one can realize

$$\mathbb{CP}^m = \left(\mathbb{C}^{m+1} \setminus \{0\}\right)/\mathbb{C}^*.$$

Using $\mathbb{R}^{2m} = \mathbb{C}^m$ we can also represent the $(2m+1)$-dimensional sphere as:

$$S^{2m+1} = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} : |z_0|^2 + \ldots + |z_m|^2 = 1\}$$

and there is an obvious map

$$H : S^{2m+1} \to \mathbb{CP}^m, \quad H(z_0, \ldots, z_m) = [z_0 : \ldots : z_m].$$

Intersecting the (complex) lines with this sphere we can realize each line as as $[z]$ with $z \in S^{2m+1}$; two like this, $[z_1]$ and $[z_2]$, are equivalent if and only if $z_2 = \lambda \cdot z_1$, where this time $\lambda \in S^1$; i.e. the group $\mathbb{Z}_2$ from the real case is replaced by the group S^1 of complex numbers of norm 1 (endowed with the usual multiplication). We obtain

$$\mathbb{CP}^m = S^{2m+1}/S^1.$$

As for the smooth structure one proceeds completely analogously, keeping in mind the identification $\mathbb{C}^m = \mathbb{R}^{2m}$: we get a (smooth) atlas made of $m+1$ charts.

$$\mathscr{A} = \{(U_0, \chi^0), \ldots, (U_m, \chi^m)\}$$

given by

$$U_i = \{[z_0 : z_1 : \ldots : z_m] \in \mathbb{CP}^m : z_i \neq 0\},$$
$$\chi^i : U_i \to \mathbb{C}^m = \mathbb{R}^{2m},$$
$$\chi^i([z_0 : z_1 : \ldots : z_m]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}\right).$$

Notice that the change of coordinates is actually holomorphic. Therefore, the structure present on $\mathbb{CP}^m$ can be recognised among the ones discussed in Section 2.2.5: $\mathbb{CP}^m$ is also a complex manifold!
ssec:Variations

Remark 2.78 (For the interested students: minimal number of charts for $\mathbb{CP}^m$). Note that the change of coordinates is actually given by holomorphic maps- therefore $\mathbb{CP}^m$ is also a complex manifold (see Section 2.2.5). Note that m is the complex dimension of $\mathbb{CP}^m$; as a manifold, it is $2m$-dimensional. As a curiosity: for the minimal number of charts needed to cover $\mathbb{CP}^m$, the answer is much simpler in the complex case:

$$N_0(\mathbb{CP}^m) = \begin{cases} m+1 & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}. \quad \square$$

2.2.4 Grassmannians

Instead of lines through the origin (i.e. 1-dimensional vector subspaces) in $\mathbb{R}^{n+1}$ like we did for $\mathbb{P}^n$, one can look at k-dimensional subspaces in $\mathbb{R}^{n+k}$. One obtains the Grassmannians:

$$G_k(\mathbb{R}^{n+k}) := \{V \subset \mathbb{R}^{n+k} : V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{R}^{n+k}\}.$$

To describe the topology and the smooth structure (well, just a sketch for this one), we fix a point $V \in G_k(\mathbb{R}^{n+k})$ and look at "nearby points". Using the standard inner product on $\mathbb{R}^{n+k}$, we consider the orthogonal $V^{\perp} \subset \mathbb{R}^{n+k}$, so that we have a direct sum decomposition

$$\mathbb{R}^{n+k} = V \oplus V^{\perp}.$$

The key idea is based on the intuition that, if V is thought of as an "OX" axis (and $V^{\perp}$ as OY), k-dimensional sub-spaces that are "close to V" look like graphs of function (from our OX to our OY). This takes the precise form of associating to any linear map $L : V \to V^{\perp}$ the k-dimensional subspace of $\mathbb{R}^{n+k}$:

$$V_L := \{v + L(v) : v \in V\} \subset \mathbb{R}^{n+k}.$$

Of course, V corresponds to $L = 0$. One has an injection

$$\mathrm{par}_V : \mathrm{Hom}\left(V, V^{\perp}\right) \to G_k(\mathbb{R}^{n+k}), \quad L \mapsto V_L,$$

where $\mathrm{Hom}\left(V, V^{\perp}\right)$ is the vector space of all linear maps from V to $V^{\perp}$, it has dimension nk and, upon choosing bases, can be identified with $\mathbb{R}^{nk}$. This allows us to put a topology on the image of par_V,

$$\mathrm{Op}_V := \mathrm{Im}(\mathrm{par}_V) \subset G_k(\mathbb{R}^{n+k}).$$

This allows us to build a topology on $G_k(\mathbb{R}^{n+k})$ by declaring all the Op_V to be open, and then the inverses $\chi_V = \mathrm{par}_V^{-1}$ will define a smooth atlas on $G_k(\mathbb{R}^{n+k})$, making it into an nk-dimensional manifold.

2.2.5 Variations on the notion of manifold; more examples

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Manifolds of class C^k

Starting from the fact that

$$\text{smooth} = \text{of class } \mathscr{C}^{\infty},$$

there are several rather obvious variations on the notion of smooth manifold. First of all one can consider a $\mathscr{C}^k$-version of the previous definitions for any $1 \leq k \leq \infty$. E.g., instead of talking about smooth compatibility of two charts, one talks about $\mathscr{C}^k$-compatibility, which means that the change of coordinates is a $\mathscr{C}^k$-diffeomorphism. One arises at the notion of **manifold of class** $\mathscr{C}^k$, or $\mathscr{C}^k$**-manifold**. For $k = \infty$ we recover smooth manifolds, while for $k = 0$ we recover topological manifolds.

Yet another possibility is to require "more than smoothness"- e.g analyticity. That gives rise to the notion of **analytic manifold**. Here is a non-trivial but fundamental result (mentioned here just because it is cool!):

Theorem 2.79 (Whitney, 1936). *Given a $\mathscr{C}^k$-manifold M with $k \in \{1, 2, \ldots, \infty, \omega\}$ and any $l \geq k$, there exist a $\mathscr{C}^l$-manifold structure on M extending the given $\mathscr{C}^k$ one. Moreover, any two such $\mathscr{C}^l$-extensions are $\mathscr{C}^l$-diffeomorphic.*

Notice however that the situation is very different when $k = 0$, i.e. when the starting point is a topological manifold. The original result that indicated this was Milonr's result on exotic spheres, mentioned above, which showed that S^7, seen as a topological manifold, has several non-diffeomorphic smooth structures ("exotic"). Among the odd dimensional spheres, it is now known that only S^1, S^3, S^5, and S^{61} do not admit such exotic spheres. On the other hand, soon after Milnor, in 1960, Kervaire found a 10-dimensional topological manifold that does not admit any differentiable structure. Nowadays many more examples are known in dimensions ≥ 4. On the other hand, in low dimensions, the situation is again different, the case $m \in \{1,2\}$ was prove by Tibor Radó in 1920, while the case $m = 3$ by Edwin E. Moise in 1952; all together:

Theorem 2.80. *Every topological manifold of dimension $m \in \{1,2,3\}$ admits one, and only one (up to diffeomorphism) smooth structure.*

In dimension $m \geq 4$ one can find both topological manifolds that do not admit a smooth structure, as well as topological manifolds that admit more than one smooth structure.

Furthermore, as we mentioned already in Remark 2.56, the discussion is very interesting even for $M = \mathbb{R}^m$: for $m \neq 4$ it admits a unique smooth structure, while $\mathbb{R}^4$ admits uncountably many non-equivalent smooth structures (this was prove in 1987 by Taubes).

Complex manifolds

And yet another possibility arises when looking at even dimensional manifolds. Indeed, when $m = 2n$ then $\mathbb{R}^m = \mathscr{C}^n$ and one can further restrict to holomorphic maps in the sense of Complex Analysis: again, the definitions are identical to what we did before, just that the change of coordinates are now required to be holomorphic. That gives rise to the notion of **complex manifold**. Whether a smooth manifold can be turned into a complex one is a very difficult problem. First of all, in case the reader wonders about an extension of Theorem 2.79 to the analytic setting, whether smooth (real) manifolds can be upgraded to a complex one and, if yes, whether it can be done in a unique way, the answer is no. In general, if a smooth manifold can be turned into a complex one, that can be done in many ways. As for the existence, many smooth manifolds are known not to come from complex ones (well, the odd dimensional ones are out from the start). But, given a smooth manifold M, deciding whether it comes from a complex one is a rather difficult question. For instance, and just for fun again, here is what is known for spheres: The only spheres that can be made into complex manifolds are S^2 and S^6. Since S^2 can be identified with the complex projective plane $\mathbb{CP}^1$ (Exercise 2.5), it can be made into a complex manifold. However the question about S^6 is still open!

Manifolds with boundary

Another possible variation is to change the "model space" $\mathbb{R}^m$. The simplest/most interesting/most standard replacement is by the upper-half planes

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

This gives rise to the notion of **smooth manifold with boundary**: the definition is precisely the same, with the only difference that the charts now are go from opens in M to opens in $\mathbb{H}^m$. Of course, it is important to notice that the usual concepts and properties of smoothness in $\mathbb{R}^m$ hold also for opens in half-planes.

Remark 2.81. Using half planes is not more restrictive but more general. Indeed, any open in $\mathbb{R}^m$, even if it does not sit inside a half plane, it is homeomorphic to one that sits inside $\mathbb{H}^m$. Hence any charts in the usual sense can be seen as a chart in the sense of manifolds with boundary (just that it does not see the boundary). The other way however is not possible- e.g. $\mathbb{H}^m$ itself can never be homeomorphic to an open inside $\mathbb{R}^m$.

As the name indicates, manifolds with boundary have ... boundary.

Definition 2.82. Given a manifold with boundary, a point $p \in M$ is called a boundary point if there exists a chart $\chi : U \to \Omega \subset \mathbb{H}^m$ around p such that $\chi(p) \in \partial(\mathbb{H}^m)$. The **boundary of** M, denoted ∂M, is the set of boundary points.

One can prove that the condition of p being a boundary point does not depend on the chart χ. For any chart (U,χ) as above, setting $\partial U := U \cap \partial M$, Furthermore, it also follows that for any χ as above,

$$\chi|_{\partial(U)} : \partial(U) \to \partial(\Omega) = \Omega \cap \partial(\mathbb{H}^m)$$

(where the last space is open in $\partial(\mathbb{H}^m) \cong \mathbb{R}^{m-1}$) can be used as a chart for ∂M.

Exercise 2.83. Prove that the resulting charts $(\partial U, \chi|_{\partial(U)})$ give rise to a smooth structure on ∂M, making it into a smooth $(m-1)$-dimensional manifold (without boundary).

We now look at some simple examples of manifolds with boundaries and methods to obtain new ones.

Example 2.84 (the closed unit disk). The closed unit disk

$$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

is a manifold with boundary, with boundary the unit circle S^1.

Example 2.85. Returning to products $M \times N$ of manifolds, as in Example2.21, it should be clear now that if M *or* N is a manifold with boundary, then $M \times N$ makes sense as a manifold with boundary (which boundary?). On the other hand, if both M and N have boundary, then $M \times N$ will be a manifold with ... corners (you can now try to make that precise!).

Example 2.86 (the full torus). As an example of products, I mention here "the full torus" $S^1 \times D^2$. As for the usual torus (see Fig 2.4), an explicit model in $\mathbb{R}^3$ can be obtained by rotating an entire disk

$$D^2_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}.$$

The outcome will be a subspace $T^2_{R,r,\text{solid}}$ of $\mathbb{R}^3$. Extending to the full torus the computation we did for the torus, one obtains a parametrization

$$\begin{aligned} T^2_{R,r,\text{solid}} &= \left\{ \left((R+ru)\cos a, (R+ru)\sin a, v \right) : a \in [0,2\pi], (u,v) \in D^2 \right\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - R \right)^2 + z^2 \leq r^2 \right\}. \end{aligned} \tag{2.2.8}$$

One can now show that $T^2_{R,r,\text{solid}}$ is a manifold with boundary and its boundary is precisely $T^2_{R,r}$. The fact that it is a can be identified with the product $S^1 \times D^2$ follows by extending the usual identification of $T^2_{R,r}$ with $S^1 \times S^1$:

$$F : S^1 \times D^2 \to T^2_{R,r,\text{solid}}, \quad \left(e^{ia}, (u,v) \right) \mapsto \left((R+ru)\cos a, (R+ru)\sin a, rv \right). \tag{2.2.9}$$

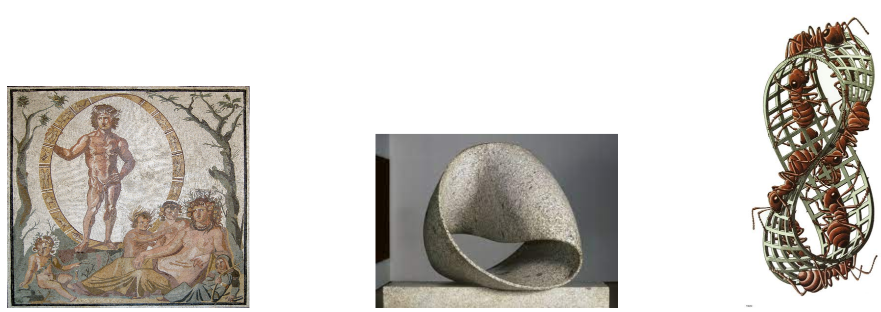
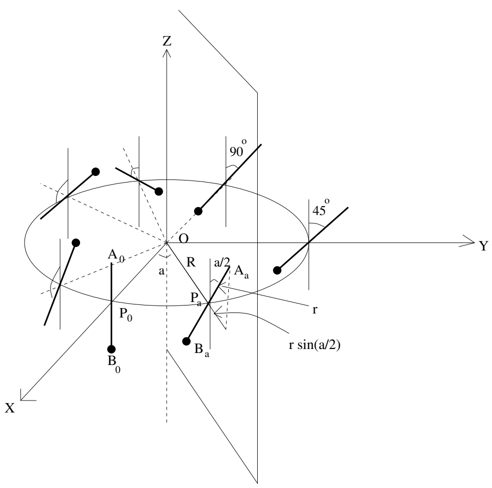


Fig. 2.10

Example 2.87 (The Moebius band). The most popular manifold with boundary is the Moebius band. According to wikipedia, it "is a surface that can be formed by attaching the ends of a strip of paper together with a half-twist. As a mathematical object, it was discovered by Johann Benedict Listing and August Ferdinand Moebius in 1858, but it had already appeared in Roman mosaics from the third century CE."

For an explicit model for the Moebius band in $\mathbb{R}^3$ we proceed like we did for the torus. We place ourselves perpendicular to the XOY plane, holding a stick by its middle point, and while rotating ourselves a full 360 degrees, we rotate the stick by 180 degrees (turning it upside down). Of course, we assume the rotations to be uniform (constant spead). See Figure 2.11. Denoting by R the length of the arm, and by r half of the length of the stick we find the model

$$\begin{aligned} M_{R,r} &= \left\{ \left(\left(R + tr\sin\frac{a}{2} \right)\cos(a), \left(R + tr\sin\frac{a}{2} \right)\sin(a), tr\cos\frac{a}{2} \right) : a \in [0, 2\pi], t \in [-1, 1] \right\} \\ &= \{ ((R + tr\sin b)\cos 2b, (R + tr\sin b)\sin 2b, tr\cos b) : b \in [0, \pi], t \in [-1, 1] \} \end{aligned} \tag{2.2.10}$$



Explicit realization of the Moebius band in $\mathbb{R}^3 : M_{R,r}$

Fig. 2.11

Exercise 2.88. Show that $M_{R,r} \subset \mathbb{R}^3$ is, indeed, a manifold with boundary. What is its boundary?

Notice that the Moebius band sits naturally inside the solid torus. Already for the concrete models, one has

$$M_{R,r} \subset T_{R,r}$$

by very construction (just have a look at figures 2.5 and 2.11!).

Example 2.89 (removing disks). The cheapest way to produce a manifold with boundary out of an usual manifold M is by removing a "small disc", by which we mean an open U which is the domain of a chart (U, χ) with the property that $\chi(U)$ is a disk in $\mathbb{R}^m$. One obtained a manifold with boundary, denoted

$$M^\circ,$$

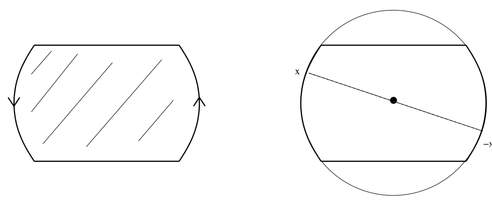
with $\partial M^\circ = S^{m-1}$ ($m = \dim M$). This notation is a bit sloppy since it makes no reference to the disk that is removed. However, one can prove that removing to two different small disks produce diffeomorphic manifolds.

Although this seems rather trivial, here is an example:

Exercise 2.90. Show that if you remove a small disk from the projective space $\mathbb{P}^2$ then you are left with ... a Moebius band! (Hint: recall from Topology that $\mathbb{P}^2$ can also be obtained from D^2 by gluing the opposite points on the boundary circle. Look now at a "band" inside the disk,

$$B = \{(x,y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\}. \subset D^2.$$

as shown in Figure 2.12. Do you see the Moebius band?



The Moebius band inside the projective plane

Fig. 2.12

Example 2.91 (connected sums). A simple but non-trivial procedure of building new examples of manifolds is via the operation of connected sum $M\#N$, whose topological version you may have seen in Topology. The input now consists of two d-dimensional manifolds M and N (say without boundary) and the outcome is another d-dimensional manifold (still without boundary). Intuitively the operation is very simple: remove a small disk from each of the two manifolds and then glue M° and N° to each other along their boundary spheres S^{d-1}:

$$M\#N = M^{\circ} \cup_{S^{d-1}} N^{\circ}.$$

Again, it requires some work to implement everything that seems so obvious intuitively, but the bottom line is that it works, and the outcome is a new d-dimensional manifold which, up to diffeomorphisms, only depends on the input manifolds M and N. In Figure 2.13, again copied from the Dictaat on Topology, you see the double torus

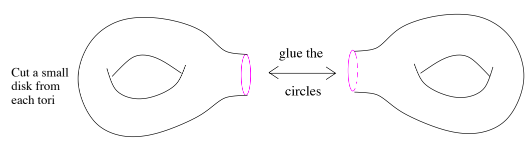


Fig. 2.13

obtained as the connected sum of the torus with itself. Repeating the process, one obtains all the compact oriented surfaces. The non-orientable ones are obtained in the same way but using the projective plane $\mathbb{P}^2$ instead of the torus. Here are a few fun facts in this direction:

- $\mathbb{P}^2 \# \mathbb{P}^2$ is ... the Klein bottle.
- ... or, since the projective plane minus a small disk is the Moebius band: the Klein bottle is obtained from two Moebius bands by gluing them along the boundary circle! This is also indicated in Figure 2.14 by appealing to the "paper model" for the Moebius band and the Klein bottle.

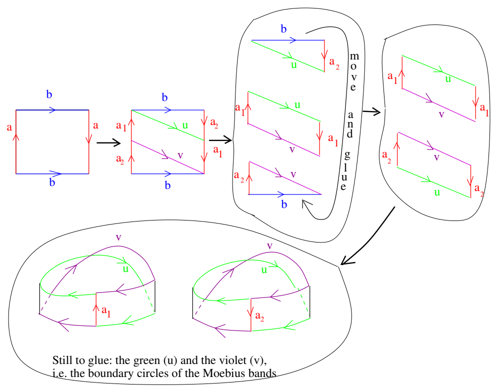


Fig. 2.14

2.2.6 More examples: classical groups and ... Lie groups

One very interesting (and special) class of manifolds are the so-called Lie groups: they are both groups, as well as manifolds, and the two structures are compatible:

Definition 2.92. A **Lie group** G is a group which is also a manifold, in a way that the multiplication $m : G \times G \to G$, $m(g,h) = gh$ and the inversion $\iota : G \to G$, $\iota(g) = g^{-1}$ are smooth maps.

A **Lie group homomorphism** between two Lie groups G and H is a group homomorphism $\phi :: G \to H$ which is also smooth. When ϕ is also a diffeomorphism, we say that ϕ is an **isomorphism of Lie groups**.

Example 2.93. The unit circle S^1, identified with the space of complex numbers of norm one,

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

is a Lie group with respect to the usual multiplication of complex numbers.

Example 2.94. Similarly, the 3-sphere S^3 can be made into a (non-commutative, this time) Lie group. For that we replace $\mathbb{C}$ by the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in $\mathbb{H}$ is uniquely determined by the fact that it is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Recall also that for

$$u = x + iy + jz + kt \in \mathbb{H}$$

one defines

$$u^* = x - iy - jz - kt \in \mathbb{H}, \ \ |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

Then, the basic property $|u \cdot v| = |u| \cdot |v|$ still holds and we see that, identifying S^3 with the space of quaternionic numbers of norm 1, S^3 becomes a Lie group.

Remark 2.95 (For the interested students: spheres, Lie groups, etc). One may wonder: which spheres can be made into Lie groups? Well, it turns out that S^0, S^1 and S^3 are the only ones!

On the other hand, it is interesting to understand what happens with the arguments we used for S^1 and S^3 (and which were very similar to each other) in higher dimensions. The main point there was the multiplication on $\mathbb{C}$ and $\mathbb{H}$ and the presence of a norm such that

$$|x \cdot y| = |x| \cdot |y| \tag{2.2.11}$$

(so that, for two elements in the sphere, i.e. of norm one, their product is again in the sphere). So, to handle S^n similarly, we would need a ”normed division algebra” structure on $\mathbb{R}^{n+1}$, by which we mean a multiplication ”$\cdot$” on $\mathbb{R}^{n+1}$ that is bilinear and a norm satisfying the previous condition. Again, it is only on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^4$ that such a multiplication exists. If we do not insist on the associativity of the multiplication, there is one more possibility: $\mathbb{R}^8$ (the so called octonions). But nothing else! And this was known since the 19th century! But why did people care about such operations in the 19th century? Well ... it was number theory and the question of which integers can be written as a sum of two, three, etc squares. For sum of two squares the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy, z_2 = a_ib$, the norm equation (2.2.11). The search for similar “magic formulas” for sum of three squares never worked, but it did for four:

$$(x^2+y^2+z^2+t^2)(a^2+b^2+c^2+d^2) =$$

$$(xa+yb+zc+td)^2+(xb-ya-zd+tc)^2+$$
$$+(xc+yd-za-tb)^2+(xd-yc+zb-ta)^2.$$

This is governed by the quaternions and its norm equation (2.2.11).

Example 2.96 (GL_n). Probably the most important example is the **general linear group** $GL_n(\mathbb{R})$, consisting of $n\times n$ invertible matrices with real entries:

$$GL_n(\mathbb{R}) = \{A \in \mathscr{M}_{n\times n}(\mathbb{R}) : \det(A) \neq 0\}.$$

This was mentioned in Example 2.17 in our discussion of disguised opens in Euclidean spaces. In particular, $GL_n(\mathbb{R})$ inherits a natural smooth structure: the one induced by the atlas consisting of one single chart:

$$GL_n(\mathbb{R}) \ni A = (A^i_j)_{ij} \quad \stackrel{\chi}{\longmapsto} \quad (A^1_1,\ldots,A^1_n,A^2_1,\ldots,A^2_n,\ldots,A^n_1,\ldots,A^n_n) \in \mathbb{R}^{n^2}.$$

Since the matrix multiplication is given by polynomial expressions in the entries, we see it is smooth. Similarly one sees that the inversion is smooth as well and, therefore, $GL_n(\mathbb{R})$ is a Lie group of dimension n^2. A similar discussion applies to matrices with complex coefficients, giving rise to a similar Lie group $GL_n(\mathbb{C})$; identifying $\mathbb{C}$ with $\mathbb{R}^2$, one has $\mathscr{M}_{\times n}(\mathbb{C}) \cong \mathbb{R}^{(2n)^2}$ and $GL_n(\mathbb{C})$ becomes a Lie group of dimension $4n^2$.

Example 2.97 (subgroups of GL_n). Inside the GL_ns one can find several interesting Lie groups such as

- **the orthogonal group**: $O(n) = \{A \in GL_n(\mathbb{R}) : A\cdot A^T = I\}$ (where A^T denotes the transpose of A).
- **the special orthogonal group**: $SO(n) = \{A \in O(n) : \det(A) = 1\}$.
- **the special linear group**: $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$.
- **the unitary group**: $U(n) = \{A \in GL_n(\mathbb{C}) : A\cdot A^* = I\}$ (where A^* is the conjugate transpose).
- **the special unitary group**: $SU(n) = \{A \in GL_n(\mathbb{C}) : A\cdot A^* = I, \det(A) = 1\}$.
- **the symplectic group**: $Sp_n(\mathbb{R}) := \{A \in GL_{2n}(\mathbb{R}) : A^T J A = J\}$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R}). \tag{2.2.12}$$

Each one of these caries a natural smooth structure; actually, they are all embedded submanifolds of the corresponding (Euclidean) spaces of matrices (see below). Since the group operations are restrictions of the ones of the GL_ns, they will continue to be smooth- so that all these groups will be Lie groups.

For the claim that they are embedded submanifolds of $GL_n \subset \mathscr{M}_{n\times n}$, since they are all given by (algebraic) equations, there is a natural to proceed: use the regular value theorem (Theorem 2.24). As an illustration, we provide the details for the case of $O(n)$.

Example 2.98 (the regular value theorem approach in the case of $O(n)$). The equation defining $O(n)$ suggest we should be looking at

$$F : GL_n(\mathbb{R}) \to \mathscr{M}_{n\times n}(\mathbb{R}), \quad F(A) = A\cdot A^T.$$

However, it is important to note that the values of this function takes values in a smaller Euclidean space. Indeed, since $(A\cdot B)^T = B^T\cdot A^T$ for all A and B, matrices of type $A\cdot A^T$ are always symmetric. Therefore, denoting by $\mathscr{S}_n$ the space of symmetric $n\times n$ matrices (again a Euclidean space, of dimension $\frac{n(n+1)}{2}$), we will view F as a map

$$F : GL_n(\mathbb{R}) \to \mathscr{S}_n.$$

(If we don't remark that F was taking values in $\mathscr{S}_n$, then we will not be able to apply the regular value theorem; however, the problem that we would encounter would indicate clearly what we have to do: return and make use of $\mathscr{S}_n$ from the beginning; so, after all, there is no mystery here!). Now, F is a map from an open in a Euclidean space (and we could even use F defined on the entire $\mathscr{M}_{n\times n}(\mathbb{R})$), with values in another Euclidean space; and it is clearly smooth since it is given by polynomial expressions. So, at each point $A \in GL_n(\mathbb{R})$, the differential of F at A,

$$(dF)_A : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathscr{S}_n$$

can be computed by the usual formula:

$$(dF)_A(X) = \frac{d}{dt}\Big|_{t=0} F(A+tX) = \frac{d}{dt}\Big|_{t=0} \left(A\cdot A^T + t(A\cdot X^T + A^T\cdot X) + t^2 X\cdot X^T\right) = A\cdot X^T + X\cdot A^T.$$

Since $O(n) = F^{-1}(\{I\})$, we have to show that $(dF)_A$ is surjective for each $A \in O(n)$. I.e., for $Y \in \mathscr{S}_n$, show that the equation $A\cdot X^T + X\cdot A^T = Y$ has a solution $X \in \mathscr{M}_{n\times n}(\mathbb{R})$. Well, it does, namely: $X = \frac{1}{2}YA$ (how did we find it?). Therefore $O(n)$ is a smooth submanifold of GL_n of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Exercise 2.99. Do the same for the other groups in the list. At least for one more.

On the other hand, instead of doing a case-by-case analysis, one can also invoke the following general result:

Theorem 2.100 (the closed subgroup theorem). *Any closed subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ is automatically an embedded submanifold and, therefore, becomes a Lie group. theorem!closed subgroup theorem*

Of course, this theorem is very useful and one should at least be aware of it. A proof is included below.

Example 2.101 (low dimensions). For n small, the classical subgroups of GL_n have can be further recognized as manifolds/groups that we have already looked at. For instance, $SO(2)$ is diffeomorphic to S^1, $SU(2)$ to S^3 and $SO(3)$ to the real projective space $\mathbb{P}^3$. See the exercises.

Proof (of Theorem 2.100 ... for the interested students). We will be using the exponential map of matrices ($\exp(A) = \sum_n \frac{A^n}{n!}$), which can be seen as a smooth map

$$\exp : \mathscr{M}_{n\times n} \to GL_n \subset \mathscr{M}_{n\times n}$$

Since its differential at the zero-matrix is

$$(d\exp)_0 : \mathscr{M}_{n\times n} \to \mathscr{M}_{n\times n}, \quad X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX) = X,$$

the inverse function theorem implies that exp restricts to a diffeomorphism between some open neighborhood of $0 \in \mathscr{M}_{n\times n}$ to an open neighborhood of $I \in \mathrm{GL}_n$. The resulting inverse, ("logarithm") can be used as chart for GL_n.

For closed subgroups $G \subset GL_n$ there is a rather simple idea to exhibit a smooth structure on G: looking for matrices whose exponential lands in G one introduces

$$\mathfrak{g} := \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G \;\forall\, t \in \mathbb{R}\} \subset \mathscr{M}_{n\times n}(\mathbb{R}), \tag{2.2.13}$$

and the idea is to use $\exp|_{\mathfrak{g}}$ to produce a chart for G around $\exp(0) = I$ (the identity matrix); for charts around other points $A \in \mathrm{GL}_n$, one uses the group structure to move from I (and around) to A. Of course, exhibiting charts on G to make G into a manifold would not be necessary if we manage to prove (as stated) that G is an embedded submanifold. The point of the simple idea above is that it can be adjusted to provide not only charts for $\mathfrak{g}$ but also adapted charts- therefore proving the stronger statement that G is an embedded submanifold. Time to start the proof!

Therefore, assume that $G \subset GL_n$; for simplicity, assume we work over $\mathbb{R}$. We introduce $\mathfrak{g}$ given by (2.2.13). It is handy to realize what G being closed implies about $\mathfrak{g}$: to ensure that $\exp(tX) \in G$ holds *for all* $t \in \mathbb{R}$, it suffices that it holds for a sequence $t_i \to 0$ of nonzero real numbers. The proof actually reveals something slightly stronger:

Lemma 2.102. *Given a matrix X, if there exists a sequence of nonzero real numbers with $t_i \to 0$, and a sequence of matrices $X_i \to X$ such that $\exp(t_i X_i) \in G$ for all i, then $\exp(tX) \in G$ for all t (i.e., $X \in \mathfrak{g}$).*

Proof. Since $\exp(-Y) = \exp(Y)^{-1}$ for all Y, after eventually changing the signs of some t_is and t, we may assume that $t_i > 0$ for all i and that we are looking at $\exp(tX)$ with $t > 0$. Fix t. For each i, the positive number t/t_i sits in an interval $[k_i, k_i + 1]$ for some integer k_i. One finds that $t - t_i \leq k_i t_i \leq t$, i.e. we managed to write our t as a limit of integer multiples of the t_is- namely $t = \lim_{i\to\infty} k_i t_i$. In turn, this allows us to write $\exp(tX)$ as a limit in G:

$$\exp(tX) = \lim_{i\to\infty} \exp(k_i t_i X_i) = \lim_{i\to\infty} \exp(t_i X_i)^{k_i}.$$

Since G is closed, we obtain $\exp(tX) \in G$. 😀

With this lemma at hand one can now show that the sum of two matrices $X, Y \in \mathfrak{g}$ is again in $\mathfrak{g}$. Indeed, writing

$$e^{tX} e^{tY} = e^{f(t)}, \quad \text{with} \quad \lim_{t\to 0} \frac{f(t)}{t} = X + Y,$$

take t_n to be any sequence of real numbers converging to 0 and $X_n = \frac{f(t_n)}{t_n}$. It follows that $\mathfrak{g}$ is a vector subspace of $\mathcal{M}_{n\times n}$. In turn, this allows us to choose a complement $\mathfrak{g}'$ of $\mathfrak{g}$ in $\mathcal{M}_{n\times n}(\mathbb{R})$- which is needed to adapt the original idea we mentioned above to proving that G is an embedded submanifold. More precisely, we aim to show that one can find an open neighborhood V of the origin in $\mathfrak{g}$, and a similar one V' in $\mathfrak{g}'$, so that

$$\phi : V \times V' \to GL_n, \phi(X, X') = e^X \cdot e^{X'}$$

is a local diffeomorphism onto an open neighborhood of the identity matrix and so that

$$G \cap \phi(V \times V') = \{\phi(v, 0) : v \in V\}. \tag{2.2.14}$$

Hence the submanifold condition is verified around the identity matrix; using left translations, it will hold at all points of G. The differential of ϕ at 0 (with ϕ viewed as a map defined on the entire $\mathfrak{g} \times \mathfrak{g}'$) is the identity map:

$$(d\phi)_0 : \mathfrak{g} \times \mathfrak{g}' \to T_I GL_n = \mathcal{M}_{n\times n}(\mathbb{R}), \quad (X, X') \mapsto \frac{d}{dt}\Big|_{t=0} e^{tX} e^{tX'} = X + X'.$$

In particular, ϕ is a local diffeomorphism around the origin. Hence the main condition that we have to take care of is (2.2.14): if $X \in V$, $X' \in V'$ satisfy $e^X \cdot e^{X'} \in G$, then $X' = 0$. It suffices to show that one can find V' so that

$$X' \in V', e^{X'} \in G \Longrightarrow X' = 0$$

(then just choose any V small enough such that ϕ is a diffeomorphism from $V \times V'$ into an open neighborhood of the identity in GL_n). For that we proceed by contradiction. If such V' would not exist, we would find a sequence $Y_n \to 0$ in $\mathfrak{g}'$ such that $e^{Y_n} \in G$. Since Y_n converges to 0, we find a sequence of integers $k_n \to \infty$ such that $X_n := k_n Y_n$ stay in a closed bounded region of $\mathfrak{g}'$ not containing the origin (e.g. on $\{X \in \mathfrak{g}' : 1 \leq ||X|| \leq 2\}$, for some norm on $\mathfrak{g}'$); then, after eventually passing to a subsequence, we may then assume that $X_n \to X \in \mathfrak{g}'$ for non-zero X. On the other hand, setting $t_n = \frac{1}{k_n}$, we see that $e^{t_n X_n} = e^{Y_n} \in G$ hence Lemma 2.102 would imply $X \in \mathfrak{g}$. Contradiction! 😀

For closed subgroups $G \subset GL_n$, the spaces $\mathfrak{g}$ used in the proof above (given by (2.2.13)) are not just a tool but, as we shall see later, are of fundamental importance: they are are the linear counterpart of Lie groups and, together with the relevant algebraic structure, they contain almost all the information on G. The relevant structure is, besides that of vector space, the one provided by the commutator of matrices: $[X, Y] = XY - YX$.

Exercise 2.103. Given $G \subset GL_n$ closed subgroup show that $\mathfrak{g}$ is stable under the commutator bracket:

$$X, Y \in \mathfrak{g} \Longrightarrow [X, Y] \in \mathfrak{g}.$$

(Hint: an argument similar to the one in the proof that $\mathfrak{g}$ is a vector subspace of $\mathcal{M}_{n\times n}$, using Lemma 2.102).

2.3 Tangent spaces

So far, the fact that we dealt with general manifolds M and not just embedded submanifolds of Euclidean spaces did not pose any serious problem: we could make sense of everything right away, by moving (via charts) to opens inside Euclidean spaces.

However, when trying to make sense of ”tangent vectors”, the situation is quite different. Indeed, already intuitively, when trying to draw a ”tangent space” to a manifold M, we are tempted to look at M as part of a bigger (Euclidean) space and draw planes that are ”tangent” (in the intuitive sense) to M. The fact that *the tangent spaces of M are intrinsic to M and not to the way it sits inside a bigger ambient space* is remarkable and may seem counterintuitive at first. And it shows that the notion of tangent vector is indeed of a geometric, intrinsic nature. Given our preconception (due to our intuition), the actual general definition may look a bit abstract. For that reason, we describe several different (but equivalent) approaches to tangent spaces. Each one of them implements one intuitive perception of tangent vectors on $\mathbb{R}^m$ and comes with its own slogan (see below).

2.3.1 Tangent vectors via curves or charts

The main slogan in this section is:

> *T_pM is made of speeds (at $t = 0$) of curves γ in M passing through p at $t = 0$:*

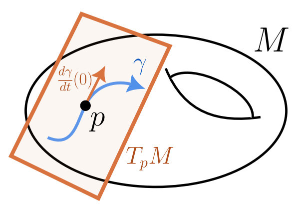


Fig. 2.15 We want to construct the tangent space in a point $p \in M$ in such a way that vectors correspond to the speeds $\frac{d\gamma}{dt}(0)$ of curves $\gamma : (-\varepsilon, \varepsilon) \to M$ that pass through p at time $t = 0$. Note that the image of the curve determines the direction of the vector, but not it's length: A reparametrized curve that passes through the same image twice as fast will correspond to a vector of twice the size.

The key remark is that, although we cannot talk yet about “the speed at $t = 0$ of a curve in M” (where would it live?), we can make sense of the statement: “two curves have the same speed at $t = 0$”. The last condition defines an equivalence relation on the collection of paths and then the tangent space will be the resulting quotient.

Definition 2.104. The **tangent space** of a manifold M at a point $p \in M$ is defined as the quotient:

$$T_pM = \text{Curves}_p(M)/\sim_p$$

where $\text{Curves}_p(M)$ is the collection of curves in M starting at p and $\sim_p$ is the equivalence relation:

$$\gamma_1 \sim_p \gamma_2 \quad \Longleftrightarrow \quad \frac{d\gamma_1^\chi}{dt}(0) = \frac{d\gamma_2^\chi}{dt}(0) \quad \forall\, \chi \in \text{Chart}_p(M). \tag{2.3.1}$$

where $\text{Chart}_p(M)$ for the collection of charts χ of M whose domain contains the point p.

The elements of T_pM are called **tangent vectors to** M **at** p. For $\gamma \in \text{Curves}_p(M)$, the **speed of** γ at $t = 0$ is defined as the equivalence class induced by γ and is denoted

$$\frac{d\gamma}{dt}(0) = \frac{d}{dt}\Big|_{t=0} \gamma(t) = \dot{\gamma}(0) \in T_pM.$$

Here are a few comments on this definition:

(1) Recall that, for any $\gamma \in \text{Curves}_p(M)$ and any $\chi \in \text{Chart}_p(M)$, the representations of γ in the chart χ is

$$\gamma^\chi = \chi \circ \gamma \in \text{Curves}_{\chi(p)}(\mathbb{R}^m).$$

In particular, the derivatives appearing in the definition (2.3.1) are usual derivatives of curves in $\mathbb{R}^m$,

$$\frac{d\gamma^\chi}{dt}(0) \in \mathbb{R}^m. \tag{2.3.2}$$

(2) The $\sim_p$ above is, indeed, an equivalence relation on $\text{Curves}_p(M)$ It basically allows us to make sense of when "two curves $\gamma_1, \gamma_2 \in \text{Curves}_p(M)$ have the same speed at $t = 0$", but before being able to talk about "the speed of an arbitrary curve $\gamma \in \text{Curves}_p(M)$" (it is actually used to make sense of those speeds).

(3) The reason the speed of a curve $\gamma \in \text{Curves}_p(M)$ cannot be defined in charts is that the quantity $\frac{d\gamma^\chi}{dt}(0) \in \mathbb{R}^m$ does depend on the chart χ. If another chart χ' is used then we can write $\gamma^{\chi'} = \chi' \circ \gamma = \chi' \circ \chi^{-1} \circ \chi \circ \gamma = c \circ \chi \circ \gamma = c \circ \gamma^\chi$, with $c = \chi' \circ \chi^{-1}$ (change of coordinates), and then the chain rule implies that

$$\frac{d\gamma^{\chi'}}{dt}(0) = (dc)_{\chi(0)}\left(\frac{d\gamma^\chi}{dt}(0)\right). \tag{2.3.3}$$

(4) On the other hand this computation shows that if the equality required in (2.3.1) holds for one χ then it holds for all others (we required it for all χ so that it is obvious that $\sim_p$ is an equivalence relation).

(5) One could say that the main slogan has been implemented by brute force since, by construction:

$$T_pM = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}_p(M) \right\}. \tag{2.3.4}$$

(6) Of course, choosing to talk about "the speed at $t = 0$" was only for convenience. More generally, whenever we have a smooth curve $\gamma : I \to M$, its speed at an arbitrary $t_0 \in I$ is defined as the speed of $s \mapsto \gamma(s + t_0)$ at $s = 0$:

$$\frac{d\gamma}{dt}(t_0) = \frac{d}{dt}\Big|_{t=t_0} \gamma(t) := \frac{d}{ds}\Big|_{s=0} \gamma(s + t_0) \in T_{\gamma(t_0)}M.$$

(7) The description (2.3.4) of the tangent space requires some care when used: it is true that any element $v \in T_pM$ can be written as $v = \frac{d\gamma}{dt}(0)$ for some $\gamma \in \text{Curves}_p(M)$, but γ is far from being unique.

(8) A tangent vector to M at p can be seen, through each chart, as a vector in the Euclidean space $\mathbb{R}^m$: while a chart χ for M allows one to represent points p by coordinates, it also allows us to represent tangent vectors $v \in T_pM$ by vectors in $\mathbb{R}^m$. To make this viewpoint more visible, we introduce the following:

Definition 2.105. For an arbitrary tangent vector $v \in T_pM$ and $\chi \in \mathrm{Chart}_p(M)$ (i.e. a chart of M around p), the **representation of** v **in the chart** χ is

$$v^{\chi} := \frac{d\gamma^{\chi}}{dt}(0) \in \mathbb{R}^m,$$

where γ is any curve representing v.

Proposition 2.106. Given $v \in T_pM$, the vectors $v^{\chi} \in \mathbb{R}^m$ are related to each other by the formula:

$$v^{\chi'} = \left(d\, c_{\chi}^{\chi'}\right)_{\chi(p)} (v^{\chi}) \quad \forall\, \chi, \chi' \in \mathrm{Chart}_p(M). \tag{2.3.5}$$

Conversely, any map $\mathrm{Chart}_p(M) \ni \chi \mapsto v^{\chi} \in \mathbb{R}^m$ satisfying (2.3.5) arises in this way and corresponds to a unique tangent vector at p.

Proof. Let us denote the map $\chi \mapsto v^{\chi}$ by v_{γ}, so that the 1-1 correspondence would be $\frac{d\gamma}{dt}(0) \mapsto v_{\gamma}$. The list of comments above shows that this well-defined and injective. Hence we are left with showing that any map $v : \mathrm{Chart}_p(M) \to \mathbb{R}^m, \quad \chi \mapsto v^{\chi}$ with the property described above is of type v_{γ} for some γ. Notice first that since our maps v are required to satisfy the same type of condition as (2.3.3), as in (4) above we see that

(4') given $v, w : \mathrm{Chart}_p(M) \to \mathbb{R}^m$ as above, if $v^{\chi_0} = w^{\chi_0}$ for one chart χ_0, then $v = w$.

Fix now a v and look for γ. Choose a chart $\chi_0 \in \mathrm{Chart}_p(M)$, with $\chi_0 : U_0 \to \Omega_0 \subset \mathbb{R}^m$, choose any path

$$\alpha : (-\varepsilon, \varepsilon) \to \Omega_0 \quad \text{satisfying} \quad \alpha(0) = \chi_0(p), \alpha'(0) = v^{\chi_0}.$$

and set $\gamma = \chi_0^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to M$. Since by construction $\gamma^{\chi_0} = \alpha$ we have

$$(v_{\gamma})^{\chi_0} = \frac{d\gamma^{\chi_0}}{dt}(0) = \frac{d\alpha}{dt}(0) = v^{\chi_0},$$

remark (4') above allows us to conclude that $v = v_{\gamma}$, which proves the desired surjectivity.

2.3.2 *The vector space structure*

The advantage of the last viewpoint is that the algebraic structure on T_pM becomes transparent: for $v, w \in T_pM$, and a scalar $\lambda \in \mathbb{R}$, there is a natural way to add up v and w, and to multiply v by λ, producing new tangent vectors

$$v + w, \quad \lambda \cdot v \in T_pM. \tag{2.3.6}$$

Explicitly, they are obtained via the usual vector space operations on $\mathbb{R}^m$ by requiring that, for all $\chi \in \mathrm{Chart}_p(M)$,

$$(v + w)^{\chi} := v^{\chi} + w^{\chi}, \quad (\lambda \cdot v)^{\chi} := \lambda \cdot v^{\chi}. \tag{2.3.7}$$

Exercise 2.107. Check that, indeed, for any two $\chi, \chi' \in \mathrm{Chart}_p(M)$, denoting $c = c_{\chi}^{\chi'}$,

$$(dc)_{\chi(p)}\Big((v+w)^{\chi}\Big) = (v+w)^{\chi'}, \quad (dc)_{\chi(p)}\Big((\lambda\cdot v)^{\chi}\Big) = (\lambda\cdot v)^{\chi'}.$$

Lemma 2.108. *For $\lambda \in \mathbb{R}$, $v, w \in T_pM$ written as $v = \frac{d\gamma_1}{dt}(0)$, $w = \frac{d\gamma_2}{dt}(0)$, the vectors $v+w$, $\lambda\cdot v$ are also given by:*

$$v+w = \frac{d}{dt}\Big|_{t=0} \chi^{-1}\left(\gamma_1^{\chi}(t) + \gamma_2^{\chi}(t)\right), \quad \lambda\cdot v = \frac{d}{dt}\Big|_{t=0} \chi^{-1}\left(\lambda \cdot \gamma_1^{\chi}(t)\right) = \frac{d}{dt}\Big|_{t=0} \gamma_1(\lambda\cdot t).$$

Proof. We first look at $v+w$. In the search for a curve γ such that $\frac{d\gamma}{dt} = v+w$ we know that the last equality is *equivalent*, one we fixed a chart χ, to the corresponding equality in the chart χ, i.e., to

$$\frac{d\gamma_1^{\chi}}{dt}(0) + \frac{d\gamma_2^{\chi}}{dt}(0) = \frac{d\alpha^{\chi}}{dt}(0).$$

The formula from the statement picks the most natural choice for α, defined by the condition $\alpha^{\chi} = \gamma_1^{\chi} + \gamma_2^{\chi}$. Similarly for $\lambda\cdot v$, where there are two very natural choices for α, as indicated by the two formulas for $\lambda\cdot v$. 😀

2.3.3 Bases and the standard identifications

We now show that the tangent spaces T_pM are finite dimensional vector spaces, of dimension equal to $m = \dim M$. Actually, we will show that any chart around p gives rise to a basis of of the vector space T_pM.

Lemma 2.109. *Any chart $\chi_0 \in \mathrm{Chart}_p(M)$ gives rise to an isomorphism of vector spaces:*

$$T_pM \xrightarrow{\sim} \mathbb{R}^m, \quad v \mapsto v^{\chi_0}.$$

Proof. As pointed out in remark (4') in the proof of Proposition 2.106, once we fixed χ_0, and given $v_0 \in \mathbb{R}^m$, the element $v \in T_pM$ such that $v^{\chi_0} = v_0$ is unique and it must be given by the formula

$$v^{\chi} = (d\,c_{\chi_0}^{\chi})_{\chi_0(p)}(v_0),$$

for any other chart χ around p. Strictly speaking we still have to prove (2.3.5) but that follows from the chain rule and the remark that $c_{\chi_0}^{\chi'} = c_{\chi}^{\chi'} \circ c_{\chi_0}^{\chi}$. 😀

Therefore the dimension of each vector space T_pM is precisely the dimension of M. We also deduce that any chart $\chi \in \mathrm{Chart}_p(M)$ gives rise to a basis, called **the canonical basis of T_pM induced by χ** and denoted

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_pM, \tag{2.3.8}$$

Via the isomorphism $v \mapsto v^{\chi}$ it corresponds to the canonical basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$.

Exercise 2.110. Show that the members of (2.3.8) can be described as speeds of curves as follows:

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \frac{d}{dt}\Big|_{t=0} \chi^{-1}\Big(\chi(p) + te_i\Big) = \frac{d}{dt}\Big|_{t=0} \chi^{-1}\Big(\chi_1(p), \ldots, \chi_{i-1}(p), t + \chi_i(p), \chi_{i+1}(p), \ldots, \chi_m(p)\Big).$$

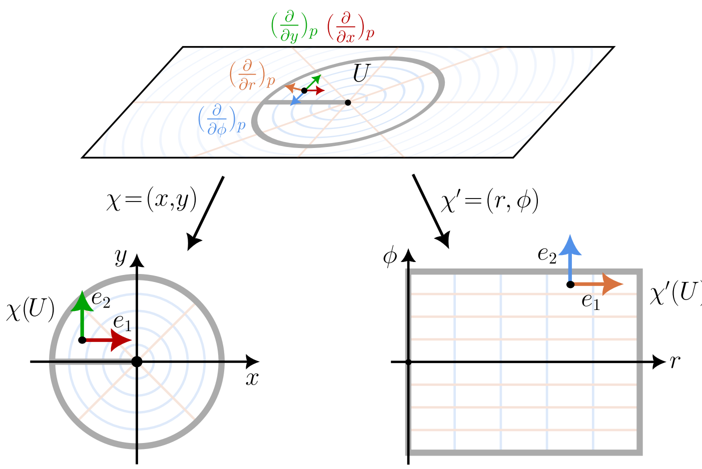


Fig. 2.16 In the situation of Fig. 2.3, we fix a point $p \in U$ and can consider the vectors based at p that are represented by unit vectors within the identity and polar coordinates. They always point in the direction in which the corresponding coordinate increases.

Exercise 2.111. If χ and χ' are two charts of M around p, $c = c_{\chi}^{\chi'}$, show that

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_j \frac{\partial c_j}{\partial x_i}(\chi(p)) \cdot \left(\frac{\partial}{\partial \chi'_j}\right)_p . \tag{2.3.9}$$

For M is $\mathbb{R}^m$ or, more generally, an open $\Omega \subset \mathbb{R}^m$, one can apply Lemma 2.109 to the identity chart Id; one obtains

- an isomorphism

$$\text{standard}_p : T_p\Omega \xrightarrow{\sim} \mathbb{R}^m, \quad v \mapsto v^{\text{Id}}. \tag{2.3.10}$$

  In terms of speeds it is rather tautological: it sends the abstract $\frac{d\gamma}{dt}(0) \in T_p\Omega$ to the standard derivative.
- a canonical basis of $T_p\Omega$, denoted

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p . \tag{2.3.11}$$

These will be called **the standard identification** of the abstract tangent space $T_p\Omega$ with $\mathbb{R}^m$, and **standard basis** of $T_p\Omega$. Hence arbitrary vectors $v \in T_p\Omega$ can be written as

$$v = \sum_{i=1}^{n} \lambda_i \cdot \left(\frac{\partial}{\partial x_i}\right)_p \in T_p\Omega \tag{2.3.12}$$

with $\lambda_i \in \mathbb{R}$, and the standard identification just picks up the coefficients:

$$\text{standard}_p(v) = (\lambda_1, \ldots, \lambda_n).$$

Very often we identify $T_p\Omega$ with $\mathbb{R}^m$. However, even when doing that, **we will still keep the notation (2.3.8) for the canonical basis**. This is in order to indicate that we are thinking about/using tangent vectors.

Exercise 2.112. Show that the inverse of the standard identification can be described as follows:

$$\text{standard}_p^{-1} : \mathbb{R}^m \to T_p\Omega, \quad \text{standard}_p^{-1}(w) = \frac{d}{dt}\Big|_{t=0} (p+tw) \in T_p\mathbb{R}^m.$$

We can go on further and consider embedded submanifolds of Euclidean spaces $M \subset \mathbb{R}^n$. Then the curves in M are also curves in R^n and then the resulting inclusion $\text{Curves}_p(M) \subset \text{Curves}_p(\mathbb{R}^n)$ induces a similar inclusion

$$T_pM \subset T_p\mathbb{R}^n.$$

This is a vector subspace inclusion since it can be interpreted as the differential of $M \hookrightarrow \mathbb{R}^n$. More on such inclusions will be discussed in the next section. The standard identification (2.3.10) restricts now to an injection

$$\text{standard}_p|_{T_pM} : T_pM \hookrightarrow \mathbb{R}^n. \tag{2.3.13}$$

Lemma 2.113. *For any embedded submanifold $M \subset \mathbb{R}^n$, the map (2.3.13) is injective and its image is precisely the classical tangent space $T_p^{class}M$ from Chapter 1 (Definition 1.35). In other words, it becomes an isomorphism*

$$\textit{standard}_p^M : T_pM \xrightarrow{\sim} T_p^{class}M.$$

Proof. standard_p^M is injective because it is defined as a composition of an injective map with an isomorphism. To identify its image in $\mathbb{R}^m$ we put together the fact that T_pM consists of speeds of curves in M, $T_p^{\text{class}}M$ consists of usual derivatives at $t=0$ of curves $\gamma \in \text{Curves}_p(M)$, and the standard identification (2.3.10) sends $\frac{d\gamma}{dt}(0)$ to the usual derivative of γ at $t=0$.

Similarly, speeds/derivatives of curves in $\mathbb{R}^m$ can be dealt with in the "usual way":

Exercise 2.114. Please remember/make sense of the fact that, for arbitrary curves $\gamma : I \to M$ in a manifold M defined in some interval $I \subset \mathbb{R}$, it makes sense of talk about the derivative/speed of γ at arbitrary time as an element $\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$. Then, when $M = \mathbb{R}^n$, show that

$$\frac{d\gamma}{dt}(t) = \gamma_1'(t) \cdot \left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \ldots + \gamma_n'(t) \cdot \left(\frac{\partial}{\partial x_n}\right)_{\gamma(t)}, \tag{2.3.14}$$

where $\gamma_i : I \to \mathbb{R}$ are the components of γ and $\gamma_i'(t) \in \mathbb{R}$ the usual derivatives.

2.3.4 Differentials

Tangent space are useful especially because they are part of a "functor": we do not only associate vector spaces to manifolds but also linear maps to smooth maps between manifolds. To do this we start with a smooth map $F: M \to N$ and we fix a point $p \in M$. The function F allows us to move curves:

$$\gamma \in \mathrm{Curves}_p(M) \Longrightarrow F \circ \gamma \in \mathrm{Curves}_{F(p)}(N),$$

hence, there is a rather obvious way to move from the tangent spaces of M to the ones of N, by sending

$$\frac{d\gamma}{dt}(0) \mapsto \frac{dF \circ \gamma}{dt}(0). \tag{2.3.15}$$

Definition 2.115. For a smooth map $F: M \to N$ and $p \in M$, the map

$$(dF)_p : T_pM \to T_{F(p)}N$$

given by (2.3.15) is called **the differential of F at the point p**.

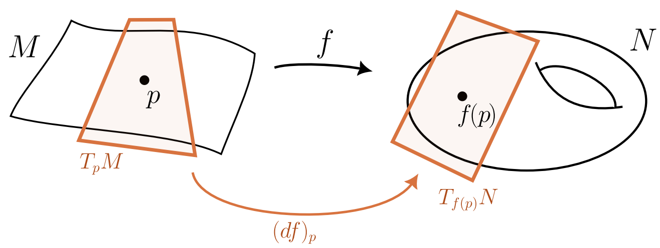


Fig. 2.17 The differential $(dF)_p$ of F at p, a linear map from the tangent space at p of the domain to the one at $F(p)$ of the codomain.

One does have however to make sure that (2.3.15) does define a map.

Lemma 2.116. *The map $(dF)_p$ given by (2.3.15) is well-defined and is linear. Furthermore, w.r.t. any charts*

$$\chi \in \mathrm{Chart}_p(M), \quad \chi' \in \mathrm{Chart}_{F(p)}(N),$$

it is given by the usual total derivative of the resulting representation of F, $F_\chi^{\chi'} = \chi' \circ F \circ \chi^{-1}$:

$$\left((dF)_p(v)\right)^{\chi'} = \left(dF_\chi^{\chi'}\right)_{\chi(p)}(v^\chi) \qquad \forall\, v \in T_pM.$$

Proof. We first have to show that

$$\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0) \quad \Longrightarrow \quad \frac{dF \circ \gamma_1}{dt}(0) = \frac{dF \circ \gamma_2}{dt}(0).$$

whenever $\gamma_1, \gamma_2 \in \mathrm{Curves}_p(M)$. Or, in terms of equivalence relations, that $\gamma_1 \sim_p \gamma_2 \Longrightarrow F \circ \gamma_1 \sim_p F \circ \gamma_2$. To prove this we choose coordinate charts $\chi \in \mathrm{Charts}_p(M)$, $\chi' \in \mathrm{Charts}_{F(p)}(N)$ and we use the representation of F w.r.t. these charts, $F_\chi^{\chi'}$ (a smooth map between open in Euclidean space). Notice that $(F \circ \gamma)^{\chi'} = F_\chi^{\chi'} \circ \gamma^\chi$ and, therefore, by the standard chain rule in Euclidean spaces,

$$\frac{d(F\circ\gamma)^{\chi'}}{dt}(0) = (dF_{\chi}^{\chi'})_{\gamma^{\chi}(0)}\Big(\frac{d\gamma^{\chi}}{dt}(0)\Big) \tag{2.3.16}$$

for all $\gamma \in \text{Curves}_p(M)$. Notice that this is precisely the formula from the statement. It has two consequences. First of all, $(dF)_p$ is linear. Secondly, it allows us to conclude the proof:

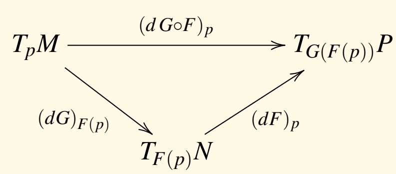
$$\gamma_1 \sim_p \gamma_2 \implies \frac{d\gamma_1^{\chi}}{dt}(0) = \frac{d\gamma_2^{\chi}}{dt}(0) \overset{dF_{\chi}^{\chi'}}{\implies} \frac{d(F\circ\gamma_1)^{\chi'}}{dt}(0) = \frac{d(F\circ\gamma_2)^{\chi'}}{dt}(0) \implies F\circ\gamma_1 \sim_p F\circ\gamma_2.$$

And here is a simple but very important result, which follows by simply applying the definitions (exercise!).

Proposition 2.117. The differentials defined above have the following properties:

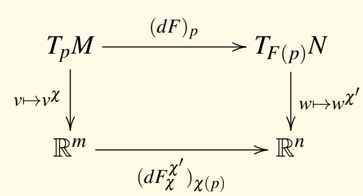
i) If $F = \text{Id}_M : M \to M$ is the identity map, $(dF)_p$ is the identity map of T_pM.

ii) The chain rule: for two smooth maps $M \overset{F}{\to} N \overset{G}{\to} P$ and $p \in M$, $(d\,G\circ F)_p = (dG)_{F(p)} \circ (dF)_p$.



Next, one can adapt subsection 2.3.3 to a similar discussion for smooth maps $F : M \to N$ and their differentials. To that end, we fix charts $\chi \in \text{Charts}_p(M)$, $\chi' \in \text{Charts}_{F(p)}(N)$.

(1) First of all, Lemma 2.109 allowz us to identify the domain and codomain of the differentials $(dF)_p$ with Euclidean spaces. Lemma 2.116 tells us that the outcome is precisely the usual differential (total derivative) of the representation of F with respect to the charts χ and χ'; or, as a commutative diagram:



(2) Secondly, the resulting standard bases

$$\Big(\frac{\partial}{\partial\chi_1}\Big)_p, \ldots, \Big(\frac{\partial}{\partial\chi_m}\Big)_p \text{ for } T_pM, \quad \Big(\frac{\partial}{\partial\chi'_1}\Big)_p, \ldots, \Big(\frac{\partial}{\partial\chi'_n}\Big)_{F(p)} \text{ for } T_{f(p)}N$$

allow us to encode the linear map $(dF)_p$ as a matrix. The diagram above shows that the resulting matrix is precisely the Jacobian of $F_{\chi}^{\chi'}$ at $\chi(p))$; explicitly:

$$(dF)_p\Big(\Big(\frac{\partial}{\partial\chi_i}\Big)_p\Big) = \sum_j \frac{\partial F_{\chi}^{\chi',j}}{\partial\chi'_i}(\chi(p))\cdot\Big(\Big(\frac{\partial}{\partial\chi'_j}\Big)_{F(p)}\Big) \tag{2.3.17}$$

(3) Again, this discussion can be applied when χ and χ' are identity charts for opens in Euclidean spaces:

$$F : \Omega \to \Omega' \quad (\text{with } \Omega \subset \mathbb{R}^m, \Omega' \subset \mathbb{R}^n \text{ open}),$$

the vertical maps in the last diagram become the standard identifications (2.3.10), via which $(dF)_p$ is identified with the usual differential (total derivative) from Analysis:

$$\begin{array}{ccc} T_p\Omega & \xrightarrow{(dF)_p} & T_{F(p)}\Omega' \\ {\scriptstyle \text{standard}_p^{\Omega}} \downarrow & & \downarrow {\scriptstyle \text{standard}_{F(p)}^{\Omega'}} \\ \mathbb{R}^m & \xrightarrow[(DF)_{\chi(p)}]{} & \mathbb{R}^n \end{array}, \qquad (dF)_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_j \frac{\partial F^j}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial x'_j}\right)_{F(p)}$$

Another particularly interesting case in the discussion above is obtained when $N = \mathbb{R}$, i.e. when we deal with real-valued smooth functions

$$f : M \to \mathbb{R}.$$

With the identifications $T_x\mathbb{R} = \mathbb{R}$, the differentials of f are always considered as linear maps

$$(df)_p : T_pM \to \mathbb{R}.$$

Changing the focus from f to v, we see that any tangent vector $v \in T_pM$ induces a linear map

$$\partial_v : \mathscr{C}^\infty(M) \to \mathbb{R}, \quad \partial_v(f) := (df)_p(v).$$

For $v = \frac{d\gamma}{dt}(0)$, $\partial_v(f) = \frac{d}{dt}\big|_{t=0} f(\gamma(t))$, hence $\partial_v(f)$ deserves the name of **derivative of f in the direction of v**. In particular, when $v = \left(\frac{\partial}{\partial \chi_i}\right)_p$ corresponding to a chart $\chi \in \text{Charts}_p(M)$, $\partial_v(f)$ will be called **the partial derivatives of f at p w.r.t. the chart χ** and denoted $\frac{\partial f}{\partial \chi_i}(p)$. Hence:

$$\frac{\partial f}{\partial \chi_i}(p) := \frac{d}{dt}\bigg|_{t=0} f\left(\chi^{-1}(\chi(p) + te_i)\right) = (df)_p\left(\left(\frac{\partial}{\partial \chi_j}\right)_p\right) = \frac{\partial f_\chi}{\partial \chi_j}(\chi(p)) \tag{2.3.18}$$

This explains the notation used for the canonical basis (2.3.8). Furthermore, it allows one to rewrite some of the formulas in a somewhat more simplified form. For instance, for the change of coordinates formula (2.3.9), we can now write

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_{j=1}^{m} \frac{\partial \chi'_j}{\partial \chi_i}(p) \cdot \left(\frac{\partial}{\partial \chi'_j}\right)_p \quad \left(or : \frac{\partial}{\partial \chi_i} = \sum_{j=1}^{m} \frac{\partial \chi'_j}{\partial \chi_i} \cdot \frac{\partial}{\partial \chi'_j}\right)$$

Returning to opens $\Omega \subset \mathbb{R}^n$, writing $v \in T_p\Omega$ as in (2.3.12), we have:

$$\partial_v = \sum_{i=1}^{n} \lambda_i \cdot \left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{C}^\infty(\Omega) \to \mathbb{R},$$

2.3.5 Tangent spaces of submanifolds

The construction of the differential can be applied, in particular, to inclusions $i: M \hookrightarrow N$ of embedded submanifolds $M \subset N$ of a manifold N. In that case the situation looks rather tautological, since smooth curves in M can be seen, themselves, as smooth curves in N:

$$\mathrm{Curves}_p(M) \subset \mathrm{Curves}_p(N), \qquad (p \in M).$$

The resulting map $T_pM \ni \frac{d\gamma}{dt}(0) \mapsto \frac{d\gamma}{dt}(0) \in T_pN$ can now be seen as an inclusion

$$\boxed{T_pM \subset T_pN}.$$

However, it is instructive to realise that this is not completely obvious beforehand, despite the fact one does have something which "obviously looks like an inclusion":

$$\left\{\frac{d\gamma}{dt}(0) : \gamma \in \mathrm{Curves}_p(M)\right\} \subset \left\{\frac{d\gamma}{dt}(0) : \gamma \in \mathrm{Curves}_p(N)\right\}$$

Remembering what the definition of speeds was, we see that this being an inclusion (injective) translates into the fact that for any $\gamma_1, \gamma_2 \in \mathrm{Curves}_p(M)$ one has the equivalence

$$\gamma_1 \sim_p \gamma_2 \text{ in } \mathrm{Curves}_p(M) \Longleftrightarrow \gamma_1 \sim_p \gamma_2 \text{ in } \mathrm{Curves}_p(N).$$

To prove it one uses a chart χ of N around p adapted to M and the induced chart $\chi_M = \chi|_M \in \mathrm{Charts}_p(M)$. Since

$$\Big(\gamma_1 \sim_p \gamma_2 \text{ in } \mathrm{Curves}_p(M)\Big) \Longleftrightarrow \Big(\frac{\gamma_1^{\chi_M}}{dt}(0) = \frac{\gamma_2^{\chi_M}}{dt}(0)\Big) \quad \text{and} \quad \Big(\gamma_1 \sim_p \gamma_2 \text{ in } \mathrm{Curves}_p(N)\Big) \Longleftrightarrow \Big(\frac{\gamma_1^{\chi}}{dt}(0) = \frac{\gamma_2^{\chi}}{dt}(0)\Big)$$

it suffices to remark now that each $\gamma_i^{\chi_M}$ is just γ_i^{χ} viewed as a curve inside $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$, hence the right hand sides of the previous two lines are actually identical. The argument above can be shortened a bit by making use of the bases (2.3.8) associated to charts, where we use charts $\chi \in \mathrm{Charts}_p(N)$ adapted to M as above. One then checks that $\left(\frac{\partial}{\partial \chi_i^M}\right)_p \in T_pM$, when seen in T_pN, are precisely $\left(\frac{\partial}{\partial \chi_i}\right)_p$, for each $1 \leq i \leq m$. Since these are linearly independent, the map tautological map $T_pM \to T_pN$ is indeed injective. Exercise: fill in the details!

Exercise 2.118. For $v \in T_pM$, interpreting $v \in T_pN$ as a function $\mathrm{Charts}_p(N) \to \mathbb{R}^n$, as in Proposition 2.106, show that for any chart χ of N adapted to M, $v(\chi) = v(\chi|_M) \in \mathbb{R}^m$, where $\mathbb{R}^m$ is viewed as a subspace of $\mathbb{R}^n$ by adding zeros in the last coordinates.

In the case of embedded submanifolds of Euclidean spaces, $M \subset \mathbb{R}^n$, the resulting inclusion

$$T_pM \subset T_p\mathbb{R}^n = \mathbb{R}^n$$

becomes the inclusion (2.3.13) inducing the standard identification $T_pM \cong T_p^{\mathrm{class}}M$. In practice, when working with tangent vectors in $\mathbb{R}^m$, we write them as sums (2.3.12). When working on $M \subset \mathbb{R}^m$, it is important to know how to recognize when such a vector v actually belongs to T_pM. For instance, when M is just a point (a 0-dimensional submanifold!), the answer is: only when all λ_i vanish! Of course, one can try to use the definition (and the standard identifications), and then one has to make sure that v can be realised as $\frac{d}{dt}(0)$ where γ is now a curve that sits in M. To avoid doing that each time, it is useful to know that, when M is given by a set of equations $F(x) = (F_1(x), \ldots, F_k(x)) = 0$ as in the regular value theorem (Theorem 2.24), then all one has to do is to check that the coefficients λ_i from (2.3.12) satisfy

$$\sum_{i=1}^{n} \lambda_i \frac{\partial F_j}{\partial x_i}(p) = 0, \quad 1 \leq j \leq k.$$

Indeed, one has the following general criteria:

Proposition 2.119 (the regular value theorem in $\mathbb{R}^n$, upgraded). *Let $M \subset \mathbb{R}^n$ be as in Theorem 2.24, i.e., the zero-set of a smooth map $F : \Omega \to \mathbb{R}^k$ with the property that F is a submersion at each point $p \in M$. Then M is an $m = n-k$ dimensional embedded submanifold of $\mathbb{R}^n$ and, for each $p \in M$,*

$$T_pM = \Big\{ v \in T_p\mathbb{R}^n : (dF)_p(v) = 0 \Big\}.$$

Proof. We know that T_pM is $m = n-k$ dimensional. Also, since $(dF)_p$ is a surjective linear map from an n-dimensional to a k-dimensional vector space, its kernel is m dimensional as well. Therefore it suffices to prove that T_pM sits inside of the kernel. This follows by writing arbitrary $v \in T_pM$ as $\frac{d\gamma}{dt}(0)$ where γ is a curve in M. According to the definition of M we must have $F(\gamma(t)) = 0$ for all t. Taking the derivatives w.r.t. t at $t = 0$ we obtain that $(dF)_p(v) = 0$, as desired. 😀

Example 2.120. For $S^1 \subset \mathbb{R}^2$, $p = (a,b) \in S^1$, to see that $v_p := b \cdot \left(\frac{\partial}{\partial x}\right)_p - a \cdot \left(\frac{\partial}{\partial y}\right)_p$ is tangent to S^1 one can either:

(1) use (2.3.14) to see that it can be described as the speed at $t = 0$ of

$$\gamma(t) = (\cos(t), \sin(t)) \in S^1.$$

(2) use that M is the zero set of $F : \mathbb{R}^2 \to \mathbb{R}, F(x,y) = x^2 + y^2 - 1$ with $0 \in \mathbb{R}$ as regular value, look at

$$(dF)_{(a,b)} : \; r \cdot \left(\frac{\partial}{\partial x}\right)_{(a,b)} + s \cdot \left(\frac{\partial}{\partial y}\right)_{(a,b)} \longmapsto r\,\frac{\partial F}{\partial x}(a,b) + s\,\frac{\partial F}{\partial y}(a,b) = 2ra + 2sb$$

and check that this annihilates v_p:

$$(dF)_{(a,b)}(v_p) = 2b \cdot a + 2(-a) \cdot b = 0.$$

In general, when having to compute differentials of smooth maps between submanifolds of Euclidean spaces,

$$F : M \to N, \quad M \subset \mathbb{R}^{\widetilde{m}}, \quad N \subset \mathbb{R}^{\widetilde{n}}, \tag{2.3.19}$$

the compute is simpler if it happens that F is the restriction to M of a smooth function on $\mathbb{R}^{\widetilde{m}}$ or an open inside it

$$\widetilde{F} : \Omega \to \mathbb{R}^{\widetilde{n}}, \quad \text{with } \Omega \subset \mathbb{R}^{\widetilde{m}} \text{ open containing } M. \tag{2.3.20}$$

Exercise 2.121. In the situation (2.3.19), if $F = \widetilde{F}|_M$ with $\widetilde{F}$ as in (2.3.20) then, for any $p \in M$,

$$(dF)_p : T_pM \to T_{F(p)}N$$

is the restriction of $(d\widetilde{F})_p : T_p\Omega \to T_{F(p)}\mathbb{R}^{\widetilde{n}}$ to $T_pM \subset T_p\mathbb{R}^{\widetilde{m}}$. Or, on a commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{(dF)_p} & T_{F(p)}N \\ \downarrow & & \downarrow \\ T_p\Omega & \xrightarrow[(d\widetilde{F})_p]{} & T_{F(p)}\mathbb{R}^{\widetilde{n}} \end{array}$$

(hint: the chain rule).

Exercise 2.122. Please do the following

(1) For the sphere $S^2 \subset \mathbb{R}^3$, show that for any $q = (x,y,z) \in S^2$,

$$X^1_q := z\left(\frac{\partial}{\partial y}\right)_q - y\left(\frac{\partial}{\partial z}\right)_q,$$

$$X^2_q := x\left(\frac{\partial}{\partial z}\right)_q - z\left(\frac{\partial}{\partial x}\right)_q,$$

$$X^3_q := y\left(\frac{\partial}{\partial x}\right)_q - x\left(\frac{\partial}{\partial y}\right)_q$$

belong to T_qS^2. Find two different arguments.

(2) For the sphere $S^3 \subset \mathbb{R}^4$, at an arbitrary $p = (x,y,z,t) \in S^3$, show that the following form a basis of T_pS^3:

$$V^1_p := \frac{1}{2}\left(-y\left(\frac{\partial}{\partial x}\right)_p + x\left(\frac{\partial}{\partial y}\right)_p + t\left(\frac{\partial}{\partial z}\right)_p - z\left(\frac{\partial}{\partial t}\right)_p\right),$$

$$V^2_p := \frac{1}{2}\left(-z\left(\frac{\partial}{\partial x}\right)_p - t\left(\frac{\partial}{\partial y}\right)_p + x\left(\frac{\partial}{\partial z}\right)_p + y\left(\frac{\partial}{\partial t}\right)_p\right),$$

$$V^3_p := \frac{1}{2}\left(-t\left(\frac{\partial}{\partial x}\right)_p + z\left(\frac{\partial}{\partial y}\right)_p - y\left(\frac{\partial}{\partial z}\right)_p + x\left(\frac{\partial}{\partial t}\right)_p\right).$$

(3) Consider now the so called Hopf map $h : S^3 \to S^2$,

$$h(x,y,z,t) = (x^2+y^2-z^2-t^2, 2(yz-xt), 2(xz+yt))$$

and show that, for $p \in S^3$ and $q = h(p) \in S^2$, $(dh)_p : T_pS^3 \to T_qS^2$ sends V^1_p to X^1_q, V^2_p to X^2_q and V^3_p to X^3_q.

2.4 Submersions and immersions

2.4.1 Submersions

Next, we export the notion of submersion form Euclidean spaces to the general setting.

Definition 2.123. Let $F : M \to N$ be a smooth map between two manifolds. We say that:

- F is an **submersion at** p if its differential $(dF)_p : T_pM \to T_{F(p)}N$ is surjective.
- F is an **submersion** if F is a submersion at all $p \in M$.

Using the definition and the chain rule, you should now work out the following

Exercise 2.124. Show that the condition that F is a submersion at p is equivalent to any of the following:

(2) its representation $F_{\chi}^{\chi'}$ with respect to any chart χ of M around p and χ' of N around $F(p)$ is a submersion in the sense of Analysis (Definition 1.47).
(3) there exists a chart χ of M around p and one chart χ' of M' around $F(p)$ such that $F_{\chi}^{\chi'}$ is a submersion.

The submersion theorem on Euclidean spaces, i.e. Theorem 1.42, can now be restated as follows:

Theorem 2.125 (the submersion theorem). *A smooth map $F : M \to N$ is a submersion at $p \in M$ if and only if there exist charts χ of M around p and χ' of N around $F(p)$ such that $F_{\chi}^{\chi'} = \chi' \circ F \circ \chi^{-1}$ is given by*

$$F_{\chi}^{\chi'}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n).$$

Example 2.126 (projections). For any two manifolds M and N, the canonical projections

$$\mathrm{pr}_M : M \times N \to M, \quad \mathrm{pr}_N : M \times N \to N$$

are submersions. The submersion theorem tells us that submersions are the smooth maps which locally (in the appropriate coordinates) look like projections.

Example 2.127 (local diffeomorphisms). A simple example of a submersion which is not a projection is the exponential map $\exp : \mathbb{R} \to S^1$, $t \mapsto e^{it} = (\cos t, \sin t)$. More generally, any local diffeomorphism is a submersion. Actually, in the border-case $\dim M = \dim N$ one has:

$$\text{border case } \dim M = \dim N : \qquad \text{submersion at } p \iff \text{local diffeomorphism around } p \tag{2.4.1}$$

(for the notion of local diffeomorphism around p see the discussion preceding Lemma 2.58). Further combining with Lemma 2.58, one obtains another interesting conclusion:

$$\left.\begin{array}{r} F : M \to N \text{ bijective submersion} \\ \dim M = \dim N \end{array}\right\} \Longrightarrow F \text{ is a diffeomorphism.}$$

To prove (2.4.1), after using charts we may assume that $M = N = \mathbb{R}^n$. Using the description of immersions in terms of total derivatives, as well as the inverse function theorem, it all boils to a property of linear maps: a linear map $A : V \to W$ between two vector spaces of the same dimension is injective if and only if it is bijective. 😀

Example 2.128 (The Hopf fibration). An even more interesting example is the Hopf map $h : S^3 \to S^2$ from Exercise 2.122. Indeed, that is one of the main but immediate consequences of that exercise.

As we have already mentioned in the case of Euclidean spaces, submersions enter one of the most useful criteria to check whether a set of equations gives rise to a submanifold: the regular value theorem. We can now state it in its full generality. The setting is as follows: we are looking at a smooth map $F : M \to N$ and we are interested in its fiber above a point $q \in N$, $F^{-1}(q) \subset M$. The condition that we need here is that q is a **regular value of** F by which we mean that F is a submersion at all points $p \in F^{-1}(q)$.

Theorem 2.129 (the regular value theorem). *If $q \in N$ is a regular value of a smooth map*

$$F : M \to N,$$

then the fiber above q, $F^{-1}(q)$, is an embedded submanifold of M of dimension

$$dim(F^{-1}(q)) = dim(M) - dim(N).$$

Furthermore, for each $p \in M$, the tangent space of M at p can be computed as:

$$T_pF^{-1}(q) = \left\{ v \in T_pM : (dF)_p(v) = 0 \right\},$$

the kernel of the differential $(dF)_p : T_pM \to T_{F(p)}N$.

Proof. In principle, this is just another facet of the submersion theorem. Let $d = m - n$, where m and n are the dimension of M and N, respectively. We check the submanifold condition around an arbitrary point $p \in M_0 := F^{-1}(q)$. For that we apply the submersion theorem to F near p to find charts $\chi : U \to \Omega$ and $\chi' : U' \to \Omega'$ of M around p and of N around $F(p)$, respectively, such that $F(U) \subset U'$ and

$$F_{\chi}^{\chi'}(x) = (x_1, \dots, x_n) \quad \text{for all } x \in \Omega.$$

After changing χ and χ' by a translation, we may assume that $\chi(p) = 0$ and $\chi'(q) = 0$. We claim that, up to a reindexing of the coordinates, χ is a chart of M adapted to M_0. Indeed, we have

$$\chi(U \cap M_0) = \{\chi(p') : p' \in U, F(p') = q\} = \{x \in \Omega : F_{\chi}^{\chi'}(x) = 0\},$$

or, using the form of $F_{\chi}^{\chi'}$,

$$\chi(U \cap M_0) = \{x = (x_1, \dots, x_m) \in \Omega : x_1 = \dots = x_n = 0\},$$

proving that M_0 is an $m-n$-dimensional embedded submanifold of M. 😀

On the other hand, one can wonder whether Theorem 2.129 has some sort of converse: is it true that any embedded submanifold $S \subset M$ can be described as the level set of such a submersion? For instance, one can use arguments from topology to show that S can be written as the zero-set of a function $f : M \to \mathbb{R}$. Even more, one can adapt those arguments to ensure a smooth f. However, such arguments fail to produce submersions (notice that, in general, there would be problems even with the dimensions). The point is that the question doesn't have a positive answer in general. However, the failure is very well understood and insightful: it is due to the way that S may sit inside M, possibly "very twisted", which would exclude the existence of such fs. The way to make everything precise is by appealing to normal bundles of submanifolds ... but let's take one step at a time. For now, let us however point out that the theorem does have a converse locally, as it follows from our discussion of (local) criteria for submanifolds of $\mathbb{R}^n$, Theorem 1.50.

Corollary 2.130. *A subset $M \subset N$ of a manifold M is an embedded m-dimensional submanifold if and only if for every $p \in M$ there exists $U \subset M$ open containing p and a submersion $\pi : U \to \mathbb{R}^{n-m}$ such that $M \cap U = \pi^{-1}(0)$.*

The regular value theorem has several variations. Here we discuss the one that is obtained by replacing the point $q \in N$ by a more general submanifold $N_0 \subset N$ so that, instead of $F^{-1}(q)$ we are now interested in

$$M_0 = F^{-1}(N_0) := \{x \in M : F(x) \in N_0\}.$$

The condition that q is a regular value is now replaced by the condition that F **is transverse to** N_0 in the sense that

$$(dF)_p(T_pM) + T_{F(p)}N_0 = T_{F(p)}N.$$

In general the left hand side is a subspace of the right hand side, and the equality means that any element $w \in T_{F(p)}N$ can be written as

$$w = (dF)_p(v) + w_0, \quad \text{with } v \in T_pM, w_0 \in T_{F(p)}N_0. \tag{2.4.2}$$

Corollary 2.131 (the transversality theorem). *If $F : M \to N$ is a smooth map that is transverse to a given embedded submanifold $N_0 \subset N$, then $M_0 := F^{-1}(N_0)$ is an embedded submanifold of M and*

$$T_pM_0 = \left\{v \in T_pM : (dF)_p(v) \in T_{F(p)}M_0\right\}.$$

Proof. The idea is rather simple. First notice that it suffices to prove it locally, i.e., that any $p_0 \in M_0$ admits a neighborhood U such that the conclusions holds for $F|_U$ (with the conclusion on $(F|_U)^{-1}(N_0) = U \cap M_0$ as a submanifold of U). Then, fixing $p_0 \in M$, we search for U such that $U \cap M_0 = \widetilde{F}^{-1}(0)$ for some map

$$\widetilde{F} : U \to \mathbb{R}^k \qquad \text{where } k = \dim N - \dim N_0,$$

for which 0 is a regular value. By Theorem 2.129 $M_0 \cap U$ will, indeed, be a submanifold; the claim about tangent spaces will follow along with the construction of U- see below. Now, to find U:

1. first apply the last corollary to $F(p_0) \in N_0 \subset N$ to obtain an open neighborhood V of $F(p_0)$ in N_0 and a submersion $\pi : V \to \mathbb{R}^k$ such that $N_0 \cap V = \pi^{-1}(0)$,
2. then set $U := F^{-1}(V)$, $\widetilde{F} := \pi \circ F|_U$.

The equation $U \cap M_0 = \widetilde{F}^{-1}(0)$ is now clearly satisfied hence we are left with proving that 0 is a regular value of $\widetilde{F}$. We fix $p \in \widetilde{F}^{-1}(0) = U \cap M_0$ and we prove that $(d\widetilde{F})_p : T_pM \to T_0\mathbb{R}^k$ is surjective. Let $u \in T_0\mathbb{R}^k$. Since π is a submersion at $F(p)$, by transversality we can write w as in (2.4.2); we then have

$$u = (d\pi)_{F(p)}(dF)_p(v) + (d\pi)_{F(p)}(w_0)$$

where the last term is 0 (again by the theorem applied to π). Using also the chain rule, we have found v such that $u = (d\widetilde{F})_p(u)$. This concludes the proof that U and $\widetilde{F}$ have the desired properties. Finally, notice that this implies not only that $U \cap M_0 = \widetilde{F}^{-1}(0)$ is a submanifold, but also that $T_p(U \cap M_0)$ is the kernel of $(d\widetilde{F})_p$. Using the chain rule for $\widetilde{F} = F \circ \pi$ and the fact that the kernel of $(d\pi)_{F(p)}$, the claim about the tangent spaces follows as well:

$$\begin{aligned} T_p(U \cap M_0) &= \{v \in T_pM : (d\widetilde{F})_p = 0\} \\ &= \{v \in T_pM : (d\pi)_{F(p)}(dF)_p(v) = 0\} \\ &= \{v \in T_pM : (dF)_p(v) \in T_pN_0\}. \end{aligned}$$ 😀

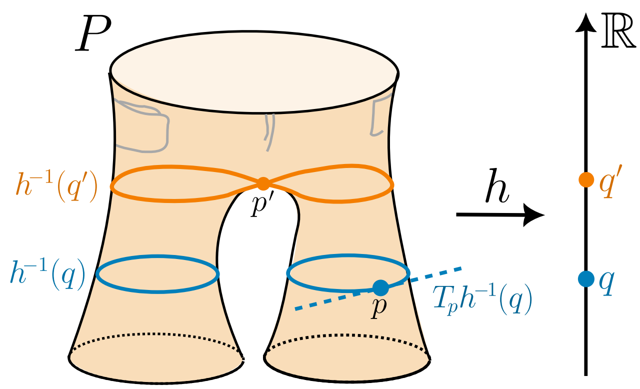


Fig. 2.18 The regular value theorem visualized for the case of the *pair of pants* manifold P, together with the map $h : P \to \mathbb{R}$ that assigns to each point its height over the ground. For the regular value q, the *level set* $h^{-1}(q)$ is an embedded submanifold of codimension one. The tangent space at $p \in h^{-1}(q)$ consists exactly of $\mathrm{Ker}\,(dh)_p$, i.e. those directions in P that get smashed together by h. After all, small movement in these directions does not change the value of h, so we remain in $h^{-1}(q)$. The set $h^{-1}(q')$ fails to be a manifold since h is not a submersion at p' (so q' is not a regular value).

Remark 2.132 (For the interested students: Sard's theorem). The regular value theorem tells us that, given a smooth map $F : M \to N$, the regular values of F are the points that interact nicest with F. The points in N that are not regular are called **singular values of** F. The good news is that the collection of singular values is "not so large when compared with N" (where the quotes can be made precise in several ways). This is Sard's theorem which, in its strongest form reads as follows:

Theorem 2.133 (Sard's theorem). *For any smooth map $F : M \to N$, the set of singular values of F has Lebesgue measure zero in N.*

If one wants a statement that does not appeal to measure theory, here is a consequence which us actually good enough for most applications of Sard's theorem.

Corollary 2.134. *For any smooth map $F : M \to N$, the set of regular values of F is dense in N.*

The proof of these results go beyond the scope of these lecture notes. However, this is a result that is worth being aware of. To appreciate it, let us have a look at the case that seems lest interesting: $\dim M < \dim N$. That doesn't look interesting since in that case F is not a submersion at any point. However, the set or regular values is still non-empty: it is actually the complement of the image $F(M)$. Hence the conclusion is that $M \setminus F(M)$ is dense in M. Notice that even saying that $M \setminus F(M)$ is nonempty is actually interesting as it boils down to:

$$\left.\begin{array}{r} F : M \to N \text{ smooth} \\ \dim M < \dim N \end{array}\right\} \Longrightarrow F \text{ cannot be surjective.} \tag{2.4.3}$$

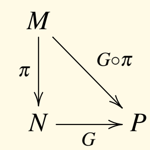
These may seem all very intuitive, but you may have seen (or look it up now) that, in topology, one can find "space-filling curves" (curves that fill us an entire square). The discussion here shows that smooth version of such curves.

2.4.2 Submersions and quotients

Another way to think about submersions is as the smooth analogue of the quotient maps from topology. To explain this, let us start with the property that is at the heart of the discussion.

Proposition 2.135. *If $\pi : M \to N$ is a surjective submersion then, for any function $G : N \to P$ with values in another manifold P one has:*

$$(G : N \to P \text{ is smooth}) \iff (G \circ \pi : M \to P \text{ is smooth})$$



In particular, for real-valued functions $g : N \to \mathbb{R}$, one has:

$$g \in C^\infty(N) \Longleftrightarrow g \circ \pi \in C^\infty(M).$$

Proof. The direct implication is clear, hence we concentrate on the reverse one. We will make use of the fact that smoothness of a map is a local property: for our F, it suffices to show that for any point $y \in N$, there exists a neighborhood V_y of y such that $F|_{V_y}$ is smooth. Fixing $y \in N$, we choose the coordinate charts around x and y that follow from the submersion theorem, and we set V_y to be the domain of the chart around y. Moving by those charts to opens inside Euclidean spaces, we may assume that π is the projection onto the first n coordinates-situation in which the fact that $F \circ \pi$-smooth implies that F is smooth is clear. The (pedantic) details are left as an exercise.

Remark 2.136 (relationship with topological quotient maps). Recall that a map $\pi : M \to N$ between topological spaces is called a topological quotient map if, for $V \subset N$ one has:

$$V - \text{open in } N \iff \pi^{-1}(V) - \text{open in } M. \tag{2.4.4}$$

In turn, this is equivalent to the fact that, for maps "out of N", continuity can be checked on M by composing to π. I.e., precisely the property described in the previous proposition, but with "smooth" replaced by "continuous".

Let us also specify the topological properties that submersions have. To that end we recall that a map $\pi : M \to N$ is called an **open map** if the following holds:

$$U - \text{open in } M \implies \pi(U) - \text{open in } N.$$

In general, topological quotient maps need not be open, hence this addition below is interesting.

Lemma 2.137. *Any submersion $\pi : M \to N$ is both a topological quotient map as well as an open map.*

Let us further exploit the analogy with the topological theory. One of the advantages of the characterization (2.4.4) is that, if we have a topological quotient map $\pi : M \to N$, then the topology on N is determined completely by the one of M and the map π (and it is called the **quotient topology** induced by π). In the smooth setting the situation is similar, modulo one problem: "the" smooth structure does not always exist- compare with the similar discussion of inducing topologies on subsets (always possible) versus inducing smooth structures (working only under the sub-manifold condition). The previous lemma implies that, for submersions $\pi : M \to N$, the topology on N is determined by the one on M and π. For the smooth structure we have:

Corollary 2.138. *If $\pi : M \to N$ is a surjective map from a manifold M to a set N, there is at most one way to make N into a manifold such that $\pi : M \to N$ is a submersion.*

Proof. One way to proceed is to make use of real-valued smooth functions and invoke Proposition 2.63.

Here is a a more direct argument: assume that N admits two such structures, and call the two resulting manifolds N_1 and N_2 (but the set is the same). Apply the proposition to the submersion $\pi : M \to N_1$ and $g = \mathrm{Id} : N_1 \to N_2$; since $g \circ \pi$ is π viewed as a map from M to N_2, hence smooth, $g = \mathrm{Id} : N_1 \to N_2$ has to be smooth. Hence the smooth charts of N_1 and N_2 are smoothly compatible and, therefore, $N_1 = N_2$ as manifolds. 😀

We now turn what we have just learned into a construction of new manifolds: quotients modulo group actions. By an action of a group $(\Gamma, \cdot)$ on a manifold M, also called a smooth action, we mean an action of Γ on the set M,

$$\Gamma \times M \to M, \ (\gamma, x) \to \gamma \cdot x$$

such that for each $\gamma \in \Gamma$, the map $\phi_\gamma : M \to M$, $\phi_\gamma(x) = \gamma \cdot x$ is smooth. Equivalently, a smooth action is a group homomorphism $\phi : \Gamma \to \mathrm{Diff}(M)$, $\gamma \mapsto \phi_\gamma$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M. For $x \in M$ one then talks about the Γ-orbit through x

$$\Gamma \cdot x :== \{\gamma \cdot x : \gamma \in \Gamma\},$$

Any two such orbits are either equal or disjoint:

$$(\Gamma \cdot x) \cap (\Gamma \cdot y) \neq \emptyset \quad \Longleftrightarrow \quad \Gamma \cdot x = \Gamma \cdot y \Longleftrightarrow \quad y = \gamma \cdot x \quad \text{for some} \quad \gamma \in \Gamma. \tag{2.4.5}$$

hence the orbits define a partition of M. The quotient M/Γ is the set of all such Γ-orbits and, at least tacitly, it is always considered together with the so-called quotient map π_{can} relating it to the original M:

$$M/\Gamma := \Big\{\Gamma \cdot x : x \in M\Big\}, \quad \pi_{\mathrm{can}} : M \to M/\Gamma, \ x \mapsto \Gamma \cdot x.$$

The role of π_{can} is to transfer structures from M to the quotient. For instance we always endow M/Γ with the quotient topology induced by π_{can}. Similarly, we are interested in smooth structures that make π_{can} into a submersion.

Definition 2.139. Given an action of a group Γ on a manifold M, we say that **the quotient M/Γ is smooth**, if can be made into a manifold in such a way that the quotient map $\pi_{\mathrm{can}} : M \to M/\Gamma$ is a submersion. The resulting smooth structure on M/Γ, unique by Corollary 2.138, is called the **quotient smooth structure**.

The previous lemma confirms that the quotient topology on M/Γ is the correct one to use. Another good sign in view of Lemma 2.137 is that π_{can} is automatically an open map (and this is rather specific to group actions!). Indeed, for any $U \subset M$ open, $\pi^{-1}(\pi(U)) = \cup_{\gamma \in \Gamma} \phi_\gamma(U)$ is open in M hence, indeed, $\pi(U)$ is open in M/Γ. And yet another result that you may remember from Topology is that for quotients modulo finite group actions, the Hausdorffness is preserved when passing to the quotients. The smoothness however requires more conditions.

Recall that an action is said to be **free**action!free if, for $\gamma \in \Gamma$ and $x \in M$, one has:

$$\gamma \cdot x = x \implies \gamma = e_\Gamma \ \ \text{(the identity element of } \Gamma).$$

Theorem 2.140 (the quotient manifold theorem). *For any free action of a finite group Γ on a manifold M, the quotient M/Γ is smooth.*

As we shall see, π_{can} will actually be a local diffeomorphism.

Proof. It is interesting to point out the following consequences to the condition that Γ is finite:

1. if the orbits through $x, y \in M$ do not coincide, then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. if the action is also free then, around any $x \in M$ there exists an open neighborhood W which is Γ-small in the sense that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma$ distinct from the unit element e.

The first item is proven/used in Topology, when proving the quotients modulo finite groups are Hausdorff; briefly, it goes as follow: the orbits must be disjoint, hence $x \neq \gamma \cdot y$ for any γ; using that M is Hausdorff one finds neighborhoods U_γ of x and V_γ of y such that $U_\gamma \cap \gamma \cdot V_\gamma = \emptyset$; finally, one sets $U = \cap_\gamma U_\gamma$, and similarly for V, which, since Γ is finite, is a finite intersection of opens hence open. This proves the existence of the desired U and V. Notice that, while the condition on x and y is precisely that they induce two distinct points in M/Γ, the resulting U and V give rise to two disjoint subsets $\pi(U)$, $\pi(V)$. Since when endowing M/Γ with the quotient topology, $\pi_{\mathrm{can}} : M \to M/\Gamma$ becomes an open map- i.e. sends opens to opens (why?), $\pi(U)$ and $\pi(V)$ will be disjoint opens and, therefore, M/Γ is Hausdorff.

For the second item we proceed similarly, but using that $x \neq \gamma \cdot x$ for all $\gamma \neq e$ (freeness). For each such γ we find neighborhoods W_γ and W'_γ of x such that $W_\gamma \cap \gamma \cdot W'_\gamma = \emptyset$ and consider

$$W = \left(\cap_{\gamma \neq e} W_\gamma\right) \cap \left(\cap_{\gamma \neq e} W'_\gamma\right).$$

It contains x, it is open (as a finite intersection of opens) and is Γ-small. Notice that being Γ-small is equivalent to the fact that $\pi|_W : W \to \pi(W)$ being a bijection and then, since π is continuous and open, to $\pi|_W$ being a local homeomorphism. Making use of this we can now exhibit the desired smooth structure on M/Γ: we consider charts (W, χ) on M that are Γ-small (in the sense that W is) and we combine χ with the inverse of $\pi|_W$ to build a homeomorphism

$$\chi_0 = \chi \circ (\pi|_W)^{-1} : \pi(W) \to \chi(W) \subset \mathbb{R}^m.$$

We claim that these fit into a smooth atlas for M/Γ. So, let us fix two Γ-small charts on M, and prove that the induced charts (W, χ) and (W', χ') on M/Γ are smoothly compatible. We have to investigate the smoothness of

$$\chi'_0 \circ \chi_0^{-1} = \chi' \circ (\pi|_{W'})^{-1} \circ (\pi|_W) \circ \chi,$$

on the domains where it is defined. Since χ and χ' are already smooth, we have to investigate the smoothness of

$$\tau_{W,W'} := (\pi|_{W'})^{-1} \circ (\pi|_W) : D \to D'$$

where $D \subset W$ and $D' \subset W'$ are the preimages of the overlap $\pi(W) \cap \pi(W')$ by $\pi|_W$ and $\pi|_{W'}$, respectively:

$$D = \{x \in W : \pi(x) \in \pi(W')\} \subset W, \quad D' = \{x \in W' : \pi(x) \in \pi(W)\} \subset W'.$$

We now check the smoothness of $\tau = \tau_{W,W'}$ around an arbitrary point $x_0 \in D$. Let $y_0 = \tau(x_0)$. Since x_0 and y_0 are mapped by π to the same point in M/Γ, we find γ_0 such that $y_0 = \gamma_0 \cdot x_0$. One then finds a neighborhood $D_0 \subset D$ such that $D'_0 := \gamma_0 \cdot D_0$ sits inside D' and the, on D_0, τ must be given by the multiplication by γ_0 (hence smooth!).

In conclusion, we have a smooth structure on M/Γ which, by construction, makes the projection into a local diffeomorphism. Moreover, M/Γ is smooth. We leave it to the reader to prove that M/Γ is also 2nd countable.

Remark 2.141. The proof above only used the properties mentioned in items 1. and 2. (well, 2. does imply that the action is free, but 1. and 2. may hold also when Γ is infinite):

1. if $(\Gamma \cdot x) \neq (\Gamma \cdot y)$ then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. around any $x \in M$ there exists an open neighborhood W such that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma \setminus \{e\}$.

An action with these properties is called a **free and proper**action!free and proper action. As the name indicates, it may be seen as a combination of two independent properties: being a free action (as discussed above) and being a **proper action**. The latter means that for any $K \subset M$ compact, the set of elements $\gamma \in \Gamma$ with the property that $\gamma \cdot K \cap K \neq \emptyset$ is finite. Together, they are equivalent to 1. and 2. above (exercise!).

Corollary 2.142. *The conclusion of the theorem holds more generally, for free and proper actions.*

Example 2.143 ($P^m = S^m/\mathbb{Z}_2$). We can now return to the projective space $\mathbb{P}^m$ and its description from Remark 2.75. In the equation (2.2.6) we immediately recognize an action the group $(\mathbb{Z}_2, +)$ on S^m, namely

$$\widehat{0} \cdot x = x, \quad \widehat{1} \cdot x = -x,$$

and the description of $\mathbb{P}^m$ as $S^m/\mathbb{Z}_2$. The theorem provides $\mathbb{P}^m$ with a smooth structure on $\mathbb{P}^n$ uniquely characterized by the condition that the map

$$H : S^m \to \mathbb{P}^m, \quad x \mapsto l_x$$

is a submersion.

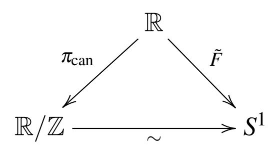
Exercise 2.144. Show that the smooth structure on $\mathbb{P}^m$ discussed in section 2.2.2 (induced by the charts (U_k, χ_k) there) coincides with the one obtained by applying Theorem 2.140 to the action of $\mathbb{Z}_2$ on S^m.

Example 2.145 ($\mathbb{P}^m = \mathbb{R}^{m+1} \setminus \{0\}/\mathbb{R}^*$). On the other hand, the original description of $\mathbb{P}^m$ as, see (2.2.3), reveals the action of $(\mathbb{R}^*, \cdot)$ on $\mathbb{R}^{m+1} \setminus \{0\}$ given by scalar multiplication and the idnetification of $\mathbb{P}^m$ with the resulting quotient $\mathbb{R}^{m+1} \setminus \{0\}/\mathbb{R}^*$. For the smooth structure, notice that the group is no longer finite, hence the theorem cannot be applied as stated. However, the action is proper, hence the corollary can be applied.

Example 2.146 ($S^1 = \mathbb{R}/\mathbb{Z}$). Similarly, for the action of $(\mathbb{Z}, +)$ on $\mathbb{R}$ given by

$$\mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \quad (r, n) \mapsto \phi_n(r) := r + n,$$

the group is not finite but the action is free and proper. The quotient can be identified with S^1, where the identification $\mathbb{R}/\mathbb{Z} \to S^1$ is induced by the map $\tilde{F} : \mathbb{R} \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$.



The fact that the map $\tilde{F}$ is a submersion implies that the smooth structure on S^1 that results from the theorem coincides with the standard one. Fill in the details!

Example 2.147 (the torus again). Consider the following action of the group $(\mathbb{Z}^2, +)$ on the plane $\mathbb{R}^2$

$$(n, m) \cdot (x, y) = (x + n, y + m).$$

The action is free and proper, hence one obtains a smooth structure on $\mathbb{R}^2/\mathbb{Z}^2$. Notice now that the corresponding equivalence relation $R_{\mathbb{Z}^2}$ on $\mathbb{R}^2$ has the following properties:

- each point in $\mathbb{R}^2$ is equivalent with at least one point in the square $[0, 1] \times [0, 1]$;
- the equivalence relation described in (2.4.5), when restricted to the unit square, is precisely the equivalence relation that describes the model for T^2 obtained by gluing the opposite sides of the torus: $(t, 0)$ becomes equivalent to $(t, 1)$ and $(0, s)$ to $(s, 1)$.

This shows that the resulting quotient $\mathbb{R}^2/\mathbb{Z}^2$ can be identified with the 2-torus. The identification can be made explicit for any of the models that we used for the torus:

$$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1 \cong T^2_{\hat{r}, r}, \quad (t, s) + \mathbb{Z}^2 \mapsto \left(e^{2i\pi t}, e^{2i\pi s}\right) \mapsto \Big((R + r \cdot \cos 2\pi t) sin 2\pi s,\ r \cdot \sin 2\pi t\Big).$$

Since all these formulas define submersions as maps defined on $\mathbb{R}^2$, it follows that these identifications preserve the smooth structure, i.e., they are diffeomorphisms.

Example 2.148 (the Klein bottle). Here is another way to describe the Klein bottle (see Example 2.27) as a smooth manifold, in a way that is quite close to the original "paper model". The idea is very simple and is similar to what we have seen for the torus: realise that the equivalence relation on the square underlying the construction of the model actually is induced by a group action. This time the group is $\Gamma = \mathbb{Z} \times \mathbb{Z}$ endowed with a new operation:

$$(n,m)\circ(n',m') = (n+n', m+(-1)^n m').$$

This acts on the plane $\mathbb{R}^2$ as follows:

$$(n,m)\circ(x,y) = (x+n, (-1)^n y+m).$$

It is useful to notice that Γ is generated by $a = (1,0)$ and $b = (0,1)$ (actually admits the presentation $\Gamma = \langle a,b; bab = a\rangle$): the general element (n,m) is simply $b^m a^n$. Furthermore, the action of the two generators on the plane is:

$$\phi_a(x,y) = (x+1,-y),\ \phi_b(x,y) = (x,y+1).$$

We see that any point in the plane can be moved, after acting by an element of Γ, to a point inside the unit square, while there are several "special points" inside the unit square that are still mapped to each other by certain group elements. Working this out one recognises right away the gluing involved in the paper model. Hence

$$K \cong \mathbb{R}^2/\Gamma$$

where an expression as in (2.1.8) corresponds to the class of (t,s) in R^2/Γ.

Exercise 2.149. Explain/show why the concrete model $K \subset \mathbb{R}^4$ is diffeomorphic to $\mathbb{R}^2/\Gamma$ with the resulting quotient smooth structure (b.t.w.: check also that the action is free and proper!).

Remark 2.150. The Klein bottle is to the torus what the projective space is to the sphere: a quotient which is obtained by gluing "antipodal points". To see this, we go back to the square used to obtain the Klein bottle. Since the gluing is so problematic, let us mirror the square as in the picture; to get the Klein bottle we would first have to fold the longer square back (a gluing process itself) and perform the original gluing. However, in the longer square, the gluing that of the opposite side (which was problematic at the beginning) disappears: we can perform it and we get precisely the torus! However, to get the Klein bottle we see we have to keep on going and finish gluing the rest. Inspecting the picture, it will eventually be clear that what we still have to do is to glue points in the torus which are "antipodal" (reflections of each other with respect to the origin).

This interpretation can be interpreted more concisely using, again, group actions: we have an action of the group $\mathbb{Z}_2$ on the 2-torus $T^2 = S^1 \times S^1$ given by $(z_1,z_2) \mapsto (\bar{z}_1, -z_2)$ (on the picture of $T^2_{R,r}$, that means we rotate on the meridian by 180 degrees (the $-z_2$) then we reflect w.r.t. the XOY plane (the $\bar{z}_1$ component). The conclusion will be that the Klein bottle can be identified with the resulting quotient $T/\mathbb{Z}_2$.

Remark 2.151 (For the interested students: general quotients). As you may remember from Topology, general quotients are best handled using the the language of equivalence relations. We will refer to an equivalence relation on M as the corresponding graph $R \subset M \times M$ (pairs of equivalent elements). Any group action of Γ on M gives rise to an equivalence relation consisting of pairs (x,y) satisfying the equivalent conditions from (2.4.5). For a general equivalence relation R on M one can still talk about R-orbits (or equivalence classes) $R(x) := \{y \in M : (x,y) \in R\}$, and then consider the quotient and the quotient map

$$M/R = \Big\{R(x) : x \in M\Big\}, \quad \pi_R : M \to M/R, \quad \pi_R(x) = R(x).$$

We say that **the quotient** M/R **is smooth**, or that R is a **regular equivalence relation**, if the quotient M/R can be made into a manifold M/R so that the quotient map

$$\pi_R : M \to M/R$$

is a submersion. The resulting smooth structure on M/R (unique by Corollary 2.138) is called the **quotient smooth structure**. What we also know from the previous discussion is that, for M/R to be smooth, π_R must be an open map- situation that is often referred to as: R is an **open equivalence relation**. Directly in terms of R that means for any open U in M, its R-saturation $R(U) := \bigcup_{x\in U} R(x)$ is open in M. As mentioned above, this is automatic in the case of group actions, but not in general. On the other hand, notice that, if $R \subset M \times M$ is an open equivalence relation, then:

$$M/R \text{ is Hausdorff} \iff R \subset M \times M \text{ is closed.}$$

Hence being closed is yet another necessary condition for the smoothness of R. Actually, we have the following:

Theorem 2.152 (Criteria for smoothness of quotients). *Given an equivalence relation R on a manifold M, M/R is smooth if and only if the following two conditions are satisfied:*

1. *$R \subset M \times M$ is a closed embedded submanifold;*
2. *The restriction of the first projection to R, $pr_1|_R : R \to M$, is a submersion.*

The direct implication can be taken as a "doable exercise". A simpler but instructive exercise is to show that item 2. implies that R is open. The converse implication in the theorem is more involved and will not be proven here.

2.4.3 Immersions

Definition 2.153. Let $F : M \to N$ be a smooth map between two manifolds. We say that:

- F is an **immersion at** p if its differential $(dF)_p : T_pM \to T_{F(p)}N$ is injective.
- F is an **immersion** if the previous condition holds at all $p \in M$.

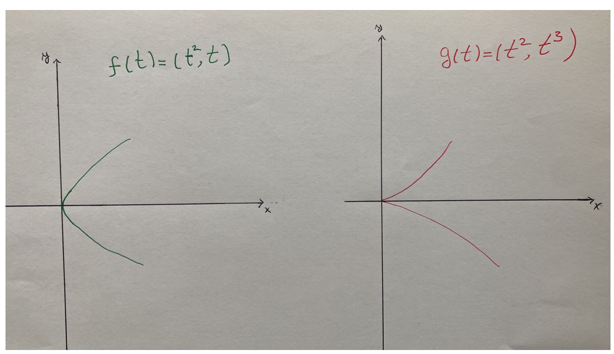


Fig. 2.19 Immersion on the left, non-immersion on the right

Using the definition and the chain rule, you should now work out the following

Exercise 2.154. Show that the following are equivalent:

(1) F is an immersion at p.
(2) its representation $F_{\chi}^{\chi'}$ with respect to any chart χ of M around p and χ' of N around $F(p)$ is an immersion in the sense of Analysis (Definition 1.47).
(3) there exists a chart χ of M around p and one chart χ' of M' around $F(p)$ such that $F_{\chi}^{\chi'}$ is an immersion.

The immersion theorem on Euclidean spaces, i.e. Theorem 1.46, can now be restated as follows:

Theorem 2.155 (the immersion theorem). *A smooth map $F: M \to N$ is an immersion at $p \in M$ of and only if there exist charts χ of M around p and χ' of N around $F(p)$ such that $F_{\chi}^{\chi'} = \chi' \circ F \circ \chi^{-1}$ is given by*

$$F_{\chi}^{\chi'}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Moreover, the charts $\chi: U \to \Omega \subset \mathbb{R}^m$ and $\chi': U' \to \Omega' \subset \mathbb{R}^n$ can be chosen so that

$$F(U) = \{q \in U' : \chi'_{m+1}(q) = \ldots = \chi'_n(q) = 0\}.$$

Example 2.156 (embeddings). The definition of embedded submanifolds $M \subset N$ shows right away that the resulting inclusions $M \subset N$ are immersions. Using Definition 2.59, this remark can be reformulated as saying that:

$$F \text{ embedding} \Longrightarrow F \text{ immersion}.$$

Notice that this is the analogue, for immersions, Example 2.126 that was describing projections as basic examples of submersions. To extend the analogy, let us also point out that immersion theorem tells us that immersions are the smooth maps which locally (in the appropriate coordinates) look like embeddings.

Example 2.157 (local diffeomorphisms). Also Example 2.127 has an immediate analogue for immersions. First of all $\exp: \mathbb{R} \to S^1$ provides a simple example of immersion that is not an embedding. Secondly, in the border case $\dim M = \dim N$ immersions are, again, the same as local diffeomorphism. And thirdly, combined with Lemma 2.58, one obtains:

$$\left.\begin{array}{c} F: M \to N \text{ bijective immersion} \\ \dim M = \dim N \end{array}\right\} \Longrightarrow F \text{ is a diffeomorphism.}$$

Exercise 2.158 (composition). Given two smooth maps $F: M \to N$, $G: N \to P$ and $p \in M$, show that:

1. if F and G are immersions at p and $F(p)$, respectively, then $G \circ F$ is an immersion at p.
2. if $G \circ F$ is an immersion at p, then F is an immersion at p

Example 2.159 (other non-embeddings). The example of $\exp: \mathbb{R} \to S^1$ shows that immersions may fail to be embeddings for a rather "stupid reason": they may fail to be even injective. However, the difference between the two concepts is deeper/more subtle, as we shall see in the next section. For now, let us point out that, given an immersion $F: M \to N$, even if it is injective hence a bijection on its image, that bijection need not be a homeomorphism and the image need not be an (embedded) submanifold. An example is shown in Figure 2.20. The next theorem (as well as Theorem 2.167) clarifies what is going on and provides an useful criteria for checking that a map is an embedding.

Theorem 2.160. *A map $F: M \to N$ between two manifolds is a smooth embedding if and only if it is both an immersion as well as a topological embedding.*

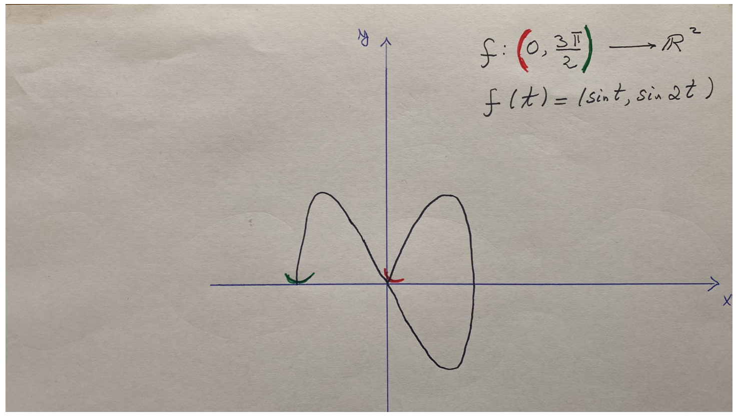


Fig. 2.20

Proof. The direct implication should be clear, so we concentrate on the converse. We first show that $F(M)$ is an embedded submanifold. We check the required condition at an arbitrary point $q = F(p) \in N$, with $p \in M$. For that use the immersion theorem (Theorem 2.155) around p; we consider the resulting charts $\chi : U \to \Omega \subset \mathbb{R}^m$ for M around p and $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$ for N around q; in particular,

$$F(U) = \{q \in \widetilde{U} : \widetilde{\chi}_{m+1}(q) = \ldots = \widetilde{\chi}_n(q) = 0\}.$$

Since F is a topological embedding, $F(U)$ will be open in $F(M)$, i.e. of type

$$F(U) = F(M) \cap W$$

for some open neighborhood W of $q \in N$. It should be clear now that $\widehat{U} := \widetilde{U} \cap W$ and $\widehat{\chi} := \widetilde{\chi}|_{\widehat{U}}$ defines a chart for N adapted to M.

We still have to show that, as a map $F : M \to F(M)$, F is a diffeomorphism (where $F(M)$ is with the smooth structure induced from N). It should be clear that this map continues to be an immersion. Since M and $F(M)$ have the same dimension we can now use Remark 2.157 to conclude that $F : M \to F(M)$ is a diffeomorphism.

And here is a very useful consequence:

Corollary 2.161. *Let $F : M \to N$ be a smooth map between two manifolds M and N, with the domain M being compact. Then F is a smooth embedding if and only if it is an injective immersion.*

Proof. Use one of the main properties of compact spaces: injective maps from compacts to Hausdorff spaces are automatically topological embeddings!

Remark 2.162 (For the interested student: embeddings in Euclidean spaces). With the previous corollary at hand, one can now adapt the proof of Theorem 1.15 from Section 1.1 to the smooth context:

Theorem 2.163. *Any (smooth) compact manifold can be smoothly embedded into some Euclidean space.*

Proof. We use the same map as in the proof of Theorem 1.15 from Section 1.1 of Chapter 1, just that we now make use of smooth partitions of unity (which exist by Theorem 2.66). It suffices to show that the resulting topological embedding

$$i = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

is also an immersion. We check this at an arbitrary point $p \in M$. Since $\sum_i \eta_i = 1$, we find an i such that $\eta_i(p) \neq 0$. We may assume that $i = 1$. In particular, p must be in U_1, and then we can choose the chart χ_1 around p and look at the representation i_{χ_1} with respect to this chart; we will check that it is an immersion at $x := \chi_1(p)$. Hence assume that the differential of i_{χ_1} at x kills some vector $v \in \mathbb{R}^d$ (and we want to prove that $v = 0$). Then the differential of all the components of i_{χ_1} (taken at x) must kill v. But looking at those components, we remark that:

- the first $\mathbb{R}$-component is $\eta := \eta_1 \circ \chi_1^{-1} : U_1 \to \mathbb{R}$.
- the first $\mathbb{R}^d$ component is the linear map $F : U_1 \to \mathbb{R}^d, F(u) = \eta(u) \cdot u$.

Hence we must have in particular:

$$(d\eta)_x(v) = 0, \quad (df)_x(v) = 0.$$

From the formula of F we see that $(df)_x(v) = (d\eta)_x(v) \cdot x + \eta(x) \cdot v$; hence, using that previous equations we find that $\eta(x) \cdot v = 0$. But $\eta(x) = \eta_1(p)$ was assumed to be non-zero, hence $v = 0$ as desired.

Finally, let us return to the remark we made on the nonexistence of smooth space filling curves or, more precisely, to the consequence of Sard's theorem described in (2.4.3). We would like to point out that, in the case of immersions, that statement can be obtained by a more direct, topological, argument.

Lemma 2.164. *If $F : M \to N$ is an immersion and $\dim M < \dim N$ then F cannot be surjective.*

Proof. We will actually show that the image of F must have empty interior in M. To that end we will use the fact that, since N is locally compact, it is a Baire space (see Theorem 1.8 from the reminder on Topology). Stated in terms of closed subsets, that means: if a subset $A \subset N$ is a countable union of closed subsets of N that have empty interior, then A must have empty interior Therefore, it suffices to show that one can find a countable cover $\mathscr{C}$ by compact subspaces of M such that each $F(C)$ with $C \in \mathscr{C}$ has empty interior in N. To achieve this we choose, for any point $p \in M$, charts for M and N as in the immersion theorem around p, and we transfer (via the chart on M) a small closed ball in $\mathbb{R}^m$ to a compact subspace $C_p \subset M$ containing p such that $F(C_p)$ has empty interior. That is possible because $m > n$ (and $\mathbb{R}^m \times \{0\}$ has empty interior in $\mathbb{R}^n$). Notice that the interiors of the C_p's form an open cover of M and, since M is 2nd countable, that open cover contains a countable subcover, corresponding to some sequence of points $p_1, p_2, \ldots$. Then $\{C_{p_1}, C_{p_2}, \ldots\}$ will have the desired properties.

2.4.4 Immersions and submanifolds

We now move to the relevance of immersions to (embedded) submanifolds. Let us start by reformulating the Theorem 2.160 from the last section.

Corollary 2.165. *If M is an embedded submanifold of the manifold N, then the set M endowed with the induced topology admits precisely one smooth structure such that the inclusion into M becomes an immersion.*

Proof. This is the theorem applied to the inclusion $i : (M, \mathscr{A}) \to N$, where $\mathscr{A}$ is any smooth structure on M as in the statement, possibly different than the induced smooth structure $\mathscr{A}_{\text{ind}}$. The hypothesis that the underlying topology of $\mathscr{A}$ is the induced one is equivalent to saying that $i : (M, \mathscr{A}) \to N$ is an embedding. The theorem implies that the inclusion $i : (M, \mathscr{A}) \to N$ must be a smooth embedding, i.e., the identity of M is a diffeomorphism between $(M, \mathscr{A})$ and $(M, \mathscr{A}_{\text{ind}})$. This implies $\mathscr{A} = \mathscr{A}_{\text{ind}}$.

The previous reformulation, and the further relevance of immersions to submanifolds, can be best understood as part of the following perspective. Assume that we have a manifold N, we start with a subset $M \subset N$ and we are looking for manifold structures on M which are related, in some sense, with the manifold structure on N. Immersions allow us to make sense of "in some sense" by imposing the condition that the inclusion $i : M \hookrightarrow N$ is smooth. Next, there are several possibilities that one may expect/hope for:

(1) try to make the space M (endowed with the topology induced from N) into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion. Is that possible to find more than one such smooth structure?

(2) forget about the induced topology on M and try to make the set M into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion. Can that be realised in more than one way. Is there a "best" possible choice? (yes, M will have a topology underlying the smooth structure, but it does not have to be the induced one).

Example 2.166. For instance, in Figure 2.20, the function f to transfer the topology and the smooth structure from $(0, \frac{3\pi}{2})$ to a topology and a smooth structure on the curve from the picture, the inclusion is not an embedding but it is still an immersion. Completing that curve to an entire "figure eight", there are two ways to put such a smooth structure (do you see the other one?).

Theorem 2.160 and its corollary provide us the answer to (1): it works for embedded submanifolds, and only for them; furthermore, the smooth structure we were looking for in (1) is unique. However, uniqueness holds in the stronger sense of (2):

Theorem 2.167. *If M is an embedded submanifold of the manifold N, then the set M can me made into a manifold in one and only one way such that the inclusion into M becomes an immersion.*

Proof. As before, let $\mathscr{A}_{\text{ind}}$ be the induced smooth structure on M and let $\mathscr{A}$ be (a possibly different) one such that $i : (M, \mathscr{A}) \to N$ is an immersion. We now use that embedded submanifolds endowed with $\mathscr{A}_{\text{ind}}$ satisfy property (2.1.11). Applied to

$$\begin{array}{ccc} & & N \\ & \nearrow & \uparrow \\ (M, \mathscr{A}) & \xrightarrow[\text{id}]{} & (M, \mathscr{A}_{\text{ind}}) \end{array}$$

with deduce that the identity map of M is smooth as a map from $(M, \mathscr{A})$ to $(M, \mathscr{A}_{\text{ind}})$. (That means that $\chi' \circ \chi^{-1}$ are smooth for all $\chi \in \mathscr{A}$, $\chi' \in \mathscr{A}_{\text{ind}}$ but, to be able to conclude that $\mathscr{A} = \mathscr{A}_{\text{ind}}$, we need to know that they are diffeomorphisms. Of course, this is the same as saying that the identity map from the diagram is a diffeomorphism). It is time to use now the fact that $i : (M, \mathscr{A}) \to N$ is an immersion; this, together with the behaviour of immersions w.r.t. compositions (Exercise 2.158) applied to the previous diagram, implies that the identity map from the diagram is an immersion. Given Remark 2.157, it suffices now to prove that $\dim M = \dim N$. But $\dim M < \dim N$ is excluded by Lemma 2.164.

Remark 2.168. It is now interesting to look back at the previous proof and ask ourselves: what did we really use about embeddings? what did we really prove? is there more to learn? Here are two observations regarding the proof of the theorem:

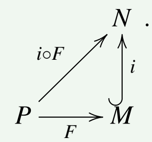
(a) First of all, the only thing we used about the inclusion $i : M \hookrightarrow N$ of embedded submanifolds is (2.1.11).
(b) ... actually we only used the reverse implication in (2.1.11) (well, the other one holds automatically anyway)
(c) On the other hand, even for the reverse implication in (2.1.11), if we add the assumption that F is continuous then the conclusion still follows.

All together, this discussion reveals the following concepts that help with understanding item (2) above.

Definition 2.169. Given a manifold N,

1. An **immersed submanifold** of N is a subset $M \subset N$ together with a structure of smooth manifold on the set M such that the inclusion $i : M \to N$ is an immersion.
2. An **initial submanifold** of N is an immersed submanifold M that satisfies property (2.1.11): for any other manifold P and any map $F : P \to M$,

$$F \text{ is smooth} \iff i \circ F \text{ is smoooth}$$



3. We say that a subset $M \subset N$ has **the unique smooth structure property** if it can be made into an immersed submanifold of M in one, and only one, way.

In item 3. above, to make M into an immersed submanifold means choosing a topology and a smooth structure on M such that the inclusion is an immersion.

Remark 2.170 (comparing immersed with embedded). To see the difference between "immersed" and "embedded" one just has to stare at the meaning of "immersions". If M (... together with a smooth structure on it) is an immersed submanifold of N, the immersion theorem tells us that there exist charts (U, χ) of N around arbitrary points $p \in M$ such that $U_M := M \cap \{q \in U : \chi_{n+1}(q) = \ldots = \chi_m(q) = 0\}$ is open in M and $\chi_M = \chi|_{U_M}$ is a chart for M. However, when saying that "U_M is open in M", we are making use of the topology on M induced by the smooth structure we considered on M, and not with respect to the topology induced from N (look again at the previous exercise!). This is the difference with the notion of "embedded submanifold".

The usefulness of the initial submanifold condition is that it is easier to check than the uniques of smooth structures property. Putting everything together, we have:

Corollary 2.171. *Given a subset M of a manifold N,*

a) Given a topology $\mathscr{T}_M$ on M, $\exists$ at most one smooth structure on $(M, \mathscr{T}_M)$ making it into an immersed submanifold.
b) If M can be made into an initial submanifold, then M has the unique smooth structure property.

Proof. There isn't much new to add here: with Remark 2.168 at hand, the proof of Theorem 2.167 now goes through. For instance, for (a), the only problem is when using the diagram from the proof to deduce that the identity map there is smooth. But, by the assumption in a), that map is already continuous, hence we just use the last part of Remark 2.168. 😀

Remark 2.172 (immersed submanifolds as injective immersions). Since general immersed submanifolds $M \subset N$ carry two topologies (the one underlying the immersed smooth structure, and the one induced from the ambient space), things can get a bit confusing. With that in mind it may be useful to separate M from N and look at general injective immersions

$$F : M \to N$$

as encoding an immersed submanifold of M- namely $F(M)$ together with the smooth structure obtained by transporting the smooth structure from M via the bijection $F : M \to F(M)$, i.e., the charts of $F(M)$ are those of type $\chi \circ F^{-1}$ with χ a chart of M. All together, the message is that one can think of immersed submanifolds as images of injective immersions. This is supported not only by the fact that all immersed submanifold are of this type (just use the inclusion), but also by the fact that they almost always arise in this way (e.g. the immersed submanifolds showed in the previous two pictures, as well as the next ones, they all come with a parametrising F).

Example 2.173. Let us finish the discussion started in Example 2.166 by showing the the “figure eight” admits more than one manifold structure that make it into an immersed submanifold of the plane. With the previous remark at hand, we can now use the same formula $f(t) = (\sin t, \sin 2t)$ to produce two different functions:

$$f_0 = f|_{(0,2\pi)} : (0,2\pi) \to \mathbb{R}^2, \quad f_1 = f|_{(\pi,3\pi)} : (\pi,3\pi) \to \mathbb{R}^2.$$

They both have as image the figure eight and they provide us with two different ways of making M into an immersed submanifold of the plane. See the left hand side of Figure 2.21. We see that the figure eight does not posses the unique smooth structure property and, in particular, neither of the two manifold structure satisfies the initial manifold condition. On the other hand, this discussion is destroyed if we perturb a bit the figure eight to “the Dutch figure eight”, as in Figure 2.21, right hand side. You should prove that, indeed, that has the unique smooth structure property.

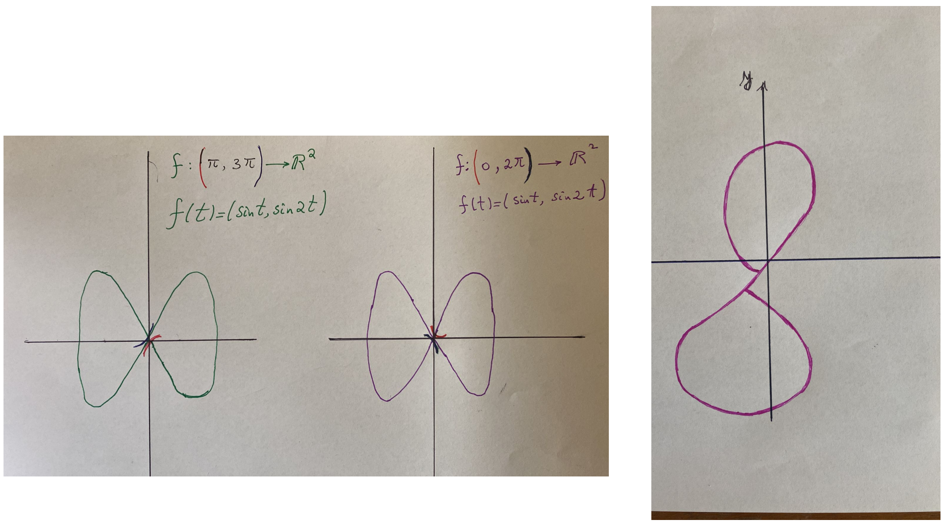


Fig. 2.21

Example 2.174. Even more interesting examples are obtained by considering curves on the torus that wind around, as shown in Figure 2.23.

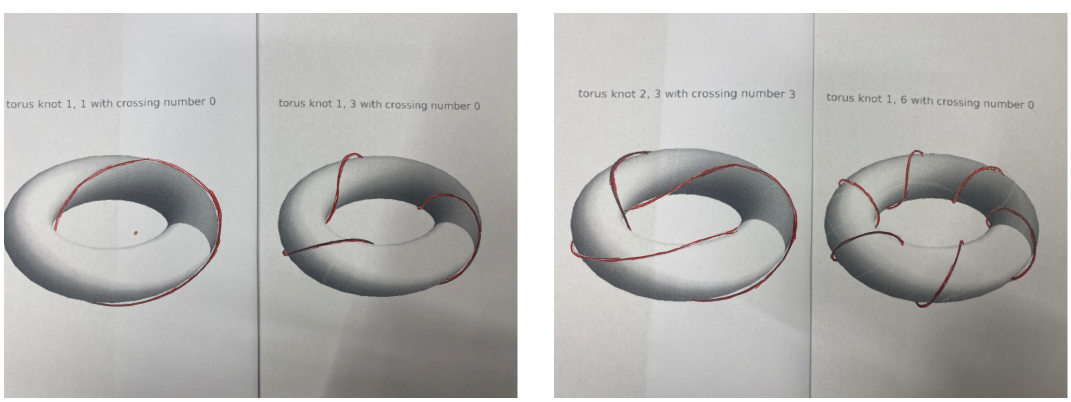


Fig. 2.22

The nicest way to describe them is by using the description of the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ (cf. Example 2.147). The lines we are talking about are obtained from lines through the origin in $\mathbb{R}^2$ by applying the projection map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. Figure 2.22 shows the circle corresponding to $y = x$, $y = 3x$, $y = \frac{3}{2}x$ and $y = 6x$, respectively. All those lines or, more generally, all the lines corresponding to $y = qx$ with q rational, close up to circles that are embedded submanifolds of the torus. However, as soon as we perturb one a bit and make q into an irrational number, the resulting curves on the torus are copies of the real lines the wind around the torus densely- see Figure 2.23.Those are no longer embedded submanifolds, but they are immersed submanifolds that satisfy the initial submanifold condition. The immersed structure is induced by the functions $\mathbb{R} \to \mathbb{R}/\mathbb{Z}^2$ that send t to the class of (t, qt). Or, using the model $T^2 = S^1 \times S^1$, we are using the injective immersions

$$F : \mathbb{R} \to S^1 \times S^1, \quad F_q(t) = (e^{2\pi i t}, e^{2\pi i q t}).$$

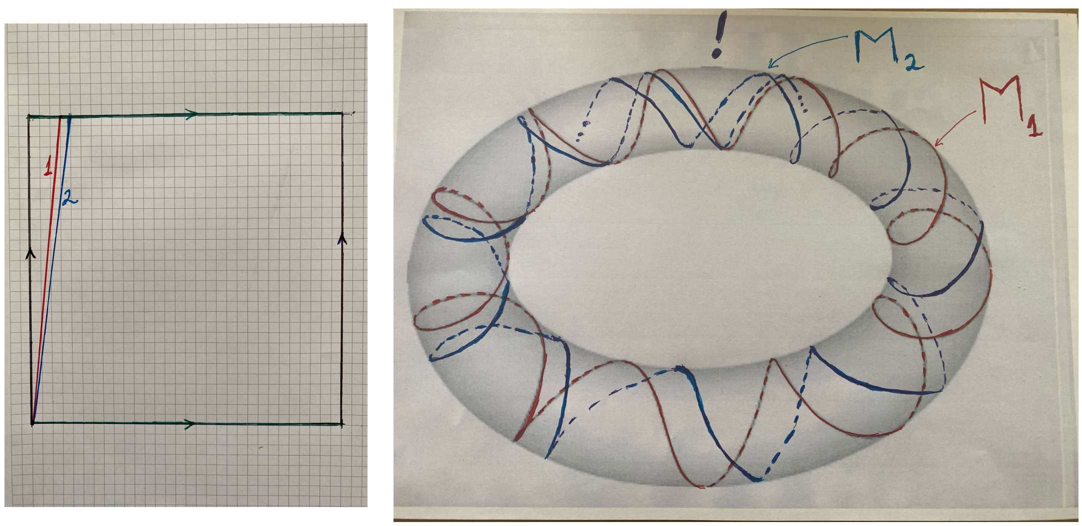


Fig. 2.23

Exercise 2.175. Work out in detail all the claims made in the last example.

Example 2.176. Finally, here is an example of a subset of the plane that has the unique smooth structure property, but is not an initial submanifold: take M to be the horizontal OX-axis together with the positive part of the OY-axis, sitting inside the plane $N = \mathbb{R}^2$. The immersed smooth structure on M is obtained using the injective immersion

$$f : \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\} \to M, \quad f(x,0) = (x,0),\ f(y,1) = (0,y).$$

Figure 2.24. To see that this is not an initial submanifold, we consider the following type of smooth function in the plane:

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} (0, e^{-\frac{1}{x^2}}) & \text{if } x < 0 \\ (0,0) & \text{if } x = 0 \\ (e^{-\frac{1}{x^2}}, 0) & \text{if } x > 0 \end{cases}$$

Since this takes value in S, if S was initial, then f would be a smooth as a function into S with the smooth structure obtained from f. In other words, the following would have to be a smooth function:

$$f^{-1} \circ g : \mathbb{R} \to \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}, \quad t \mapsto \begin{cases} (e^{-\frac{1}{t^2}}, 1) & \text{if } x < 0 \\ (0,0) & \text{if } x = 0 \\ (e^{-\frac{1}{t^2}}, 0) & \text{if } x > 0 \end{cases}$$

but that function is not even continuous (e.g. the pre-image of the open $\mathbb{R} \times \{0\}$ is $[0,\infty)$). We leave as an exercise to show that the smooth structure on M described above is the only one that makes M into an immersed submanifold.

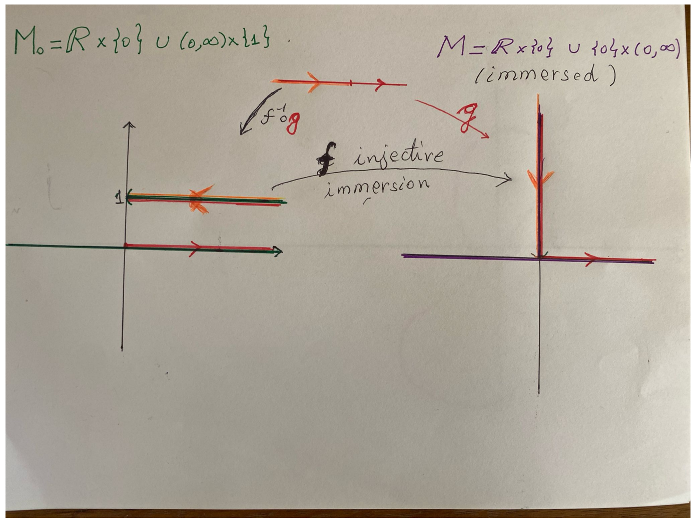


Fig. 2.24

2.5 More exercises

Smooth structures

Exercise 2.177. With the notations from Example 2.4 and Example 2.5, prove that

$$(\mathscr{A}_{\mathbb{R}^2})^{\max} = \left(\mathscr{A}_{\mathbb{R}^2}^{\text{lin}}\right)^{\max} = \left(\mathscr{A}_{\mathbb{R}^2}^{\text{polar}}\right)^{\max} = \mathscr{A}_{\mathbb{R}^2}^{\max}.$$

Exercise 2.178. Prove that, for any smooth atlas $\mathscr{A}$ on a space M one has

$$\mathscr{C}^\infty(M,\mathscr{A}) = \mathscr{C}^\infty(M,\mathscr{A}^{\max}),$$

Exercise 2.179. Give an example of a "non-Hausdorff manifold", i.e., a topological space that is not Hausdorff, together with a smooth structure. Hint: glue two lines to each other, point with point, except for one. E.g., work with $M = (-\infty,0)\times\{0\}\cup\{(0,-1),(0,1)\}\cup(\infty,0)\times\{0\}$, but be careful with the topology you consider on M.

Exercise 2.180. Given a smooth structure on a space M, the smooth structure can be defined by an atlas that is at most countable if and only if M is 2nd countable.

Exercise 2.181. Assume that N is a smooth manifold, M is a set and $\phi : M \to N$ is a bijection. We "transport" the manifold structure of N to M in the sense that, on M, we consider

(i) the topology $\mathscr{T}_M$ consisting of those subsets of M of type $\phi^{-1}(U)$ with U open in N.
(ii) the smooth structure with the property that, for any smooth chart (U,ϕ) of N, $\left(\phi^{-1}(U),\chi\circ\phi\right)$ is a smooth chart for M.

Show that

(a) (i) and (ii) do indeed make M into a manifold.
(b) $\phi : M \to N$ is a diffeomorphism.
(c) When $\phi = \chi$ a homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$, this recovers the discussion from Example 2.54.

Exercise 2.182. Assume that $F : M \to N$ is a local homeomorphism between topological spaces. We assume that N is also a manifold and M is Hausdorff and 2nd countable. Show that M admits one and only one smooth structure that makes it into a manifold with the property that F becomes a local diffeomorphism. Draw a conclusion about universal covers of manifolds.

Exercise 2.183. Consider the following subset of $\mathbb{R}^4$:

$$M = \{(x,y,z,t)\in\mathbb{R}^4 : xt = yz + 12\}.$$

Using the regular value theorem you can prove that this is an embedded submanifold. But that is another exercise. Here, you may assume you already know that that is the case, and we see M as a smooth manifold with the induced smooth structure.

1. describe a smooth chart for the manifold M of type (U,χ) with $\chi(x,y,z,t) = (x,y,z)$.
2. show that M admits a smooth atlas consisting of only two charts.

The regular value theorem, tangent spaces

Exercise 2.184. In case you did not do it already: use the regular value theorem to prove that the model $T^2_{R,r}$ of the torus (cf. Example 2.22) is an embedded submanifold of $\mathbb{R}^3$.

Exercise 2.185. Show that

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$$

is a 2-dimensional embedded submanifold of $\mathbb{R}^3$.

Exercise 2.186. For the smooth manifold M from the previous exercise (with the induced smooth structure), describe an open $U \subset M$ with the property that (U, pr_{12}) is a smooth chart for M, where $\mathrm{pr}_{12}(x,y,z) = (x,y)$.

Exercise 2.187. (from the 2017-2018 retake exam) Show that $M := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 4\}$ is an embedded submanifold of $\mathbb{R}^3$ and compute the tangent space $T_pM \subset \mathbb{R}^3$ at the point $p = (1,1,-1)$.

Exercise 2.188. Show that the function

$$F : S^2 \to S^2, \quad F(x,y,z) = (x^2 + y^2 - z^2, 2yz, 2xz)$$

is neither a submersion nor an immersion.

Exercise 2.189. Consider the height function $F : T \to \mathbb{R}$ on the torus as indicated in Fig 2.25. At which points does F fail to be a submersion?

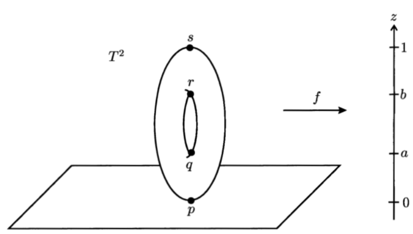


Fig. 2.25

Exercise 2.190. Show that

$$F : \mathbb{R} \to \mathbb{R}^2, \quad F(t) = (\cos(t), \sin(t))$$

is an immersion at each point. Then, looking around $t = 0$, find a chart χ' of $\mathbb{R}^2$ around $F(0) = (1,0)$ such that, w.r.t. this chart,

$$F_{\chi'}(t) = (t, 0).$$

Exercise 2.191. (part of the 2019/2020 exam) Consider

$$F : S^3 \to \mathbb{R}, \quad F(x,y,z,t) = x^2 + y^2 - z^2 - t^2.$$

(a) show that the zero-set $M_0 := F^{-1}(0)$ is an embedded submanifold of S^3.
(b) show that M_0 is diffeomorphic to the 2-torus.
(c) find all the points at which F is a submersion.

Exercise 2.192. We consider the space of two by two matrices with real coefficients, identified to the Euclidean space $\mathbb{R}^4$ with coordinate functions denoted by x, y, z, t:

$$N := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{R} \right\}$$

and, inside it, the space of matrices of determinant 1:

$$M := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : xt - yz = 1 \right\}.$$

(a) Show that M is an embedded submanifold of N.
(b) Consider the following re-parametrization of M:

$$\begin{cases} x = r \cdot \cos\alpha - \frac{u}{r}\sin\alpha & y = r \cdot \sin\alpha + \frac{u}{r}\cos\alpha \\ z = -\frac{1}{r}\sin\alpha & t = \frac{1}{r}\cos\alpha \end{cases} \qquad u \in \mathbb{R},\ r \in (0, \infty),\ \alpha \in (0, 2\pi)$$

which we interpret as the inverse χ^{-1} of a chart of M. Hence the u, r, α determined by the equations above (all functions of (x, y, z)) are precisely the components $\chi_1(x, y, z)$, $\chi_2(x, y, z)$ and $\chi_3(x, y, z)$ of $\chi(x, y, z)$. What is the domain U of χ?

Exercise 2.193. For $M = S^2 \subset \mathbb{R}^3$, $p = (1, 0, 0) \in S^2$,

a) Both vectors $\left(\frac{\partial}{\partial y}\right)_p, \left(\frac{\partial}{\partial z}\right)_p \in T_p\mathbb{R}^3$ belong to T_pS^2.
b) The two tangent vectors will be send by the differential $(dF)_p$ of the map

$$F : S^2 \to \mathbb{R}^2, \quad F(x, y, z) = (xy, xz)$$

to two tangent vectors to $\mathbb{R}^2$ (at what point?). Compute them!
c) Show that F is a submersion at $p = (1, 0, 0)$.
d) Find all the points in S^2 at which F is a submersion.

Exercise 2.194. Define

$$M = \{(x, y, z, t) \in \mathbb{R}^4 : xt = yz + 12\}.$$

(a) Show that M is an embedded submanifold of $\mathbb{R}^4$.
(b) Compute the tangent space of M at the point $p_0 = (4, 0, 0, 3)$.
(e) Consider the spheres centred at the origin

$$S^3_r = \{(x, y, z, t) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + t^2 = r^2\}$$

(one for each $r > 0$), and show that their intersections with M

$$M_r := M \cap S^3_r,$$

are embedded submanifolds of $\mathbb{R}^4$ (be aware that, depending on how you approach this, you may end up doing some very time consuming computations ...).
(f) When $r = 5$ (so that $p_0 \in M_r$) compute the tangent space of M_5 at p_0.

(g) Show that the following defines a submersion from M_5 to S^1:

$$F : M_5 \to S^1, \quad F(x,y,z,t) = (x-t, y+z).$$

(h) Show that M_5 is diffeomorphic to a torus.

Exercise 2.195 (from the 2018-2019 exam). Consider

$$M_4 := \{(x,y,z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3,$$

and the map from M_4 to the 2-sphere S^2 given by

$$F : M_4 \to S^2, \quad F(x,y,z) = (x^2, y^2, z^2)$$

a) Show that M_4 is a submanifold of $\mathbb{R}^3$ and F is a smooth map.
b) Compute the tangent space of M_4 at the point $p = (\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM_4.$$

Similarly at the point $q = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$.
c) Show that F is not an immersion at p, but it is a local diffeomorphism around q.
d) Show that M_4 is not diffeomorphic to

$$M_3 := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3.$$

but it is diffeomorphic to S^2.

Exercise 2.196 (from the 2017-2018 exam). Consider

$$M := \{(x,y,z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 - 5)^2 + 16z^2 = 16\} \subset \mathbb{R}^3.$$

a) Show that M is a submanifold of $\mathbb{R}^3$.
b) Compute the tangent space of M at the point $p = (3,0,0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM.$$

Similarly, compute the tangent space at the point $q = (2,0,1)$.
c) Draw a picture of M (hint: if you do not see how M looks like, maybe compute T_pM at one more point).
d) Prove that M is diffeomorphic to $S^1 \times S^1$.

Embeddings, immersions, submersions

Recall that, while $S^1 \times S^1$ naturally embeds in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, the interpretation of $S^1 \times S^1$ as a torus provides an even better embedding: inside $\mathbb{R}^3$. In the following you are asked to show that this can be propagated to higher dimensional tori. Here we use products of manifolds as discussed in Exercise 2.21

Exercise 2.197. Show that the n-dimensional torus,

$$T^n := \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$$

(sitting canonically inside $\mathbb{R}^{2n}$) can actually be embedded inside $\mathbb{R}^{n+1}$.

Exercise 2.198. Consider S^1 endowed with the standard smooth structure. Prove that there are no submersions or immersions $F : S^1 \to \mathbb{R}$.

Exercise 2.199. Assume that $F : M \to N$, $G : N \to P$ are smooth maps between manifolds. Show that:

a) If F and G are submersions then so is $G \circ F$. Similarly for immersions and for local diffeomorphisms.
(b) Show that if $G \circ F$ is a submersion then so is G. Furthermore, give an example which shows that, in general, F need not be a submersion.
(c) Show that if $G \circ F$ is an immersion then so is F. Furthermore, give an example which shows that, in general, G need not be an immersion.

Exercise 2.200. Consider a smooth function $F : M \to N$, $p \in M$. Show that the following are equivalent:

(1) F is a local diffeomorphism around p.
(2) F is both a submersion as well as an immersion at p.
(3) F is a submersion at p and $\dim(M) = \dim(N)$.
(4) F is an immersion at p and $\dim(M) = \dim(N)$.

Exercise 2.201. (form the 2023/2024 exam) Show that, if $\pi : P \to M$ is a surjective submersion then:

(a) For any $x \in M$ there exists an open $U \subset M$ and a smooth function $\sigma : U \to P$ with the property that $\pi(\sigma(y)) = x$ for all $y \in U$.
(b) A function $f : M \to \mathbb{R}$ is smooth if and only if $f \circ \pi : P \to \mathbb{R}$ is smooth.

Exercise 2.202. Let M be a manifold and consider the product $M \times M$ (with the product smooth structure as in Exercise 2.21). Show that the diagonal inclusion $\Delta : M \longrightarrow M \times M$, $\Delta(x) = (x,x)$, is an immersion.

Exercise 2.203. (form the 2022/2022 retake) Prove that $N = \{(x,y) \in \mathbb{R}^2 : x^3 = y^2\}$ is not an embedded submanifold of $\mathbb{R}^2$.

Exercise 2.204. Which of the subsets of $\mathbb{R}^2$ from the Figure 2.26 describes an embedded submanifold of $\mathbb{R}^2$. But an immersed submanifold? In how many ways?

Exercise 2.205. (form the 2022/2022 exam) You know already, from one of the homeworks, that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$. Show now that $\mathbb{P}^2$ can be embedded in S^4. (Hint: do not look for complicated formulas).

Projective spaces

The following exercise show that, when $m = 1$, $\mathbb{P}^1$ is not really new: it is diffeomorphic to the circle $S^1 \subset \mathbb{R}^2$.

Exercise 2.206. Consider the function

$$F : \mathbb{P}^1 \to S^1, \quad F([x:y]) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right)$$

and do the following:

1. show that F is well-defined and smooth.

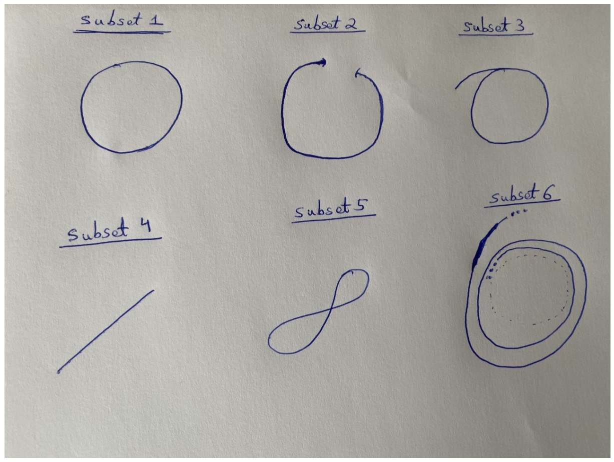


Fig. 2.26

2. show hat F is actually a diffeomorphism.
3. after identifying $\mathbb{P}^1$ with S^1 using F, what does the map $H : S^1 \to \mathbb{P}^1$ become?
4. explain F with a picture.

Exercise 2.207. Let $F : \mathbb{P}^2 \to \mathbb{R}^4$ be the function given by

$$F([x : y : z]) = (xy,\ yz,\ zx,\ y^2 - z^2) \quad \text{for } (x,y,z) \in S^2.$$

Please do the following:

1. check that F is well-defined and write the general formula for F (not only for (x,y,z) in the sphere).
2. compute the representation of F with respect to the charts χ^i of $\mathbb{P}^2$ (and the identity chart for $\mathbb{R}^4$).
3. show that F is an immersion.
4. show that F is an embedding.
5. we also compose F with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ on the first three coordinates,

$$G := \mathrm{pr} \circ F : \mathbb{P}^2 \to \mathbb{R}^3;$$

   Show that its image is the following explicit subspace of $\mathbb{R}^3$:

$$R := \left\{ (X,Y,Z) \in \mathbb{R}^3 : |X|, |Y|, |Z| \leq \frac{1}{2},\ (XY)^2 + (YZ)^2 + (ZX)^2 = XYZ \right\} \ \ldots$$

6. ... but this is not an embedded submanifold of $\mathbb{R}^3$.
7. show that, however, $G : \mathbb{P}^2 \to \mathbb{R}^3$ is an immersion everywhere except for six points $p_1, \ldots, p_6$ ...
8. ... but even after removing those six points, $g(\mathbb{P}^2 \setminus \{p_1, \ldots, p_6\})$ is not an embedded submanifold of $\mathbb{R}^3$.
9. show that, on the other hand,

$$R_0 = \{(X,Y,Z) \in R : XYZ \neq 0\}$$

   is an open dense subset of R which is an embedded submanifold of $\mathbb{R}^3$, and there is open dense subset $\mathbb{P}^2_0 \subset \mathbb{P}^2$ such that $g|_{\mathbb{P}^2_0} : \mathbb{P}^2_0 \to R_0$ is a diffeomorphism.

Remark 2.208. The image R of g is know as "the Roman surface", or the "Steiner surface" (discovered by Steiner in Rome in 1844, according to Wikipedia). One can actually show that $\mathbb{P}^2$ cannot be embedded in $\mathbb{R}^3$. The g and R above can be seen as an attempt to find an immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ (non-injective, of course). You may be surprised to hear that such immersions actually exist. Finding an explicit one is quite a bit more difficult but also very interesting, and gives rise to Boy's surface in $\mathbb{R}^3$ ... but I let you google this one ...

Similar to Exercise 2.206, the following exercise show that, when $m = 1$, $\mathbb{CP}^1$ is not really new: it is diffeomorphic to the 2-sphere. This will be discussed together with the map

$$H : S^3 \to \mathbb{CP}^1, \quad H(z_0, z_1) = [z_0 : z_1].$$

The notation H (and h for the map in the next exercise) are related to the name of Hopf (... fibration).

Exercise 2.209. Consider the map

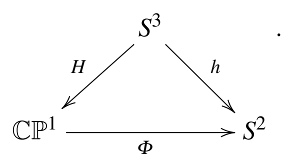
$$h : S^3 \to S^2, \quad h(z_0, z_1) := (|z_0|^2 - |z_1|^2, 2i \cdot \overline{z_0} \cdot z_1)$$

or, using real coordinates, h sends $(x, y, z, t) \equiv (x + i \cdot y, z + i \cdot t)$ to

$$h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

Show that:

1. h is well defined and it is a smooth submersion.
2. H is a smooth submersion.
3. There exists and is unique a map $\Phi : \mathbb{CP}^1 \to S^2$ such that $h = \Phi \circ H$ i.e. a commutative diagram:



4. Φ is smooth.
5. Φ is actually a diffeomorphism.

Exercise 2.210. With the same notations as in Exercise 2.209, show that each fiber of the Hopf map $h : S^3 \to S^2$ is an embedded submanifold of S^3 which is diffeomorphic to a circle.

The net exercise reveals a very nice relationship between the 3-sphere and the torus: the 3-sphere can be obtained by gluing together two solid tori along their boundary. The following exercise explains this process and its interaction with the Hopf map from Exercise 2.209.

Exercise 2.211. Consider again the Hopf map in the explicit form

$$h : S^3 \to S^2, \quad h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

Consider also the decomposition of S^2 into the open upper and lower hemispheres,

$$S^2_+ = \{(u, v, w) \in S^2 : u \geq 0\}, \quad S^2_- = \{(u, v, w) \in S^2 : u \leq 0\}$$

with the common intersection the circle identified with

$$S^1 = S^2_+ \cap S^2_+ = \{(u, v, w) \in S^2 : u = 0\}.$$

Show that:

1. each of the hemispheres are manifolds with boundary diffeomorphic to the unit disk D^2.

2. the pre-image of the common circle S^1 via h is diffeomorphic to the torus.
3. the pre-image of the upper/lower hemisphere via h is diffeomorphic to the solid torus.

Therefore, the pre-image via h of the decomposition $S^2 = S^2_+ \cup S^2_-$ becomes a decomposition of S^3 into two copies of the solid torus.

Exercise 2.212. Show that the smooth structure on $\mathbb{P}^m$ discussed here makes the canonical map

$$H : S^m \to \mathbb{P}^m, \quad H(x_0, \ldots, x_m) = [x_0 : \ldots : x_m]$$

into a submersion. Then prove the same for the similarly map $H : S^{2m+1} \to \mathbb{CP}^m$.

Lie groups

Exercise 2.213. Show that if G is a Lie group, then for any $a \in G$ the map

$$L_a : G \to G, \quad L_a(g) = ag$$

is a diffeomorphism. Then show that

$$G \ni a \longmapsto L_a \in \mathrm{Diff}(G)$$

is a group homomorphism from G to the group of diffeomorphisms of G.

Exercise 2.214. Have a look at Example 2.98. Then prove that $SL_n(\mathbb{R})$ and $SU(n)$ are Lie groups. What is their dimensions?

Exercise 2.215. Show that

$$\Phi : S^1 \to SO(2), \ \ \Phi(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is an isomorphism of Lie groups.

Exercise 2.216. Show that

$$\Phi : S^3 \to SU(2), \ \ \Phi(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where we interpret S^3 as $\{(\alpha,\beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}$, is an isomorphism of Lie groups.

By analogy with the last example, one may expect that S^3 is isomorphic also to $SO(3)$ (at least the dimensions match!). However, the relation between these two is more subtle:

Exercise 2.217. Recall that we view S^3 inside $\mathbb{H}$. We also identify $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\}, \ (a,b,c) \mapsto ai + bj + ck.$$

For each $u \in S^3$, show that $A_u(v) := u^* v u$ defines a linear map $A_u : \mathbb{R}^3 \to \mathbb{R}^3$ which, as a matrix, gives an element $A_u \in SO(3)$. Then show that the resulting map

$$\phi : S^3 \to SO(3), \ u \mapsto A_u$$

is smooth, is a group homomorphism, is a surjective local diffeomorphism, but each fiber has two elements (it is a 2-1 cover). Finally, deduce that $SO(3)$ is diffeomorphic to the real projective space $\mathbb{P}^3$.

Exercise 2.218. The aim of this exercise is to show how $GL_n(\mathbb{C})$ sits inside $GL_{2n}(\mathbb{R})$, and similarly for the other complex groups. Well, there is an obvious map

$$j : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Show that:

1. j is an embedding, with image $\{X \in GL_{2n}(\mathbb{R}) : J \cdot X = X \cdot J\}$, where J is the matrix (2.2.12).
2. j takes $U(n)$ into $O(2n)$.
3. actually j takes $U(n)$ into $SO(2n)$.

(Hint for (3): prove that $\det(j(Z)) = |\det(Z)|^2$ for all $Z \in GL_n(\mathbb{C})$. For this: show that for B invertible, there exists a matrix $X \in GL_n(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Chapter 3
Vector fields

3.1 Tangent vectors as directional derivatives

Here is another approach to the tangent spaces; the brief philosophy of this approach is:

> *tangent vectors at p allow us to take directional derivatives (at p) of smooth functions*

3.1.1 The general notion

To implement the previous "slogan" we return to the differential

$$(df)_p(v)$$

of a smooth function $f : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^m$ open) at a point $p \in \Omega$, applied to a (tangent) vector $v \in \mathbb{R}^m$, that we reinterpreted it as a v-derivative at p

$$(df)_p(v) = \frac{\partial f}{\partial v}(p) = \partial_v(f)(p).$$

Hence, if we want to understand what the (tangent) vector v really does (at p), one may say that it defines a function

$$\partial_v = \frac{\partial}{\partial v} : \mathscr{C}^\infty(\Omega) \to \mathbb{R};$$

What are the main properties of this function? Well, it is clearly linear and, moreover, it acts on the product of functions according to the Leibniz rule at p:

$$\partial_v(fg) = f(p) \cdot \partial_v(g) + g(p) \cdot \partial_v(g) \tag{3.1.1}$$

for all $f, g \in \mathscr{C}^\infty(\Omega)$. Note that here we are making use of the following algebraic structures on $C^\infty(\Omega)$: the vector space structure (for linearity) and the product of functions (to make sense of the Leibniz identity at p); all together, we are making use of the "algebra structure" on $C^\infty(\Omega)$. The main point is that, conversely, any map

$$\partial : \mathscr{C}^\infty(\Omega) \to \mathbb{R}$$

which is linear and satisfies the Leibniz identity at p is of type ∂_v for a unique vector $v \in \mathbb{R}^m$ (this will follow from the discussion below). This gives another perspective/possible approach to tangent spaces.

Definition 3.1. Given an m-dimensional manifold M and $p \in M$, a **derivation of $C^\infty(M)$ at** p is a map

$$\partial : \mathscr{C}^\infty(M) \to \mathbb{R}$$

which is linear and which satisfies the Leibniz identity (3.1.1) at p.

We denote by $T_p^{\mathbf{deriv}}M$ the collection of all such derivations at p. We endow it with the following vector space structure: for $\partial, \partial' \in T_p^{\text{deriv}}M$, and a scalar $\lambda \in \mathbb{R}$, the sum and the multiplication by scalar

$$\partial + \partial' \in T_p^{\text{deriv}}M, \quad \lambda \cdot \partial \in T_p^{\text{deriv}}M$$

are defined using the similar operations $+$ and $\cdot$ from $\mathbb{R}$:

$$(\partial + \partial')(f) := \partial(f) + \partial'(f), \quad (\lambda \cdot \partial)(f) := \lambda \cdot \partial(f).$$

Let us point out right away that any tangent vector in the sense of the previous sections, $v \in T_pM$, gives rise to a derivation at p, which we will denote by

$$\partial_v : \mathscr{C}^\infty(M) \to \mathbb{R}.$$

Assuming $v = \frac{d\gamma}{dt}(0)$, one can take inspiration from Lemma 1.34 and define

$$\boxed{\partial_v(f) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)).} \tag{3.1.2}$$

Lemma 3.2. *One obtains a well-defined linear map*

$$\boxed{I_p^M : T_pM \to T_p^{deriv}M, \quad v \mapsto \partial_v}\,.$$

Proof. Using a chart $\chi \in \text{Charts}_p(M)$ we can write $f \circ \gamma = (f \circ \chi) \circ (\chi^{-1} \circ \gamma) = f_\chi \circ \gamma^\chi$, and then

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = (df_\chi)_{\chi(p)}\left(\frac{d\gamma^\chi}{dt}(0)\right)$$

This shows that the right hand side of (3.1.2) doesn't depend on the entire γ but only on $\frac{d\gamma^\chi}{dt}(0)$, i.e. only on the equivalence class of γ w.r.t. $\sim_p$, i.e. only on $v = \frac{d\gamma}{dt}(0)$. In other words, I_p^M is well-defined. The last formula also gives the description of I using the representation of v in charts:

$$\partial_v(f) = (df_\chi)_{\chi(p)}(v^\chi).$$

Since the vector space structure of T_pM was defined by (2.3.7) and each $(df_\chi)_{\chi(p)}$ is linear this formula implies

$$\partial_{v+w}(f) = \partial_v(f) + \partial_w(f), \quad \partial_{\lambda \cdot v}(f) = \lambda \cdot \partial_v(f)$$

for all $v, w \in T_pM$, $\lambda \in \mathbb{R}$, and for all $f \in \mathscr{C}^\infty(M)$. Hence $\partial_{v+w} = \partial_v + \partial_w$, $\partial_{\lambda \cdot v} = \lambda \cdot \partial_v$ for all v, w, λ, which is precisely the linearity of I_p^M. 😀

In some sense, the rest of this chapter is devoted to the proof and the discussion of the following:

Theorem 3.3. *The map $I_p^M : T_pM \to T_p^{deriv}M$ is an isomorphism of vector spaces.*

3.1.2 Extra-for the interested student: the algebraic nature of the construction

It is clear that the model for the tangent space provided by $T_p^{\mathrm{deriv}}M$ is more algebraic. Let us make this a bit more explicit. This discussion fits in very well with the previous discussions on the algebra of smooth functions (Remark 2.68), going back all the way to the "Gelfand-Naimark philosophy" (see also Remark 1.11).

Looking at the definition of $T_p^{\mathrm{deriv}}M$ and remembering that evaluation at p can be seen as a character χ_p of $\mathscr{C}^\infty(M)$, there is a clear way one can proceed in general: for any algebra A and any character χ on A we introduce the notion of χ-derivation on A, by which we mean a linear map $\chi : A \to \mathbb{R}$ satisfying the derivation identity:

$$\partial(ab) = \chi(a)\partial(b) + \partial(a)\chi(b)$$

for all $a, b \in A$. We denote by $\mathrm{Der}_\chi(A)$ the (vector) space of all such derivations. Intuitively, this may be interpreted as "the tangent space of $X(A)$ at χ".

When applied to $A = \mathscr{C}^\infty(M)$, while we know (when M is compact) that any character on A is necessarily of type χ_p for some p (unique), the corresponding derivations give

$$\mathrm{Der}_{\chi_p}(\mathscr{C}^\infty(M)) = T_p^{\mathrm{deriv}}M.$$

For this reason, for a general algebra A, one may think of its characters as "points", while for a character χ one may think of $\mathrm{Der}_\chi(A)$ as "the tangent space of $X(A)$ at χ".

Exercise 3.4. When $A = \mathbb{R}[X_1, \ldots, X_m]$ is the algebra of polynomials in m variables show that any character is the evaluation χ_x at some point $x \in \mathbb{R}^m$, and then show that $\mathrm{Der}_{\chi_x}(A)$ is a finite dimensional vector space and exhibit a basis.

Then consider another A: the algebra of polynomial functions on the circle S^1. And, again, consider the characters χ_p for $p \in S^1$ and compute $\mathrm{Der}_{\chi_p}(A)$.

One warning however: while when looking at characters/points it does not make a difference whether we use the algebra $\mathscr{C}^\infty(M)$ or the one of continuous functions, the situation is dramatically different when it comes to tangent vectors:

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Exercise 3.5. Show that $\mathrm{Der}_{\chi_p}(\mathscr{C}(M)) = 0$. (Hint: the proof is much shorter than the bla-bla preceeding the exercise)

3.1.3 The case of Euclidean spaces

When $M = \Omega$ is an open in $\mathbb{R}^m$ (endowed with the canonical smooth structure), $p \in \Omega$, then the operation of taking the usual partial derivatives at p,

$$\left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{C}^\infty(\Omega) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x_i}(p),$$

are derivations at p- hence they can be interpreted as vectors

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{\mathrm{deriv}}\Omega. \tag{3.1.3}$$

Exercise 3.6. Check that these vectors coincide with the vectors obtained by applying $I_p^\Omega : T_p\Omega \to T_p^{\mathrm{deriv}}\Omega$ to the standard basis (2.3.11)., The exercise shows that Theorem 3.3 for $M = \Omega$ is equivalent to the fact that (3.1.3) is a basis of $T_p^{\text{deriv}}\Omega$. Recall that, for any manifold M, a chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of M around p induces similar **partial derivatives w.r.t.** χ,

$$\frac{\partial f}{\partial \chi_i}(p) := \frac{\partial f_\chi}{\partial x_i}(\chi(p)),$$

where $f_\chi = f \circ \chi^{-1} : \Omega \to \mathbb{R}$. This defines vectors

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_p^{\text{deriv}} M \tag{3.1.4}$$

and Theorem 3.3 is equivalent to saying that this is a basis of $T_p^{\text{deriv}} M$. Here we take care of the case $M = \mathbb{R}^m$:

Proposition 3.7. *For any $p \in \mathbb{R}^m$, the vectors (3.1.4) form a basis of $T_p^{deriv}\mathbb{R}^m$.*

Proof. For notational simplicity we assume that $p = 0$. For the linear independence we assume that $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ are scalars with the property that

$$\lambda_1 \cdot \left(\frac{\partial}{\partial x_1}\right)_0 + \ldots + \lambda_m \cdot \left(\frac{\partial}{\partial x_m}\right)_0 = 0$$

and we have to prove that each λ_i vanishes. For that one applies the left hand side of the previous equality (a derivation at p!) to the coordinate function $x \mapsto x_i$ and we obtain that, indeed, $\lambda_i = 0$.

We still have to check that any derivation at 0,

$$\partial : \mathscr{C}^\infty(\mathbb{R}^m) \to \mathbb{R},$$

is a linear combination of the standard partial derivatives at 0:

$$\partial = \sum_i \lambda_i \cdot \left(\frac{\partial}{\partial x_i}\right)_0 \quad \text{with } \lambda_i \in \mathbb{R}.$$

We will show that $\lambda_i := \partial(x_i) \in \mathbb{R}$ do the job. To show this, we use the fact that any $f \in \mathscr{C}^\infty(\mathbb{R}^m)$ can be written as

$$f(x) = f(0) + \sum_i x_i g_i(x), \quad \text{with } g_i \in \mathscr{C}^\infty(\mathbb{R}^m) \tag{3.1.5}$$

(see below). Using the Leibniz identity (which also implies that ∂ is 0 on constant functions) we obtain

$$\partial(f) = \sum_i \lambda_i g_i(0),$$

while $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. Hence we are left with proving that any f can be written in the form (3.1.5). For this we use the identity

$$h(1) - h(0) = \int_0^1 \frac{dh}{dt}(t)dt$$

for all smooth functions $h : [0,1] \to \mathbb{R}$; of course, we want to choose h such that $h(1) = f(x)$ and $h(0) = f(0)$. Choose then $h(t) = f(tx)$ and we obtain the desired identity (3.1.5) with

$$g_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt. \qquad 😀$$

3.1.4 The local nature of derivations

We now discuss the local nature of derivations, which allows us to related $T_p^{\text{deriv}}M$ to the similar tangent spaces of open neighborhoods U of p, and then reduce the proof of Theorem 3.3 to the case of domains of coordinate charts.

Lemma 3.8. *Any derivation of $C^\infty(M)$ at p, $\partial \in T_p^{deriv}M$, has the following local property: for any $f_1, f_2 \in \mathscr{C}^\infty(M)$:*

$$f_1 = f_2 \text{ in a neighborhood of } p \Longrightarrow \partial(f_1) = \partial(f_2).$$

Proof. Let $f = f_1 - f_2$. We know that $f = 0$ in a neighborhood U of p and we want to prove that $\partial(f) = 0$. We may assume that U is chosen so that it is diffeomorphic to a ball $B(0,\varepsilon) \subset \mathbb{R}^m$, by a diffeomorphism that takes p to 0. As we have already noticed, one can find a smooth function on $\mathbb{R}^m$ that is supported inside $B(0,\varepsilon)$ and is non-zero at 0 (e.g. take $x \mapsto g(\frac{1}{\varepsilon^2}||x||^2)$, where $g : \mathbb{R} \to \mathbb{R}$ is the function from Exercise 1.29). Moving from $B(0,\varepsilon)$ to U, and extending by zero outside U, we find a function $\eta \in C^\infty(M)$ which is supported inside U and $\eta(p) \neq 0$. Then $\eta f = 0$; applying the Leibniz identity for ηf we find $\eta(p)\partial(f) = 0$. Since $\eta(p) \neq 0$, $\partial(f) = 0$ follows. 😀

If $U \subset M$ is an open containing p. Then any derivation $\partial \in T_p^{\text{deriv}}U$ gives a derivation $\tilde{\partial} \in T_p^{\text{deriv}}M$ simply by $\tilde{\partial}(f) := \partial(f|_U)$. This gives rise to a map denoted

$$(di)_p : T_p^{\text{deriv}}U \to T_p^{\text{deriv}}M, \quad \partial \mapsto \tilde{\partial}$$

which, as indicated by the notation, can be thought of as the map induced on the T^{deriv}-tangent spaces by the inclusion $i : U \hookrightarrow M$. Notice that the similar map $(di)_p$ between the usual tangent spaces was "an obvious isomorphism"- so obvious that we actually used it to identify T_pU with T_pM.

Lemma 3.9. *Given $p \in U \subset M$ with U open, the map $(di)_p : T_p^{deriv}U \to T_p^{deriv}M$ is a linear isomorphism.*

Proof. For injectivity, let $\partial : \mathscr{C}^\infty(U) \to \mathbb{R}$ be a derivation at p such that $\partial(\tilde{f}|_U) = 0$ for all $\tilde{f} \in \mathscr{C}^\infty(M)$; we must show that $\partial = 0$. This follows immediately from the following:

- $\partial(f)$ depends only on f in an arbitrarily small neighborhood of p. This is the content of the previous lemma.
- for any $f \in \mathscr{C}^\infty(U)$, there exists $\tilde{f} \in \mathscr{C}^\infty(M)$ s.t. $f = \tilde{f}$ in some neighborhood of p.

To prove the last item we take $\tilde{f} := \eta \cdot f$ where $\eta \in \mathscr{C}^\infty(M)$ is supported inside U and $\eta = 1$ in some (smaller) neighborhood of p. The existence of η is a local problem- therefore it suffice to build a smooth function η on $\mathbb{R}^m$ that is supported in the ball $B(0,1)$ and is 1 in a neighborhood of 0 (say on $B(0,\frac{1}{3})$); for that one takes again η of type $\eta(x) = g(||x||^2)$ where $g : \mathbb{R} \to \mathbb{R}$ is any smooth function that is 1 when $|t| < \frac{1}{3}$ and is 0 when $t \geq \frac{1}{2}$.

For the surjectivity, we start with $\tilde{\partial} : \mathscr{C}^\infty(M) \to \mathbb{R}$ and we want to build ∂. By now the definition should be clear: or $f \in \mathscr{C}^\infty(U)$, choose any $\tilde{f} \in \mathscr{C}^\infty(M)$ that coincides with f near p (possible by the first part) and set $\partial(f) := \tilde{\partial}(\tilde{f})$. The previous lemma shows that the definition of $\partial(f)$ does not depend on the choice of $\tilde{f}$. And this can used also to check that ∂ is still a derivation at p. E.g., for the Leibniz identity, choosing extensions $\tilde{f}$ or f and $\tilde{g}$ or g, then use the extension $\tilde{f}\tilde{g}$ or fg and use the Leibniz identity for $\tilde{\partial}(\tilde{f}\tilde{g})$. 😀

Corollary 3.10. *Given $p \in U \subset M$ with U open, the map from Theorem 3.3,*

$$I_p^M : T_pM \to T_p^{deriv}M,$$

is an isomorphism if and only if I_p^U is. In particular, Theorem 3.3 holds for all $M = \Omega$ open in $\mathbb{R}^m$.

Proof. The key remark is that the objects that we deal with fit into a commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{\ I_p^M\ } & T_p^{\mathrm{deriv}}M \\ {\scriptstyle (di)_p}\uparrow & & \uparrow{\scriptstyle (di)_p} \\ T_pU & \xrightarrow{\ I_p^U\ } & T_p^{\mathrm{deriv}}U \end{array}$$

The commutativity follows by a direct check using the definitions of the objects involved. The first part follows from the remark that in a commutative square in which the vertical maps are isomorphisms, one of the horizontal maps is an isomorphism if and only if the other one is. For the second part we further invoke Proposition 3.7.

Remark 3.11 (Extra- for the interested student). Lemma 3.8 can be packed into a more conceptual conclusion: any derivations ∂ at p descends to the quotient space

$$\mathscr{C}_p^\infty(M) := \mathscr{C}^\infty(M)/I_p = C^\infty(M)/\sim_p .$$

where I_p is the space of functions that vanish around p and $\sim_p$ is the associated equivalence relation:

$$f_1 \sim_p f_2 \Longleftrightarrow f_1 = f_2 \text{ in a neighborhood of } p.$$

For $f \in \mathscr{C}^\infty(M)$, its equivalence class is denoted

$$\mathrm{germ}_p(f) \in \mathscr{C}_p^\infty(M)$$

and is called **the germ of f at p**. The space of germs at p, i.e. our quotient $\mathscr{C}_p^\infty(M)$, is still an algebra (since I_p is an ideal of $\mathscr{C}^\infty(M)$); explicitly, the operations are induced from $\mathscr{C}^\infty(M)$: $\mathrm{germ}_p(f+g) = \mathrm{germ}_p(f) + \mathrm{germ}_p(g)$ etc. The locality property from the previous lemma can now be interpreted as an isomorphism

$$T_p^{\mathrm{deriv}}M \cong \mathrm{Der}_p(\mathscr{C}_p^\infty(M))$$

between the tangent space and the space of all derivations of $\mathscr{C}_p^\infty(M)$ w.r.t. the evaluation at p; The main difference with the similar discussion using $\mathscr{C}^\infty(M)$ is that, this time, $\mathscr{C}_p^\infty(M)$ has a unique character (the evaluation at p). □

3.1.5 Naturality, and end of proof of Theorem 3.3

Corollary 3.10 tells us that if we want to prove Theorem 3.3 by using a chart $\chi : U \to \Omega$ around p, all we have to do now is to show that we can move from Ω to U. More generally, we show the following "naturality" property:

Lemma 3.12. *If $F : M \to N$ is a diffeomorphism, $p \in M$, then I_p^M is an isomorphism if and only if $I_{F(p)}^N$ is.*

The proof will be similar to that of Corollay 3.10. While in the previous section the spaces $T_p^{\mathrm{deriv}}U$ and $T_p^{\mathrm{deriv}}M$ were compared using the map $(di)_p$, we now need similar maps

$$\boxed{(dF)_p : T_p^{\mathrm{deriv}}M \to T_{F(p)}^{\mathrm{deriv}}N}$$

associated to smooth maps $F : M \to N$. We define them by sending $\partial \in T_p^{\mathrm{deriv}}M$ to $(dF)_p(\partial) \in T_{F(p}^{\mathrm{deriv}}N$ given by

$$\mathscr{C}^\infty(N) \longrightarrow \mathbb{R}, \quad g \mapsto \partial(g \circ F).$$

Exercise 3.13. Check that, indeed, $(dF)_p(\partial)$ is a derivation at $F(p)$, and the resulting map $(dF)_p$ is linear.

Lemma 3.14. *This construction has the same properties as that of the usual differential from Proposition 2.117:*

i) If $F = Id_M : M \to M$ is the identity map, $(dF)_p$ is the identity map of $T_p^{deriv}M$.

ii) The chain rule: for two smooth maps $M \xrightarrow{F} N \xrightarrow{G} P$ and $p \in M$, $(dG \circ F)_p = (dG)_{f(p)} \circ (dF)_p$.

Furthermore, the maps from Theorem 3.3 relate $(dF)_p$ with the usual differentials in the sense that:

$$\begin{array}{ccc} T_pM & \xrightarrow{(dF)_p} & T_{F(p)}N \\ \downarrow{\scriptstyle I_p^M} & & \downarrow{\scriptstyle I_{F(p)}^N} \\ T_p^{deriv}M & \xrightarrow[(dF)_p]{} & T_{F(p)}^{deriv}N \end{array} \quad , \quad (dF)_p \circ I_p^M = I_{F(p)}^N \circ (dF)_p \quad .$$

Proof. This follows by direct checking using the definitions. For instance, for the last part, we start with a vector in the upper left corner $v = \frac{d\gamma}{dt}(0) \in T_pM$. We first compute $(dF)_p \circ I_p^M(\partial)$: it is the derivation of $C^\infty(N)$ given by

$$g \mapsto \partial_v(g \circ F) = \frac{d\gamma}{dt}\bigg|_{t=0} g(F(\gamma(t)).$$

Then we write out the derivation $I_{F(p)}^N \circ (dF)_p(\partial)$, for which we find

$$g \mapsto \partial_{(dF)_p(v)}(g).$$

Since $(dF)_p(v) = \frac{d}{dt}\big|_{t=0} F(\gamma(t))$, we find that the right hand sider of the previous two equations coincide. 😀

Proof (Proof of Lemma 3.12:). The chain rule applied to F and F^{-1} implies that $(dF)_p$ is an isomorphism. Hence we end up again with a commutative square in which two opposite sides (now the horizontal ones) are isomorphisms, and we are saying that one of the remaining sides is an isomorphism if and only if the other one is. 😀

Proof (Proof of Theorem 3.3). The theorem now follows from Proposition 3.7, Corollary 3.10 and Lemma 3.12 rather directly. Let us however go slower and illustrate the entire proof on a diagram in which M:

$$\begin{array}{ccc} T_pM & \xrightarrow{I_p^M} & T_p^{\text{deriv}}M \\ \uparrow{\scriptstyle (di)_p} & & \uparrow{\scriptstyle (di)_p} \\ T_pU & \xrightarrow{I_p^U} & T_p^{\text{deriv}}U \\ \downarrow{\scriptstyle (d\chi)_p} & & \downarrow{\scriptstyle (d\chi)_p} \\ T_0\Omega & \xrightarrow{I_0^\Omega} & T_0^{\text{deriv}}\Omega \\ \downarrow{\scriptstyle (dj)_0} & & \downarrow{\scriptstyle (dj)_0} \\ T_0\mathbb{R}^m & \xrightarrow{I_0^{\mathbb{R}^m}} & T_0^{\text{deriv}}\mathbb{R}^m \end{array}$$

Here M is the original manifold and we fix a chart $\chi : U \to \Omega \subset \mathbb{R}^m$ around p sending p to $0 \in \mathbb{R}^m$. Furthermore, $i : U \to M$ and $j : \Omega \to \mathbb{R}^m$ are the inclusions. By the last lemma, the diagram is commutative. By Lemma 3.9 and Lemma 3.12, the vertical maps on the right side are isomorphisms. By the properties of the usual tangent spaces and differentials, so are the ones vertical maps on the left. Also notice that, in a commutative square, if three of the maps are isomorphisms then so is the fourth. This allows us to move up on the diagram, provided the bottom horizontal map is an isomorphism. But, since that map sends the basis $\left(\frac{\partial}{\partial x_i}\right)_p$ to (3.1.4) in T^{deriv}, Proposition 3.7 implies that the map is indeed an isomorphism. 😀

3.2 Vector fields

We now discuss vector fields. The brief philosophy is:

> *While for embedded submanifolds $M \subset \mathbb{R}^k$ each single tangent space $T_pM \subset \mathbb{R}^k$ was interesting (e.g. because it reflects the position of M inside $\mathbb{R}^k$ near p), for a general manifold M this is less so. Instead, it is the entire family $\{T_pM\}_{p\in M}$ that is interesting. I.e. not one single tangent vector, but a family $\{v_p\}_{p\in M}$ of vectors (of course, "varying smoothly" with respect to p). I.e. vector fields.*

Looking back at the previous sections of this chapter, we see that we insisted in making sense of T_pM as vector spaces independently of the way that M may sit in some larger Euclidean space. Now, just as a vector space, T_pM is rather boring: it is isomorphic to $\mathbb{R}^m$. And the same is true for any finite dimensional vector space V; however, one should keep in mind that, for a bare m-dimensional vector space V, realizing an explicit isomorphism with $\mathbb{R}^m$ amounts to extra choices (namely a basis of V). In particular, for the tangent spaces T_pM, we obtained identifications with $\mathbb{R}^m$ once we fixed a chart χ (and these identifications work for p in the domain of χ only!). Actually this is a "problem" (or better: "interesting phenomena") not only for general Ms, but even for embedded submanifolds $M \subset \mathbb{R}^k$: while isomorphisms

$$\mathbb{R}^m \cong T_pM \subset \mathbb{R}^k$$

can be chosen for each p, often cannot be chosen so that they depend smoothly on $p \in M$. The "hairy ball theorem" implies that this is the case already for $M = S^2 \subset \mathbb{R}^3$:

Exercise 3.15. Assume that for each $p \in S^2$ one can find a linear isomorphism

$$\Phi_p : \mathbb{R}^2 \to T_pS^2 \subset \mathbb{R}^3$$

so that the Φ_ps vary smoothly with respect to p, in the sense that the map

$$\Phi : S^2 \times \mathbb{R}^2 \to \mathbb{R}^3, \quad (p,v) \mapsto \Phi_p(v)$$

is smooth. Show that that would imply the existence of a nowhere vanishing vector field on S^2, i.e. a smooth map

$$X : S^2 \to \mathbb{R}^3,$$

nowhere vanishing, such that $X(p) \in T_pS^2$ for all $p \in S^2$. Try now to construct such an X! Then state the "hairy ball theorem" (or look it up).

More generally, for an embedded submanifold $M \subset \mathbb{R}^k$, a vector field on M is a function

$$M \ni p \mapsto X(p) \in T_pM \subset \mathbb{R}^k$$

which, when interpreted as a function with values in $\mathbb{R}^k$, is smooth. What if an embedding into some Euclidean space is not fixed (or if we use a different one)?

3.2.1 Smooth vector fields

Continuing the last discussion, for general manifolds M: first of all, we can still look at maps

$$X : M \ni p \mapsto X_p \in T_pM$$

(interpreted also as families $X = \{X_p\}_{p\in M}$). These will be called **set-theoretical vector fields** on M. To make sense of their smoothness, we use charts $\chi : U \to \Omega \subset \mathbb{R}^m$ of M. Recall that any such chart induces a basis $\left(\frac{\partial}{\partial \chi_i}\right)_p$ of T_pM hence any set-theoretical vector field X can be written as

$$X_p = \sum_{i=1}^{m} X_\chi^i(\chi(p)) \left(\frac{\partial}{\partial \chi_i}\right)_p \tag{3.2.1}$$

for all $p \in M$, where each X_χ^i is a function

$$X_\chi^i : \Omega \to \mathbb{R}.$$

These will be called **the coordinate functions of** X **w.r.t. the chart** χ.

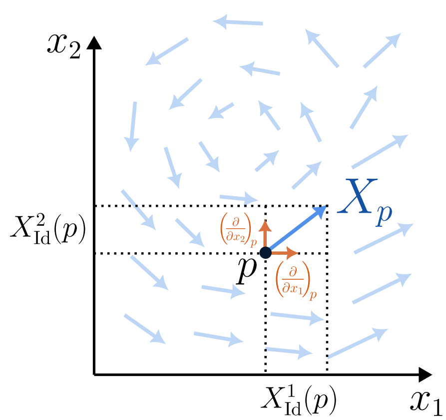


Fig. 3.1 Let M be the Euclidean plane drawn here with the identity chart $\mathrm{Id} = (x_1, x_2)$. The vector field X assigns a vector X_p to every point p. The components of X_p with respect to the basis vectors $\left(\frac{\partial}{\partial x_i}\right)_p$ of the identity chart are its coordinate functions X^i_{Id}.

Definition 3.16. A vector field on a manifold is any set-theoretical vector field X,

$$X : M \ni p \mapsto X_p \in T_pM,$$

with the property that its coordinate functions X_χ^i w.r.t any chart χ are smooth.

We denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

Exercise 3.17. Show that, for the smoothness condition, it is enough to check it only for charts in a/any atlas of M inducing the smooth structure of M.

Note the algebraic structure on $\mathfrak{X}(M)$ that is present right away:

- it is a vector space, with the addition $+$ and multiplication by scalars $\cdot$ defined pointwise:

$$(X+Y)_p := X_p + Y_p, \quad (\lambda \cdot X)_p := \lambda \cdot X_p.$$

- it is a $\mathscr{C}^\infty(M)$-module, i.e. there is an operation of multiplication of vector fields by smooth functions,

$$\mathscr{C}^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (f,X) \mapsto f \cdot X.$$

Again, this is defined pointwise by the obvious formula:

$$(f \cdot X)_p := f(p) \cdot X_p.$$

Example 3.18. (in $\mathbb{R}^m$) Using the standard basis (2.3.8) and its blue version (3.1.3), at each point, one obtains the corresponding standard vector fields on $\mathbb{R}^m$:

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m); \tag{3.2.2}$$

and, using the basic operations described above, one obtains vector fields

$$f_1 \cdot \frac{\partial}{\partial x_1} + \ldots + f_m \cdot \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$$

for any smooth functions f_i on $\mathbb{R}^m$. And the fact that the standard basis (2.3.8) is actually a basis implies that any vector field on $\mathbb{R}^m$ is of this type. In a more algebraic language: (3.2.2) forms a basis of $\mathfrak{X}(\mathbb{R}^m)$ as a $C^\infty(\mathbb{R}^m)$-module. Of course, the same discussion applies to any open $\Omega \subset \mathbb{R}^m$.

Remark 3.19 (the chart representation). The discussion from the previous example allows one to slightly re-interpret the coordinate functions of a vector field $X \in \mathfrak{X}(M)$ w.r.t. a chart $\chi : U \to \Omega$ discussed above: they will be precisely the coefficients of a vector field X_χ on Ω obtained by pushing forward X (restricted to U) via Ω:

$$\chi_*(X) \in \mathfrak{X}(\Omega), \quad \chi_*(X)_x := (d\chi)_p(X_p) \in T_x\Omega \quad \text{where } p = \chi^{-1}(x).$$

This is called **the representation of X in the chart χ**. More on "push-forwards" will be discussed in section 3.2.5.

Exercise 3.20. In $\mathbb{R}^2$ we consider the vector fields

$$X := \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad E := x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

And we also consider the polar coordinates chart $\chi : (x,y) \mapsto (r,\theta)$ where

$$r = r(x,y) \in \mathbb{R}_{>0}, \quad \theta = \theta(x,y) \in (0,2\pi)$$

are determined by the usual formulas $x = r \cdot \cos\theta, \quad y = r \cdot \sin\theta$, on $U = \mathbb{R}^2 \setminus \{0\}$. Compute the representation of X, Y and E in the chart χ. (since the outcome are vector fields in $\mathbb{R}_{>0} \times (0,2\pi)$ where we use r and θ for the coordinates, please use the notations $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ for the standard basis there).

Example 3.21. (on submanifolds $M \subset \mathbb{R}^n$) When working in an embedded submanifold $M \subset \mathbb{R}^n$ it is, of course, natural to use the coordinates $(x_1, \ldots, x_n)$ that are there anyway, from the ambient space. Actually, there may even no be any other coordinates available without any extra-choices (think e.g. of the spheres). Similarly, since $T_pM \subset T_p\mathbb{R}^n$, it is natural to write the tangent vector to M as

$$X_p = f_1(p) \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \ldots + f_n(p) \cdot \left(\frac{\partial}{\partial x_n}\right)_p.$$

But one should be aware that the vectors $\frac{\partial}{\partial x_i}$ are, in general, not tangent to M. Actually only some expressions of this type are tangent to M (also, in principle, the functions f_i are defined only for $p \in M$).

A very good example is the one of the spheres $S^m \subset \mathbb{R}^{m+1}$. To have actual manifold coordinates one needs to choose some smooth chart (e.g. stereographic projection), but one also use the coordinates $(x_0, \ldots, x_m)$ of the ambient space. And then, when looking at tangent vectors (and vector fields) one deals with expressions

$$\lambda_0 \cdot \left(\frac{\partial}{\partial x_0}\right)_x + \ldots + \lambda_m \cdot \left(\frac{\partial}{\partial x_m}\right)_x$$

and remembers that such a vector is tangent to the sphere if and only if

$$\lambda_0 \cdot x_0 + \ldots + \lambda_m \cdot x_m = 0.$$

Exercise 3.22. Show that the following defines a smooth vector field on S^1:

$$V_{(x,y)} := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

And make a picture. And try to see the relationship with complex numbers.

Exercise 3.23. Show that the following defines a smooth vector field V on S^3:

$$V_{(x,y,z,t)} := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

Hence a "hairy ball theorem" does not hold on S^3 (see Theorem 5.55). Can you go further and find two more, similar and equally interesting vector fields on S^3? (hint: rebaptise V to $\mathbb{I}$, and call the other two $\mathbb{J}$ and $\mathbb{K}$ ...)

Exercise 3.24. In the last two exercises, in principle, to show that those vector fields are smooth you applied the definition and looked into charts. However, isn't there something more general going on? What if $M \subset N$ is an embedded submanifold and $X \in \mathfrak{X}(N)$ has the property that $X_p \in T_pM$ for all $p \in M$?

Exercise 3.25. On S^m consider the stereographic projection w.r.t. the north pole:

$$\chi : S^m \setminus \{p_N\} \to \mathbb{R}^m, \quad (x_0, \ldots, x_m) \mapsto (u_1, \ldots, u_m)$$

with

$$u_1 = \frac{x_0}{1 - x_m}, \ldots, \; u_m = \frac{x_{m-1}}{1 - x_m}.$$

Find a vector field $X \in \mathfrak{X}(S^m)$ which, represented in this chart, becomes

$$X_\chi = u_1 \cdot \frac{\partial}{\partial u_1} + \ldots + u_m \cdot \frac{\partial}{\partial u_m}.$$

3.2.2 Vector fields as derivations

Next, we look for alternative ways to characterize the smoothness of vector fields; as a bonus, these will reveal more of the structure that vector fields carry. First of all, while tangent vectors could be interpreted as derivations (at given points $p \in M$), for any $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ one has $\partial_{X_p}(f) \in \mathbb{R}$ for each $p \in M$, therefore a function

$$L_X(f) : M \to \mathbb{R}, \quad p \mapsto \partial_{X_p}(f) = (df)_p(X_p).$$

This is called **the Lie derivative of f along the vector field X**. Of course, we expect that $L_X(f)$ is again smooth.

Proposition 3.26. *A set-theoretical vector field X on M is smooth if and only if*

$$L_X(f) \in C^\infty(M) \text{ for all } f \in C^\infty(M).$$

Proof. Assume first that X is smooth in sense of the definition. For $f \in C^\infty(M)$, we can check the smoothness of $L_X(f)$ locally: for an arbirary chart χ we have to make sure that $L_X(f) \circ \chi^{-1}$ is smooth. But applying the definitions, we see that this expression is precisely $\sum_i X^i_\chi \frac{\partial f_\chi}{\partial x_i}$ and then smoothness follows.

For the converse, by the type of arguments we have already seen, it suffices to show that for any point $p \in M$ there exists a chart around p such that the coefficients X_i^χ are smooth. For an arbitrary chart $\chi : U \to \Omega \subset \mathbb{R}^m$ around p, we choose $f_i \in \mathscr{C}^\infty(M)$ such that, in a smaller neighborhood $U_0 \subset U$ of p, f_i coincides with χ_i. By hypothesis, $L_X(f_i)$ is smooth; but over U_0, this function is precisely X_i^χ. Hence taking $\chi_0 := \chi|_{U_0}$, the coefficients $X_i^{\chi_0}$ will be smooth.

Hence each vector field X gives rise to an operation

$$L_X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad L_X(f)(p) = (df)_p(X_p). \tag{3.2.3}$$

The interpretation of tangent vectors at p as derivations at p can be pushed further to a similar characterization of vector fields. The algebraic objects are

derivations of $\mathscr{C}^\infty(M)$, by which we mean maps

$$L : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$$

which are linear and satisfy the derivation low (Leibniz identity):

$$L(fg) = L(f)g + fL(g) \quad \text{for all } f, g \in \mathscr{C}^\infty(M).$$

We denote by $\mathrm{Der}(\mathscr{C}^\infty(M))$ the (vector) space of such derivations.

Note: again, this is a purely algebraic construction, that applies to any algebra.

Theorem 3.27. *For any $X \in \mathfrak{X}(M)$, the Lie derivative along X,*

$$L_X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad f \mapsto L_X(f)$$

is a derivation. Moreover,

$$I : \mathfrak{X}(M) \to Der(\mathscr{C}^\infty(M)), \quad X \mapsto L_X$$

is an isomorphism of vector spaces.

Proof. The fact that L_X is a derivation follows right away from the similar property of the ∂_{X_p}s. The injectivity of I follows from the fact that, if a tangent vector $v_p \in T_pM$ is non-zero, there exists a smooth function f on M such that $\partial_{v_p}(f) \neq 0$ (why?). For the surjectivity of I, let L be a derivation. Then, for each $p \in M$,

$$X_p : \mathscr{C}^\infty(M) \to \mathbb{R}, \quad f \mapsto L(f)(p)$$

is a derivation at p, hence defines a tangent vector $X_p \in T_pM$. Since $L_X(f) = L(f)$ is smooth for any smooth f, the previous proposition implies that X is smooth. 😀

Exercise 3.28. For the vector field $X \in \mathfrak{X}(S^2)$ that you found in Exercise 3.25, compute $L_X(f)$ for the function

$$f : S^2 \to \mathbb{R}, \quad f(x,y,z) = x + y^2 + z^3.$$

Again, you will probably use here the fact that, although we make reference to f only as a function on S^2, you will be using its extension to the entire ambient space $\mathbb{R}^3$. Is there something more general to learn from this?

Exercise 3.29. Consider the vector field $V \in \mathfrak{X}(S^3)$ from Exercise 3.23, the Hopf map $h : S^3 \to S^2$ (Exercise 2.209) and denote by h_1, h_2 and h_3 its components. Show that L_V kills all these components. Can you reformulate this without referring to the components of h, but to h itself?

3.2.3 The Lie bracket

And here is one clear advantage of the point of view of derivations: the presence of yet another structure on $\mathfrak{X}(M)$. More precisely, given two derivations on $\mathscr{C}^\infty(M)$, necessarily of type L_X and L_Y, their composition

$$L_X \circ L_Y : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$$

is not a derivation anymore. Indeed, what we have is:

$$(L_X \circ L_Y)(fg) = (L_X \circ L_Y)(f)g + f(L_X \circ L_Y)(g) + L_X(f)L_Y(g) + L_Y(f)L_X(g),$$

i.e. a "defect term" $L_X(f)L_Y(g) + L_Y(f)L_X(g)$ which spoils the Leibniz identity. However, the other composition $L_Y \circ L_X$ comes with the same "defect term"; therefore, their difference

$$[L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$$

is again a derivation.

Definition 3.30. Given $X, Y \in \mathfrak{X}(M)$, we denote by $[X,Y] \in \mathfrak{X}(M)$, called **the Lie bracket of X and Y**, the (unique) vector field on M with the property that

$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X.$$

The following exercise should convince you of the advantage of the point of view of derivations.

Exercise 3.31. Compute the coordinate functions of $[X,Y]$ with respect to a chart χ, in terms of the coordinates functions of X and Y defined by (3.2.1).

Exercise 3.32. In this exercise we point out the main properties of the Lie bracket operation $[\cdot,\cdot]$ on $\mathfrak{X}(M)$. First of all, the way it interacts with the other structure present on on $\mathfrak{X}(M)$- show that:

- w.r.t. the vector space structure: it is a bilinear and skew-symmetric operation
$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$
- w.r.t. the $\mathscr{C}^\infty(M)$-module structure: it satisfies the derivation rule:
$$[X, f\cdot Y] = f\cdot[X,Y] + L_X(f)\cdot Y,$$
for all $X,Y \in \mathfrak{X}(M)$, $f \in \mathscr{C}^\infty(M)$.

And then a property that $[\cdot,\cdot]$ has on its own- **the Jacobi identity**:
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \quad \text{for all } X,Y,Z \in \mathfrak{X}(M).$$

Exercise 3.33. On $\mathbb{R}^3$ we consider the vector fields
$$X^1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad X^2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$
Show that:
$$[X^1, X^2] = X^3, \quad [X^2, X^3] = X^1, \quad [X^3, X^1] = X^2$$
in two different ways:

- using the definition of the Lie bracket via commutators of derivations.
- using the properties of the Lie bracket described in Exercise 3.32 (and the fact that the Lie bracket between any two of the standard vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ is zero).

Exercise 3.34 (from the 2019-2020 retake exam). On $\mathbb{R}^2$ compute the Lie bracket $[X,Y]$ of the following vector fields:
$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Exercise 3.35. Consider the following three vector fields on the sphere S^3:
$$V^1_{(x,y,z)} := \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}\right),$$
$$V^2_{(x,y,z)} := \frac{1}{2}\left(-z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t}\right),$$

$$V^3_{(x,y,z)} := \frac{1}{2}\left(-t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}\right).$$

Compute $[V^1,V^2]$, $[V^2,V^3]$ and $[V^3,V^1]$. Also, as a test, check the Jacobi identity for V^1,V^2 and V^3. (But please be aware that we ask you for a computation with vector fields on S^3; so just working formally with symbols may require some explanations- e.g. because simple vector fields like $\frac{\partial}{\partial x}$ are not tangent to S^3. We are not saying the result is not correct at the end, but please explain why what you do is correct. Any way, you should be getting $[V^1,V^2] = V^3$, $[V^2,V^3] = V^1$, $[V^3,V^1] = V^2$.)

Exercise 3.36 (from the 2020-2021 retake exam). Let $E : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, we denote by E_x, E_y and E_z the partial derivatives of E with respect to the variables x,y and z (each a smooth function on $\mathbb{R}^3$), and we construct the following three vector fields on $\mathbb{R}^3$:

$$\widetilde{U} = E_z\cdot\frac{\partial}{\partial y} - E_y\cdot\frac{\partial}{\partial z}, \quad \widetilde{V} = E_x\cdot\frac{\partial}{\partial z} - E_z\cdot\frac{\partial}{\partial x}, \quad \widetilde{W} = E_y\cdot\frac{\partial}{\partial x} - E_x\cdot\frac{\partial}{\partial y}.$$

We also consider

$$M_E := \{(x,y,z) \in \mathbb{R}^3 : E(x,y,z) = 0\}$$

and we assume that, at every point $p \in M_E$, at least one of the three vector fields above does not vanish. Show that:

(a) M_E is a 2-dimensional submanifold of $\mathbb{R}^3$.
(b) $\widetilde{U},\widetilde{V}$ and $\widetilde{W}$ are tangent to M_E at each point $p \in M_E$. Hence they define three vector fields on M_E- and those will be denoted

$$U,V,W \in \mathfrak{X}(M_E).$$

(c) U,V and W span the tangent space T_pM_E at each $p \in M_E$.
(d) as a consequence of (c), at every point $p \in M$, the Lie bracket $[V,W]_p$ and the similar ones are linear combinations of U_p, V_p and W_p. Show that there are smooth functions f,g and h on M_E such that

$$[V,W] = f\cdot U + g\cdot V + h\cdot W,$$

and similarly for the other two Lie brackets.
(e) Consider the projection on the first two factors

$$\mathrm{pr}_{1,2} : M_E \to \mathbb{R}^2, \quad \mathrm{pr}_{1,2}(x,y,z) = (x,y).$$

Describe the points at which $\mathrm{pr}_{1,2}$ fails to be a submersion. And the same for the similar projections $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$.
(f) Show that, around each point $p \in M_E$, at least one of the projections $\mathrm{pr}_{1,2}$, $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$ can be used as a chart, i.e. there exists an open neighborhood U of p and a chart of type of type (U,χ), with χ being the restriction to U of one of those three projections.

3.2.4 *The tangent bundle/the configuration space*

Back to the the very definition of vector fields, here is another way to characterize smoothness. The idea is to

$$TM := \bigsqcup_{p\in M} T_pM = \Big\{(p,v_p) : p \in M, v_p \in T_pM\Big\}$$

and make this into a manifold, so that the smoothness of a set-theoretical vector field X is the same as the smoothness of X viewed as a map between manifolds:

$$X : M \to TM, \quad p \mapsto X_p.$$

To put a smooth structure on TM, i.e. to exhibit charts, we need to understand how to parametrize the points of TM

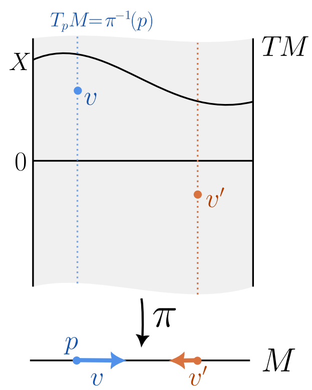


Fig. 3.2 Visualization of the tangent bundle TM of a manifold M. Points of TM correspond to vectors in M. The vertical projection map $\pi : TM \to M$ assigns to every vector $v \in T_pM$ the point p at which it is based; hence each tangent space T_pM is the preimage $\pi^{-1}(p)$, also called the *fiber over* p. The curve in the picture describes a vector field: it pics up a vector in each fiber.

by coordinates. Of course, we should start from a chart

$$\chi : U \to \Omega \subset \mathbb{R}^m$$

for M. The point is that such a chart also allows us to parametrize tangent vectors $v_p \in T_pM$, for $p \in U$, by coordinates $\hat{\chi}_i(v_p)$ w.r.t. to the induced basis $\left(\frac{\partial}{\partial \chi_i}\right)_p$ of T_pM: just decompose v_p as

$$v_p = \sum_i \lambda_i \left(\frac{\partial}{\partial \chi_i}\right)_p \quad \text{and set } \hat{\chi}_i(v_p) := \lambda_i. \tag{3.2.4}$$

Therefore, on the subset $TU \subset TM$, we have natural coordinates, hence a chart:

$$\tilde{\chi} : TU \to \Omega \times \mathbb{R}^m \subset \mathbb{R}^{2m}, \qquad \tilde{\chi}(p,v_p) = (\chi_1(p),\ldots,\chi_m(p),\hat{\chi}_1(v_p),\ldots,\hat{\chi}_m(v_p)).$$

Proposition 3.37. *TM can be made into a manifold in a unique way so that, for any chart (U,χ) of M, $(TU,\tilde{\chi})$ is a chart of TM.*
Moreover, this is also the unique smooth structure with the property that a set-theoretical vector field X on M is smooth if and only if it is smooth as a map $X : M \to TM$.

Proof. We first compute the change of coordinates between charts of type $(TU, \tilde{\chi})$. We start with two charts (U, χ) and (U', χ') for M, with change of coordinates $c = \chi' \circ \chi^{-1}$. We want to compute $\tilde{c} := \tilde{\chi}' \circ \tilde{\chi}^{-1}$. Hence we try to write $\tilde{\chi}'(p)$ as a function depending on $\tilde{\chi}(p)$. While the first m coordinates are $c(\chi(p))$, we need to understand the last m. For that we use:

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_j \frac{\partial c_j}{\partial x_i}(\chi(p)) \left(\frac{\partial}{\partial \chi_i'}\right)_p.$$

(see (2.3.9)). Then, for a tangent vector $p \in M$, replacing this in (3.2.4), we find that

$$\hat{\chi'}_j(v_p) = \sum_i \hat{\chi}_i(v_p) \frac{\partial c_j}{\partial x_i}(\chi(p)).$$

Denoting by $(x, \hat{x})$ the coordinates in $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$, we find that

$$\tilde{c}(x, \hat{x}) = (c(x), \sum_i \hat{x}_i \frac{\partial c}{\partial x_i}(x).$$

Hence the changes of coordinates is smooth.

However, strictly speaking, to show that TM is a manifold, there are still a few things to be done. First of all, we should have made it into a topological space. But, as pointed out in Remark 2.33 from Chapter 2, the topology can actually be recovered from an atlas. Let us give the details, independently of our earlier discussion. Given

$$\text{a chart } \chi : U \to \Omega \text{ for } M, \quad \tilde{\Omega} - \text{open in } \Omega \times \mathbb{R}^m \tag{3.2.5}$$

we would like

$$\tilde{\chi}^{-1}(\tilde{\Omega}) \subset TM$$

to be open in M. We denote by $\mathscr{B}$ the collection of subsets of TM of this type. We will declare a subset $D \subset TM$ to be open if for each $(p, v_p) \in D$ there exists $B \in \mathscr{B}$ such that

$$(p, v_p) \in B \subset D.$$

Of course, what is happening here is that $\mathscr{B}$ is a topology basis, and we consider the associated topology (hence one can also say that a subset of TM is open iff it is an union of members of $\mathscr{B}$).

It should be clear that the resulting topology is Hausdorff. For 2nd countability, note that, in (3.2.5), we can restrict ourselves to χ belonging to an atlas inducing the smooth structure on M and, for each χ, to $\tilde{\Omega}$ belonging to a basis of the Euclidean topology on $\Omega \times \mathbb{R}^m$. As long as the domains of the charts from $\mathscr{A}$ for a basis for the topology of M, the resulting $\mathscr{B}$ would still be a topology basis for the topology we defined on TM. Since M and the $\Omega \times \mathbb{R}^m$s are 2nd countable, we are able to produce a countable basis for the topology on TM.

The fact that a set-theoretical vector field is smooth if and only if it is smooth as a map $X : M \to TM$ should be clear (one just has to check it locally, and then it is really the definition of the smoothness of X). The uniqueness is left as an exercise for the interested student. 😀

Note the object that results from this discussion: TM. It is now a manifold that relates to M by an obvious "projection map"

$$\pi : TM \to M, \quad (p, v_p) \mapsto p.$$

And the fibers $\pi^{-1}(p)$ are vector spaces (precisely the tangent spaces of M). This is precisely what vector bundles are- but we will return to them later on.

3.2.5 *Push-forwards of vector fields*

We now discuss how vector fields can be pushed forward from one manifold to another. The best scenario is when we start with a diffeomorphism $F : M \to N$ between two manifolds; then it induces a **push-forward** operation:

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N), \quad X \mapsto F_*(X)$$

where, for $X \in \mathfrak{X}(M)$, $F_*(X) \in \mathfrak{X}(N)$ is defined by describing how it acts on functions $f \in \mathscr{C}^\infty(N)$:

$$L_{F_*(X)}(f) := L_X(f \circ F) \circ F^{-1} \quad \text{for all } f \in \mathscr{C}^\infty(N). \tag{3.2.6}$$

Or, with a bit more insight: F induces a pull-back operation on functions

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M), \quad f \mapsto f \circ F;$$

this works for any smooth F but, for diffeomorphisms, this is an isomorphism (which is the inverse?). And then one can just use it to transport derivations on $\mathscr{C}^\infty(M)$ to derivations on $\mathscr{C}^\infty(N)$

$$\begin{array}{ccc} \mathscr{C}^\infty(M) & \xrightarrow{L_X} & \mathscr{C}^\infty(M) \\ \uparrow F^* & & \uparrow F^* \\ \mathscr{C}^\infty(N) & \xrightarrow[L_{F_*X}]{\cdots\cdots} & \mathscr{C}^\infty(N) \end{array}$$

And that is precisely what formula (3.2.6) does. You should check yourself that the resulting dotted arrow is, indeed, a derivation, i.e. (3.2.6) does define a vector field $F_*(X) \in \mathfrak{X}(N)$. A more explicit description of $F_*(X)$ (that gives $F_*(X)_q \in T_qN$ for each $q \in N$) is obtained using formula (3.2.3).

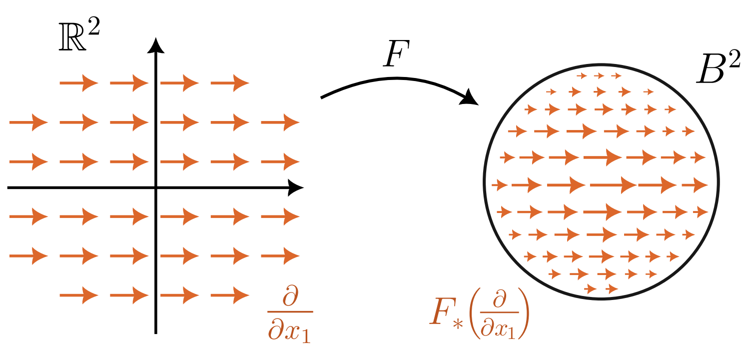


Fig. 3.3 Consider a map $F : \mathbb{R}^2 \to B^2$ that diffeomorphically compresses $\mathbb{R}^2$ into the open disc B^2. The push-forward of the canonical coordinate vector field $\frac{\partial}{\partial x_1}$ along F is exactly the vector field that intuitively gets compressed along with the surrounding space. Can you choose an explicit map F and calculate the components of $F_*(\frac{\partial}{\partial x_1})$ with regard to the canonical identity chart of $B^2 \subseteq \mathbb{R}^2$?

Exercise 3.38. Deduce that, for all $q \in N$,

$$F_*(X)_q = (dF)_p(X_p) \quad \text{where } p = F^{-1}(q). \tag{3.2.7}$$

Also show that, if $G : N \to P$ is another diffeomorphism, then $(G \circ F)_* = G_* \circ F_*$.

Proposition 3.39. *The push forward operation preserves the Lie bracket for any $X_1, X_2 \in \mathfrak{X}(M)$, one has*

$$F_*([X_1, X_2]) = [F_*(X_1), F_*(X_2)].$$

Proof. We start with the definition of $[F_*(X_1), F_*(X_2)]$:

$$L_{[F_*(X_1),F_*(X_2)]}(f) = L_{F_*(X_1)}\left(L_{F_*(X_2)}(f)\right) - L_{F_*(X_2)}\left(L_{F_*(X_1)}(f)\right)$$

(for all $f \in \mathscr{C}^\infty(N)$, in which we plug in the definition of $F_*(X_i)$ to obtain

$$L_{F_*(X_1)}\left(L_{X_2}(f \circ F) \circ F^{-1}\right) - \text{ the similar one} = L_{X_1}\left(L_{X_2}(f \circ F)\right) \circ F^{-1} - \text{ the similar one ;}$$

using again the formula defining $[X_1, X_2]$, and then the one for F_*, the last expression is

$$L_{[X_1,X_2]}(f \circ F) \circ F^{-1} = L_{F_*([X_1,X_2])}(f).$$

Exercise 3.40. By similar arguments define also a pull-back operation $F^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$. And show that this makes sense not only for diffeomorphisms $F : M \to N$, but for all local diffeomorphisms. Also: is it true that $(G \circ F)^* = G^* \circ F^*$?

The construction of push-forwards cannot be extended for arbitrary smooth maps $F : M \to N$. Instead, one can talk about a vector field $X \in \mathfrak{X}(M)$ being "related via F" to a vector field $Y \in \mathfrak{X}(N)$:

Definition 3.41. For a smooth map $F : M \to N$, we say that $X \in \mathfrak{X}(M)$ is **F-projectable** to $Y \in \mathfrak{X}(N)$ if

$$(dF)_p(X_p) = Y_{F(p)} \quad \text{for all } p \in M.$$

Note that, when F is a diffeomorphism, this corresponds precisely to $Y = F_*(X)$. However, in general, there may be different vector fields on M that are F-projectable to the same vector field on N (think e.g. of the case when N is a point) or there may be vector fields on M that are not projectable to any vector field on N. However, Proposition 3.39 still has a version that applies to this general setting:

Exercise 3.42. For any smooth map $F : M \to N$, if $X_i \in \mathfrak{X}(M)$ are F-projectable to $Y_i \in \mathfrak{X}(N)$ for $i \in \{1, 2\}$, then $[X_1, X_2]$ is F-projectable to $[Y_1, Y_2]$. Please prove this locally, using the formula that you obtained when you solved Exercise 3.31.

Exercise 3.43 (from the 2021-2022 exam). Here is a more elegant way to prove the statement from the previous exercise. Show that:

(a) If $X \in \mathfrak{X}(M)$ if F-projectable to $V \in \mathfrak{X}(N)$, then $L_X(F^*(f)) = F^*(L_V(f))$ for all $f \in C^\infty(N)$, where $F^* : C^\infty(N) \to C^\infty(M)$, $F^*(f) = f \circ F$.
(b) Then show that the converse of (a) holds as well.
(c) Deduce that, if $X \in \mathfrak{X}(M)$ is F-projectable to $V \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ is F-projectable to $W \in \mathfrak{X}(N)$ then $[X, Y]$ is F-projectable to $[V, W]$.

Exercise 3.44. Exercise 3.33 and Exercise 3.35 give you three vector fields X^i on S^2, and V^i on S^3, respectively; you have surely noticed that, mysteriously enough, they satisfy the same types of formulas for their Lie brackets. How are they related? Well, here is one more piece of data: to move from S^3 to S^2 you can use the Hopf map h from Exercise 2.209. Do some computations and explain what is going on.
(warning: not to get confused by the symbols, better use different letter for the coordinates in the two spheres- e.g. (x,y,z,t) in S^3 and (u,v,w) in S^2; in particular, the vector fields from Exercise 3.33 will be $V^1 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}$ etc.)

3.2.6 *Extra-for the interested student: cool stuff with vector fields*

Here are some very simple questions about vector fields whose answer is non-trivial question but very exciting. Let us start from the "hairy ball theorem" which can be seen as an interesting property of the two-sphere S^2: it does not admit a nowhere vanishing vector field. The first question in our list is now obvious:

Question: *Which manifolds admit a nowhere vanishing vector field?*

It is worth looking at this question even for the spheres $S^m \subset \mathbb{R}^{m+1}$. Then the question becomes pretty elementary: does there exist a smooth function

$$X : S^m \to \mathbb{R}^{m+1} \setminus \{0\}$$

with the property that

$$x_0 \cdot X_0(x) + x_1 \cdot X_1(X) + \ldots + X_m \cdot X_m(x) = 0 \quad \forall\, x = (x_0, x_1, \ldots, x_m) \in S^{m+1}?$$

For S^1 the answer is clearly yes (think on the picture!) and, as we have already mentioned, for S^2 the answer is no. What about S^3? Well, using quaternions as in Example 2.94 we see that the answer is yes. More precisely, viewing $S^3 \subset \mathbb{H} \cong \mathbb{R}^4$, the multiplication by i, j and k provide three nowhere vanishing (and even linearly independent!) vector fields on S^3:

$$X^i(x) = i \cdot x, \quad X^j(x) = j \cdot x, \quad X^k(x) = k \cdot x.$$

(why are these vector fields on spheres?).

It turns out that, among the spheres, it is precisely the odd dimensional ones that do admit such vector fields. This will be Theorem 5.55.

For general manifolds the answer to the previous question is closely related to the topology of M- namely to the Euler number $\chi(M)$ of M. We will briefly discuss the Euler number later on when we will introduce DeRham cohomology and we will see that $\chi(S^m) = 1 + (-1)^m$. For now, let us mention that this number is, in principle, easy to compute using a triangulation of M, by the Euler formula

$$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces} - \ldots.$$

(using this you should convince yourself that, indeed, $\chi(S^m) = 1 + (-1)^m$). The answer to the previous question is: a compact manifold M admits a no-where vanishing vector field if and only $\chi(M) = 0$!

What about more than one no-where vanishing vector fields? Of course, the question is interesting only if we look for linearly independent ones- and then the number of such vector fields will be bounded from above by the dimension of the manifold. E.g., on S^3, there are at most three such; and the vector field X^i, X^j and X^k described above show that the upper bound can be achieved. More generally:

Question: *An m-dimensional manifold M is said to be parallelizable if it admits m vector fields which, at each point, are linearly independent.*

For instance, we already know that S^1 and S^3 are parallelizable. One possible explanation for this is the fact that both S^1 as well as S^3 are Lie groups (and they are the only spheres that can be made into Lie groups- see Example 2.94 from Chapter 2). More precisely:

Exercise 3.45. Show that all Lie groups are parallelizable. Here is how to do it. Let G be a k-dimensional Lie group. Consider its tangent space at the identity, $\mathfrak{g} := T_eG$ (a k-dimensional vector space). For $v \in \mathfrak{g}$, we define the vector field $\vec{v}$ on G by

$$\vec{v}_g := (dL_g)_e(v)$$

where $L_g : G \to G$ is given by $L_g(h) = gh$ and $(dL_g)_e : \mathfrak{g} \to T_gG$ is its differential.

Show that if $\{v_1, \ldots, v_k\}$ is a basis of the vector space $\mathfrak{g}$, then $\vec{v}_1, \ldots, \vec{v}_k$ are k-linearly independent vector fields on G.

While the only spheres that can be made into Lie groups are S^0, S^1 and S^3, are there any other spheres that are parallelizable? Can one reproduce the argument that we used for S^3 so that it applies to other spheres S^m? We see that we need some sort of "product operation" on $\mathbb{R}^{m+1}$, so that we can define vector fields X^l by

$$X^l(x) = e_l \cdot x, \quad l \in \{1, \ldots, m\}$$

where $\{e_0, e_1, \ldots, e_m\}$ is the canonical basis. We end up again with the question mentioned in Example 2.94 from Chapter 2, of whether $\mathbb{R}^{m+1}$ can be made into a normed division algebra. And, as we mentioned there, the only possibilities were $\mathbb{R}, \mathbb{R}^2$ and R^4 for associative products, and also $\mathbb{R}^8$(octonions) in full generality. However, while the failure of associativity was a problem in making the corresponding sphere into a Lie group, it is not difficult to see that it does not pose a problem in producing the desired vector fields. Therefore: yes, the arguments that used $\mathbb{H}$ works also for octonions, hence also S^7 is parallelizable.

And, as you may expect by now, it turns out that S^0, S^1, S^3 and S^7 are the only spheres that are parallelizable!

So far we have looked at the extreme cases: one single nowhere vanishing vector field, or the maximal number of linearly independent vector fields. Of course, the most general question is:

Question: *Given a manifold M, what is the maximal number of linearly independent vector fields that one can find on M?*

Already for the case of the spheres S^m, this deceivingly simple question turns out to be highly non-trivial. The solution (found half way in the 20th century) requires again the machinery of Algebraic Topology. But here is the answer: write $m+1 = 2^{4a+r}m'$ with m' odd, $a \geq 0$ integer, $r \in \{0,1,2,3\}$. Then the maximal number of linearly independent vector fields on S^m is

$$r(m) = 8a + 2^r - 1.$$

Note that $r(m) = 0$ if m is even- and that corresponds to the fact that there are no nowhere vanishing vector fields on even dimensional spheres. The parallelizability of S^m is equivalent to $r(m) = m$, i.e. $8a + 2^r = 2^{4a+r}m'$, which is easily seen to have the only solutions

$$m = 1, \quad a = 0, \quad r \in \{0,1,2,3\},$$

giving $m = 0, 2, 4, 8$. Hence again the spheres S^0, S^1, S^3 and S^7.

3.2.7 *Application: the Lie algebra of a Lie group*

We now discuss the infinitesimal counterpart of Lie groups.

Definition 3.46. A Lie algebra is a vector space $\mathfrak{a}$ endowed with an operation

$$[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$$

which is bi-linear, antisymmetric and which satisfies the Jacobi identity:

$$[[u,v],w] + [[v,w],u] + [[w,u],v = 0 \quad \text{for all } u,v,w \in \mathfrak{a}.$$

Example 3.47 (commutators of matrices). The space $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices, together with the commutator

$$[A,B] := AB - BA$$

is a Lie algebra. More generally, for any vector space V, the space $\mathrm{Lin}(V,V)$ of linear maps from V can be made into a Lie algebra by the same formula (just that AB becomes the composition of A and B).

Example 3.48 (vector fields). As we have already pointed out, for any manifold M, the space of vector fields $\mathfrak{X}(M)$ endowed with the Lie bracket of vector fields becomes a (infinite dimensional) Lie algebra.

Exercise 3.49. What is the relationship between the previous two examples?

We now explain how a Lie group G gives rise to a finite dimensional Lie algebra $\mathfrak{g}$. As a linear space, it is defined as the tangent space of G at the identity element $e \in G$:

$$\mathfrak{g} := T_eG$$

For the Lie bracket, we will relate $\mathfrak{g}$ to vector fields on G, and then use the Lie bracket of vector fields. Hence, as a first step, we note that any $v \in \mathfrak{g}$ gives rise to a vector field $\vec{v} \in \mathfrak{X}(G)$ defined as follows. For $g \in G$, to define $\vec{v}_g \in T_gG$, we use the left translation by g:

$$L_g : G \to G, \quad L_g(h) = gh.$$

This is smooth (and even a diffeomorphism!) and make use of its differential at e, $(dL_g)_e : \mathfrak{g} \to T_gG$. Define then

$$\vec{v}_g := (dL_g)_e(v) \in T_gG.$$

Exercise 3.50. Show that, indeed, $\vec{v}$ is smooth.

Proposition 3.51. *With the same notations as above (a Lie group G and $\mathfrak{g} := T_eG$), for any $u,v \in \mathfrak{g}$, the Lie bracket $[\vec{u},\vec{v}] \in \mathfrak{X}(G)$ is again of type $\vec{w}$ for some $w \in \mathfrak{g}$. Denoting the resulting w by $[u,v] \in \mathfrak{g}$, we obtain an operation*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that makes $\mathfrak{g}$ into a Lie algebra.

Proof. Note that the map

$$\mathrm{j} : \mathfrak{g} \to \mathfrak{X}(G), \quad v \mapsto \overrightarrow{v} \tag{3.2.8}$$

is injective. That is simply because $\overrightarrow{v}_e = v$. This implies the uniqueness of w; and it also indicates how to define it:

$$w := [\overrightarrow{u}, \overrightarrow{v}]_e.$$

We still have to show that $[\overrightarrow{u}, \overrightarrow{v}] = \overrightarrow{w}$. For that we note that

$$(L_a)_*(\overrightarrow{v}) = \overrightarrow{v}$$

for all $a \in G$ and all $v \in \mathfrak{g}$, where we use the push-forward as defined in subsection 3.2.5. Indeed, using the formula (3.2.7) for the push-forward and then the definition of $\overrightarrow{v}$:

$$(L_a)_*(\overrightarrow{v})_g = (dL_a)_{a^{-1}g}\left((dL_{a^{-1}g})_e(v)\right)$$

and then, using the chain rule and $L_a \circ L_{a^{-1}g} = L_g$, we end up with $\overrightarrow{v}_g$.

Since $(L_a)_*(\overrightarrow{v}) = \overrightarrow{v}$ and similarly for u, Proposition 3.39 implies that $[\overrightarrow{u}, \overrightarrow{v}] = (L_a)_*([\overrightarrow{u}, \overrightarrow{v}])$. Evaluating this at $a \in G$ and using again formula formula (3.2.7) (where $F = L_a, q = a$ so that $p = L_a^{-1}(a) = e$):

$$[\overrightarrow{u}, \overrightarrow{u}]_a = (dL_a)_e([\overrightarrow{u}, \overrightarrow{u}]_e) = (dL_a)_e(w_e) = \overrightarrow{w}_a\ .$$

To fact that the resulting operation $[\cdot,\cdot]$ on $\mathfrak{g}$ is a bilinear and skew-symmetric is immediate. For the Jacobi identity, since the map j from (3.2.8) is injective and satisfies $j([v,w]) = [j(v), j(w)]$, it suffices to remember that the Lie bracket of vector fields does satisfy the Jacobi identity. 😀

> **Definition 3.52. The Lie algebra of the Lie group** G is defined as $\mathfrak{g} = T_eG$ endowed with the Lie algebra structure from the previous proposition.

We now compute some examples.

Example 3.53. We start with the general linear group $GL_n(\mathbb{R})$, and we claim that its Lie algebra is just the algebra $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices endowed with the commutator bracket.

Proof. As we already mentioned, $\mathscr{M}_{n\times n}(\mathbb{R})$ is seen as a Euclidean space; we denote coordinates functions by

$$x^i_j : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

(sending a matrix to the element on the position (i, j)). We also remarked already that, while $GL_n = GL_n(\mathbb{R})$ sits openly inside $\mathscr{M}_{n\times n}(\mathbb{R})$, its tangent space at the identity matrix I (and similarly at any point) is canonically identified with $\mathscr{M}_{n\times n}(\mathbb{R})$: any $X \in \mathscr{M}_{n\times n}(\mathbb{R})$ is identified with the speed at $t = 0$ of $t \mapsto (I + tX)$. After left translating, we find that the corresponding vector field $\overrightarrow{X} \in \mathfrak{X}(GL_n)$ is given, at an arbitrary point $A \in GL_n$, by

$$\overrightarrow{X}_A = \frac{d}{dt}\Big|_{t=0} A \cdot (I + tX) \in T_A GL_n. \tag{3.2.9}$$

Let $X, Y \in \mathscr{M}_{n\times n}(\mathbb{R})$ and let $[X,Y] \in \mathscr{M}_{n\times n}(\mathbb{R})$ be the Lie algebra bracket- hence defined by $\overrightarrow{[X,Y]} = [\overrightarrow{X}, \overrightarrow{Y}]$, where the last bracket is the Lie bracket of vector fields on GL_n. To compute it, we use the defining equation for the Lie bracket of two vector fields:

$$L_{[\overrightarrow{X},\overrightarrow{Y}]}(f) = L_{\overrightarrow{X}} L_{\overrightarrow{Y}}(f) - L_{\overrightarrow{Y}} L_{\overrightarrow{X}}(f) \tag{3.2.10}$$

for all smooth functions

$$f : GL_n \to \mathbb{R}.$$

From the previous description of $\vec{X}$ we deduce that

$$L_{\vec{X}}(f)(A) = \frac{d}{dt}\bigg|_{t=0} f(A\cdot(I+tX)) = \sum_{i,j,k} \frac{\partial f}{\partial x^i_j}(A) A^i_k X^k_j.$$

In particular, applied to a coordinate function $f = x^i_j$, we find

$$L_{\vec{X}}(x^i_j)(A) = \sum_k A^i_k X^k_j$$

or, equivalently, $L_{\vec{X}}(x^i_j) = \sum_k x^i_k X^k_j$. Therefore

$$L_{\vec{Y}}(L_{\vec{X}}(x^i_j)) = \sum_k L_{\vec{Y}}(x^i_k) X^k_j = \sum_{k,l} x^i_l Y^l_k X^k_j = \sum_l x^i_l (Y\cdot X)^l_j,$$

which is precisely $L_{\overrightarrow{Y\cdot X}}(x^i_j)$. Since this holds for all coordinate functions, we have

$$L_{\vec{Y}} \circ L_{\vec{X}} = L_{\overrightarrow{Y\cdot X}};$$

we deduce that $[\vec{X}, \vec{Y}] = \overrightarrow{Y\cdot X} - \overrightarrow{X\cdot Y}$ and then that $[X,Y]$ is $X\cdot Y - Y\cdot X$, i.e. the usual commutator of matrices. 😀

Example 3.54. For the classical subgroups of GL_n from Example 2.97, since $T_I\mathrm{GL}_n = \mathscr{M}_{n\times n}$, their Lie algebras will be subspaces of $\mathscr{M}_{n\times n}(\mathbb{R})$. They can be described explicitly in each case. Let us look here at $O(n)$. When we discuss its smooth structure (in Example 2.98) we made use of the regular value theorem writing $O(n) = F^{-1}(\{I\})$ and computing df at arbitrary points. In particular, since $(dF)_I(X) = X + X^T$, one finds that

$$T_I O(n) = \{X \in \mathscr{M}_{n\times n}(\mathbb{R}) : X + X^T = 0\},$$

the space of skew-symmetric matrices, usually denoted $o(n)$. Similar computations can be done for the other groups, leading to the following list:

- for $O(n)$ and $SO(n)$ we obtain
$$o(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{R}) : X + X^T = 0\}.$$
- for $SL_n(\mathbb{R})$ we obtain
$$sl_n(\mathbb{R}) := \{A \in \mathscr{M}_{n\times n}(\mathbb{R}) : Tr(X) = 0\}.$$
- $U(n)$ we obtain
$$u(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{C}) : X + X^* = 0\}.$$
- for $SU(n)$ we obtain
$$su(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{C}) : X + X^* = 0, Tr(X) = 0\}.$$

For more general closed subgroups $G \subset \mathrm{GL}_n$ (see Theorem 2.100) their Lie algebras appeared implicitly in the proof of Theorem 2.100): it follows that they are precisely the spaces (2.2.13) that were used there

$$T_I G = \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G \text{ for all } t\}. \tag{3.2.11}$$

What about the Lie brackets? We claim that, in all these examples, i.e. when G is one of the classical groups from Example 2.97 or, if you are comfortable with Theorem 2.100, then for all closed subgroups of GL_n, the resulting Lie bracket on $\mathfrak{g} = T_I G$ is still the commutator bracket. Notice that the fact that $\mathfrak{g}$ was closed under the commutator bracket operation was already pointed out in the previous chapter, in Exercise 2.103.

Proof. Note that for $X \in \mathfrak{g}$ we have two induced vector fields: one on G, denoted as above by $\overrightarrow{X} \in \mathfrak{X}(G)$ and then, since $\mathfrak{g} \subset \mathscr{M}_{n\times n}(\mathbb{R})$ we will have a similar one on GL_n; to distinguish the two we will denote the second one by $\overset{\longmapsto}{X} \in \mathfrak{X}(GL_n)$. Note that, denoting by $i : G \hookrightarrow GL_n$ the inclusion, we are in the situation of Definition 3.41: $\overrightarrow{X}$ is i-projectable to $\overset{\longmapsto}{X}$. Hence, by Exercise 3.42, $[\overrightarrow{X}, \overrightarrow{Y}]$ is i-projectable to $[\overset{\longmapsto}{X}, \overset{\longmapsto}{Y}]$ (for all $X, Y \in \mathfrak{g}$). By the previous example, the last expression is $\overset{\longmapsto}{[X,Y]_{\text{comm}}}$ where, for clarity, we denote by $[X,Y]_{\text{comm}}$ the commutator bracket of the matrices X and Y. Hence, for the Lie bracket of X and Y as elements of the Lie algebra $\mathfrak{g}$ of G we find

$$[X,Y] = [\overrightarrow{X}, \overrightarrow{Y}]_I = \left(\overset{\longmapsto}{[X,Y]_{\text{comm}}}\right)_I = [X,Y]_{\text{comm}}.$$ 😀

Remark 3.55 (For the interested students: Lie's fundamental theorems). The Lie algebra $\mathfrak{g}$ of a Lie group G contains (almost) all the information about G! This may sound surprising at first, since $\mathfrak{g}$ is only "the linear approximation" of G and only around the identity $e \in G$. The fact that it is enough to consider only the identity e can be explained by the fact that, using the group structure (and the resulting left translations) one can move around from e to any other element in g. The fact that "the linear approximation" contains (almost) all the information is more subtle and the real content of this is to be found on the Lie bracket structure of $\mathfrak{g}$.

And here is what happens precisely:

- As we have seen, any Lie group has an associated (finite dimensional) Lie algebra.
- Any finite dimensional Lie algebra $\mathfrak{g}$ comes from a Lie group.
- For any finite dimensional Lie algebra $\mathfrak{g}$ there exists and is unique a Lie group $G(\mathfrak{g})$ whose Lie algebra is $\mathfrak{g}$ and which is 1-connected (i.e. both connected as well as simply-connected). The Lie group $G(\mathfrak{g})$ can be constructed explicitly out of $\mathfrak{g}$. And any other connected Lie group which has $\mathfrak{g}$ as Lie algebra is a quotient of $G(\mathfrak{g})$ by a discrete subgroup.
- Explanation for the 1-connectedness condition: starting with a Lie group G, the connected component of the identity element, is itself a Lie group with the same Lie algebra as G. And so is the universal cover of G. I.e. one can always replace G by a Lie group that is 1-connected without changing its Lie algebra.
- the construction $G \mapsto \mathfrak{g}$ gives an equivalence between (the category of) 1-connected Lie groups and (the category of) finite dimensional Lie algebras. At the level of morphisms even more is true: while a morphism of Lie groups $\Phi : G \to H$ induces a morphism of the corresponding Lie algebras, $\phi : \mathfrak{g} \to \mathfrak{h}$, if G is 1-connected (but H is arbitrary), then Φ can be recovered from ϕ (i.e.: any morphism ϕ of Lie algebras comes from a unique morphism Φ of Lie groups!). That is remarkable since ϕ is just the linearization of Φ at the identity $\phi = (d\Phi)_e$.

3.3 The flow of a vector field

The brief philosophy is:

> *If you live on a manifold M (... not necessarily the sphere) and your are sitting at a point p, then a tangent vector $X_p \in T_pM$ gives you a direction in which you can start walking. A vector field however gives you an entire path to walk on. In the ideal situation (no friction etc), and if you think of a vector field as describing the wind blowing, its integral curve at p is the trajectory that you will follow, blown by the wind.*

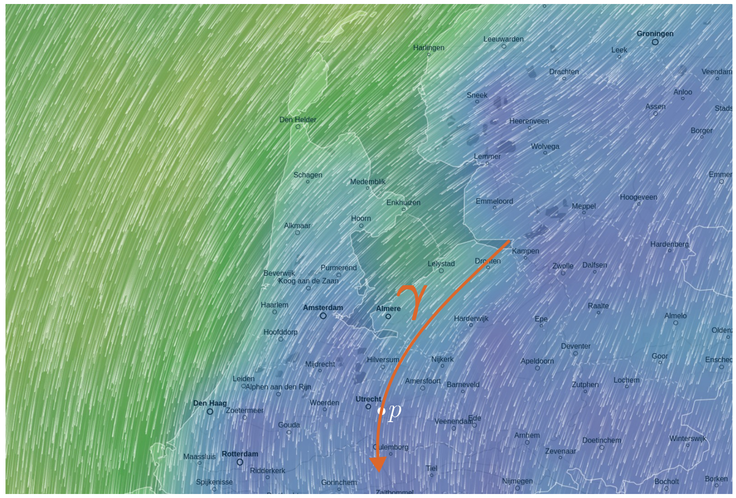


Fig. 3.4 Wind speeds as a vector field together with an integral curve γ passing through the Uithof p.

3.3.1 Integral curves

We first make precise what it means for ”a trajectory (path) to follow a vector field”.

Definition 3.56. Given a vector field $X \in \mathfrak{X}(M)$, **an integral curve of** X is any curve $\gamma : I \to M$ defined on some open interval $I \subset \mathbb{R}$ ($I = \mathbb{R}$ not excluded) such that

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad \text{for all } t \in I.$$

We say that γ **starts at** p if $0 \in I$ and $\gamma(0) = p$.

Example 3.57. Consider $M = \mathbb{R}^2$ and the vector field X given by

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Then a curve in $\mathbb{R}^2$, written as $\gamma(t) = (x(t), y(t))$, is an integral curve if and only if

$$\dot{x}(t) = 1, \quad \dot{y}(t) = 1.$$

Hence we find that the integral curves of X are those of type

$$\gamma(t) = (t + a, t + b)$$

with $a, b \in \mathbb{R}$ constants. Prescribing the starting point of γ is the same thing as fixing the constants a and b.

A more interesting vector field is:

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Then the equations to solve are $\dot{x} = y$ and $\dot{y} = -x$. Hence $\ddot{x} = -x$ from which one can deduce (or guess, and then use the uniqueness recalled below) that

$$x(t) = a\cos t + b\sin t, \quad y(t) = b\cos t - a\sin t,$$

for some (arbitrary) constants $a, b \in \mathbb{R}$; again, prescribing the starting point of γ is the same thing as fixing a and b.

Note that in both examples the integral curves that we were obtaining are defined on the entire $I = \mathbb{R}$. However, this need not always be the case. For instance, for

$$X = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y},$$

starting with any constants $a, b \in \mathbb{R}$ one has the integral curve given by

$$\gamma(t) = \left(\frac{a}{at+1},\ be^{-t} \right);$$

but, for $a > 0$, the largest interval containing 0 on which this is defined is $(-\frac{1}{a}, \infty)$. However, still as above, for any point $p = (a, b) \in \mathbb{R}^2$, one finds an integral curve that starts at p.

In the previous examples, the existence of integral curves starting at a given (arbitrary) point is no accident:

Proposition 3.58 (local existence and uniqueness). *Given a vector field $X \in \mathfrak{X}(M)$:*

1. *for any $p \in M$ there exists an integral curve $\gamma : I \to M$ of X that starts at p.*
2. *any two integral curves of X that coincide at a certain time t_0 must coincide in a neighborhood of t_0.*

Since this is a local statement, we just have to look in a chart. Then, as we shall see, what we end up with is a system of ODEs and the previous proposition becomes the standard local existence/uniqueness result for ODEs. Here are the details. Fix a chart $\chi : U \to \Omega \subset M$ of M around p. Then, as in (3.2.1), X can be written on U as

$$X_p = \sum_{i=1}^{m} X_\chi^i(\chi(p)) \left(\frac{\partial}{\partial \chi_i} \right)_p$$

with the coordinate functions X_χ^i of X w.r.t. χ being smooth functions on Ω. Also, a curve γ in U becomes, via χ, a curve γ_χ in Ω. Since $\gamma(t) = \chi^{-1}(\gamma_\chi(t))$, its derivatives becomes

$$\frac{d\gamma}{dt}(t) = \frac{d\gamma_\chi^i}{dt}(t) \left(\frac{\partial}{\partial \chi_i} \right)_{\gamma(t)}.$$

Hence the integral curve condition becomes the system of ODEs:

$$\frac{d\gamma_\chi^i}{dt}(t) = X_\chi^i(\gamma_\chi^1(t), \ldots, \gamma_\chi^m(t)) \quad i \in \{1, \ldots, m\}$$

or, more compactly,

$$\frac{d\gamma_\chi}{dt}(t) = F_\chi(\gamma_\chi(t)),$$

where $F_\chi = (X_\chi^1, \ldots, X_\chi^m) : \Omega \to \mathbb{R}^m$. And here is the standard result on such ODEs:

Theorem 3.59. *Let $F : \Omega \to \mathbb{R}^m$ be a smooth function. Then, for any $x \in \Omega$, the following ordinary differential equation with initial condition:*

$$\frac{d\gamma}{dt}(t) = F(\gamma(t)), \quad \gamma(0) = x \tag{3.3.1}$$

has a solution γ defined on an open interval containing $0 \in \mathbb{R}$, and any two such solutions must coincide in a neighborhood of 0. Furthermore, for any $x_0 \in \Omega$ there exists an open neighborhood Ω_0 of x_0 in Ω, $\varepsilon > 0$ and a smooth map

$$\phi : (-\varepsilon, \varepsilon) \times \Omega_{x_0} \to \Omega,$$

such that, for any $x \in \Omega_0$, $\phi(\cdot, x) : (-\varepsilon, \varepsilon) \to \Omega$ is a solution of (3.3.1).

We see that this the first part of the theorem immediately implies (well, it is actually the same as) the previous proposition. For a more global version of the proposition, we have to look at maximal integral curves.

Definition 3.60. Given a vector field $X \in \mathfrak{X}(M)$, a **maximal integral curve** of X is any integral curve $\gamma : I \to M$ which admits no extension to a strictly larger interval which is still an integral curve of X.

With this, the local result implies quite easily the following one, with a more global flavour.

Corollary 3.61. *Given a vector field $X \in \mathfrak{X}(M)$, for any $p \in M$ there exists a unique maximal integral curve of X $\gamma_p : I_p \to M$ that starts at p.*

Proof. The main remark is that if γ_1 and γ_2 are two integral curves defined on intervals I_1 and I_2, respectively, then the set I of points in $I_1 \cap I_2$ on which the two coincide is closed in $I_1 \cap I_2$ (because it is defined by an equation) and open by the second part of the previous proposition. Hence, since $I_1 \cap I_2$ is connected, if $I \neq \emptyset$ (i.e. the two coincide at some t), then $I = I_1 \cap I_2$ (i.e. the two must coincide on their common domain of definition). Therefore, to obtain I_p and γ_p we can just put together all the integral curves that start at p. 😀

The best possible scenario is:

Definition 3.62. A vector field $X \in \mathfrak{X}(M)$ is said to be **complete** if all its maximal integral curves are defined on the entire $\mathbb{R}$.

As we shall see below, all the vector fields that are compactly supported (i.e. which vanish outside a compact subset of M) are complete. In particular, when M is compact, all vector fields on M are complete.

Exercise 3.63. Consider the vector fields $X, Y \in \mathfrak{X}(\mathbb{R})$ defined by:

$$X_x = x \left.\frac{\partial}{\partial x}\right|_x \quad \text{and} \quad Y_x = e^x \left.\frac{\partial}{\partial x}\right|_x \quad \text{for } x \in \mathbb{R}.$$

(i) Compute the maximal integral curves of X and conclude that X is complete.
(ii) Verify that

$$\gamma :]-\infty, 1[\to \mathbb{R}, \quad \gamma(t) = -\log(1-t)$$

defines an integral curve of Y. Show that it is maximal and conclude that Y is not complete.
(iii) Consider the diffeomorphism

$$\phi : \mathbb{R} \to \mathbb{R}_{>0}, \quad x \mapsto e^{-x}$$

and compute $\phi_*(Y)$. Use this to again conclude that Y is not complete.

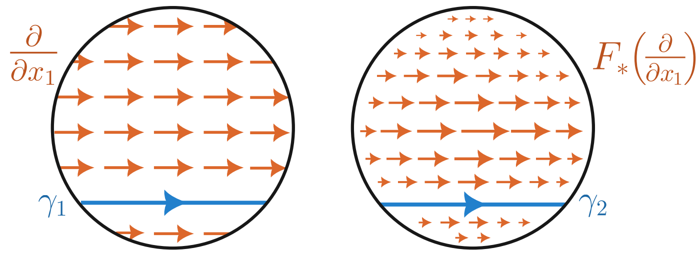


Fig. 3.5 Take M to be the open unit disc in $\mathbb{R}^2$ and consider the vector fields $\frac{\partial}{\partial x_1}$ and $F_*(\frac{\partial}{\partial x_1})$, where F is as in Figure 3.3. Note that the marked maximal integral curves γ_1 and γ_2 with respect to each of these fields have the same image, but must be parametrized differently: γ_1 moves at constant (unit) speed and is defined on a finite open interval, while γ_2 is defined on all of $\mathbb{R}$ and moves slower and slower as one tends towards the boundary (within the ambient $\mathbb{R}^2$) of M. Since we can't extend γ_1, $\frac{\partial}{\partial x_1}$ is not a complete vector field while $F_*(\frac{\partial}{\partial x_1})$ is. As pushing a vector field along a diffeomorphism preserves completeness, $\frac{\partial}{\partial x_1}$ *is* complete when regarded as a vector field on all of $\mathbb{R}^2$. Note that an open boundary within an ambient space from within a manifold looks like it stretches towards infinity, and intuitively, a vector field is complete if it doesn't grow so fast when moving towards that infinity that you reach it in a finite time when you follow along.

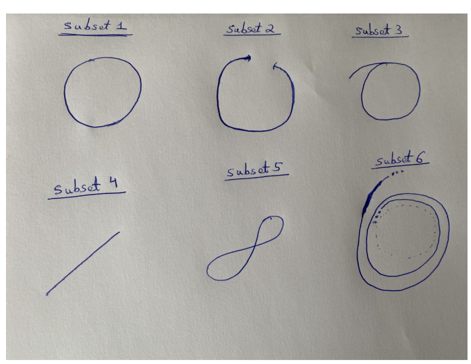
Exercise 3.64. Consider the vector field on the sphere $X \in \mathfrak{X}(S^2)$ given by

$$X_{(x,y,z)} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Draw it in a picture. Then compute all its integral curves and show them in the picture as well. Is there something more general happening (with the integral curves) at points where a vector field vanishes?

Exercise 3.65. If $\gamma : \mathbb{R} \to M$ is an integral curve of a vector field $X \in \mathfrak{X}(M)$ such that there exists $t_0, t_1 \in \mathbb{R}$ distinct such that $\gamma(t_0) = \gamma(t_1)$, show that γ is periodic, i.e. there exists $T \in \mathbb{R}_{>0}$ such that $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$. (Hint: uniqueness of integral curves).

Exercise 3.66 (from the 2019 exam). Which of the following subsets of $\mathbb{R}^2$ can be the image of an integral curve of a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. (Explanation: subset 1 is a circle, subset 2 is an open part of a circle, subset 4 is an infinite line, subset 6 describes a spiral approaching a circle. Please explain you answer; but when you describe a vector field, you do not have to write down a formula- just draw it on the picture)



Exercise 3.67 (from the 2019 exam). On the 2-sphere we consider the following curve:

$$\gamma : \mathbb{R} \to S^2, \quad \gamma(t) = \left(\frac{t}{\sqrt{1+t^2}}, \frac{\cos t}{\sqrt{1+t^2}}, \frac{\sin t}{\sqrt{1+t^2}} \right).$$

(a) draw a picture of γ.
(b) find a vector field X on S^2 (explicit formulas!) for which γ is an integral curve.
(c) did you check how smooth your vector field is? please do!

Exercise 3.68 (from the 2018-2019 exam). This is about vector fields on $\mathbb{R}^3$. Do the following:

(a) find a vector field X for which $\gamma(t) = (e^{2t}, e^t \sin t, e^t \cos t)$ is an integral curve.
(b) find one more such vector field, but different from X.

Exercise 3.69 (from the 2018-2019 retake exam). Let c be an irrational real number and consider the vector field X on T^2 defined as:

$$X = \frac{\partial}{\partial \theta_1} + c \frac{\partial}{\partial \theta_2}.$$

a) Compute the maximal integral curve γ of X starting at $(1,1) \in T^2$.
b) Prove that the image of γ is an immersed submanifold of T^2.
c) Prove that the image of γ is not an embedded submanifold of T^2.

3.3.2 Flows

We now put together all the maximal integral curves in one object: the flow of X. Therefore, we define:

- **the domain of the flow** of X as

$$\mathscr{D}(X) := \{(p,t) \in M \times \mathbb{R} : t \in I_p\} \subset M \times \mathbb{R}.$$

- **the flow of** X is defined as the resulting map

$$\phi_X : \mathscr{D}(X) \to M, \quad \phi_X(p,t) := \gamma_p(t).$$

For complete vector fields one has $\mathscr{D}(X) = M \times \mathbb{R}$. What we can say in general is that

$$M \times \{0\} \subset \mathscr{D}(X) \subset M \times \mathbb{R}.$$

and $\mathscr{D}(X)$ is open in $M \times \mathbb{R}$. Actually, applying the second part of the Theorem 3.59, we deduce:

Corollary 3.70. *For any $X \in \mathfrak{X}(M)$, the domain $\mathscr{D}(X)$ is open in $M \times \mathbb{R}$ and the flow ϕ_X is a smooth map.*

One of the main uses of the flow of a vector field X comes from the maps one obtains whenever one fixes a time t; we will use the notation:

$$\phi_X^t(\cdot) := \phi_X(\cdot, t)$$

The best scenario is when X is complete, when each ϕ^t will be defined on the entire M:

Theorem 3.71. *If $X \in \mathfrak{X}(M)$ is complete then $\phi_X^t : M \times \mathbb{R} \to M$ satisfy:*

$$\phi_X^t \circ \phi_X^s = \phi_X^{t+s}, \quad \phi_X^0 = Id$$

(for all $t, s \in \mathbb{R}$). In particular, each ϕ_X^t is a diffeomorphism and

$$(\phi_X^t)^{-1} = \phi_X^{-t}.$$

Proof. Fixing $x \in M$ and $s \in \mathbb{R}$, the two curves

$$t \mapsto \phi_X^{t+s}(p), \quad t \mapsto \phi_X^t(\phi_X^s(p))$$

are both integral curves of X and they coincide at $t = 0$- hence they coincide everywhere. That ϕ_X^0 is the identity is clear; we see that the last part follows by taking $s = -t$ in the first part.

For general (possibly non-complete) vector fields one has to be a bit more careful.

Definition 3.72. For $X \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$, define **the flow of X at time t** as the map

$$\phi_X^t : \mathscr{D}_t(X) \to M, \quad p \mapsto \phi_X^t(p) := \phi_X(p,t) = \gamma_p(t),$$

defined on:

$$\mathscr{D}_t(X) := \{p \in M : t \in I_p\}.$$

Theorem 3.73. *For any $X \in \mathfrak{X}(M)$:*

1. *for any $t \in \mathbb{R}$, the domain $\mathscr{D}_t(X)$ of ϕ_X^t is open in M, ϕ_X^t takes values in $\mathscr{D}_{-t}(X)$, and*
$$\phi_X^t : \mathscr{D}_t(X) \to \mathscr{D}_{-t}(X)$$
is a diffeomorphism.
2. *For $t, s \in \mathbb{R}$ one has*
$$\phi^t \circ \phi^s = \phi^{t+s}$$
in the sense that, for each $p \in M$ on which the left hand side is defined, also $\phi^{t+s}(p)$ is defined and the two expressions coincide.

Proof. The fact that $\mathscr{D}_t(X)$ is open and ϕ_X^t is smooth follows right away from the previous the standard local result (Theorem 3.59). For the last part we consider again the two curves from the proof of Theorem 3.71; this time we have to be more careful with their domain of definition. Hence we fix $p \in M$, $s \in I(p)$ and we look at the curves:

$$a : -s + I(p) \to M, \quad a(t) = \phi_X^{t+s}(p),$$

$$b : I(\phi_X^s(p)) \to M, \quad b(t) = \phi_X^t(\phi_X^s(p)).$$

Note that both a and b are integral curves of X, and their are both maximal (the second one is maximal from its very definition; for the first one, if $\tilde{a}$ was an extension to $J \supset -s + I(p)$, then $s + J \ni t \mapsto \tilde{a}(t-s)$ would be an integral curve defined on $s + J \supset I(p)$, and starting at p). We then obtain the last part of the theorem together with the equality

$$I(\phi_X^s(p)) = I(p) - s$$

for all s. Since $0 \in I(p)$ we find that $-s \in I(\phi_X^s(p))$, hence

$$\phi_X^s(\mathscr{D}_s) \subset \mathscr{D}_{-s}$$

and, for $p \in \mathscr{D}_s$, $\phi_X^{-s}(\phi_X^s(p)) = p$. We still have to show that, in the previously centered inclusion, equality holds. Using the inclusion for $-s$ instead of s and applying ϕ_X^s we obtain

$$\phi_X^s\left(\phi_X^{-s}(\mathscr{D}_{-s})\right) \subset (\phi_X^s(\mathscr{D}_s);$$

but the first term equals to $\phi_X^0(\mathscr{D}_{-s}) = \mathscr{D}_{-s}$, hence we obtain the reverse inclusion and this finishes the proof. 😀

Exercise 3.74. Show that after multiplying a vector field $X \in \mathfrak{X}(M)$ by a real number λ, one has

$$\mathscr{D}(\lambda \cdot X) = \{(x,t) \in M \times \mathbb{R} : (x, \lambda t) \in \mathscr{D}(M)\}, \quad \phi_{\lambda \cdot X}(x,t) = \phi_X(x, \lambda \cdot t).$$

Exercise 3.75 (from the 2020-2021 retake exam). Does there exists a vector field X on $\mathbb{R}^3$ with flow given by

$$\phi_X^t(x,y,z) = (x+t, y+tz, z)?$$

But a vector field Y with

$$\phi_Y^t(x,y,z) = (x+t, y+tz, z+t)?$$

Exercise 3.76. For the vector field $X \in \mathfrak{X}(S^2)$ that you found in Exercise 3.25, compute its flow. Again, you will probably use the ambient space $\mathbb{R}^3$ to carry out some of the computations (e.g. you may end up computing flows in $\mathbb{R}^3$ and then noticing that they do not leave the sphere). So, again: is there something more general to learn from this?

Exercise 3.77. Consider the vector field V from Exercise 3.23 and the Hopf map $h : S^3 \to S^2$ from Exercise 2.209. Do the following:

(a) Compute the flow of V.
(b) If you did not do it already, write the flow making use of the fact that $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$.
(c) Check that h is constant on the integral curves of V, i.e. $h(\phi_V^t(p)) = h(p)$ for all $p \in S^2$ and all $t \in \mathbb{R}$.
(d) Derive now the main conclusion of Exercise 3.29: that the differential of h kills V.
(e) What can you say about the fibers of the Hopf map h (for each $q \in S^2$ one talks about the fiber of h above q which is simply the pre-image $h^{-1}(p) = \{p \in S^3 : h(p) = q\}$)?

Exercise 3.78 (from the 2018-2019 exam, continuation of Exercise 3.68). Return to Exercise 3.68 and compute the flow of one of the vector fields you found.

Exercise 3.79. If you want more computations of flows as for V in the previous exercise: look also at the vector fields V^1, V^2 and V^3 from Exercise 3.35.

Exercise 3.80 (from the 2018 exam). This is about vector fields on $\mathbb{R}^3$. Do the following:

(a) find a vector field X for which $\gamma(t) = (e^{2t}, e^t \sin t, e^t \cos t)$ is an integral curve.
(b) find one more such vector field, but different from X.
(c) for one of the vector fields you found, compute its flow.

Exercise 3.81. Do again 3.65, but using now the properties of the flow ϕ_X^t.

> **Exercise 3.82 (from the 2020-2021 retake exam).** Show that there exists a vector field $X \in \mathscr{X}(S^2)$ with flow given by
>
> $$\phi_X^t(x,y,z) = \left(\frac{(1+x)e^t - (1-x)e^{-t}}{(1+x)e^t + (1-x)e^{-t}}, \frac{2y}{(1+x)e^t + (1-x)e^{-t}}, \frac{2z}{(1+x)e^t + (1-x)e^{-t}} \right).$$
>
> Make sure you give all the details.

3.3.3 Completeness

The notion of completeness was already mentioned before as "the best possible scenario": $X \in \mathfrak{X}(M)$ is complete if all its maximal integral curves are defined on $\mathbb{R}$ or, equivalently, the domain of its flow is the entire $M \times \mathbb{R}$:

$$\mathscr{D}(X) = M \times \mathbb{R}.$$

Theorem 3.83. *If M is compact then any vector field $X \in \mathfrak{X}(M)$ is complete. More generally, for any M, any $X \in \mathfrak{X}(M)$ that is compactly supported (i.e. is zero outside some compact subset of M) is complete.*

Proof. When M is compact, since $\mathscr{D}(X)$ is an open in $M \times \mathbb{R}$ containing $M \times \{0\}$, the tube lemma implies that there exists $\varepsilon > 0$ such that:

$$M \times (-\varepsilon, \varepsilon) \subset \mathscr{D}(X). \tag{3.3.2}$$

(proof: for any $p \in M$, using that ϕ_X is continuous at $(p,0)$ and that $\mathscr{D}(X)$ is open, find $\varepsilon_p > 0$ and U_p open containing p with $U_p \times (-\varepsilon_p, \varepsilon_p) \subset \mathscr{D}(X)$. Then $\{U_p : p \in M\}$ is an open cover of M hence, by compactness, M is covered by a finite number of them, say the ones corresponding to $p_1, \ldots, p_k \in M$; set ε as the smallest of the epsilons corresponding to the points p_i.)

We claim that the existence of ε such that the inclusion (3.3.2) holds implies (without further using the compactness of M!) that X is complete. Indeed, we would have that $\phi_X^t(p)$ is defined for all $t \in (-\varepsilon, \varepsilon)$ and all $p \in M$. But then so would be $\phi_X^t(\phi_X^t(p))$, hence $\phi_X^{2t}(p)$, hence (3.3.2) holds also with 2ε instead of ε. Repeating the process, we find that $\mathbb{R} \times M \subset \mathscr{D}(X)$, hence X must be complete.

For the last part (M general and X is compactly supported) it is enough to remark that an inclusion of type (3.3.2) can still be achieved: if X is zero outside the compact K then the tube lemma implies that $(-\varepsilon, \varepsilon) \times K \subset \mathscr{D}(X)$ for some $\varepsilon > 0$, while all the points (t, p) with $p \notin K$ are clearly in $\mathscr{D}(X)$ since X vanishes at p (hence the integral curve of X starting at p is simply $\gamma(t) = p$, defined for all t). 😀

For latter reference, let us also cast the last part of the argument into:

Corollary 3.84. *If $X \in \mathfrak{X}(M)$ has the property that $M \times (-\varepsilon, \varepsilon) \subset \mathscr{D}(X)$ for some $\varepsilon > 0$, then X is complete.*

> **Exercise 3.85.** Is the sum of two complete vector fields on $\mathbb{R}^m$ again complete? Or the multiplication of a complete vector field by an arbitrary smooth function?

> **Exercise 3.86 (from the 2020-2021 retake exam, continuation of Exercise 3.82).** Return to Exercise 3.82. Is the vector field that you found complete?

> **Exercise 3.87.** Let $F : M \to N$ be a smooth function. If $X \in \mathfrak{X}(M)$ is F-projectable to $Y \in \mathfrak{X}(N)$ (see Definition 3.41) show that, for any $t \in \mathbb{R}$ one has
>
> $$F(\mathscr{D}_t(X)) \subset \mathscr{D}_t(Y)$$
>
> and, on $\mathscr{D}_t(X)$,
>
> $$F \circ \phi_X^t = \phi_Y^t \circ F.$$

3.3.4 Lie derivatives along vector fields

The general philosophy of taking Lie derivatives L_X along vector fields is rather simple:

for any "type of objects" ξ on manifolds M (functions, vector fields, etc etc), which are natural (in the sense that a diffeomorphism $\phi : M \to N$ allows one pull-back the objects from N to ones on M), the Lie derivative $\mathscr{L}_X(\xi)$ of ξ along X measures the variation of ξ along the flow of X:

$$\mathscr{L}_X(\xi) := \left.\frac{d}{dt}\right|_{t=0} (\phi_X^t)^*(\xi) = \lim_{t\to 0} \frac{(\phi_X^t)^*(\xi) - \xi}{t}. \tag{3.3.3}$$

This is a very general principle that is applied over and over in Differential Geometry, depending on the "type of objects" one is interested in. Here we illustrate this principle for functions and vector fields, and compute the outcome. For functions, there is an obvious pull-back operation associated to any smooth function $F : M \to N$ (diffeomorphism or not):

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M), \quad F^*(f) = f \circ F.$$

Therefore, when $F = \phi_X^t : M \to M$ is the flow of a complete vector field, the guiding equation (3.3.3) for $\xi = f \in \mathscr{C}^\infty(M)$ makes sense pointwise as

$$\mathscr{L}_X(f)(p) := \left.\frac{d}{dt}\right|_{t=0} f(\phi_X^t(p)). \tag{3.3.4}$$

Even more, written in this way, the completeness of X is not even necessary for this formula to make sense (indeed, we only need $\phi_X^t(p)$ to be defined for t near 0, which is always the case).

Exercise 3.88. For the vector field $X \in \mathfrak{X}(S^2)$ from Exercise 3.25 you have computed its flow in Exercise 3.76. Take $f(x,y,z) = x + y^2 + z^3$ interpreted as a smooth function on S^2 and compute now $\mathscr{L}_X(f)$. Did you get the same result as in Exercise 3.28?

Proposition 3.89. *For $X \in \mathfrak{X}(M)$ and $f \in \mathscr{C}^\infty(M)$, $\mathscr{L}_X(f)$ defined by (3.3.4) is precisely $L_X(f) \in \mathscr{C}^\infty(M)$.*

Proof. The expression in the right hand side of (3.3.4) is precisely

$$(df)_p \left(\left.\frac{d}{dt}\right|_{t=0} \phi_X^t(p) \right) = (df)_p(X_p) = L_X(f)(p).$$

The discussion is similar for vector fields. In this case, a diffeomorphism $F : M \to N$ allows one to pull-back vector fields on N to vector fields on M,

$$F^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$$

where, for $Y \in \mathfrak{X}(N)$, its pull-back $F^*(Y) \in \mathfrak{X}(M)$ is defined by

$$F^*(Y)_p := (dF^{-1})_{F(p)}(Y_{F(p)}).$$

With this, when $F = \phi_X^t : M \to M$ is the flow of a complete vector field, the guiding equation (3.3.3) for $\xi = Y \in \mathfrak{X}(M)$ makes sense pointwise as

$$\mathscr{L}_X(Y)(p) := \frac{d}{dt}\bigg|_{t=0} (d\phi_X^{-t})_{\phi_X^t(p)}(Y_{\phi_X^t(p)}). \tag{3.3.5}$$

And, as for functions, this formula makes sense without any completeness assumption on X.

Proposition 3.90. *For $X \in \mathfrak{X}(M)$ and any other $Y \in \mathfrak{X}(M)$, $\mathscr{L}_X(Y)$ defined by (3.3.5) is precisely the Lie bracket $[X,Y] \in \mathfrak{X}(M)$.*

Proof. Let $Z = \mathscr{L}_X(Y)$. For $f \in \mathscr{C}^\infty(M)$ we compute $L_Z(f)(p) = (df)_p(Z_p)$ and we find

$$\frac{d}{dt}\bigg|_{t=0} (df)_p\left((d\phi_X^{-t})_{\phi_X^t(p)}(Y_{\phi_X^t(p)})\right) = \frac{d}{dt}\bigg|_{t=0} d\left(f\circ\phi_X^{-t}\right)_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right).$$

Note that $f\circ\Phi_X^{-t}$ is f at $t=0$ and has the derivative w.r.t. t at $t=0$ equal to $-L_X(f)$, hence

$$f\circ\Phi_X^{-t} = f - tL_X(f) + t^2\cdot ?, \tag{3.3.6}$$

where ? is smooth. Continuing the computation started above, we find

$$\frac{d}{dt}\bigg|_{t=0} (df - td\,(L_X(f)))_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right) = \frac{d}{dt}\bigg|_{t=0} (df)_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right) - (dL_X(f))\,(Y_p) =$$

$$\frac{d}{dt}\bigg|_{t=0} L_Y(f)(\phi_X^t(p)) - L_Y(L_X(f))(p) = L_X(L_Y(f))(p) - L_Y(L_X(f))(p),$$

i.e. $L_Z(f)(p) = L_{[X,Y]}(f)(p)$ for all p and f. This implies that $Z = [X,Y]$. 😀

We obtain the following characterization of the commutation relation $[X,Y]=0$; for simplicity, we restrict here to the case of complete vector fields.

Corollary 3.91 (pairwise commuting vector fields). *For $X,Y \in \mathfrak{X}(M)$ complete, one has:*

$$[X,Y]=0 \quad\Longleftrightarrow\quad \Phi_X^t\circ\Phi_Y^s = \Phi_Y^s\circ\Phi_X^t \quad \forall\, t,s\in\mathbb{R}.$$

Exercise 3.92. Returning to Exercise 3.33 compute $[X^1,X^2]$ in yet another way, now using flows (that you have to compute!).

Exercise 3.93 (from the 2023/2024 exam). If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, show that $\mathscr{L}_X(f)=0$ if and only if f is constant along the integral curves of X, i.e., for any integral curve $\gamma: I \to M$ of X, $f\circ\gamma$ is constant.

Exercise 3.94. Using the previous Proposition and Exercise 3.87 prove again that if $F: M\to N$ is smooth and $X_1,X_2 \in \mathfrak{X}(M)$ are F-projectable to Y_1 and $Y_2 \in \mathfrak{X}(N)$, respectively, then $[X_1,X_2]$ is F-projectable to $[Y_1,Y_2]$.

3.3.5 Application: the exponential map for Lie groups

Proposition 3.95. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $u \in \mathfrak{g}$, the vector field $\vec{u}$ is complete and its flow $\phi^t_{\vec{u}}$ satisfies*

$$\phi^t_{\vec{u}}(ag) = a\phi^t_{\vec{u}}(g).$$

In particular, the flow can be reconstructed from what it does at time $t = 1$, at the identity element of G, i.e. from

$$exp : \mathfrak{g} \to G, \quad exp(u) := \phi^1_{\vec{u}}(e),$$

by the formula

$$\phi^t_{\vec{u}}(a) = a\, exp(t\, u).$$

Moreover, the exponential is a local diffeomorphism around the origin: it sends some open neighborhood of the origin in $\mathfrak{g}$ diffeomorphically into an open neighborhood of the identity matrix in G.

Proof. For the first part the key remark is that if $\gamma : I \to G$ is an integral curve of $\vec{u}$ defined on some interval I, i.e. if

$$\frac{d\gamma}{dt}(t) = \vec{u}\,(\gamma(t)) \quad \forall\, t \in I$$

then, for any $a \in G$, the left translate $a \cdot \gamma : t \mapsto a \cdot \gamma(t)$ is again an integral curve (of the same $\vec{u}$); this follows by writing $a \cdot \gamma = L_a \circ \gamma$ and using the chain rule. Hence, if γ_g is the maximal integral curve starting at $g \in G$ then $a \cdot \gamma_g$ will be the one starting at $a \cdot g$- and that proves the first identity (for as long as both terms are defined). And we also obtain that $I_g = I_{ag}$ for all $a, g \in G$, hence $I_g = I_e$ for all $g \in G$. Using Corollary 3.84, we obtain that $\vec{u}$ is complete.

We are left with the last part. For that it suffices to show that $(d\exp)_0$ is an isomorphism. But

$$(d\exp)_0 : \mathfrak{g} \to T_eG = \mathfrak{g}$$

sends $u \in \mathfrak{g}$ to

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tu) = \left.\frac{d}{dt}\right|_{t=0} \phi^t_{\vec{u}}(e) = \vec{u}_e = u,$$

i.e. the differential is actually the identity map. 😀

Example 3.96. In Example 3.53 we have seen that the Lie algebra of the general linear group $GL_n(\mathbb{R})$ is just the algebra $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices endowed with the commutator bracket. We claim that the resulting exponential map is just the usual exponential map for matrices:

$$\mathrm{Exp} : \mathscr{M}_{n\times n}(\mathbb{R}) \to GL_n(\mathbb{R}),\ X \mapsto e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

(where we work with the conventions that $X^0 = I$ the identity matrix and $0! = 1$). Let us compute $\exp(X)$ for an arbitrary $X \in \mathscr{M}_{n\times n}(\mathbb{R}) = T_I GL_n$; according to the definition, it is $\gamma(1)$, where γ is the unique path in GL_n satisfying

$$\gamma(0) = I,\ \frac{d\gamma}{dt}(t) = \vec{X}\,(\gamma(t)).$$

The last equality takes place in $T_{\gamma(t)}GL_n$, which is canonically identified with $\mathscr{M}_{n\times n}(\mathbb{R})$ (as in Example 3.53 , $Y \in \mathscr{M}_{n\times n}(\mathbb{R})$ is identified with the speed at $\varepsilon = 0$ of $\varepsilon \mapsto \gamma(t) + \varepsilon Y$). By this identification, $\frac{d\gamma}{dt}(t)$ goes to the usual derivative of γ (as a path in the Euclidean space $\mathscr{M}_{n\times n}(\mathbb{R})$) and $\vec{X}\,(\gamma(t))$ goes to $\gamma(t) \cdot X$. Hence γ is the solution of

$$\gamma(0) = I,\ \gamma'(t) = X \cdot \gamma(t),$$

which is precisely what $t \mapsto e^{tX}$ does. We deduce that $\exp(X) = e^X$.

Given the fact that the main ingredient in proving Theorem 2.100 was the exponential for matrices, which we have just generalized, it is now just a simple observation that the same proof applied in the greater generality, giving rise to:

Theorem 3.97. *Any closed subgroup of a Lie group is automatically an embedded submanifold and, therefore, becomes a Lie group.*

3.4 More exercises

Exercise 3.98 (from the 2023/2024 exam). Let M be a manifold. Recall that, when X is a vector field, ϕ_X^t denotes the flow of X. Show that for any vector field X one has

(a) $\phi_{cX}^t = \phi_X^{ct}$ for any constant $c \in \mathbb{R}$ and all t for which one of the sides is defined.
(b) if M is compact then $\phi_{5X}^t\left(\phi_X^{-t}(p)\right) = \phi_{2X}^t\left(\phi_X^{2t}(p)\right)$ for all $p \in M$ and all $t \in \mathbb{R}$.

Exercise 3.99. In this exercise $\gamma : \mathbb{R} \to M$ is an integral curve of a vector field $X \in \mathfrak{X}(M)$ and we assume that γ is periodic, in the sense that there exists $T \in \mathbb{R}_{>0}$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Such a T will be called a period of γ.

1. Show that any closed subgroup Γ of $(\mathbb{R},+)$, different from $\mathbb{R}$, must be of type $\mathbb{Z}\cdot r_0$ for some $r_0 \in \mathbb{R}$.
2. Show that one can talk about "the period of γ", i.e. there exists a smallest period T_0 of γ.
3. With T_0 as above, show that
$$\overline{\gamma} : S^1 \to M, \quad e^{i\cdot\theta} \mapsto \gamma\left(\frac{T\cdot\theta}{2\pi}\right)$$
is an embedding of S^1 in M.

Exercise 3.100. Assume that $\{X^1,\dots,X^n\}$ are n vector fields on the n-dimensional manifold M, which are pairwise commuting (i.e. $[X^i,X^j] = 0$ for all i and j) , and which are linearly independent at some point $x_0 \in M$. Show that there exists a chart (U,χ) near x_0 such that

$$X_x^1 = \left(\frac{\partial}{\partial\chi_1}\right)_x, \quad \dots \quad, \left(\frac{\partial}{\partial\chi_n}\right)_x \quad \text{for all } x \in U.$$

More precisely, show that:

(a) for $\varepsilon > 0$ small enough,
$$F : (-\varepsilon,\varepsilon)^n \to M, \quad F(t_1,\dots,t_n) = \Phi_{X^1}^{t_1}\left(\Phi_{X^2}^{t_2}\left(\dots\left(\Phi_{X^n}^{t_n}(x_0)\right)\dots\right)\right)$$
is smooth and its differential at 0 sends $\left(\frac{\partial}{\partial x_i}\right)_0$ to $X^i(x_0)$.
(b) for ε small enough, F is a diffeomorphism into an open $U \subset M$.
(c) the inverse χ of F defines a chart with the desired property.

Exercise 3.101 (from the 2021/2022 retake). Assume that M is a compact manifold, $F : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is F-projectable to $\frac{\partial}{\partial\varphi} = -y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$. Show that:

(a) For each $\alpha \in \mathbb{R}$, the fiber

$$M_\alpha := \{p \in M : F(p) = e^{i\alpha}\}$$

is an embedded submanifold of M.

(e) $\phi_V^t(p) \in M_{\alpha+t}$ for any $p \in M_\alpha$, and that the flow of V gives rise to diffeomorphisms

$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad (\text{for all } t, \alpha \in \mathbb{R}).$$

Exercise 3.102 (from the 2023/2024 exam). Consider

$$M := \{(x,y,z,u) \in \mathbb{R}^4 : (x^2+y^2+z^2-1)^2 + (xy-u)^2 = 0\}$$

(a) Show that M is a 2-dimensional embedded submanifold of $\mathbb{R}^4$

(b) Show that the following define vector fields on M:

$$V_1 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + yz\frac{\partial}{\partial u}$$

$$V_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + xz\frac{\partial}{\partial u}$$

(c) Compute $[V_1, V_2]$

(d) Compute the flow of the vector field on M given by

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (y^2 - x^2)\frac{\partial}{\partial u}.$$

(i) (1pt) Find a diffeomorphism $F : S^2 \to M$ and compute the pull-backs by F of V_1, V_2 and V.

Exercise 3.103 (from the 2022-2023 exam). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x,y,z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.

d) Here are three more subspaces of $\mathbb{R}^3$ that contain γ:

$$\{(x,y,z) \in \mathbb{R}^3 : x = z^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : y = z^3\}.$$

Among them, only one is not a submanifold of $\mathbb{R}^3$. Which one, and why is it not?

e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.

f) Compute $[X, Y]$.

g) $^{1\text{pt}}$ Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.

Exercise 3.104 (from the 2019-2020 exam, continuation). Return to M from Exercise 2.194 and please do the following:

a) Find a vector field $V \in \mathfrak{X}(M)$ with the property that its flow satisfies

$$\phi_V^{2s}(p_0) = (4\cos s, 4\sin s, -3\sin s, 3\cos s).$$

b) Show that also

$$X^1 := \frac{1}{2}\left(y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} + t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right)$$

defines a vector field tangent to M and compute $X^2 := [X^1, V], X^3 := [X^1, X^2]$.

c) Consider the spheres centred at the origin

$$S_r^3 = \{(x,y,z,t) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + t^2 = r^2\}$$

(one for each $r > 0$), and show that their intersections with M

$$M_r := M \cap S_r^3,$$

are embedded submanifolds of $\mathbb{R}^4$ (be aware that, depending on how you approach this, you may end up doing some very time consuming computations ...).

d) When $r = 5$ (so that $p_0 \in M_r$) compute the tangent space of M_5 at p_0.

e) Show that the following defines a submersion from M_5 to S^1:

$$F : M_5 \to S^1, \quad F(x,y,z,t) = (x - t, y + z).$$

f) Show that M_5 is diffeomorphic to a torus.

Exercise 3.105 (from the 2023/2024 retake; adaptation). For any $= (x,y,z) \in \mathbb{R}^3$ we consider the following vector subspace of $T_p\mathbb{R}^3$:

$$\mathscr{F}_p := \left\{ a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p + c\left(\frac{\partial}{\partial z}\right)_p : a,b,c \in \mathbb{R} \text{ satisfying } 2x^3a + yb + zc = 0 \right\}.$$

Please do the following:

(a) Show that there exist 2-dimensional embedded submanifolds $M \subset \mathbb{R}^3$ satisfying $T_pM \subset \mathscr{F}_p$ for all $p \in M$.

From now on we assume that such a manifold M is fixed.

(b) Show that, for any $p \in M$, one has $T_pM = \mathscr{F}_p$.

(c) Show that the following define vector fields on M:

$$Y_p = y\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial y}, \quad (\text{for } p = (x,y,z) \in M).$$

$$Z_p = z\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial z}, \quad (\text{for } p = (x,y,z) \in M).$$

(d) Compute $V = [Y,Z]$. More precisely, show that V is of type $cx^n\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)$, where the constants c and n have to be computed.

(e) Compute the integral curves of V.

(f) Is V necessarily complete?

Exercise 3.106 (from the 2021-2022 exam). We consider the space of two by two matrices with real coefficients, identified to the Euclidean space $\mathbb{R}^4$ with coordinate functions denoted by x,y,z,t:

$$N := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x,y,z,t \in \mathbb{R} \right\}$$

and, inside it, the space of matrices of determinant 1:

$$M := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : xt - yz = 1 \right\}.$$

First show that:

(a) M is an embedded submanifold of N.
(b) The following define vector fields tangent to M:

$$V^1 := z\frac{\partial}{\partial x} + t \cdot \frac{\partial}{\partial y}$$

$$V^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}.$$

$$V^3 = -y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$$

(c) $F : M \to \mathbb{R}^2, \quad f\begin{pmatrix} x & y \\ z & t \end{pmatrix} = (z,t)$ is a submersion.
(d) Because t is used to denote the last coordinate in M, the time-parameter of curves will be denoted by s. Compute the flows of the vector fields V_1, V_2 and V_3 and show they are of type

$$\phi^s_{V_1}(A) = A_1(s) \cdot A, \quad \phi^s_{V_2}(A) = A_2(s) \cdot A, \quad \phi^s_{V_3}(A) = A \cdot A_3(s)$$

where $A_i(s)$ are matrices that depend on s. For instance, you should get

$$\phi^s_{V_1} = \begin{pmatrix} x + s \cdot z & y + s \cdot t \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad \text{hence } A_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

(e) Show that $[V_1, V_3] = 0$ and $[V_2, V_3] = 0$.
(f) Show that $[V_1, V_2] = 2V_1$.
(g) Find another vector field V on M which admits the following integral curve:

$$\gamma(s) = A_1(s) \cdot A_3(s) = \begin{pmatrix} \cos s - s \cdot \sin s & \sin s + s \cdot \cos s \\ -\sin s & \cos s \end{pmatrix}$$

(n) Consider the following parametrization of M:

$$\begin{cases} x = r \cdot \cos\alpha - \frac{u}{r}\sin\alpha & y = r \cdot \sin\alpha + \frac{u}{r}\cos\alpha \\ z = -\frac{1}{r}\sin\alpha & t = \frac{1}{r}\cos\alpha \end{cases} \qquad u \in \mathbb{R},\ r \in (0,\infty),\ \alpha \in (0,2\pi)$$

which we interpret as the inverse χ^{-1} of a chart of M. What is the domain U of χ?
(o) Hence the u,r,α determined by the equations above (all functions of (x,y,z)) are precisely the components $\chi_1(x,y,z)$, $\chi_2(x,y,z)$ and $\chi_3(x,y,z)$ of $\chi(x,y,z)$. Show that the resulting vector field $\frac{\partial}{\partial \chi_1}$ coincides with V_1 at all points $p \in U$.
(p) Find a function

$$f^\mu_\chi \in C^\infty(\mathbb{R} \times (0,\infty) \times (0,2\pi))$$

such that $\theta_1 \wedge \theta_1 \wedge \theta_2 = \chi^*(f^\mu_\chi \cdot du \wedge dr \wedge d\alpha)$.

Exercise 3.107 (from the 2017/2018 retake). Assume that M is an S^1-manifold, i.e. a manifold together with an action of the circle S^1 from the right- i.e. a smooth map

$$M \times S^1 \to M, \quad (\lambda, p) \mapsto \lambda \cdot p$$

such that $p \cdot 1 = p$ and $(p \cdot \lambda) \cdot \lambda' = p \cdot (\lambda\lambda')$ for all $\lambda, \lambda' \in S^1$, $p \in M$. We define the following vector field on M:

$$V_p = \frac{d}{dt}\Big|_{t=0} p \cdot e^{2\pi it}. \tag{3.4.1}$$

(a) Write down the flow of V using the action of S^1 on M.
(b) Deduce that a point $p \in M$ is a fixed point of the action, i.e. satisfies $p \cdot \lambda = p$ for all $\lambda \in S^1$, if and only if p is a zero of V, i.e. satisfies $V_p = 0$.
(c) Assume now that the action is free i.e. that it has no fixed points (or, cf. the previous point, V has no zeroes). Show that for each $p \in M$, the orbit through p:

$$\mathscr{O}_p := \{p \cdot \lambda : \lambda \in S^1\} \tag{3.4.2}$$

is an embedded submanifold of M diffeomorphic to S^1.

Exercise 3.108 (from the 2021/2022 retake). On $\mathbb{R}^3$ we define the following operation ("product"):

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and we consider the associated left translations, one for each $p \in \mathbb{R}^3$,

$$L_p : \mathbb{R}^3 \to \mathbb{R}^3, \quad L_p(q) := p \star q.$$

We define the vector fields $X^1, X^2, X^3 \in \mathfrak{X}(\mathbb{R}^3)$ which, at $p \in \mathbb{R}^3$, are obtained by applying the differential of L_p at the origin, $(dL_p)_0 : T_0\mathbb{R}^3 \to T_p\mathbb{R}^3$, to

$$\left(\frac{\partial}{\partial x}\right)_0, \quad \left(\frac{\partial}{\partial y}\right)_0, \quad \left(\frac{\partial}{\partial z}\right)_0.$$

(a) Compute X^1, X^2, X^3 explicitly and show that they are dual to

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = -xdy + dz,$$

i.e. $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise).
(b) Show that $[X^1, X^2] = X^3, [X^2, X^3] = 0, [X^3, X^1] = 0$.
(c) Find a vector field V such that $\theta^1(V) = 1$, $\theta^2(V) = 0$ and which commutes with X^1, X^2, X^3 (i.e. $[V, X^1] = 0$, $[V, X^2] = 0$, $[V, X^3] = 0$).
(d) For V that you found in (c), compute the flow ϕ_V^t of V explicitly.
(e) Show that ϕ_V^t preserves θ_3, i.e. $(\phi_V^t)^*\theta_3 = \theta_3$.
(f) Deduce that $\mathscr{L}_V(\theta_3) = 0$.
(g) Prove directly (by an algebraic computation) that $\mathscr{L}_V(\theta_3) = 0$.

Exercise 3.109 (part of the 2019/2020 retake exam). The 2019/2020 retake exam consisted on 20 questions on the 2-torus

$$\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$$

(equipped with the product manifold structure). Here one considers the vector field $\frac{\partial}{\partial \theta_1}$ given by

$$\left(\frac{\partial}{\partial\theta_1}\right)_{(z_1,z_2)} := \frac{d}{dt}\Big|_{t=0}(e^{i\cdot t}\cdot z_1, z_2) \in T_{(z_1,z_2)}\mathbb{T}^2,$$

and similarly we define $\frac{\partial}{\partial\theta_2}$. For any smooth function $f : \mathbb{T}^2 \to \mathbb{R}$, we use the notations $\frac{\partial f}{\partial\theta_i}$ for the resulting Lie derivatives (again functions on the torus):

$$\frac{\partial f}{\partial\theta_1}(z_1,z_2) = \frac{d}{dt}\Big|_{t=0} f(e^{i\cdot t}\cdot z_1, z_2), \quad \text{etc.}$$

We will also consider the more general vector fields on the torus:

$$X^f := \frac{\partial}{\partial\theta_1} + f\cdot\frac{\partial}{\partial\theta_2} \in \mathfrak{X}(\mathbb{T}^2). \qquad (*)$$

defined for any smooth function f on $\mathbb{T}^2$.

Recall also that for any $R > r > 0$, one has a "concrete model" of $\mathbb{T}^2$:

$$T^2_{R,r} := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\},$$

which is the image of the map

$$F : \mathbb{T}^2 \to \mathbb{R}^3, \quad (e^{i\theta_1}, e^{i\theta_2}) \mapsto ((R + r\cdot\cos\theta_1)\cdot\cos\theta_2, (R + r\cdot\cos\theta_1)\cdot\sin\theta_2, r\cdot\sin\theta_1).$$

You may assume (e.g. from Topology) that you know already that F is a homeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$. Here are eight questions from the exam:

(a) For $i \in \{1,2\}$ compute $(dF)_p(\frac{\partial}{\partial\theta_i})$ at arbitrary points $p = (e^{i\cdot\theta_1}, e^{i\cdot\theta_2}) \in \mathbb{T}^2$ and show that F is an immersion at each point.

(b) Deduce that F is a diffeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.

(c) We want to compute the vector fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ by moving them to $T^2_{R,r}$ (via F) and decomposing them w.r.t. the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ of the tangent spaces of $\mathbb{R}^3$. In other words, we want to compute $F_*\left(\frac{\partial}{\partial\theta_1}\right)$ and $F_*\left(\frac{\partial}{\partial\theta_2}\right)$. Show that

$$F_*\left(\frac{\partial}{\partial\theta_2}\right)_q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \text{for all } q = (x,y,z) \in T^2_{R,r},$$

and find a similar formula for $F_*\left(\frac{\partial}{\partial\theta_1}\right)$.

(d) Compute the flows of $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ and then deduce that $[\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}] = 0$.

(e) Looking at the vector fields of type $(*)$, show that for any two smooth functions f, g on $\mathbb{T}^2$, there exists another smooth function h such that $\frac{\partial}{\partial\theta_1} + [X^f, X^g] = X^h$.

(f) For each real number $\lambda \in \mathbb{R}$ consider the vector field X^λ (i.e. $(*)$ obtained when f is the constant function λ). Compute the maximal integral curve γ of X^λ starting at $(1,1) \in \mathbb{T}^2$.

(g) With γ as above (and λ arbitrary constant) show that:

  (h) the image of γ is always an immersed submanifold of $\mathbb{T}^2$
  (i) but, if λ is irrational, then that image is not an embedded submanifold of $\mathbb{T}^2$.

(j) Show that there do not exist smooth functions f, g on $\mathbb{T}^2$ such that $\frac{\partial g}{\partial\theta_1} - \frac{\partial f}{\partial\theta_2} = 1$.

Chapter 4
Differential forms

4.1 Differential forms of degree one (1-forms)

4.1.1 Cotangent vectors

For a (real) vector space V, its dual V^* is the vector space consisting of all linear maps $\xi : V \to \mathbb{R}$, also called covectors of V. Vectors and covectors pair via the evaluation map

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}, \quad (\xi, v) \mapsto \langle \xi, v \rangle := \xi(v).$$

Moreover, any basis $e_1, \ldots, e_m$ of V gives rise to the **dual basis**- the basis $e^1, \ldots, e^m$ of V^* determined by

$$\langle e^i, e_j \rangle = \delta^i_j.$$

Explicitly, $e^i : V \to \mathbb{R}$ takes $v \in V$ to the i-the coefficient of v w.r.t. the given basis. In particular, $\dim V^* = \dim V$. Differential 1-forms on a manifold M arise similarly, in duality with vector fields. First of all, for each $p \in M$,

> we define **the cotangent space** of M at p as the dual of the tangent space
>
> $$T^*_p M := (T_p M)^*.$$
>
> The elements of $T^*_p M$ are called **cotangent vectors at** p.

Example 4.1. For any smooth function $f : M \to \mathbb{R}$ its differential at any point $p \in M$,

$$(df)_p : T_p M \to \mathbb{R}, \quad v = \frac{d\gamma}{dt}(0) \mapsto (df)_p(v) = \partial_v(f) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)),$$

can now be interpreted as a cotangent vector at p, $\boxed{(df)_p \in T^*_p M}$.

While a chart χ around p gives rise to a basis of $T_p M$,

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \tag{4.1.1}$$

it will also give rise to a basis of $T^*_p M$, denoted

$$(d\chi_1)_p, \ldots, (d\chi_m)_p. \tag{4.1.2}$$

As the notation suggests, each member $(d\chi_i)_p \in T^*_p M$ is obtained by applying the previous example to the component $\chi_i : U \to \mathbb{R}$ of χ. Unraveling the definitions one obtains:

$$\left\langle (d\chi_i)_p, \left(\frac{\partial}{\partial \chi_j}\right)_p \right\rangle = \delta^i_j$$

for all i and j. This implies that (4.1.2) is indeed a basis of $T^*_p M$- actually the dual basis to (4.1.1).

4.1.2 Differential 1-forms

Similar to set-theoretical vector fields on M (subsection 3.2.1), but dually, we now look at maps

$$\boxed{\omega : M \ni p \mapsto \omega_p \in T_p^* M}$$

and, to make sense of their smoothness, we look in arbitrary charts $\chi : U \to \Omega \subset \mathbb{R}^m$, we decompose ω as

$$\omega_p = \sum_i \omega_\chi^i(\chi(p))(d\chi_i)_p \quad (\text{for } p \in U),$$

and we consider of the resulting **coordinate functions of** ω w.r.t. the chart χ

$$\omega_\chi^i : \Omega \to \mathbb{R}.$$

Definition 4.2. A **differential** 1**-form** on M is any map

$$\omega : M \ni p \mapsto \omega_p \in T_p^* M$$

whose coordinate functions w.r.t. any chart are smooth.
We denote by $\Omega^1(M)$ the set of such 1-forms.

As for vector fields (the discussion following Definition 3.16), $\Omega^1(M)$ carry the following algebraic structure:

- it is a vector space, with the addition $+$ and multiplication by scalars $\cdot$ defined pointwise:

$$(\omega + \omega')_p := \omega_p + \omega'_p, \quad (\lambda \cdot \omega)_p := \lambda \cdot \omega_p.$$

- it is a $\mathscr{C}^\infty(M)$-module, i.e. there is an operation of multiplication of vector fields by smooth functions,

$$\mathscr{C}^\infty(M) \times \Omega^1(M) \to \Omega^1(M), \quad (f, \omega) \mapsto f \cdot \omega.$$

where $(f \cdot \omega)_p := f(p) \cdot \omega_p$.

Example 4.3 (differentials of functions). Example 4.1 can now be upgraded: we see that for any function $f \in \mathscr{C}^\infty(M)$ its differential now becomes a 1-form

$$\boxed{df \in \Omega^1(M)}$$

(why is it smooth?). In this way, the differentiation of $\mathbb{R}$-valued functions becomes a linear operator

$$\boxed{d : \mathscr{C}^\infty(M) \to \Omega^1(M).} \tag{4.1.3}$$

Exercise 4.4. Show the Leibniz derivation identity: for any $f, g \in C^\infty(M)$, one has:

$$\boxed{d(fg) = f \cdot dg + g \cdot df.} \tag{4.1.4}$$

Exercise 4.5. Show that, for a 1-form $\omega \in \Omega^1(M)$ to be exact, i.e. to be of type df for some smooth function f, a necessary (but not sufficient) condition is that

$$\omega([X,Y]) = L_X(\omega(Y)) - L_Y(\omega(X)) \quad \text{for all } X,Y \in \mathfrak{X}(M).$$

Example 4.6 (expressions of type $\sum f_i dg_i$). Using the vector space and C^∞-module operations on $\Omega^1(M)$, as well as differentials of functions, we see that any expressions of type

$$\sum_{i=1}^{k} f_i \cdot dg_i \qquad \text{with } f_i, g_i \in C^\infty(M)$$

now make sense as 1-forms. One can actually show that any 1-form can be written like this, but one should be careful: such a writing is not unique (this is clearly shown already by the derivation property from Exercise 4.4).

On $M = \mathbb{R}^m$ one obtains the analogue of Example 3.18: any 1-form ω on $\mathbb{R}^m$ can be uniquely written as

$$\omega = \sum f_i \cdot dx_i \tag{4.1.5}$$

with $f_i \in \mathscr{C}^\infty(\mathbb{R}^m)$, and where x_i are the coordinate functions seens as elements of $\mathscr{C}^\infty(\mathbb{R}^m)$. Similarly in charts.

Exercise 4.7. Show that:

(a) For any function $f \in \mathscr{C}^\infty(\mathbb{R}^m)$, one has

$$df = \sum_i \frac{\partial f}{\partial x_i} \cdot dx_i.$$

(b) Show that the 1-form on $\mathbb{R}^2$ given by

$$\omega = ydx - xdy$$

is not exact, i.e. cannot be written as df for some smooth function f.

Exercise 4.8. Show that for any $f \in \mathscr{C}^\infty(M)$ and any chart (U, χ) one has, on U (with the notations (2.3.18)):

$$\boxed{df = \sum_i \frac{\partial f}{\partial \chi_i} \cdot d\chi_i.}$$

Deduce that for any two coordinate charts (U, χ) and (U', χ') one has, on $U \cap U'$,

$$d\chi_i' = \sum_j \frac{\partial \chi_i'}{\partial \chi_j} \cdot d\chi_j. \tag{4.1.6}$$

Then, using the duality between $\frac{\partial}{\partial \chi_i}$ and $d\chi_i$ show again that, over $U \cap U'$, one has:

$$\frac{\partial}{\partial \chi_i} = \sum \frac{\partial \chi_j'}{\partial \chi_i} \cdot \frac{\partial}{\partial \chi_j'}.$$

Exercise 4.9. Show that there is a unique 1-form on S^1, denoted by $d\theta$, such that $p^* d\theta = dt \in \Omega^1(\mathbb{R})$, where $p : \mathbb{R} \to S^1$, $t \mapsto e^{it}$. Then try to prove that, despite the notation that we use, $d\theta$ is actually not exact.
(the most elegant argument for the last part is via Stokes theorem, to be discussed later).

4.1.3 Pull-backs of 1-forms

And here is one very nice feature of 1-forms: unlike vector fields, 1-forms can be pulled-back along *any* smooth map. We start here with the simplest instance of this operation: the restriction to submanifolds.

Example 4.10 (restrictions to submanifolds). For a submanifold $M \subset N$, dual to the inclusions $T_pM \subset T_pN$, there is a restriction operation of 1-forms on N to 1-forms on M,

$$\mathrm{restr}: \Omega^1(N) \to \Omega^1(M), \quad \omega \mapsto \omega|_M,$$

where $\omega|_M$ is defined by

$$(\omega|_M)_p(X_p) := \omega_p(X_p)$$

for all $X_p \in T_pM$. Note that the right hand side makes sense precisely because $T_pM \subset T_pN$.

This operation is particularly useful when M is an embedded submanifold of a Euclidean space $\mathbb{R}^k =: N$: one can produce 1-forms on M by starting with 1-forms on $\mathbb{R}^k$ (necessarily of type (4.1.5)); and, although it is not completely obvious, one can prove that any 1-form on M arises in this way. In other words, one can "represent" 1-forms on M by 1-forms (4.1.5) on $\mathbb{R}^k$. E.g., for $M = S^2$, we can now talk about

$$(x^2+y)\cdot dx + dy + xyz\cdot dz \quad \text{on } S^2,$$

by which we mean the restriction of $(x^2+y)\cdot dx + dy + xyz\cdot dz$ from $\mathbb{R}^3$ to $S^2 \subset \mathbb{R}^3$.

However, one should be aware that *different forms on* $\mathbb{R}^k$ *may give the same restriction to* M. E.g., for $S^2 \subset \mathbb{R}^3$, there are expressions that look like being non-zero (and they are non-zero as 1-forms on $\mathbb{R}^3$) but which represent the zero form on S^2. For instance,

$$x\cdot dx + y\cdot dy + z\cdot dz \quad \text{on } S^2$$

is zero. Indeed, viewing it as a 1-form on S^2 means that we evaluate it only on vectors tangent to S^2; and, for $p = (x,y,z) \in S^2$ and an arbitrary tangent vector

$$V_p = a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p + c\left(\frac{\partial}{\partial z}\right)_p \in T_pS^2$$

one has $(x\cdot dx + y\cdot dy + z\cdot dz)(V_p) = ax + by + cz = 0$ since V was tangent to S^2. In other words,

$$(x\cdot dx + y\cdot dy + z\cdot dz)|_{S^2} = 0.$$

Exercise 4.11. Prove the last formula starting from $x^2+y^2+z^2 = 1$ and using the derivation property (4.1.4).

Exercise 4.12. Show that, conversely, if $M \subset \mathbb{R}^3$ is a compact connected submanifold with the property that

$$(xdx + ydy + zdz)|_M = 0,$$

then M is one of the spheres centred at the origin.

Exercise 4.13. Show that, for the 1-form $d\theta$ on S^1 discussed in Exercise 4.9, $d\theta = (ydx - xdy)|_{S^1}$.

Exercise 4.14. For the torus $T_{R,r}$ as in (2.1.6), show that the restrictions of dx, dy, dz from $\mathbb{R}^3$ to $T^2_{R,r}$ satisfy:

$$x\cdot dx + y\cdot dy + z\cdot dz = \frac{R}{\sqrt{x^2+y^2}}(x\cdot dx + y\cdot dy) \quad (\text{on } T^2_{R,r}).$$

Exercise 4.15 (part of the 2019/2020 exam). Do now the last part of the exam Exercise 2.191:

(d) give an example of a 1-form θ on S^3, different from the zero form, such that $\theta|_{M_0} = 0$.

The restriction operation is a particular case of pull-backs, corresponding to inclusions $i : M \hookrightarrow N$. To explain this, we start with an arbitrary smooth map $F : M \to N$ and $\omega \in \Omega^1(N)$. Then at each point $p \in M$ we can apply the differential $(dF)_p : T_pM \to T_{F(p)}N$ followed by $\omega_{F(p)} : T_{F(p)}N \to \mathbb{R}$ to obtain a covector on M at p:

$$T_pM \ni X_p \mapsto \omega_{F(p)}((dF)_p(X_p)) \in \mathbb{R}$$

(compare with vector fields!). The resulting 1-form on M is denoted by

$$F^*(\omega) \in \Omega^1(M)$$

and is called **the pull-back of ω by** F,. Hence any smooth map $F : M \to N$ gives rise to a pull-back operation

$$\boxed{F^* : \Omega^1(N) \to \Omega^1(M)} \quad \text{given by} \quad \boxed{F^*(\omega)_p(X_p) = \omega_{F(p)}((dF)_p(X_p))}.$$

Exercise 4.16. Show that:

(a) indeed, $F^*(\omega)$ is smooth.
(b) when $M \subset N$ and $F = i$ is the inclusion $i^*(\omega) = \omega|_M$.
(c) if $G : N \to P$ is another smooth map, $\boxed{(G \circ F)^* = F^* \circ G^*}$.

Exercise 4.17. Show that for any $F : M \to N$ smooth, the following diagram is commutative:

$$\boxed{\begin{array}{ccc} \mathscr{C}^\infty(N) & \xrightarrow{d} & \Omega^1(N) \\ \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} \\ \mathscr{C}^\infty(M) & \xrightarrow{d} & \Omega^1(M) \end{array} \qquad F^* \circ d = d \circ F^*.}$$

Exercise 4.18. Since $\mathbb{P}^n$ does not sit canonically in any Euclidean space, the principle of representing 1-forms on $\mathbb{P}^n$ by 1-forms on Euclidean spaces (via restrictions) does not work right away. However, one can relate $\mathbb{P}^n$ to a nicer space, namely to the sphere, via the canonical map (2.2.7),

$$H : S^n \to \mathbb{P}^n, \quad x \mapsto l_x.$$

In this way, any 1-form $\omega \in \Omega^1(\mathbb{P}^n)$ can be "lifted" to a form on S^n:

$$\widetilde{\omega} := H^*(\omega) \in \Omega^1(S^n).$$

We will also make use of the map $\tau : S^n \to S^n$, $\tau(x) = -x$ and the space of τ-invariant 1-forms,

$$\Omega^1(S^n)^\tau := \{\eta \in \Omega^1(S^n) : \tau^*\eta = \eta\}.$$

Show that:

(a) for any $\omega \in \Omega^1(S^n)$, $\widetilde{\omega}$ is τ-invariant.
(b) Conversely, any $\eta \in \Omega^1(S^n)^\tau$, is necessarily of type $\widetilde{\omega}$ for some $\omega \in \Omega^1(\mathbb{P}^n)$.
(c) Conclude that one obtains a linear isomorphism $H^* : \Omega^1(\mathbb{P}^n) \xrightarrow{\sim} \Omega^1(S^n)^\tau$.

4.1.4 1-forms as $C^\infty(M)$-linear functionals eating vector fields

While covectors of a vector space V take vectors to real numbers, 1-forms take vector fields to smooth functions: given ω as above and $X \in \mathfrak{X}(M)$, evaluating ω_p on X_p for each $p \in M$ we obtain a smooth function

$$\omega(X) \in \mathscr{C}^\infty(M);$$

(why smooth?). When we vary X it is clear that the resulting map, still denoted by ω,

$$\omega : \mathfrak{X}(M) \to \mathscr{C}^\infty(M)$$

is $C^\infty(M)$-linear, i.e. it is linear and

$$\omega(f \cdot X) = f \cdot \omega(X) \quad \text{for all } f \in \mathscr{C}^\infty(M), X \in \mathfrak{X}(M).$$

The space of all such $\mathscr{C}^\infty(M)$-linear maps from $\mathfrak{X}(M)$ to $\mathscr{C}^\infty(M)$ will be denoted

$$\mathrm{Hom}_{\mathscr{C}^\infty}(\mathfrak{X}(M), \mathscr{C}^\infty(M)).$$

Theorem 4.19. *Interpreting any $\omega \in \Omega^1(M)$ as a map $\mathfrak{X}(M) \to \mathscr{C}^\infty(M)$, we obtain an isomorphism*

$$\Omega^1(M) \cong Hom_{\mathscr{C}^\infty}(\mathfrak{X}(M), \mathscr{C}^\infty(M)).$$

Proof. Note first that, by the same arguments as in Proposition 3.26 from Chapter 3, for a set-theoretical 1-form ω its smoothness is equivalent to the fact that $\omega(X)$ is a smooth function on M for all $X \in \mathfrak{X}(M)$.

Another ingredient that we need in the proof is the fact that, for each $p \in M$, the evaluation of vector fields at p,

$$\mathrm{ev}_p : \mathfrak{X}(M) \to T_pM, \quad X \mapsto X_p, \tag{4.1.7}$$

is surjective. To show this, we start with an arbitrary tangent vector $v \in M$ (p is fixed now). Using a coordinate chart around p, we find a vector field Y defined on an open neighborhood U of p such that $Y_p = v$. As in the proof of Lemma 3.8 from Chapter 3, choose a smooth function η on M, supported inside U and with $\eta(p) = 1$. Then $X := \eta \cdot Y$ is well-defined and smooth on M and $X_p = \eta(p) \cdot v = v$.

We now move to the actual proof. For the injectivity of the map we have to show that, if $\omega_p(X_p) = 0$ for all $X \in \mathfrak{X}(M)$ and all $p \in M$, then $\omega_p = 0$ for all $p \in M$. But this clearly follows from the surjectivity of ev_p.

We now prove the surjectivity of the map in the statement. Let $A : \mathfrak{X}(M) \to \mathscr{C}^\infty(M)$ be a $\mathscr{C}^\infty(M)$-linear map. We fix $p \in M$ arbitrary and we define

$$\omega_p : T_pM \to \mathbb{R}, \quad \omega_p(v) := A(Y)(p)$$

where A is any vector field with $Y_p = v$. If we check that this definition does not depend on the choice of Y then we are done: linearity of each ω_p follows right away, while the resulting set-theoretical 1-form is also smooth since, by construction, $\omega(X) = A$.

To see that $A(Y)(p)$ does not depend on the choice of Y, assume that Y' is another vector field with $Y'_p = v$. Then $X := Y - Y'$ vanishes at p and we have to show that $A(X)(p) = 0$. Using the $\mathscr{C}^\infty(M)$-linearity of A, this will follow from the following:

Lemma 4.20. *If a vector field $X \in \mathfrak{X}(M)$ vanishes at a point $p \in M$, then we can write*

$$X = \sum_{j=0}^{k} f_j \cdot X^j$$

with $k \geq 1$ integer, $X_j \in \mathfrak{X}(M)$ and f_j smooth functions that vanish at p.

Proof. We first look in a coordinate chart (U, χ) around p which sends p to $\chi(p) = 0 \in \mathbb{R}^m$. Then writing

$$X|_U = \sum_{i=1}^{m} F_i \cdot \frac{\partial}{\partial \chi_i},$$

the coefficients F_i are smooth functions on U that vanish at p. By the the proof of Theorem 3.3 from Chapter 3 applied to $F_i \circ \chi^{-1}$, we see that each F_i can be written as a finite sum $\sum_j \chi_j \cdot g_i^j$ with $g_i^j : U \to \mathbb{R}$ smooth. From this we immediately deduce that we can find some smooth functions $f_1, \ldots, f_k$ on U and vector fields $Y_1, \ldots, Y_k$ (on U) such that

$$X|_U = \sum_{j=1}^{k} f_j \cdot Y^j.$$

To go from U to the entire M, the main idea is to write $X = \eta \cdot X + (1 - \eta) \cdot X$, for some carefully chosen smooth function η.

Note that, after eventually making U smaller, we may assume that $f_i = \tilde{f}_i|_U$ for some smooth function $\tilde{f}_i \in \mathscr{C}^\infty(M)$; to see this, just use the second dotted item in the proof of Lemma 3.9 from Chapter 3. Furthermore, as in that proof, we choose a smooth function $\eta \in C^\infty(M)$ which is supported inside U and is 1 in some neighborhood V of p. Since η is supported inside U, each $\eta|_U \cdot Y^j \in \mathfrak{X}(U)$ can be promoted to a vector field $X^j \in \mathfrak{X}(M)$ by declaring it to be zero outside U. With the inspiration from the "main idea" mentioned above, we now remark that

$$X = (1 - \eta) \cdot X + \sum_{j=1}^{k} \tilde{f}_j \cdot \tilde{X}^j$$

on the entire M. Indeed, at $p \in U$ the right hand side is

$$(1 - \eta(p)) \cdot X_p + \eta(p) \cdot \sum_{j=1}^{k} f_j(p) \cdot X_p^j = (1 - \eta(p)) \cdot X_p + \eta(p) \cdot X_p = X_p,$$

while at p outside U we obtain $(1 - 0) \cdot X_p + \sum_j \tilde{f}_j(p) \cdot 0 = X_p$.

Therefore the desired decomposition is achieved if we take $f_0 = 1 - \eta$ and $X^0 = X$.

And, still as in the case of tangent vectors, one can form the cotangent bundle

$$T^*M := \bigsqcup_{p \in M} T_p^*M = \{(p, \omega_p) : p \in M, \omega_p \in T_p^*M\}$$

and put a smooth structure on T^*M so that the smoothness of a set-theoretical 1-form ω is equivalent to the smoothness of ω as a map from M to T^*M.

Exercise 4.21. Fill in the details, i.e. state and prove a version of Proposition 3.37 for T^*M instead of TM.

4.2 Differential forms of arbitrary degree

4.2.1 The relevant linear algebra

The passage from 1-forms to arbitrary k-forms is basically a passage from linear maps $T_pM \to \mathbb{R}$ to k-multilinear skew-symmetric maps. Here we discuss the linear algebra involved.

Definition 4.22. Given a vector space V and $k \in \mathbb{N}$ a natural number, a k-covector on V is any map

$$\omega : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$$

that is multilinear (i.e. linear in each argument). We say that ω if a linear k-form on V if

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \text{sign}(\sigma)\omega(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in V$ and any permutation $\sigma \in S_k$. We denote by $\Lambda^k V^*$ the (vector) space of all such linear k-forms on V.

Exercise 4.23. Show that ω is skew-symmetric if and only if the expression $\omega(v_1, \ldots, v_k)$ changes sign whenever two entries v_i and v_j are interchanged.

Remark 4.24. One way to think about a k-form is as a "k-dimensional (signed) volume function" on V. More precisely, one should interpret the expressions $\omega(v_1, \ldots, v_k)$ as the volume that ω assigns to the parallelipiped

$$P(v_1, \ldots, v_k) = \left\{ t_1 \cdot v_1 + \ldots + t_k \cdot v_k : t_1, \ldots, t_k \in [0,1], \sum_i t_i = 1 \right\}.$$

In other words, the ω-volume of $P(v_1, \ldots, v_k)$ is $\omega(v_1, \ldots, v_k)$. One could further go on and use the absolute value to produce positive volumes, or go on and talk about the ω-volume for more complicated "k-dimensonal bodies" inside V. Please be aware that these comments here are not very precise and are meant just to help with the intuition.

Although our interest is $\Lambda^k V^*$, it will be useful to consider more general k-multilinear maps (without being skew-symmetric); we denote the space of such by $T^k V^*$. Note that, while

$$\Lambda^k V^* \subset T^k V^*,$$

there is also a skew-symmetrization map going the other way,

$$\text{Alt} : T^k V^* \to \Lambda^k V^*$$

where for $\omega \in T^k V^*$, $\text{Alt}(\omega)$ is given by

$$\text{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Exercise 4.25. Show that, for $\omega \in T^k V^*$, one has:

$$\omega \in \Lambda^k V^* \iff \text{Alt}(\omega) = \omega.$$

The main operation on forms is that of wedge-product. Before giving the precise definition/formula, note first that there a pretty obvious operation involving the spaces T^kV^*, namely:

$$T^kV^* \times T^lV^* \to T^{k+l}V^*, \quad (\omega, \eta) \mapsto \omega \bullet \eta,$$

where $\omega \bullet \eta$ is given by

$$(\omega \bullet \eta)(v_1, \ldots, v_{k+l}) := \omega(v_1, \ldots, v_k) \cdot \eta(v_{k+1}, \ldots, v_{k+l}).$$

Exercise 4.26. Do the following:

1. If $\omega \in T^2V^*$ is symmetric and $\eta \in T^lV^*$ is arbitrary, show that $\mathrm{Alt}(\omega \bullet \eta) = 0$.
2. Prove the same but now assuming that $\omega \in T^kV^*$ satisfies $\mathrm{Alt}(\omega) = 0$.
3. Deduce that for any $\omega \in T^kV^*$ and $\eta \in T^lV^*$ one has $\mathrm{Alt}(\omega \bullet \eta) = \mathrm{Alt}(\mathrm{Alt}(\omega) \bullet \eta)$.

However, if ω and η are skew-symmetric, $\omega \bullet \eta$ may fail to be skew-symmetric. Therefore, to obtain a skew-symmetric element, we will apply the skew-symmetrization map Alt. To avoid un-necessary coefficients in the final formula we will do a small rescaling (see also below):

Definition 4.27. Given $\omega \in \Lambda^kV^*$ and $\eta \in \Lambda^lV^*$, define the element

$$\omega \wedge \eta \in \Lambda^{k+l}V^*,$$

called the **wedge product of ω and η**, by

$$\omega \wedge \eta = \frac{(k+l)!}{k! \cdot l!} \mathrm{Alt}(\omega \bullet \eta).$$

Remark 4.28. When computing $\mathrm{Alt}(\omega \cdot \eta)$, the fact that ω and η are skew-symmetric will be reflected in the fact that many of the terms will repeat. When we avoid repetitions, we get sums not over all permutations $\sigma \in S_{k+l}$ but only over those with the property that

$$\sigma(1) < \ldots < \sigma(k), \quad \sigma(k+1) < \ldots < \sigma(k+l).$$

Such permutations will be called (k,l)**-shuffles** and we denote by $S_{k,l} \subset S_{k+l}$ the resulting set. All together, we find the following explicit formula for the wedge product:

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+l}) := \sum_{\sigma \in S_{k,l}} \mathrm{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

For instance, for $\omega, \eta \in V^*$ one obtains $\omega \wedge \eta \in \Lambda^2V^*$ given by

$$(\omega \wedge \eta)(v, w) = \omega(v)\eta(w) - \omega(w)\eta(v).$$

Or, for $\omega, \in \Lambda^2V^*, \eta \in V^*$,

$$(\omega \wedge \eta)(u, v, w) = \omega(u, v)\eta(w) + \omega(v, w)\eta(u) - \omega(u, w)\eta(v).$$

The reason for the coefficients $k!$ used in the definition of $\wedge$ is precisely obtain such formulas without repetitions. Actually, replacing $k!$ by any coefficients c_k would still produce (modified) wedge-products with similar properties (see the next proposition) but, in some (precise) sense, they are all "isomorphic". However, simple choices of c_ks, such as $c_k = 1$, would produce uglier final formulas.

In this way we obtain operations

$$\cdot \wedge \cdot : \Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^*.$$

Here are the main properties of these operations:

Proposition 4.29. *The wedge operation is bilinear, and:*

1. *graded-commutativity: for any $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$,*

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

2. *associativity: for any linear forms ω, η and ζ one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

Proof. For graded-commutativity, we look at $\eta \wedge \omega(v_1, \ldots, v_{k+l})$. The sum defining it has expressions of type

$$\operatorname{sign}(\sigma) \eta(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \cdot \omega(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+k)})$$

where σ is an (l,k)-shuffle. Now, any such σ defines another permutation $\hat{\sigma}$ by

$$\hat{\sigma}(1) = \sigma(l+1), \ldots, \hat{\sigma}(k) = \sigma(l+k), \quad \hat{\sigma}(k+1) = \sigma(1), \ldots, \hat{\sigma}(k+l) = \sigma(l)$$

which is a (k,l)-shuffle. A moment of reflection shows that

$$\operatorname{sign}(\hat{\sigma}) = (-1)^{kl} \operatorname{sign}(\sigma),$$

and graded-commutativity follows. For associativity, removing a common constant, we have to prove

$$\operatorname{Alt}(\operatorname{Alt}(\omega \bullet \theta) \bullet \zeta) = \operatorname{Alt}(\omega \bullet \operatorname{Alt}(\theta \bullet \zeta)).$$

Using the last part of Exercise 4.26 to rewrite the left hand side, and the similar argument for the right hand side, what we have to prove becomes:

$$\operatorname{Alt}((\omega \bullet \theta) \bullet \zeta) = \operatorname{Alt}(\omega \bullet (\theta \bullet \zeta)).$$

This follows right away since the operation $\bullet$ is clearly associative. 😀

Example 4.30. For any linear 1-forms $\xi_1, \ldots, \xi_k \in V^*$, one obtains a k-form

$$\xi_1 \wedge \ldots \wedge \xi_k \in \Lambda^k V^*.$$

As we shall soon see, any linear k-form can be written as a sum of expressions of this type. Even more, this operation allows us to produce an explicit basis for the vector space $\Lambda^k V^*$ by starting with a basis

$$e_1, \ldots, e_m \in V$$

of V. First of all, we consider the dual basis

$$e^1, \ldots, e^m \in V^*.$$

Then we start taking wedges of such elements; to obtain k-forms, we need to use k such (co)vectors. Given the graded commutativity of the wedge-operations, we can always re-order the indices in an increasing fashion. Therefore, we will consider sets of increasing indices

$$I = (i_1, \ldots, i_k), \quad \text{with} 1 \leq i_1 < \ldots < i_k \leq m;$$

the set of such Is will be denoted by $\mathrm{Ord}_k(m)$. And, for $I \in \mathrm{Ord}_k(m)$, consider

$$e^I := e^{i_1} \wedge \ldots \wedge e^{i_k} \in \Lambda^k V^*. \tag{4.2.1}$$

Lemma 4.31. *The vectors (4.2.1) indexed by $I \in Ord_k(m)$ form a basis of $\Lambda^k V^*$. In particular, for $k \leq m$ (where m is the dimension of V) one has*

$$dim(\Lambda^k V^*) = Card\,(Ord_k(m)) = \frac{m!}{k!(m-k)!},$$

while for $k > m$ one has $\Lambda^k V^* = 0$.

Proof. Note that, for any I as above and any other $J = (j_1, \ldots, j_k) \in \mathrm{ord}_k(M)$ one has

$$e^I(e_{j_1}, \ldots, e_{j_k}) = \delta^I_J$$

(i.e. 1 when $I = J$ and zero otherwise). From this we deduce right away that the e^Is are linearly independent: if $\sum_I \lambda_I e^I = 0$ with $\lambda_I \in \mathbb{R}$, evaluating on $(e_{j_1}, \ldots, e_{j_k})$ we find that $\lambda_J = 0$ for all J.

To see that our family spans the entire $\Lambda^k V^*$, it suffices to show that, for any $\omega \in \Lambda^k V^*$),

$$\omega = \sum_{I \in \mathrm{Ord}_k(m)} \omega(e_{i_1}, \ldots, e_{i_k}) \cdot e^I.$$

To check this equality, we first note that the difference of the two sides, call it $\eta \in \Lambda^k V^*$, is zero when evaluated on all elements of type $(e_{i_1}, \ldots, e_{i_k})$. But this is easily seen to imply that $\eta = 0$: for an arbitrary expression $\eta(v_1, \ldots, v_k)$, decomposing each v_i with respect to the basis $\{e_1, \ldots, e_m\}$ and using the multilinearity of η and then the antisymmetry, we obtain a sum of expressions of type $\eta(e_{i_1}, \ldots, e_{i_k})$. 😀

4.2.2 *Differential forms*

We now pass to differential forms on an arbitrary m-dimensional manifold M. As for 1-forms, one first talks about **set-theoretical k-forms**, by which we mean maps

$$\omega : M \ni p \mapsto \omega_p \in \Lambda^k T^*_p M. \tag{4.2.2}$$

And, still as for 1-forms, one makes sense of their smoothness using charts (U, χ): any chart gives rise to the basis

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$

of $T_p M$, the dual basis

$$(d\chi_1)_p, \ldots, (d\chi_m)_p$$

of $T^*_p M$ and then, using Lemma 4.31, to the basis of $\Lambda^k T^*_p M$ given by

$$(d\chi)^I_p = (d\chi_{i_1})_p \wedge \ldots \wedge (d\chi_{i_k})_p$$

with $I \in \mathrm{Ord}_k(1, \ldots, m)$. Therefore, we can write our ω as

$$\omega_p = \sum_{I \in \mathrm{Ord}_k(m)} \omega^I_\chi(\chi(p))(d\chi)^I_p,$$

where each coefficient ω^I_χ is a function on the codomain of the chart.

Definition 4.32. A **differential k-form** on M, or a **differential form of degree** k on M, is any set-theoretical k-form (4.2.2) that is smooth in the sense that its coordinate functions ω^I_χ w.r.t. any chart χ are smooth. We denote by $\Omega^k(M)$ the space of all such k-forms.

It should be clear now that $\Omega^k(M)$ is a vector space, and it comes with a structure of $\mathscr{C}^\infty(M)$-module,

$$\mathscr{C}^\infty(M) \times \Omega^k(M) \to \Omega^k(M), \quad (f, \omega) \mapsto f \cdot \omega.$$

While, point-wise (at some $p \in M$), ω takes k tangent vectors (at p) to produce numbers, globally (when varying p), it takes k vector fields and produces a function. More explicitly, for $X^1, \ldots, X^k \in \mathfrak{X}(M)$, one obtains the function

$$\omega(X^1, \ldots, X^k) : M \to \mathbb{R}, \quad p \mapsto \omega_p(X^1_p, \ldots, X^k_p).$$

In other words, ω can be re-interpreted as a map

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-\text{times}} \to \mathscr{C}^\infty(M).$$

Exercise 4.33. Check that the smoothness of ω is equivalent to the fact that $\omega(X^1, \ldots, X^k)$ is smooth for any $X^1, \ldots, X^k \in \mathfrak{X}(M)$.

Of course, with the case $k = 1$ already discussed in Theorem 4.19, the following does not come as a surprise.

Theorem 4.34. *The previous interpretation of k-forms gives a 1-1 correspondence between k-forms on M and multilinear maps*

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-times} \to \mathscr{C}^\infty(M)$$

with the following two properties:

- *ω is skew-symmetric.*
- *ω is $\mathscr{C}^\infty(M)$-linear in each argument X_i:*

$$\omega(X_1, \ldots, X_{i_1}, X_i + X_i', X_{i+1}, \ldots, X_k) = \omega(X_1, \ldots, X_{i_1}, X_i, X_{i+1}, \ldots, X_k) + \\ + \omega(X_1, \ldots, X_{i_1}, X_i', X_{i+1}, \ldots, X_k),$$

$$\omega(X_1, \ldots, X_{i_1}, f \cdot X_i, X_{i+1}, \ldots, X_k) = f \cdot \omega(X_1, \ldots, X_k).$$

Proof. For the moment, let denoted by $\hat{\omega}$ the version of ω that acts on vector fields. It is clear that it is skew-symmetric and $\mathscr{C}^\infty(M)$-multilinear. For the converse, let

$$A : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-\text{times}} \to \mathscr{C}^\infty(M)$$

a map with the same properties and we want to show that $A = \hat{\omega}$ for some differential k-form ω. As in the previous sections, for $p \in M$ we define ω_p on tangent vectors $v^1_p, \ldots, v^k_p \in T_pM$ by:

$$\omega_p(v^1_p, \ldots, v^k_p) := A(X^1, \ldots, X^k)(p),$$

where $X^i \in \mathfrak{X}(M)$ are chosen so that, at p, they give back the vectors v_p^i. The main issue is, as for the similar discussion for 1-forms, to see that this definition does not depend on the choice of the X^is. But that follows exactly as for 1-forms (or one could even use that result inductively). The rest should be clear.

Example 4.35. In Exercise 4.5 we have seen that, for a 1-form $\omega \in \Omega^1(M)$ to be exact one would need in particular that the expression

$$-\omega([X,Y]) + L_X(\omega(Y)) - L_Y(\omega(X))$$

to be zero for all $X, Y \in \mathfrak{X}(M)$. Looking at the expression above one can check right away that it is skew-symmetric and $\mathscr{C}^\infty(M)$-bilinear in X and Y. Therefore it defines a 2-form on M- that is usually denoted by

$$d\omega \in \Omega^2(M);$$

Exercise 4.5 tells us that $d\omega$ arises as an obstruction to ω being exact; one says that ω is **closed** if $d\omega = 0$. Note that, in this way, we have complemented the differential of functions from (4.1.3) with its version on 1-forms:

$$\mathscr{C}^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M).$$

This, and its further continuation will be discussed in more detail in the next section (”De Rham differential”).

The wedge-product discussed in the previous section applied at each point $p \in M$ gives rise to similar operations

$$\cdot \wedge \cdot : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \quad (\omega, \eta) \mapsto \omega \wedge \eta,$$

i.e. defined by $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$. Or, using the interpretation of k-forms as in the previous proposition,

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S_{k,l}} \operatorname{sign}(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\eta(V_{\sigma(k+1)}, \ldots, V_{\sigma(k+l)}). \tag{4.2.3}$$

Proposition 4.36. *The wedge operation is bilinear, and:*

1. *graded-commutativity: for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,*

$$\eta \wedge \omega = (-1)^{kl}\omega \wedge \eta.$$

2. *associativity: for any differential forms ω, η and ζ on M one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

Proof. Just apply point-wise the similar linear algebra result.

And, still as for 1-forms, forms of arbitrary degree can be pulled-back via any smooth map $F : M \to N$, giving rise to operations:

$$F^* : \Omega^k(N) \to \Omega^k(M).$$

Explicitly, for $\omega \in \Omega^k(N)$, **the pull-back of ω by** F, $F^*\omega \in \Omega^k(M)$, is given by

$$F^*(\omega)_p(X_p^1, \ldots, X_p^k) := \omega_{F(p)}\left((dF)_p(X_p^1), \ldots, (dF)_p(X_p^k)\right).$$

Exercise 4.37. Show that the pull-back operation preserves the wedge-products: if $F : M \to N$ is smooth then, for any ω and η differential forms on N, one has

$$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta).$$

When M is a submanifold of N and F is the inclusion map, the pull-back of ω via the inclusion will be denoted

$$\omega|_M \in \Omega^k(M)$$

and will be called **the restriction of ω to M** . And of particular interest is the case when M is a submanifold of some Euclidean space $\mathbb{R}^d$; then:

- k-forms on $\mathbb{R}^d$ are always of type

$$\sum_{1 \leq i_1 < \ldots < i_k \leq d} f_{i_1,\ldots,i_k} \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

  with $f_{i_1,\ldots,i_k}$ smooth functions on $\mathbb{R}^d$.
- one obtains k-forms on M by restricting such k-forms on $\mathbb{R}^d$ to M.
- with the same warning as for 1-forms: *different forms on $\mathbb{R}^k$ may give the same restriction to M.*

Exercise 4.38 (from the 2018/2019 exam). Show that, on any manifold, for any $\theta \in \Omega^k(M)$ with k odd, one has $\theta \wedge \theta = 0$. Then find an example of a manifold M and a 2-form M such that $\theta \wedge \theta \neq 0$.

Exercise 4.39. On the following subspace of the 2-sphere

$$N := \{(x,y,z) \in S^2 : z \neq 0\}$$

consider the restriction of $\frac{1}{z} \cdot dx \wedge dy$ to N:

$$\sigma_0 := \frac{1}{z} \cdot dx \wedge dy\Big|_N .$$

Show that σ_0 can be extended to the entire S^2, i.e., $\exists$ a (smooth) 2-form $\sigma \in \Omega^2(S^2)$ such that $\sigma|_N = \sigma_0$.

Exercise 4.40. Consider the so-called volume form of S^2,

$$\sigma := (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)|_{S^2} \in \Omega^2(S^2).$$

Since $xdx + ydy + zdz = 0$ on S^2, on the part $z \neq 0$ of S^2 the 1-forms dz can be expressed in terms of dx and dy and, therefore, σ can be written as $f \cdot dx \wedge dy$ for some function f. Compute f.

Exercise 4.41. Show that Exercise 4.18 works exactly the same for forms of arbitrary degree, so that k-forms on $\mathbb{P}^n$ are, via the pull-back to S^n, in 1-1 correspondence to τ-invariant k-forms on S^n:

$$\Omega^k(S^n)^\tau := \left\{\eta \in \Omega^k(S^n) : \tau^*\eta = \eta\right\}.$$

Remark 4.42 (The graded world). It is customary to formally put together all the spaces $\Omega^k(M)$ and consider

$$\Omega^{\bullet}(M) := \bigoplus_k \Omega^k(M).$$

The notation "$\bullet$" is to indicate that we deal with a graded object; its elements are (finite) sums of homogeneous ones (the ones that live in one single degree). With this, the wedge product becomes now (after extending bi-linearly) a product

$$\cdot \wedge \cdot : \Omega^{\bullet}(M) \times \Omega^{\bullet}(M) \to \Omega^{\bullet}(M)$$

making $\Omega(M)$ into a (graded) algebra. Note the similarity with the algebra of polynomials in several variables, where each polynomial is a sum of homogeneous ones. And, as there, on sums of homogeneous elements, the wedge product is:

$$(\omega_0 + \omega_1 + \ldots) \wedge (\eta_0 + \eta_1 + \ldots) = (\omega_0 \wedge \eta_0) + (\omega_0 \wedge \eta_1 + \omega_1 \wedge \eta_0) + (\omega_0 \wedge \eta_2 + \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_0) + \ldots$$

And here is one very useful slogan/principle when working with graded objects:

The sign rule: *looking at an expression involving elements that have a degree, when interchanging two such elements, say of degrees k and l, the expression changes sign by* $(-1)^{kl}$.

E.g., graded commutativity becomes precisely the one from Proposition 4.36.

4.2.3 The exterior (DeRham) differential

In Example 4.3 we mentioned the differential of functions (0-forms) as 1-forms, while in Example 4.35 we indicated a similar operation on from 1-forms to 2-forms. We now extend those to arbitrary degrees, i.e. a sequence of operators d:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \ldots$$

or, equivalently, an operator on the full algebra of differential forms (cf. Remark 4.42),

$$\boxed{d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)}$$

which rises the degrees by 1 (as indicated by the notation $\bullet + 1$"). From the graded point of view (Remark 4.42) one should think of the symbol d as having degree 1 (and this will dictate some signs later on).

There are quite a few different ways to proceed to introduce d. But, before anything, let us point out the properties that we may want to be satisfied by d; we will see that some of them already force what d should be, or at least its uniqueness. Already at the start, we have to decide whether we want to discuss d (properties, definitions, existence, etc) on a fixed manifold M, or for all manifolds at once.

Of course, first of all we want:

DeRham-0: *d is $\mathbb{R}$-linear and, on 0-forms (functions), it is the usual differential of functions.*

Next, by thinking about the behaviour of d on the product of functions, we would like a similar Leibniz identity w.r.t. the wedge product $\omega \wedge \eta$:

DeRham-1: *d satisfies the (graded) Leibniz identity:*

$$\boxed{d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta).}$$

for all $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$.

For an explanation of the sign, please see the sign rule from Remark 4.42 and remember that the symbol d is assigned degree 1.

One way to come across d is by applying inductively the reasoning described in Example 4.35: if d was defined on forms up to degree $k-1$, and we wonder when a k-form $\omega \in \Omega^k(M)$ is exact (i.e. of the type $d\eta$ for some $\eta \in \Omega^{k-1}(M)$), looking for "the natural conditions" that ω must satisfy, one discovers an expression $d\omega$ whose non-vanishing is an obstruction to ω being exact; this will be d on the k-forms. Of course, underlying the entire idea is one very simple property that d will have:

DeRham-2: $d \circ d = 0$.

Example 4.43. On $M = \mathbb{R}^m$, if we assume these properties for d, we already know how to compute d on arbitrary forms. E.g., for

$$\omega = (x^2 + yz)dx \wedge dy,$$

applying the Leibniz rule we find

$$d\omega = d(x^2 + yz) \wedge dx \wedge dy + (x^2 + yz)d(dx \wedge dz).$$

For the term $d(x^2 + yz)$ we get, using "DeRham-0", $2xdx + ydz + zdy$. The term $d(dx \wedge dz)$ is, by Leibniz and $d^2 = 0$, zero. We obtain

$$d\omega = 2xdx \wedge dx \wedge dy + ydz \wedge dx \wedge dy + zdy \wedge dx \wedge dy.$$

Remembering that $dx \wedge dx = 0$ (by graded commutativity of the wedge), and similarly $zdy \wedge dx \wedge dy = 0$, we are left with

$$d\omega = ydz \wedge dx \wedge dy,$$

or, to keep the natural order of the variables,

$$d\omega = ydx \wedge dy \wedge dz.$$

Actually, a similar reasoning forces the definition of d on all k-forms on $\mathbb{R}^m$: any k-form is a sum of forms of type

$$fdx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

and, by the same arguments as above,

$$d(fdx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

The fact that the d defined in this way does indeed satisfy (DeRham-1) and (DeRham-2) is now a simple check forms of this type.

Of course, exactly the same applies to any open subset $U \subset \mathbb{R}^m$ or to any domain U of a coordinate chart (U, χ) in any manifold M (we would use then the 1-forms $d\chi_i$ instead of dx_i); given the uniqueness of d satisfying the axioms, it follows that d doesn't even depend on the actual χ that we use to write down the formulas.

Corollary 4.44. *If M is a manifold, then for any $U \subset M$ that is the domain of a coordinate chart, there is and is unique an operator*

$$d_U : \Omega^{\bullet}(U) \to \Omega^{\bullet+1}(U)$$

satisfying (DeRham-0), (DeRham-1) and (DeRham-2). It is given by

$$d_U(fd\chi_{i_1} \wedge \ldots \wedge d\chi_{i_k}) = \sum_i \frac{\partial f}{\partial \chi_i} d\chi_i \wedge d\chi_{i_1} \wedge \ldots \wedge d\chi_{i_k}$$

where $\chi : U \to \Omega \subset \mathbb{R}^m$ is any coordinate chart defined on U (but independent of the actual χ).

Therefore, an obvious way to proceed to define d on a general M is to ”proceed locally”. That would mean that we would be looking for operators d that are local in the following sense:

DeRham-3: *Locality: if $U \subset M$ is open and $\omega \in \Omega^\bullet(M)$ is supported inside U, then so is $d\omega$.*

As for functions, the support of ω is the closure of the set of points $p \in M$ at which $\omega_p \neq 0$.

Exercise 4.45. Show that a linear operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is local if and only if for any open $U \subset M$ there is a linear operator $d|_U : \Omega^\bullet(U) \to \Omega^{\bullet+1}(U)$ with the property that

$$(d\,\omega)|_U = d|_U(\omega|_U) \quad \text{for all } \omega \in \Omega^\bullet(U).$$

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{\ d\ } & \Omega^{\bullet+1}(M) \ . \\ {\scriptstyle \text{restr}}\downarrow & & \downarrow{\scriptstyle \text{restr}} \\ \Omega^\bullet(U) & \xrightarrow{\ d|_U\ } & \Omega^{\bullet+1}(U) \end{array}$$

Then show that, given two local operators d and d', $d = d'$ if and only if M can be covered by opens U on which $d|_U = d'|_U$.

Corollary 4.46. *On any manifold M there exists and is unique an operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ which is local and so that for any domain U of a coordinate chart, $d|_U$ coincides with d_U from the previous corollary. Moreover, d will automatically satisfy (DeRham-0), (DeRham-1) and (DeRham-2).*

Proof. The definition is forced: for $p \in M$, choosing a coordinate chart (U, χ) around p we must have

$$(d\omega)_p = (d_U\omega|_U)_p.$$

Given the uniqueness property of the d_Us it follows that the right hand side does not depend on the coordinate chart that we use, hence it can be used to define $d\omega$ unambiguously. Since the conditions (DeRham-1) and (DeRham-2) are local and they hold for the d_Us, it follows that they hold for d as well. 😀

It is interesting to note that locality can be obtained from the rest of the axioms:

Lemma 4.47. *On a manifold M, if a linear map $D : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ satisfies the condition that*

$$D(f \cdot \omega) = fD(\omega) + df \wedge \omega \quad \textit{for all } f \in \mathscr{C}^\infty(M), \omega \in \Omega^\bullet(M)$$

(in particular if D satisfies (DeRham-0) and (DeRham-1)) then D is local.

Proof. We fix ω, we denote by A its support and we claim that the support of $d\omega$ is inside A. Note that A is the largest closed subset of M outside which ω vanishes. Therefore suffices to show that $d(\omega)$ vanishes outside A. Let $p \in M \setminus A$; since A is closed, $M \setminus A$ is open, hence we find a smooth function $\eta \in \mathscr{C}^\infty(M)$ that is zero on A and is 1 in a neighborhood of p. But then $\omega = (1-\eta) \cdot \omega$ and, using the condition from the statement we obtain

$$d\omega = (1-\eta)d\omega - d\eta \wedge d\omega.$$

Evaluating at the point p we obtain that $d\omega$ vanishes at p. 😀

In particular,

Corollary 4.48. *On any manifold M there exists and is unique an operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ which satisfies (DeRham-0) , (DeRham-1) and (DeRham-2).*

Proof. One just needs to remark that, if d satisfies (DeRham-1) and (DeRham-2) then so do all its restrictions $d|_U$; for domains U of coordinate charts, due to the uniqueness of d_U, we obtain that $d|_U = d_U$.

Definition 4.49. The resulting operator $d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$ is called the **DeRham operator on** M (or the exterior differential).

While so far we concentrated on building $d = d_M$ on a given manifold M, one can proceed slightly differently and handle all the manifolds at once; then the natural axiom to impose is the behaviour of d_M when we change the manifold, i.e. its compatibility with pull-backs. In low degree this was seen in Exercise 4.17. It means that, for any smooth map $F : M \to N$ one has a commutative diagram

$$\begin{array}{ccc} \Omega^{\bullet}(N) & \xrightarrow{d_N} & \Omega^{\bullet+1}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega^{\bullet}(M) & \xrightarrow{d_M} & \Omega^{\bullet+1}(M) \end{array} ,$$

i.e. DeRham differentials commute with pull-backs:

$$F^* \circ d_N = d_M \circ F^*.$$

Note that this condition forces automatically that each d is local. Therefore:

Corollary 4.50. *If we are looking for operators $d_M : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$, one for each manifold M, such that they are compatible with pull-backs, we have uniqueness and existence under any of the following requirements:*

1. *on $\mathbb{R}^m$, $d_{\mathbb{R}^m}$ is the operator previously discussed.*
2. *all d_M satisfy (DeRham-0) and (DeRham-1).*

Of course, for any manifold M, d_M will be precisely the DeRham operator.

And here is another way to introduce the DeRham operator: just right away, by explicit formulas. One way to find those formulas would be by returning to the previous idea of proceeding like in Example 4.35 inductively, by looking for the natural conditions for a form to be exact.

Proposition 4.51 (the Koszul formula). *One has the following explicit formula for the DeRham operator on any manifold M: for $\omega \in \Omega^k(M)$, $d\omega \in \Omega^{k+1}(M)$ is given by*

$$\begin{aligned} d\omega(X_0,\ldots,X_k) = & \sum_{i=0}^{k} (-1)^i L_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_k)) \end{aligned} \tag{4.2.4}$$

where the elements under the hat are deleted.

Proof. Note that the previous formula is skew-symmetric and $\mathscr{C}^{\infty}(M)$-multilinear (check that!). Hence it defines an operator d_M for each M. Looking back at the previous corollary, we see that all that is used is the compatibility with pull-backs by diffeomorphisms and by inclusions of opens inside manifolds. That is clearly true for the d_Ms. Hence we are left with checking that, for $\mathbb{R}^m$, this formula gives back the standard DeRham operator. Moreover, a simple check shows that $d(f \cdot \omega) = f \cdot d\omega + df \wedge \omega$ for any function f, hence it suffices to restrict our attention to forms on $\mathbb{R}^m$ of type $\omega = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and we are left with showing that

$$d_{\mathbb{R}^m}(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = 0$$

(where recall that $d_{\mathbb{R}^m}$ now denotes the operator from the statement). This can be done in several ways:

- direct computation (straightforward but not so nice).
- check it first for $k = 1$ (obvious) and then check the Leibniz identity on expressions of form $\omega \wedge \eta$ where ω is a 1-form (and this computation is not that ugly and can easily be carried out on a general manifold M). 😀

Exercise 4.52. Check by direct computation that, defining d by the formula (4.2.4), then it satisfies the Leibniz identity on expressions of form $\omega \wedge \eta$ where ω is a 1-form.

Exercise 4.53. Which one of the following 2-forms on S^2:

$$\theta_1 = xdy - ydx + zdz, \quad \theta_2 = xdy + ydx + zdz \quad (\text{restricted to } S^2)$$

is closed? But exact?

Exercise 4.54 (from the 2023/2024 exam). Let M and V_1, V_2, V be the manifold and vector fields appearing in Exercise 3.102 and consider the form

$$\omega = 2xdy \wedge dz + zdx \wedge dy - du \wedge dz \in \Omega^2(M).$$

(e) Compute $\omega(V_1, V_2)$.
(f) Show that ω is closed.
(g) compute $i_V(\omega)$ and $L_V(\omega)$.
(h) Show that ω is nowhere zero.
(i) For the diffeomorphism $F : S^2 \to M$ the you found compute $F^*\omega$.

Exercise 4.55 (from the 2021-2022 exam). Let M and V_1, V_2, V_3 be the manifold and vector fields appearing in Exercise 3.106.

(h) Given an example of a 1-form θ on N which is nonzero but such that $\theta|_M = 0$.
(i) Using the 1-forms on M:

$$\theta_1 = -y \cdot dx + x \cdot dy + \frac{x^2 + y^2}{z^2 + t^2}(t \cdot dz - z \cdot dt),$$

$$\theta_2 = -\frac{1}{z^2 + t^2}(z \cdot dz + t \cdot dt), \quad \theta_3 = \frac{1}{z^2 + t^2}(-t \cdot dz + z \cdot dt),$$

show that V^1, V^2, V^3 induce a basis of T_pM at each $p \in M$.
(j) Show that $d\theta_2 = 0$ and $d\theta_3 = 0$.
(k) Consider now $\omega := dx \wedge dy \wedge dz$ as a form on M (the restriction to M). Compute $i_{V^1}(\omega)$, and then $\omega(V^1, V^2, V^3)$.
(l) Find all $p \in M$ at which $\omega_p = 0$.
(m) Show that $L_{V^1}(\omega) = -dx \wedge dz \wedge dt$. After discussing Cartan's magic formula, return to this item and do it simpler.

Exercise 4.56 (from the 2023/2024 exam). Let M be an arbitrary manifold. Recall that a differential form ω on M is called closed if $d\omega = 0$, and it is called exact if there exists a form η such that $\omega = d\eta$.

(a) Show that the wedge product of two closed forms is again closed.
(b) Show that for any form ω of even degree, $\omega \wedge d\omega$ is closed.
(c) In (b), is it true that $\omega \wedge d\omega$ is actually exact?
(d) Is the conclusion from (b) true also for forms of odd degrees?

Exercise 4.57 (from the 2018/2019 exam). Let $\theta_1,\ldots,\theta_n \in \Omega^1(M)$ be $n = \dim M$ one-forms on M which are dual to a set $\{X^1,\ldots,X^n\}$ of vector fields on M, in the sense that

$$\theta^i(X_j) = \delta_i^j \quad (1 \text{ if } i = j \text{ and } 0 \text{ otherwise}).$$

(a) show that the 1-forms $\theta_1,\ldots,\theta_n$ are closed if and only if $[X_i,X_j] = 0$ for all i and j.
(b*) show that for any family of vector fields $X^1,\ldots,X^n$ satisfying $[X_i,X_j] = 0$ for all i and j and which are linearly independent at some point $x_0 \in M$, there exists a chart (U,χ) near x_0 such that

$$X_x^1 = \left(\frac{\partial}{\partial\chi_1}\right)_x, \quad \ldots \quad, X_x^n = \left(\frac{\partial}{\partial\chi_n}\right)_x \quad \text{for all } x \in U.$$

(c) Deduce that if $\{\theta_1,\ldots,\theta_n\}$ is a set of closed 1-forms that are linearly independent at some point $x_0 \in M$, then there exists a chart (U,χ) of M around x_0 such that $\theta_i|_U = d\chi_i$ on U, for all $i \in \{1,\ldots,n\}$.

4.2.4 Cartan's magic formula $di_X + i_X d = \mathscr{L}_X$

We first discuss the Lie derivative of forms $\omega \in \Omega^\bullet(M)$ along vector fields $V \in \mathfrak{X}(M)$, giving rise to operations

$$\mathscr{L}_V : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

These Lie derivatives fit in the general philosophy explained and illustrated in Section 3.3.4 of the previous chapter:

Definition 4.58. For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the Lie derivative of ω along** V as

$$\mathscr{L}_V(\omega) := \left.\frac{d}{dt}\right|_{t=0} (\phi_V^t)^*\omega \in \Omega^k(M).$$

As before, when V is not complete, one has to look at each $p \in M$ and note that the resulting formula

$$\mathscr{L}_V(\omega)(X_p^1,\ldots,X_p^k) = \left.\frac{d}{dt}\right|_{t=0} \omega_{\phi_V^t(p)}\left((d\phi_V^t)_p(X_p^1),\ldots,(d\phi_V^t)_p(X_p^k)\right) \tag{4.2.5}$$

makes sense independent of whether V is complete or not. From the graded point of view, $\mathscr{L}_V$ has degree 0.

Proposition 4.59. *For any vector field $V \in \mathfrak{X}(M)$:*

1. *$\mathscr{L}_V : \Omega^\bullet \to \Omega^\bullet(M)$ is a degree zero derivation, i.e. it is linear and satisfies*

$$\mathscr{L}_V(\omega \wedge \eta) = \mathscr{L}_V(\omega) \wedge \eta + \omega \wedge \mathscr{L}_V(\eta)$$

   for any two differential forms ω and η.
2. *$\mathscr{L}_V$ commutes with d: $\mathscr{L}_V \circ d = d \circ \mathscr{L}_V$.*
3. *$\mathscr{L}_V$ is the only degree zero derivation on $\Omega^\bullet(M)$ which commutes with d and which, on functions, it is the usual derivative L_V.*

Moreover, it can be described by the explicit formula:

$$\mathscr{L}_V(\omega)(X_1,\ldots,X_k) = L_V(\omega(X_1,\ldots,X_k)) - \sum_{i=1}^k \omega(X_1,\ldots,X_{i-1},[V,X_i],X_{i+1},\ldots,X_k).$$

Proof. The derivation identity follows from the standard one and the fact that $(\phi_V^t)^*$ is compatible with the wedge operation:

$$\frac{d}{dt}\Big|_{t=0}(\phi_V^t)^*(\omega\wedge\eta) = \frac{d}{dt}\Big|_{t=0}\big((\phi_V^t)^*\omega\wedge(\phi_V^t)^*\eta\big) = \left(\frac{d}{dt}\Big|_{t=0}(\phi_V^t)^*\omega\right)\wedge\eta + \omega\wedge\left(\frac{d}{dt}\Big|_{t=0}(\phi_V^t)^*\eta\right).$$

That $\mathscr{L}_V$ commutes with d follows from the similar property of the pull-backs $(\phi_V^t)^*$.

For the uniqueness in (3), assume that D is another operators with the same properties. It follows that $\mathscr{L}_V(df) = dL_V(f)$, and we obtain that $D(\omega) = \mathscr{L}_V(\omega)$ on differential forms of type

$$\omega = \sum_i f_i \wedge dg_i^1 \wedge \ldots \wedge dg_i^k.$$

It would suffices to show that any k-form can be written in this way. This is true and it would follow for instance if we proved that M can be embedded in some Euclidean space (why would that be enough?). To avoid using such a result, one can first restrict to manifolds on which it is certainly true: domain of coordinate charts. In more detail: due to the Leibniz identity it follows (as in the discussions about d) that D is local, we can talk about $D|_U$ and, on domains of coordinate charts, $D|_U$ still has the same properties as D. Hence $D|_U = \mathscr{L}_V|_U$, and then $D = \mathscr{L}_V$.

For the final formula, we start from the explicit formula (4.2.5) and we use that $\frac{d}{dt}|_{t=0}(\phi_V^t)^*(X) = [V,X]$ (Proposition 3.90 in the previous chapter). Or, equivalently,

$$\frac{d}{dt}\Big|_{t=0}(d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = [V,X]_p$$

for all X. Similar to (3.3.6) (and for the same reasons)

$$(d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = X_p + t[V,X]_p + o(t^2)$$

for t near 0, and then

$$(d\phi_V^t)_p(X_p) = X_{\phi_V^t(p)} - t[V,X]_{\phi_V^t(p)} + o(t^2).$$

Inserting this in (4.2.5) and using also Proposition 3.89 from the previous chapter (applied to V and $f = \omega(X_1,\ldots,X_k)$ we find the formula from the statement. 😀

Another interesting operators induced by a vector field $V \in \mathfrak{X}(M)$, but simpler this time, is the interior product:

Definition 4.60. For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the interior product ω by V**, denoted $i_V(\omega)$ as the differential form of degree $k-1$ given by

$$i_V(\omega)(X_1,\ldots,X_{k-1}) = \omega(V,X_1,\ldots,X_{k_1}).$$

This defines an operator

$$i_V : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M);$$

since it lowers the degree by 1, we say it is of degree -1.

Proposition 4.61. *For any vector field $V \in \mathfrak{X}(M)$,i_V is a degree -1 derivation, i.e. it is linear and satisfies*

$$i_V(\omega\wedge\eta) = i_V(\omega)\wedge\eta + (-1)^k\omega\wedge i_V(\eta)$$

for any two differential forms $\omega \in \Omega^k(M)$ and η. Actually, it is the only degree -1 derivation on $\Omega^\bullet(M)$ which, on 1-forms, is the evaluation on V.

Proof. The derivation formula is a completely straightforward computation. For the uniqueness, one can just imitate the arguments used for the uniqueness of $\mathscr{L}_V$, just that it is a bit simpler now (being degree -1, it kills of the function, and the condition from the statement prescribes i_V on 1-forms hence, by the Leibniz identity, on all the sums of wedges of 1-forms. One can either show that any differential form can be written as such a sum, or just provide a local argument completely similar to that for $\mathscr{L}_V$. 😄

Next: the Cartan magic formula, which says that $\mathscr{L}_V$ is the graded commutator of d and i_V; given that d has degree 1 and i_V has degree -1, that means:

Theorem 4.62 (Cartan's magic formula). *For any vector field $V \in \mathfrak{X}(M)$ one has*

$$\mathscr{L}_V = d \circ i_V + i_V \circ d.$$

Proof. A simple computation using the Leibniz identities for d and i_V (with special care at the signs involved) shows that $d \circ i_V + i_V \circ d$ is a derivation of degree zero. Since $d \circ d = 0$, we see that it also commutes with d. Moreover, on functions f it gives $i_V d(f) = L_V(f)$. Hence, by the uniqueness from Proposition 4.59, it must coincide with $\mathscr{L}_V$. 😄

Exercise 4.63. Return to Exercise 4.54 and prove again that $L_V(\omega)$, now using Cartan's magic formula.

Exercise 4.64. Return now to item (m) of Exercise 4.55 and do it again, now using Cartan's magic formula.

Exercise 4.65 (from the 2020/2021 exam). Assume that M is a compact manifold, $F: M \to S^1$ is a smooth map and that we have a vector field $V \in \mathfrak{X}(M)$ that is projectable to $\frac{\partial}{\partial \theta}$, i.e.

$$(dF)_p(V_p) = \left(\frac{\partial}{\partial \theta}\right)_{F(p)} \quad \text{for all } p \in M.$$

We also consider the pull-back via f of the canonical 1-form $d\theta \in \Omega^1(S^1)$,

$$\omega := F^*(d\theta) \in \Omega^1(M).$$

Show that: ω is closed, $\omega(V) = 1$ and $\mathscr{L}_V(\omega) = 0$.

Exercise 4.66 (from the 2018/2019 exam). Let M be a manifold, and $X, Y \in \mathfrak{X}(M)$ two vector fields. We look at

$$L_X \circ i_Y - i_Y \circ L_X : \Omega^k(M) \to \Omega^{k-1}(M) \quad (\text{for all } k \geq 0).$$

Show that:

(a) on 1-forms $\theta \in \Omega^1(M)$, it coincides with $i_{[X,Y]}$.
(b) it satisfies a Leibniz-type identity that you have to write down yourself.
(c) $L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]}$.

4.3 More exercises

Exercise 4.67 (from the 2019/2020 exam). Let $\theta \in \Omega^1(M)$ be a closed 1-form and define

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta(\omega) := d\omega + \theta \wedge \omega.$$

Show that $d_\theta \circ d_\theta = 0$ if and only if θ is closed.

Exercise 4.68 (from the 2021/2022 exam). Show that, on any manifold M,

$$L_{f\cdot X}(\omega) = f \cdot L_X(\omega) + df \wedge i_X(\omega)$$

for any smooth function $f \in C^\infty(M)$, any vector field $X \in \mathfrak{X}(M)$ and any differential form $\omega \in \Omega^k(M)$

Exercise 4.69 (from the 2023/2024 retake). Assume that $F : P \to M$ is a surjective submersion and we consider the induced pull-back operations (one for each natural number k)

$$F^* : \Omega^k(M) \to \Omega^k(P).$$

The forms on P that are in the image of this map will be called F-basic.

(a) Show that F^* is an injective map.
(b) Show that if $\eta \in \Omega^k(M)$ has the property that $F^*\eta$ is exact, then η is closed.
(c) Assume that $\theta \in \Omega^1(M)$ is such that $F^*\theta = df$ for some function $f \in C^\infty(P)$ which is not F-basic (recall the functions are 0-forms!). Show that θ is not exact.
(d) Deduce, using the previous items, that the 1-form $xdy - ydx \in \Omega^1(S^1)$ is not exact.

Exercise 4.70. Consider the sphere $S^2 \subset \mathbb{R}^3$ and we use (x, y, z) to denote the standard coordinates in $\mathbb{R}^3$. We consider the following vector field tangent to the sphere

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$$

as well as the volume form on the sphere:

$$\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy \in \Omega^2(S^2)$$

(as before, while the previous formula defines a 2-form on $\mathbb{R}^3$, σ is the restriction to S^2).

(a) Compute $i_X(\sigma)$ and $d(i_X(\sigma))$.
(b) Compute $d\sigma$ and $i_X(d\sigma))$.
(c) Compute $L_X(\sigma)$ in two ways: one using the Cartan formula, and one using the properties of L_X (being a derivation, and commuting with d).
(d) Compute the flow ϕ^t of X.
(e) Show that $(\phi^t)^*\sigma = \sigma$ for all $t \in \mathbb{R}$.

Exercise 4.71. Consider the following 1-form on $\mathbb{R}^3$:

$$\theta = (z^2 + 2xy) \cdot dx + (x^2 + 2yz) \cdot dy + (y^2 + 2zx) \cdot dz.$$

Do the following:

1. compute $\theta \wedge dy \wedge dz$ (i.e. write it as $? \cdot dx \wedge dy \wedge dz$, with ? to be computed).
2. show that $d\theta = 0$.
3. find a smooth function $f \in C^\infty(\mathbb{R}^3)$ such that $\theta = df$.
4. for $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi(x, y, z) = (y, z, x)$ show that $\phi^*(\theta) = \theta$.
5. show that for any smooth map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$, the 2-form $\theta \wedge \psi^*(\theta)$ is closed.
6. For the vector field $X = x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}$ compute $i_X(\theta)$ and $L_X(\theta)$.
7. Compute the flow ϕ_X^t of X.

8. With the formula for the ϕ_X^t and $L_X(\theta)$ that you found at the previous two points check directly that, indeed,
$$L_X(\theta) = \frac{d}{dt}\Big|_{t=0} (\phi_X^t)^*(\theta).$$
9. show that $\theta|_{S^2} \neq 0$.
10. Find/describe a 2-dimensional submanifold $M \subset \mathbb{R}^3$ such that $\theta|_M = 0$.

Exercise 4.72 (from the 2023/2024 retake). Consider
$$\theta := 2x^3 dx + ydy + zdz \in \Omega^1(\mathbb{R}^3).$$
Please do the following:

(a) Show that there exist 2-dimensional embedded submanifolds $M \subset \mathbb{R}^3$ with the property:
$$\theta|_M = 0.$$

From now on we assume that such a manifold M is fixed.

(b) Show that, for any $p \in M$, one has $T_pM = \mathrm{Ker}\theta_p$.
(c) Show that the following define vector fields on M:
$$Y_p = y\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial y}, \quad (\text{for } p = (x,y,z) \in M).$$
$$Z_p = z\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial z}, \quad (\text{for } p = (x,y,z) \in M).$$
(d) Compute $V = [Y,Z]$. More precisely, show that V is of type $cx^n\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)$, where the constants c and n have to be computed.
(e) Compute the integral curves of V.
(f) Is V necessarily complete?
(g) Compute $i_V(\mu)$ and $L_V(\mu)$ where μ is the form on M given by
$$\mu := \frac{1}{3}xdy\wedge dz + ydz\wedge dx + zdx\wedge dy \in \Omega^2(M).$$
(h) Show that μ is nowhere vanishing (as a 2-form on M).

Exercise 4.73. As a continuation of Exercise 3.109:

8. Show that there exist, and they are unique, two 1-forms $d\theta_1$ and $d\theta_2$ on $\mathbb{T}^2$ with the property that
$$d\theta_i\left(\frac{\partial}{\partial\theta_j}\right) = \delta_{i,j} \text{ (1 if } i = j \text{ and 0 otherwise)}.$$
9. Show that for any smooth function f on $\mathbb{T}^2$ one has
$$df = \frac{\partial f}{\partial\theta_1}\cdot d\theta_1 + \frac{\partial f}{\partial\theta_2}\cdot d\theta_2.$$

10. In contrast with what the notation may suggest, $d\theta_1$ and $d\theta_2$ are not exact forms. To see this, you are asked to prove something a bit more general: on any compact manifold M, any 1-form that is exact must vanish at at least one point in M.
11. However, show that $d\theta_1$ and $d\theta_2$ are both closed.

Exercise 4.74. Consider the following three 1-forms on the sphere S^3:

$$\theta^1 := 2\,(-ydx + xdy + tdz - zdt)\,,$$

$$\theta^2 := 2\,(-zdx - tdy + xdz + ydt)\,,$$

$$\theta^3 := 2\,(-tdx + zdy - ydz + xdt)\,.$$

Show that:

(a) the differentials of these 1-forms satisfy:

$$d\theta_1 = -\theta_2 \wedge \theta_3, \quad d\theta_2 = -\theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2$$

(b) that, for all $f \in \mathscr{C}^\infty(S^3)$, one has:

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V_3}(f) \cdot \theta_3,$$

where V^1, V^2 and V^3 are the vector fields from Exercise 3.35.

(c) For the vector field V from Exercise 3.23 one has

$$L_V\theta_1 = L_V\theta_2 = L_V\theta_3 = 0.$$

Exercise 4.75. Let us first show that, in the previous exercise, all that matters are the main properties of the vector fields involved (one doesn't even know that we are working on the sphere). More precisely, assume that M is a connected 3-dimensional manifold, and $V, V^1, V^2, V^3 \in \mathfrak{X}(M)$ are vector fields on M with the property that V_p^1, V_p^2, V_p^3 form a basis of T_pM for all $p \in M$ and

$$[V^1, V^2] = V^3, \quad [V^2, V^3] = V^1, \quad [V^3, V^1] = V^2.$$

$$[V, V_1] = [V, V_2] = [V, V_3] = 0.$$

Let $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ be the 1-forms that are dual to V^1, V^2, V^3, i.e. satisfying

$$\theta_i(V^j) = \delta^i_j \quad (1 \text{ if } i = j \text{ and } 0 \text{ otherwise}).$$

Show that:

(a) for any $f \in \mathscr{C}^\infty(M)$ one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3.$$

(b) the 1-forms θ_i satisfy

$$d\theta_1 = -\theta_2 \wedge \theta_3, \quad d\theta_2 = -\theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2.$$

$$L_V\theta_1 = L_V\theta_2 = L_V\theta_3 = 0.$$

On the other hand, despite the apparent greater generality of the setting, we are actually getting closer to ... the Hopf fibration. More precisely, defining

$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3),$$

show that:

(c) the differentials of these functions satisfy:

$$dh_1 = h_2 \cdot \theta_3 - h_3 \cdot \theta_2,$$
$$dh_2 = h_3 \cdot \theta_1 - h_1 \cdot \theta_3,$$
$$dh_3 = h_1 \cdot \theta_2 - h_2 \cdot \theta_1$$

(d) the function

$$h = (h_1, h_2, h_3) : M \to \mathbb{R}^3$$

takes values in a sphere S^2_r (of some radius $r > 0$).

(e) furthermore, h must be a submersion.

(f) What happens when $M = S^3$ and V, V^1, V^2 and V^3 are the vector fields from Exercise 3.23 and Exercise 3.35?

Exercise 4.76. Return to M from Exercise 2.194,

$$M = \{(x,y,z,t) \in \mathbb{R}^4 : xt = yz + 12\}.$$

Consider the following vector fields

$$X^1 := \frac{1}{2}\left(y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} + t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right),$$

$$X^2 := \frac{1}{2}\left(-x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} - z \cdot \frac{\partial}{\partial z} + t \cdot \frac{\partial}{\partial t}\right),$$

$$X^3 := \frac{1}{2}\left(-y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} - t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right)$$

and, in case you haven't done Exercise 3.104, jut assume that you know already that these are vector fields on M satisfying: $[X^1, X^2] = X^3$, $[X^2, X^3] = -X^1$, $[X^3, X^1] = -X^2$. Show that:

(a) The 1-forms

$$\theta_1 = (-z \cdot dx + t \cdot dy + x \cdot dz - y \cdot dt)/12,$$
$$\theta_2 = (-t \cdot dx - z \cdot dy + y \cdot dz + x \cdot dt)/12,$$
$$\theta_3 = (\ z \cdot dx + t \cdot dy - x \cdot dz - y \cdot dt)/12$$

satisfy $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise.

(b) Deduce that X^1_p, X^2_p, X^3_p form a basis of T_pM for all $p \in M$.

(c) Deduce that two 2-forms $\eta, \xi \in \Omega^2(M)$ coincide if and only if

$$\eta(X^1, X^2) = \xi(X^1, X^2), \quad \eta(X^2, X^3) = \xi'(X^2, X^3), \quad \eta(X^3, X^1) = \xi(X^3, X^1).$$

(d) Show that $d\theta_1 = \theta_2 \wedge \theta_3, \quad d\theta_2 = \theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2.$

Chapter 5
Integration on manifolds, Stokes theorem, DeRham cohomology

5.1 Integration on manifolds

5.1.1 Integration on $\mathbb{R}^m$

In this section we explain how to extend standard integration from $\mathbb{R}^m$ to more general manifolds. By standard integration we mean the integral

$$\int : \mathscr{C}_c^\infty(\mathbb{R}^m) \to \mathbb{R}, \quad f \mapsto \int f = \int_{\mathbb{R}^m} f(x)dx = \int_{\mathbb{R}^m} f(x_1, \dots, x_m)dx_1 \dots dx_m$$

defined (only) on the space of compactly supported smooth functions. For $\Omega \subset \mathbb{R}^m$ open and $f \in \mathscr{C}_c^\infty(\Omega)$, we will use the notation

$$\int_\Omega f = \int_\Omega f(x)dx$$

for the integral $\int \tilde{f}$ where $\tilde{f}$ is f extended by zero outside Ω.

Actually, what we need to know about the standard integration is that:

- it is linear in f.
- change of variables formula: if $c : \Omega \to \Omega'$ is a diffeomorphism between two opens in $\mathbb{R}^m$ then

$$\int_{\Omega'} f(x')dx' = \int_\Omega |\text{Jac}_x(c)| \cdot f(c(x))dx$$

  for any $f \in \mathscr{C}_c^\infty(\Omega')$. Or, more compactly,

$$\int_{\Omega'} f = \int_\Omega |\text{Jac}(c)| \cdot f \circ c$$

Recall that the Jacobian of c is the function

$$\text{Jac}(c) : \Omega \to \mathbb{R}, \quad x \mapsto \text{Jac}_x(c) := \det\left(\frac{\partial c_i}{\partial x_j}(x)\right).$$

Note that its presence in the change of variable formula shows that the integral of a function f is not independent of the coordinates. For that reason, the objects that can (naturally) be integrated on manifolds will not be functions (we will be able to integrate functions only when a certain extra-choice will be made). Instead, one has to look for objects which locally (in coordinate charts) can be represented by smooth functions, and so that when we change the coordinates the representing functions change by a Jacobian factor. This brings us to differential forms of top degree m (also making more sense of the symbol of the expression "$f(x_1, \dots, x_m)dx_1 \dots dx_m$" in the integration).

5.1.2 Top degree forms; orientations

Definition 5.1. Given an m-dimensional manifold M, the differential forms of degree precisely m,

$$\omega \in \Omega^m(M),$$

are called **top-differential form** on M. It is called a **volume form** on M if it is nowhere vanishing:

$$\mu_p \neq 0 \quad \forall\, p \in M.$$

Any chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of M induces a volume form on U, namely

$$\mu_\chi := d\chi_1 \wedge \ldots \wedge d\chi_m. \tag{5.1.1}$$

On the other hand, a top-differential form $\omega \in \Omega^m(M)$ can be written locally, w.r.t. any coordinate chart χ, as

$$\omega|_U = \chi^*\left(f_\chi^\omega \cdot dx_1 \wedge \ldots \wedge dx_m\right) = \chi^*(f_\chi^\omega) \cdot d\chi_1 \wedge \ldots \wedge d\chi_m == \left(f_\chi^\omega \circ \chi\right) \cdot \mu_\chi$$

for some function $f_\chi^\omega \in \mathscr{C}^\infty(\Omega)$.

Definition 5.2. The function f_χ^ω is called **the representing function of ω w.r.t. the chart** (U, χ).

How does the representing function change when we change the chart? Well, if $\chi' : U' \to \Omega'$ is another chart, denoting by $c = \chi' \circ \chi^{-1} : \Omega \to \Omega'$ the change of coordinates, we know that

$$d\chi_i' = \sum_j \frac{\partial \chi_i'}{\partial \chi_j} \cdot d\chi_j = \sum_j \frac{\partial c_i}{\partial x_j} \circ \chi \cdot d\chi_j.$$

From this, and the graded commutativity of the wedge-product, we deduce that

$$d\chi_1' \wedge \ldots \wedge d\chi_m' = \det\left(\frac{\partial \chi_i'}{\partial \chi_j}\right) \cdot d\chi_1 \wedge \ldots \wedge d\chi_m,$$

where the coefficient is

$$\det\left(\frac{\partial \chi_i'}{\partial \chi_j}\right) = \det\left(\frac{\partial c_i}{\partial x_j} \circ \chi\right) = \mathrm{Jac}(c \circ \chi).$$

Lemma 5.3. *The representing functions with respect to any two charts (U, χ) and (U', χ') are related by*

$$f_\chi^\omega = Jac(c) \cdot f_{\chi'}^\omega \circ c, \quad \text{where } c = \chi' \circ \chi^{-1} \text{ (the change of coordinates map).}$$

This is "almost" the expression in the change of variables formula- only the absolute value is missing. For that reason, we need the notion of orientation.They will be defined so that:

- any volume form μ induces an orientation $\mathscr{O}_\mu$, and any orientation arises like this.
- two volume forms μ and μ' induce the same orientation if and only if (5.1.2) holds.

Definition 5.4. Two volume forms μ and μ' are said to be orientation equivalent, denoted $\mu \sim \mu'$, if

$$\mu' = f \cdot \mu, \quad \text{with } f > 0. \tag{5.1.2}$$

An **orientation** $\mathscr{O}$ on M is an equivalence class w.r.t. $\sim$. We denote by $\mathscr{O}_\mu := [\mu]$ the class (hence orientation) corresponding to a volume form μ.

An **oriented manifold** is a pair $(M, \mathscr{O})$ consisting of a manifold together with an orientation.

Example 5.5. On $\mathbb{R}^{m+1}$ one has "the standard volume form"

$$\mu_{\mathbb{R}^{m+1}} := dx_0 \wedge dx_1 \wedge \ldots \wedge dx_m.$$

Taking the interior product with the radial vector field $R = \sum_i x_i \cdot \frac{\partial}{\partial x_i}$ we obtain an m-form

$$i_R(\mu_{\mathbb{R}^{m+1}}) = \sum_{i=0}^{m} (-1)^i x_i \cdot dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m \in \Omega^m(\mathbb{R}^m).$$

By restricting to the sphere $S^m \subset \mathbb{R}^{m+1}$ one obtains a volume form on S^m (exercise!):

$$\mu_{S^m} := \left(\sum_{i=0}^{m} (-1)^i x_i \cdot dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m \right)\Bigg|_{S^m} \in \Omega^m(S^m)$$

Example 5.6. A simple example of compact manifold that is not orientable is $\mathbb{R}P^2$. To show that one proceeds by contradiction. Assume that $\exists$ a volume-form $\omega \in \Omega^2(\mathbb{R}P^2)$. To reach a contradiction one can proceed as follows:

(a) show that the pull-back of ω via the canonical projection $\pi : S^2 \to \mathbb{R}P^2$ (section 2.2.2) is a volume-form on S^2.
(b) deduce that $f \cdot \pi^* \omega = z dx \wedge dy + y dz \wedge dx + x dy \wedge dz$ for some nowhere vanishing function $f \in C^\infty(S^2)$.
(c) Compute the pull-back of both sides by $\tau : S^2 \to S^2, \quad p \mapsto -p$ to conclude that $f(-p) = -f(p)$ for all $p \in S^2$.
(d) Using that S^2 is connected, deduce that f must vanish at some point.

Exercise 5.7. Prove that $\mathbb{R}P^2$ is not orientable following the steps described above.

Orientations allows one to talk about positively oriented bases and charts. To that end, it is useful to notice first that a volume form μ on M can be used to test when a collection of m tangent vectors at some $p \in M$ forms a basis:

$$v_1, \ldots, v_m \text{ form a basis of } T_pM \Longleftrightarrow \mu_p(v_1, \ldots, v_m) \neq 0. \tag{5.1.3}$$

The key point is that the sign of the right hand side does not depend on μ but only on the orientation that it induces.

Definition 5.8. Given a manifold M endowed with an orientation $\mathscr{O} = [\mu]$, a basis of tangent vectors at a point $p \in M$, i.e. a basis $v_1, \ldots, v_m$ of T_pM, is called **positively oriented** w.r.t. $\mathscr{O}$ if

$$\mu_p(v_1, \ldots, v_m) > 0.$$

A **positively oriented chart** is any chart (U, χ) such that the corresponding bases are positively oriented:

$$\mu_p\left(\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \right) > 0 \quad \forall\, p \in U.$$

Slightly differently: any chart (U,χ) induces an orientation on U, namely the one corresponding to the volume form (5.1.1), and called **the chart orientation of** U; this characterises positive orientability as follows:

Exercise 5.9. In an arbitrary oriented manifold $(M,\mathscr{O})$, show that a chart (U,χ) is positively oriented if and only if the chart orientation of U coincides with the orientation induced by $\mathscr{O}$ (by restriction).

On an oriented manifold we will only use positively oriented charts. The main gain for us is the resulting positivity

$$\mathrm{Jac}(\chi'\circ\chi^{-1})>0 \quad \text{for all } \chi,\chi' - \text{positively oriented.}$$

Actually, fixing an atlas with this property is equivalent to fixing an orientation on M:

Exercise 5.10. Let M be a manifold. Given an atlas $\mathscr{A}$ on M we wonder about the existence of an orientation $\mathscr{O}_{\mathscr{A}}$ on M with the property that each $\chi\in\mathscr{A}$ is adapted to $\mathscr{O}_{\mathscr{A}}$. Show that:

1. given $\mathscr{A}$, if such an orientation exists then it is unique.
2. such an orientation exists if and only if $\mathscr{A}$ is a positive atlas in the sense that:
$$\mathrm{Jac}(\chi'\circ\chi^{-1})>0 \quad \text{for all } \chi,\chi'\in\mathscr{A}. \tag{5.1.4}$$
3. given two positive atlases $\mathscr{A}$ and $\mathscr{A}'$, the corresponding orientations $\mathscr{O}_{\mathscr{A}}$ and $\mathscr{O}_{\mathscr{A}'}$ coincide if and only if
$$\mathrm{Jac}(\chi'\circ\chi^{-1})>0 \quad \text{for all } \chi\in\mathscr{A},\chi'\in\mathscr{A}'. \tag{5.1.5}$$

In other words, to chose an orientation on M is the same thing as choosing an atlas $\mathscr{A}$ satisfying (5.1.4), and two such atlases $\mathscr{A}$ and $\mathscr{A}'$ induce the same orientation on M if an only if (5.1.5) holds.

Remark 5.11 (For the interested students: orientations done a bit better). First of all, one can talk about orientations on any finite dimensional vector space V. To that end, we denote by $\mathrm{Fr}(V)$ the set of ordered bases of V, also called **frames of** V. For $u,u'\in\mathrm{Fr}(V)$ we denote by $[u:u']$ the matrix of coordinate changes from u to u' (an $m\times m$ invertible matrix with entires real numbers, where $m=\dim(V)$) and by $\mathrm{sign}(u:u')$ the sign of its determinant:

$$\mathrm{sign}(u:u'):=\mathrm{sign}\left(\det([u:u'])\right)\in\{-1,+1\}.$$

An orientation on V is basically a partition of the set of frames into two: positively and negatively oriented frames, so that two frames u and u' are of the same type if and only if $\mathrm{sign}(u:u')>0$. Abstractly:

An **orientation** on V is a map
$$\mathscr{O}:\mathrm{Fr}(V)\to\{-1,+1\}$$
with the property that
$$\mathscr{O}(u')=\mathrm{sign}(u:u')\cdot\mathscr{O}(u) \quad \text{for all } u,u'\in\mathrm{Fr}(V).$$
We also say that $(V,\mathscr{O})$ is an oriented vector space; given $(V,\mathscr{O})$ one talks about **positively oriented frames**, or negatively oriented ones, depending on whether $\mathscr{O}(u)=1$ or -1, respectively.

Any $u\in\mathrm{Fr}(V)$ gives rise to a unique orientation $\mathscr{O}_u$ by the condition that u is positively oriented for $\mathscr{O}_u$; explicitly,

$$\mathscr{O}_u:\mathrm{Fr}(V)\to\{-1,+1\}, \quad u'\mapsto\mathrm{sign}(u:u').$$

Moreover, two frames $u,u'\in\mathrm{Fr}(V)$ induce the same orientation, i.e. $\mathscr{O}_u=\mathscr{O}_{u'}$, if and only if $\mathrm{sign}(u:u')>0$. We deduce that an orientation on V can also be seen as an equivalence class of frames:

$$\mathrm{Orient}(V)\cong\mathrm{Fr}(V)/\sim \qquad \text{where } u\sim u' \text{ iff } \mathrm{sign}(u:u')>0.$$

Moving to a manifold M, we look at families

$$\mathscr{O} = \{\mathscr{O}_p : p \in M\} \quad \text{with } \mathscr{O}_p - \text{orientation on } T_pM. \tag{5.1.6}$$

We say that $\mathscr{O}$ is **smooth** if for any $p_0 \in M$ there is a chart (U, χ) around p_0 such that, for any $p \in U$,

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$

is a (positively) oriented frame of $(T_pM, \mathscr{O}_p)$. Such a chart is said to be **adapted to** $\mathscr{O}$.

Definition 5.12 (alternative definition of orientability). An **orientation on the manifold** M is a family (5.1.6) of orientations on the tangent spaces which is smooth in the previous sense. We call $(M, \mathscr{O})$ an oriented manifold, and the charts of M adapted to $\mathscr{O}$ are called **positively oriented charts** of $(M, \mathscr{O})$.

A less abstract (but also less intuitive) description of orientations builds on the remark that frames $u = (u_1, \ldots, u_m)$ of a vector space V give rise to non-zero elements in the top exterior power::

$$\mu_u := u^1 \wedge \ldots \wedge u^m \in \Lambda^m V^* \quad (m = \dim V),$$

where we are making use of the dual basis associated to u. Conversely, any non-zero element in the top exterior power arises in this way, and the condition that two frames induce the same orientation is equivalent to:

$$u \sim u' \Longleftrightarrow \mu_{u'} = \lambda \cdot \mu_u \text{ for some } \lambda > 0.$$

Passing to manifold, this amounts to looking at volume forms and we end up precisely with the equivalence relation and the notion of orientations from Definition 5.4 and the notion of positive chart from Definition 5.8:

Lemma 5.13. *A manifold M is orientable in the sense just defined (Definition 5.12) if and only if it admits a volume form $\mu \in \Omega^m(M)$ (i.e. it is orientable in the sense of Definition 5.12). Moreover, any such volume form μ induces an orientation $\mathscr{O}_\mu$ on M, uniquely determined by the condition that, at any $p \in M$, a frame*

$$v_p^1, \ldots, v_p^m$$

of T_pM is positively oriented if and only if $\mu_p(v_p^1, \ldots, v_p^m) > 0$.

Proof. The orientation $\mathscr{O}_\mu$ can be defined, at each $p \in M$, as the map

$$\mathscr{O}_{\mu,p} : \mathrm{Fr}(T_pM) \to \{-1, +1\}, \quad \mathscr{O}_{\mu,p}(v_p^1, \ldots, v_p^m) := \mathrm{sign}(\mu(v_p^1, \ldots, v_p^m)).$$

This is well-defined because of (5.1.3). We still to do the converse: if M is orientable (in the sense of Definition 5.12) show that it admits a volume form. While volume forms exist locally, the question is how to "glue them" to global volume forms. In principle, we would consider a family $\{\omega_i\}$ of volume forms on domains U_i of coordinate charts covering M indexed by a set I, a partition of unity $\{\eta_i : i \in I\}$ subordinated to the cover and set $\omega := \sum_i \eta_i \cdot \omega_i \in \Omega^m(M)$. We have to ensure that $\omega_p \neq 0$ at all p. Denoting by $J \subset I$ the set of indices j with the property that $p \in \mathrm{supp}(\eta_j)$ (a finite number of them!), $c_j = \eta_j(p) \in \mathbb{R}$, $V = T_pM$, $\mu_j = \omega_{j,p} \in \Lambda^m V^*$, we end with a finite sum

$$\sum_{j \in J} c_j \cdot \mu_j \in \Lambda^m V^*, \quad \text{with} \quad \sum_j c_j = 1, \quad \mu_j \neq 0$$

(an affine combination of non-zero elements in the top exterior power). The problem is that this sum may be zero- and it is here that we should the use the orientation $\mathscr{O}$ of M. Indeed, if we knew that V comes with an orientation and each μ_j was not only non-zero, but positively oriented, then the above sum is clearly still positively oriented, hence non-zero. Hence, all we have to do in the previous argument is to start right from the beginning with a family $\{\omega_i\}$ of volume forms on domains U_i which are positively oriented there with respect to the original orientation (i.e, at each $p \in U_i$, $\omega_{i,p}$ is so w.r.t. the orientation $\mathscr{O}_p$). 😀

5.1.3 Integration on manifolds

Returning to our discussion on integration, it should be clear now that the choice of an orientation on M is precisely what we need to solve our small problem with the absolute value of the Jacobians; here is the final outcome:

Theorem 5.14. *On any oriented manifold $(M, \mathscr{O})$, there is a well-defined map*

$$\int_M : \Omega_c^m(M) \to \mathbb{R}$$

defined on the space $\Omega_c^m(M)$ of m-forms on M that are compactly supported (i.e. which vanish outside a compact subspace of M), which is uniquely determined by the following two properties:

(1) $\int_M$ is linear.
(2) for any positively oriented chart $\chi : U \to \Omega$, if $\omega \in \Omega_c^m(M)$ is supported in U then

$$\int_M \omega = \int_\Omega f_\chi^\omega(x)dx \quad (\textit{standard integration over } \Omega \subset \mathbb{R}^m), \tag{5.1.7}$$

where f_χ^ω is the representing function of ω w.r.t. the chart (U, χ).

Proof. We divide the proof into three steps:

Step 1: integrating over domains of charts: It should now be clear from the previous discussions that for any $U \subset M$ which is inside the domain of some chart of M one obtains an integration map

$$\int_U : \Omega_c^m(U) \to \mathbb{R}$$

(defined on the space of m-forms whose support is compact and included in U); it is defined simply by the right hand side of (5.1.7), but the point is that it does not depend on the choice of a positively oriented chart χ defined on U.

Note also that for any open $V \subset U$, one has a commutative diagram

$$\begin{array}{ccc} \Omega_c^m(V) & \hookrightarrow & \Omega_c^m(U) \\ & {\scriptstyle \int_V} \searrow \quad \swarrow {\scriptstyle \int_U} & \\ & \mathbb{R} & \end{array}, \text{ i.e. } \quad \int_V \omega = \int_U \omega \quad \text{for all } \omega \in \Omega_c^m(V).$$

This implies that we can safely write $\int_M \omega$ for any $\omega \in \Omega_c^m(M)$ which is supported inside the domain of a chart: it actually does not depend on which domain we use.

Step 2: from domains of charts to the entire M: Now we claim that any $\omega \in \Omega_c^m(M)$ can be written as a finite sum

$$\omega = \sum_{i=1}^k \omega^i$$

where each $\omega_i \in \Omega_c^m(M)$ is supported in the domain of a chart (and k is some integer). Once we prove this, we will define

$$\int_M \omega := \sum_i \int_M \omega_i. \tag{5.1.8}$$

To prove the claim one uses a partition of unity argument: if K is the support of ω, since K is compact it can be covered by a finite number of opens $U_1, \ldots, U_k$, each of them a domain of a coordinate chart, and each one of them relatively compact (in the sense that they have compact closures) has compact closure. Then $\{M \setminus K, U_1, \ldots, U_k\}$ is an open cover of M and we choose a partition of unity $\eta_0, \eta_1, \ldots, \eta_k$ with η_0 supported inside $M \setminus K$ and η_i

supported inside U_i. Note that, since U_i is relatively compact, each η_i is compactly supported. Also, since η_0 is supported in $M \setminus K$, one has $\eta_0 \cdot \omega = 0$. Therefore, multiplying the identity $1 = \sum_{i=0}^{k} \eta_i$ by ω we obtain

$$\omega = \sum_{i=1}^{k} \omega_i \quad \text{where } \omega_i = \eta_i \cdot \omega \in \Omega_c^m(M).$$

Step 3: $\int_M \omega$ does not depend on the decomposition $\omega = \sum_i \omega_i$: Taking now (5.1.8) as the definition of $\int_M$, we have to check that the outcome does not depend on the way we write ω as a sum of forms supported inside domains of charts.

To prove this, assume that we have two such writings

$$\omega = \sum_{i=1}^{k} \omega_i = \sum_{j=1}^{k'} \omega_j'$$

Let K be a compact that contains the supports of all ω_is and ω_j's. As before, we can find a finite family $\{\eta_\alpha\}_{\alpha \in I}$ of compactly supported smooth functions, each supported inside a chart domain V_α, such that $\sum_\alpha \eta_\alpha = 1$. Then $\omega_i = \sum_\alpha \eta_\alpha \cdot \omega_i$. Note also that we already know that $\int_M$ behaves linearly when looking at forms that are supported inside a coordinate chat. Applying this for the coordinate charts U_i, and then V_α, we find that $\sum_{i=1}^{k} \int_M \omega_i$ equals to

$$\sum_{i=1}^{k} \sum_\alpha \int_M \eta_\alpha \cdot \omega_i = \sum_\alpha \sum_{i=1}^{k} \int_M \eta_\alpha \cdot \omega_i = \sum_\alpha \int_M \eta_\alpha \cdot \omega.$$

Of course, the same applies to $\sum_{j=1}^{k'} \omega_j'$, therefore proving the claim.

Finally, note also that $\int_M$ defined in this way is obviously linear. 😀

Theorem 5.15 (Stokes- version without bounday). *If M is an m-dimensional oriented manifold then, for any $\eta \in \Omega_c^{m-1}(M)$ one has*

$$\int_M d\eta = 0.$$

In particular, on a compact oriented manifold, $\int_M \omega = 0$ whenever ω is exact.

Proof. Writing (as above) η as a sum of forms supported in domains of coordinate charts, we may assume that $M = \mathbb{R}^m$. Then η is of type

$$\eta = \sum_i f_i \cdot dx_1 \wedge \ldots dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_m, \tag{5.1.9}$$

so that we can write

$$d\eta = \left(\sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \ldots \wedge dx_m.$$

Hence it suffices to show that, for any $f \in \mathscr{C}^\infty(\mathbb{R}^m)$ compactly supported and any i one has

$$\int_{\mathbb{R}^m} \frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge dx_m = 0.$$

Choosing an interval $[a,b]$ such that the support of f is strictly inside $[a,b]^m$, everything follows from

$$\int_a^b \frac{\partial f_i}{\partial x_i} dx_i = f(b) - f(a) = 0. \quad 😀 \tag{5.1.10}$$

Exercise 5.16. Do again Exercise 4.9. 4.13.

Finally, let us point out that, while the choice of an orientation $\mathcal{O}$ on M allows us to talk about the integral of m-forms (m being the dimension of M), and while the choice of a volume form does give rise to an orientation, once a volume form μ on M is fixed we can actually talks about integration of compactly supported functions on M (or, more precisely, on (M,μ)):

$$\mathscr{C}_c^\infty(M) \to \mathbb{R}, \quad f \mapsto \int_M f \cdot \mu,$$

the integral (w.r.t. the orientation induced by μ) of the top-form $f \cdot \mu$. In some sense, this is precisely how the standard integration on $\mathbb{R}^m$ (of functions!) is obtained: it is associated to the choice of the the top-form

$$\mu = dx_1 \wedge \ldots \wedge dx_m$$

(note the compatibility with the standard notation $\int_{R^m} f(x)dx_1 \ldots dx_m$ for the integration of functions!).

In particular, on compact manifolds, the integral of the constant function $1 \in C^\infty(M)$ deserves the name of "volume".

Definition 5.17. Given a compact manifold M endowed with a volume form μ, the **volume of M with respect to** μ (or: the volume of (M,μ)) is defined as the following integral

$$\mathrm{Vol}(M,\mu) := \int_M \mu,$$

where the integration is with respect to the orientation induced by μ.

Exercise 5.18. Show that the volume is always positive (assuming that M is non-empty ...). Deduce that, on compact manifolds, there do not exist volume forms that are exact.

5.1.4 Stokes theorem on manifolds with boundary

The Stokes formula discussed above, when applied to functions on $M = \mathbb{R}$, or on intervals $(a,b) \subset \mathbb{R}$, boils down to the fact that if $f : \mathbb{R} \to \mathbb{R}$ is supported inside the interval (a,b) then

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x}(x)dx = \int_a^b \frac{\partial f}{\partial x}(x)dx = f(b) - f(a) = 0.$$

However, such formulas are especially useful when f is supported inside (or just defined on) a closed interval $[a,b]$, without assuming that $f(a) = f(b) = 0$. This shows that a Stokes theorem would be (even) more interesting when formulated on manifolds with boundary.

The notion of manifold with boundary was briefly mentioned in subsection 2.2.5 of Chapter 2: the main point was to allow charts that compare opens in M with opens inside the half space

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

Since any open inside $\mathbb{R}^m$ is diffeomorphic to an open inside

$$\mathrm{Int}(\mathbb{H}^m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m > 0\},$$

that means that we are allowing more local models. Of course, the new addition is due to the presence of the boundary

$$\partial(\mathbb{H}^m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 0\}.$$

Definition 5.19. Given a manifold with boundary, a point $p \in M$ is called a boundary point if there exists a chart $\chi : U \to \Omega \subset \mathbb{H}^m$ around p such that $\chi(p) \in \partial(\mathbb{H}^m)$. The **boundary of** M, denoted ∂M, is the set of boundary points.

One can check that, in the previous definition, the condition $\chi(p) \in \partial(\mathbb{H}^m)$ does not depend on the choice of χ. Also, it follows that

$$\chi|_{\partial(U)} : \partial(U) \to \partial(\Omega) = \Omega \cap \partial(\mathbb{H}^m)$$

(where the last space is open in $\partial(\mathbb{H}^m) \cong \mathbb{R}^{m-1}$) can be used as a chart for ∂M. These make ∂M into an $(m-1)$-dimensional manifold (fill in the details!).

The definitions we gave in the previous lectures for:

- smooth maps,
- tangent vectors and tangent spaces,
- differential forms and DeRham differential,
- top-degree differential forms, volume forms and orientations,
- integration of top-degree differential forms on oriented manifolds with boundary.

apply word by word to all manifolds with boundary.

Remark 5.20 (positive charts when $m = 1$). One concept that does require more attention is that of positively oriented charts in dimension one. What happens for $m \geq 2$ (or when $\partial M = \emptyset$), but brakes down for 1-dimensional manifolds with boundary, is the fact that, around any point p in an oriented manifold, one can always find a positively oriented charts. Choosing a connected chart (U, χ) around p, either it is already positively oriented, or we can change χ_1 by $-\chi_1$ to force positivity. Just that this does not work when $m = 1$ and p is a boundary point (... finally paying the price for choosing the upper half plane when defining the notion of manifold with boundary).

This problem is relevant to us just because our description/construction of the integral makes use of positive charts only. However, the "problematic case" $m = 1$ can be easily fixed: just integrate over the interior $M \setminus \partial M$ [1].

The other point that requires a bit more care when moving to manifolds with boundary is the representation of tangent vectors as speeds of curves: this time, for points $p \in \partial M$, one has to allow also curves $\gamma : [0, \varepsilon) \to M$ (not necessarily defined on an entire open interval containing the origin). Note that, at points $p \in \partial M$, the tangent space $T_p\partial M$ (a tangent space of a manifold without boundary!) sits inside T_pM:

$$T_p\partial M \subset T_pM.$$

Even more, among the tangent vectors that are not tangent to the boundary, there is a special class:

Definition 5.21. For $p \in \partial M$, a tangent vector $v \in T_pM$ is said to be an **outward** tangent vector if $v \notin T_p\partial M$ and if there exists a curve $\gamma : (-\varepsilon, 0] \to M$ with $\gamma(0) = p$ and such that $v = \frac{d\gamma}{dt}(0)$.

In terms of coordinate charts, these are the tangent vectors $v \in T_pM$ with the property that, for any coordinate chart $\chi : U \to \Omega \subset \mathbb{H}^m$ around p, the corresponding vector

[1] in principle, this trick can be used for arbitrary m. But one should be aware that that would not be so much of a "free ride" as it sounds. That is because, when M is not a compact manifold (without boundary), $M \setminus \partial M$ is not compact, hence the forms on $M \setminus \partial M$ that we do know (by the previous section) how to integrate are the compactly supported ones. Therefore, one has to ensure that for those (top) forms on $M \setminus \partial M$ that, even if not compactly supported, have the (special!) property that they are restrictions of forms on M, their integral can still be defined. And, in principle, that would force us look at looking at more general integrable functions on $\mathbb{R}^m$, not only with compact supports. Of course, when $m = 1$ there isn't so much trouble.

$$v^{\chi} \in \mathbb{R}^m$$

has strictly negative last coordinate (think on a picture!).

One may say that the "extra-care" to be payed when looking at tangent vectors on manifolds with boundary is, after all, not really a "problem", but rather a new and interesting feature of manifolds with boundary- and an useful one. For instamnce, the notion of outward pointing vector allows us to obtain an orientation of ∂M out of an orientation of M. Here is the characterisation in terms of positive bases in the sense of Definition 5.8.

Lemma 5.22. Given an oriented manifold with boundary $(M, \mathscr{O})$, there exists an orientation $\partial\mathscr{O}$ on the boundary ∂M that is uniquely determined by the property that, at any boundary point $p \in \partial M$,

$$\left.\begin{array}{l} v_1, \ldots, v_{m-1} - \text{basis of } T_p\partial M \text{ positive w.r.t. } \partial\mathscr{O} \\ v \in T_pM - \text{outer tangent vector} \end{array}\right\} \Longrightarrow v, v_1, \ldots, v_{m-1} - (\text{a basis of } T_pM!) \text{ is positive w.r.t. } \mathscr{O}.$$

Furthermore, $\partial\mathscr{O}$ can be represented by a volume form of type

$$i_V(\mu)|_{\partial M} \in \Omega^{m-1}(\partial M),$$

where $\mu \in \Omega^m(M)$ is some volume form on M representing $\mathscr{O}$ and V is a (arbitrary) vector field that is outwards pointing at all points in ∂M.

Definition 5.23. Given an oriented manifold with boundary $(M, \mathscr{O})$, the orientation $\partial\mathscr{O}$ resulting from the lemma above is called **the induced orientation** on the boundary.

Remark 5.24 (Continuation of Remark 5.11). In the spirit of Remark 5.11, an orinetation of a manifold with boundary can also be described as a family $\mathscr{O} = \{\mathscr{O}_p\}$ of orientations on the tangent spaces T_pM. Just that, this time, we say that $\mathscr{O}$ **is a smooth orientation on** M if, around each point $p_0 \in M$ one can find a chart (U, χ) satisfying one of the following two conditions:

- for any $p \in U$,
$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$
is a positively oriented frame of $(T_pM, \mathscr{O}_p)$.
- for any $p \in U$,
$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$
is a negatively oriented frame of $(T_pM, \mathscr{O}_p)$.

In this way we avoid the problem pointed out in Remark 5.20.

Notice that, using this point of view on orientations, inducing an orientation on the boundary can be done right away:

Definition 5.25. Given an oriented manifold with boundary $(M, \mathscr{O})$ in the sense of the alternative definition 5.12, **the induced orientation** $\partial\mathscr{O}$ on the boundary is defined as follows: for $p \in \partial M$, declare a frame $(v_1, \ldots, v_{m-1})$ of $T_p\partial M$ to be positively oriented if, for a (or, equivalently, for any) outwards pointing tangent vector $v \in T_pM$, $(v, v_1, \ldots, v_{m-1})$ is a positively oriented frame of $(T_pM, \mathscr{O}_p)$.

From now on, whenever we have an oriented manifold M with boundary, its boundary ∂M will be automatically considered to be oriented with the induced orientation. Therefore, the integrations $\int_M$ and $\int_{\partial M}$ are unambiguously defined.

Theorem 5.26 (Stokes on manifolds with boundary). *For any m-dimensional oriented compact manifold with boundary M one has*

$$\int_M d\eta = \int_{\partial M} \eta|_{\partial M} \quad \text{for all } \eta \in \Omega^{m-1}(M).$$

Proof. We will prove the previous formula without assuming that M is compact, but under the condition that η is compactly supported. Covering the support of η by a finite number of coordinate charts $\chi_i : U_i \xrightarrow{\sim} \Omega_i$ where each Ω_i is either $\mathbb{R}^m$ or $\mathbb{H}^m$, and decomposing η as a sum of forms supported inside U_i, we may assume that $M = \mathbb{R}^m$ or $M = \mathbb{H}^m$. The first situation was already treated, so let us assume that $M = \mathbb{H}^m$. As in the case without boundary, we write η in the fom (5.1.9) and see that the terms corresponding to $i \neq m$ do not contribute to $\int d\eta$, and neither to $\eta|_{\mathbb{H}^m}$. Hence we may assume that

$$\eta = f \cdot dx_1 \wedge \ldots \wedge dx_{m-1},$$

so that we can write

$$\int_{\mathbb{H}^m} d\eta = (-1)^{m-1} \int_{\mathbb{H}^m} \frac{\partial f}{\partial x_m}(x_1, \ldots, x_m) dx_1 \ldots dx_m.$$

Writing

$$\int_{\mathbb{H}^m} \frac{\partial f}{\partial x_m}(x_1, \ldots, x_m) dx_1 \ldots dx_m = \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f}{\partial x_m}(x_1, \ldots, x_m) dx_m \right) dx_1 \ldots dx_{m-1}$$

we find that

$$\int_{\mathbb{H}^m} d\eta = (-1)^m \int_{\mathbb{R}^{m-1}} f(x_1, \ldots x_{m-1}, 0) dx_1 \ldots dx_{m-1}.$$

Note that, in the last equality, $\int_{R^{m-1}}$ is the standard integration- i.e. it uses the standard orientation on $\mathbb{R}^{m-1}$, for which the canonical basis $\{e_1, \ldots, e_{m-1}\}$ is positively oriented. However, the orientation induced by the one on $\mathbb{H}^m$ is defined by the condition that $\{-e_m, e_1, \ldots, e_{m-1}\}$ is positively oriented; hence the induced orientation differs from the canonical one by a factor of $(-1)^m$; hence

$$\int_{\mathbb{H}^m} d\eta = \int_{\partial \mathbb{H}^m} f(x_1, \ldots x_{m-1}, 0) dx_1 \ldots dx_{m-1} = \int_{\partial \mathbb{H}^m} \eta|_{\partial \mathbb{H}^m}.$$

Exercise 5.27. Compute

$$\int_{S^2} (x+y+z)\sigma$$

where σ is the volume for on the sphere (as mentioned for instance in Exercise 4.40).

5.2 More exercises

Exercise 5.28 (from the 2023-2024 exam). Let's wonder whether a manifold M can admit a volume form μ of type $\mu = \theta \wedge \theta$, for some other differential form θ on M. Show that if this happens then the dimension of M is divisible by 4. Then show that this can happen already on $M = \mathbb{R}^4$ (provide an explicit example!).

Exercise 5.29 (from the 2022-2023 exam). Return to Exercise 3.103 and continue with the following:

h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.

j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 5.30. Show that, for any closed differential form

$$\omega \in \Omega^k(\mathbb{R}^n)$$

and any compact k-dimensional submanifold $\Sigma \subset \mathbb{R}^n$, the restriction $\omega|_\Sigma \in \Omega^k(\Sigma)$ vanishes in at least one point $x \in \Sigma$.

Exercise 5.31. Do again Exercise 5.27, but making use of Exercise 4.40) (and the fact that the answer to the question there was $f = \frac{1}{z}$).

Exercise 5.32 (from the 2018/2019 exam). Compute the following integral

$$\int_{S^2} \left((x+y^{2017})dy \wedge dz + (y+z^{2018})dz \wedge dx + (z+x^{2019})dx \wedge dy\right)$$

(you can use any orientation on S^2 that you want).

Exercise 5.33 (from the 2021/2022 retake). On the 2-sphere S^2, on which we use the coordinate (x,y,z):

(a) Show that the function

$$F : S^2 \to S^2, \quad F(x,y,z) = (x^2+y^2-z^2, 2yz, 2xz)$$

is not a submersion.
(b) Show that $(xz \cdot dy - yz \cdot dx) \wedge dz = (1-z^2) \cdot dx \wedge dy$ (an equality of 2-forms on S^2).
(c) Show that the pull-back via f of the volume form $\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy$ satisfies

$$F^*\sigma = 4dx \wedge dy = 4z \cdot \sigma.$$

(d) Find the points $p \in S^2$ at which F fails to be a submersion.
(e) Compute $\int_{S^2} F^*\sigma$ (where you can use any orientation on S^2 that you want).

Exercise 5.34 (from the 2020/2021 exam). Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute

$$\int_{\mathbb{T}^2} \omega, \quad \text{where } \omega = (xdy \wedge dz)|_{\mathbb{T}^2}.$$

Exercise 5.35 (from the 2020/2021 retake). Consider the 3-dimensional projective space $\mathbb{P}^3$.

(a) Show that

$$p : (0,\pi) \times (0,\pi) \times (0,\pi) \to \mathbb{P}^3,$$
$$(\phi_1, \phi_2, \phi_3) \mapsto [\cos\phi_1 \;:\; \sin\phi_1 \cdot \cos\phi_2 \;:\; \sin\phi_1 \cdot \sin\phi_2 \cdot \cos\phi_3 \;:\; \sin\phi_1 \cdot \sin\phi_2 \cdot \sin\phi_3]$$

is a parametrization of $\mathbb{P}^3$, i.e. a diffeomorphism into an open $U \subset \mathbb{P}^3$.

(b) Assuming that you know already that $\mathbb{P}^3$ is orientable, and using any orientation that you prefer, compute

$$\int_{\mathbb{P}^3} f \cdot df_0 \wedge df_1 \wedge df_2,$$

where $f, f_0, f_1, f_2 : \mathbb{P}^3 \to \mathbb{R}$ are the functions which, for $(x_0, x_1, x_2, x_3) \in S^2$, send:

$$[x_0 : x_1 : x_2 : x_3] \xrightarrow{f_i} (x_i)^2, \quad [x_0 : x_1 : x_2 : x_3] \xrightarrow{f} x_0 \cdot x_1 \cdot x_2 \cdot x_3.$$

You can express the result in terms of $I_k = \int_0^\pi \sin^k \phi$ without further numerical computations (those would be obtained by further remarking that $I_k = \frac{k-1}{k} I_{k-2}$ and $I_0 = \pi$, $I_1 = 2$.)

Exercise 5.36 (part of the 2019/2020 retake exam). Return to Exercise 3.109 and Exercise 4.73, and continue with the remaining questions of the exam:

12. Show that $d\theta_2 \wedge d\theta_1$ is a volume form on $\mathbb{T}^2$ and compute the corresponding volume.
13. Show that there do not exist smooth functions f, g on $\mathbb{T}^2$ such that

$$\frac{\partial g}{\partial \theta_1} - \frac{\partial f}{\partial \theta_2} = 1.$$

(hint: volumes are strictly positive, hence non-zero).

14. Consider again the vector field $(*)$ (with f again an arbitrary smooth function). Since $L_{X^f}(d\theta_2 \wedge d\theta_1)$ is again a 2-form on $\mathbb{T}^2$, it is a function times the volume form $d\theta_2 \wedge d\theta_1$. Compute that function.
15. Deduce that the flow of X^f preserves our volume form, i.e.

$$\left(\phi_{X^f}^t\right)^* (d\theta_2 \wedge d\theta_1) = d\theta_2 \wedge d\theta_1 \quad \text{for all } t,$$

if and only if $f = f(z_1, z_2)$ depends only on the first coordinate z_1.

16. Show that there is no closed 2-form $\tilde{\eta}$ on $\mathbb{R}^3$ such that $F^*\tilde{\eta} = d\theta_2 \wedge d\theta_1$. (hint: think of a more general statement, for any volume form on a compact embedded submanifold of a Euclidean space).
17. Show that the restrictions of the standard 1-forms dx, dy, dz from $\mathbb{R}^3$ to $T^2_{R,r}$ satisfy:

$$x \cdot dx + y \cdot dy + z \cdot dz = \frac{R}{\sqrt{x^2 + y^2}} (x \cdot dx + y \cdot dy) \quad (\text{on } T^2_{R,r}).$$

18. Compute the pull-backs of dx, dy and dz to $\mathbb{T}^2$ (in terms of θ_1 and θ_2).
19. For $\sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3)$, restrict it to $T_{R,r}$ and compute

$$\int_{T_{R,r}} \sigma|_{T_{R,r}} \quad \text{(use any orientation you prefer)}.$$

(hint: you may want to compute $F^*\sigma$, and hopefully you will obtain the expression $r(R + r \cdot \cos\theta_1)(r + R\cos\theta_1) \cdot d\theta_2 \wedge d\theta_1$).

20. We are now trying to write the volume form $d\theta_2 \wedge d\theta_1$, moved to $T^2_{R,r}$ (via the diffeomorphism F), in terms of the standard coordinates x, y, z on $\mathbb{R}^3$ Show that

$$\eta := \frac{1}{r^2(x^2 + y^2)} \left[\left(\sqrt{x^2 + y^2} - R \right) \cdot \sigma + Rz \cdot dx \wedge dy \right]$$

works. In other words, interpreting η as a 2-form on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$, show that

$$F^*\eta = d\theta_2 \wedge d\theta_1.$$

5.3 DeRham cohomology

5.3.1 Cohomology; DeRham cohomology

We now return to the DeRham differential and its property that $d \circ d = 0$. The principle underlying "Homological algebra" tells us that, whenever we have a sequence of operators d satisfying $d \circ d = 0$, the first thing we do is to look at the resulting cohomology.

Definition 5.37. A cochain complex of vector spaces is a sequence

$$\xrightarrow{d} \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1 \xrightarrow{d} \mathscr{C}^2 \xrightarrow{d} \dots$$

consisting of vector spaces and linear maps between them satisfying $d \circ d = 0$. We say, in short, that $(\mathscr{C}^\bullet, d)$ **is a complex**.

Given such a complex, an element $\omega \in \mathscr{C}^k$ is called

- **closed** if $d\omega = 0$; denote by Z^k the set of such.
- **exact** if $\omega = d\eta$ for some $\eta \in \mathscr{C}^{k-1}$; denote by B^k the resulting set.

While the condition $d \circ d = 0$ means simply that $B^k \subset Z^k$, we define the k-th cohomology space of $(\mathscr{C}^\bullet, d)$ as the resulting quotient vector space

$$H^k(\mathscr{C}^\bullet, d) := Z^k / B^k.$$

Of course, the central example for us is provided by differential forms and DeRham differential,

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

(with 0 in negative degrees) which is called the **DeRham complex of the manifold** M.

Definition 5.38. The **DeRham cohomology spaces of** M, denoted

$$H^k(M) \quad \text{(one for each } k \geq 0\text{)},$$

are the cohomology spaces of the DeRham complex $(\Omega^\bullet(M), d)$. For a closed form $\omega \in \Omega^k(M)$,

$$[\omega] \in H^k(M)$$

will denote the induced cohomology class.

When all the cohomology spaces are finite dimensional (e.g. when M is compact-see below), define **the Euler characteristic** of M as

$$\chi(M) = \sum_i (-1)^i \dim(H^i(M)).$$

Exercise 5.39. Show that for any connected compact manifold M, one has $H^k(M) = 0$ if $k > \dim(M)$ and $H^0(M) \cong \mathbb{R}$ if M is connected. What if M is not connected?

Exercise 5.40. Show that the wedge product of forms induces a well-defined operation

$$H^k(M) \times H^l(M) \to H^{k+l}(M), \quad ([\omega],[\eta]) \mapsto [\omega]\cdot[\eta] := [\omega \wedge \eta].$$

Deduce that

$$H^\bullet(M) := \bigoplus_k H^k(M)$$

has the structure of a ring (actually an algebra).

Next, the fact that differential forms can be pulled-back allows us to compare the cohomology of different manifolds. Underlying this discussion is the notion of chain map: given two complexes $(\mathscr{C}^\bullet, d_{\mathscr{C}})$ and $(\mathscr{D}^\bullet, d_{\mathscr{D}})$, a **chain map** between them is a sequence of linear maps $f : \mathscr{C}^k \to \mathscr{D}^k$ which commute with the operators d:

$$f \circ d_{\mathscr{C}} = d_{\mathscr{D}} \circ f.$$

In this situation it is an easy (but must do!) exercise to see that f induces maps

$$[f] : H^k(\mathscr{C}, d_{\mathscr{C}}) \to H^k(\mathscr{D}, d_{\mathscr{D}})$$

by sending the class of an element ω into the class of $f(\omega)$.

Again, the basic example is the pull-back map induced by a smooth map $F : M \to N$ between two manifolds,

$$F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

We use the same notation for the maps induced in cohomology. Hence:

$$F^* : H^\bullet(N) \to H^\bullet(M), \quad [\omega] \mapsto [F^*(\omega)]$$

Exercise 5.41. Chek that $(G \circ F)^* = F^* \circ G^*$ in cohomology; deduce that if $F : M \to N$ is a diffeomorphism, than the map induced by F in cohomology is an isomorphism of vector spaces.

Exercise 5.42. With respect to the ring/algebra structure on cohomology from Exercise 5.40, show that the map $F^* : H^\bullet(N) \to H^\bullet(M)$ is a ring/algebra homomorphism.

So far we have discussed the cohomology groups H^k as "functors" that allow us to pass from the world of manifolds (with smooth maps between them) to that of vector spaces (with linear maps between them) (or, with the previous exercises in mind, to the category of algebras with algebra homomorphisms between them). We now look at some of the more specific properties, properties that are extremely useful for explicit computations.

5.3.2 Poincare lemma

In short, the Poincare lemma says that any closed form on $\mathbb{R}^m$ of strictly positive degree must be exact. In other words, it provides the first complete computation of cohomology:

Theorem 5.43 (Poincare lemma). *One has*

$$H^k(\mathbb{R}^m) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{R} & \text{if } k = 0 \end{cases}.$$

Moreover, we will see there is even more to learn from its proof.

Proof. We proceed by induction on m. For $m = 0$ this is clearly true; we assume it to hold for m and prove it for $m+1$. Write

$$\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$$

with coordinates denoted by $(t, x_1, \ldots, x_m)$. We will treat t as a special coordinate that we try to eliminate: we will show that any closed k-form on $\mathbb{R}^{m+1}$ is an exact one plus one that depends only on $(x_1, \ldots, x_m)$ (so that we can apply the induction hypothesis).

An arbitrary k-form on $\mathbb{R}^{m+1}$ can be decomposed as

$$\omega = \alpha_t + dt \wedge \beta_t \tag{5.3.1}$$

where

$$\mathbb{R} \ni t \mapsto \alpha_t \in \Omega^k(\mathbb{R}^m), \quad \mathbb{R} \ni t \mapsto \beta_t \in \Omega^{k-1}(\mathbb{R}^m)$$

are smooth families of forms on $\mathbb{R}^m$ and they are interpreted as forms on $\mathbb{R}^m \times \mathbb{R}$ without dt-terms. To such families one can apply the DeRham differential in two ways:

- apply the DeRham differential in $\mathbb{R}^m$ to each α_t. In this case we use the notation $d\alpha_t$- again a smooth family of forms, and again interpreted as a form on $\mathbb{R}^{m+1}$ without dt-component.
- interpret from the beginning α_t as a form on $\mathbb{R}^{m+1}$ and apply the DeRham differential for $\mathbb{R}^{m+1}$.

To distinguish the two, we will denote by d_{tot} the DeRham differential for $\mathbb{R}^{m+1}$. The relationship between the two is simply:

$$d_{\text{tot}}(\alpha_t) = d(\alpha_t) + dt \wedge \dot{\alpha}_t, \tag{5.3.2}$$

where $\dot{\alpha}_t$ is the derivative with respect to t (of $t \mapsto \alpha_t$). We now start with a $\omega \in \Omega^k(\mathbb{R}^{m+1})$ closed; writing it as (5.3.1), we find (using also that $dt \wedge dt = 0$)

$$0 = d_{\text{tot}}(\omega) = d(\alpha_t) + dt \wedge (\dot{\alpha}_t - d\beta_t)).$$

So, first of all,

$$d\alpha_t = 0 \quad \text{for all } t.$$

We will be using this only for $t = 0$ (i.e. α_0 will be the closed form on $\mathbb{R}^m$ we are looking for). Secondly, we obtain

$$\dot{\alpha}_t = d\beta_t \tag{5.3.3}$$

for all t and, therefore, we can write

$$\alpha_t = \alpha_0 + \int_0^t d\beta_t. \tag{5.3.4}$$

. Denoting $\gamma_t = \int_0^t \beta_t$ (again a family of forms on $\mathbb{R}^m$, interpreted as a form on $\mathbb{R}^{m+1}$ with no dt-term), the general formula (5.3.2) gives us

$$d_{\text{tot}}(\gamma_t) = \int_0^t d\beta_t + dt \wedge \beta_t.$$

Plugging the resulting formula for $dt \wedge \beta_t$, as well as the formula (5.3.4) for α_t, in (5.3.1), we find:

$$\omega = \alpha_0 + d_{\text{tot}}(\gamma_t).$$

And this is precisely what we wanted to achieve (just to make sure: α_0 is a closed form on $\mathbb{R}^m$ hence, by induction hypothesis, it is $d\eta$ for some $\eta \in \Omega^k(\mathbb{R}^m)$; of course, since η has no dependence on t, one has $d\eta = d_{\text{tot}}\eta$, hence $\omega = d_{\text{tot}}(\eta + \gamma_t)$ will be exact.) 😀

Remark 5.44. The previous proof is written in such a way that it is not so important that we work with $\mathbb{R}^m$: instead, we can replace it by an arbitrary manifold M and $\mathbb{R}^{m+1}$ by $M \times \mathbb{R}$; hence the argument above shows that if a manifold M has no cohomology in degree k, then the same is true for $M \times \mathbb{R}$; and with a bit more effort one can prove that, for any M, $M \times \mathbb{R}$ and M have isomorphic cohomologies (see below). However, let us state here what we get for general M without any effort (and which is at the heart of the homotopy invariance discussed below).

Corollary 5.45 (of the proof). *For any manifold M, the maps*

$$i_0^*, i_1^* : H^\bullet(M \times \mathbb{R}) \to H^\bullet(M)$$

induced by the "inclusions" $i_0, i_1 : M \to M \times \mathbb{R}$, $i_\varepsilon(p) = (p, \varepsilon)$ ($\varepsilon \in \{0,1\}$), coincide.

Proof. Decomposing a k-form on $M \times \mathbb{R}$ as before, i.e. as in (5.3.1), then $i_\varepsilon^*\omega = \alpha_\varepsilon$. Hence the statement is that, if ω is closed, then $\alpha_0 = \alpha_1$ is an exact form on M; but that is clear from (5.3.4) since we get

$$\alpha_1 - \alpha_0 = \int_0^1 d\beta_t = d\left(\int_0^1 \beta_t\right).$$ 😀

5.3.3 Homotopy invariance

We now push the previous discussion to a more conceptual level. While the heard work has already been done with the proof of the Poincare lemma, the conceptual gain is substantial: it says that $H^\bullet(M)$ depends only on the "homotopy type" of M (which is much weaker than the diffeomorphism type).

To explain this, we first recall the notion of homotopy and homotopy equivalence. These notions are usually introduced in the general context of topological spaces; of ocurse, here we restrict to the smooth context. Moreover, while for topological spaces (and also intuitively) it is natural to use the interval

$$I = [0,1]$$

(e.g. to parametrize deformations), in the smooth context that would force us right away to allow for manifolds with boundary. Allowing manifold with boundary is not a real problem, but not absolutely necessary in order to achieve the main result; more precisely, we can just use $\mathbb{R}$ instead of $[0,1]$.

Therefore, in what follows, one may assume that I is either $[0,1]$ (if you are ok with manifolds with boundary), or $I = \mathbb{R}$ (if you want to stick to manifolds without boundary but being willing to a accept a less natural framework).

Definition 5.46. Two smooth maps $F_0, F_1 : M \to N$ between two smooth manifolds M and N are said to be **homotopic** if there exists a smooth map

$$\tilde{F} : M \times I \to N$$

such that $F_0(\cdot) = \tilde{F}(\cdot, 0)$, $F_1(\cdot) = \tilde{F}(\cdot, 1)$.

We say that a smooth map $F : M \to N$ is a **homotopy equivalence** if there exists a smooth map $G : N \to M$ with the property that $G \circ F$ is homotopic to Id_M and $F \circ G$ is homotopic to Id_N. When such a map exists, we say that M and N are **homotopy equivalent**.

While Topology studies topological spaces and continuous maps between them (the ”category” of topological spaces) and Differential Geometry studies manifolds and smooth maps between them, Homotopy Theory studies topological spaces and homotopy classes of maps between them. ”Isomorphisms” are: homeomorphisms in Topology, diffeomorphisms in Differential Geometry and ... homotopy equivalences in Homotopy Theory. Therefore, the next result is called ”homotopy invariance”:

Theorem 5.47 (homotopy invariance). *For any two manifolds M and N:*

1. *If $F_0, F_1 : M \to N$ are homotopic, then they induce the same maps in cohomology.*
2. *If $F : M \to N$ is a homotopy equivalence, then the maps induced in cohomology, $F^* : H^\bullet(N) \to H^\bullet(M)$, are isomorphisms.*

Proof. The first part follows right away from Corollary 5.45: if $\tilde{F} : M \times I \to N$ relates F_0 and F_1 as in the definition, we can write $F_0 = \tilde{F} \circ i_0$ and $F_1 = \tilde{F} \circ i_1$, where i_ε are the inclusions from the corollary. Therefore, in cohomology,

$$F_0^* = i_0^* \circ \tilde{F}^* = i_1^* \circ \tilde{F}^* = F_1^*.$$

And the second part follows easily from the first part: if $G : N \to M$ is as in the definition of homotopy equivalence we get

$$F^* \circ G^* = (G \circ F)^* = \mathrm{Id}$$

and similarly for $G^* \circ F^*$, hence F^* is an isomorohism. 😀

Example 5.48. $M = \mathbb{R}^m$ and $N = \{0\}$ are homotopic equivalent. In one direction, there is only one map $F : \mathbb{R}^m \to \{0\}$; in the other direction, we use the map G that maps N to the origin. The only thing we have to show is that

$$F : \mathbb{R}^m \to \mathbb{R}^m, \quad F(x) = 0,$$

is homotopic to the identity. For that take $\tilde{F}(x,t) = t \cdot x$.

Therefore, the previous theorem will imply the Poincare lemma.

Exercise 5.49. By a similar argument show that, for any manifold M, the projection $\mathrm{pr} : M \times I \to M$ induces isomorphisms in cohomology

$$\mathrm{pr}^* : H^\bullet(M) \xrightarrow{\sim} H^\bullet(M \times I).$$

Example 5.50. Similarly, one gets

$$H^\bullet(\mathbb{R}^m \setminus \{0\}) \cong H^\bullet(S^{m-1})$$

(and the last group will be computed in the next subsection). In other words, we claim that the two spaces involved are homotopy equivalent. To see this, we consider the inclusion

$$i : S^{m-1} \hookrightarrow \mathbb{R}^m \setminus \{0\}$$

and, in the other direction, the rescaling

$$r : \mathbb{R}^m \setminus \{0\} \to S^{m-1}, \quad r(x) = \frac{1}{||x||} \cdot x.$$

While $r \circ i$ is the identity map of S^{m-1}, we claim that $i \circ r$ is homotopic to the identity of $\mathbb{R}^m \setminus \{0\}$. Indeed,

$$F : \mathbb{R}^m \setminus \{0\} \times I \to \mathbb{R}^m \setminus \{0\}, \quad F(x,t) = t \cdot x + (1-t) \cdot i(r(x))$$

defined the desired homotopy.

5.3.4 Stokes theorem revisited; applications: hairy ball and Brower's fixed pont

Stokes' theorem tells us that integration of top-forms interacts nicely with cohomology.

Corollary 5.51 (Stokes' theorem reformulated). *For a compact oriented manifold M, the integration $\int_M : \Omega^m(M) \to \mathbb{R}$ descends to a linear map*

$$\int_M : H^m(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

Using orientations induced by a volume form μ (Lemma 5.13), the resulting integration gives $\int_M \mu > 0$ (why?), therefore:

Corollary 5.52. *On a compact manifold, any volume form $\mu \in \Omega^m(M)$ defines a non-trivial cohomology class*

$$[\mu] \neq 0 \in H^m(M).$$

Example 5.53. In particular, we see that the m-th cohomology group of the sphere S^m is non-zero, with with the volume form μ_{S^m} from Example 5.5 inducing a non-zero cohomology class.

We no show that this fact alone has some very interesting applications.

Corollary 5.54 (Brower's fixed point theorem). *Any smooth map from the closed unit disk to itself, $F : D^m \to D^m$, has at least one fixed point.*

Proof. Assume there is a map F with no fixed point. Then for each $x \in D^m$, the line through x and $F(x)$ intersects the boundary sphere S^{m-1} in two points. Let $r(x)$ the point with the property that x belongs to the segment $[F(x), r(x)]$. Then we obtain a smooth map

$$r : D^m \to S^{m-1} \quad \text{with } r|_{S^{m-1}} = \mathrm{Id}$$

(why is the map r smooth)? Denoting by $i : S^{m-1} \hookrightarrow D^m$ the inclusion, we have $r \circ i = \mathrm{Id}$ hence, if the degree $m-1$ cohomology, the composition

$$H^{m-1}(S^{m-1}) \xrightarrow{r^*} H^m(D^m) \xrightarrow{i^*} H^{m-1}(S^{m-1})$$

is the identity map. But, as $H^{m-1}(D^m) = 0$ (why?), that would imply $H^{m-1}(S^{m-1}) = 0$. So no such F exists.

Theorem 5.55 (the hairy ball theorem). *The sphere S^m admits a non-where vanishing vector field if and only if m is odd.*

Proof. Assume that such a vector field exists V exists. Hence

$$V : S^m \to \mathbb{R}^m, \quad V(p) \perp p \text{ for all } p \in S^m.$$

By rescaling, we may assume $||V(p)|| = 1$ at all p. But then

$$H(p,t) = \cos(\pi t)\, p + \sin(\pi t)\, V(p) \in S^m \quad \text{for all } p \in S^m, t \in I$$

and we obtain a homotopy

$$H : S^m \times I \to S^m$$

between Id and $\tau := -\mathrm{Id}$. Hence, by homotopy invariance, $-\tau$ will induce the identity map in the m-cohomology group of S^m; in particular,

$$[\tau^*(\mu_{S^m})] = [\mu_{S^m}].$$

But, given the explicit formula of μ_{S^m}, it is clear that

$$\tau^*(\mu_{S^m}) = (-1)^{m+1}\mu_{S^m}$$

already as forms (hence also at the level of cohomology classes). Since the cohomology class of μ_{S^m} is non-zero, we deduce that $1 = (-1)^{m+1}$, hence m must be odd. Of course, when m is odd, one can just build a simple nowhere vaninshing vector fields on S^m- e.g.

$$V := \left(x_0\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_0}\right) + \left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) + \ldots$$ 😀

Remark 5.56 (for the interested students). For any oriented compact manifold M, using the product operation on cohomology induced by the wedge-product of forms (cf. e.g. Exercise 5.40) and the integration, we obtain a map

$$\beta : H^k(M)\times H^{m-k}(M) \to \mathbb{R}, \quad \beta([\omega],[\eta]) = \int_M \omega\wedge\beta$$

(for each k, where $m = \dim(M)$). This is know under the name of **the Poincare pairing**. The Poincare duality theorem says that this map is non-degenerate or, equivalently, that the induced map

$$H^k(M) \to H^{m-k}(M)^*, \quad u \mapsto \beta(u,\cdot)$$

is an isomorphism. Using the fact that the cohomology spaces of compact manifolds are finite dimensional (see below), this will imply that the groups $H^k(M)$ and $H^{m-k}(M)$ have the same dimension, for all k. In particular, we deduce that $\chi(M) = 0$ for all odd dimensional compact oriented manifolds M.

While the non-vanishing of $\int_M$ was enough to get such results, let us now describe the full information that it provides on the top cohomology spaces.

Theorem 5.57. *For any connected compact oriented manifold M,*

$$\int_M : H^m(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

is an isomorphism.

Proof. We have to show that a top-form $\omega\in\Omega^m(M)$ is zero in cohomology (exact) if and only if $\int_M\omega = 0$. Actually a version of this is true on any oriented M (compact or not), but for compactly supported form $\omega\in\Omega^m_c(M)$: $\omega = d\eta$ for some compactly supported η if and only if $\int_M\omega = 0$. We first prove that this is true for Euclidean spaces. On $\mathbb{R}$ the statement is immediate. In general, we proceed by induction on m; we assume it to hold for m and we prove it for $m+1$. Of course, this is similar to the proof of the Poincare lemma (Theorem 5.43), and we use the same notations as there. Hence let $\omega\in\Omega^{m+1}_c(\mathbb{R}^{m+1})$ whose integral is zero and write it in the form (5.3.1). Note that, from (5.3.3) and the fact that ω has compact support (hence $\alpha_t = 0$ for t very large or very small), it follows that

$$\int_{\mathbb{R}} d\beta_s = 0.$$

Hence the form $\int_{\mathbb{R}}\beta_s\in\Omega^m(\mathbb{R}^m)$ is closed, and clearly with compact supports. Moreover, its integral over $\mathbb{R}^m$ equals $\int_{\mathbb{R}^{m+1}}\omega = 0$. By the induction hypothesis, we can write it as

$$\int_{\mathbb{R}}\beta_s\,ds = d\gamma \quad \text{for some } \gamma\in\Omega^{m-1}_c(\mathbb{R}^{m-1}).$$

Consider now a function $\phi\in\mathscr{C}^\infty_c(\mathbb{R})$ with $\int_{\mathbb{R}}\phi(s)\,ds = 1$ and consider $\psi(t) = \int_{-\infty}^t\phi(s)\,ds$. And define the following m-form on $\mathbb{R}^{m+1}$:

$$\eta := \int_{-\infty}^{t} \beta_s\, ds - \psi(t) \cdot \int_{-\infty}^{\infty} \beta_s\, ds - \phi \cdot dt \wedge \gamma.$$

By the choice of ϕ and the fact that γ has compact support in $\mathbb{R}^m$, η is compactly supported. We now compute its differential; using $\psi'(t) = \phi(t)$ and $\int_{\mathbb{R}} d\beta_s = 0$ (see also 5.3.2), we find

$$d_{\text{tot}}\eta = \int_{-\infty}^{t} d\beta_s\, ds + dt \wedge \left(\beta_t - \phi(t) \int_{-\infty}^{\infty} \beta_s ds + \phi(t) d\gamma\right).$$

From the choice of γ, the last two terms that are wedged with dt cancel. And, using (5.3.3) again, $\int_{-\infty}^{t} d\beta_s\, ds = \alpha_t$. Hence we see that $d_{\text{tot}}\eta = \alpha_t + dt \wedge \beta_t = \omega$.

We now return to the original proof. Hence M is compact and oriented, $\int_M \omega = 0$ and we want to prove that ω is exact. The strategy is to write ω as a sum of forms, each supported in a coordinate chart $U_i \cong \mathbb{R}^m$, and each with zero integral; by the previous result for $\mathbb{R}^m$, we will be done. To implement this idea, we first cover M with a finite number of domains of coordinate charts $U_i \cong \mathbb{R}^m$, $i \in \{0, 1, \ldots, k\}$. After eventually adding some more opens, we may assume that

$$U_0 \cap U_1 \neq \emptyset, \quad U_1 \cap U_2 \neq \emptyset, \quad \ldots U_{k-1} \cap U_k \neq \emptyset.$$

(this is an exercise; hint: for U and V two consecutive domains, if they do not intersect, choose a path that starts in U and ends in V, then cover the path carefully so that each two consecutive charts intersect).

Choose now a partition of unity $\{\phi_0, \phi_1, \ldots, \phi_k\}$ subordinated to this cover and let $c_i = \int_M \phi_i \cdot \omega$. Choose also forms $\alpha^i \in \Omega^m(M)$, supported inside $U_i \cap U_{i+1}$, with $\int_M \alpha_i = 1$ and consider the following forms

$$\begin{aligned}
\omega_0' &= \phi_0 \cdot \omega - c_0\, \alpha_0 \\
\omega_1' &= \phi_1 \cdot \omega - (c_0 + c_1)\, \alpha_1 + c_0\, \alpha_0 \\
\omega_2' &= \phi_2 \cdot \omega - (c_0 + c_1 + c_2)\, \alpha_2 + (c_0 + c_1)\, \alpha_1, \quad \text{etc}
\end{aligned}$$

Note that, since $\sum_i c_i = \int_M \omega = 0$, and $\sum \phi_i = 1$, the sum of these forms is precisely ω. Moreover, by construction, each ω_i' is supported inside U_i. Hence this provides the desired decomposition of ω. 😀

5.3.5 Another useful tool for computations: Mayer-Vietoris

We now discuss another useful tool for computing cohomology- the Mayer-Vietoris sequence. Roughly speaking, it provides a way to compute that cohomology of a manifold M by breaking M into pieces. It is formulated in terms of long exact sequences in the following sense.

Definition 5.58. A sequence of vector spaces with linear maps between them,

$$\ldots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \ldots$$

is said to be **exact at** B if $\mathrm{Ker}(g) = \mathrm{Im}(g)$. We say that it is an **exact sequence** is it is exact at all entries.

Example 5.59. A sequence of vector spaces of type

$$A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f is surjective, while a sequence of type

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only of f is injective.

Exercise 5.60. Exact sequences of type

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

are called **short exact sequences**. Show that, for such a short exact sequence, B is finite dimensional if and only if A and C are. Moreover, in this case prove that

$$\dim(B) = \dim(A) + \dim(C).$$

Exercise 5.61. Show that a sequence of vector spaces with linear maps between them,

$$\xrightarrow{d} C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \dots,$$

is exact if and only if it is a cochain complex with vanishing cohomology.

Moreover, when this happens, if the vector spaces C^i are finite dimensional and only a finite number of them are non-zero, show that

$$\sum_i (-1)^i \dim(C^i) = 0.$$

The Mayer-Vietoris theorem (and its application to the computation of cohomology of spheres) is probably the best illustration of the usefulness of long exact sequences.

To state the main result, assume we have a manifold M written as

$$M = U \cup V, \quad \text{with } U, V \subset M \text{ opens.}$$

There are some natural maps relating the cohomologies of M, U, V and $U \cap V$; they are built from the inclusions

$$i_U : U \to M, \quad i_V : U \to M,$$

$$j_U : U \cap V \to U, \quad j_U : U \cap V \to V.$$

Using the maps induced in cohomology, we consider

$$H^\bullet(M) \xrightarrow{(i_U^*, i_V^*)} H^\bullet(U) \oplus H^\bullet(V) \xrightarrow{j_U^* - j_V^*} H^\bullet(U \cap V).$$

Note that the composition of these two maps is zero.

Theorem 5.62. *There exist linear maps*

$$\delta : H^\bullet(U \cap V) \to H^{\bullet+1}(M)$$

such that the following sequence is exact

$$\dots \xrightarrow{\delta} H^k(M) \xrightarrow{(i_U^*, i_V^*)} H^k(U) \oplus H^\bullet(V) \xrightarrow{j_U^* - j_V^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \longrightarrow \dots.$$

Proof. We first claim that, for each k, the sequence

$$0 \to \Omega^k(M) \xrightarrow{(i_U^*, i_V^*)} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_U^* - j_V^*} \Omega^k(U \cap V) \to 0 \tag{5.3.5}$$

is exact. Exactness at $\Omega^k(M)$ means the injectivity of the map (i_U^*, i_V^*), i.e. that, for $\omega \in \Omega^k(M)$, if $\omega|_U = 0$ and $\omega|_V = 0$ then $\omega = 0$. And this is clear.

Next, the exactness in the middle. Since the composition of the two maps is clearly zero, the image of the first is inside the kernel of the second. To prove the reverse inclusion, let

$$(\omega_U, \omega_V) \in \mathrm{Ker}(j_U^* - j_V^*).$$

That means that

$$\omega_U|_{U\cap V} = \omega_V|_{U\cap V}.$$

We have to show that there exists $\omega \in \Omega^k(M)$ whose restrictions to U and V is ω_U and ω_V, respectively. But that should be clear (the desired condition tells us how to define ω).

We are left with showing the exactness at $\Omega^k(U\cap V)$, i.e. the surjectivity of the map $j_U^* - j_V^*$. So, let $\omega \in \Omega^k(U\cap V)$ and we are looking for $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ so that

$$\omega = \omega_U|_{U\cap V} - \omega_V|_{U\cap V}.$$

For that choose partition of unity $\{\eta_U, \eta_V\}$ subordinated to $\{U, V\}$ and set $\omega_U = \eta_U \cdot \omega$, $\omega_V = -\eta_V \cdot \omega$.

Therefore (5.3.5) is exact. We now claim that the rest is a consequence of a general property of short exact sequence of complexes:

Lemma 5.63. *Consider a short exact sequence of chain complexes*

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0, \tag{5.3.6}$$

i.e. $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ *and* $(C^\bullet, d_C)$ *are chain complexes,* f *and* g *are chain maps, and the previous sequence is exact in each degree. Then there exist linear maps*

$$\delta : H^\bullet(C, d_C) \to H^{\bullet+1}(A, d_A)$$

such that the following sequence is exact

$$\ldots \xrightarrow{\delta} H^k(A, d_A) \xrightarrow{[f]} H^k(B, d_B) \xrightarrow{[g]} H^k(C, d_C) \xrightarrow{\delta} H^{k+1}(A, d_A) \longrightarrow \ldots.$$

Proof. Let us describe the map δ on an arbitrary element $[c] \in H^k(C, d_C)$. Hence $c \in C^k$,

$$d_C(c) = 0.$$

The exactness of (5.3.6) at C ensures that g is surjective, hence we can write

$$c = g(b), \quad \text{with } b \in B^k.$$

Now, since $d_C(c) = 0$ and g is a chain map, we obtain $g(d_B(b)) = 0$, i.e. $d_B(b) \in \mathrm{Ker}(g)$. Using again the exactness of (5.3.6), this time at B (and in degree $k+1$) we can write

$$d_B(b) = f(a), \quad \text{with } a \in A^{k+1}.$$

Applying d_B we also find $f(d_A(a)) = 0$ and, since f is injective (the exactness of (5.3.6) at A), one has $d_A(a) = 0$. Hence a defines an element in the cohomology of A and we set:

$$\delta([c]) := [a].$$

We have to check that $[a]$ not depend on the choices involved: if $[c] = [c']$ and we also write

$$c' = g(b'), \quad d_B(b') = f(a'),$$

then a and a' represent the same cohomology class. To see this, since $[c] = [c']$, we can write $c' = c + d_C(z)$ with $z \in C^{k-1}$. Using again the surjectivity of g, write $z = g(y)$ for some $y \in B^{k-1}$. Then $c' = c + g(d_B(y))$ and, using the choice of b and b' we obtain

$$b' - b = d_B(y) \in \mathrm{Ker}(g)$$

hence we can write

$$b' = b + d_B(y) + f(x), \quad \text{with } x \in A^k.$$

Then $d_B(b') = d_B(b) + f(d_A(x))$ hence, using the way we chose a and a', $f(a') = f(a) + f(d_A(x))$. Hence, by the injectivity of f again, $a' = a + d_A(x)$, therefore $[a'] = [a]$ in cohomology.

With all the maps defined now, one still has to prove the exactness of the sequence. This is not so difficult and it is an extremely useful exercise (a really "must do" one!). Therefore, we check here the exactness at one spot only. Say at $H^k(C, d_C)$:

$$\mathrm{Im}[g] = \mathrm{Ker}(\delta).$$

(but we insist: it may be more instructive, and even easier, to do it yourself than reading the argument below!).

For the direct inclusion, we start with $[c] \in H^k(C, d_C)$ with the property that $c = g(b_0)$ for some $b_0 \in B^k$ closed. Recall that $\delta([c]) = [a]$ where a is the result of the choices of b with $c = g(b)$ and of a with $d_B(b) = f(a)$. But we can just choose $b = b_0$ and then, since $d_B(b_0) = 0$, one can choose $a = 0$. Hence $\delta([c]) = 0$. For the reverse inclusion, let $[c] \in C^k(C, d_C)$ such that $\delta([c]) = 0$. That means that we can write

$$c = g(b), \quad d_B(b) = f(a), \quad a = d_A(a_0)$$

for some $b \in B^k$, $a \in A^{k+1}$, $a_0 \in A^k$. The last two equations give $d_B(b) = f d_A(a_0) = d_B f(a_0)$ hence, denoting $b' = b - f(a_0)$, we have

$$b = b' + f(a_0) \quad \mathrm{withd_B(b') = 0/}$$

Applying g and using that $g \circ f = 0$ and $c = g(b)$, we find

$$c = g(b') \quad \text{with } d_B(b') = 0.$$

Hence $[b'] \in H^k(B, d_B)$ is defined and is sent by $[g]$ to $[c]$. 😀 😀

Example 5.64. Let us compute the cohomology of the spheres S^m. We claim that

$$\boxed{H^k(S^m) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = m \\ 0 & \text{otherwise} \end{cases}.}$$

The strategy is to decompose

$$S^m = U \cup V$$

where U and V are basically the half hemispheres, extended so that they become open. E.g. they could be taken the entire domains of the stereographic projections with respect to the north and south pole, or just

$$U = \{(x_0, \ldots, x_m) \in S^m : x_m > -\varepsilon\}, \quad V = \{(x_0, \ldots, x_m) \in S^m : x_m < -\varepsilon\},$$

with $\varepsilon \in (0, 1)$ any number. Note that both U and V are diffeomorphic to a disk (hence also to a Euclidean space), hence

$$H^k(U) \cong H^k(V) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, $U \cap V$ is homotopy equivalent to S^{m-1}- more precisely, the inclusion

$$i : S^{m-1} \hookrightarrow U \cap V$$

is a homotopy equivalence (prove this, if necessary with some inspiration from Example 5.50). Hence

$$H^k(U \cap V) \cong H^k(S^{m-1}).$$

Let us look at the case $m = 1$ first. This is slightly different than the rest because, in this case, $S^{m-1} = S^0 = \{-1, 1\}$ has two connected components, hence $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$ (and zero in other degrees), hence the first part of the Mayer-Vietoris sequence,

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(S^1) \to 0$$

will become (isomorphic to)

$$0 \to \mathbb{R} \xrightarrow{i} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j} \mathbb{R}^2 \to H^1(S^1) \to 0$$

Independent of what the maps i and j are, this can happen if and only of $H^1(S^1) \cong \mathbb{R}$ (exercise!). While $H^k(S^1) = 0$ for $k > 1$ (why?), we are done with the case $m = 1$. For $m \geq 2$, the Mayer-Vietoris will become

$$\xrightarrow{\delta} H^1(S^m) \longrightarrow 0 \longrightarrow H^1(S^{m-1}) \xrightarrow{\delta} H^2(S^m) \longrightarrow 0 \longrightarrow \dots$$

$$\dots$$

$$\dots \xrightarrow{\delta} H^{k-1}(S^m) \to 0 \longrightarrow H^{k-1}(S^{m-1}) \xrightarrow{\delta} H^k(S^m) \longrightarrow 0 \longrightarrow \dots$$

from which we deduce that

$$H^k(S^m) \cong H^{k-1}(S^{m-1}) \quad \forall k \geq 2, m \geq 1.$$

Hence

$$H^m(S^m) \cong H^{m-1}(S^{m-1}) \cong \dots \cong H^1(S^1) \cong \mathbb{R}$$

and, similarly, $H^k(S^m) = 0$ for $k < m$, while for $k > 0$ we already know that the cohomology vanishes.

And here is another application of the Mayer-Vietoris sequence: the finite dimensionality of the cohomology groups of any manifold. However, for the proof we will need one extra-ingredient that will not be proven here:

Proposition 5.65. *Any compact manifold admits a* ***finite good cover****, i.e. a finite open cover $\mathscr{U}$ with the property that, for each opens $U_1, \dots, U_k \in \mathscr{U}$, the intersection $U_1 \cap \dots \cap U_k$ is either empty or diffeomorphic to $\mathbb{R}^m$.*

Appealing to your intuition for imagining what a triangulation of a manifold is, a triangulation (which always exists) gives rise to such a finite good cover (one just has to ”fatten” each simplex a bit). The standard way to prove the proposition above is by using a ”Riemannian metric” on the manifold and taking geodesically convex neighborhoods of points. What we are able to prove here is:

Corollary 5.66. *Any manifold M which admits a finite good cover (in particular, any compact manifold) has finite dimensional cohomology group).*

This corollary allows us to define **the Euler characteristic** of any compact manifold M:

$$\chi(M) := \sum_i (-1)^i \dim \left(H^i(M)\right).$$

Proof. We prove by induction on k the following statement: if M has a good cover consisting of k opens, then its cohomology is finite dimensional. The situation is clear when $k = 1$, since then $M \cong \mathbb{R}^m$. Assume that the statement is true for k and we prove it for $k+1$. Hence assume that M admits a good cover $\{U_1, \dots, U_{k+1}\}$. Consider

$$U = U_1 \cup \dots \cup U_k, \quad V = U_{k+1}.$$

Note that the induction hypothesis can be applied to U, V and $U \cap V$ (for the last one note that $\{U_1 \cap V, \dots, U_k \cap V\}$ is a good cover); hence their cohomologies are finite dimensional. Given the Mayer-Vietoris, it suffices to show that, if one has an exact sequence

$$A \xrightarrow{f} B \xrightarrow{\delta} C \xrightarrow{g} D \xrightarrow{h} E$$

where A, B, D and E are finite dimensional, then so is C. To see this, note that g takes values in

$$D' = \mathrm{Ker}(h)$$

δ descends to the quotient

$$B' := B/\mathrm{Im}(f),$$

and the exactness of the sequence gives a short exact sequence

$$0 \to B' \xrightarrow{\delta} C \xrightarrow{g} D' \to 0,$$

where B' and D' are finite dimensional (as a subspaces of the finite dimensional B, and a quotient of the finite dimensional D). Form this is follows easily that C is finite dimensional, of dimension equal to the sums of the dimensions of B' and D'. 😀

Exercise 5.67. Using the last part of Exercise 5.61, shows that the Euler characteristic $\chi(M)$ of a compact manifold M can be computed with the help of a finite good cover $\mathscr{U} = \{U_i : i \in I\}$ as follows:

$$\chi(M) = \sum_k (-1)^k n_k(\mathscr{U})$$

where, for each k, $n_k(\mathscr{U})$ is the cardinality of the set

$$\{J \subset I : |J| = k, \bigcap_{j \in J} U_j \neq \emptyset\}.$$

Then use this to compute the Euler characteristic of S^2.

Chapter 6
Previous exams

Here are some of the last exams- but you may have seen already quite a few of the questions already in the previous exercises. Anyway, here are the complete exams.

6.1 Retake 2023/2024, December 19, 2023

Exercise 6.1 (2pt). Show that, if $\pi : P \to M$ is a surjective submersion then:

(a) For any $x \in M$ there exists an open $U \subset M$ and a smooth function $\sigma : U \to P$ with the property that $\pi(\sigma(y)) = x$ for all $y \in U$.
(b) A function $f : M \to \mathbb{R}$ is smooth if and only if $f \circ \pi : P \to \mathbb{R}$ is smooth.

Exercise 6.2 (2.5pt). Assume that $F : P \to M$ is a surjective submersion and we consider the induced pull-back operations (one for each natural number k)

$$F^* : \Omega^k(M) \to \Omega^k(P).$$

The forms on P that are in the image of this map will be called F-basic.

(a) Show that F^* is an injective map.
(b) Show that if $\eta \in \Omega^k(M)$ has the property that $F^*\eta$ is exact, then η is closed.
(c) Assume that $\theta \in \Omega^1(M)$ is such that $F^*\theta = df$ for some function $f \in C^\infty(P)$ which is not F-basic (recall the functions are 0-forms!). Show that θ is not exact.
(d) Deduce, using the previous items, that the 1-form $xdy - ydx \in \Omega^1(S^1)$ is not exact.

Exercise 6.3 (6pt). Consider

$$\theta := 2x^3dx + ydy + zdz \in \Omega^1(\mathbb{R}^3).$$

Please do the following:

(a) Show that there exist 2-dimensional embedded submanifolds $M \subset \mathbb{R}^3$ with the property:

$$\theta|_M = 0.$$

From now on we assume that such a manifold M is fixed.

(b) Show that, for any $p \in M$, one has $T_pM = \mathrm{Ker}\theta_p$.
(c) Show that the following define vector fields on M:

$$Y_p = y\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial y}, \quad (\text{for } p = (x,y,z) \in M).$$

$$Z_p = z\frac{\partial}{\partial x} - 2x^3\frac{\partial}{\partial z}, \quad (\text{for } p = (x,y,z) \in M).$$

(d) Compute $V = [Y,Z]$. More precisely, show that V is of type $cx^n\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)$, where the constants c and n have to be computed.
(e) Compute the integral curves of V.
(f) Is V necessarily complete?
(g) Compute $i_V(\mu)$ and $L_V(\mu)$ where μ is the form on M given by

$$\mu := \frac{1}{3}xdy\wedge dz + ydz\wedge dx + zdx\wedge dy \in \Omega^2(M).$$

(h) Show that μ is nowhere vanishing (as a 2-form on M).

NOTES:

- Please write down your name and the student no
- The order of the exercises is completely unrelated to their difficulty.
- The underlined items are worth 1 point (but that does not mean they are more difficult!). The rest are worth 0.5 points.
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please: include all your computations that support your claims. E.g. when you have to compute something please do not just write down the final result (that will not count!) but explain how you got it/provide the details.

6.2 The 2023/2024 exam, November 7, 2023

Exercise 6.4 (1pt). Show that the composition of two immersions is again an immersion.

Exercise 6.5 (1pt). If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, show that $\mathscr{L}_X(f) = 0$ if and only if f is constant along the integral curves of X, i.e., for any integral curve $\gamma : I \to M$ of X, $f \circ \gamma$ is constant.

Exercise 6.6. Let M be an arbitrary manifold. Recall that a differential form ω on M is called closed if $d\omega = 0$, and it is called exact if there exists a form η such that $\omega = d\eta$.

(a) (0.5pt) Show that the wedge product of two closed forms is again closed.
(b) (0.5pt) Show that for any form ω of even degree, $\omega \wedge d\omega$ is closed.
(c) (0.5pt) In (b), is it true that $\omega \wedge d\omega$ is actually exact?
(d) (0.5pt) Is the conclusion from (b) true also for forms of odd degrees?

Exercise 6.7 (6.5pt). Consider

$$M := \{(x,y,z,u) \in \mathbb{R}^4 : (x^2+y^2+z^2-1)^2 + (xy-u)^2 = 0\}$$

(a) (0.5pt) Show that M is a 2-dimensional embedded submanifold of $\mathbb{R}^4$
(b) (0.5pt) Show that the following define vector fields on M:

$$V_1 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + yz\frac{\partial}{\partial u}$$

$$V_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + xz\frac{\partial}{\partial u}$$

(c) (0.5pt) Compute $[V_1, V_2]$
(d) (1pt) Compute the flow of the vector field on M given by

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (y^2 - x^2)\frac{\partial}{\partial u}.$$

(e) (1pt) Consider the form

$$\omega = 2x dy \wedge dz + z dx \wedge dy - du \wedge dz \in \Omega^2(M)$$

and compute $\omega(V_1, V_2)$.
(f) (0.5pt) Show that ω is closed.
(g) (1pt) compute $i_V(\omega)$ and $L_V(\omega)$.
(h) (0.5p) Show that ω is nowhere zero.
(i) (1pt) Find a diffeomorphism $F : S^2 \to M$ and compute the pull-backs by F of ω, V_1, V_2 and V.

Exercise 6.8 (2pt). Let M be a manifold. Recall that, when X is a vector field, ϕ_X^t denotes the flow of X. Show that for any vector field X one has

(a) $\phi_{cX}^t = \phi_X^{ct}$ for any constant $c \in \mathbb{R}$ and all t for which one of the sides is defined.
(b) if M is compact then $\phi_{5X}^t\left(\phi_X^{-t}(p)\right) = \phi_{2X}^t\left(\phi_X^{2t}(p)\right)$ for all $p \in M$ and all $t \in \mathbb{R}$.

NOTES:

- Please write down your name and the student no
- The order of the exercises is completely unrelated to their difficulty.
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please:
  - include all your computations that support your claims. E.g. in items (c), (d), (e), (g) of Exercise 4 (and similar items where you have to compute something) please do not just write down the final result (that will not count!) but explain how you got it/provide the details.
  - in items (c) and (d) of Exercise 3 please do not just answer with ”yes” or ”no” (if it ”yes” provide a proof, if it is ”no” provide a proof as well (e.g. find a counterexample)). Also, in item (i) of Exercise 4, please explain (i.e. prove) why the F that you describe is a diffeomorphism.

6.3 Retake 2022/2023 (December 20, 2022)

Exercise 6.9 (1pt). Prove that $N = \{(x,y) \in \mathbb{R}^2 : x^3 = y^2\}$ is not an embedded submanifold of $\mathbb{R}^2$.

Exercise 6.10 (2pt). Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n-m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_xM_0 = \{v \in T_xM : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_xM) + T_{f(x)}N_0 = T_{f(x)}N$ or, more explicitly: any element $w \in T_{f(x)}N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_xM, w_0 \in T_{f(x)}N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 6.11 (6.5pts). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x,y,z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X,Y]$.
g) $^{1\text{pt}}$ Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 6.12 (1pt). Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2}-2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute

$$\int_{\mathbb{T}^2} \omega, \quad \text{where } \omega = (x^{2022}y^{2023}dx \wedge dy)|_{\mathbb{T}^2}.$$

NOTES:

- Please write down your name CLEARLY (this is important for me to be able to give you feed-back via Microsoft Teams), and do not forget to mention also the student no!
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please include all your computations that support your claims.
- Each subquestion (labelled by a letter) is worth 0.5 points, except for g) of Exercise 3 (1pt, if you provide all details, as mentioned above).
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- The order of the exercises is completely unrelated to their difficulty.

6.4 Exam 2022/2023 (November 8-th, 2022)

Exercise 6.13 (1pt). You know already, from one of the homeworks, that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$. Show now that $\mathbb{P}^2$ can be embedded in S^4. (Hint: do not look for complicated formulas).

Exercise 6.14 (2pt). Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n-m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_xM_0 = \{v \in T_xM : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_xM) + T_{f(x)}N_0 = T_{f(x)}N$ or, more explicitly: any element $w \in T_{f(x)}N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_xM, w_0 \in T_{f(x)}N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 6.15 (7pts). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x,y,z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
d) Here are three more subspaces of $\mathbb{R}^3$ that contain γ:

$$\{(x,y,z) \in \mathbb{R}^3 : x = z^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : y = z^3\}.$$

Among them, only one is not a submanifold of $\mathbb{R}^3$. Which one, and why is it not?
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X,Y]$.
g) ${}^{1\text{pt}}$ Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.

k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 6.16 (1pt). Let's wonder whether a manifold M can admit a volume form μ of type $\mu = \theta \wedge \theta$, for some other differential form θ on M. Show that if this happens then the dimension of M is divisible by 4. Then show that this can happen already on $M = \mathbb{R}^4$ (provide an explicit example!).

NOTES:

- Please write down your name CLEARLY (this is important for me to be able to give you feed-back via Microsoft Teams), and do not forget to mention also the student no!
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please include all your computations that support your claims.
- Each subquestion (labelled by a letter) is worth 0.5 points, except for g) of Exercise 3 (1pt, if you provide all details, as mentioned above).
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- The order of the exercises is completely unrelated to their difficulty (in particular, there is absolutely no reason to be scared about the first or the last exercise!).

6.5 Retake 2021/2022 (December, 2021)

Note: Please give all details. All the questions below are worth 0.5 points except for the ones in blue which are worth 1 pt (with a grand total of 11.5pt, of which only 10 are needed to receive the maximum mark).

Exercise 6.17. On the 2-sphere S^2, on which we use the coordinate functions $(x,y,z) \in \mathbb{R}^3$:

(a) Show that the function
$$f : S^2 \to S^2, \quad f(x,y,z) = (x^2+y^2-z^2, 2yz, 2xz)$$
is not a submersion.
(b) Show that $(xz \cdot dy - yz \cdot dx) \wedge dz = (1-z^2) \cdot dx \wedge dy$ (an equality of 2-forms on S^2).
(c) Show that the pull-back via f of the volume form $\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy$ satisfies
$$f^*\sigma = 4dx \wedge dy = 4z \cdot \sigma.$$
(d) Find the points $p \in S^2$ at which f fails to be a submersion.
(e) Compute $\int_{S^2} f^*\sigma$ (where you can use any orientation on S^2 that you want).

Exercise 6.18. Let $F : M \to N$ be a smooth map between two manifolds. Recall that, for vector fields $X \in \mathfrak{X}(M)$, $V \in \mathfrak{X}(N)$, one says that X is F-projectable to V if
$$(dF)_p(X_p) = V_{F(p)}$$
for all $p \in M$. Show that:

(a) If $X \in \mathfrak{X}(M)$ if F-projectable to $V \in \mathfrak{X}(N)$, then $L_X(F^*(f)) = F^*(L_V(f))$ for all $f \in C^\infty(N)$, where $F^* : C^\infty(N) \to C^\infty(M)$, $F^*(f) = f \circ F$.
(b) Then show that the converse of (a) holds as well.
(c) Deduce that, if $X \in \mathfrak{X}(M)$ is F-projectable to $V \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ is F-projectable to $W \in \mathfrak{X}(N)$ then $[X,Y]$ is F-projectable to $[V,W]$.

Exercise 6.19. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is f-projectable to $\frac{\partial}{\partial \varphi} = -y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$. We also consider the pull-back via f of the canonical 1-form $d\varphi = -y \cdot dx + x \cdot dy \in \Omega^1(S^1)$,
$$\omega := f^*(d\varphi) \in \Omega^1(M).$$
Show that:

(a) For each $\alpha \in \mathbb{R}$, the fiber
$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$
is an embedded submanifold of M.
(b) ω is closed.
(c) $\omega(V) = 1$.
(d) $\mathscr{L}_V(\omega) = 0$.
(e) $\phi_V^t(p) \in M_{\alpha+t}$ for any $p \in M_\alpha$, and that the flow of V gives rise to diffeomorphisms
$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad \text{(for all } t, \alpha \in \mathbb{R}).$$

Exercise 4. On $\mathbb{R}^3$ we define the following operation ("product"):

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and we consider the associated left translations, one for each $p \in \mathbb{R}^3$,

$$L_p : \mathbb{R}^3 \to \mathbb{R}^3, \quad L_p(q) := p \star q.$$

We define the vector fields $X^1, X^2, X^3 \in \mathfrak{X}(\mathbb{R}^3)$ which, at $p \in \mathbb{R}^3$, are obtained by applying the differential of L_p at the origin, $(dL_p)_0 : T_0\mathbb{R}^3 \to T_p\mathbb{R}^3$, to

$$\left(\frac{\partial}{\partial x}\right)_0, \quad \left(\frac{\partial}{\partial y}\right)_0, \quad \left(\frac{\partial}{\partial z}\right)_0.$$

(a) Compute X^1, X^2, X^3 explicitly and show that they are dual to

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = -xdy + dz,$$

i.e. $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise).

(b) Show that $[X^1, X^2] = X^3, [X^2, X^3] = 0, [X^3, X^1] = 0$.

(c) Find a vector field V such that $\theta^1(V) = 1$, $\theta^2(V) = 0$ and which commutes with X^1, X^2, X^3 (i.e. $[V, X^1] = 0$, $[V, X^2] = 0$, $[V, X^3] = 0$).

(d) For V that you found in (c), compute the flow ϕ_V^t of V explicitly.

(e) Show that ϕ_V^t preserves θ_3, i.e. $(\phi_V^t)^* \theta_3 = \theta_3$.

(f) Deduce that $\mathscr{L}_V(\theta_3) = 0$.

(g) Prove directly (by an algebraic computation) that $\mathscr{L}_V(\theta_3) = 0$.

6.6 Exam 2021/2022 (November 8-th, 2021)

Note: please do not forget to write down your name and student number. Also, please motivate your answers. The first exercise is worth 1.5 point, while in the second exercise all the questions (a)-(p) are worth 0.5p except for the ones in blue which are worth 1p. The mark for the exam is the minimum between 10 and the total number of points you collect.

Exercise 6.20. (*1.5 pt*) Show that, on any manifold M,

$$L_{f\cdot X}(\omega) = f\cdot L_X(\omega) + df\wedge i_X(\omega)$$

for any smooth function $f\in C^\infty(M)$, any vector field $X\in\mathfrak{X}(M)$ and any differential form $\omega\in\Omega^k(M)$

Exercise 6.21. We consider the space of two by two matrices with real coefficients, identified to the Euclidean space $\mathbb{R}^4$ with coordinate functions denoted by x,y,z,t:

$$N := \left\{A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x,y,z,t\in\mathbb{R}\right\}$$

and, inside it, the space of matrices of determinant 1:

$$M := \left\{A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : xt - yz = 1\right\}.$$

First show that:

(a) M is an embedded submanifold of N.
(b) The following define vector fields tangent to M:

$$V^1 := z\frac{\partial}{\partial x} + t\cdot\frac{\partial}{\partial y}$$

$$V^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}.$$

$$V^3 = -y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$$

(c) $f: M\to\mathbb{R}^2,\quad f\begin{pmatrix} x & y \\ z & t \end{pmatrix} = (z,t)$ is a submersion.
(d) Because t is used to denote the last coordinate in M, the time-parameter of curves will be denoted by s. Compute the flows of the vector fields V_1, V_2 and V_3 and show they are of type

$$\phi^s_{V_1}(A) = A_1(s)\cdot A,\quad \phi^s_{V_2}(A) = A_2(s)\cdot A,\quad \phi^s_{V_3}(A) = A\cdot A_3(s)$$

where $A_i(s)$ are matrices that depend on s. For instance, you should get

$$\phi^s_{V_1} = \begin{pmatrix} x+s\cdot z & y+s\cdot t \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & t \end{pmatrix},\quad \text{hence } A_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

(e) Show that $[V_1,V_3] = 0$ and $[V_2,V_3] = 0$.
(f) Show that $[V_1,V_2] = 2V_1$.
(g) Find another vector field V on M which admits the following integral curve:

$$\gamma(s) = A_1(s)\cdot A_3(s) = \begin{pmatrix} \cos s - s\cdot\sin s & \sin s + s\cdot\cos s \\ -\sin s & \cos s \end{pmatrix}$$

(h) Given an example of a 1-form θ on N which is nonzero but such that $\theta|_M = 0$.

(i) Using the 1-forms on M:

$$\theta_1 = -y\cdot dx + x\cdot dy + \frac{x^2+y^2}{z^2+t^2}\left(t\cdot dz - z\cdot dt\right),$$

$$\theta_2 = -\frac{1}{z^2+t^2}(z\cdot dz + t\cdot dt), \quad \theta_3 = \frac{1}{z^2+t^2}(-t\cdot dz + z\cdot dt),$$

show that V^1, V^2, V^3 induce a basis of T_pM at each $p \in M$.

(j) Show that $d\theta_2 = 0$ and $d\theta_3 = 0$.

(k) Consider now $\omega := dx\wedge dy\wedge dz$ as a form on M (the restriction to M). Compute $i_{V^1}(\omega)$, and then $\omega(V^1, V^2, V^3)$.

(l) Show that ω is not a volume form on M and find all $p\in M$ at which $\omega_p = 0$.

(m) Show that $L_{V^1}(\omega) = -dx\wedge dz\wedge dt$ in two ways: one using Cartan's formula, and one using flows.

(n) Consider the following parametrization of M:

$$\begin{cases} x = r\cdot\cos\alpha - \frac{u}{r}\sin\alpha & y = r\cdot\sin\alpha + \frac{u}{r}\cos\alpha \\ z = -\frac{1}{r}\sin\alpha & t = \frac{1}{r}\cos\alpha \end{cases} \qquad u\in\mathbb{R},\ r\in(0,\infty),\ \alpha\in(0,2\pi)$$

which we interpret as the inverse χ^{-1} of a chart of M. What is the domain U of χ?

(o) Hence the u, r, α determined by the equations above (all functions of (x,y,z)) are precisely the components $\chi_1(x,y,z)$, $\chi_2(x,y,z)$ and $\chi_3(x,y,z)$ of $\chi(x,y,z)$. Show that the resulting vector field $\frac{\partial}{\partial\chi_1}$ coincides with V_1 at all points $p\in U$.

(p) Let $\mu := \theta_1\wedge\theta_1\wedge\theta_2$. Compute the coefficient of μ w.r.t. the chart χ,

$$f_\chi^\mu \in C^\infty(\mathbb{R}\times(0,\infty)\times(0,2\pi)).$$

6.7 Retake 2020/2021 (January 8-th, 2021)

Note: The points below add up together to 12 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with "yes" or "no" but also provide arguments; do not just write down the final result, but also explain how you found it.

Exercise 6.22. (*1 pt*) If $\omega \in \Omega^k(M)$ is a closed differential form on a manifold M show that the operation

$$\Omega^l(M) \to \Omega^{k+l}(M), \quad \theta \mapsto \omega \wedge \theta$$

takes closed forms to closed forms, and exact forms to exact forms.

Exercise 6.23. (*1.5 pt*) Does there exists a vector field X on $\mathbb{R}^3$ with flow given by

$$\phi_X^t(x,y,z) = (x+t, y+tz, z)?$$

But a vector field Y with

$$\phi_Y^t(x,y,z) = (x+t, y+tz, z+t)?$$

Exercise 6.24. Consider the following differential 1-forms on S^2:

$$\theta_1 = x \cdot dy - y \cdot dx + z \cdot dz, \quad \theta_2 = x \cdot dy + y \cdot dx + z \cdot dz \quad (\text{restricted to } S^2!)$$

and the vector field $V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \in \mathfrak{X}(S^2)$.

(a) (*0.5 pt*) Which one of the 1-forms above is closed and which not?
(b) (*0.5 pt*) Which one of the 1-forms above is exact and which not?
(c) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ a volume form?
(d) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ exact?
(e) (*0.5 pt*) Compute $i_V(\theta_1 \wedge \theta_2)$.
(f) (*0.5 pt*) Show that $L_V(\theta_1) = 0$.
(g) (*1 pt*) Compute $(\phi_V^t)^* \theta_1$, where ϕ_V^t is the flow of V.

Exercise 6.25. Let $E : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, we denote by E_x, E_y and E_z the partial derivatives of E with respect to the variables x, y and z (each a smooth function on $\mathbb{R}^3$), and we construct the following three vector fields on $\mathbb{R}^3$:

$$\widetilde{U} = E_z \cdot \frac{\partial}{\partial y} - E_y \cdot \frac{\partial}{\partial z}, \quad \widetilde{V} = E_x \cdot \frac{\partial}{\partial z} - E_z \cdot \frac{\partial}{\partial x}, \quad \widetilde{W} = E_y \cdot \frac{\partial}{\partial x} - E_x \cdot \frac{\partial}{\partial y}.$$

We also consider

$$M_E := \{(x,y,z) \in \mathbb{R}^3 : E(x,y,z) = 0\}$$

and we assume that, at every point $p \in M_E$, at least one of the three vector fields above does not vanish. Show that:

(a) (*0.5 pt*) M_E is a 2-dimensional submanifold of $\mathbb{R}^3$.
(b) (*0.5 pt*) $\widetilde{U}, \widetilde{V}$ and $\widetilde{W}$ are tangent to M_E at each point $p \in M_E$. Hence they define three vector fields on M_E- and those will be denoted

$$U, V, W \in \mathfrak{X}(M_E).$$

(c) (*0.5 pt*) U, V and W span the tangent space $T_p M_E$ at each $p \in M_E$.

(d) (*1 pt*) as a consequence of (c), at every point $p \in M$, the Lie bracket $[V,W]_p$ and the similar ones are linear combinations of U_p, V_p and W_p. Show that there are smooth functions f, g and h on M_E such that

$$[V,W] = f \cdot U + g \cdot V + h \cdot W,$$

and similarly for the other two Lie brackets.

(e) (*0.5 pt*) Consider the projection on the first two factors

$$\mathrm{pr}_{1,2} : M_E \to \mathbb{R}^2, \quad \mathrm{pr}_{1,2}(x,y,z) = (x,y).$$

Describe the points at which $\mathrm{pr}_{1,2}$ fails to be a submersion. And the same for the similar projections $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$.

(f) (*0.5 pt*) Show that, around each point $p \in M_E$, at least one of the projections $\mathrm{pr}_{1,2}$, $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$ can be used as a chart, i.e. there exists an open neighborhood U of p and a chart of type of type (U, χ), with χ being the restriction to U of one of those three projections.

Exercise 6.26. Consider the 3-dimensional projective space $\mathbb{P}^3$.

(a) (*0.5 pt*) Show that

$$p : (0,\pi) \times (0,\pi) \times (0,\pi) \to \mathbb{P}^3,$$

$$(\phi_1, \phi_2, \phi_3) \mapsto [\cos\phi_1 \ : \ \sin\phi_1 \cdot \cos\phi_2 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \cos\phi_3 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \sin\phi_3]$$

is a parametrization of $\mathbb{P}^3$, i.e. a diffeomorphism into an open $U \subset \mathbb{P}^3$.

(b) (*1.5 pt*) Assuming that you know already that $\mathbb{P}^3$ is orientable, and using any orientation that you prefer, compute

$$\int_{\mathbb{P}^3} f \cdot df_0 \wedge df_1 \wedge df_2,$$

where $f, f_0, f_1, f_2 : \mathbb{P}^3 \to \mathbb{R}$ are the functions which, for $(x_0, x_1, x_2, x_3) \in S^2$, send:

$$[x_0 : x_1 : x_2 : x_3] \xrightarrow{f_i} (x_i)^2, \quad [x_0 : x_1 : x_2 : x_3] \xrightarrow{f} x_0 \cdot x_1 \cdot x_2 \cdot x_3.$$

You can express the result in terms of $I_k = \int_0^\pi \sin^k \phi$ without further numerical computations (those would be obtained by further remarking that $I_k = \frac{k-1}{k} I_{k-2}$ and $I_0 = \pi$, $I_1 = 2$.)

6.8 Exam 2020/2021 (November 6th, 2020)

Note: The points below add up together to 13 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with "yes" or "no" but also provide arguments; do not just write down the final result, but also explain how you found it. If you can't answer a question, move on to the next one. You may use information from any skipped questions.

Exercise 6.27. Define

$$M = \{(x,y,z,t) \in \mathbb{R}^4 : xt = yz + 12\}.$$

(a) (*0.5 pt*) Show that M is an embedded submanifold of $\mathbb{R}^4$.
(b) (*0.5 pt*) Compute the tangent space of M at the point $p_0 = (4,0,0,3)$.
(c) (*0.5 pt*) Find a vector field $V \in \mathfrak{X}(M)$ with the property that its flow satisfies

$$\phi_V^{2s}(p_0) = (4\cos s, 4\sin s, -3\sin s, 3\cos s).$$

(d) (*1 pt*) Show that also

$$X^1 := \frac{1}{2}\left(y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} + t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right)$$

defines a vector field tangent to M and compute $X^2 := [X^1, V], X^3 := [X^1, X^2]$.
(e) (*1 pt*) Consider the spheres centred at the origin

$$S_r^3 = \{(x,y,z,t) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + t^2 = r^2\}$$

(one for each $r > 0$), and show that their intersections with M

$$M_r := M \cap S_r^3,$$

are embedded submanifolds of $\mathbb{R}^4$ (be aware that, depending on how you approach this, you may end up doing some very time consuming computations ...).
(f) (*0.5 pt*) When $r = 5$ (so that $p_0 \in M_r$) compute the tangent space of M_5 at p_0.
(g) (*0.5 pt*) Show that the following defines a submersion from M_5 to S^1:

$$f : M_5 \to S^1, \quad f(x,y,z,t) = (x - t, y + z).$$

(h) (*0.5 pt*) Show that M_5 is diffeomorphic to a torus.

Exercise 6.28. Return to the last exercise and, in (d) above, just assume that

$$X^1 := \frac{1}{2}\left(y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} + t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right),$$

$$X^2 := \frac{1}{2}\left(-x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} - z \cdot \frac{\partial}{\partial z} + t \cdot \frac{\partial}{\partial t}\right),$$

$$X^3 := \frac{1}{2}\left(-y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} - t \cdot \frac{\partial}{\partial z} + z \cdot \frac{\partial}{\partial t}\right)$$

and you may assume that you now they satisfy:

$$[X^1, X^2] = X^3, \quad [X^2, X^3] = -X^1, \quad [X^3, X^1] = -X^2$$

Show that:

(a) (*0.5 pt*) The 1-forms

$$\theta_1 = (-z\cdot dx + t\cdot dy + x\cdot dz - y\cdot dt)/12,$$
$$\theta_2 = (-t\cdot dx - z\cdot dy + y\cdot dz + x\cdot dt)/12,$$
$$\theta_3 = (\ z\cdot dx + t\cdot dy - x\cdot dz - y\cdot dt)/12$$

satisfy $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise.

(b) (*0.5 pt*) Deduce that X_p^1, X_p^2, X_p^3 form a basis of T_pM for all $p \in M$.

(c) (*0.5 pt*) Deduce that two 2-forms $\eta, \xi \in \Omega^2(M)$ coincide if and only if

$$\eta(X^1,X^2) = \xi(X^1,X^2), \quad \eta(X^2,X^3) = \xi'(X^2,X^3), \quad \eta(X^3,X^1) = \xi(X^3,X^1).$$

(d) (*0.5 pt*) Show that $d\theta_1 = \theta_2 \wedge \theta_3, \quad d\theta_2 = \theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2.$

Exercise 6.29. (*1.5 pt*) Show that there exists a vector field $X \in \mathscr{X}(S^2)$ with flow given by

$$\phi_X^t(x,y,z) = \left(\frac{(1+x)e^t - (1-x)e^{-t}}{(1+x)e^t + (1-x)e^{-t}}, \frac{2y}{(1+x)e^t + (1-x)e^{-t}}, \frac{2z}{(1+x)e^t + (1-x)e^{-t}} \right).$$

Make sure you give all the details. Is the vector field that you found complete?

Exercise 6.30. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is projectable to $\frac{\partial}{\partial\theta}$, i.e.

$$(df)_p(V_p) = \left(\frac{\partial}{\partial\theta}\right)_{f(p)} \quad \text{for all } p \in M.$$

We also consider the pull-back via f of the canonical 1-form $d\theta \in \Omega^1(S^1)$,

$$\omega := f^*(d\theta) \in \Omega^1(M).$$

Show that:

(a) (*0.5 pt*) for each $\alpha \in \mathbb{R}$, the fiber

$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$

is an embedded submanifold of M.

(b) (*1.5 pt*) ω is closed, $\omega(V) = 1$ and $\mathscr{L}_V(\omega) = 0$.

(c) (*1 pt*) for any $p \in M_\alpha$, $\phi_V^t(p) \in M_{\alpha+t}$ and the flow of V gives rise to diffeomorphisms

$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad (\text{for all } t, \alpha \in \mathbb{R}).$$

Exercise 6.31. (*1.5 pt*) Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute $\int_{\mathbb{T}^2} \omega$ where $\omega = (xdy \wedge dz)|_{\mathbb{T}^2}$.

6.9 Retake 2019/2020 (January 6-th, 2020)

Consider the 2-dimensional torus

$$\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$$

(equipped with the product manifold structure) and the vector field $\frac{\partial}{\partial \theta_1}$ given by

$$\left(\frac{\partial}{\partial \theta_1}\right)_{(z_1,z_2)} := \left.\frac{d}{dt}\right|_{t=0} (e^{i\cdot t} z_1, z_2) \in T_{(z_1,z_2)}\mathbb{T}^2,$$

and similarly we define $\frac{\partial}{\partial \theta_2}$. For any smooth function $f : \mathbb{T}^2 \to \mathbb{R}$, we use the notations $\frac{\partial f}{\partial \theta_i}$ for the resulting Lie derivatives (again functions on the torus):

$$\frac{\partial f}{\partial \theta_1}(z_1,z_2) = \left.\frac{d}{dt}\right|_{t=0} f(e^{i\cdot t} z_1, z_2), \quad \text{etc.}$$

We will also consider the more general vector fields on the torus:

$$X^f := \frac{\partial}{\partial \theta_1} + f \cdot \frac{\partial}{\partial \theta_2} \in \mathfrak{X}(\mathbb{T}^2). \qquad (*)$$

defined for any smooth function f on T^2. Recall also that for any $R > r > 0$, one has a "concrete model" of $\mathbb{T}^2$ inside $\mathbb{R}^3$, namely

$$T^2_{R,r} := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\},$$

which is the image of the map

$$F : \mathbb{T}^2 \to \mathbb{R}^3, \quad (e^{i\theta_1}, e^{i\theta_2}) \mapsto ((R + r\cdot\cos\theta_1)\cdot\cos\theta_2,\ (R + r\cdot\cos\theta_1)\cdot\sin\theta_2,\ r\cdot\sin\theta_1).$$

You may assume (e.g. from Topology) that you know already that F is a homeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.

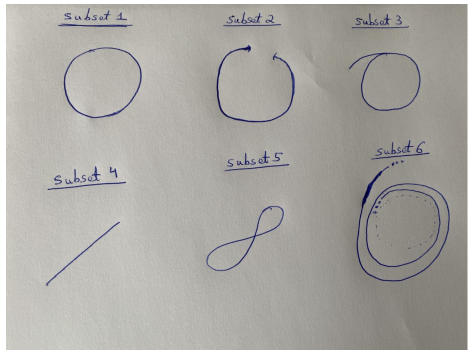
Below is a list of questions for you. The list is rather long, but that should make the questions easier to answer. Each question is worth 0.5 pt, except for those from numbers 1, 7, 15, 16 and 19 which are 1 pt. In particular, you do not have to answer all the questions in order to obtain the maximum of points (10 pt). And here are the questions:

1. For $i \in \{1,2\}$ compute $(dF)_p(\frac{\partial}{\partial \theta_i})$ at arbitrary points $p = (e^{i\cdot\theta_1}, e^{i\cdot\theta_2}) \in \mathbb{T}^2$ and show that F is an immersion.
2. Deduce that F is a diffeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.
3. We want to compute the vector fields $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$ by moving them to $T^2_{R,r}$ (via F) and decomposing them w.r.t. the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ of the tangent spaces of $\mathbb{R}^3$. In other words, we want to compute $F_*\left(\frac{\partial}{\partial \theta_1}\right)$ and $F_*\left(\frac{\partial}{\partial \theta_2}\right)$. Show that
$$F_*\left(\frac{\partial}{\partial \theta_2}\right)_q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \text{for all } q = (x,y,z) \in T^2_{R,r},$$
and find a similar formula for $F_*\left(\frac{\partial}{\partial \theta_1}\right)$.
4. Compute the flows of $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$ and then deduce that $[\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}] = 0$.
5. Looking at the vector fields of type $(*)$, show that for any two smooth functions f, g on $\mathbb{T}^2$, there exists another smooth function h such that $\frac{\partial}{\partial \theta_1} + [X^f, X^g] = X^h$.
6. For each real number $\lambda \in \mathbb{R}$ consider the vector field X^λ (i.e. $(*)$ obtained when f is the constant function λ). Compute the maximal integral curve γ of X^λ starting at $(1,1) \in \mathbb{T}^2$.
7. With γ as above (and λ arbitrary constant) show that:
   - the image of γ is always an immersed submanifold of $\mathbb{T}^2$
   - but, if λ is irrational, then that image is not an embedded submanifold of $\mathbb{T}^2$.

8. Show that there exist, and they are unique, two 1-forms $d\theta_1$ and $d\theta_2$ on $\mathbb{T}^2$ with the property that
$$d\theta_i\left(\frac{\partial}{\partial\theta_j}\right) = \delta_{i,j} \text{ (1 if } i = j \text{ and 0 otherwise).}$$
9. Show that for any smooth function f on $\mathbb{T}^2$ one has
$$df = \frac{\partial f}{\partial\theta_1}\cdot d\theta_1 + \frac{\partial f}{\partial\theta_2}\cdot d\theta_2.$$
10. In contrast with what the notation may suggest, $d\theta_1$ and $d\theta_2$ are not exact forms. To see this, you are asked to prove something a bit more general: on any compact manifold M, any 1-form that is exact must vanish at at least one point in M.
11. However, show that $d\theta_1$ and $d\theta_2$ are both closed.
12. Show that $d\theta_2 \wedge d\theta_1$ is a volume form on $\mathbb{T}^2$ and compute the corresponding volume.
13. Show that there do not exist smooth functions f, g on $\mathbb{T}^2$ such that
$$\frac{\partial g}{\partial\theta_1} - \frac{\partial f}{\partial\theta_2} = 1.$$
(hint: volumes are strictly positive, hence non-zero).
14. Consider again the vector field $(*)$ (with f again an arbitrary smooth function). Since $L_{X^f}(d\theta_2 \wedge d\theta_1)$ is again a 2-form on $\mathbb{T}^2$, it is a function times the volume form $d\theta_2 \wedge d\theta_1$. Compute that function.
15. Deduce that the flow of X^f preserves our volume form, i.e.
$$\left(\phi^t_{X^f}\right)^*(d\theta_2 \wedge d\theta_1) = d\theta_2 \wedge d\theta_1 \quad \text{for all } t,$$
if and only if $f = f(z_1, z_2)$ depends only on the first coordinate z_1.
16. Show that there is no closed 2-form $\tilde{\eta}$ on $\mathbb{R}^3$ such that $F^*\tilde{\eta} = d\theta_2 \wedge d\theta_1$. (hint: think of a more general statement, for any volume form on a compact embedded submanifold of a Euclidean space).
17. Show that the restrictions of the standard 1-forms dx, dy, dz from $\mathbb{R}^3$ to $T^2_{R,r}$ satisfy:
$$x\cdot dx + y\cdot dy + z\cdot dz = \frac{R}{\sqrt{x^2+y^2}}(x\cdot dx + y\cdot dy) \quad (\text{on } T^2_{R,r}).$$
18. Compute the pull-backs of dx, dy and dz to $\mathbb{T}^2$ (in terms of θ_1 and θ_2).
19. For $\sigma = xdy\wedge dz + ydz\wedge dx + zdx\wedge dy \in \Omega^2(\mathbb{R}^3)$, restrict it to $T_{R,r}$ and compute
$$\int_{T_{R,r}} \sigma|_{T_{R,r}} \quad (\text{use any orientation you prefer}).$$
(hint: you may want to compute $F^*\sigma$, and hopefully you will obtain the expression $r(R + r\cdot\cos\theta_1)(r + R\cos\theta_1)\cdot d\theta_2 \wedge d\theta_1$).
20. We are now trying to write the volume form $d\theta_2 \wedge d\theta_1$, moved to $T^2_{R,r}$ (via the diffeomorphism F), in terms of the standard coordinates x, y, z on $\mathbb{R}^3$ Show that
$$\eta := \frac{1}{r^2(x^2+y^2)}\left[\left(\sqrt{x^2+y^2} - R\right)\cdot\sigma + Rz\cdot dx\wedge dy\right]$$
works. In other words, interpreting η as a 2-form on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$, show that
$$F^*\eta = d\theta_2 \wedge d\theta_1.$$

6.10 Exam 2019/2020 (November 6th, 2019)

Exercise 6.32. (*1.5 pt*) Which of the following subsets of $\mathbb{R}^2$ can be the image of an integral curve of a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. (Explanation: subset 1 is a circle, subset 2 is an open part of a circle, subset 4 is an infinite line, subset 6 describes a spiral approaching a circle. Please explain you answer; but when you describe a vector field, you do not have to write down a formula- just draw it on the picture)



Exercise 6.33. (*1.5 pt*) Which of the subsets of $\mathbb{R}^2$ from the previous picture describes an embedded submanifold of $\mathbb{R}^2$. But an immersed submanifold? (please motivate your answer, but you can use the picture instead of explicit formulas)

Exercise 6.34. (*1.5 pt*) Let $\theta \in \Omega^1(M)$ be a closed 1-form and define

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta(\omega) := d\omega + \theta \wedge \omega.$$

Show that $d_\theta \circ d_\theta = 0$ if and only if θ is closed.

Exercise 6.35. ($4 \times 0.5 = 2$ *pt*) Consider

$$f : S^3 \to \mathbb{R}, \quad f(x,y,z,t) = x^2 + y^2 - z^2 - t^2.$$

(a) show that the zero-set $M_0 := f^{-1}(0)$ is an embedded submanifold of S^3.
(b) show that M_0 is diffeomorphic to the 2-torus.
(c) find all the points at which f is a submersion.
(d) give an example of a 1-form θ on S^3, different from the zero form, such that $\theta|_{M_0} = 0$.

Exercise 6.36. ($1+1+1 = 3$ *pt*) On the following subspace of the 2-sphere

$$N := \{(x,y,z) \in S^2 : z \neq 0\}$$

consider the restriction of $\frac{1}{z} \cdot dx \wedge dy$ to N:

$$\sigma_0 := \frac{1}{z} \cdot dx \wedge dy\Big|_N .$$

(a) show that σ_0 can be extended to the entire S^2, i.e. there exists a (smooth) 2-form $\sigma \in \Omega^2(S^2)$ such that $\sigma|_N = \sigma_0$.
(b) compute $i_X(\sigma)$ and $L_X(\sigma)$ for $X = y \cdot \frac{\partial}{\partial x} - x \cdot \frac{\partial}{\partial y}$.
(c) compute $\int_{S^2}(x+y+z) \cdot \sigma$ (use your favourite orientation of S^2).

Exercise 6.37. ($0.5 + 1.5 + 0.5 = 2.5$ *pt*) On the 2-sphere we consider the following curve:

$$\gamma : \mathbb{R} \to S^2, \quad \gamma(t) = \left(\frac{t}{\sqrt{1+t^2}}, \frac{\cos t}{\sqrt{1+t^2}}, \frac{\sin t}{\sqrt{1+t^2}} \right).$$

(a) draw a picture of γ.
(b) find a vector field X on S^2 (explicit formulas!) for which γ is an integral curve.
(c) did you check how smooth your vector field is? please do!

6.11 Retake 2018/2019 (January 2019)

Exercise 6.38. Consider the unit circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $x+iy \mapsto (x,y)$ we give $\mathbb{C}$ its standard manifold structure.

a) Give a smooth map $f : \mathbb{C} \to \mathbb{R}$ which has $1 \in \mathbb{R}$ as a regular value. Conclude that S^1 is an embedded submanifold of $\mathbb{C}$.
b) By inspecting its differential, show that the map

$$p : \mathbb{R} \to S^1, \quad t \mapsto e^{it}$$

is a local diffeomorphism.
c) Show that the vector field $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R})$ is p-projectable to a unique smooth vector field $\frac{\partial}{\partial \theta}$ on S^1.
d) Show that there is a unique smooth 1-form on S^1, denoted by $d\theta$, such that $p^*d\theta = dt \in \Omega^1(\mathbb{R})$.
e) Is $d\theta$ closed? Is it a volume form?
f) Compute the integral

$$\int_{S^1} d\theta.$$

g) Is $d\theta$ exact?

In the coming exercise we consider the 2-dimensional torus:

$$T^2 := S^1 \times S^1$$

equipped with the product manifold structure. Write (t_1,t_2) for the standard coordinates in $\mathbb{R}^2$. By part (a) of the previous exercise, the map

$$q : \mathbb{R}^2 \to T^2, \quad (t_1,t_2) \mapsto (e^{it_1}, e^{it_2})$$

is a local diffeomorphism and by part (c) of that same exercise, the vector fields $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$ on $\mathbb{R}^2$ are q-projectable to unique vector fields on T^2, denoted by $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$.

Exercise 6.39. Let c be an irrational real number and consider the vector field X on T^2 defined as:

$$X = \frac{\partial}{\partial \theta_1} + c\frac{\partial}{\partial \theta_2}.$$

a) Compute the maximal integral curve γ of X starting at $(1,1) \in T^2$.
b) Prove that the image of γ is an immersed submanifold of T^2.
c) Prove that the image of γ is not an embedded submanifold of T^2.

Exercise 6.40. This exercise is about orientability. Recall that a manifold M is orientable if it admits an atlas $\mathscr{A}$ such that:

$$\mathrm{Jac}(\chi_1 \circ \chi_2^{-1})_x > 0, \quad \forall x \in \chi_2(U_1 \cap U_2)$$

and all charts $(U_1,\chi_1),(U_2,\chi_2) \in \mathscr{A}$.

a) Show that if M is an orientable n-dimensional manifold, then there is a volume form $\omega \in \Omega^n(M)$.
b) Show that if ω_1 and ω_2 are two volume forms, then there is a unique no-where vanishing smooth function $f \in C^\infty(M)$ such that

$$f \cdot \omega_1 = \omega_2.$$

In the rest of this exercise we aim to prove that $\mathbb{R}P^2$ is not orientable. Suppose by contradiction that it is orientable. Then it admits a volume-form $\omega \in \Omega^2(\mathbb{R}P^2)$.

c) Consider the map

$$\pi : S^2 \to \mathbb{R}P^2, \quad (x,y,z) \mapsto [x:y:z].$$

Show that π is a local diffeomorphism and conclude that $\pi^*\omega$ is a volume-form on S^2.

d) Show that there is a no-where vanishing function $f \in C^\infty(S^2)$ such that:

$$f \cdot (\pi^*\omega) = z dx \wedge dy + y dz \wedge dx + x dy \wedge dz,$$

where $x,y,z : S^2 \to \mathbb{R}$ are the projections onto the three components in $\mathbb{R}^3$.

e) Compute the pull-back of both sides with the anti-podal map

$$\iota : S^2 \to S^2, \quad p \mapsto -p$$

and derive a contradiction.

6.12 Exam 2018/2019 (November 6th, 2018)

Exercise 6.41. (*1 pt*) Show that, on any manifold, for any $\theta \in \Omega^k(M)$ with k odd, one has $\theta \wedge \theta = 0$. Then find an example of a manifold M and a 2-form M such that $\theta \wedge \theta \neq 0$.

Exercise 6.42. (*3× 0.5= 1.5 pt*) This is about vector fields on $\mathbb{R}^3$. Do the following:

(a) find a vector field X for which $\gamma(t) = (e^{2t}, e^t \sin t, e^t \cos t)$ is an integral curve.
(b) find one more such vector field, but different from X.
(c) for one the vector fields you found, compute its flow.

Exercise 6.43. (*4× 1= 4 pt*) Consider

$$M_4 := \{(x,y,z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3,$$

and the map from M_4 to the 2-sphere S^2 given by

$$f : M_4 \to S^2, \quad f(x,y,z) = (x^2, y^2, z^2)$$

(a) Show that M_4 is a submanifold of $\mathbb{R}^3$ and f is a smooth map.
(b) Compute the tangent space of M_4 at the point $p = (\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_p M_4.$$

Similarly at the point $q = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$.
(c) Show that f is not an immersion at p, but it is a local diffeomorphism around q.
(d) Show that M_4 is not diffeomorphic to

$$M_3 := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3.$$

but it is diffeomorphic to S^2.

Exercise 6.44. Let $\theta_1, \ldots, \theta_n \in \Omega^1(M)$ be $n = \dim M$ one-forms on M which are dual to a set $\{X^1, \ldots, X^n\}$ of vector fields on M, in the sense that

$$\theta^i(X_j) = \delta_i^j \quad (1 \text{ if } i = j \text{ and } 0 \text{ otherwise}).$$

(a) (*1 pt*) show that all the 1-forms $\theta_1, \ldots, \theta_n$ are closed if and only if the vector fields $X^1, \ldots, X^n$ are pairwise commuting, i.e. $[X_i, X_j] = 0$ for all i and j.
(b) (*1.5 pt*; the most difficult- you may want to leave it for later) show that for any family of vector fields $X^1, \ldots, X^n$ that are pairwise commuting and which are linearly independent at some point $x_0 \in M$, there exists a chart (U, χ) near x_0 such that

$$X_x^1 = \left(\frac{\partial}{\partial \chi_1}\right)_x, \quad \ldots \quad , X_x^n = \left(\frac{\partial}{\partial \chi_n}\right)_x \quad \text{for all } x \in U.$$

(c) (*0.5 pt*) Deduce that if $\{\theta_1,\dots,\theta_n\}$ is a set of closed 1-forms on the n-dimensional manifold M, that are linearly independent at some point $x_0 \in M$, then there exists a chart (U,χ) of M around x_0 such that $\theta_i|_U = d\chi_i$ on U, for all $i \in \{1,\dots,n\}$.

Exercise 6.45. Let M be a manifold, and $X,Y \in \mathfrak{X}(M)$ two vector fields. We look at

$$L_X \circ i_Y - i_Y \circ L_X : \Omega^k(M) \to \Omega^{k-1}(M) \quad \text{(for all } k \geq 0\text{)}.$$

Show that:

(a) (*0.5 pt*) on 1-forms $\theta \in \Omega^1(M)$, it coincides with $i_{[X,Y]}$.
(b) (*1 pt*) it satisfies a Leibniz-type identity that you have to write down yourself.
(c) (*1 pt*) $L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]}$.

Exercise 6.46. (*2.5 pt*) Compute the following integral

$$\int_{S^2} \big((x+y^{2017})dy\wedge dz + (y+z^{2018})dz\wedge dx + (z+x^{2019})dx\wedge dy\big)$$

(you can use any orientation on S^2 that you want).

6.13 Retake 2017/2018 (January 3rd, 2018)

Note: The questions in blue are worth 1 point, and the rest 0.5 points. The points below add up together to 13.5, so you do not have to solve everything to get the maximum mark of 10! Also, please be aware: blue really means that "it is worth more points", and not that "it is more difficult".

Exercise 6.47. Show that $M := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 4\}$ is an embedded submanifold of $\mathbb{R}^3$ and compute the tangent space $T_pM \subset \mathbb{R}^3$ at the point $p = (1,1,-1)$.

Exercise 6.48. Assume that M is an S^1-manifold, i.e. a manifold together with an action of the circle S^1- i.e. a smooth map

$$M \times S^1 \to M, \quad (\lambda, p) \mapsto \lambda \cdot p$$

such that $p \cdot 1 = p$ and $(p\cdot\lambda)\cdot\lambda' = p\cdot(\lambda\lambda')$ for all $\lambda, \lambda' \in S^1$, $p \in M$. We define the following vector field on M:

$$V_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{2\pi i t}. \tag{6.13.1}$$

(a) Write down the flow of V using the action of S^1 on M.
(b) Deduce that a point $p \in M$ is a fixed point of the action, i.e. satisfies $p \cdot \lambda = p$ for all $\lambda \in S^1$, if and only if p is a zero of V, i.e. satisfies $V_p = 0$.
(c) Assume now that the action is <u>free</u> i.e. that it has no fixed points (or, cf. the previous point, V has no zeroes). Show that for each $p \in M$, <u>the orbit through p</u>:

$$\mathscr{O}_p := \{p \cdot \lambda : \lambda \in S^1\} \tag{6.13.2}$$

is an embedded submanifold of M diffeomorphic to S^1.

Exercise 6.49. Consider

$$M := \{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2-5)^2 + 16z^2 = 16\} \subset \mathbb{R}^3.$$

(a) (*1 pt*) Show that M is a submanifold of $\mathbb{R}^3$.
(b) (*1 pt*) Compute the tangent space of M at the point $p = (3,0,0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM.$$

Similarly, compute the tangent space at the point $q = (2,0,1)$.
(c) (*0.5 pt*) Draw a picture of M (hint: if you do not see how M looks like, maybe compute the tangent space at one more point).
(d) (*0.5 pt*) Prove that M is diffeomorphic to $S^1 \times S^1$.

Exercise 6.50. Consider again an S^1-manifold M. A differential form $\omega \in \Omega^\bullet(M)$ is called:

- equivariant: if $\mathrm{m}_\lambda^*(\omega) = \omega$ for all $\lambda \in S^1$, where $\mathrm{m}_\lambda : M \to M$ is the map $p \mapsto p \cdot \lambda$.
- horizontal: if $i_V(\omega) = 0$.
- <u>basic form</u>: if it is both horizontal as well as equivariant.

Show that:

(a) if ω is equivariant, then so is $d\omega$.
(b) ω is equivariant if and only if $L_V(\omega) = 0$.

(c) if ω is basic, then so is $d\omega$.
(d) in general, it is not true that if ω is horizontal than so is $d\omega$ (provide a counter-example, e.g. using the last homework).

Exercise 6.51. Assume now that M is a principal S^1-bundle, i.e. an S^1-manifold with the property that the action is free and that the orbits (6.13.2) of the action are precisely the fibers of a submersion

$$h : M \to B,$$

with values in some other manifold B (usually called the base of the bundle). Show that

(a) The pull-back map $h^* : \Omega^\bullet(B) \longrightarrow \Omega^\bullet(M)$ is injective.
(b) For any differential form $\eta \in \Omega^k(B)$, its pull-back $\omega := h^*(\eta)$ is a basic form on M.
(c) h^* is a linear isomorphism between $\Omega^\bullet(B)$ and the space of basic forms on M.

Exercise 6.52. Assume again that M is a principal S^1-bundle, with corresponding submersion $h : M \to B$, and corresponding vector field (6.13.1).

We choose a 1-form $\omega \in \Omega^1(M)$ which is equivariant and satisfies $\omega(V) = 1$. Show that

(a) $d\omega \in \Omega^2(M)$ is a basic 2-form.
(b) denoting by $\eta \in \Omega^2(B)$ the 2-form on characterized by

$$d\omega = h^*(\eta)$$

(which exists and is unique by the previous exercise), η is a closed 2-form on B.
(c) while η is built using any ω which was equivariant and satisfied $\omega(V) = 1$, prove that the cohomology class $[\eta] \in H^2(B)$ does not depend on the choice of ω.

(this cohomology class is known as "the first Chern class" of the principal S^1-bundle).

Exercise 6.53. Returning now to the Hopf fibration and the last homework(s),

$$h : S^3 \to S^2,$$

we are in the case of a principal S^1-bundle with the action given by $(z_0, z_1) \cdot \lambda = (\lambda z_0, \lambda z_1)$, and with corresponding vector field (as in the last homework):

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

Show that

(a) $\omega = -y \cdot dx + x \cdot dy - t \cdot dz + z \cdot dt \in \Omega^1(S^3)$ is equivariant and satisfies $\omega(V) = 1$.
(b) compute the resulting form $\eta \in \Omega^2(S^2)$ (cf. the previous exercise).
(c) show that $[\eta] \in H^2(S^2)$ is non-zero and try to draw conclusions on the Hopf fibration (e.g.: can it be isomorphic to the product fibration $S^1 \times S^2$?)

6.14 Exam 2017/2018 (November 8th, 2017)

Exercise 6.54. (*1 pt*) Show that, for a vector field X on a manifold M and $f \in C^\infty(M)$, one has $L_X(f) = 0$ if and only if f is constant on the integral curves of X.

Exercise 6.55. Consider the sphere $S^2 \subset \mathbb{R}^3$ and we use (x,y,z) to denote the standard coordinates in $\mathbb{R}^3$. We consider the following vector field tangent to the sphere

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$$

as well as the volume form on the sphere:

$$\theta = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy \in \Omega^2(S^2)$$

(as before, while the previous formula defines a 2-form on $\mathbb{R}^3$, θ is the restriction to S^2).

(a) (*0.5 pt*) Compute $i_X(\theta)$ and $d(i_X(\theta))$.
(b) (*0.5 pt*) Compute $d\theta$ and $i_X(d\theta))$.
(c) (*0.5 pt*) Compute $L_X(\theta)$ in two ways: one using the Cartan formula, and one using the properties of L_X (being a derivation, and commuting with d).
(d) (*0.5 pt*) Compute the flow ϕ^t of X.
(e) (*0.5 pt*) Show that $(\phi^t)^*\theta = \theta$ for all $t \in \mathbb{R}$.

Exercise 6.56. Assume that M is a connected 3-dimensional manifold, and $V^1, V^2, V^3 \in \mathfrak{X}(M)$ are vector fields on M with the property that V^1_p, V^2_p, V^3_p form a basis of T_pM for all $p \in M$ and let $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ be the 1-forms that are dual to V^1, V^2, V^3, i.e. satisfying

$$\theta_i(V^j) = \delta^i_j \text{ (1 if } i = j \text{ and 0 otherwise)}.$$

(a) (*0.5 pt*) Show that, for any $f \in \mathscr{C}^\infty(M)$ one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3.$$

(b) (*1 pt*) Show that the vector fields V^i satisfy

$$[V^1, V^2] = 2V^3, \quad [V^2, V^3] = 2V^1, \quad [V^3, V^1] = 2V^2.$$

if and only if the 1-forms θ_i satisfy

$$d\theta_1 = -2\theta_2 \wedge \theta_3, \quad d\theta_2 = -2\theta_3 \wedge \theta_1, \quad d\theta_3 = -2\theta_1 \wedge \theta_2.$$

From now on we assume that all these are satisfied. Assume furthermore that the 1-forms are invariant with respect to a vector field $V \in \mathfrak{X}(M)$ in the sense that

$$L_V(\theta_1) = L_V(\theta_2) = L_V(\theta_3) = 0.$$

Introduce the following real-valued functions on M:

$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3).$$

(c) (*0.5 pt*) Prove that

$$dh_1 = 2h_2 \cdot \theta_3 - 2h_3 \cdot \theta_2,$$
$$dh_2 = 2h_3 \cdot \theta_1 - 2h_1 \cdot \theta_3,$$
$$dh_3 = 2h_1 \cdot \theta_2 - 2h_2 \cdot \theta_1$$

(d) (*0.5 pt*) Deduce that

$$h = (h_1, h_2, h_3) : M \to \mathbb{R}^3$$

takes values in a sphere S^2_r (of some radius $r \geq 0$).

<u>*From now on we assume that* $r = 1$</u> (i.e. h takes values in S^2).

(e) (*0.5 pt*) Show that h is constant on each integral curve of V.

(f) (*0.5 pt*) Show that V^1 is h-projectable to the following tangent vector on S^2:

$$E^1 = 2\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \in \mathfrak{X}(S^2),$$

i.e. $(dh)_p(V^1_p) = E^1_{h(p)}$ for all $p \in M$. And similarly for V^2 and V^3.

(g) (*0.5 pt*) Show that if M is compact then h is surjective submersion onto S^2 and any fiber $h^{-1}(q)$ (with $q \in S^2$) which is connected is diffeomorphic to a circle.

(h) (*0.5 pt*) The homeworks show that the previous scenario can arise on $M = S^3$. Find another example, with M still connected, but for which the fibers of h are not connected.

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