

Manus Crainic

Manifolds 2026/27

	Objects	Maps/morphisms	Isomorphisms	Key words
Linear algebra	Vector spaces	Linear maps	Linear isom.	Matrices, bases, dimension
Group theory	Groups	Group homomorphisms	Isomorphisms of groups	Subgroups, order, order of an element
Topology	Topological spaces	Continuous maps	Homeomorphism	Convergence, compactness
Analysis	Opens in Euclidean $\mathbb{R}^n$	Smooth/diff maps between open sets in $\mathbb{R}^n$	Diffeomorphism	...
Manifolds / diff. geometry	Manifolds = "spaces for which smoothness is defined"	Smooth maps between manifolds	Diffeomorphism	all from topology & tangent vectors, differentials of maps

"Concrete manifolds": (embedded) submanifolds of $\mathbb{R}^n$.

* 1-dimensional: curves

* 2-dim: surfaces

(Abstract) manifolds: topological spaces & extra-structure that allow us to talk about smoothness.

"smooth atlas" = (to be discussed).

NB: all manifolds can be realised as concrete manifolds.

important manifolds sit in $\mathbb{R}^n$'s in many possible ways, none natural.

Some which do not sit naturally in some $\mathbb{R}^n$

Ex: projective space

Rk: Often one is interested in manifolds M + extra-structure reflects one's interest

Ex of extra-structures:

- (pseudo) Riemannian metric $\Rightarrow$ Riemannian manifolds: playground for relativity
- Symplectic structures $\Rightarrow$ symplectic manifolds: playground for classical mechanics

Webpage: science.uu.nl/~crain101/manifolds2026/

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Manifolds 2026/27

①

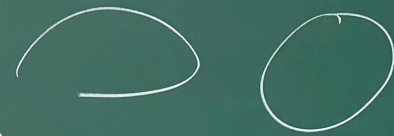
Key words.

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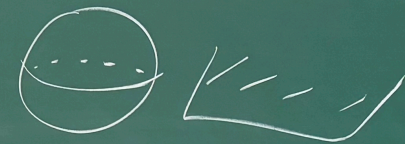
"Embeddable manifolds" (embedded) ...

"Concrete manifolds": (embedded) submanifolds of $\mathbb{R}^n$. ②

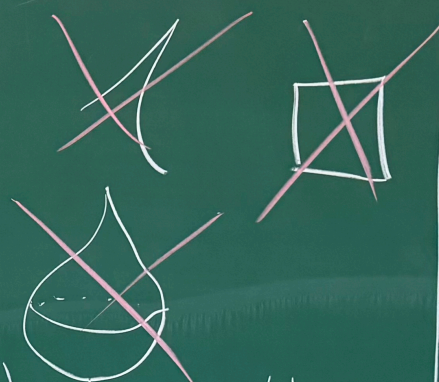
* 1-dimensional: curves



* 2-dim: surfaces



Smooth:



(Abstract) manifolds: topological spaces

& extra-structure that allow us to talk about smoothness.
 "smooth atlas" (to be discussed).

NB: all manifolds can be realised as concrete manifolds.

important manifolds M embed M into some $\mathbb{R}^n$
 sit in $\mathbb{R}^n$'s in many possible ways, none natural as a concrete manifolds.
 Some which do not sit naturally in some $\mathbb{R}^n$ Ex: projective spaces.

Webpage: science.uu.nl/~c

map.

Smooth.



about smoothness.

Rk: Often one is interested in ③
manifolds M + extra-structure
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Ex of extra-structures:

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(1)

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Manifolds/ diff. geometry	Manifolds = "spaces for which smoothness is defined"	Smooth maps between manifolds	diffeomorphism	all from topology & tangent vectors, differentials of maps

Ex: $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ ⁽¹⁾ equation
 $= \{z \in \mathbb{C} : |z| = 1\}$ ^{equation}
 $= \{(e^{it}, i e^{it}) : t \in \mathbb{R}\}$ ^{parametrization}
 $= \bigcirc$ ^{topology} $\rightarrow$ $\bigcirc$ ^{topology} $\rightarrow$ $\square$ ^{topology}

Ex: $M = \{(x, y) : x^2 = y^2\}$
 $= \{(t^2, t^2) : t \in \mathbb{R}\}$ ^{Not a manifold}
 $= \star$

Digression: the inverse function theorem: (6/7)

$F: U \xrightarrow{\text{smooth}} V$
 $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$

J.J. $(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $\Rightarrow$ $\left\{ \begin{array}{l} F \text{ is a local} \\ \text{diffeomorphism around } p \\ \text{a linear isomorphism} \end{array} \right.$

$\begin{array}{ccc} U_0 & \xrightarrow{F|_{U_0}} & V_0 \\ \uparrow & & \uparrow \\ p & & q \end{array}$
 $U_0 \subset U, V_0 \subset V$
 $\leftarrow$ "easy" from chain rule.

$$F(x, y) = e^{\sin(e^{xy})} + \ln(\sin(e^{xy})) + x^2$$

Theorem 1.50: $M \subset \mathbb{R}^n, p \in M$. The following are equivalent:

- $\exists$ m -dimensional equation describing M around p :
 $U \subset \mathbb{R}^n$ open, $\tilde{U} \subset U$ open, $\tilde{U} \xrightarrow{\text{smooth}} \mathbb{R}^n$ submersion, $M \cap \tilde{U} = \{q \in \tilde{U} : g(q) = 0\}$, $p \in \tilde{U}$.
- $\exists$ m -dimensional parametrization of M around p :
 $U \subset \mathbb{R}^n$ open, $\tilde{U} \subset U$ open, $\tilde{U} \xrightarrow{\text{smooth}} \mathbb{R}^n$ immersion, $M \cap \tilde{U} = \text{image of } \tilde{U}$.
- $\exists$ classical-smooth chart for M around p :
 $U \subset \mathbb{R}^n$ open, $M \cap U \xrightarrow{\text{classical-diffeomorphism}} \mathbb{R}^m$, $p \in U$.

(5)

any M satisfying these for all $p \in M$ is called an embedded (smooth) submanifold of $\mathbb{R}^n$.

Smooth map (6/7)

$$F: U \xrightarrow{\text{smooth}} V$$

Call it:

① a submersion at $p \in U$ if $\text{Jac}_p(F) = \left(\frac{\partial F_i}{\partial x_j}(p) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ has rank m .

or, equivalently, the differential/total derivative of F at p .

much simpler than F .

$(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ usually: it is the linear approximation of F around p is surjective.

② an immersion at $p \in U$ if $(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is injective or, equivalently, $\text{Jac}_p(F)$ has rank n .

$$\lim_{v \rightarrow 0} \frac{F(p+v) - F(p) - (dF)_p(v)}{\|v\|} = 0$$

$$(dF)_p(v) = \lim_{t \rightarrow 0} \frac{F(p+tv) - F(p)}{t}$$

Ex: $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ ⁽⁴⁾

$= \{z \in \mathbb{C} : |z| = 1\}$

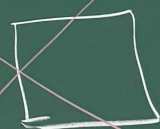
$= \{(\cos(t), \sin(t)) : t \in \mathbb{R}\}$



" $\cong$ "
topology



" $\cong$ "
topology



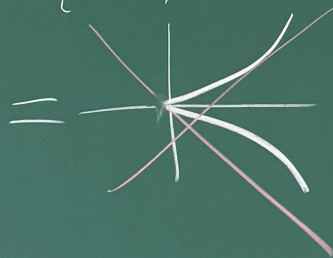
equation

equation

parametrisation

Ex: $M = \{(x, y) : x^3 = y^2\}$

$= \{(t^2, t^3) : t \in \mathbb{R}\}$



NOT a
manifold

Smooth map

(6/7)

$$F : \begin{array}{c} U \\ \cong \\ \mathbb{R}^n \end{array} \longrightarrow \begin{array}{c} V \\ \cong \\ \mathbb{R}^m \end{array}$$

Call it:

diff. geometry "spaces for which smoothness is defined" manifolds

differentials of maps

Theorem 1.50: $M \subseteq \mathbb{R}^n$, $p \in M$. The following are equivalent:
 $m \in \mathbb{N}$

(1) $\exists$ m-dimensional equation describing M around p :

i.e.: $\left\{ \begin{array}{l} \text{eq: } \tilde{U} \rightarrow \mathbb{R}^{n-m} \\ \text{N open} \\ \mathbb{R}^n \\ \text{smooth submersion} \end{array} \right.$ s.t. $M \cap \tilde{U} = \{q \in \tilde{U} : \ell(q) = 0\}$
 $p \in \tilde{U}$.

(2) $\exists$ m-dimensional parametrization of M around p

i.e.: $\left\{ \begin{array}{l} \text{par: } \Omega \rightarrow \mathbb{R}^n \\ \text{N open} \\ \mathbb{R}^m \\ \text{smooth immersion} \end{array} \right.$ s.t. $\text{par}(\Omega) \subseteq M$ open containing p .

(3) $\exists$ classical-smooth chart for M around p

i.e. $\chi: U \rightarrow \Omega$
 N open N open $\text{classical-diffeomorphism}$
 M $\mathbb{R}^m$ s.t. $p \in U$.

(5)

any M satisfying these for all $p \in M$ is called:

an embedded (smooth) submanifold of $\mathbb{R}^n$

much much simpler than F .

Smooth F

Call it:

(1) a submanifold
 $\text{Jac}_p(F)$

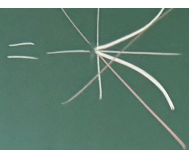
or, equivalently

(2) an immersion
 or, equi

is su

$\rightarrow (dF)$

all from topology
 &
 tangent vectors
 &
 differentials of maps



Chain rule

$$F(x, y) = e^{\sin(e^{xy})} + \ln(\sin(e^{xy})) + x^2$$

Smooth map

(6/7)

$$F: \underset{\mathbb{R}^n}{U} \longrightarrow \underset{\mathbb{R}^m}{V}$$

Call it:

① a submersion at $p \in U$ if

$$\text{Jac}_p(F) = \left(\frac{\partial F_i}{\partial x_j}(p) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \text{ has rank } m$$

or, equivalently, the differential / total derivative of F at p .

much much
 simpler than F .

is surjective.

② an immersion at $p \in U$ if

or, equivalently, $\text{Jac}_p(F)$ has rank n .

intually: it is the linear approximation of F around p

$$\lim_{v \rightarrow 0} \frac{F(p+v) - F(p) - (dF)_p(v)}{\|v\|} = 0$$

$$(dF)_p(v) = \lim_{t \rightarrow 0} \frac{F(p+tv) - F(p)}{t}$$

equation
equation
parametrization
 $\square$
topology

OT a
manifold

Digression: the inverse function theorem: (6/7)

$$F: \underset{\substack{\cap \\ \mathbb{R}^n}}{U} \xrightarrow{\text{smooth}} \underset{\substack{\cap \\ \mathbb{R}^m}}{V}$$

$$\text{If } (dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ is a linear isomorphism } \Rightarrow \begin{cases} F \text{ is a local} \\ \text{diffeomorphism around } p \end{cases}$$

"easy"
from chain rule.

$$\begin{array}{ccc} U_0 & \xrightarrow{F|_{U_0}} & V_0 \\ \cap & & \cap \\ U & & V \end{array} \quad \text{a diffeom.}$$

$$F(x) = y$$

$$F(x, y) = t^{\sin(e^{xy})} + \ln(\sin(e^{xy})) + x^2$$

$$= (\cos t, \sin t)$$

$$f(x, y) = l + \ln(\sinh(e^{xy}) + x^2)$$

(6/7)

$$\bigvee_{\mathbb{R}^m}$$

if
has rank m

partial/total derivative

$$\mathbb{R}^m$$

if $(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$
has rank n

C -notions : the usual notions in $\mathbb{R}^m$
↑
classical

C -smoothness of functions f

C -diffeomorphisms f (f -bijective,
 $f, f^{-1} = C$ -smooth)

(8)

Ok for $f: U \rightarrow V$ - These concepts can
 $\substack{\text{open} \\ \mathbb{R}^m}$ $\substack{\text{open} \\ \mathbb{R}^n}$

be extended when U is an arbitrary subset $M \subseteq \mathbb{R}^m$
and/or $V \subseteq \mathbb{R}^n$

ex: $f: U \rightarrow N$ C -smooth if C -smooth as
 $\substack{\subseteq \mathbb{R}^n} \substack{\subseteq \mathbb{R}^m}$

ex: $f: M \rightarrow \mathbb{R}^n$ if $\exists U \subseteq \mathbb{R}^m$ open and $\tilde{f}: U \rightarrow \mathbb{R}^n$ smooth
s.t. $f = \tilde{f}$ on $M \cap U$
 $p \in U$



$\Rightarrow$ Smoothness makes sense for $f: M \rightarrow N \subseteq \mathbb{R}^n$ on $M \cap U_p$.

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 $= \{z \in \mathbb{C} : |z|=1\}$ equation
 $= \{(e^{it}, i e^{it}) : t \in \mathbb{R}\}$ parametrization
 $= \bigcirc \xrightarrow[\text{topology}]{} \bigcirc \xrightarrow[\text{topology}]{} \square$

Ex: $M = \{(x,y) : x^2 = y^2\}$ Not a manifold
 $= \{(t^2, t^3) : t \in \mathbb{R}\}$
 $= \star$

Digression: the inverse function theorem: (6/7)

$F: U \xrightarrow[\mathbb{R}^n]{} V \xrightarrow[\mathbb{R}^m]$ smooth

$JF_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear isomorphism $\Rightarrow$ $\left\{ \begin{array}{l} F \text{ is a local diffeomorphism around } p \\ \text{if } JF_p \text{ is a linear isomorphism} \end{array} \right.$

$\sum \frac{\partial F_i}{\partial x_j} v_j$ from chain rule. $F(x,y) = y$
 $F(x,y) = e^{\sin(x,y)} + \ln(\sin(e^{xy})) + x^2$

$f: \mathbb{R} \rightarrow S^1$ $f(t) = (\cos t, \sin t)$ $\xrightarrow[\mathbb{R}^2]{} \mathbb{R}^2$
 $f(x,y) = x$
 $f(x,y) = x^2$
 $(df)_p$ $(df)_{(x,y)}$

Theorem 1.50: $M \subseteq \mathbb{R}^n$, $p \in M$. The following are equivalent: (5)

- $\exists$ m -dimensional equation describing M around p .
 i.e. $\exists \tilde{U} \xrightarrow[\mathbb{R}^n]{\text{open}} \mathbb{R}^{n-m}$ s.t. $M \cap \tilde{U} = \{x \in \tilde{U} : g(x) = 0\}$
 $\hookrightarrow$ smooth submanifold
- $\exists$ m -dimensional parametrization of M around p .
 i.e. $\exists \text{ par: } \mathbb{R}^m \xrightarrow[\mathbb{R}^n]{\text{open}} \mathbb{R}^n$ s.t. $\text{par}(\mathbb{R}^m) = M$ open containing p .
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- $\exists$ classical-smooth chart for M around p .
 i.e. $\chi: U \xrightarrow[\mathbb{R}^m]{\text{open}} \mathbb{R}^n$ classical-diffeomorphism s.t. $p \in U$.

any M satisfying these for all $p \in M$ is called an embedded (smooth) submanifold of $\mathbb{R}^n$.

much simpler than F .

Smooth map (6/7)

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Call it:

- a submersion at $p \in U$ if $Jac_p(F) = \left(\frac{\partial F_i}{\partial x_j}(p) \right)_{1 \leq i \leq m, 1 \leq j \leq n}$ has rank m
 or, equivalently, the differential/total derivative $(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is surjective.
- an immersion at $p \in U$ if $(dF)_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has rank n .

C -notions: the usual notions in $\mathbb{R}^n$ (8)

classical C -smoothness of functions f
 C -diffeomorphisms f (f, f^{-1} C -smooth)

Ok for $f: U \xrightarrow[\mathbb{R}^m]{\text{open}} V \xrightarrow[\mathbb{R}^n]$. These concepts can be extended when U is an arbitrary subset $M \subseteq \mathbb{R}^n$ and/or $V \subseteq \mathbb{R}^m$.

ex: $f: U \rightarrow N$ C -smooth if C -smooth as a map $U \rightarrow \mathbb{R}^n$
 ex: $f: M \rightarrow \mathbb{R}^n$ if $\exists U \subseteq \mathbb{R}^n$ open and $\tilde{f}: U \rightarrow \mathbb{R}^n$ smooth s.t. $f = \tilde{f}|_M$ $p \in U$.

$\Rightarrow$ smoothness makes sense for $f: M \rightarrow N$ on $M \cap U$.