

(Smooth) manifolds

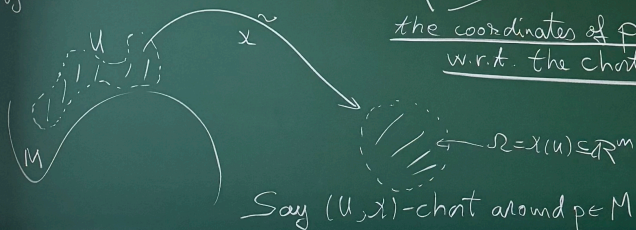
$$m = \dim(M) \quad 1$$

CHARTS:

Def: Given $M = (\text{topological})$ space, $m \in \mathbb{N}$,
a (m -dimensional) topological chart on M
is a pair (U, χ) with $U \subseteq M$ open and

$$\begin{array}{ccc} \chi: \underbrace{U}_{\substack{\text{the domain} \\ \text{of the chart}}} \xrightarrow{\text{homeomorphism}} \underbrace{\chi(U) \subseteq \mathbb{R}^m}_{\substack{\text{open} \\ \text{call it } \Omega}} & \text{a homeomorphism} & \\ & \text{bijective,} & \\ & \text{continuous \& inverse} & \\ & \text{continuous} & \end{array}$$

$p \mapsto (\chi_1(p), \dots, \chi_m(p))$
the coordinates of p
w.r.t. the chart (U, χ) .



SMOOTH

does

Def:

'

Def:

is a

Ex:

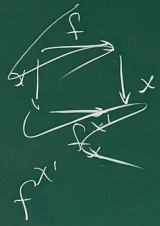
Chart

WHAT ARE CHARTS GOOD FOR? $\hookrightarrow$ Allows one to
represent "objects" that live on M , locally, as more
familiar objects on $\mathbb{R}^m$ (or on opens $\subseteq \mathbb{R}^m$).

E.g. for functions

$$\begin{array}{ccc} f: M & \xrightarrow{f} & N' \\ \downarrow \chi & & \downarrow \chi' \\ U & \xrightarrow{f_x} & U' \subseteq \mathbb{R}^m \\ \downarrow \chi & & \downarrow \chi' \\ \mathbb{R}^m & \xrightarrow{f_x} & \mathbb{R}^m \end{array}$$

$x' = \chi'(x)$
 $x' \in U' \subseteq \mathbb{R}^m$
 $x' = \chi'(f(x))$



$$f_x^{x'} = \chi' \circ f \circ \chi^{-1} : \text{domain } U \rightarrow \text{codomain } U' \subseteq \mathbb{R}^m$$

the representation of f w.r.t. $(U, \chi), (U', \chi')$

$\chi'(f(p)) = f_x^{x'}(\chi(p)) = f_x^{x'}(\chi_1(p), \dots, \chi_m(p))$

$$\begin{aligned} f_x^{x'} \circ \chi &= \chi' \circ f \\ (f_x^{x'} \circ \chi)(p) &= (\chi' \circ f)(p) \quad \forall p \\ f_x^{x'}(\chi(p)) &= \chi'(f(p)) \end{aligned}$$

(Smooth) manifolds

CHARTS:

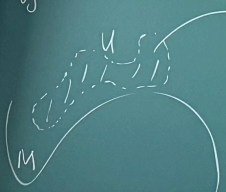
Def: Given $M = (\text{topological space}, m \in \mathbb{N})$,
a $(m\text{-dimensional})$ topological chart on M
is a pair (U, χ) with $U \subseteq M$ open and

$$\chi: \underset{\substack{\uparrow \\ \text{the domain} \\ \text{of the chart}}}{U} \xrightarrow{\text{homeo}} \underset{\substack{\uparrow \\ \text{call it } \mathcal{R}}}{\chi(U) \subseteq \mathbb{R}^m} \text{ a homeomorphism.}$$

the domain
of the chart

$$p \mapsto (\chi_1(p), \dots, \chi_m(p)) \quad \begin{matrix} \text{bijective,} \\ \text{continuous} \leftarrow \text{inverse} \\ \text{continuous} \end{matrix}$$

the coordinates of p
w.r.t. the chart (U, χ) .



$$\mathcal{R} = \chi(U) \subseteq \mathbb{R}^m$$

Say (U, χ) -chart around $p \in M$ if $p \in U$.

$$(m = \dim(M)) \quad 1$$

SMOOTH CHARTS?!!?

Look when smoothness
does make sense classically (in the sense of calculus)

Def: An $\mathcal{C}^k$ -smooth chart on $\mathbb{R}^m$ is a pair (U, χ) :

$$\chi: \underset{\substack{\uparrow \\ \mathbb{R}^m}}{U} \xrightarrow{\sim} \underset{\substack{\uparrow \\ \mathbb{R}^m}}{\mathcal{R}} \text{ } \mathcal{C}^k\text{-diffeomorphism.}$$

Def: For $M \subseteq \mathbb{R}^n$ an $(m\text{-dimensional})$ $\mathcal{C}^k$ -smooth chart on M
is a (U, χ) with $\chi: \underset{\substack{\uparrow \\ \mathcal{R}^m}}{U} \xrightarrow{\sim} \underset{\substack{\uparrow \\ \mathcal{R}^m}}{\mathcal{R}} \text{ } \mathcal{C}^k\text{-diffeomorphism.}$

Do these
even exist?
(on a given M)

Ex: $\underbrace{x^2 - 4xy + 13y^2}_{(x-2y)^2 + (3y)^2} = f(x, y)$

Chart $\chi(x, y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f_x^{(id)}(u, v) = u^2 + v^2$$

WHAT ARE CHARTS GOOD FOR? $\hookrightarrow$ Allows one to
represent "objects" that live on M , locally, as more
familiar objects on $\mathbb{R}^m$ (or on opens $\subseteq \mathbb{R}^m$).

Ex: $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ submersion at p if $(df)_p: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective

Theorem 1.42: this is the case
iff: $\exists \mathcal{C}^k$ -chart of $\mathbb{R}^n$ around p s.t.

Thm 1.50 $M \subseteq \mathbb{R}^n$,

3. $\exists$ par: $\mathcal{R} \xrightarrow{\sim} \mathbb{R}^m$

$m = \dim(M)$ 1

SMOOTH CHARTS??!

Look when smoothness does make sense classically (in the sense of calculus)

Def: An $\mathbb{C}$ -smooth chart on $\mathbb{R}^m$ is a pair (U, χ) :

$$\chi: \underset{\substack{\cap \\ \mathbb{R}^m}}{U} \xrightarrow{\sim} \underset{\substack{\cap \\ \mathbb{R}^m}}{\mathbb{R}^m} \text{ } \mathbb{C}\text{-diffeomorphism}$$

Def: For $M \subseteq \mathbb{R}^n$ an $(m\text{-dimensional})$ $\mathbb{C}$ -smooth chart on M is a (U, χ) with $\chi: \underset{\substack{\cap \\ \mathbb{R}^n}}{U} \xrightarrow{\sim} \underset{\substack{\cap \\ \mathbb{R}^m}}{\mathbb{R}^m}$ $\mathbb{C}$ -diffeomorphism.

Do these even exist? (on a given M)

Ex: $\underbrace{x^2 - 4xy + 13y^2}_{(x-2y)^2 + 3y^2} = f(x, y)$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $\uparrow$ here $\uparrow$ id chart

Chart $\chi(x, y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$

$f_x^{\text{id}}(u, v) = u^2 + v^2$

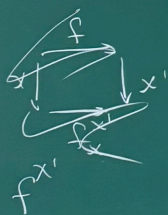
morphism.

jective,
continuous & inverse
continuous

(U, χ) .

if $p \in U$.

one to
as many



1)

$f_x^{\chi'} \cdot \chi = \chi' \circ f$

$(f_x^{\chi'} \cdot \chi)(p) = (\chi' \circ f)(p) \quad \forall p$

$\underbrace{f_x^{\chi'}}_{\sim}(\chi(p)) = \chi'(f(p))$

Ex: $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$

submersion at p if

$n \geq k$

Theorem 1.42: this is the case $(df)_p: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective

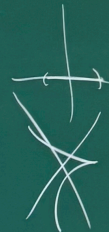
iff $\exists \mathbb{C}$ -chart of $\mathbb{R}^n$ around p s.t.

$f_x^{\text{id}}(x_1, \dots, x_k, \dots, x_n) = (x_1, \dots, x_k)$

Ex: $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$. Thm 1.46: g is an immersion at p

iff $\exists \mathbb{C}$ -smooth chart of $\mathbb{R}^n$ around $g(p)$

s.t. $g_x^{\chi'}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0) \iff$



Thm 1.50

3. $\exists$ par

imm

4. $\exists$ eg: $\mathbb{R}^n$

$\mathbb{R}^n$

Subn

2. $\exists$ $\mathbb{C}$ -smo

1. $\exists$ $\mathbb{C}$ -smo

Def: Say
of $\mathbb{R}^n$

inverse continuous

Ex: $\underbrace{x^2 - 4xy + 13y^2}_{(x-2y)^2 + 9y^2} = f(x,y)$

Chart $\chi(x,y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$

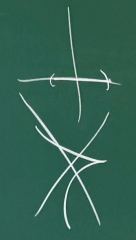
$f_x^{id}(u,v) = u^2 + v^2$

$\mathbb{R}^n \xrightarrow{\chi} \mathbb{R}^m$ C -diffeomorphism
Do these even exist? (on a given M)

Ex: $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ submersion at p if $n \geq k$
Thm 1.42: this is the case $(df)_p: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective

iff: $\exists$ C -chart of $\mathbb{R}^n$ around p s.t.
 $f_x^{id}(x_1, \dots, x_n) = (x_1, \dots, x_k)$

Ex: $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$ Thm 1.46: g is an immersion at p
iff $\exists$ C -smooth chart of $\mathbb{R}^n$ around $g(p)$
s.t. $g_x^{x'}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$



Thm 1.50 $M \subseteq \mathbb{R}^m, m \in \mathbb{N}, p \in M$. The following are equivalent: 5

3. $\exists$ par: $\underbrace{\mathcal{O}_p}_{\substack{\mathbb{R}^m \text{ open} \\ \text{immersion}}} \rightarrow \mathbb{R}^n$ s.t. $U := \text{par}(\mathcal{O}) \subseteq M$ open
 $p \in U$

4. $\exists$ eg: $\underbrace{\tilde{U}}_{\substack{\mathbb{R}^n \text{ open} \\ \text{submersion}}} \rightarrow \mathbb{R}^{n-m}$ s.t. $M \cap \tilde{U} = \{g \in \tilde{U} : g(g) = 0\}$
 $p \in \tilde{U}$

2. $\exists$ C -smooth $(m\text{-dimensional})$ chart of M around $p, \chi: \underbrace{\tilde{U}}_{\substack{\mathbb{R}^m \\ M}} \xrightarrow{\sim} \underbrace{\mathcal{O}_p}_{\substack{\mathbb{R}^n \\ M}}$

1. $\exists$ C -smooth chart of $\mathbb{R}^n$ around p ,
 $\tilde{\chi}: \underbrace{\tilde{U}}_{\substack{\mathbb{R}^n \\ M}} \rightarrow \underbrace{\tilde{\mathcal{O}}_p}_{\substack{\mathbb{R}^n \\ M}} \text{ s.t. } M \cap \tilde{U} = \{g \in \tilde{U} : \chi_{m+1}(g) = \dots = \chi_n(g) = 0\}$
(or: $\chi(M \cap \tilde{U}) = (\mathbb{R}^m \times \{0\}) \cap \tilde{\mathcal{O}}$)

Def: Say M - an embedded $(m\text{-dimensional})$ smooth submanifold of $\mathbb{R}^n$ if the conditions above hold $\forall p \in M$.

(Smooth) manifolds

($m = \dim(M)$)

CHARTS:

Def: Given $M = (\text{topological space}, m \in \mathbb{N})$,
a (m -dimensional) topological chart on M
is a pair (U, χ) with $U \subseteq M$ open and
 $\chi: U \xrightarrow{\sim} \chi(U) \subseteq \mathbb{R}^m$ a homeomorphism.
call it χ

the domain of the chart $p \mapsto (\chi(p), j_M(p))$ bijective, continuous & inverse continuous
the coordinates of p wrt. the chart (U, χ) .
Say (U, χ) -chart around $p \in M$ if $p \in U$.



SMOOTH CHARTS?!!? Look when smoothness does make sense classically (in the sense of calculus)

Def: An C^∞ -smooth chart on $\mathbb{R}^m$ is a pair (U, χ) :

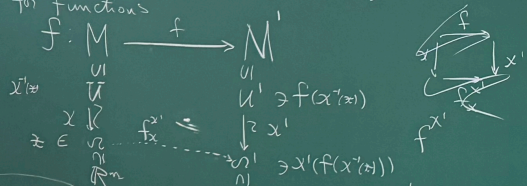
$$\chi: U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^m \text{ } C^\infty\text{-diffeomorphism}$$

Def: For $M \subseteq \mathbb{R}^n$ an (m -dimensional) C^∞ -smooth chart on M is a (U, χ) with $\chi: U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^m$ C^∞ -diffeomorphism.
Do these exist? (on a given M)

Ex: $\chi^2 - 4xy + 13y^2 = f(x, y)$
 $(x-2y)^2 + 9y^2$
Chart $\chi(x, y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$
 $f_x^{cd}(u, v) = u^2 + v^2$

WHAT ARE CHARTS GOOD FOR?

Allows one to represent "objects" that live on M , locally, as more familiar objects on $\mathbb{R}^m$ (or on opens $\subseteq \mathbb{R}^m$).
E.g. for functions



$f_x^{x'} = x' \circ f \circ \chi^{-1}$: domain $\chi(U)$ open in $\mathbb{R}^m$ $\rightarrow$ codomain $\chi'(V)$ open in $\mathbb{R}^n$
the representation of f wrt. $(U, \chi), (V, \chi')$
 $(f(p)) = (x')^{-1}(\chi'(f(p))) = (x')^{-1}(\chi'(f(\chi^{-1}(\chi(p)))) = (x')^{-1}(\chi'(f \circ \chi^{-1}(\chi(p)))) = (x')^{-1}(\chi'(f_x^{x'}(\chi(p))))$
 $f_x^{x'}(\chi(p)) = \chi'(f(p))$

Ex: $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ submersion at p if $n \geq k$

Theorem 1.42: this is the case $\text{Id}_p: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective
iff $\exists C^\infty$ -chart of $\mathbb{R}^k$ around p s.t.
 $f_x^{cd}(x_1, \dots, x_n) = (x_1, \dots, x_k)$

Ex: $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$ Thm 1.46: g is an immersion at p
iff $\exists C^\infty$ -smooth chart of $\mathbb{R}^n$ around $g(p)$
s.t. $g_x^{x'}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$



Thm 1.50 $M \subseteq \mathbb{R}^n, m \in \mathbb{N}, p \in M$. The following are equivalent:

- $\exists \text{ par: } \Omega \xrightarrow{\sim} \mathbb{R}^n$ s.t. $U := p \circ \chi(\Omega) \subseteq M$ open & $p \in U$.
immersion
- $\exists \text{ eg: } \tilde{U} \xrightarrow{\sim} \mathbb{R}^{n-m}$ s.t. $M \cap \tilde{U} = \{g \in \tilde{U} : g_2(g) = 0\}$ & $p \in \tilde{U}$.
submersion

2. $\exists C^\infty$ -smooth (m -dimensional) chart of M around $p, \chi: U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^m$

1. $\exists C^\infty$ -smooth chart of $\mathbb{R}^n$ around $p, \tilde{\chi}: \tilde{U} \xrightarrow{\sim} \tilde{\Omega} \subseteq \mathbb{R}^n$ s.t. $M \cap \tilde{U} = \{g \in \tilde{U} : \chi_m(g) = 0\}$ (or: $\chi(M \cap \tilde{U}) = (\mathbb{R}^m \times \{0\}) \cap \tilde{\Omega}$)

Def: Say M - an embedded (m -dimensional) smooth submanifold of $\mathbb{R}^n$ if the conditions above hold $\forall p \in M$.

ness

se of calculus

(U, χ)

charts on M
diffeomorphism.

se exist?
on a given (x) .

$\mathbb{R}^m$
id chart

3 Def: $M = \text{topological space}, m \in \mathbb{N}$.

- given $(U, \chi), (U', \chi')$ two topological charts of M ,

6 the change of coordinates between the two charts is the map

$$\underbrace{c_x^{x'} := \chi' \circ \chi^{-1} : \chi(U \cap U')}_{\substack{\cap \text{ open} \\ \mathbb{R}^m}} \longrightarrow \substack{\cap \text{ open} \\ \mathbb{R}^m} \chi'(U \cap U')$$

- we say that (U, χ) and (U', χ') are smoothly compatible if $c_x^{x'}$ is C^∞ -diffeomorphism.

- an $(m\text{-dimensional})$ smooth atlas on M is a collection A of ——— topological charts of M s.t.

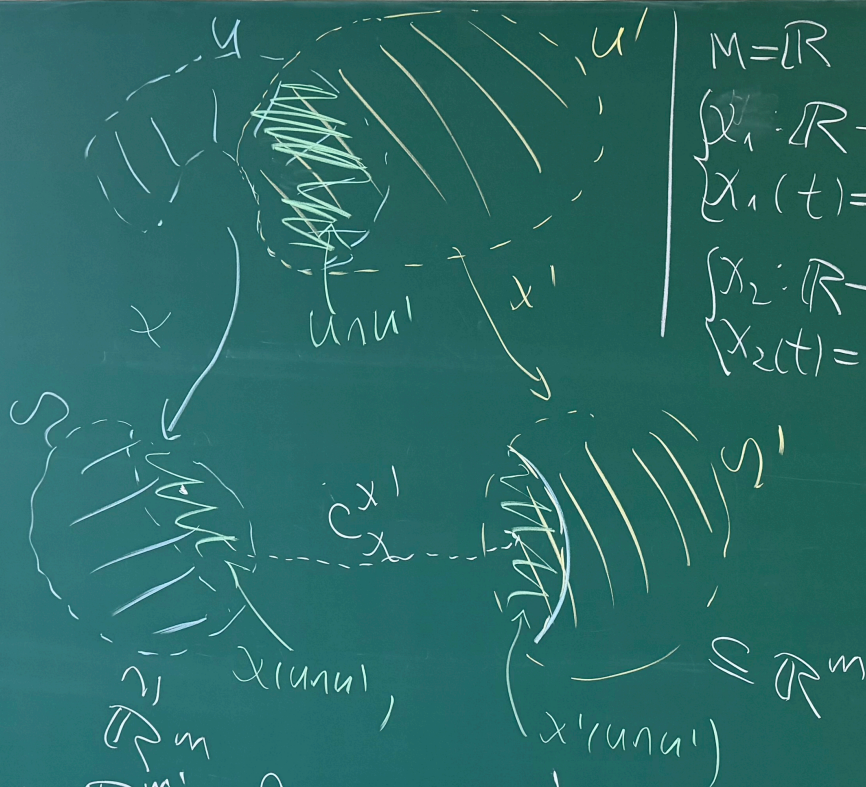
$$(A1) \quad (\forall) p \in M \quad \exists (U, \chi) \in A \quad \text{s.t.} \quad p \in U$$

$$(A2) \quad (\forall) (U, \chi), (U', \chi') \in A \quad \text{are smoothly compatible}$$

the structure on M is

~~40~~

$$c_{x'}^{x'} = x' \circ x^{-1}$$



$$M = \mathbb{R}$$

$$\{x_1: \mathbb{R} \rightarrow \mathbb{R}\}$$

$$x_1(t) = t$$

$$\{x_2: \mathbb{R} \rightarrow \mathbb{R}\}$$

$$x_2(t) = t^3$$

both
topological
charts

but not
smoothly
comp. because
 $x_1 \circ x_2^{-1}(t) = \sqrt[3]{t}$
not smooth

~~7~~

an (m-dimensional)

a "maximality condition"

(A3)
 Sm
 fro

Rk1: Any smooth

$$A^{\max} := \{ \dots \}$$

Exercise: $A^{\max} =$

& the

Compare: $\mathbb{R}^m, \mathbb{R}^{m'}$ for $m \neq m'$

as vector spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be isom. as vector spaces for $m \neq m'$? Easy NO

analysis: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be diffeom. for $m \neq m'$?

as top. spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be homeom. as top. spaces for $m \neq m'$? COMPLICATED NO

$f: \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ diffeom

(M, τ_M)
is the

top. space M

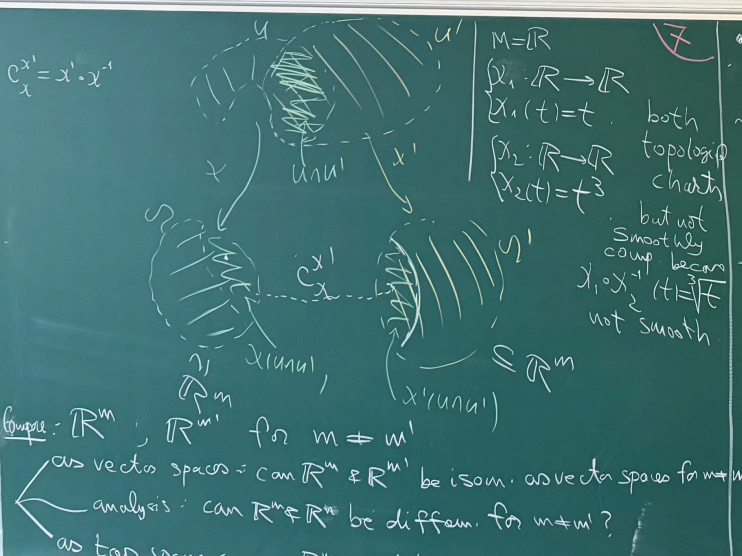
Hausdorff and
Second countable
manifold M

smoothly with each other
compatible charts
a smooth atlas
as smooth atlases
no classes
smooth atlases
chart on $\mathbb{R}^n$

SMOOTH CHARTS??! ? Look when smoothness does make sense classically (in the sense of calculus)
Def: An $\mathbb{C}$ -smooth chart on $\mathbb{R}^m$ is a pair (U, χ)
 $\chi: U \xrightarrow{\sim} \Omega \subset \mathbb{R}^m$ $\mathbb{C}$ -diffeomorphism
Def: For $M \in \mathbb{R}^m$ an $(m$ -dimensional) $\mathbb{C}$ -smooth chart on M is a (U, χ) with $\chi: U \xrightarrow{\sim} \Omega \subset \mathbb{R}^m$ $\mathbb{C}$ -diffeomorphism
 Ex: $\chi^2 - 4xy + 13y^2 = f(x, y)$ $f: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $(x-2y)^2 + 9y^2$ χ here $\uparrow$ d chart
 Chart $\chi(x, y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$ $f^{\text{id}}(u, v) = u^2 + v^2$

Def: M = topological space, $m \in \mathbb{N}$.
 • given $(U, \chi), (U', \chi')$ two topological charts of M ,
 the change of coordinates between the two charts is the map
 $\chi'_x := \chi' \circ \chi^{-1} = \chi'(U \cap U')$ $\xrightarrow{\sim}$ $\chi'(U \cap U')$
 $\uparrow$ $\uparrow$
 $\Omega \subset \mathbb{R}^m$ $\Omega' \subset \mathbb{R}^m$
 • we say that (U, χ) and (U', χ') are smoothly compatible if χ'_x is $\mathbb{C}$ -diffeomorphism.
 • an $(m$ -dimensional) smooth atlas on M is a collection A of m -topological charts of M s.t.
 (A1) $\forall p \in M \exists (U, \chi) \in A$ s.t. $p \in U$
 (A2) $\forall (U, \chi), (U', \chi') \in A$ are smoothly compatible

$\chi'_x = \chi' \circ \chi^{-1}$
 $(M, \mathcal{A})$
 is the
 top space M
 manifold and
 second countable
 (Hausdorff)



• an $(m$ -dimensional) smooth structure on M is a m -smooth atlas on M s.t.
 (A3) $\forall (U, \chi)$ -top chart of M which is smoothly compatible to all the charts from A must be in A .
Prop: Any smooth atlas A induces a smooth structure A^{max}
 $A^{\text{max}} := \{ (U, \chi) \text{-top chart of } M \mid \chi \text{ is smoothly compatible to all charts in } A \}$
Exercise: A^{max} = smooth structure indeed.
 = the only smooth structure containing A .
Easy No
 $f: \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ diffeo. $(df)_p: \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ is a linear map isomorphism
 $(df)_p \circ (df')_{f(p)} = d(f \circ f') = \text{id}$

as vector spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be isom. as vector spaces for $m \neq m'$?
 analysis: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be diffeo. for $m \neq m'$?
 as top. spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be homeom. as top. spaces for $m \neq m'$? COMPLICATED No

Rk2: If $\mathcal{A}, \mathcal{B}$ -smooth atlases then

$$\mathcal{A}^{\max} = \mathcal{B}^{\max} \iff \begin{array}{l} \text{each chart from } \mathcal{A} \text{ is smoothly compatible} \\ \text{with each chart from } \mathcal{B} \end{array}$$

$\iff \mathcal{A} \cup \mathcal{B}$ is still a smooth atlas.

When this happens we say $\mathcal{A} \sim \mathcal{B}$ as smooth atlases.

Hence: Smooth structures $\xleftrightarrow{1-1}$ equiv. classes of smooth atlases.

$$\mathcal{A}^{\max} \longleftarrow \mathcal{A}$$

Ex: The $\mathcal{C}$ -smooth structure on $\mathbb{R}^m$:

$$\mathcal{A}_{\mathbb{R}^m}^{\max} = \{ (U, \chi) : (U, \chi) = \text{a } \mathcal{C}\text{-smooth chart on } \mathbb{R}^m \}$$

Many other interesting smooth atlases:

$$\text{say } (U, \chi)\text{-chart s.t. } p \in M \iff p \in U$$

SMOOTH

does

Def:

Def: is a

Ex:

Chart χ

$$\mathcal{A}^1 = \{ (\mathbb{R}^m, \text{id}_{\mathbb{R}^m}) \} \quad (\text{One chart only!})$$

$$\mathcal{A}^2 = \{ (U, \text{id}_U) : U \text{ any open in } \mathbb{R}^m \}$$

$$\mathcal{A}^3 = \{ (B_r(0), \text{id}_{B_r(0)}) : r > 0 \}$$

they are all smooth atlases!

$$\text{Notice: } (\mathcal{A}^1)^{\max} = (\mathcal{A}^2)^{\max} = (\mathcal{A}^3)^{\max} = \mathcal{A}_{\mathbb{R}^m}^{\max}$$

$(M, \mathcal{A}^1)$

Interesting question: given a smooth st $\mathcal{B}$ what is the smallest smooth atlas inducing it (s.t. $\mathcal{B} = \mathcal{A}^{\max}$)

Def: A smooth (m-dim) manifold is a top. space M together with a smooth structure on M s.t. M is Hausdorff and second countable.

$$(M, \mathcal{A})$$

its members are called

the smooth charts of the manifold M

$$c_x^{x'} = x'$$

Compare: $\mathcal{A}$

$\swarrow$
 $\searrow$

Rk2: If $\mathcal{A}, \mathcal{B}$ - smooth atlases then
 $\mathcal{A}^{\max} = \mathcal{B}^{\max} \iff$ each chart is smoothly compatible with each chart from $\mathcal{A}$ and $\mathcal{B}$

$\iff \mathcal{A} \cup \mathcal{B}$ is still a smooth atlas.
 When this happens we say $\mathcal{A} \sim \mathcal{B}$ as smooth atlases.

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Many other interesting smooth atlases.

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$$\chi: \underset{\substack{U \\ \text{open in } \mathbb{R}^m}}{\mathbb{R}^m} \xrightarrow{\sim} \underset{\substack{\mathbb{R}^m \\ \text{open}}} \mathbb{R}^m \text{ } \mathbb{C}\text{-diffeomorphism}$$

Def: For $M \subseteq \mathbb{R}^n$ an $(m\text{-dimensional})$ $\mathbb{C}$ -smooth chart on M is a (U, χ) with $\chi: \underset{\substack{U \text{ open in } M \\ \text{open in } \mathbb{R}^n}}{\mathbb{R}^m} \xrightarrow{\sim} \underset{\substack{\mathbb{R}^m \\ \text{open}}} \mathbb{R}^m$ $\mathbb{C}$ -diffeomorphism.

Ex: $\underbrace{x^2 - 4xy + 13y^2}_{(x-2y)^2 + 9y^2} = f(x, y)$

Chart $\chi(x, y) = (\underbrace{x-2y}_u, \underbrace{3y}_v)$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $\chi \text{ home}$ $\uparrow$ chart
 $\underset{f \circ \chi}{f_x} (u, v) = u^2 + v^2$

Def: M = topological space, $m \in \mathbb{N}$.

• given $(U, \chi), (U', \chi')$ two topological charts of M , the change of coordinates between the two charts is the map

$$\underbrace{\chi_x^{-1} := \chi^{-1} \circ \chi'}_{\substack{\text{open in } \mathbb{R}^m \\ \text{open in } \mathbb{R}^m}}: \underbrace{\chi'(U \cap U')}_{\substack{\text{open in } \mathbb{R}^m \\ \text{open in } \mathbb{R}^m}} \rightarrow \underbrace{\chi(U \cap U')}_{\substack{\text{open in } \mathbb{R}^m \\ \text{open in } \mathbb{R}^m}}$$

• we say that (U, χ) and (U', χ') are smoothly compatible if χ_x^{-1} is $\mathbb{C}$ -diffeomorphism.

• an $(m\text{-dimensional})$ smooth atlas on M is a collection $\mathcal{A}$ of topological charts of M s.t.

(A1) $\forall p \in M \exists (U, \chi) \in \mathcal{A} \text{ s.t. } p \in U$

(A2) $\forall (U, \chi), (U', \chi') \in \mathcal{A} \text{ are smoothly compatible}$

$\mathcal{A}^1 = \{ (\mathbb{R}^m, \text{id}_{\mathbb{R}^m}) \}$ (One chart only!)

$\mathcal{A}^2 = \{ (U, \text{id}_U) : U \text{ any open in } \mathbb{R}^m \}$

$\mathcal{A}^3 = \{ (B_r(0), \text{id}_{B_r(0)}) : r > 0 \}$

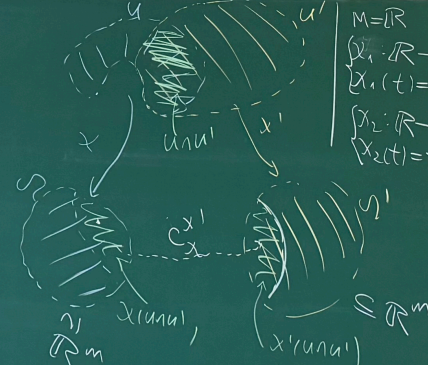
they are all smooth atlases!

Notice: $(\mathcal{A}^1)^{\max} = (\mathcal{A}^2)^{\max} = (\mathcal{A}^3)^{\max} = \mathcal{A}_{\mathbb{R}^m}^{\max}$

Interesting question: given a smooth st. $\mathcal{B}$ what is the smallest smooth atlas inducing it (s.t. $\mathcal{B} = \mathcal{A}^{\max}$)

Def: A smooth $(m\text{-dim})$ manifold is a top. space M together with a smooth structure on M s.t. M is Hausdorff and second countable.
 it manifolds are called the smooth charts of the manifold M

$\chi_x^{-1} = \chi^{-1} \circ \chi'$



$M = \mathbb{R}$
 $\chi_1: \mathbb{R} \rightarrow \mathbb{R}, \chi_1(t) = t$
 $\chi_2: \mathbb{R} \rightarrow \mathbb{R}, \chi_2(t) = t^3$
 both topological charts
 but not smoothly comp. because $\chi_1 \circ \chi_2^{-1}(t) = t^{1/3}$ not smooth

• an $(m\text{-dimensional})$ smooth structure on M is a smooth atlas on M s.t.

(A3) $\forall (U, \chi)$ -top chart of M which is smoothly compatible to all the charts from $\mathcal{A}$ must be in $\mathcal{A}$

Rk1: Any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{\max} := \{ (U, \chi) \text{-top chart of } M \mid \chi \text{ smoothly compatible to all charts in } \mathcal{A} \}$

Exercise: $\mathcal{A}^{\max}$ = smooth structure indeed. & the only smooth structure containing $\mathcal{A}$.

Easy NO
 $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ diff'able. $(df)_p: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a linear map isomorphism
 $(df)_p \circ (df)_{f(p)} = d(f \circ f^{-1}) = \text{id}$

Compare: $\mathbb{R}^m, \mathbb{R}^{m'}$ for $m \neq m'$
 as vector spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be isom. as vector spaces for $m \neq m'$?
 analysis: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be diff'able for $m \neq m'$?
 as top. spaces: can $\mathbb{R}^m \neq \mathbb{R}^{m'}$ be homeom. as top. spaces for $m \neq m'$? COMPLICATED NO