

REMINDER: $M = \text{topological space}$, $m \in \mathbb{N}$.

- an m -dimensional topological chart of M

$$x: U \xrightarrow[\substack{\text{Nopen} \\ M}]{\substack{x(U) \\ \text{Nopen} \\ \mathbb{R}^m}} \text{homeomorphism}$$

- two charts $(U, x), (U', x')$

Smoothly compatible if

$$x' \circ x^{-1}: x(U \cap U') \rightarrow x'(U \cap U') \text{ classical diffeomorphism}$$

- smooth m -dim. atlas on M : collection $\mathcal{A}$ of topological charts of M s.t.

(A1) $(\forall) p \in M, \exists (U, x) \in \mathcal{A}$ s.t. $p \in U$

(A2) each $(U, x), (U', x') \in \mathcal{A}$ are smoothly compatible.

- smooth structure on M : any smooth atlas $\mathcal{A}$ s.t., furthermore:

(A3) any topological chart of M that is smoothly compatible with all $(U, x) \in \mathcal{A}$ must be in $\mathcal{A}$.

- Smooth (m -dimensional) manifold M :

a topological space that is Hausdorff & second countable

+ a smooth structure $\mathcal{A}$ on M .

Ex 1: $\mathbb{R}^m$

In others

Smaller

Ex 2: Open

induced

the class

smooth s

on $\mathbb{R}^m$

Ex 3: $\mathbb{R}^m$

$\mathcal{O} = \{x \in \mathbb{R}^m : x_1 > 0\}$

$M_{n,m}(\mathbb{R})$

$GL_n(\mathbb{R})$

Terminology: $(M, \mathcal{A})$ - smooth manifold

the charts $(U, x) \in \mathcal{A}$: the smooth charts of the smooth manifold $(M, \mathcal{A})$

Rk 1: Any smooth atlas $\mathcal{A}$ induces a smooth structure

Many examples are constructed like this

$\mathcal{A} \subseteq \mathcal{A}^{\max} := \{ (U, x) \text{ - top. chart of } M \mid \text{it is smoothly compatible with all } (U', x') \in \mathcal{A} \}$

Q: Given a smooth st. $\mathcal{A}$ find $\mathcal{A}^{\max}$ s.t. $\mathcal{A}^{\max} \neq \mathcal{A}$ - very small

2 Two smooth atlases

$\mathcal{A}$ and $\mathcal{B}$ induce same smooth st. (i.e. $\mathcal{A}^{\max} = \mathcal{B}^{\max}$)

$\Leftrightarrow$ any $(U, x) \in \mathcal{A}$ is smoothly compatible with any $(U', x') \in \mathcal{B}$

Smooth maps: $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ smooth manifolds

$f: M \rightarrow N$ called smooth if:

(*) $(\forall) (U, x) \in \mathcal{A}_M, (\forall) (U', x') \in \mathcal{A}_N \Rightarrow f = x' \circ f \circ x^{-1}: \substack{x(U) \\ \mathbb{R}^m} \xrightarrow[\text{open}]{\substack{\text{open} \\ \mathbb{R}^n}} \substack{x'(U') \\ \mathbb{R}^m}$ is classically smooth

Rk 3 / Exercise: If $\mathcal{A}_M = \mathcal{A}^{\max}, \mathcal{A}_N = \mathcal{B}^{\max}$

with $\mathcal{A} / \mathcal{B}$ smooth atlas on M / N then: (*) $\Leftrightarrow$ same could but use $\mathcal{A}$ and $\mathcal{B}$ instead

Specia

1 Real

when

$f: \mathbb{R}^m \rightarrow \mathbb{R}^n$

Denote

C^∞

Use no

Rk: C^∞

$J_0, \dots, J_k$

A nice

2 Sm

γ :

REMINDER: M = topological space, $m \in \mathbb{N}$

- an m -dimensional topological chart of M

- two charts $(U, x), (U', x')$ smoothly compatible if

$$x: \underset{\substack{U \\ \text{open} \\ M}}{\longrightarrow} \underset{\substack{x(U) \\ \text{open} \\ \mathbb{R}^m}}{\longrightarrow} \text{homeomorphism}$$

$$x' \circ x^{-1}: x(U \cap U') \longrightarrow x'(U \cap U') \text{ classical diffeomorphism}$$

- smooth m -dim. atlas on M : collection A of topological charts of M s.t.
 - (A1) $(\forall) p \in M, \exists (U, x) \in A$ s.t. $p \in U$
 - (A2) each $(U, x), (U', x') \in A$ are smoothly compatible

- smooth structure on M : any smooth atlas A s.t. furthermore:
 - (A3) any topological chart of M that is smoothly compatible with all $(U, x) \in A$ must be in A .

- smooth (m -dimensional) manifold M : a topological space that is Hausdorff & second countable + a smooth structure A on M .

Ex 1: $\mathbb{R}^m$ with $A_{\mathbb{R}^m}^{\max} = \{ (U, x) \text{ - top. charts of } \mathbb{R}^m \mid (U, x) = \text{c-smooth} \}$
 In other words: the smooth charts of $(\mathbb{R}^m, A_{\mathbb{R}^m}^{\max}) =$ the c-smooth charts of $\mathbb{R}^m$.

Ex 2: Opens $U \subseteq \mathbb{R}^m$ has a canonical smooth structure induced by $\{ (U, id_U) \}$. $\mathbb{R}^m, \mathbb{R}^n$ opens in $\mathbb{R}^n$ are the classical smooth structure $\rightarrow$ are endowed with this smooth structure unless otherwise specified.

Ex 3: $\mathbb{R}^m$ / opens of it in disguise.
 $\mathbb{C} = \{ x + iy, x, y \in \mathbb{R} \} \leftarrow \text{copy of } \mathbb{R}^2$
 $M_{n,m}(\mathbb{R}) \leftarrow \text{copy of } \mathbb{R}^{n \times m}$ chart $A \rightarrow (A_1^1, \dots, A_m^1, \dots, A_1^n, \dots, A_m^n)$
 $GL_n(\mathbb{R}) = \{ A \in M_{n,n}(\mathbb{R}) \mid A \text{ invertible} \} \leftarrow \text{open in } M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 $\det(A) \in (-\infty, 0) \cup (0, \infty), \det: M_{n,n}(\mathbb{R}) \rightarrow \mathbb{R}$ continuous

Terminology: (M, A) - smooth manifold
 the charts $(U, x) \in A$: the smooth charts of the smooth

Special smooth maps:
 ① Real valued ones / (smooth) observables in M :
 ② Call

+ a smooth structure A on M .

Terminology: (M, A) - smooth manifold

the charts $(U, \chi) \in A$: the smooth charts of the smooth manifold (M, A)

Rk: ① Any smooth atlas A induces a smooth structure
 Many examples are constructed like this
 $A \subseteq A^{\max} := \{ (U, \chi) \text{ - top. chart of } M \mid \text{it is smoothly compatible with } A \}$
 Q: Given a smooth str $\mathcal{S}$ find A st. $\mathcal{S} = A^{\max}$ $\neq A$ - very small all $(U, \chi) \in A$

② Two smooth atlases A and B induce same smooth str. (i.e. $A^{\max} = B^{\max}$) $\Leftrightarrow$ any $(U, \chi) \in A$ is smoothly compatible with any $(U', \chi') \in B$

Smooth maps: $(M, A_M), (N, A_N)$ smooth manifolds

$f: M \rightarrow N$ called smooth if:
 $(*) \quad (\forall) (U, \chi) \in A_M, (\forall) (U', \chi') \in A_N \Rightarrow f_{\chi'}^{\chi} = \chi' \circ f \circ \chi^{-1}$ is classically smooth
 (open $\mathbb{R}^m \rightarrow \mathbb{R}^n$)

Rk ③ / Exercise: If $A_M = A^{\max}, A_N = B^{\max}$ with A/B smooth atlas on M/N then: $(*) \Leftrightarrow$ same cond but using A and B instead of A_M and A_N

chart $A \rightarrow (A_1, \dots, A_m, \dots, A_1^n, \dots, A_m^n)$
 $GL_n(\mathbb{R}) = \{ A \in M_{n,n}(\mathbb{R}) \mid A \text{ invertible} \}$ - open in $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 $\det(A) \in (-\infty, 0) \cup (0, \infty), \det: M_{n,n}(\mathbb{R}) \rightarrow \mathbb{R}$ cont.

Special smooth maps:

① Real valued ones / (smooth) observables in M :
 when $N = \mathbb{R}$ with the classical smooth str.
 $f: M \rightarrow \mathbb{R}$ smooth $\Leftrightarrow f_{\chi} = f \circ \chi^{-1}: \chi(U) \rightarrow \mathbb{R}$

Denote by $C^{\infty}(M) = \{ f: M \rightarrow \mathbb{R} \mid f \text{ smooth} \}$ classically smooth
 $(\mathbb{R}) (U, \chi)$ - smooth chart of the manifold M

Use notation $C^{\infty}(M, A)$ when we want to specify the smooth structure.

Rk: $C^{\infty}(M, A) = C^{\infty}(M, B) \Leftrightarrow A = B$.
 I.e.: knowing which observables are smooth determines uniquely the smooth structure

A nice exercise
 ② Smooth curves in a manifold N : when $M = I \subseteq \mathbb{R}$ open interval
 $\gamma: I \rightarrow N$
 $\gamma_{\chi'}^{\chi} = \chi' \circ \gamma \circ \chi^{-1}$ domain $I \rightarrow \mathbb{R}^n$ classically smooth


$M = \{ \dots \}$
 $m=1$

Special smooth maps:

① Real valued ones / (smooth) observables in M :
 when $N = \mathbb{R}$ with the classical smooth str.
 $f: M \rightarrow \mathbb{R}$ smooth $\Leftrightarrow f_x = f \circ x^{-1}: x^{-1}(U) \rightarrow \mathbb{R}$

Denote by

$$C^\infty(M) = \{f: M \rightarrow \mathbb{R} / f = \text{smooth}\}$$

classically smooth
 (4) (U, x) - smooth chart of the manifold M

Use notation $C^\infty(M, A)$ when we want to specify the smooth structure.

Def: $C^\infty(M, A) = C^\infty(M, B) \Leftrightarrow A = B$.

J.o.: knowing which observables are smooth determines uniquely the smooth structure.

A nice exercise

② Smooth curves in a manifold N :
 $\gamma: I \rightarrow N$

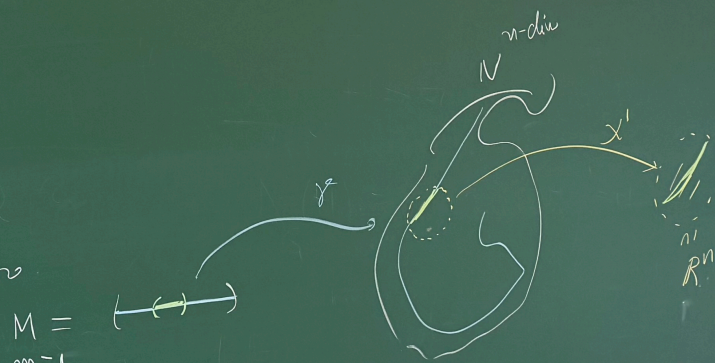
$\gamma' = x'_* \gamma: \text{domain } I \rightarrow \mathbb{R}^n$ classically smooth

and B instead of A_M and A_N

③ $M = N$, $f = \text{bijective}$, f, f^{-1} - smooth

Call $f: M \rightarrow N$ a diffeomorphism

Call M, N - diffeomorphic manifolds if $\exists f: M \rightarrow N$ diffeomorphism



REMINDEIS: M = topological space, $m \in \mathbb{N}$

- an m -dimensional topological chart of M

$$\begin{array}{ccc} X: U & \xrightarrow{\text{homeomorphism}} & X(U) \\ \text{Nopen} & & \text{Nopen} \\ M & & \mathbb{R}^m \end{array}$$
- two charts $(U, X), (U', X')$
 Smoothly compatible if $X \cdot X'^{-1}: X(U \cap U') \rightarrow X'(U \cap U')$ classical diffeomorphism
- smooth m -dim. atlas on M : collection $\mathcal{A}$ of topological charts of M st
 - (A1) $(\forall) p \in M, \exists (U, X) \in \mathcal{A}$ s.t. $p \in U$
 - (A2) each $(U, X), (U', X') \in \mathcal{A}$ are smoothly compatible
- smooth structure on M : any smooth atlas $\mathcal{A}$ s.t., furthermore:
 - (A3) any topological chart of M that is smoothly compatible with all $(U, X) \in \mathcal{A}$ must be in $\mathcal{A}$
- smooth (m -dimensional) manifold M : a topological space that is Hausdorff & second countable
- $\oplus$ a smooth structure $\mathcal{A}$ on M

Ex 1: $\mathbb{R}^m$ with $\mathcal{A}_{\mathbb{R}^m}^{\max} = \{ (U, X) \text{-top chart of } \mathbb{R}^m / (U, X) = \text{c-smooth} \}$
 In other words: the smooth charts of $(\mathbb{R}^m, \mathcal{A}_{\mathbb{R}^m}^{\max}) =$ the c-smooth charts of $\mathbb{R}^m$
 $=$ the c-smooth charts of $\mathbb{R}^m$ $\mathcal{A} = \{ (U, \text{id}_U) \}$

Smaller atlas inducing this smooth structure
 Ex 2: Opens $U \subseteq \mathbb{R}^m$ has a canonical smooth structure induced by $\{ (U, \text{id}_U) \}$
 $\mathbb{R}^m$ is open in $\mathbb{R}^m$ so the canonical smooth structure on $\mathbb{R}^m / U$ is the smooth structure unless otherwise specified

Ex 3: $\mathbb{R}^m$ / opens of it in disguise
 $C = \{ x \mapsto y, x, y \in \mathbb{R}^n \} \leftarrow$ copy of $\mathbb{R}^n$
 $M_{n,m}(\mathbb{R}) \leftarrow$ copy of $\mathbb{R}^{nm}$ chart $A \rightarrow (A_1, \dots, A_m, \dots, A_1, \dots, A_m)$
 $GL_n(\mathbb{R}) = \{ A \in M_{n,n}(\mathbb{R}) / A \text{ invertible} \}$ - open in $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 $\det(A) \in \mathbb{R} \setminus \{0\}$, $\det: M_{n,n}(\mathbb{R}) \rightarrow \mathbb{R}$ continuous

Terminology: $(M, \mathcal{A})$ - smooth manifold
 the charts $(U, X) \in \mathcal{A}$: the smooth charts of the smooth manifold $(M, \mathcal{A})$

Rk 1: Any smooth atlas $\mathcal{A}$ induces a smooth structure
 $\mathcal{A} \subseteq \mathcal{A}^{\max}$
 1. $\mathcal{A} := \{ (U, X) \text{-top chart of } M / \text{it is smoothly compatible with all } (U', X') \in \mathcal{A} \}$
 2. Given a smooth st. $\mathcal{A}$ find $\mathcal{A}^{\max}$ s.t. $\mathcal{A} \subseteq \mathcal{A}^{\max}$ & $\mathcal{A}$ -very small

2) Two smooth atlases
 $\mathcal{A}$ and $\mathcal{B}$ induce same smooth st. $\iff$ any $(U, X) \in \mathcal{A}$ is smoothly compatible with any $(U', X') \in \mathcal{B}$

Smooth maps $f: (M, \mathcal{A}_M) \rightarrow (N, \mathcal{A}_N)$ smooth manifolds
 $f: M \rightarrow N$ called smooth if:

(A) $(\forall) (U, X) \in \mathcal{A}_M, (V, Y) \in \mathcal{A}_N \Rightarrow f \circ X^{-1} \circ Y \circ X'^{-1}: X(U \cap U') \rightarrow Y(V \cap V')$ is classically smooth

Rk 2: Exercise: If $\mathcal{A}_M = \mathcal{A}_M^{\max}, \mathcal{A}_N = \mathcal{A}_N^{\max}$ then f is smooth iff f is classically smooth

Special smooth maps:

1) Real valued over / (smooth) observables in M :
 when $N = \mathbb{R}$ with the classical smooth st.
 $f: M \rightarrow \mathbb{R}$ smooth $\iff f_x = f \circ X^{-1}: X(U) \rightarrow \mathbb{R}$

Denote by $C^\infty(M) = \{ f: M \rightarrow \mathbb{R} / f \text{ smooth} \}$ (classically smooth)
 (U, X) - smooth chart of the manifold M

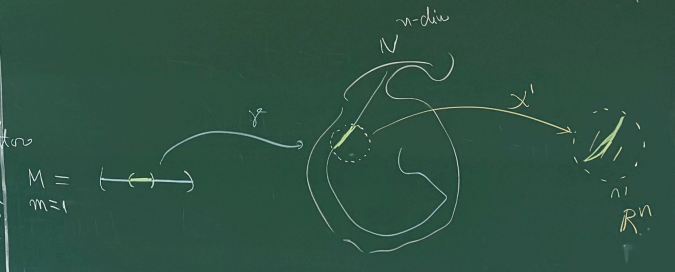
Use notation $C^\infty(M, \mathcal{A})$ when we want to specify the smooth structure

Rk: $C^\infty(M, \mathcal{A}) = C^\infty(M, \mathcal{B}) \iff \mathcal{A} = \mathcal{B}$

To: knowing which observables are smooth determines uniquely the smooth structure

A nice exercise
 2) Smooth curves in a manifold N : when $M = I \subseteq \mathbb{R}$ open interval
 $\gamma: I \rightarrow N$
 $\gamma' = X' \circ \gamma: \text{domain } I \rightarrow \mathbb{R}^m$ classically smooth

3) $M \xrightarrow{f} N$, f = bijective, f, f^{-1} - smooth
 Call $f: M \rightarrow N$ a diffeomorphism
 Call M, N - diffeomorphic manifolds if $\exists f: M \rightarrow N$ diffeomorphism



④

M : st.

$\rightarrow \mathbb{R}$

smooth

chart of manifold M

by the structure

determines

smooth structure

$\subseteq \mathbb{R}$

open interval

classically smooth

$\mathbb{R}^m$

③ $M=N$, f = bijective, f, f^{-1} - smooth

Call $f: M \rightarrow N$ a diffeomorphism

Call M, N - diffeomorphic manifolds if $\exists f: M \rightarrow N$ diffeomorphism

$C(M) = \{f: M \rightarrow \mathbb{R} \mid f = \text{cont}\}$

n -dim

N

x'

n

$\mathbb{R}^n$

$M = \text{---} \text{---} \text{---}$

$m=1$

⑤

$(U, x) = \textcircled{5}$

C -smooth

$\mathbb{R}^m$

$\mathbb{R}^n$

$\mathbb{R}^n$ are

this smooth

manifold specified

$A_1, \dots, A_n$

$\mathbb{R}^n \cong \mathbb{R}^n$

$(0, \infty)$, det M

Ex 4: $M = \mathbb{R}$, $A_h = \{(\mathbb{R}, h)\}$ here $h: \mathbb{R} \rightarrow \mathbb{R}$ or any $\textcircled{6}$

$h(t) = t^3$ homeo which is not a classical diffe.

$\Rightarrow$ get a smooth structure on $\mathbb{R}$, different from the classical smooth structure.

$(\mathbb{R}, id_{\mathbb{R}}), (\mathbb{R}, h)$ are not smoothly compatible.

Lots of different smooth structures on $\mathbb{R}$! !

$h \circ id_{\mathbb{R}}$ smooth $\checkmark$

But: $(\mathbb{R}, d_h^{\max})$ is diffeomorphic to $(\mathbb{R}, d_{\mathbb{R}}^{\max})$.

$id_{\mathbb{R}} \circ h^{-1}$ smooth $\times$

$f = h: (\mathbb{R}, d_h^{\max}) \rightarrow (\mathbb{R}, d_{\mathbb{R}}^{\max})$ smooth

$(U, x) = (\mathbb{R}, h)$ $f_{x'}^{-1}$ $(U', x') = (\mathbb{R}, id_{\mathbb{R}})$

$x' \circ f \circ x^{-1} = id_{\mathbb{R}} \circ h \circ h^{-1} = id_{\mathbb{R}}$

$h^{-1}: (\mathbb{R}, d_{\mathbb{R}}^{\max}) \rightarrow (\mathbb{R}, d_h^{\max})$ is smooth.

Terminology: $(M, \mathcal{A})$ - smooth manifold

the charts $(U, \chi) \in \mathcal{A}$: the smooth charts of the smooth manifold $(M, \mathcal{A})$

Def: Any smooth atlas $\mathcal{A}$ induces a smooth structure

1. $\mathcal{A} \in \mathcal{A}^{\max} := \{ (U, \chi) \text{-top chart of } M \mid \text{it is smoothly compatible with all } (U', \chi') \in \mathcal{A} \}$

2. Two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ induce same smooth st. (i.e. $\mathcal{A}^{\max} = \mathcal{B}^{\max}$) $\Leftrightarrow$ any $(U, \chi) \in \mathcal{A}$ is smoothly compatible with any $(U', \chi') \in \mathcal{B}$

Smooth maps $f: M \rightarrow N$ called smooth if:

(*) $(U, \chi) \in \mathcal{A}_M, (V, \chi') \in \mathcal{A}_N \Rightarrow f \circ \chi' \circ \chi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is classically smooth

Rk: Exercise: If $\mathcal{A}_M = \mathcal{A}_M^{\max}, \mathcal{A}_N = \mathcal{A}_N^{\max}$ then: (*) $\Leftrightarrow$ same could be said with $\mathcal{A}$ and $\mathcal{B}$ instead of $\mathcal{A}_M$ and $\mathcal{A}_N$

Special smooth maps:

1. Real valued over (smooth) observables in M : when $N = \mathbb{R}$ with the classical smooth st.

$f: M \rightarrow \mathbb{R}$ smooth $\Leftrightarrow f_{\chi} = f \circ \chi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}$

Denote by $C^{\infty}(M) = \{ f: M \rightarrow \mathbb{R} \mid f \text{ is smooth} \}$ classically smooth

Use notation $C^{\infty}(M, \mathcal{A})$ when we want to specify the smooth structure.

Rk: $C^{\infty}(M, \mathcal{A}) = C^{\infty}(M, \mathcal{B}) \Leftrightarrow \mathcal{A} = \mathcal{B}$.

2. Smooth curves in a manifold N : when $M = I \subseteq \mathbb{R}$ open interval

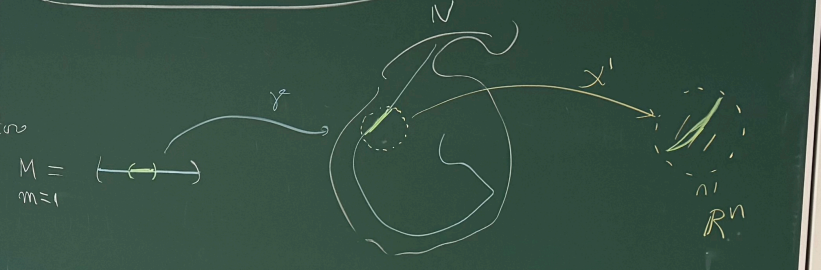
$\gamma: I \rightarrow N$ $\gamma' = \gamma_* \cdot \dot{\gamma}: \mathbb{R} \rightarrow \mathbb{R}^n$ is classically smooth

3. $M = N, f = \text{bijective}, f, f^{-1}$ - smooth

Call $f: M \rightarrow N$ a diffeomorphism

Call M, N - diffeomorphic manifolds if $\exists f: M \rightarrow N$ diffeomorphism

$C(M) = \{ f: M \rightarrow \mathbb{R} \mid f = \text{const} \}$



REMINDER: M = topological space, $m \in \mathbb{N}$

- an m -dimensional topological chart of M : $\chi: U \rightarrow \chi(U)$ homeomorphism
- two charts $(U, \chi), (U', \chi')$ smoothly compatible if $\chi' \circ \chi^{-1}: \chi(U \cap U') \rightarrow \chi'(U \cap U')$ classical diffeomorphism
- smooth m -dim atlas on M : collection $\mathcal{A}$ of topological charts of M st.
 - (A1) $(\gamma) \in M, \exists (U, \chi) \in \mathcal{A}$ st $\gamma \in U$
 - (A2) each $(U, \chi), (U', \chi') \in \mathcal{A}$ are smoothly compatible
- smooth structure on M : any smooth atlas $\mathcal{A}$ st, furthermore:
 - (A3) any topological chart of M that is smoothly compatible with all $(U, \chi) \in \mathcal{A}$ must be in $\mathcal{A}$.
- smooth $(m$ -dimensional) manifold M : that is Hausdorff & second countable with a smooth structure $\mathcal{A}$ on M .

Ex 1: $\mathbb{R}^n$ with $\mathcal{A}_{\mathbb{R}^n}^{\max} = \{ (U, \chi) \text{-top charts of } \mathbb{R}^n \mid (U, \chi) = (U, \text{id}_U) \}$

In other words: the smooth charts of $(\mathbb{R}^n, \mathcal{A}_{\mathbb{R}^n}^{\max})$ = the C^{∞} -smooth charts of $\mathbb{R}^n$

Smaller atlas inducing this smooth structure: $\mathcal{A}_0 = \{ (\mathbb{R}^n, \text{id}_{\mathbb{R}^n}) \}$

Ex 2: Opens $U \subseteq \mathbb{R}^m$ has a canonical smooth structure induced by $\{ (U, \text{id}_U) \}$. $\mathbb{R}^m, \mathbb{R}^n$ opens in $\mathbb{R}^n$ are endowed with this smooth structure unless otherwise specified

Ex 3: $\mathbb{R}^m$ / opens of it in disguise

$\mathbb{C} = \{ z = x + iy \mid x, y \in \mathbb{R} \} \leftarrow \text{copy of } \mathbb{R}^2$

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Ex 4: $M = \mathbb{R}, \mathcal{A}_h = \{ (\mathbb{R}, h) \}$ here $h: \mathbb{R} \rightarrow \mathbb{R}$ or any homeo which is not a classical diffeomorphism

$h(t) = t^3$

$\Rightarrow$ get a smooth structure on $\mathbb{R}$, different from the classical smooth structure

Lots of different smooth structures on $\mathbb{R}$

But: $(\mathbb{R}, \mathcal{A}_h^{\max})$ is diffeomorphic to $(\mathbb{R}, \mathcal{A}_{\mathbb{R}}^{\max})$

$f = h: (\mathbb{R}, \mathcal{A}_h^{\max}) \rightarrow (\mathbb{R}, \mathcal{A}_{\mathbb{R}}^{\max})$ is smooth

$(U, \chi) = (\mathbb{R}, h) \rightarrow (U', \chi') = (\mathbb{R}, \text{id}_{\mathbb{R}})$

$\chi' \circ f \circ \chi^{-1} = \text{id}_{\mathbb{R}} \circ h \circ h^{-1} = \text{id}_{\mathbb{R}}$

$n \in \mathbb{N}$

$\chi(U) \xrightarrow{\chi|_{\text{open}}} \chi(U)$ homeomorphism

$\chi(U \cap U') \xrightarrow{\chi|_{\text{open}}} \chi(U \cap U')$ classical diffeomorphism

$\mathcal{A}$ of topological charts of M s.t. $\chi \in \mathcal{A}$

smoothly compatible

th atlas $\mathcal{A}$ s.t. furthermore:

that is smoothly compatible in $\mathcal{A}$.

Let M :

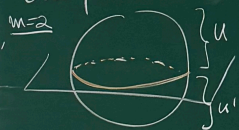
Hausdorff & second countable

M .

①

Ex 5: $S^m = \{ (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} : \sum_{k=0}^m (x_k)^2 = 1 \}$

One possible smooth atlas: use $2(m+1)$ charts



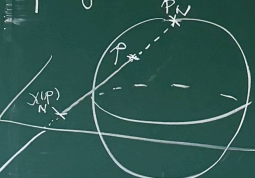
Smooth atlas $\mathcal{A}$.

$\chi_U: U \rightarrow \mathbb{R}^2, \chi_U(x, y, z) = (x, y)$

$\chi_{U'}: U' \rightarrow \mathbb{R}^2, \chi_{U'}(x, y, z) = (x, y)$

use any of the coordinate planes $\Rightarrow 4$ more.

Another possible atlas: use the stereographic projections w.r.t. $p_N = (0, 0, 1)$ and $p_S = (0, 0, -1)$



$\chi_N: S^2 \setminus \{p_N\} \rightarrow \mathbb{R}^2$

$\chi_S: S^2 \setminus \{p_S\} \rightarrow \mathbb{R}^2$

Smooth atlas $\mathcal{B}$ with only two charts

②

both charts of the smooth manifold $(M, \mathcal{A})$

both structure of M / it is smoothly compatible with all $(U, \chi) \in \mathcal{A}$

any $(U, \chi) \in \mathcal{A}$ is smoothly compatible with any $(U', \chi') \in \mathcal{B}$

Smooth manifolds

$\Rightarrow f = \chi' \circ \chi^{-1}: \text{open} \rightarrow \text{open}$
 $\chi: \mathbb{R}^m \rightarrow \mathbb{R}^m$
 is classically smooth

same could but w.r.t. $\mathcal{A}$ and

Ex: $A^{\max} = \mathbb{B}^{\max}$

Rk: Milnor: S^7 has precisely 28 (nondiffeom) Smooth structures!

Ex 6: $\mathbb{P}^m = \{ \ell / \ell\text{-line in } \mathbb{R}^{m+1} \text{ through the origin} \}$
 $\ell \in \mathbb{R}^{m+1}$ one-dim. vector subspace

Any $x \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to a line $\ell_x = \mathbb{R} \cdot x = \{ \lambda x : \lambda \in \mathbb{R} \} = [x_0 : x_1 : \dots : x_m]$

Hence: $\mathbb{P}^m = \{ [x_0 : x_1 : \dots : x_m] : (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\} \}$

$[x_0 : x_1 : \dots : x_m] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_m] \Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{ s.t.}$

$x_k = \lambda x_k$

$\forall k \in \{0, 1, \dots, m\}$

③

$[x_0 : x_1 : \dots : x_m]$

$U_m = \{ \text{domain} \}$
 $x_m = \dots$

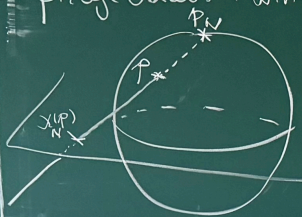
Top

sat

compatible:
furthermore:
only compatible
second countable

Smooth atlas $\mathcal{A}$. Use any of the coordinate planes $\Rightarrow 4$ maps.

Another possible atlas: use the stereographic projections with $p_N = (0,0,1)$ and $p_S = (0,0,-1)$



$$x_N: S^2 \setminus \{p_N\} \rightarrow \mathbb{R}^2$$

$$x_S: S^2 \setminus \{p_S\} \rightarrow \mathbb{R}^2$$

Smooth atlas $\mathcal{B}$ with only two charts



the smooth
(M, $\mathcal{A}$)
smoothly
compatible with
(U, x) $\in \mathcal{A}$
A is
compatible
(U, x) $\in \mathcal{B}$
manifolds
f.o.x: $\frac{\text{open}}{\mathbb{R}^m} \rightarrow \frac{\text{open}}{\mathbb{R}^n}$
classically smooth
but using $\mathcal{A}$ and

$$\text{Ex: } \mathcal{A}^{\max} = \mathcal{B}^{\max}$$

Rk: Milnor: S^7 has precisely 28 (non-diffeom) smooth structures!

Ex 6: $\mathbb{P}^m = \{ \ell \mid \ell \text{ - line in } \mathbb{R}^{m+1} \text{ through the origin} \}$
 $\ell \in \mathbb{R}^{m+1}$ one-dim. vector subspace

Any $x \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to a line:
 $\ell_x = \mathbb{R} \cdot x = \{ \lambda x \mid \lambda \in \mathbb{R} \} = [x_0: x_1: \dots: x_m]$

Hence: $\mathbb{P}^m = \{ [x_0: x_1: \dots: x_m] : (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\} \}$

$$[x_0: x_1: \dots: x_m] = [\lambda x'_0: \lambda x'_1: \dots: \lambda x'_m] \Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{ st } x'_i = \lambda x_i$$

($\forall$) $i \in \{0, 1, \dots, m\}$

$$[x_0: x_1: \dots: x_{m-1}: x_m] = \left[\frac{x_0}{x_m} : \dots : \frac{x_{m-1}}{x_m} : 1 \right] \quad [y_0, \dots, y_{m-1}, 1] \text{ when } x_m \neq 0.$$

$U_m = \{ [x_0: x_1: \dots: x_m] \in \mathbb{P}^m : x_m \neq 0 \} \subseteq \mathbb{P}^m$
domain of a chart:

$$x_m: U_m \xrightarrow{\cap / \mathbb{P}^m} \mathbb{R}^m, [x_0: x_1: \dots: x_m] \mapsto \left(\frac{x_0}{x_m}, \dots, \frac{x_{m-1}}{x_m} \right)$$

$$[y_0, \dots, y_{m-1}, 1] \longleftarrow (y_0, \dots, y_{m-1})$$

Top chart (U_m, x_m) "of dividing by x_m "

$\Downarrow$
at $\{ (x_0, x_0), \dots, (x_m, x_m) \}$ a smooth atlas of $\mathbb{P}^m$

REMINDER: M = topological space, $m \in \mathbb{N}$

• an m -dimensional topological chart of M

• two charts $(U, x), (U', x')$ smoothly compatible if

• smooth m -dim. atlas on M : collection $\mathcal{A}$ of topological charts of M st.

(A1) $(\forall) p \in M, \exists (U, x) \in \mathcal{A}$ s.t. $p \in U$
 (A2) each $(U, x), (U', x') \in \mathcal{A}$ are smoothly compatible.
 • smooth structure on M : any smooth atlas $\mathcal{A}$ s.t. furthermore:
 (A3) any topological chart of M that is smoothly compatible with all $(U, x) \in \mathcal{A}$ must be in $\mathcal{A}$.

• smooth (m -dimensional) manifold M :
 a topological space that is Hausdorff & second countable
 ⊕ a smooth structure $\mathcal{A}$ on M .

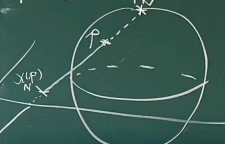
$$x: U \xrightarrow[\text{Nlopen}]{\text{Nlopen}} \underbrace{X(U)}_{\mathbb{R}^m} \text{ homeomorphism}$$

$$x'x^{-1}: x(U \cap U') \rightarrow x'(U \cap U') \text{ classical diffeomorphism}$$

Ex 5: $S^m = \{ (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1} : \sum_{k=0}^m (x_k)^2 = 1 \}$

One possible smooth atlas: use $2(m+1)$ charts
 $m=2$

 $x_U: U \rightarrow \mathbb{R}^2, x_U(x, y, z) = (x, y)$
 $x_{U'}: U' \rightarrow \mathbb{R}^2, x_{U'}(x, y, z) = (x, y)$
 use any of the coordinate planes $\Rightarrow 4$ more.
 Smooth atlas $\mathcal{A}$.

Another possible atlas: use the stereographic projections w.r.t. $P_N = (0, 0, 1)$ and $P_S = (0, 0, -1)$

 $x_N: S^2 \setminus \{P_N\} \rightarrow \mathbb{R}^2$
 $x_S: S^2 \setminus \{P_S\} \rightarrow \mathbb{R}^2$
 Smooth atlas $\mathcal{B}$ with only two charts.

Terminology: $(M, \mathcal{A})$ - smooth manifold

the charts $(U, x) \in \mathcal{A}$: the smooth charts of the smooth manifold $(M, \mathcal{A})$

Rk: Any smooth atlas $\mathcal{A}$ induces a smooth structure
 $\mathcal{A} \subseteq \mathcal{A}^{\max} := \{ (U, x) \text{ - top chart of } M \mid \text{it is smoothly compatible with all } (U', x') \in \mathcal{A} \}$
 Q: Given a smooth st. $\mathcal{A}$ find $\mathcal{A}^{\max}$ st. compatible with all $(U', x') \in \mathcal{A}$

Two smooth atlases $\mathcal{A} = \mathcal{A}^{\max} \neq \mathcal{A}$ - very small
 $\mathcal{A}$ and $\mathcal{B}$ induce same smooth st. $\Leftrightarrow$ any $(U, x) \in \mathcal{A}$ is smoothly compatible with any $(U', x') \in \mathcal{B}$
 (i.e. $\mathcal{A}^{\max} = \mathcal{B}^{\max}$)

Smooth maps: $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ smooth manifolds
 $f: M \rightarrow N$ called smooth if:

(X) $(\forall) (U, x) \in \mathcal{A}_M, (\forall) (U', x') \in \mathcal{A}_N \Rightarrow f = x' \circ f \circ x^{-1}$
 Rk: Exercise: If $\mathcal{A}_M = \mathcal{A}_M^{\max}, \mathcal{A}_N = \mathcal{A}_N^{\max}$
 with $\mathcal{A}_M, \mathcal{A}_N$ smooth atlas on M, N then: $(*) \Leftrightarrow$ same cond but w.r.t. $\mathcal{A}_M^{\max}$ and $\mathcal{A}_N^{\max}$

Ex: $\mathcal{A}^{\max} = \mathcal{B}^{\max}$

Rk: Multiv: S^7 has precisely 28 (nondiff) smooth structures!

Ex 6: $\mathbb{P}^m = \{ \ell \mid \ell \text{ - line in } \mathbb{R}^{m+1} \text{ through the origin} \}$
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Any $x \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to a line
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Hence: $\mathbb{P}^m = \{ [x_0 : x_1 : \dots : x_m] : (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\} \}$
 $[x_0 : x_1 : \dots : x_m] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_m] \Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{ s.t. } x_k = \lambda x'_k$
 $(\forall) k \in \{0, 1, \dots, m\}$

$$[x_0 : x_1 : \dots : x_{m-1} : x_m] = \left[\frac{x_0}{x_m} : \dots : \frac{x_{m-1}}{x_m} : 1 \right] \text{ when } x_m \neq 0$$

$U_m = \{ [x_0 : x_1 : \dots : x_m] \in \mathbb{P}^m : x_m \neq 0 \} \subseteq \mathbb{P}^m$
 domain of a chart:

$$x_m: U_m \rightarrow \mathbb{R}^m, [x_0 : x_1 : \dots : x_m] \mapsto \left(\frac{x_0}{x_m}, \dots, \frac{x_{m-1}}{x_m} \right)$$

Top chart (U_m, x_m) "of dividing by x_m "
 $\Downarrow$
 set $\{ (u_0, x_0), \dots, (u_m, x_m) \}$ a smooth atlas of $\mathbb{P}^m$