

Reminder: Thm 1.50: For $M \subseteq \mathbb{R}^n$, $p \in M$, the following are equiv.:

3. $\exists$ parametrization of M around p
4. $\exists$ equation defining M around p (eg: $\tilde{U} \rightarrow \mathbb{R}^{n-m}$ subm. $M \cap \tilde{U} = \text{Zero}(g)$)
2. $\exists$ an m -dimensional chart of M around p which is C^∞ -smooth
1. $\exists \tilde{\chi}: \tilde{U} \rightarrow \mathbb{R}^n$ C^∞ -smooth chart of $\mathbb{R}^n$ around p s.t.

adapted to M
 $(\chi = \tilde{\chi}|_{M \cap \tilde{U}})$

$M \cap \tilde{U} = \{q \in \tilde{U} : \tilde{\chi}_{n+1}(q) = \dots = \tilde{\chi}_n(q) = 0\} \subset \mathbb{R}^n$
 (OR: $\tilde{\chi}(M \cap \tilde{U}) = \tilde{J}_m(\tilde{\chi}) \cap \underbrace{\mathbb{R}^m \times \{0, \dots, 0\}}_{n-m} \subset \mathbb{R}^n$)

When this holds ($\forall$) $p \in M$: say $M =$ embedded submanifold of $\mathbb{R}^n$

Theorem (RVT): Let $F: \Omega \rightarrow \mathbb{R}^k$, $n \geq k$, $F =$ smooth
 Ω open $\mathbb{R}^n$

Consider $M = F^{-1}(0) = \{x \in \Omega : F(x) = 0\}$.

Assume $0 \in \mathbb{R}^k$ is a regular value of F , i.e.

$(\forall) p \in F^{-1}(0)$, $(dF)_p$ has rank k .

Then: $M =$ smooth submfd of $\mathbb{R}^n$ of dimension $n-k$.

Ex: $S^m \subseteq \mathbb{R}^{m+1}$, $F: \mathbb{R}^{m+1} \rightarrow \mathbb{R}$, $F(x_0, \dots, x_m) = \sum_{k=0}^m (x_k)^2 - 1$
 $\text{Zero}(F) = S^m$

$p = (x_0, \dots, x_m) : (dF)_p : (2x_0, \dots, 2x_m) \neq 0$ ($\forall$) $(x_0, \dots, x_m) \in F^{-1}(0)$
 $\Rightarrow$ carries an induced smooth structure.

Exercise: Prove that this coincides with the previously def. $\odot$
 Smooth str. on S^m

Reminder: smooth manifold = nice topological space M + a smooth structure on M

Hausdorff & countable

smooth atlas

i.e. a collection $\mathcal{A}$ of top. charts $X: U \xrightarrow{\sim} \mathbb{R}^m$

Satisfying:

- (A1) $(\forall) p \in M \exists (U, X)$ s.t. $p \in U$
- (A2) $(\forall) (U, X), (U', X') \in \mathcal{A} \Rightarrow$ they are smoothly compatible
- (A3) maximality

Important: any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{\max}$.

Examples: $\mathbb{R}^m$, opens (using one chart only!)

- copies of these, such as $\mathbb{C}$, $n \times n$ matrices, $GL_n(\mathbb{R})$
- S^m (possible with two charts!), $\mathbb{P}^m$ (the projective space)
- for any smooth submanifold M of $\mathbb{R}^m \Rightarrow$

$\Rightarrow$ we have an induced smooth structure: the one

defined by the atlas of C^∞ -smooth charts. $M = \text{manifold}$ (1)

products $M \times N = \{ (x, y) \mid x \in M, y \in N \}$

Smooth between manifolds M, N : $f: M \rightarrow N$ continuous s.t. map

$(\forall) (U, X) \in \mathcal{A}_M, (U', X') \in \mathcal{A}_N \Rightarrow$ the map $\frac{X'}{X} := X' \circ f \circ X^{-1}: \text{domain} \rightarrow \mathbb{R}^n$ is C^∞ -smooth.

Important: Here $\mathcal{A}_M$ can be any smooth atlas inducing the (given) smooth structure on M ; similarly $\mathcal{A}_N$ for N .

Examples / special cases:

- when $N = \mathbb{R}$: $f: M \rightarrow \mathbb{R}$ "(smooth) observables" on M $C^\infty(M)$
- when $M = I \subseteq \mathbb{R}$ interval: $\gamma: I \rightarrow N$ (smooth) curves in N .
- diffeomorphisms
- Compositions of smooth maps is smooth: $M \xrightarrow{f} N \xrightarrow{g} P$ $f, g = \text{smooth} \Rightarrow g \circ f$ smooth
- $M = \text{smooth submanifold of } \mathbb{R}^n \Rightarrow i: M \rightarrow \mathbb{R}^n, i(x) = x$ smooth
- $M, N = \text{manifolds} \Rightarrow \text{pr}_M: M \times N \rightarrow M, \text{pr}_N$ smooth

Reminder: Thm 1.50

3. $\exists$ parametrization

4. $\exists$ equation

2. $\exists$ an m -dim

1. $\exists \tilde{X}: \tilde{U} \rightarrow M$ adapted to M ($\tilde{X} = \tilde{X}|_{\tilde{U}}$) (OR: $M \cap \tilde{U}$)

When this h

Theorem

Reminder: smooth manifold = nice topological space M + a smooth structure on M

Hausdorff
2nd countable

smooth atlas

i.e. a collection $\mathcal{A}$ of top. charts
 $\chi: \underset{M}{U} \xrightarrow{\sim} \underset{\mathbb{R}^m}{\Omega}$

satisfying:
(A1) $(\forall) p \in M \exists (U, \chi)$ s.t. $p \in U$
(A2) $(\forall) (U, \chi), (U', \chi') \in \mathcal{A} \Rightarrow$ they are smoothly compatible
(A3) maximality

Important: any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{\max}$.

Examples: $\bullet \mathbb{R}^m$, opens (using one chart only!)

- copies of these, such as C , $n \times n$ matrices, $GL_n(\mathbb{R})$
- S^m (possible with two charts!), $\mathbb{P}^m$ (the projective space)
- for any smooth submanifold M of $\mathbb{R}^m \Rightarrow$

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defined by the atlas of c -smooth charts. $M = \text{manifold}$ (1')

• products $M \times N = \{ (x, y) \mid x \in M, y \in N \}$

manif. with atlas $\mathcal{A}_M, \mathcal{A}_N \rightarrow$ combined into an atlas for $M \times N$

$\left. \begin{array}{l} \chi: \underset{M}{U} \xrightarrow{\sim} \underset{\mathbb{R}^m}{\Omega} \\ \chi': \underset{N}{U'} \xrightarrow{\sim} \underset{\mathbb{R}^n}{\Omega'} \end{array} \right\} \text{ produce } (\chi, \chi'): \underset{M \times N}{U \times U'} \xrightarrow{\sim} \underset{\mathbb{R}^{m+n}}{\Omega \times \Omega'}$

and $\{ (\chi, \chi') : \chi \in \mathcal{A}_M, \chi' \in \mathcal{A}_N \}$ - smooth atlas $\Rightarrow$ smooth str. on $M \times N$.

Exercise: $S^1 \times S^1$ homeomorphic to  diffeomorphism.

- $N = \text{manifold}$
 $U \subseteq N$ open $\Rightarrow U$ carries itself a smooth str. $\leftarrow$ submanifold (use RVT)
whose smooth charts are those $\chi_{\alpha'}(U' \cap U)$ of N with $U' \subseteq U$.

• for $N = \text{any manifold}$: an (embedded) submanifold M of N is any $M \subseteq N$ such that: read cond 1. in Thm 1.50 with $\mathbb{R}^n$ replaced by $\mathbb{R}^m$
 $(\forall) p \in M$
Then M carries an induced smooth structure.

Reminder: smooth manifold = nice topological space M + a smooth structure on M

Hausdorff
& countable

• a collection $\mathcal{A}$ of top. data
$$\chi: \begin{matrix} U \\ M \end{matrix} \xrightarrow{\sim} \begin{matrix} \mathbb{R}^n \\ \mathbb{R}^m \end{matrix}$$

satisfying:
(A1) $(\forall) p \in M \exists (U, \chi)$ st $p \in U$
(A2) $(\forall) (U, \chi), (U', \chi')$ st $U \cap U' \neq \emptyset \Rightarrow$ they are smoothly compatible
(A3) maximality

Important: any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{\max}$

Examples: $\bullet \mathbb{R}^n$, opens (using one chart only!)
 $\bullet$ copies of these, such as $\mathbb{C}$, $n \times n$ matrices, $GL_n(\mathbb{R})$
 $\bullet S^1$ (possible with two charts!), $\mathbb{P}^n$ (the projective space)
 $\bullet$ for any smooth submanifold M of $\mathbb{R}^n \Rightarrow$
 $\Rightarrow$ we have an induced smooth structure: the one

Smooth between manifolds M, N : $f: M \rightarrow N$ continuous st. map

$(\forall) (U, \chi) \in \mathcal{A}_M, (U', \chi') \in \mathcal{A}_N \Rightarrow$ the map $\tilde{f}_{\chi, \chi'} = \chi' \circ f \circ \chi^{-1}: \text{domain} \rightarrow \mathbb{R}^n$ is C^∞ -smooth

Important: Here $\mathcal{A}_M$ can be any smooth atlas inducing the (given) smooth structure on M ; similarly $\mathcal{A}_N$ for N

Example/special cases:

- when $N = \mathbb{R}$: $f: M \rightarrow \mathbb{R}$ "smooth" observable on M $(C^\infty(M))$
- when $M = [a, b] \subset \mathbb{R}$ interval: $\gamma: I \rightarrow N$ (smooth) curves in N
- diffeomorphisms
- Compositions of smooth maps is smooth: $M \xrightarrow{f} N \xrightarrow{g} P$
 f, g smooth $\Rightarrow$ $g \circ f$ smooth
- M = smooth submanifold of $\mathbb{R}^n \Rightarrow i: M \rightarrow \mathbb{R}^n, i(x) = x$ smooth
- M, N = manifolds $\Rightarrow \text{pr}_M: M \times N \rightarrow M, \text{pr}_N$ smooth

Reminder: Thm 1.50: For $M \subset \mathbb{R}^n, p \in M$, the following are equiv:

- $\exists$ parametrization of M around p $\begin{matrix} \mathbb{R}^k \\ U \end{matrix} \xrightarrow{\sim} \begin{matrix} \mathbb{R}^n \\ M \end{matrix}$ s.t. $p \in U$
- $\exists$ equation defining M around p $(\exists) \tilde{U} \subset \mathbb{R}^n$ s.t. $M \cap \tilde{U} = \text{Zero}(F)$
- $\exists$ an m -dimensional chart of M around p which is C^∞ -smooth
- $\exists \tilde{\chi}: \tilde{U} \rightarrow \mathbb{R}^n$ C^∞ smooth chart of $\mathbb{R}^n$ around p s.t. adapted to M $(\exists) \tilde{\chi}(M \cap \tilde{U}) = \{ \tilde{x} \in \tilde{U} : \tilde{x}_n = 0 \}$ (OR: $\tilde{\chi}(M \cap \tilde{U}) = \tilde{J}_n(\tilde{x}) \cap \mathbb{R}^{n-k}$)

When this holds $(\forall) p \in M$: say M = embedded submanifold of $\mathbb{R}^n$

defined by the atlas of C^∞ -smooth charts. M = manifold

- products $M \times N = \{ (x, y) \mid x \in M, y \in N \}$
manif. with atlas $\mathcal{A}_M \times \mathcal{A}_N$ combined into an atlas for $M \times N$

$\chi: \begin{matrix} U \\ M \end{matrix} \xrightarrow{\sim} \begin{matrix} \mathbb{R}^n \\ \mathbb{R}^n \end{matrix}, \chi': \begin{matrix} U' \\ N \end{matrix} \xrightarrow{\sim} \begin{matrix} \mathbb{R}^m \\ \mathbb{R}^m \end{matrix}$ products: $(\chi, \chi'): \begin{matrix} U \times U' \\ M \times N \end{matrix} \xrightarrow{\sim} \begin{matrix} \mathbb{R}^n \times \mathbb{R}^m \\ \mathbb{R}^{n+m} \end{matrix}$

and $\{ (\chi, \chi') : x \in U, x' \in U' \}$ - smooth atlas $\Rightarrow$ smooth st. on $M \times N$

Exercise: $S^1 \times S^1$ homeomorphic to T^2 (torus) diffeomorphism

- N = manifold $\Rightarrow U$ carries itself a smooth st. $(U \subset N \text{ open})$ where smooth charts are those charts (U, χ) of N with $U \subset U$.

• for N = any manifold: an (embedded) submanifold M of N is any $M \subset N$ such that: read cond 4 in Thm 1.50 with $\mathbb{R}^n$ replaced by N $(\forall) p \in M$

Then M carries an induced smooth structure.



Theorem (RVT): Let $F: \Omega \rightarrow \mathbb{R}^k, n \geq k, F$ = smooth $(\Omega \text{ open in } \mathbb{R}^n)$

Consider $M = F^{-1}(a) = \{ x \in \Omega : F(x) = a \}$

Assume $0 \in \mathbb{R}^k$ is a regular value of F , i.e.

$(\forall) p \in F^{-1}(0), (dF)_p$ has rank k .

Then: M = smooth submanif. of $\mathbb{R}^n$ of dimension $n - k$

Ex: $S^m \subset \mathbb{R}^{m+1}, F: \mathbb{R}^{m+1} \rightarrow \mathbb{R}, F(x_1, \dots, x_{m+1}) = \sum_{k=1}^{m+1} x_k^2 - 1$

$\text{Zero}(F) = S^m$

$p = (x_1, \dots, x_m) : (dF)_p : (\mathbb{R}^2, \dots, \mathbb{R}^2) \rightarrow \mathbb{R} \neq 0$ $(\forall) (x_1, \dots, x_m) \in F^{-1}(0)$

$\Rightarrow$ carries an induced smooth structure

Exercise: Prove that this coincides with the previously def. smooth st. on S^m

(A3) maximality
 atlas $\mathcal{A}$ induces a smooth structure $A^{\max}$
 (using one chart only!)
 of these, such as C , $n \times n$ matrices, $GL_n(\mathbb{R})$
 possible with two charts!), $\mathbb{P}^n$ (the projective space)
 smooth submanifold M of $\mathbb{R}^m \Rightarrow$
 induced smooth structure: the one

- when $M = I \subseteq \mathbb{R}$ interval: $\gamma: I \rightarrow N$ (smooth) curves
- diffeomorphisms
- Compositions of smooth maps is smooth: $M \xrightarrow{f} N \xrightarrow{g} P$
 $f, g = \text{smooth} \Rightarrow g \circ f$ smooth
- $M = \text{smooth submanifold of } \mathbb{R}^n \Rightarrow i: M \rightarrow \mathbb{R}^n, i(x) = x$ smooth
- $M, N = \text{manifolds} \Rightarrow \text{pr}_M: M \times N \rightarrow M, \text{pr}_N$ smooth

When this holds $(\forall) p \in M$: say $M = \text{emb}$

charts of C -smooth charts. $M = \text{manifold}$ (1)
 $N = \{ (x, y) \mid x \in M, y \in N \}$
 N combined into an atlas for $M \times N$
 $U' \rightarrow x'(U')$
 N in d_N
 $M \times N$
 $\{ (x, x') \mid (x, x') \in U' \times U' \}$ produce:
 $(x, x') \mapsto (x'(U') \times x'(U'))$
 $M \times N$
 $\{ (x, x') \in U' \times U' \}$ - smooth atlas $\Rightarrow$ smooth str. on $M \times N$
 isomorphic to $\mathbb{R}^{m+n}$ diffeomorphism
 submanifold (use RVT)
 $\Rightarrow U$ carries itself a smooth str. whose smooth charts are those charts (U', x')
 of N with $U' \subseteq U$.
 manifold: an (embedded) submanifold M
 N such that: real cond 4 in Thm 1.50 with $\mathbb{R}^m$ replaced by N
 structure:

$M = \text{submanifold of } N \Rightarrow$ the inclusion $i: M \rightarrow N$ smooth (2) (Exercise!)

$\tilde{f}: N \rightarrow P$ smooth map $\Rightarrow \tilde{f}|_M: M \rightarrow P$ smooth map

$M \subseteq N$ submanifold

very useful way to prove that maps $f: M \rightarrow P$ are smooth.

all $\{ \tilde{x} \mid \tilde{x} \in M \}$ $\tilde{x}$ -chart of N
 Smooth atlas on M .

$F(x) = x^2$

$(df)_{x,y} = (y, x)$
 $(df)_{(0,0)} = 0$

$xy=0$
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x,y) = x^2 y$
 $M = \{ (x,y) \mid x=0 \text{ or } y=0 \}$

Ex: 

examples of submanifolds.

Theorem (RVT): Let $F: \Omega \rightarrow \mathbb{R}^k$

Consider $M = F^{-1}(0) = \{ x \in \Omega \mid F(x) = 0 \}$
 Assume $0 \in \mathbb{R}^k$ is a regular value
 $(\forall) p \in F^{-1}(0), (dF)_p$

Then: $M = \text{smooth submanifold of } \Omega$

Ex: $S^m \subseteq \mathbb{R}^{m+1}, F: \mathbb{R}^{m+1} \rightarrow \mathbb{R}, F(x) = |x|^2 - 1$
 $\text{Zero}(F) = S^m$
 $p = (x_0, \dots, x_m): (dF)_p = (2x_0, \dots, 2x_m) \neq 0$
 $\Rightarrow$ carries an induced smooth structure
 Exercise: Prove that this coincides with

Reminder: smooth manifold = nice topological space M

1. a collection $\mathcal{A}$ of top. charts $\chi: U \xrightarrow{\sim} \mathbb{R}^m$

2. satisfying:

- (A1) $(\forall) p \in M \exists (U, \chi) \in \mathcal{A}$ s.t. $p \in U$
- (A2) $(\forall) (U, \chi), (V, \psi) \in \mathcal{A} \Rightarrow \chi^{-1} \cap \psi^{-1} \neq \emptyset \Rightarrow \chi \circ \psi^{-1}$ is smoothly compatible
- (A3) maximality

Important: any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{\max}$

Examples: $\mathbb{R}^n$, opens (using one chart only!)

- copies of these, such as $C, n \times n$ matrices, $GL_n(\mathbb{R})$
- S^n (possible with two charts!), $\mathbb{P}^n$ (the projective space)
- for any smooth submanifold M of $\mathbb{R}^n \Rightarrow$

$\Rightarrow$ we have an induced smooth structure: the one

Smooth between manifolds $M, N: f: M \rightarrow N$ continuous s.t.

$(\forall) (U, \chi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N \Rightarrow$ the map $\tilde{f} := \psi \circ f \circ \chi^{-1}: \chi(U) \rightarrow \psi(V) \subset \mathbb{R}^n$ is C -smooth

Important: Here $\mathcal{A}_M$ can be any smooth atlas inducing the (given) smooth structure on M ; similarly $\mathcal{A}_N$ for N .

- Examples / special cases:
- when $N = \mathbb{R}: f: M \rightarrow \mathbb{R}$ "(smooth) observable" on M $C^\infty(M)$
 - when $M = I \subset \mathbb{R}$ interval: $\gamma: I \rightarrow N$ (smooth) curves in N
 - diff. homeomorphisms
 - Compositions of smooth maps is smooth: $M \xrightarrow{f} N \xrightarrow{g} P$
 $f, g = \text{smooth} \Rightarrow g \circ f$ smooth
 - $M = \text{smooth submanifold of } \mathbb{R}^n \Rightarrow i: M \rightarrow \mathbb{R}^n, i(x) = x$ smooth
 - $M, N = \text{manifolds} \Rightarrow \text{pr}_M: M \times N \rightarrow M, \text{pr}_N$ smooth

Reminder: Thm 1.50: For $M \subset \mathbb{R}^n, p \in M$, the following are equiv:

- $\exists$ parametrization of M around p (eg: $\tilde{U} \rightarrow \mathbb{R}^{n-m}$ subm.)
- $\exists$ equation defining M around p ($M \cap \tilde{U} = \text{Zero}(g)$)
- $\exists$ an m -dimensional chart of M around p which is C -smooth
- $\exists \tilde{\chi}: \tilde{U} \rightarrow \mathbb{R}^n$ smooth chart of $\mathbb{R}^n$ around p s.t. adapted to M ($\chi = \tilde{\chi}|_{M \cap \tilde{U}}$) (OR: $\tilde{\chi}(M \cap \tilde{U}) = \tilde{J}_m(\tilde{\chi}) \cap \overline{\mathbb{R}^m \times \{0, \dots, 0\}} \subset \mathbb{R}^n$)

When this holds $(\forall) p \in M$: say $M = \text{embedded submanifold of } \mathbb{R}^n$

defined by the atlas of C -smooth charts. $M = \text{manifold}$

- products $M \times N = \{ (x, y) \mid x \in M, y \in N \}$
- manifolds with atlas $\mathcal{A}_M, \mathcal{A}_N$ combined into an atlas for $M \times N$
- $\chi: U \rightarrow \mathbb{R}^m, \chi': U' \rightarrow \mathbb{R}^{m'}$ in $\mathcal{A}_M$ and $\psi: V \rightarrow \mathbb{R}^n, \psi': V' \rightarrow \mathbb{R}^{n'}$ in $\mathcal{A}_N$
- product: $(\chi, \chi'): U \times U' \rightarrow \mathbb{R}^m \times \mathbb{R}^{m'}$
- and $\{ (x, x'): x \in U, x' \in U' \}$ - smooth atlas $\Rightarrow$ smooth str on $M \times N$

Exercise: $S^1 \times S^1$ homeomorphic to T^2 (torus) diffeomorphism

- $N = \text{manifold}$ $\Rightarrow U$ carries itself a smooth str. $\Rightarrow$ submanifold (use RVT)
- $U \subset M$ open whose smooth charts are those charts (U', χ') of M with $U' \subset U$.

Def 2.30: for $N = \text{any manifold}$: an (embedded) submanifold M of N is any $M \subset N$ such that: read cond 4 in Thm 1.50 with $\mathbb{R}^n$ replaced by N .

$M = \text{submanifold of } N \Rightarrow$ the inclusion $i: M \rightarrow N$ smooth (Exercise!)

$\tilde{f}: N \rightarrow P$ smooth map $\Rightarrow \tilde{f}|_M: M \rightarrow P$ smooth map

$M \subset N$ submanifold

very useful way to prove that maps $f: M \rightarrow P$ are smooth.

all $\tilde{\chi}: \tilde{U} \rightarrow \mathbb{R}^n$ of N which is adapted to M : $M \cap \tilde{U} \xrightarrow{\tilde{\chi}} \mathbb{R}^m \times \{0, \dots, 0\} \subset \mathbb{R}^n$

$\tilde{\chi}|_{M \cap \tilde{U}}: M \cap \tilde{U} \rightarrow \mathbb{R}^m$ is chart of M

$\tilde{\chi}(x) = x$ Smooth atlas on M

$F(x) = x$ Smooth str.

$\text{Zero}(F)$

$xy=0$
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = xy$
 $M = \{ (x, y) \mid x=0 \text{ or } y=0 \}$

$(df)_{(0,0)}(y, x) = (df)_{(0,0)} = 0$

Ex: $S^1 \subset \mathbb{R}^2$ $S^2 \subset \mathbb{R}^3$

examples of submanifolds.

Theorem (RVT): Let $F: \Omega \rightarrow \mathbb{R}^k, n \geq k, F = \text{smooth}$

$\Omega \subset \mathbb{R}^n$ open

Consider $M = F^{-1}(0) = \{ x \in \Omega : F(x) = 0 \}$

Assume $0 \in \mathbb{R}^k$ is a regular value of F , i.e.

$(\forall) p \in F^{-1}(0), (dF)_p$ has rank k .

Then: $M = \text{smooth submanifold of } \mathbb{R}^n$ of dimension $n-k$.

Ex: $S^n \subset \mathbb{R}^{n+1}, F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}, F(x_0, \dots, x_n) = \sum_{k=0}^n x_k^2 - 1$

$\text{Zero}(F) = S^n$

$p = (x_0, \dots, x_n): (dF)_p = (2x_0, \dots, 2x_n) \neq 0$ $(\forall) (x_0, \dots, x_n) \in F^{-1}(0)$

$\Rightarrow$ carries an induced smooth structure.

Exercise: Prove that this coincides with the previously def smooth str on S^n

Def: A lie group consists of a set G together with
 1) a group structure $\cdot$ on G making $(G, \cdot)$ into a group
 2) a manifold structure on G $G = \text{manifold}$

s.t. these structure are compatible; i.e.:

$$m: G \times G \rightarrow G, \quad m(g, h) = gh \quad \text{smooth maps}$$

$$i: G \rightarrow G, \quad i(g) = g^{-1}$$

Ex 1: $(\mathbb{R}, +), (\mathbb{R}^*, \cdot)$

Ex 2: $S^1 = \{z \in \mathbb{C} / |z| = 1\}$ $|z_1 z_2| = |z_1| \cdot |z_2|$
 $(S^1, \cdot) = \text{group} \dots \text{lie group}$ $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2$

Ex 3: $S^3 = \{z \in \mathbb{H} / |z| = 1\}$ $|z_1 z_2| = |z_1| |z_2|$
 with quaternionic multiplication $\rightarrow$ lie group

Ex? : Other spheres that are lie groups?
 All!

Ex 4: $GL_n(\mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) : \det A \neq 0\} \subseteq M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n \times n}$
 $\hookrightarrow$ manifold & matrix multiplication makes it into a group
 $\Rightarrow (GL_n(\mathbb{R}), \cdot) = \text{a lie group}$

Check that directly for $A = I_n$
 $(dF)_{I_n}: M_{n,n}(\mathbb{R}) \rightarrow M_{n,n}^{sym}(\mathbb{R})$ $\uparrow$ surjective?
 $(dF)_X(X) = \lim_{t \rightarrow 0} (I_n + tX)(I_n + tX)^T - I_n$

Theorem

Def: A Lie group consists of a set G together with

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s.t. these structure are compatible; i.e.:

$$m: G \times G \rightarrow G, \quad m(g, h) = gh \quad \text{smooth map}$$

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Ex 1: $(\mathbb{R}, +)$

Ex 2: $S^1 = (S^1, \cdot)$

Ex 3: $S^3 =$ with

Ex?: Other All!

Ex 4: $GL_n(\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \det A \neq 0\} \subseteq M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n \times n}$
 $\hookrightarrow$ manifold & matrix multiplication makes it into a group
 $\Rightarrow (GL_n(\mathbb{R}), \cdot) = \text{a Lie group}$

Ex 5: $O(n) = \{A \in M_{n \times n}(\mathbb{R}) : A \cdot A^T = I_n\}$ the orthogonal group

not an open! $M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 Submanifold

$F: M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}^{sym}(\mathbb{R}), \quad F(A) = A \cdot A^T - I_n$
 $O(n) = \text{Zero}(F)$

To check: $(\forall) A \in O(n), \begin{cases} (dF)_A: M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}^{sym}(\mathbb{R}) \\ (dF)_A(X) = \lim_{t \rightarrow 0} \frac{F(A+tx) - F(A)}{t} \end{cases}$ surjective?

Check the

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Def: A lie group consists of a set G together with
 1) a group structure on G making $(G, \cdot)$ into a group
 2) a manifold structure on G $G = \text{manifold}$
 s.t. these structure are compatible; i.e.:
 $m: G \times G \rightarrow G, m(g, h) = gh$ smooth maps.
 $i: G \rightarrow G, i(g) = g^{-1}$

Ex 2: $S^1 = \{z \in \mathbb{C} / |z| = 1\}$ $|z_1 z_2| = |z_1| \cdot |z_2|$
 $(S^1, \cdot) = \text{group} \dots \text{lie group.}$ $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2$

Ex 3: $S^3 = \{z \in \mathbb{H} / |z| = 1\}$ $|z_1 z_2| = |z_1| |z_2|$
 with quaternionic multiplication $\Rightarrow \dots \text{lie group.}$

Ex? other spheres that are lie groups?
 All!

Ex 4: $GL_n(\mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) : \det A \neq 0\} \subseteq M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 $\hookrightarrow$ manifold & matrix multiplication makes it into a group
 $\Rightarrow (GL_n(\mathbb{R}), \cdot) = \text{a lie group.}$

Ex 5: $O(n) = \{A \in M_{n,n}(\mathbb{R}) : A \cdot A^T = I_n\}$ the orthogonal group
 "isometries"
 $\mathbb{R}^{n(n+1)/2}$
 $\{A : \det A = 1\}$
 $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$
 $\mathbb{R}^{n^2}$
 $\mathbb{M}_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 submanifold
 $F: \mathbb{M}_{n,n}(\mathbb{R}) \rightarrow \mathbb{M}_{n,n}^{sym}(\mathbb{R}), F(A) = A \cdot A^T - I_n$
 $F(a_1, a_2, \dots) = (\sum a_i a_i^T, \dots)$
 $O(n) = \text{Zero}(F)$

To check: (i) $A \in O(n), \begin{cases} (dF)_A: \mathbb{M}_{n,n}(\mathbb{R}) \rightarrow \mathbb{M}_{n,n}^{sym}(\mathbb{R}) \\ (dF)_A(X) = \lim_{t \rightarrow 0} \frac{F(A+tX) - F(A)}{t} \end{cases}$ surjective?

check that directly for $A = I_n$
 $(dF)_{I_n}: \mathbb{M}_{n,n}(\mathbb{R}) \rightarrow \mathbb{M}_{n,n}^{sym}(\mathbb{R})$ surjective?

$$(dF)_{I_n}(X) = \lim_{t \rightarrow 0} \frac{(I_n + tX)(I_n + tX)^T - I_n}{t}$$

$$= X + X^T$$

$(dF)_{I_n}(X) = X + X^T$ True that any $Y \in \mathbb{M}_{n,n}^{sym}(\mathbb{R})$ can be written as $Y = X + X^T$?

$O(n)$ smooth submanifold of $\mathbb{M}_{n,n}(\mathbb{R})$
 $O(n) \times O(m)$
 $\mathbb{M}_{n,n}(\mathbb{R}) \times \mathbb{M}_{m,m}(\mathbb{R}) \xrightarrow{\text{mult.}} O(n) = \text{lie group!}$

Theorem (RVT)

Consider $M =$
 Assume $0 \in \mathbb{R}$
 (type)

Then: $M = \text{smooth}$

Ex: $S^m \subseteq \mathbb{R}^m, F: \mathbb{R}^m \rightarrow \mathbb{R}^m$
 $p = (x_0, \dots, x_m): (dF)_p: \mathbb{R}^m \rightarrow \mathbb{R}^m$
 $\Rightarrow$ carries an induced s
 Exercise: Prove that this

Def A Lie group consists of a set G together with
 1) a group structure on G making $(G, \cdot)$ into a group
 2) a manifold structure on G
 + these structures are compatible, i.e.:
 $m: G \times G \rightarrow G, m(g, h) = gh$ smooth map.
 $i: G \rightarrow G, i(g) = g^{-1}$ smooth map.

Ex 1: $(\mathbb{R}, +), (\mathbb{R}^n, \cdot)$

Ex 2: $S^1 = \{z \in \mathbb{C} / |z| = 1\}$ $i(z) = |z| \cdot |z|$
 $(S^1, \cdot)$ = group ... Lie group $(x^{-1} \cdot y)(x \cdot y)^{-1} = (xy^{-1})^{-1} = yx^{-1}$
Ex 3: $S^3 = \{z \in \mathbb{H} / |z| = 1\}$ $i(z) = |z| \cdot |z|$
 with quaternionic multiplication $\rightarrow$ Lie group.
Ex 4: Other spheres that are Lie groups?
 All!

Ex 4: $GL_n(\mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
 $\hookrightarrow$ manifold & matrix multiplication makes it into a group
 $\Rightarrow (GL_n(\mathbb{R}), \cdot)$ = a Lie group.

Ex 5: $O(n) = \{A \in M_{n,n}(\mathbb{R}) : A \cdot A^T = -I_n\}$ the orthogonal group
 "isometries"
 $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ $\xrightarrow{\text{linear}} \mathbb{R}^{n^2}$
 $\{A : \det A = 1\}$
 $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$
 $F: M_{n,n}(\mathbb{R}) \rightarrow M_{n,n}(\mathbb{R}), F(A) = A \cdot A^T - I_n$
 $O(n) = \text{Zero}(F)$

To check: (i) $A \in O(n)$, $(dF)_A: M_{n,n}(\mathbb{R}) \rightarrow M_{n,n}(\mathbb{R})$ surjective?
 $(dF)_A(X) = \lim_{t \rightarrow 0} \frac{F(A+tX) - F(A)}{t}$

Check that directly: for $A = I_n$
 $(dF)_{I_n}: M_{n,n}(\mathbb{R}) \rightarrow M_{n,n}(\mathbb{R})$ surjective?

$(dF)_{I_n}(X) = \lim_{t \rightarrow 0} \frac{(I_n + tX)(I_n + tX)^T - I_n}{t}$
 $= X + X^T$
 $(dF)_{I_n}(X) = X + X^T$. True that any $Y \in M_{n,n}(\mathbb{R})$ can be written as $Y = X + X^T$?

$O(n)$ smooth submanifold of $M_{n,n}(\mathbb{R})$
 $O(n) \times O(m) \xrightarrow{\text{mult}} O(n+m)$
 $M_{n,n}(\mathbb{R}) \times M_{m,m}(\mathbb{R}) \xrightarrow{\text{mult}} M_{n+m, n+m}(\mathbb{R})$
 $O(n)$ = Lie group!

Theorem (RVT): Let $F: \Omega \rightarrow \mathbb{R}^k, \Omega \subseteq \mathbb{R}^n, F$ smooth

Consider $M = F^{-1}(c) = \{x \in \Omega : F(x) = c\}$
 Assume $c \in \mathbb{R}^k$ is a regular value of F , i.e.
 $(v) p \in F^{-1}(c), (dF)_p$ has rank k .

Then: M = smooth submanifold of $\mathbb{R}^n$ of dimension $n-k$
Ex: $S^n \subseteq \mathbb{R}^{n+1}, F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}, F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2 - 1$
 $\text{Zero}(F) = S^n$

$p(x_1, \dots, x_{n+1}) : (dF)_p : (\mathbb{R}^{n+1}) \rightarrow \mathbb{R} \neq 0$ (i) $(x_1, \dots, x_{n+1}) \in F^{-1}(0)$
 $\Rightarrow$ carries on induced smooth structure
 Elsewise: Prove that this coincides with the previously def smooth str on S^n