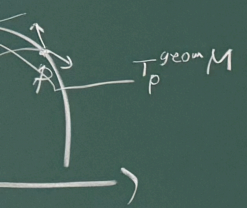


$\subseteq \mathbb{R}^n$ 1 /
 $p \in M$


Reminder: Submanifold M of $\mathbb{R}^n$: if
 $(\forall) p \in M, \exists \tilde{\alpha}: \underbrace{\tilde{U}}_{\mathbb{R}^n} \rightarrow \underbrace{\tilde{\mathcal{R}}}_{\mathbb{R}^n}$ $\hookleftarrow$ smooth chart around p st.
 $M \cap \tilde{U} = \{q \in \tilde{U} : \tilde{\alpha}_{m+1}(q) = \dots = \tilde{\alpha}_n(q) = 0\}$
 $\mathbb{R}^n$ can be replaced by an arbitrary N (or: $\chi|_U: U \rightarrow \mathcal{R}$ chart of M , where $\mathcal{R} = \tilde{\mathcal{R}} \cap (\mathbb{R}^m \times \{0\})$)

Thm (RVT): $F: \mathcal{R} \rightarrow \mathbb{R}^k$ smooth, $\mathcal{R} \subseteq \mathbb{R}^n, n \geq k$.
 Consider: $M = \text{Zero}(F) = \{x \in \mathcal{R} : F(x) = 0\}$
 Assume: 0 is a regular value of F (i.e. $(dF)_x = \text{surjective}$ $(\forall) x \in M$)
 Conclusion: M is a submanifold of $\mathbb{R}^n$ of dimension $m = n - k$.
 : and its tangent spaces at $p \in M$ are

l manifolds

$T_p M$?

$\Rightarrow$

$\text{res}_p(M)$

x to $\mathbb{R}^m$

$s' \rightarrow \mathbb{R}^n$

$\mathbb{R}^m$

$\frac{dx}{dt}(0)$

$x(0)$ using a/some chart

d can
th

Tangent spaces 1: for submanifolds $M \subseteq \mathbb{R}^n$

$$T_p^{\text{geom}} M = \left\{ \frac{du}{dt}(0) : u: (-\varepsilon, \varepsilon) \rightarrow M \text{ smooth, } u(0) = p \right\} \subseteq \mathbb{R}^n \quad p \in M$$

$$\left(\frac{du}{dt}(0) \in \mathbb{R}^n \right)$$

Rk: we picture $T_p^{\text{geom}} M$ as passing through p as origin

$$T^{\text{geom}} M = \{ (p, u) : p \in M, u \in T_p^{\text{geom}} M \}$$

Rk: For subsets $M \subseteq \mathbb{R}^n$, $T_p^{\text{geom}} M$ may fail to be a vector space.

$$M = \{ (x, y) : x y = 0 \}$$

Example: If M is as in the RVT, then:

$$T_p^{\text{geom}} M = \text{Ker} (dF)_* : \mathbb{R}^n \rightarrow \mathbb{R}^k$$

Why? " $\subseteq$ " $\frac{du}{dt}(0) \xrightarrow{(dF)_*} \frac{d(F \circ u)}{dt}(0) = 0$

Compare dimensions $(dF)_* = \text{surj} \Rightarrow$ the RHS has dimension $n - k = m$.

Exercise: $M, M' \subseteq \text{Euclidean spaces}$, $F: M \rightarrow M'$ smooth ... what would be $(dF)_*$? $\rightarrow ??$

Reminder: Sur

$$(\forall) p \in M, \exists \tilde{x}$$

$\mathbb{R}^n$ can be re
by an arb

Thm(RVT):

Consider: M

Assume: c

Conclusion:

and its t

$T_p^{\text{geom}} M$

$T_p^{\text{geom}} M$

$T_p^{\text{geom}} M$

$T_p^{\text{geom}} M$

$T_p^{\text{geom}} M$

$\mathbb{R}^n$
 $\mathbb{R}^n$
 $p \in M$
 $T_p^{\text{geom}} M$
 $T_p M$
 $T_{F(p)} M$
 $(dF)_x$
 $be (dF)_x$

Reminder: Submanifold M of $\mathbb{R}^n$: if
 $(\forall) p \in M, \exists \tilde{\alpha}: \underbrace{\tilde{U}}_{\mathbb{R}^n} \rightarrow \underbrace{\tilde{\Omega}}_{\mathbb{R}^n}$ $\hookleftarrow$ smooth chart around p st.
 $M \cap \tilde{U} = \{q \in \tilde{U} : \tilde{\alpha}_{m+1}(q) = \dots = \tilde{\alpha}_n(q) = 0\}$

$\mathbb{R}^n$ can be replaced by an arbitrary N
 (or: $\chi|_U: U \rightarrow \Omega$ chart of M , where $\Omega = \tilde{\Omega}(\mathbb{R}^m \times \{0\})$)

Thm(RVT): $F: \Omega \rightarrow \mathbb{R}^k$ smooth, $\Omega \subseteq \mathbb{R}^n, n \geq k$.

Consider: $M = \text{Zero}(F) = \{x \in \Omega : F(x) = 0\}$

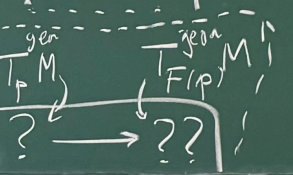
Assume: 0 is a regular value of F (i.e. $(dF)_x = \text{surjective}$) $(\forall) x \in M$

Conclusion: M is a submanifold of $\mathbb{R}^n$ of dimension $m = n - k$.

and its tangent spaces at $p \in M$ are

$$T_p^{\text{geom}} M = \text{Ker}(dF)_p$$

$$(dF)_x: \mathbb{R}^n \rightarrow \mathbb{R}^k$$



olds

2

Tangent spaces 1: for submanifolds $M \subseteq \mathbb{R}^n$

1

$$T_p^{\text{geom}} M = \left\{ \frac{du}{dt}(0) : u: (-\varepsilon, \varepsilon) \rightarrow M \text{ smooth, } u(0)=p \right\} \subseteq \mathbb{R}^n \quad p \in M$$

$\left(\frac{du}{dt}(0) \in \mathbb{R}^n \right)$

Rk: we picture $T_p^{\text{geom}} M$ as passing through p as origin

$$T_p^{\text{geom}} M = \{ (p, u) : p \in M, u \in T_p^{\text{geom}} M \}$$

Rk: For subsets $M \subseteq \mathbb{R}^n$, $T_p^{\text{geom}} M$ may fail to be a vector space

$$M = \{ (x, y) : y \geq 0 \}$$

Example: If M is as in the RVT, then

$$T_p^{\text{geom}} M = \text{Ker}((dF)_* : \mathbb{R}^n \rightarrow \mathbb{R}^k)$$

Why? " $\subseteq$ " $\frac{du}{dt}(0) \xrightarrow{(dF)_*} \frac{d(F \circ u)}{dt}(0) = 0$

Compare dimensions $(dF)_* = \text{surj} \Rightarrow$ the RHS has dimension $n-k=m$

Exercise: $M, M' \subseteq \text{Euclidean spaces}$, $F: M \rightarrow M'$ smooth ... what would be $(dF)_* : ? \rightarrow ??$

Reminder: Submanifold M of $\mathbb{R}^n$: if

$(\forall) p \in M, \exists \tilde{\alpha} : \tilde{U} \rightarrow \tilde{\mathbb{R}}^n$ $\tilde{\alpha}$ -smooth chart around p st

$$M \cap \tilde{U} = \{ q \in \tilde{U} : \tilde{\alpha}_{m+1}(q) = \dots = \tilde{\alpha}_n(q) = 0 \}$$

$\mathbb{R}^n$ can be replaced by an arbitrary N

(or $\chi|_U : U \rightarrow \mathbb{R}^k$ chart of M , where $\mathbb{R}^n = \tilde{\mathbb{R}}^n \times \mathbb{R}^k$)

Thm(RVT): $F: \mathcal{N} \rightarrow \mathbb{R}^k$ smooth, $\mathcal{N} \subseteq \mathbb{R}^n, n \geq k$

Consider: $M = \text{Zero}(F) = \{ x \in \mathcal{N} : F(x) = 0 \}$

Assume: 0 is a regular value of F (i.e. $(dF)_* = \text{surjective}$)

Conclusion: M is a submanifold of $\mathbb{R}^n$ of dimension $m = n-k$

and its tangent spaces at $p \in M$ are

$$T_p^{\text{geom}} M = \text{Ker}(dF)_p$$

$$(dF)_* : \mathbb{R}^n \rightarrow \mathbb{R}^k$$

using a/s chart

can

Tangent spaces for general manifolds

M = general manifold, $p \in M$

$\text{Curv}_p(M) = \{ \gamma: (-\epsilon, \epsilon) \rightarrow M \mid \gamma \text{ smooth, } \gamma(0) = p \}$

Aim: make sense of $\frac{d\gamma}{dt}(0)$ for $\gamma \in \text{Curv}_p(M)$

① Use charts (U, χ) of M around p , move γ to $\mathbb{R}^m$

$$\begin{array}{ccc} \chi & \xrightarrow{\chi} & \mathbb{R}^m \\ M & & \mathbb{R}^m \end{array}$$

$$\gamma^x = \chi \circ \gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^m$$

$$\text{and consider } \boxed{\frac{d\gamma^x}{dt}(0) \in \mathbb{R}^m}$$

Problem: this depends on the chart (U, χ)

$$\gamma^x = \chi \circ \gamma = \chi \circ \chi^{-1} \circ \chi \circ \gamma \Rightarrow \frac{d\gamma^x}{dt}(0) = (d\chi)_{\gamma(0)} \left(\frac{d\gamma}{dt}(0) \right)$$

calculus

Hence I cannot define / make sense of $\frac{d\gamma}{dt}(0)$ as $\frac{d\gamma^x}{dt}(0)$ using a local chart

② But: saying that $\gamma_1, \gamma_2 \in \text{Curv}_p(M)$ have the same speed can be defined using charts

$$\begin{aligned} \text{if for charts } (U, \chi) \text{ around } p \quad \frac{d\gamma_1^x}{dt}(0) &= \frac{d\gamma_2^x}{dt}(0) \\ \text{if } \chi \text{ another chart } \frac{d\gamma_1^{x'}}{dt}(0) &= (d\chi')_{\gamma(0)} \left(\frac{d\gamma_1}{dt}(0) \right) \\ \frac{d\gamma_1^{x'}}{dt}(0) &= (d\chi')_{\gamma(0)} \left(\frac{d\gamma_2}{dt}(0) \right) \end{aligned}$$

③ Define the equivalence relation on $\text{Curv}_p(M)$

$\gamma_1 \sim \gamma_2$ iff they have the same speed w.r.t. a / any chart of M around p

For $\gamma \in \text{Curv}_p(M)$ denote by $\frac{d\gamma}{dt}(0)$ the equivalence class of γ w.r.t. $\sim$

Define the tangent space of M at p

$$T_p M = \text{Curv}_p(M) / \sim = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curv}_p(M) \right\}$$

abstract for now

Re: γ_1, γ_2
 $\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0)$
 $\Downarrow$
 the same is true in charts

T_p

Rk: we have T_p

Rk: For a vector

Example

Why?

Compare Exercise

Tangent spaces 2: for general manifolds

M = general manifold, $p \in M$? $T_p M$?

$\text{Curv}_p(M) = \{ \gamma: (-\epsilon, \epsilon) \rightarrow M \mid \gamma \text{ smooth, } \gamma(0) = p \}$

Aim: make sense of $\frac{d\gamma}{dt}(0)$ for $\gamma \in \text{Curv}_p(M)$

① Use charts (U, χ) of M around p , move γ to $\mathbb{R}^m$

$$\begin{array}{ccc} \gamma: \mathbb{R} \rightarrow M & \xrightarrow{\chi} & \chi \circ \gamma: \mathbb{R} \rightarrow \mathbb{R}^m \\ \gamma^x = \chi \circ \gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^m & & \end{array}$$

and consider $\frac{d\gamma^x}{dt}(0) \in \mathbb{R}^m$

Problem: this depends on the chart (U, χ)

$$\gamma^y = \chi' \circ \gamma = \chi' \circ \chi^{-1} \circ \gamma^x \Rightarrow \frac{d\gamma^y}{dt}(0) = (d\chi')_{\gamma^x(0)} \left(\frac{d\gamma^x}{dt}(0) \right)$$

Hence I cannot define/make sense of $\frac{d\gamma}{dt}(0)$ using a/some chart.

Tangent spaces 1: for submanifolds $M \subseteq \mathbb{R}^n$

$$T_p^{\text{geom}} M = \left\{ \frac{du}{dt}(0) : u: (-\epsilon, \epsilon) \rightarrow M \text{ smooth, } u(0) = p \right\} \subseteq \mathbb{R}^n \quad p \in M$$

Rk: we picture $T_p^{\text{geom}} M$ as passing through p on origin

$T_p^{\text{geom}} M = \{ (p, u) : p \in M, u \in T_p^{\text{geom}} M \}$

Rk: For subsets $M \subseteq \mathbb{R}^n$, $T_p^{\text{geom}} M$ may fail to be a vector space

Example: If M is as in the RVT, then

$$T_p^{\text{geom}} M = \text{Ker}((dF)_x: \mathbb{R}^n \rightarrow \mathbb{R}^k)$$

Why? "≤" $\frac{du}{dt}(0) \mid (dF)_x \rightarrow \frac{d(F \circ u)}{dt}(0) = 0$

Compare dimensions $(dF)_x = \text{surj} \Rightarrow$ the RHS has dimension $n-k=m$

Exercise: $M, M' \subseteq \mathbb{R}^n$ Euclidean spaces, $F: M \rightarrow M'$ smooth ... what would be $(dF)_x$?

Reminder: Submanifold M of $\mathbb{R}^n$: if

(V) $p \in M, \exists \tilde{\chi}: \tilde{U} \rightarrow \tilde{\mathbb{R}}^n$ $\hookleftarrow$ smooth chart around p st

$M \cap \tilde{U} = \{ q \in \tilde{U} : \tilde{\chi}(q) = \tilde{\chi}(p) + \tilde{\chi}(q) - \tilde{\chi}(p) \}$

$\mathbb{R}^n$ can be replaced by an arbitrary N (or $\chi|_U: U \rightarrow \mathbb{R}^n$ chart of M , where $\mathbb{R}^n = \tilde{\mathbb{R}}^n \times \{0\}$)

Thm(RVT): $F: \mathcal{N} \rightarrow \mathbb{R}^k$ smooth, $\mathcal{N} \subseteq \mathbb{R}^n, n \geq k$

Consider: $M = \text{Zero}(F) = \{ x \in \mathcal{N} : F(x) = 0 \}$

Assume: 0 is a regular value of F (i.e. $(dF)_x = \text{surjective}$)

Conclusion: M is a submanifold of $\mathbb{R}^n$ of dimension $m = n - k$

and its tangent spaces at $p \in M$ are

$$T_p^{\text{geom}} M = \text{Ker}(dF_p) \quad (dF)_x: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

② But: saying that $\gamma, \gamma' \in \text{Curv}_p(M)$ have the same speed can

be defined using charts

if for charts (U, χ) around p $\frac{d\gamma^x}{dt}(0) = \frac{d\gamma'^x}{dt}(0)$

If χ' another chart $\frac{d\gamma'^x}{dt}(0) = (d\chi')_{\gamma^x(0)} \left(\frac{d\gamma^x}{dt}(0) \right)$

$$\frac{d\gamma'^x}{dt}(0) = (d\chi')_{\gamma^x(0)} \left(\frac{d\gamma^x}{dt}(0) \right)$$

③ Define the equivalence relation on $\text{Curv}_p(M)$

$\gamma_1 \sim \gamma_2$ iff they have the same speed w.r.t. a/any chart of M around p

For $\gamma \in \text{Curv}_p(M)$ denote by $\frac{d\gamma}{dt}(0)$ the equivalence class of γ w.r.t. $\sim$

Define the tangent space of M at p

$$T_p M = \text{Curv}_p(M) / \sim = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curv}_p(M) \right\}$$

abstract for now

Rk: γ_1, γ_2
 $\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0)$
 $\Downarrow$
the same is true in charts

Problem: this depends on the chart (U, χ)

$$\gamma^{\chi'} = \chi' \circ \gamma = \chi' \circ \chi^{-1} \circ \chi \circ \gamma \Rightarrow \frac{d\gamma^{\chi'}}{dt}(0) = (dc)_{\chi(p)} \left(\frac{d\gamma^{\chi}}{dt}(0) \right)$$

calculus

Hence γ cannot define/make sense of $\frac{d\gamma}{dt}(0)$ using charts

Why? " $\leq$ " $\frac{du}{dt}(0) \xrightarrow{(dF)_*} \frac{d(F \circ u)}{dt}(0) = 0$

Compare dimensions: $(dF)_* = \text{surj} \Rightarrow$ the RHS has dimension $n-k=m$.

Exercise: $M, M' \subseteq \text{Euclidean spaces}$, $F: M \rightarrow M'$ smooth ... what would be $(dF)_*$

(2) But: Saying that $\gamma_1, \gamma_2 \in \text{Curves}_p(M)$ have the same speed can be defined using charts

if for charts (U, χ) around p $\frac{d\gamma_1^{\chi}}{dt}(0) = \frac{d\gamma_2^{\chi}}{dt}(0)$

If χ' -another chart $\frac{d\gamma_1^{\chi'}}{dt}(0) = (dc)_{\chi(p)} \left(\frac{d\gamma_1^{\chi}}{dt}(0) \right)$

$\frac{d\gamma_2^{\chi'}}{dt}(0) = (dc)_{\chi(p)} \left(\frac{d\gamma_2^{\chi}}{dt}(0) \right)$

(3) Define the equivalence relation on $\text{Curves}_p(M)$

$\gamma_1 \sim \gamma_2$ iff they have the same speed w.r.t. a / any chart of M around p

For $\gamma \in \text{Curves}_p(M)$ denote by $\frac{d\gamma}{dt}(0)$ the equivalence class of γ w.r.t. $\sim$

Define the tangent space of M at p

$$T_p M = \text{Curves}_p(M) / \sim = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}_p(M) \right\}$$

the same is true in charts

Def: For $v \in T_p M$, (U, χ) -chart around p the representation of v w.r.t. χ in the chart χ is

$$v^{\chi} := \frac{d(\gamma^{\chi})}{dt}(0) \in \mathbb{R}^m \quad (\text{usual } \frac{d}{dt} \text{ of curves in } \mathbb{R}^m)$$

Have seen

$$v^{\chi'} = (dc)_{\chi(p)}(v^{\chi}) \quad \text{where } c = \chi' \circ \chi^{-1} : \text{open } \mathbb{R}^m \rightarrow \text{open } \mathbb{R}^m$$

Prop: Denoting by $\text{Chart}_p(M) \leftarrow$ all charts around p then for any map $V: \text{Chart}_p(M) \rightarrow \mathbb{R}^m$, $\chi \mapsto V(\chi)$

with the properties: $V(\chi') = (dc)_{\chi(p)}(V(\chi))$ $(\forall \chi, \chi' \in \text{Chart}_p(M))$ $(c = \chi' \circ \chi^{-1})$

$\exists! v \in T_p M$ such that $V(\chi) = v^{\chi}$ $(\forall \chi \in \text{Chart}_p(M))$

proof? $\exists$: Fix $\chi_0 \in \text{Chart}_p(M)$. Choose any $v \in T_p M$ s.t. $v^{\chi_0} = V(\chi_0) \in \mathbb{R}^m$

For any other χ $v^{\chi} = (dc)_{\chi_0(p)}(v^{\chi_0}) = V(\chi)$

$c = \chi \circ \chi_0^{-1}$

$$v_1 = \frac{d\gamma_1}{dt}(0) \quad v_2 = \frac{d\gamma_2}{dt}(0)$$

Another Prop: V is a linear map. By def. Useful basis

using charts
Compare dimensions: $(dF)_* = \text{surj} \Rightarrow$ the RHS has dimension $n-k=m$
Exercise: $M, M' \subseteq \text{Euclidean spaces}$, $F: M \rightarrow M'$ smooth ... what would be $(dF)_*$?

Standard for $\mathbb{R}^n$ restrict to $\text{Standard}_p^M: T_p M \rightarrow T_p M$

Def: For $v \in T_p M$, (U, χ) -chart around p (g^x)
the representation of v w.r.t. χ is

Have seen $v^x := \frac{d}{dt} \gamma^x(0) \in \mathbb{R}^m$ (usual $\frac{d}{dt}$ of curves in $\mathbb{R}^m$)
 $v^x = (dc)_{\chi(p)}(v^x)$ where $c = c_\chi^x = \chi \circ \chi^{-1}$ open $\rightarrow$ open $\mathbb{R}^m \rightarrow \mathbb{R}^m$

Prop: Denoting by $\text{Chart}_p(M) \leftarrow$ all charts around p
then any map $V: \text{Chart}_p(M) \rightarrow \mathbb{R}^m$, $\chi \mapsto V(\chi)$
with the properties: $V(\chi') = (dc)_{\chi(p)}(V(\chi))$ $(\forall \chi, \chi' \in \text{Chart}_p(M))$
 $(c = \chi' \circ \chi^{-1})$

$\exists!$ $v \in T_p M$ such that $V(\chi) = v^x$ $(\forall \chi \in \text{Chart}_p(M))$
 proof? $\exists$: Fix $\chi_0 \in \text{Chart}_p(M)$. Choose any $v \in T_p M$ s.t. $v^{x_0} = V(\chi_0) \in \mathbb{R}^m$
 For any other χ $v^x = (dc)_{\chi_0(p)}(v^{x_0}) = V(\chi)$ $v_1 = \frac{dx_1}{dt}(0)$
 $v_2 = \frac{dx_2}{dt}(0)$
 $c = \chi \circ \chi_0^{-1}$

One interesting consequence: we can make $T_p M$ into $\text{a } \mathbb{R}^m$ space with $+$ and $\cdot$ defined so that
 $v, w \in T_p M$ $(v+w)^x = v^x + w^x$ the addition in $\mathbb{R}^m$
 $\lambda \in \mathbb{R}, v \in T_p M$ $(\lambda \cdot v)^x = \lambda \cdot v^x$ $(\forall \chi \in \text{Chart}_p(M))$

Another one:
 Prop: For any $\chi \in \text{Chart}_p(M)$ the map
 $T_p M \rightarrow \mathbb{R}^m, v \mapsto v^x$
 is a linear isomorphism.

The map is linear by def of $+$ and $\cdot$.
 By before: it is injective, surjective.
 Useful:

① $\Rightarrow \dim(T_p M) = m$ and any $\chi \in \text{Chart}_p(M)$ induces a basis of $T_p M$. Denote it by $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p$.

$\text{Chart}_p(M) \rightarrow \mathbb{R}^m$
 $\chi \mapsto v^x + w^x$
 Use this as $V \rightsquigarrow$
 $\rightsquigarrow$ an elt in $T_p M$
 call it $v+w$

$\{x\}$

One interesting consequence: we can make $T_p M$ into a $\mathbb{R}^m$ space with $+$ and $\cdot$ defined so that

$$v, w \in T_p M \quad (v+w)^x = v^x + w^x \quad \text{the addition in } \mathbb{R}^m$$

$$\lambda \in \mathbb{R}, v \in T_p M \quad (\lambda \cdot v)^x = \lambda \cdot v^x \quad (\forall x \in \text{Chart}_p(M))$$

Another one:

Prop: For any $x \in \text{Chart}_p(M)$ the map

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is a linear isomorphism.

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$$\text{Chart}_p(M) \rightarrow \mathbb{R}^m$$

$$x \mapsto v^x + w^x$$

Use this as $V \rightarrow$
 $\rightarrow$ an elt in $T_p M$
call it $v+w$

①

$p \in M$

$T_p^{\text{geom}} M$

In other words, they are the unique

$$(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p \in T_p M$$

Such that: $(\frac{\partial}{\partial x_i})_p^x = (0, \dots, 0, 1, 0, \dots, 0)$
with place i .

Exercise: $\gamma_i: (-\varepsilon, \varepsilon) \rightarrow M, \gamma_i(t) = x^{-1}(0, \dots, 0, t, 0, \dots, 0) \mid x(p) = 0$

represent $(\frac{\partial}{\partial x_i})_p$ i.e.: $(\frac{\partial}{\partial x_i})_p = \frac{d\gamma_i}{dt}(0)$.

② For $M = \mathbb{R}^m, x = \text{id} \Rightarrow$ get an

isomorphism $\text{Standard}_p: T_p \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$ the standard identification

Notice that $\frac{dx}{dt}(0) \mapsto$ the classical $\frac{dx}{dt}(0) \in \mathbb{R}^n$ (calculus)

For $M \subseteq \mathbb{R}^n$ submanifold the equiv. class of

Standard for $\mathbb{R}^n$ restrict to $\text{Standard}_p^M: T_p M \rightarrow T_p^{\text{geom}} M$