

Reminder: $M = m$ dim. manifold, $p \in M \rightsquigarrow$ the tangent space $T_p M$ (of M at p)

→ it is an m -dimensional vector space

→ what are the elements $v \in T_p M$?

① speeds $\frac{dx}{dt}(0)$ with $\gamma \in \text{Curves}_p(M)$

Keep in mind: $\frac{dx_1}{dt}(0) = \frac{dx_2}{dt}(0) \iff \frac{dx_1^x}{dt}(0) = \frac{dx_2^x}{dt}(0)$ for a/any $x \in \text{Chart}_p(M)$ (*)

② maps $\text{Chart}_p(M) \rightarrow \mathbb{R}^m$, $x \mapsto v^x$ satisfying

$$v^{x'} = (dc)_{x(p)}(v^x) \quad (\forall x, x' \in \text{Chart}_p(M)) \quad (\text{where } c = c_{x'}^x = x' \circ x^{-1})$$

→ one (any) chart $x \in \text{Chart}_p(M)$ induces

• a linear isomorphism $I^x: T_p M \xrightarrow{\sim} \mathbb{R}^m$, $v \mapsto v^x$

• in particular: a basis of $T_p M$, denoted

$$\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p \quad \left(\begin{array}{l} \text{the canonical basis of } T_p M \\ \text{w.r.t. the chart } x \end{array} \right)$$

$\text{Curves}_p(M)$ = the collection of (smooth) curves in M starting at p ($\gamma(0) = p$)

$\text{Chart}_p(M)$ = the collection of all charts of M around p

$$C^\infty(M) = \{ f: M \rightarrow \mathbb{R} \mid f = \text{smooth} \}$$

"speed" introduced by brute

force so that the left part holds true:

$$T_p M = \text{Curves}_p(M) / \sim_p \quad \& \quad \frac{dx}{dt}(0) := \text{the equivalence class of } \gamma \text{ w.r.t. } \sim_p$$

where the equiv. rel. is defined by (*).

if $v = \frac{dx}{dt}(0)$ then $v^x = \text{usual } \frac{dx^x}{dt}(0) \in \mathbb{R}^m$

"the tangent vector v (represented) in the chart x "

how does this basis change if we change x ?

EXERCISE

→ in particular, when $M = \mathbb{R}^n$, $x = \text{Id}_{\mathbb{R}^n} \Rightarrow$ get and basis $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_n} \right)_p$ of $T_p \mathbb{R}^n$

→ when $M \subseteq \mathbb{R}^n$ embedded

one has $T_p M \subseteq T_p \mathbb{R}^n$, $\forall p \in M$

& Standard_p restricts to

$$\text{Standard}_p: T_p \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$$

$$\left(\frac{\partial}{\partial x_i} \right)_p \longleftrightarrow e_i = (0, \dots, 1, \dots, 0)$$

the standard identifications

subspace

Reminder: $M = m$ dim. manifold, $p \in M \rightsquigarrow$ the tangent space $T_p M$ (of M at p)

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① speeds $\frac{dx}{dt}(0)$ with $\gamma \in \text{Curves}_p(M)$

& keep in mind: $\frac{dx_1}{dt}(0) = \frac{dx_2}{dt}(0) \iff \boxed{\frac{dx_1^x}{dt}(0) = \frac{dx_2^x}{dt}(0)}$ for a/any $\chi \in \text{chart}_p(M)$ (*)

② maps $\text{chart}_p(M) \rightarrow \mathbb{R}^m, \chi \mapsto \chi^x$ satisfying

$$\chi^x = (dc)_{\chi(p)}(\chi^x) \quad (\forall) \chi, \chi' \in \text{chart}_p(M) \quad (\text{where } c = c_{\chi'}^{\chi} = \chi' \circ \chi^{-1})$$

$$(\chi + \chi')^x = \chi^x + \chi'^x$$

$\rightarrow$ one (any) chart $\chi \in \text{chart}_p(M)$ induces

• a linear isomorphism $I^x: T_p M \xrightarrow{\sim} \mathbb{R}^m, v \mapsto v^x$

• in particular: a basis of $T_p M$, denoted

$$\left(\frac{\partial}{\partial x_i} \right)_p, \left(\frac{\partial}{\partial x_m} \right)_p \quad \left(\text{the canonical basis of } T_p M \right) \quad \left(\text{w.r.t. the chart } \chi \right)$$

$\text{Curves}_p(M)$ = the collection

$\text{chart}_p(M)$ = the collection

$C^\infty(M) = \{ f: M \rightarrow \mathbb{R} \}$

"speed" introduced by brut force so that the left

$T_p M = \text{Curves}_p(M)$

where the equiv. rel.

if $v = \frac{dx}{dt}(0)$ the

how does this change if we

EXERCISE

$\rightarrow$ in particular, when $M = \mathbb{R}^n, \chi = \text{Id}_{\mathbb{R}^n} \Rightarrow$ get

and basis $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p$ of $T_p \mathbb{R}^n$

$\rightarrow$ when $M \subseteq \mathbb{R}^n$ embedded

one has $T_p M \subseteq T_p \mathbb{R}^n, \forall p \in M$

& Standard_p restricts to

$$\text{Standard}_p|_{T_p M}: T_p M \xrightarrow{\sim} T_p^{\text{geom}} M$$

Standard_p^M

Rk: Why the $\left(\frac{\partial}{\partial x_i} \right)_p$ notation?

Key: any $v \in T_p M$ given rise to

Thind way to look at $v \in T_p M$

the standard identification

$$\left(\frac{\partial}{\partial x_i} \right)_p \longleftrightarrow e_i$$

$$e_i = \begin{pmatrix} 0, \dots, 1, \dots, 0 \end{pmatrix}$$

subspace

and basis $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p$ of $T_p \mathbb{R}^n$

→ when $M \subseteq \mathbb{R}^n$ embedded
one has $T_p M \subseteq T_p \mathbb{R}^n, \forall p \in M$

& Standard_p restricts to

Standard_p|_{T_pM} : $T_p M \xrightarrow{\sim} T_p M$

Standard_p^M

subspace

Rk: Why the $(\frac{\partial}{\partial x_i})_p$ notation? Third way to look at $v \in T_p M$.

Key: any $u \in T_p M$ gives rise to an operation

$$\partial_u : C^\infty(M) \rightarrow \mathbb{R}, f \mapsto \partial_u(f)$$

where if we use viewpoint ①
then write $u = \frac{dx}{dt}(t_0)$

if we use viewpoint ②
i.e. work with γ_x
for $x \in \text{Chart}_p(M)$

then $\partial_u(f) := \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) = (f \circ \gamma)'(0)$

for $\gamma : \mathbb{R} \rightarrow M$

then $\partial_u(f) = \text{the usual } (df_x)(\gamma'(0))$

classical

Ex: $M = \mathbb{R}^n, x = \text{Id}_{\mathbb{R}^n}$ then

$$(\frac{\partial}{\partial x_i})_p \in T_p \mathbb{R}^n$$

induce $C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}, f \mapsto \frac{\partial f}{\partial x_i}(p)$

In general: tgt vector $v \in T_p M$

↔ operations $\partial : C^\infty(M) \rightarrow \mathbb{R}$ satisfying some conditions

Reminder: $M = m$ dim. manifold, $p \in M \rightsquigarrow$ the tangent space $T_p M$ (of M at p)

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$\rightarrow$ what are the elements $v \in T_p M$?

① speeds " $\frac{dx}{dt}(0)$ " with $\gamma \in \text{Curves}_p(M)$

Keep in mind: $\frac{dx_1}{dt}(0) = \frac{dx_2}{dt}(0) \iff \frac{dx_1^x}{dt}(0) = \frac{dx_2^x}{dt}(0)$ for a/any $x \in (\text{chart}_p(M))$ (*)

② maps $(\text{chart}_p(M) \rightarrow \mathbb{R}^m, x \mapsto x^x)$ satisfying

$x^x = (dc)_{x(p)}(v^x) \quad (\forall) x, x' \in (\text{chart}_p(M)) \quad (\text{where } c = c_{x'}^x = x'^{-1} \circ x^{-1})$

$(v + w)^x = v^x + w^x$

$\rightarrow$ One (any) chart $x \in (\text{chart}_p(M))$ induces

- a linear isomorphism $I^x: T_p M \xrightarrow{\sim} \mathbb{R}^m, v \mapsto v^x$
- in particular: a basis of $T_p M$, denoted $(\frac{\partial}{\partial x_i})_p, (\frac{\partial}{\partial x_m})_p$ (the canonical basis of $T_p M$ w.r.t. the chart x)

$\text{Curves}_p(M) = \{ \gamma: \mathbb{R} \rightarrow M \mid \gamma(0) = p \}$

$\text{Chart}_p(M) = \{ (U, x) \mid p \in U \}$

$C^\infty(M) = \{ f: M \rightarrow \mathbb{R} \}$

"speed" introduced force so that

$T_p M =$

where the

if $v = \frac{dx}{dt}$

how does change

Ex:

$\rightarrow$ in particular, when $M = \mathbb{R}^n, x = \text{Id}_{\mathbb{R}^n} \Rightarrow$ get $\text{Standard}_p: T_p \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$

and basis $(\frac{\partial}{\partial x_i})_p, (\frac{\partial}{\partial x_n})_p$ of $T_p \mathbb{R}^n$

$\rightarrow$ when $M \subseteq \mathbb{R}^n$ embedded one has $T_p M \subseteq T_p \mathbb{R}^n, \forall p \in M$

& Standard_p restricts to $\text{Standard}_p|_{T_p M}: T_p M \xrightarrow{\sim} \mathbb{R}^n$

Standard_p^M

the standard identification

subspace

Re: Why the $(\frac{\partial}{\partial x_i})_p$ notation? Third way to look at $v \in T_p M$.

Key: any $v \in T_p M$ gives rise to an operation $\partial_v: C^\infty(M) \rightarrow \mathbb{R}, f \mapsto \partial_v(f)$

where

- if we use viewpoint ① then write $v = \frac{dx}{dt}(0)$ then $\partial_v(f) := \frac{d}{dt} f(x(t)) \Big|_{t=0} = (f \circ \gamma)'(0)$
- if we use viewpoint ② i.e. work with v^x for $x \in (\text{chart}_p(M))$ then $\partial_v(f) = \text{the usual } \partial_{v^x}(f_x) = (df_x)_{x(p)}(v^x)$ classical

Ex: $M = \mathbb{R}^n$

$(\frac{\partial}{\partial x_i})$

induce

In ge

$\leftarrow$

Reminder: $M = m$ dim manifold, $p \in M \mapsto$ the tangent space $T_p M$ (of M at p)

→ it is an m -dimensional vector space
 → what are the elements $v \in T_p M$?

① speeds $\frac{dx}{dt}(t)$ with $x \in \text{Curves}(M)$

Keep in mind: $\frac{dx}{dt}(t) = \frac{dx}{dt}(t) \iff \frac{dx}{dt}(t) = \frac{dx}{dt}(t)$ for a/any $x \in \text{Curves}(M)$

② maps $\text{Curves}(M) \rightarrow \mathbb{R}^m, x \mapsto x^*$ satisfying
 $x^* = (dc_x)(x^*) \iff (x)x^* \in \text{Curves}(M)$ (where $c \circ x^* = x^*, x^*$)

→ one (any) chart $X \in \text{Charts}(M)$ induces

- a linear isomorphism $I^X: T_p M \xrightarrow{\sim} \mathbb{R}^m, v \mapsto v^X$
- in particular, a basis of $T_p M$, denoted $\left(\frac{\partial}{\partial x_i}\right)_p$ (the canonical basis of $T_p M$ w.r.t the chart X)

$\text{Curves}_p(M) =$ the collection of (smooth) curves in M starting at p

$\text{Charts}_p(M) =$ the collection of all charts of M around p
 $C^\infty(M) = \{f: M \rightarrow \mathbb{R} \mid f \text{ is smooth}\}$

"speed" introduced by brute force $\Rightarrow$ that the left part holds true:
 $T_p M = \text{Curves}_p(M) / \sim_p$ & $\frac{dx}{dt}(t) =$ the equivalence class of x w.r.t $\sim_p$

where the equiv. is defined by $(x) \sim (y) \iff \frac{dx}{dt}(t) = \frac{dy}{dt}(t)$

if $v = \frac{dx}{dt}(t)$ then $v^X =$ usual $\frac{dx}{dt}(t) \in \mathbb{R}^m$
 "the tangent vector is represented in the chart X "

how does this basis change if we change X ?

EXERCISE

Induces linear map
 (for $M = \mathbb{R}^m, N = \mathbb{R}^n$, wind th

→ in particular, when $M = \mathbb{R}^n, X = \text{id}_{\mathbb{R}^n} \Rightarrow$ get $\text{Standard}_p: T_p \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$
 and basis $\left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p$ of $T_p \mathbb{R}^n$

→ when $M \subseteq \mathbb{R}^n$ embedded, one has $T_p M \subseteq T_p \mathbb{R}^n$ w.r.t $p \in M$
 & Standard_p restricts to $\text{Standard}_p|_{T_p M}: T_p M \xrightarrow{\sim} \mathbb{R}^n$

FA: Why the $\left(\frac{\partial}{\partial x_i}\right)_p$ notation? Third way to look at $v \in T_p M$

Key: any $v \in T_p M$ gives rise to an operation

$$\partial_v: C^\infty(M) \rightarrow \mathbb{R}, f \mapsto \partial_v(f)$$

where if we use vector field $\frac{d}{dt}$ then with $v = \frac{dx}{dt}(t)$ then $\partial_v(f) := \frac{d}{dt} f(x(t)) = (f \circ x)'(t)$

if we use vector field $\frac{d}{dt}$ i.e. work with real for $x \in \text{Charts}(M)$ then $\partial_v(f) =$ the usual $\frac{dx}{dt}(f_x) = \left(\frac{df_x}{dx}\right)(v^x)$

the standard identification $\left(\frac{\partial}{\partial x_i}\right)_p \mapsto e_i$

subspace $T_p M \subseteq T_p \mathbb{R}^n$

$\text{Standard}_p|_{T_p M}: T_p M \xrightarrow{\sim} \mathbb{R}^n$

Ex: $M = \mathbb{R}^n, X = \text{id}_{\mathbb{R}^n}$ then $\left(\frac{\partial}{\partial x_i}\right)_p \in T_p \mathbb{R}^n$

induce $C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}, f \mapsto \frac{\partial f}{\partial x_i}(p)$

In general: tangent vector $v \in T_p M$

$\mapsto$ operations $\partial: C^\infty(M) \rightarrow \mathbb{R}$ satisfying some conditions

Reminder: $M = m$ -dim manifold, $p \in M \mapsto$ the tangent space $T_p M$ (of M at p)

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① speeds $\frac{dx}{dt}(t)$ with $x \in \text{Curves}_p(M)$

Δ keep in mind: $\frac{dx}{dt}(t) = \frac{dx}{dt}(0) \iff \frac{dx^i}{dt}(t) = \frac{dx^i}{dt}(0)$ for all i $\iff$ $\frac{dx^i}{dt}(0) = \frac{dx^i}{dt}(0)$ for all i $\iff$ $\frac{dx^i}{dt}(0) = \frac{dx^i}{dt}(0)$ for all i

② maps $\text{Curves}_p(M) \rightarrow \mathbb{R}^n, x \mapsto x^i$ satisfying

$$x^i = (dc)(v^i) \quad (v) x^i \in \text{Curves}_p(M) \quad (\text{where } c \circ c_x^i = x^i)$$

$\rightarrow$ one (any) chart $x \in \text{Chart}_p(M)$ induces

• a linear isomorphism $I^x: T_p M \xrightarrow{\sim} \mathbb{R}^m, v \mapsto v^i$

• in particular, a basis of $T_p M$, denoted the canonical basis of $T_p M$ w.r.t the chart x

$$\left(\frac{\partial}{\partial x^i} \right)_p, \left(\frac{\partial}{\partial x^j} \right)_p$$

$\text{Curves}_p(M)$ = the collection of (smooth) curves in M starting at p $\frac{dx}{dt}(0) = p$

$\text{Chart}_p(M)$ = the collection of all charts of M around p
 $C^\infty(M) = \{ f: M \rightarrow \mathbb{R} \mid f \text{ smooth} \}$

"speed" introduced by brute force so that the left part holds true:
 $T_p M = \text{Curves}_p(M) / \sim_p$ & $\frac{dx}{dt}(0) =$ the equivalence class of x w.r.t $\sim_p$

where the equivalence is defined by $(u) \sim (v)$ iff $u = \frac{dx}{dt}(0)$ then $v^i =$ usual $\frac{dx^i}{dt}(0) \in \mathbb{R}^m$

"the tangent vector v (represented) in the chart x "

how does this basis change if we change x ?

EXERCISE

Differentials of maps: any smooth $F: M \rightarrow N$ $\frac{dF}{dt}(p)$

induces linear map $(dF)_p: T_p M \rightarrow T_p N$

(for $M = \mathbb{R}^m, N = \mathbb{R}^n$, using the standard identification $T_p \mathbb{R}^m \cong \mathbb{R}^m, T_p \mathbb{R}^n \cong \mathbb{R}^n$ this will be the usual diff from calculus)

$$(dF)_p: \mathbb{R}^m \rightarrow \mathbb{R}^n \quad (dF)_p(v)$$

$\rightarrow$ in particular, when $M = \mathbb{R}^n, x = \text{id}_{\mathbb{R}^n} \Rightarrow$ get $\text{Standard}_p: T_p \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ and basis $\left(\frac{\partial}{\partial x^i} \right)_p, \left(\frac{\partial}{\partial x^j} \right)_p$ of $T_p \mathbb{R}^n$

$\rightarrow$ when $M \subseteq \mathbb{R}^n$ embedded, one has $T_p M \subseteq T_p \mathbb{R}^n$ $\wedge$ Standard_p restricts to $\text{Standard}_p|_{T_p M}: T_p M \xrightarrow{\sim} \mathbb{R}^m$

RM: Why the $\left(\frac{\partial}{\partial x^i} \right)_p$ notation? Third way to look at $v \in T_p M$

Key: any $v \in T_p M$ gives rise to an operation

$$\partial_v: C^\infty(M) \rightarrow \mathbb{R}, f \mapsto \partial_v(f) = \frac{df}{dt}(0)$$

where if we use viewpoint ① then with $v = \frac{dx}{dt}(0)$ then $\partial_v(f) = \frac{d}{dt} f(x(t)) \big|_{t=0} = \frac{df}{dt}(0)$

if we use viewpoint ② i.e. with v with $x \in \text{Chart}_p(M)$ then $\partial_v(f) =$ the usual $\partial_{v^i} (f_x) = (df_x)_p(v^i)$ classical

the standard identification

isomorphism

Ex: $M = \mathbb{R}^n, x = \text{id}_{\mathbb{R}^n}$ then $\left(\frac{\partial}{\partial x^i} \right)_p \in T_p \mathbb{R}^n$

induce $C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}, f \mapsto \frac{\partial f}{\partial x^i}(p)$

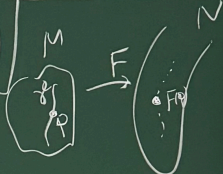
In general: tgt vector $v \in T_p M$ $\leftrightarrow$ operations $\partial: C^\infty(M) \rightarrow \mathbb{R}$ satisfying some conditions

Differentials of maps: any smooth $F: M \rightarrow N$ induces linear map $(dF)_p: T_p M \rightarrow T_{F(p)} N$.

Definition:

Well defined? I.e., for

$$\left. \frac{d\gamma}{dt}(0) \right| \mapsto \left. \frac{d(F \circ \gamma)}{dt}(0) \right|$$



$$\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0) \implies \frac{d(F \circ \gamma_1)}{dt}(0) = \frac{d(F \circ \gamma_2)}{dt}(0) \quad ??$$

Choosing $\chi \in \text{Chart}_p(M)$, $\chi' \in \text{Chart}_{F(p)}(N)$ the question is

$$\frac{d\gamma^\chi}{dt}(0) = \frac{d\gamma^{\chi'}}{dt}(0) \implies \frac{d(F \circ \gamma)^\chi}{dt}(0) = \frac{d(F \circ \gamma)^{\chi'}}{dt}(0) \quad (*)$$

$$\text{Look at } (F \circ \gamma)^{\chi'} = \chi' \circ (F \circ \gamma) = \chi' \circ F \circ \gamma = \chi' \circ F \circ \chi^{-1} \circ \chi \circ \gamma = F_{\chi}^{\chi'} \circ \gamma^\chi$$

$$\implies \frac{d(F \circ \gamma)^{\chi'}}{dt}(0) = (dF_{\chi}^{\chi'}) \left(\frac{d\gamma^\chi}{dt}(0) \right)$$

$$F_{\chi}^{\chi'} = \chi' \circ F \circ \chi^{-1}$$

(5)

Consequences

(1) $p \in M$

(*) holds i.e. $(dF)_p$ is well defined

(1) This also reads



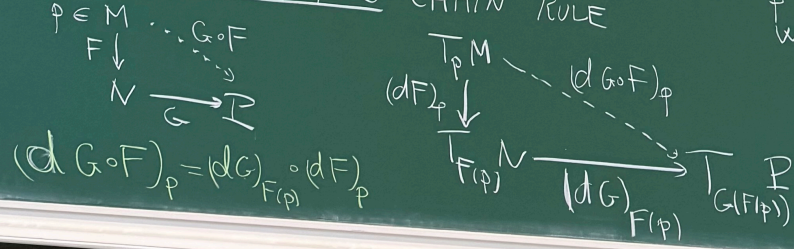
In particular, $(dF)_p$ is linear

(2) Red differently, $(**) \rightarrow$

$\Rightarrow$ another def. of $(dF)_p$ but using viewpoint (2)

(3) for $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $\chi = \text{Id}_{\mathbb{R}^m}$, $\chi' = \text{Id}_{\mathbb{R}^n} \implies (dF)_p$ becomes identified with the classical one

Another important property: CHAIN RULE



Def & chain rule in Euclidean spaces

(*)
(where $c = x_2^{x_1} = x_1^{x_2}$)
 $(v+w)^x = v^x + w^x$
 $v \mapsto v^x$
basis of $T_p M$
chart x

Choosing $x \in \text{Chart}_p(M)$, $x' \in \text{Chart}_{F(p)}(N)$ the question is

$$\frac{dx^x}{dt}(0) = \frac{dx^{x'}}{dt}(0) \implies \frac{d(F \circ x)_1^{x'}}{dt}(0) = \frac{d(F \circ x_2)^{x'}}{dt}(0) \quad (*)$$

 Look at $(F \circ x)^{x'} = x' \circ (F \circ x) = x' \circ F \circ x = x' \circ F \circ x \circ x^{-1} \circ x = F_x^{x'} \circ x^x$

$$\implies \frac{d(F \circ x)^{x'}}{dt}(0) = (dF_x^{x'}) \left(\frac{dx^x}{dt}(0) \right)$$

$$F_x^{x'} = x' \circ F \circ x^{-1}$$

Conseq. 1: $F = \text{diff} \implies (dF)_p = \text{linear isom. } \forall p \in M$ (5)
Def: $F: M \rightarrow N$ is called a submersion/immersion
 at $p \in M$ if $(dF)_p$ is surjective/injective.
 Rk: these are $\iff$ same conditions for $F_x^{x'} = x' \circ F \circ x^{-1}$
 for $x \in \text{Chart}_p(M)$, $x' \in \text{Chart}_{F(p)}(N)$
 $\iff F_x^{x'}$ is sub/imm in the usual sense.
 Conseq. 3: Assume $M \subseteq N$ submanifold & use F for the inclusion $M \hookrightarrow N$
 Notice: $T_p M \subseteq T_p N$
 $\forall p \in M$
 i.e. for $G: N \rightarrow P$ smooth $\implies d(G|_M)_p = (dG)_p|_{T_p M}$
 Part case: $N = \mathbb{R}^n$
 $G|_M: M \rightarrow P$

Consequences
 (1) (*) holds i.e. $(dF)_p$ is well defined
 (1) This also reads

$$v \in T_p M \xrightarrow{(dF)_p} T_{F(p)} N$$

 In particular,

$$T_p M \xrightarrow{I^x} \mathbb{R}^m \xrightarrow{(dF_x^{x'})} \mathbb{R}^n \xrightarrow{I^{x'}} T_{F(p)} N$$

$$I^{x'} \circ (dF)_p = (dF_x^{x'})_p \circ I^x$$

 (2) Red differentially, $(*) \rightarrow$
 $\implies$ another def. of $(dF)_p$ but using viewpoint (2)
 (3) for $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $x = \text{Id}_{\mathbb{R}^m}$, $x' = \text{Id}_{\mathbb{R}^n} \implies (dF)_p$ becomes identified with the classical one
 Another important property: CHAIN RULE

$$p \in M \xrightarrow{F} N \xrightarrow{G} P$$

$$T_p M \xrightarrow{(dF)_p} T_{F(p)} N \xrightarrow{(dG)_{F(p)}} T_{G(F(p))} P$$

$$(d(G \circ F))_p = (dG)_{F(p)} \circ (dF)_p$$

 Def & chain rule in Euclidean spaces