

Reminder: the tangent spaces $T_p M$ (for $p \in M$) ①

→ $T_p M =$ vector space of dimension $m := \dim M$

→ each $\gamma \in \text{Curves}_p(M)$ has a "speed" $\frac{d\gamma}{dt}(0) \in T_p M$ and

$$T_p M = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}_p(M) \right\}$$

→ each $v \in T_p M$ can be looked at in charts $\chi \in \text{Charts}_p(M)$:

$$v^\chi \in \mathbb{R}^m \quad \& \quad v^\chi = (dx^i \chi^i)_{\chi(p)}(v^\chi) \quad (\chi^i)_* v^\chi$$

→ each chart $\chi \in \text{Charts}_p(M)$ induces $\xrightarrow{\text{linear}}$ an isomorphism $T_p M \xrightarrow{\sim} \mathbb{R}^m$
 a basis of $T_p M$: $(\frac{\partial}{\partial x_i})_p, \dots$

→ for $M = \mathbb{R}^m$ one has/uses the standard identification $T_p \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$

①'

$$v = \frac{d\gamma}{dt}(0)$$

$$v^\chi = \frac{d\gamma^\chi}{dt}(0)$$

$$\left(\frac{d\gamma}{dt}(0) \right)^{\chi_{id}} = \left(\gamma'_1(0) \left(\frac{\partial}{\partial x_1} \right)_{\gamma(0)} + \dots \right)^{\chi_{id}} ?$$

$$\uparrow$$

usual $\frac{d\gamma}{dt}(0) = \gamma'_1(0) \cdot (1, 0, \dots, 0) + \gamma'_2(0) \cdot (0, 1, 0, \dots, 0) + \dots$
 $\gamma'(0) = (\gamma'_1(0), \gamma'_2(0), \dots)$

→ for each $F: M \rightarrow N$ smooth, $p \in M$, one can talk about its differential at p : $(dF)_p: T_p M \rightarrow T_{F(p)} N$

- linear map

②

$$\chi: \bigcup_{\substack{U \\ M}} \xrightarrow{\sim} \bigcup_{\substack{V \\ \mathbb{R}^m}}$$

proof of Lemma:

① (*) are linearly indep. ⑤

I.e.: if $\lambda_1, \dots, \lambda_m \in \mathbb{R}$

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s.t. the Lemma
 Deno

→ for each $F: M \rightarrow N$ smooth, $p \in M$, one can talk about its

differential at p : $(dF)_p: T_p M \rightarrow T_{F(p)} N$

- linear map
- sends the speed of $x \in \text{Curves}_p(M)$ to the one of $F \circ x \in \text{Curves}_{F(p)}(N)$
- chain rule holds: $(dG \circ F)_p = (dG)_{F(p)} \circ (dF)_p$

possible def of $(dF)_p$: $(dF)_p \left(\frac{dx}{dt}(0) \right) = \frac{d(F \circ x)}{dt}(0)$

in terms of charts

$x \in \text{Chart}_p(M), x' \in \text{Chart}_{F(p)}(N)$

$((dF)_p(x'))^{x'} = (dF^{x'})_{F(p)}(x)$

Particular cases:

• for curves $x: I \rightarrow M$ the differential

$(dx)_0: T_0 I \rightarrow T_{x(0)} M$
 $\mathbb{R} \xrightarrow{\frac{dx}{dt}} \mathbb{R} \xrightarrow{\frac{dx}{dt}} \mathbb{R} \xrightarrow{\frac{dx}{dt}} \mathbb{R}$

$x: \underset{\sim}{U} \xrightarrow{\sim} \underset{\sim}{\mathbb{R}^m}$

$v \in T_p M \cong T_p U$

$(dx)_p: T_p M \xrightarrow{\sim} \mathbb{R}^m$

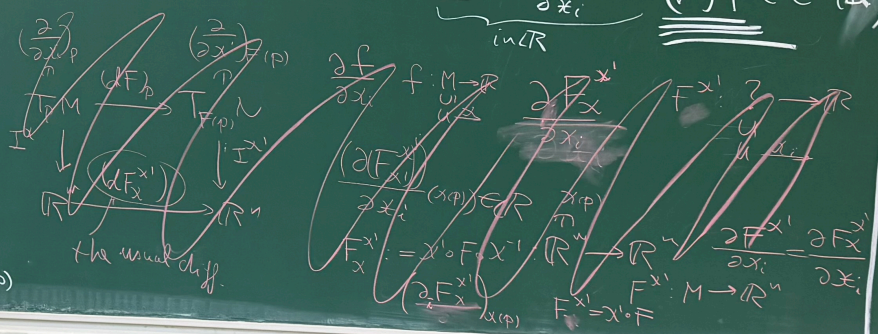
proof of Lemma:

① (*) are linearly indep.

I.e.: if $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ &

$\sum \lambda_i \frac{\partial}{\partial x_i}(0) = 0 \Rightarrow \lambda_1 = \dots = 0$
 in $T_p^{\text{deriv}} M$

i.e.: $\sum \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \quad (\forall) f \in C^\infty(\mathbb{R}^m)$
 in $\mathbb{R}$



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Ex: $M = \mathbb{R}^m$, th

Reminder: the tangent spaces $T_p M$ (for $p \in M$)

- $T_p M$ = vector space of dimension $m := \dim M$
- each $\gamma \in \text{Curves}(M)$ has a "speed" $\frac{d\gamma}{dt}(0) \in T_p M$ and $T_p M = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}(M) \right\}$
- each $v \in T_p M$ can be looked at in charts $\chi \in \text{Charts}_p(M)$: $v^x = (dx^i x^i)'(0) = (v^x)$ (v^x, x')
- each chart $\chi \in \text{Charts}_p(M)$ induces an isomorphism $T_p M \xrightarrow{\sim} \mathbb{R}^m$ a basis of $T_p M$ $\left(\frac{\partial}{\partial x^i} \right)_p, \dots$
- for $M = \mathbb{R}^m$ one has/uses the standard identification $T_p \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$

$$v = \frac{d\gamma}{dt}(0)$$

$$v^x = \frac{dx^i}{dt}(0)$$

$$\left(\frac{d\gamma}{dt}(0) \right)^x = \left(\gamma^i(0) \left(\frac{\partial}{\partial x^i} \right)_p + \dots \right)^x \quad \text{?}$$

$$\text{usual } \frac{d\gamma}{dt}(0) = \gamma^1(0) \cdot (1, 0, \dots, 0) + \gamma^2(0) \cdot (0, 1, 0, \dots, 0) + \dots$$

$$\gamma'(0) = (\gamma^1(0), \gamma^2(0), \dots)$$

for functions $f: M \rightarrow \mathbb{R}$ it's seen as $(df)_p: T_p M \rightarrow \mathbb{R}$
In other words: $(df)_p \left(\frac{d\gamma}{dt}(0) \right)$

charts: $(df)_p$
This is what, last time

Important next: any vector

Def: A derivation on $C^\infty(M)$ at p is a linear map $D_p: C^\infty(M) \rightarrow \mathbb{R}$ s.t. the Leibniz rule at p .
Denote by $T_p^{\text{deriv}} M$

for each $F: M \rightarrow N$ smooth, $p \in M$, one can talk about its differential at p : $(dF)_p: T_p M \rightarrow T_{F(p)} N$

- linear map
- sends the speed of $\gamma \in \text{Curves}(M)$ to the one of $F \circ \gamma \in \text{Curves}(N)$
- chain rule holds: $(dG \circ F)_p = (dG)_{F(p)} \circ (dF)_p$

possible def of $(dF)_p$: $(dF)_p \left(\frac{d\gamma}{dt}(0) \right) = \frac{d(F \circ \gamma)}{dt}(0)$

in terms of charts $\chi \in \text{Charts}_p(M), \chi' \in \text{Charts}_{F(p)}(N)$: $(dF)_p(v^x) = (dF^x)_\lambda(v^x)$

Particular cases:

- for curves $\gamma: I \rightarrow M$ the differential $(d\gamma)_0: T_0 I \rightarrow T_{\gamma(0)} M$

$\chi: U \xrightarrow{\sim} \mathbb{R}^m$

$v \in T_p M \cong T_p U$

$(dx)_p: T_p M \xrightarrow{\sim} \mathbb{R}^m$

proof of Lemma: ⑤

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I.e.: if $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ & $\sum \lambda_i \frac{\partial}{\partial x^i}(0) = 0 \Rightarrow \lambda_1 = \dots = 0$

in $T_p^{\text{deriv}} M$

i.e.: $\sum \lambda_i \frac{\partial f}{\partial x^i}(0) = 0 \quad (\forall) f \in C^\infty(\mathbb{R}^m)$

Prop 1: $T_p^{\text{deriv}} M$ is a

Ex: For $M = \mathbb{R}^m$

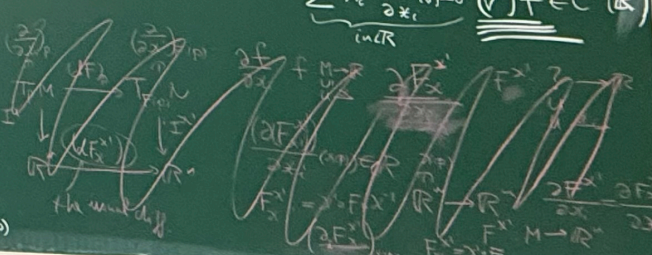
$\Rightarrow$ get elements $\frac{\partial}{\partial x^i}$

Lemma: For $M = \mathbb{R}^m$, $T_p^{\text{deriv}} M \cong \mathbb{R}^m$

Ex: For any $v \in T_p M$

Thm: The map I_p is a linear isomorphism

Ex: $M = \mathbb{R}^m$, this map I_p



- for functions $f: M \rightarrow \mathbb{R}$ its differential at $p \in M$ can be seen as $(df)_p: T_p M \rightarrow \mathbb{R}$ (strictly speaking)

In other words:

$$(df)_p \left(\frac{dx}{dt}(0) \right) = (f \circ \gamma)'(0)$$

$$\text{charts: } (df)_p(v) = (df_x)_{\gamma(p)}(v^x) \quad v \in \text{chart}_p(M)$$

This is what, last time, was denoted $\partial_v(f)$

Important next: any $v \in T_p M$ induces a map

$$\partial_v: C^\infty(M) \rightarrow \mathbb{R}, \quad \partial_v(f) = (df)_p(v)$$

Def: A derivation on $C^\infty(M)$ at the point $p \in M$ is any linear map $\partial: C^\infty(M) \rightarrow \mathbb{R}$

s.t. the Leibniz rule at p is satisfied: $\partial(fg) = f(p)\partial(g) + g(p)\partial(f)$

Denote by $T_p^{\text{der}} M$ the collection of all such ∂ ($\forall f, g \in C^\infty(M)$)

Prop 1: $T_p^{\text{der}} M$ is a vector space: $\partial, \partial': C^\infty(M) \rightarrow \mathbb{R}$ then $\partial + \partial': C^\infty(M) \rightarrow \mathbb{R}, (\partial + \partial')(f) = \partial(f) + \partial'(f)$

Ex: For $M = \Omega \subset \mathbb{R}^m$ open, $0 \in \Omega$ $\frac{\partial}{\partial x_i}(0): C^\infty(\Omega) \rightarrow \mathbb{R}$

$\Rightarrow$ get elements $\frac{\partial}{\partial x_i}(0), \dots, \frac{\partial}{\partial x_m}(0) \in T_0^{\text{der}} \Omega$ defined by $f \mapsto \frac{\partial f}{\partial x_i}(0)$ (*)

Lemma: For $M = \mathbb{R}^m$, these form a basis of $T_0^{\text{der}} \mathbb{R}^m$.

Ex: For any $v \in T_p M \Rightarrow \partial_v \in T_p^{\text{der}} M \Rightarrow$

Thm: The map $I_p^M: T_p M \rightarrow T_p^{\text{der}} M, v \mapsto \partial_v$

is a linear isomorphism.

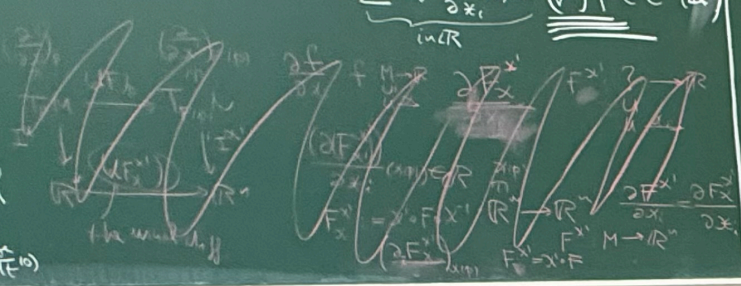
Ex: $M = \mathbb{R}^m, p=0$, this map sends $\left(\frac{\partial}{\partial x_i}\right)_0 \mapsto \frac{\partial}{\partial x_i}(0)$ hence Thm in the lemma

usual $\frac{dx}{dt}(0) = x'_1(0)(1, 0, \dots, 0) + x'_2(0)(0, 1, \dots, 0) + \dots$
 $x'(0) = (x'_1(0), x'_2(0), \dots)$

A derivation on $C^\infty(M)$ at the point $p \in M$ is any linear map $\partial: C^\infty(M) \rightarrow \mathbb{R}$ s.t. the Leibniz rule at p is satisfied: $\partial(fg) = f(p)\partial(g) + g(p)\partial(f)$
 Denote by $T_p^{\text{deriv}} M$ the collection of all such ∂ . ($\forall f, g \in C^\infty(M)$)

$\chi: U \xrightarrow{\sim} \mathbb{R}^m$
 $U \subset M$
 $v \in T_p M \cong T_p U$
 $(dx)_p: T_p M \xrightarrow{\sim} \mathbb{R}^m$

proof of Lemma: ⑤
 ① (*) are linearly indep.
 I.e.: if $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ &
 $\sum \lambda_i \frac{\partial}{\partial x_i}(0) = 0 \Rightarrow \lambda_1 = \dots = 0$
 in $T_p^{\text{deriv}} M$
 i.e.: $\sum \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \quad \forall f \in C^\infty(\mathbb{R}^m)$



Prop 1: $T_p^{\text{deriv}} M$ is a vectn space: if $\partial, \partial': C^\infty(M) \rightarrow \mathbb{R}$ then $\partial + \partial': C^\infty(M) \rightarrow \mathbb{R}, (\partial + \partial')(f) = \partial(f) + \partial'(f)$

Ex: For $M = \Omega \subset \mathbb{R}^m$ open, $0 \in \Omega$ $\frac{\partial}{\partial x_i}(0): C^\infty(\Omega) \rightarrow \mathbb{R}$
 $\Rightarrow$ get elements $\frac{\partial}{\partial x_1}(0), \dots, \frac{\partial}{\partial x_m}(0) \in T_0^{\text{deriv}} \Omega$ defined by $f \mapsto \frac{\partial f}{\partial x_i}(0)$ (*)

Lemma: For $M = \mathbb{R}^m$, these form a basis of $T_0^{\text{deriv}} \mathbb{R}^m$.

Ex: For any $u \in T_p M \Rightarrow \partial_u \in T_p^{\text{deriv}} M \Rightarrow$

Thm: The map $I_p^M: T_p M \rightarrow T_p^{\text{deriv}} M, u \mapsto \partial_u$ is a linear isomorphism.

Ex: $M = \mathbb{R}^m, p=0$, this map sends $(\frac{\partial}{\partial x_i})_0 \mapsto \frac{\partial}{\partial x_i}(0)$ hence Thm in the lemma



by Lemma $\Rightarrow$ also $f_1, \dots, f_m$ are isomorphisms

Proof of Lemma

(5)

① linear independence:

$$\sum_i \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots = 0$$

$$\text{i.e. } \sum_i \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \quad \forall f \in C^\infty(\mathbb{R}^m)$$

$$\frac{\partial p_j}{\partial x_i} = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \quad \text{in particular also for } f = p_j \quad \Rightarrow \lambda_j = 0$$

② generating property: choose arbitrary $\partial: C^\infty(\mathbb{R}^m) \rightarrow \mathbb{R}$ deriv. at 0

$$\text{To prove: } \exists \lambda_1, \dots, \lambda_m \text{ s.t. } \partial = \lambda_1 \frac{\partial}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial}{\partial x_m}(0)$$

$$\text{For } f = p_j \Rightarrow \lambda_j \text{ must be } \partial(p_j) \quad \text{i.e. } \partial(f) = \lambda_1 \frac{\partial f}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0) \quad \forall f$$

Left to prove: these λ_j do work, i.e.

(6)

$$\partial(f) = \underbrace{\partial(p_1)}_{\lambda_1} \frac{\partial f}{\partial x_1}(0) + \dots + \underbrace{\partial(p_m)}_{\lambda_m} \frac{\partial f}{\partial x_m}(0) \quad (\star)$$

Lemma: For any $f \in C^\infty(\mathbb{R}^m) \exists g_1, \dots, g_m \in C^\infty(\mathbb{R}^m)$ s.t.
 $f(x) = f(0) + x_1 g_1(x) + \dots + x_m g_m(x)$

$$f(0) = 0 \Rightarrow f(x) = x_1 g_1(x) + \dots \quad \text{i.e. } f = p_1 g_1 + \dots + p_m g_m$$

Look at LHS & RHS of $(\star)$ using this expression for f :

$$\begin{aligned} \partial(f) &= \partial(\sum p_i g_i) \stackrel{\partial = \text{linear}}{=} \sum \partial(p_i g_i) \\ &\stackrel{\text{Leibniz}}{=} \sum \underbrace{p_i(0)}_0 \partial(g_i) + g_i(0) \underbrace{\partial(p_i)}_{\lambda_i} = \sum \lambda_i g_i(0) \end{aligned}$$

$$\text{use } \frac{\partial}{\partial x_i} \Big|_0 (p_1 g_1 + \dots + p_m g_m) = g_i(0)$$

$\square$

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Lemma

EX: 1

Theorem

EX: $M = \mathbb{R}$
 $p = 0$

Is there a version of dF on these tangent spaces $T_p^{\text{der}} M$?

I.e. for $F: M \rightarrow N$... $(dF)_p: T_p^{\text{der}} M \rightarrow T_{F(p)}^{\text{der}} N$?

$$C^\infty(M) \xrightarrow{F^*} C^\infty(N) \xrightarrow{\partial} \mathbb{R}$$

$F^*(f) = f \circ F$

$$M \xrightarrow{F} N$$

$f \circ F$ and f both map to $\mathbb{R}$

$$\partial \mapsto \partial \circ F^*$$

Idea of proof of thm: use chart

$$\begin{array}{ccc} T_p M & \xrightarrow{I_M^p} & T_p^{\text{der}} M \\ \uparrow (di)_p & & \uparrow s(di)_p \\ T_p U & \xrightarrow{I_U^p} & T_p^{\text{der}} U \\ \downarrow (dx)_p & & \downarrow (dx)_p \\ T_0 \mathbb{R}^m & \xrightarrow{I_0^{\mathbb{R}^m}} & T_0^{\text{der}} \mathbb{R}^m \end{array}$$

by Lemma $\Rightarrow$ also $\dots$ are isomorphisms

TYPO IN FORMULA

$$\frac{\partial F_x^{x,i}}{\partial x_i}(x(p))$$

∂x_i (the usual)

$$M \supset U \xrightarrow{i} \mathbb{R}^m$$

$i: U \rightarrow M$

$$S^2 \subseteq \mathbb{R}^3$$

$$T_p S^2 \subseteq T_p \mathbb{R}^3$$

$x(\frac{\partial}{\partial y})_p - y(\frac{\partial}{\partial x})_p$

$p = (x, y, z)$

$(\frac{\partial}{\partial y})_p$

Lemma: $(di)_p$ is an iso

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Proof of Lemma

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① linear independence:

$$\sum \lambda_i \frac{\partial}{\partial x_i}(0) = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots$$

Left to prove: these λ_j do work, i.e.

$$\partial(f) = \partial(p_1) \cdot \frac{\partial f}{\partial x_1}(0) + \dots + \partial(p_m) \cdot \frac{\partial f}{\partial x_m}(0)$$

(6)

Rk:

$S^2 \subseteq \mathbb{R}^3$
 $T_p S^2 \subseteq T_p \mathbb{R}^3$
 $(\frac{\partial}{\partial x})_p$
 $(\frac{\partial}{\partial y})_p - y(\frac{\partial}{\partial x})_p$
 $= (x, y, z)$
 $(\frac{\partial}{\partial y})_p$

Def: A vector field X on a manifold M is (8)
 a collection $\{X_p\}_{p \in M}$ of tangent vectors $X_p \in T_p M$
 (an "assignment/function" $M \ni p \mapsto X_p \in T_p M$)
 which is smooth.

$\rightarrow (\forall) (U, \alpha)$ for $p \in U$, writing X_p out w.r.t. $(\frac{\partial}{\partial x_i})_p$

$$X_p = \underbrace{\text{coeff}^i(p)}_{\in \mathbb{R}} (\frac{\partial}{\partial x_i})_p + \dots$$
 [Exercise: there two are " $\Leftrightarrow$ "]
 then the resulting $\text{coeff}^i: U \rightarrow \mathbb{R}$ are smooth.
 $\rightarrow \partial_{X_p}: \underbrace{C^\infty(M)}_{\substack{U \\ f}} \rightarrow \mathbb{R}$ $\partial_{X_p}(f) \in \mathbb{R} \quad (\forall) p \in M \Rightarrow$ new function on M .
 $\partial_X(f)$ $\leftarrow$ smooth $(\forall) f \in C^\infty(M)$

(6) Rk 1: $T_p^{\text{dual}} M$ is a vectn space: $\partial, \partial': C^\infty(M) \rightarrow \mathbb{R}$ (4) from $\mathbb{R}$
 then $\partial + \partial': C^\infty(M) \rightarrow \mathbb{R}, (\partial + \partial')(f) = \partial(f) + \partial'(f)$

Is there a version of dF on these tangent spaces $T_p^{dov} M$?

J.e. for $F: M \rightarrow N$, $(dF)_p: T_p^{dov} M \rightarrow T_p^{dov} N$?

$$C^0(M) \xrightarrow{F} C^0(N) \xrightarrow{\partial} \mathbb{R}$$

$$M \xrightarrow{F} N$$

Idea of proof of thm use chart χ

$$T_p M \xrightarrow{I_p^M} T_p^{dov} M \xrightarrow{S(d\chi)_p} T_p^{dov} U \xrightarrow{I_p^U} T_p^{dov} \mathbb{R}^m$$

by Lemma $\Rightarrow$ also r.v.s. are isomorphisms

TIP IN FORMULA

$$\frac{\partial F_{x,i}}{\partial x_i}(x(p))$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$S^1 = \mathbb{R}^2$$

$$T_p S^1 \subseteq T_p \mathbb{R}^2$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

$$\frac{\partial x_i}{\partial x_j}(p)$$

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Def: A vector field X on a manifold M is a collection $\{X_p\}_{p \in M}$ of tangent vectors $X_p \in T_p M$ (an "assignment/function" $M \ni p \mapsto X_p \in T_p M$) which is smooth.

$$X_p = \text{coeff}^1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots$$

then the resulting $\text{coeff}^i: U \rightarrow \mathbb{R}$ are smooth. $\partial_{X_p}(f) \in \mathbb{R} \forall p \in M \Rightarrow$ new function on M smooth $\partial_X(f)$

Proof of Lemma

① linear independence:

$$\sum \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots = 0$$

$$\text{i.e. } \sum \lambda_i \frac{\partial f}{\partial x_i}(0) = 0 \quad \forall f \in C^0(\mathbb{R}^m)$$

② generating property: choose arbitrary $\partial: C^0(\mathbb{R}^m) \rightarrow \mathbb{R}$

To prove: $\exists \lambda_1, \dots, \lambda_m$ s.t.

$$\partial = \lambda_1 \frac{\partial}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial}{\partial x_m}(0)$$

$$\text{i.e. } \partial(f) = \lambda_1 \frac{\partial f}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0) \quad \forall f$$

For $f = p_j \Rightarrow \lambda_j$ must be $\partial(p_j)$.

Left to prove: these λ_j do work, i.e.

$$\partial(f) = \sum_{i=1}^m \lambda_i \frac{\partial f}{\partial x_i}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0)$$

Lemma: For any $f \in C^0(\mathbb{R}^m) \exists g_1, \dots, g_m \in C^0(\mathbb{R}^m)$ s.t.

$$f(x) = f(0) + x_1 g_1(x) + \dots + x_m g_m(x)$$

$$f(0) = 0 \Rightarrow f(x) = x_1 g_1(x) + \dots + x_m g_m(x)$$

Look at LHS & RHS of (*) using this expression for f :

$$\partial(f) = \partial(\sum p_i g_i) = \sum \partial(p_i g_i)$$

$$\text{Leibniz} \Rightarrow \sum p_i(0) \partial(g_i) + g_i(0) \partial(p_i) = \sum \lambda_i g_i(0)$$

$$\text{use } \frac{\partial}{\partial x_i} \left(p_i g_i + \dots + p_m g_m \right) = g_i(0)$$

Rk 1: $T_p^{dov} M$ is a vector space $\partial, \partial': C^0(M) \rightarrow \mathbb{R}$

then $\partial + \partial': C^0(M) \rightarrow \mathbb{R}, (\partial + \partial')(f) = \partial(f) + \partial'(f)$

Ex: For $M = \mathbb{R}^m$ open, $0 \in \mathbb{R}^m$ $\frac{\partial}{\partial x_i}(0): C^0(M) \rightarrow \mathbb{R}$

$\Rightarrow$ get elements $\frac{\partial}{\partial x_1}(0), \dots, \frac{\partial}{\partial x_m}(0) \in T_0^{dov} \mathbb{R}^m$

$$\text{defined by } f \mapsto \frac{\partial f}{\partial x_i}(0)$$

Lemma: For $M = \mathbb{R}^m$, these form a basis of $T_0^{dov} \mathbb{R}^m$.

Ex: For any $u \in T_p M \Rightarrow \partial_u \in T_p^{dov} M$

Thm: The map $I_p^M: T_p M \rightarrow T_p^{dov} M, u \mapsto \partial_u$

is a linear isomorphism.

Ex: $M = \mathbb{R}^m$, this map sends $\left(\frac{\partial}{\partial x_i} \right)_0 \mapsto \frac{\partial}{\partial x_i}(0)$ hence Thm in this case

from a basis of $T_0 \mathbb{R}^m$ the lemma.