

TANGENT SPACES Given $M, p \in M$: tangent vectors $v \in T_p M$

• $v = \frac{d\gamma}{dt}(0)$ with $\gamma \in \text{Curves}_p(M)$

• $v^X \in \mathbb{R}^m$ for $X \in \text{Chart}_p(M)$

• $\partial_v: C^\infty(M) \rightarrow \mathbb{R}$
 $f \mapsto \partial_v(f) := (df)_p(v) = \begin{cases} \frac{d f \circ \gamma}{dt}(0) & \text{if we write } v = \frac{d\gamma}{dt}(0) \\ (df_X)_{X(p)}(v^X) & \text{if we use } X \in \text{Chart}_p(M) \end{cases}$

Prop: Any linear map $\partial: C^\infty(M) \rightarrow \mathbb{R}$ satisfying Leibniz at p must be $\partial = \partial_v$ for a unique $v \in T_p M$.

$\partial(fg) = f(p) \partial(g) + g(p) \partial(f)$

VECTOR FIELD: on M is a map X
 $(X) \quad M \ni p \mapsto X_p \in T_p M$

$\{X_p\}_{p \in M}$

which is smooth in the sense that, $(\forall)$ chart (U, X) of M , writing for $p \in U$: $X_p = C_1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots + C_m(p) \left(\frac{\partial}{\partial x_m} \right)_p$ the coefficients $C_i: U \rightarrow \mathbb{R}$ are smooth.

Notation:
 $\mathfrak{X}(M)$

for the set of vector fields on M

GLOBAL VIEW-POINT

Fixing X & f =
 This function of f with

Lemma: Given

$\frac{\partial}{\partial x_i}$: locally $\frac{\partial}{\partial x_i}$
 Hence each

Def: A derivation $L: C^\infty(M) \rightarrow \mathbb{R}$

Theorem: $X \in \mathfrak{X}(M)$

Operations with vector fields / the algebraic structure on $\mathfrak{X}(M)$

• addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X + Y \in \mathfrak{X}(M)$ by: $(X + Y)_p = X_p + Y_p$ (in $T_p M$)

Examples:

1) On $M = \mathbb{R}$

(*) $M \ni p \mapsto X_p \in T_p M$ $\{X_p\}_{p \in M}$

which is smooth in the sense that, (U, χ) chart of M ,
writing for $p \in U$: $X_p = c_1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots + c_m(p) \left(\frac{\partial}{\partial x_m} \right)_p$
the coefficients $c_i: U \rightarrow \mathbb{R}$ are smooth.

Notation:
 $\mathfrak{X}(M)$
for the set
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fields on M

Hence each $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$.

Def: A derivation on $C^\infty(M)$ is any linear map
 $L: C^\infty(M) \rightarrow C^\infty(M)$, satisfying Leibniz: $L(fg) = f L(g) + g L(f)$
 $(\forall f, g \in C^\infty(M))$ } $\text{Der}(C^\infty(M))$
collection of all such

Theorem: $X \mapsto L_X$ defines a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \text{Der}(C^\infty(M))$

Operations with vector fields / the algebraic structure on $\mathfrak{X}(M)$ (2-3)

- addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X+Y \in \mathfrak{X}(M)$ by: $(X+Y)_p = X_p + Y_p$ $\leftarrow$ in $T_p M$
 - multiplication by scalars: $\lambda \in \mathbb{R}, X \in \mathfrak{X}(M) \Rightarrow$ can form $\lambda \cdot X$ by: $(\lambda \cdot X)_p = \lambda \cdot X_p$
 - multiplication by fcts: $f \in C^\infty(M), X \in \mathfrak{X}(M) \Rightarrow$ can form $f \cdot X$ by:
 $(f \cdot X)_p = f(p) \cdot X_p$
- $\mathfrak{X}(M) = \text{a } C^\infty(M)\text{-module}$

Examples:

1. On $M = \mathbb{R}^n$: $p \mapsto \left(\frac{\partial}{\partial x_i} \right)_p$, for $i \in \{1, \dots, n\}$; vector fields $\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$

$\Rightarrow$ for each $f_1, \dots, f_n \in C^\infty(\mathbb{R}^n)$: $f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n} \in \mathfrak{X}(\mathbb{R}^n)$
($\mathbb{R}$ all vector fields on $\mathbb{R}^n$ are like this)

2. Arbitrary M , (U, χ) chart $\Rightarrow \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(U)$.

3. When $M \subseteq \mathbb{R}^n$ submanifold $\Rightarrow$ any $X \in \mathfrak{X}(M)$

can be written as $X_p = f_1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots + f_m(p) \left(\frac{\partial}{\partial x_m} \right)_p$ for $p \in M$!!!

Fact: $X = \text{smooth as in def} \Leftrightarrow f_1, \dots, f_m \in C^\infty(M)$. $X = f_1 \frac{\partial}{\partial x_1} + \dots + f_m \frac{\partial}{\partial x_m}$

But: some vector fields are not tangent to M . $f_1, \dots, f_m \in C^\infty(M)$

Ex: $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ is tangent to S^1 (at points $p \in S^1$) $\Rightarrow y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$

$y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \notin \mathfrak{X}(S^1)$

GLOBAL VIEW-POINT

$$X \in \mathfrak{X}(M), f \in C^\infty(M), p \in M \Rightarrow \partial_{X_p}(f) \in \mathbb{R}$$

Fixing X & $f \rightarrow$ a function $M \rightarrow \mathbb{R}, p \mapsto \partial_{X_p}(f)$.

This function is denoted $L_X(f)$, called the Lie derivative of f with respect to X . Is it smooth?

$$L_X(f) \in C^\infty(M)??$$

Lemma: Given $(*)$, X is smooth iff $L_X(f) \in C^\infty(M) \quad (\forall) f \in C^\infty(M)$.

$\uparrow$ = locally & exercise.

Hence each $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$.

Def: A derivation on $C^\infty(M)$ is any linear map $L: C^\infty(M) \rightarrow C^\infty(M)$, satisfying Leibniz: $L(fg) = f L(g) + g L(f)$ $(\forall) f, g \in C^\infty(M)$.

$\left. \begin{array}{l} \text{Der}(C^\infty(M)) \\ \text{collection} \\ \text{of all such} \end{array} \right\}$

Theorem: $X \mapsto L_X$ defines a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \text{Der}(C^\infty(M))$.

Examples: $\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$

Application: Product of vector fields $X, Y \in \mathfrak{X}(M)$

$$L_X, L_Y: C^\infty(M) \rightarrow C^\infty(M)$$

$$\text{Consider } L_X \circ L_Y: C^\infty(M) \rightarrow C^\infty(M)$$

$\Rightarrow$ a new vector field, that deserves the name $X \cdot Y$ or $X \lrcorner Y$.

$C^\infty(M) = \{f: M \rightarrow \mathbb{R} \mid f \text{ smooth}\}$
 $\left. \begin{array}{l} \text{vector space} \\ \text{ring} \end{array} \right\}$ an algebra

TANGENT SPACES

Given $M, p \in M$: tangent vectors $v \in T_p M$

• $v = \frac{d}{dt} \gamma(t)$ with $\gamma \in \text{Curve}_p(M)$

• $v^X \in \mathbb{R}^m$ for $X \in \text{Vect}_p(M)$

• $\partial_v: C^\infty(M) \rightarrow \mathbb{R}$

$$f \mapsto \partial_v(f) = (df)_p(v) = \begin{cases} \frac{df}{dt}(\gamma(t)) & \text{if we write } v = \frac{d}{dt} \gamma(t) \\ (df_X)_{p(p)}(v^X) & \text{if we use } X \in \text{Vect}_p(M) \end{cases}$$

Prop: Any linear map $\partial: C^\infty(M) \rightarrow \mathbb{R}$ satisfying Leibniz at p must be $\partial = \partial_v$ for an unique $v \in T_p M$.

$$\partial(fg) = f(p)\partial(g) + g(p)\partial(f)$$

VECTOR FIELD

on M is a map $X: M \rightarrow TM$ such that (U, X) is a chart on M , writing for $p \in U$:

$$X_p = c_1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots + c_m(p) \left(\frac{\partial}{\partial x_m} \right)_p$$

the coefficients $c_i: U \rightarrow \mathbb{R}$ are smooth.

$$\{X_p\}_{p \in M}$$

Notation: $\mathfrak{X}(M)$ for the set of vector fields on M

GLOBAL VIEW-POINT

$X \in \mathfrak{X}(M), f \in C^\infty(M), p \in M \Rightarrow \partial_X(f) \in \mathbb{R}$

Fixing X & $f \Rightarrow$ a function $M \rightarrow \mathbb{R}, p \mapsto \partial_X(f)$

This function is denoted $L_X(f)$, called the Lie derivative of f with respect to X . Is it smooth? $L_X(f) \in C^\infty(M)$??

Lemma: Given (X, X) , X is smooth iff $L_X(f) \in C^\infty(M)$ $(\forall f \in C^\infty(M))$.

p : locally exercise

Hence each $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$.

Def: A derivation on $C^\infty(M)$ is any linear map $L: C^\infty(M) \rightarrow C^\infty(M)$, satisfying Leibniz: $L(fg) = fL(g) + gL(f)$ $(\forall f, g \in C^\infty(M))$

Theorem: $X \mapsto L_X$ defines a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \text{Der}(C^\infty(M))$

Application: Product of vector fields

$X, Y \in \mathfrak{X}(M)$

$L_X, L_Y: C^\infty(M) \rightarrow C^\infty(M)$

$\Rightarrow$ a new vector field, that deserves the name $X \cdot Y$ or $X \lrcorner Y$.

$C^\infty(M) = \{f: M \rightarrow \mathbb{R} / f \text{ smooth}\}$
vector space
ring
an algebra

Operations with vector fields / the algebraic structure on $\mathfrak{X}(M)$

- addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X+Y \in \mathfrak{X}(M)$ by: $(X+Y)_p = X_p + Y_p$
- multiplication by scalars: $\lambda \in \mathbb{R}, X \in \mathfrak{X}(M) \Rightarrow$ can form λX by: $(\lambda X)_p = \lambda X_p$
- multiplication by fct: $f \in C^\infty(M), X \in \mathfrak{X}(M) \Rightarrow$ can form $f \cdot X$ by: $(f \cdot X)_p = f(p) \cdot X_p$

$\mathfrak{X}(M)$ is a $C^\infty(M)$ -module

Examples

1) On $M = \mathbb{R}^n: p \mapsto \left(\frac{\partial}{\partial x_i} \right)_p$, for $i=1, \dots, n$, vector fields $\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$

$\Rightarrow$ for each $f_1, \dots, f_n \in C^\infty(\mathbb{R}^n): f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n} \in \mathfrak{X}(\mathbb{R}^n)$ (all vector fields on $\mathbb{R}^n$ are like this)

2) Arbitrary $M, (U, X)$ chart $\Rightarrow \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(U)$

3) When $M \subseteq \mathbb{R}^n$ submanifold $\Rightarrow$ any $X \in \mathfrak{X}(M)$

can be written as $X_p = f_1(p) \left(\frac{\partial}{\partial x_1} \right)_p + \dots + f_m(p) \left(\frac{\partial}{\partial x_m} \right)_p$ for $p \in M$!!!

Fact: X is smooth as in def $\Leftrightarrow f_1, \dots, f_m \in C^\infty(M)$

But: some vector fields are not tangent to M .

Ex: $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ is tangent to S^1 (at points $p \in S^1$) $\Rightarrow y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$

$y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \notin \mathfrak{X}(S^1)$

Application: Product of vector fields $X, Y \in \mathfrak{X}(M)$ (-5-)
 $C^\infty(M) = \{f: M \rightarrow \mathbb{R} \mid f \text{ smooth}\}$
 $L_X, L_Y: C^\infty(M) \rightarrow C^\infty(M)$ vector space
 Consider $L_X \circ L_Y: C^\infty(M) \rightarrow C^\infty(M)$ ring an algebra
 $\Rightarrow$ a new vector field, that deserves the name $X \circ Y$ or $X \cdot Y$.
 Problem: $(\circ)$ not a derivation:

$$L_X(L_Y(fg)) = L_X(f L_Y(g) + L_Y(f)g) =$$

$$= f L_X(L_Y(g)) + L_X(f) L_Y(g) + L_Y(f) L_X(g) + L_X(L_Y(f))g$$

 Notice: $L_Y(L_X(fg))$ has the same problematic terms
 Hence $L_X \circ L_Y - L_Y \circ L_X: C^\infty(M) \rightarrow C^\infty(M)$
 is again a derivation. ~~$X = L_X$ for some~~

Def: The Lie bracket of $X, Y \in \mathfrak{X}(M)$ is the new (-6-)
 vector field $[X, Y] \in \mathfrak{X}(M)$ s.t. $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$
 $(\forall f \in C^\infty(M))$
 Ex: on $M = \mathbb{R}^2$: $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$

$$\underbrace{\frac{\partial}{\partial x}}_X \underbrace{x \frac{\partial}{\partial y}}_Y = \frac{\partial}{\partial y}$$

 Better methods for computations?
 Or, using (3):

$$[\frac{\partial}{\partial x}, x \frac{\partial}{\partial y}] = x \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \right) \right) + \underbrace{L_{\frac{\partial}{\partial x}} \left(x \frac{\partial}{\partial y} \right)}_{\frac{\partial}{\partial y}}$$

$$= \frac{\partial}{\partial y}$$

the coefficients $c_i: U \rightarrow \mathbb{R}$ are smooth.

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- addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X+Y \in \mathfrak{X}(M)$ by: $(X+Y)_p := X_p + Y_p$ in $T_p M$
- multiplication by scalars: $\lambda \in \mathbb{R}, X \in \mathfrak{X}(M) \Rightarrow$ can form $\lambda \cdot X$ by: $(\lambda \cdot X)_p = \lambda \cdot X_p$
- multiplication by fcts: $f \in C^\infty(M), X \in \mathfrak{X}(M) \Rightarrow$ can form $f \cdot X$ by: $(f \cdot X)_p = f(p) \cdot X_p$

$\mathfrak{X}(M)$ is a $C^\infty(M)$ -module

• Lie bracket operation: $\mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M), (X, Y) \mapsto [X, Y]$
satisfying:

EXERCISE

$\mathfrak{X}(M)$ is a Lie algebra

- (1) $[Y, X] = -[X, Y]$ (skew symmetry)
- (2) bilinear ($[X, Y+Z] = [X, Y] + [X, Z]$ etc)
- (3) derivation rule: $[X, f \cdot Y] = f[X, Y] + L_X(f) \cdot Y$
- (2) Jacobi identity: $[X, Y], Z] + [Y, Z], X] + [Z, X], Y] = 0$
($\forall X, Y, Z \in \mathfrak{X}(M)$)

Def: A Lie alg. is a vector space $\mathfrak{g}$ & an operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying (1), (2), (3)

Examples:

1) On $M = \mathbb{R}^n$

$\mathfrak{X}(M) =$ vector space

$\Rightarrow$ for

(& all v)

2) Arbitrary M

3) When $M \subseteq \mathbb{R}^n$

can be writ

Fact: $X =$ smooth

But: some vec

Ex:

Notation: $\mathfrak{X}(M)$ for the set of vector fields on M

pl: locally exercise

Hence each $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$.

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[Theorem: $X \mapsto L_X$ defines a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \text{Der}(C^\infty(M))$]

Notice: $L_Y(L_X(fg))$ has the same problematic term

$\text{Der}(C^\infty(M))$ collection of all such

Hence $L_X \circ L_Y - L_Y \circ L_X: C^\infty(M) \rightarrow C^\infty(M)$

is again a derivation

~~$X = L_X$ for some~~

in $T_p M$
 $X_p = \lambda \cdot X_p$
 $f(p) \cdot X_p \mapsto [fX, Y] = 0$
 $Y \in \mathfrak{X}(M)$

The tangent bundle: $TM = \{(p, v) : p \in M, v \in T_p M\}$ (8)

Vector fields $\xrightarrow{\text{can be seen as}}$ functions $X: M \rightarrow TM, p \mapsto (p, X_p)$

Q: is TM a manifold s.t. smoothness of X becomes Smoothness of maps between manifolds?

possible exercise

Yes: any chart (U, χ) of M gives rise to a chart $(\tilde{U}, \tilde{\chi})$ of TM .

$$\chi: \underset{M}{U} \rightarrow \underset{\mathbb{R}^m}{\Omega}$$

$$\tilde{U} := TU = \{(p, X_p) : p \in U\}$$

$$(x_1(p), \dots, x_m(p))$$

$T_p M$

$$T_p M \quad \left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p \text{ basis}$$

$$(c_1, \dots, c_m) \in \mathbb{R}^m \text{ from the writing } X_p = \sum_i c_i \left(\frac{\partial}{\partial x_i} \right)_p$$

$$\tilde{\chi}: \tilde{U} \rightarrow \Omega \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}$$

$$\tilde{\chi}(p) = (x(p), c_1(X_p), \dots, c_m(X_p))$$

Compute $\tilde{\chi}' \circ \tilde{\chi}^{-1} \dots \Rightarrow$ Smooth changes of coordinates
 $TM = 2m$ -dimensional manifold

$$\begin{array}{ccc} TM & & (p, X_p) \\ \downarrow & & \downarrow \\ M & & p \end{array}$$