



For 3.25

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right]^n$$

Exercises for Today

- 3.22
- 3.24
- 3.25
- 3.28
- 3.34

⊗ Please do this problem for S^2 first. Then generalise.

We start w/ Homework 4 at 10:00
You can pick up homework 2

$$\begin{aligned} (\sigma_N = F \circ \sigma_N^{-1})(t) \\ \sigma_N(F(\frac{2t}{(t^2-1)}, \frac{t^2-1}{t^2+1})) \\ = \sigma_N\left(\frac{4(t^2-(t^2-1)^2)}{(t^2-1)^2}, \frac{4t(t^2-1)}{(t^2-1)^2}\right) \\ y.z.veltmaat@uu.nl \\ \frac{f(t)}{\frac{(t^2-1)^2}{(t^2-1)^2} \frac{\partial f}{\partial t}} \end{aligned}$$

Reminder:

- vector fields X on M : $M \ni p \mapsto X_p \in T_p M$ smooth

i.e., $(\forall) \chi: U \xrightarrow{\sim} \mathbb{R}^n$ chart, writing

$$X_p = \sum F_i^X(\chi(p)) \left(\frac{\partial}{\partial x_i} \right)_p$$

then $F_i^X: \mathcal{O} \rightarrow \mathbb{R}$ are smooth

the set $\mathcal{X}(M)$ of vector fields
- vector space
- $C^\infty(M)$ -module

$(X|_U \in \mathcal{X}(U) \mapsto X|_U \in \mathcal{X}(U))$ written as $\sum F_i^X \frac{\partial}{\partial x_i}$

- globally: $X \leftrightarrow$ derivation $L_X: C^\infty(M) \rightarrow C^\infty(M)$

$$L_X(f)(p) = \sum F_i^X(p) \frac{\partial f}{\partial x_i}(p) = \dots$$

- speeds of curves $\gamma: I \rightarrow M$ at $t=0$: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) \in T_{\gamma(0)} M$ (also denoted $\frac{d}{dt} \Big|_{t=0} \gamma(t)$)

At arbitrary $t_0 \in I$: $\dot{\gamma}(t_0) = \frac{d\gamma}{dt}(t_0) = \frac{d}{dt} \Big|_{t=t_0} \gamma(t+t_0) \in T_{\gamma(t_0)} M$

- Ex: For $\mathcal{O} \subseteq \mathbb{R}^m$ open $\Rightarrow$ vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathcal{X}(\mathcal{O})$

and arbitrary ones as $\sum F_i \frac{\partial}{\partial x_i}$, with $F_i \in C^\infty(\mathcal{O})$

For $\gamma: I \rightarrow \mathcal{O}$, writing $\gamma = (x_1, \dots, x_m) \Rightarrow \dot{\gamma}(t) = \sum \dot{\gamma}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}$

- On a general M , $(\forall)$ chart $\chi: U \xrightarrow{\sim} \mathcal{O} \subseteq \mathbb{R}^m$ induces $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathcal{X}(U)$

and, for $\gamma: I \rightarrow M$, at points t with $\gamma(t) \in U$:

$$\dot{\gamma}(t) = \underbrace{\dot{\gamma}_1(t)}_{\text{usual derivative of } (\gamma^1)_1} \left(\frac{\partial}{\partial x_1} \right)_{\gamma(t)} + \dots + \dot{\gamma}_m(t) \left(\frac{\partial}{\partial x_m} \right)_{\gamma(t)}$$

usual derivative of $(\gamma^1)_1: I \rightarrow \mathbb{R}$

Def: Given $X \in \mathcal{X}(M)$, an integral curve of X is any curve $\gamma: I \rightarrow M$ ($I \subseteq \mathbb{R}$ open interval) such that: $\dot{\gamma}(t) = X_{\gamma(t)} \quad (\forall) t \in I$

Say: γ starts at $p \in M$ if $0 \in I, \gamma(0) = p$ (equality in $T_{\gamma(t)} M$)

Ex 1: $M = \mathbb{R}^2, X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \in \mathcal{X}(\mathbb{R}^2), \lambda = \text{fixed constant}$

Integral curve: $\gamma(t) = (x(t), y(t))$ satisfying

$$\dot{\gamma}(t) = X_{\gamma(t)}$$

Ex 2: $M = \mathbb{R}^2, X = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$

$\gamma(t) = (x(t), y(t)) \dots$ equation $x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y} = -x(t)^2 \frac{\partial}{\partial x} - y(t) \frac{\partial}{\partial y}$

Consider

Look at

Theorem

$(\forall) X \in \mathcal{X}(M)$

& $(\forall) p \in M$

More

i.e.

More

X

i.e., $(\forall) X: U \rightarrow \mathbb{R}^2$ chart, writing

$$X_p = \sum F_i^X(x(p)) \left(\frac{\partial}{\partial x_i} \right)_p \quad \text{then } F_i^X: \mathcal{O} \rightarrow \mathbb{R} \quad \dots$$

are smooth and

$$\left(X|_U \in \mathcal{X}(U) \mapsto X_*(X|_U) \in \mathcal{X}(U) \text{ written as } \sum F_i^X \frac{\partial}{\partial x_i} \right) \quad \left. \begin{array}{l} \text{Lie bracket} \\ \text{Lie algebra} \end{array} \right\}$$

- globally: $X \leftrightarrow$ derivation $L_X: C^\infty(M) \rightarrow C^\infty(M)$
 $L_X(f)(p) = \sum x_p^i(f) \frac{\partial}{\partial x_i}(f)(p) = \dots$

- speeds of curves $\gamma: I \rightarrow M$ at $t=0$: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) \in T_{\gamma(0)}M$ (also denoted $\frac{d}{dt}|_{t=0} \gamma(t)$)

At arbitrary $t_0 \in I$: $\boxed{\dot{\gamma}(t_0) = \frac{d\gamma}{dt}(t_0) = \frac{d}{dt}|_{t=t_0} \gamma(t+t_0) \in T_{\gamma(t_0)}M}$

- Ex: For $\mathcal{O} \subseteq \mathbb{R}^m$ open $\Rightarrow$ vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathcal{X}(\mathcal{O})$
 and arbitrary ones as $\sum F_i \frac{\partial}{\partial x_i}$, with $F_i \in C^\infty(\mathcal{O})$

For $\gamma: I \rightarrow \mathcal{O}$, writing $\gamma = (x_1, \dots, x_m) \Rightarrow \boxed{\dot{\gamma}(t) = \sum \dot{\gamma}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}}$

$$\dot{\gamma}(t) = \frac{d\gamma}{dt}(t) = \frac{d}{dt} \left(\sum x_i^{\gamma(t)} \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)} \right) = \dots + \sum \dot{x}_m^{\gamma(t)} \left(\frac{\partial}{\partial x_m} \right)_{\gamma(t)}$$

usual derivative of $\left(\frac{d\gamma}{dt} \right)_1: I \rightarrow \mathbb{R}$

Def: Given $X \in \mathcal{X}(M)$, an integral curve of X is any curve $\gamma: I \rightarrow M$ ($I \subseteq \mathbb{R}$ open interval) such that: $\dot{\gamma}(t) = X_{\gamma(t)} \quad (\forall) t \in I$
 Such γ starts at $p \in M$ if $0 \in I, \gamma(0) = p$ (equality in $T_{\gamma(t)}M$).

Theorem

$(\forall) X \in \mathcal{X}(M)$

& $(\forall) p \in M$

Moreover

i.e.

Ex 1: $M = \mathbb{R}^2, X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \in \mathcal{X}(\mathbb{R}^2), \lambda = \text{fixed constant}$

Integral curve $\gamma(t) = (x(t), y(t))$ satisfying

$$\dot{\gamma}(t) = X_{\gamma(t)} = \left\{ \begin{array}{l} \dot{x}(t) = 1 \\ \dot{y}(t) = \lambda \end{array} \right.$$

$$\left\{ \begin{array}{l} \dot{x}(t) \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \dot{y}(t) \left(\frac{\partial}{\partial y} \right)_{\gamma(t)} = \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \lambda \left(\frac{\partial}{\partial y} \right)_{\gamma(t)} \end{array} \right.$$

$\Rightarrow \left\{ \begin{array}{l} x(t) = t + a \\ y(t) = \lambda t + b \end{array} \right. \quad a, b \in \mathbb{R} \text{ some constant} \Rightarrow \left\{ \begin{array}{l} (\forall) a, b \in \mathbb{R} \text{ we get} \\ \text{an integral curve } \gamma_{a,b} \\ \gamma_{a,b}(t) = (t+a, \lambda t+b) \end{array} \right.$

Rk 1: $(\forall) p = (a, b) \in \mathbb{R}^2, \exists!$ integral curve $\gamma_{a,b}: \mathbb{R} \rightarrow \mathbb{R}^2$ starting at p !

Rk 2: $\gamma_{a,b}$ is defined on the entire $\mathbb{R}$!

Exercise: Same for $X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \in \mathcal{X}(\mathbb{R}^2)$

Ex 2: $M = \mathbb{R}^2, X = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$

$\gamma(t) = (x(t), y(t)) \dots$ equation $\dot{x}(t) \frac{\partial}{\partial x} + \dot{y}(t) \frac{\partial}{\partial y} = -x(t)^2 \frac{\partial}{\partial x} - y(t) \frac{\partial}{\partial y}$

$\Rightarrow$, to solve: $\begin{cases} \dot{x} = -x^2 \\ \dot{y} = -y \end{cases} \Rightarrow$

$\Rightarrow$ solutions that depend on two constants $a, b \in \mathbb{R}$:

$\gamma_{a,b}(t) = \left(\frac{a}{at+1}, be^{-t} \right) \dots$ again $\gamma_{a,b}(0) = (a, b)$

Rk 1: we do find unique sol. with given starting point

Rk 2: For $a \neq 0, \gamma_{a,b}$ is not defined on $\mathbb{R}$. $p = (a, b)$

Instead: on $I \subseteq (-\infty, -\frac{1}{a})$ or $I \subseteq (-\frac{1}{a}, \infty)$

Def: X called complete if $(\forall) p \in M \exists$ integral curve $\gamma: \mathbb{R} \rightarrow M$ starting at p .

Thm: If $M = \text{compact} \Rightarrow$ all $X \in \mathcal{X}(M)$ are complete

More general

$X \in \mathcal{X}$

Curve γ

Integral equation

Initial

Reminder:
 - vector fields X on $M: M \ni p \mapsto X_p \in T_p M$ smooth
 i.e., $(\forall) \chi: U \rightarrow \mathbb{R}^n$ chart, writing

$$X_p = \sum F_i(\chi(p)) \left(\frac{\partial}{\partial x_i} \right)_p$$
 then $F_i: U \rightarrow \mathbb{R}$ are smooth
 the set $\mathcal{X}(M)$ of vector fields on M is a $C^\infty(M)$ -module and a Lie algebra

($X_h \in \mathcal{X}(U) \mapsto X_h(X_h) \in \mathcal{X}(U)$) written as $\sum F_i \frac{\partial}{\partial x_i}$

- globally $X \leftrightarrow$ derivation $L_X: C^\infty(M) \rightarrow C^\infty(M)$
 $L_X(f)(p) = \lim_{t \rightarrow 0} \frac{f(\gamma_t(p)) - f(p)}{t}$

- speeds of curves $\gamma: I \rightarrow M$ at $t=0: \dot{\gamma}(0) = \frac{d\gamma}{dt}(0) \in T_{\gamma(0)} M$ (also denoted $\frac{d}{dt} \Big|_{t=0} \gamma(t)$)
 At arbitrary $t \in I: \dot{\gamma}(t) = \frac{d\gamma}{dt}(t) = \frac{d}{dt} \Big|_{t=t_0} \gamma(t) \in T_{\gamma(t)} M$

- Ex: $\text{For } \Omega \subseteq \mathbb{R}^n \text{ open} \Rightarrow$ vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in \mathcal{X}(\Omega)$
 and arbitrary ones as $\sum F_i \frac{\partial}{\partial x_i}$, with $F_i \in C^\infty(\Omega)$
 For $\gamma: I \rightarrow \Omega$, writing $\gamma = (x_1, \dots, x_n) \Rightarrow \dot{\gamma}(t) = \sum \dot{\gamma}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}$

- On a general M , $(\forall)$ chart $\chi: U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^n$ induces $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in \mathcal{X}(U)$
 and, for $\gamma: I \rightarrow M$, at points t with $\gamma(t) \in U$

$$\dot{\gamma}(t) = \sum_{i=1}^n \dot{\gamma}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}$$
 normal derivative of $(\gamma^i)_1: I \rightarrow \mathbb{R}$

Def: Given $X \in \mathcal{X}(M)$, an integral curve of X is any curve $\gamma: I \rightarrow M$ ($I \subseteq \mathbb{R}$ open interval) such that: $\dot{\gamma}(t) = X_{\gamma(t)}$ ($\forall t \in I$)
 Says: γ starts at $p \in M$ if $0 \in I, \gamma(0) = p$

Consider $F = (F_1, \dots, F_m): \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$
 Look at the equation $\dot{\gamma}(t) = F(\gamma(t)), \gamma(0) = p$
 Theorem (Cauchy's local $\exists$ and $!$)
 ($\forall$) $x_0 \in \Omega$ (*) has solution $\gamma: I \rightarrow \Omega$ starting at x_0
 & ($\forall$) two solutions $\gamma_1, \gamma_2: I_1 \rightarrow \Omega$ of (*)
 Moreover, the solution of (*) i.e. ...

Ex 1: $M = \mathbb{R}^2, X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \in \mathcal{X}(\mathbb{R}^2), \lambda = \text{fixed constant}$
 Integral curve $\gamma(t) = (x(t), y(t))$ satisfying $\dot{\gamma}(t) = X_{\gamma(t)} \Rightarrow \begin{cases} \dot{x}(t) = 1 \\ \dot{y}(t) = \lambda \end{cases}$

$$\dot{x}(t) \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \dot{y}(t) \left(\frac{\partial}{\partial y} \right)_{\gamma(t)} = \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \lambda \left(\frac{\partial}{\partial y} \right)_{\gamma(t)}$$

 $\Rightarrow \begin{cases} x(t) = t + a \\ y(t) = \lambda t + b \end{cases}, a, b \in \mathbb{R} \text{ some const.} \Rightarrow \begin{cases} (\forall) a, b \in \mathbb{R} \text{ we get an integral curve } \gamma_{a,b} \\ \gamma_{a,b}(t) = (t+a, \lambda t+b) \end{cases}$
 Rk1: $(\forall) p = (a, b) \in \mathbb{R}^2, \exists!$ integral curve $\gamma_{a,b}: \mathbb{R} \rightarrow \mathbb{R}^2$ starting at p
 Rk2: $\gamma_{a,b}$ is defined on the entire $\mathbb{R}$!
 Ex 2: Same for $X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \in \mathcal{X}(\mathbb{R}^2)$

Ex 2: $M = \mathbb{R}^2, X = -y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$
 $\gamma(t) = (x(t), y(t)) \dots$ equation $\dot{\gamma}(t) \frac{\partial}{\partial x} + \dot{y}(t) \frac{\partial}{\partial y} = -x(t) \frac{\partial}{\partial x} - y(t) \frac{\partial}{\partial y}$
 $\Rightarrow$ to solve $\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases} \Rightarrow$ solutions that depend on two constants $a, b \in \mathbb{R}$
 $\gamma_{a,b}(t) = \left(\frac{a}{a^2+b^2} \cos t, \frac{b}{a^2+b^2} \sin t \right)$ again $\gamma_{a,b}(0) = (a, b)$
 Rk1: we do find unique sol. with given starting point $p = (a, b)$
 Rk2: For $a \neq 0, \gamma_{a,b}$ is not defined on $\mathbb{R}$. Instead on $I = (-\infty, -\frac{1}{a})$ or $I = (-\frac{1}{a}, \infty)$
 Def: X called complete if $(\forall) p \in M \exists$ integral curve $\gamma: \mathbb{R} \rightarrow M$ starting at p .
 Thm: If M is compact $\Rightarrow$ all $X \in \mathcal{X}(M)$ are complete

More generally, for $\Omega \subseteq \mathbb{R}^n$ open
 $X \in \mathcal{X}(\Omega), X = \sum F_i \frac{\partial}{\partial x_i}$
 (curve $\gamma = (\gamma_1, \dots, \gamma_m)$)
 Integral curve equations $\begin{cases} \dot{\gamma}_i(t) = F_i(\gamma(t)) \end{cases}$
 Initial condition $\gamma_i(0) = c_i$

Consider $F = (F_1, \dots, F_m): \underset{\mathbb{R}^m}{\mathcal{U}} \rightarrow \mathbb{R}^m$ smooth. (6)

Look at the equation

$$\dot{\gamma}(t) = F(\gamma(t)), \quad \gamma(0) = \gamma_0$$

Theorem (Cauchy's local $\exists$ and $!$)

(*) $\gamma_0 \in \mathcal{U}$ (*) has solutions $\gamma: I \rightarrow \mathcal{U}$

& (V) two solutions $\begin{cases} \gamma_1: I_1 \rightarrow \mathcal{U} \\ \gamma_2: I_2 \rightarrow \mathcal{U} \end{cases}$ of (*), they must coincide on some $I_0 \subseteq I_1 \cap I_2$

Moreover, the solution of (*) depends smoothly in γ_0 , i.e. ...

More generally, on $\mathcal{U} \subseteq \mathbb{R}^m$ open (5)

$$X \in \mathcal{X}(\mathcal{U}), \quad X = \sum F_i \frac{\partial}{\partial x_i}$$

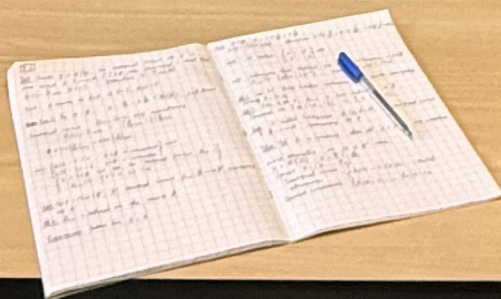
Curve $\gamma = (\gamma_1, \dots, \gamma_m)$

Integral curve equations $\begin{cases} \dot{\gamma}_i(t) = F_i(\gamma_1(t), \dots, \gamma_m(t)) \end{cases}$

Initial condition $\begin{cases} \gamma_i(0) = c_i, \dots, \gamma_m(0) = c_m \end{cases}$

starting point $p = (c_1, \dots, c_m)$

curve $\gamma: \mathbb{R} \rightarrow M$ starting at p



- Reminder
 - vector fields X on M : $M \ni p \mapsto X_p \in T_p M$ smooth
 i.e. $(M, X, U) \hookrightarrow \mathbb{R}^2$ chart, writing

$$X_p = \sum_i F_i(x(p)) \left(\frac{\partial}{\partial x_i} \right)_p$$
 then $F_i: U \rightarrow \mathbb{R}$ are smooth and $\dots$
 $\left(x_h \mapsto X_h(x_h) \right) \in \mathbb{R}(U)$ written as $\sum_i F_i \frac{\partial}{\partial x_i}$ } - a Lie bracket
 - globally $X \leftrightarrow$ derivation $L_X: C^\infty(M) \rightarrow C^\infty(M)$
 $L_X(f)(p) = \sum_i F_i(p) \frac{\partial f}{\partial x_i}(p)$
 - speeds of curves $\gamma: I \rightarrow M$ at $t=0$: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) \in T_{\gamma(0)} M$ (also denoted
 At arbitrary $t_0 \in I$ $\left. \dot{\gamma}(t_0) = \frac{d}{dt} \gamma(t_0) = \frac{d}{dt} \Big|_{t=t_0} \gamma(t) \right\} \in T_{\gamma(t_0)} M$
 - Ex. $F_i: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ open $\rightarrow$ vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in \mathbb{R}(U)$
 and arbitrary ones $\alpha = \sum_i F_i \frac{\partial}{\partial x_i}$, with $F_i \in C^\infty(U)$

$$F_i \gamma: I \rightarrow \mathbb{R}, \text{ writing } \gamma = (x_1, \dots, x_n) \mapsto \dot{\gamma}(t) = \sum_i \dot{x}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}$$

and, for $\gamma: I \rightarrow M$, at points t with $\gamma(t) \in U$

$$\dot{\gamma}(t) = \underbrace{\dot{\gamma}_1(t)}_{\text{usual derivative of } \gamma_1} + \dots + \dot{\gamma}_m(t) \left(\frac{\partial}{\partial x_m} \right)_{\gamma(t)}$$

derivative of (γ_1)

Def: Given $X \in \mathcal{X}(M)$, an integral curve of X is any curve $\gamma: I \rightarrow M$ ($I \subset \mathbb{R}$ open interval) such that: $\dot{\gamma}(t) = X_{\gamma(t)}$ ($\forall t \in I$)
Says: γ starts at (equality in $T_{\gamma(t)} M$)
 $p \in M$ if $0 \in I, \gamma(0) = p$

Look at the equation
 $\dot{x}(t) = F(x(t)), x(0) = x_0$

$$(\forall) x_0 \in \mathcal{R} \quad (*) \text{ has solutions } \gamma: I \rightarrow \mathcal{R}$$

& $\forall$ two solutions $\begin{cases} I_1: I_1 \rightarrow \mathcal{R} \\ I_2: I_2 \rightarrow \mathcal{N} \end{cases}$, they must coincide on some $I_0 \subseteq I_1 \cap I_2$

Moreover, the solution of (4) depends smoothly in x_0 ,
i.e. ...

Integral curve $\gamma(t) = (x(t), y(t))$ satisfies $\left\{ \begin{array}{l} \dot{x}'(t) = \lambda \\ \dot{y}'(t) = \lambda \end{array} \right.$

$$\dot{\gamma}(t) = \chi_{\gamma(t)} = \left\{ \begin{array}{l} \lambda \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \lambda \left(\frac{\partial}{\partial y} \right)_{\gamma(t)} \end{array} \right.$$

$\Rightarrow \left\{ \begin{array}{l} x(t) = t + a \\ y(t) = \lambda t + b \end{array} \right. \quad a, b \in \mathbb{R} \text{ some const.} \Rightarrow \left\{ \begin{array}{l} \forall a, b \in \mathbb{R} \text{ we get} \\ \text{an integral curve } \gamma_{a,b} \\ \gamma_{a,b}(t) = (t+a, \lambda t+b) \end{array} \right.$

Rk2: $\gamma_{a,b} \mapsto \gamma_{a,b} \in \mathbb{R}^2$, $\exists!$ integral curve $\gamma_{a,b}: \mathbb{R} \rightarrow \mathbb{R}^2$ starting at q
Rk2: $\gamma_{a,b}$ is defined on the entire $\mathbb{R}$!

Exercise: Show for $X = y \frac{\partial}{\partial x} - z \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3)$

$r(t) = (x(t), y(t)) \dots$ Equation $x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y} = -\text{div} \frac{\partial}{\partial x} - \text{div} \frac{\partial}{\partial y}$
 $\rightarrow$ to solve $\begin{cases} x' = -x^2 \\ y' = -y \end{cases} \rightarrow$
 $\Rightarrow$ solutions that depend on two constants $a, b \in \mathbb{R}$:
 $x_b(t) = \left(\frac{a}{a+t}, b e^t \right) \dots$ again $f_{a,b}(0) = (a, b)$

Prk1: we do find unique no. with given starting point
Prk2: For $a \neq 0$, x_1, x_2 is not defined. $p = (a, b)$

Instead on $I \subseteq (-\infty, -\frac{1}{a})$ or $I \subseteq (-\frac{1}{a}, \infty)$

Def: X called complete if $(\forall) p \in M \exists$ integral curve $r: \mathbb{R} \rightarrow M$
starting at p

Thm: $\mathcal{J} \mathcal{M} = \text{compact} \Rightarrow$ all $X \in \mathcal{X}(\mathcal{M})$ are complete

$$X \in \mathcal{K}(n), \quad X = \sum F_i \frac{\partial}{\partial x_i}$$

(curve $\gamma = (\gamma_1, \dots, \gamma_m)$)

Integral curve equations $\left\{ \begin{aligned} \dot{x}_i(t) &= F_i(x_1(t), \dots, x_n(t)) \end{aligned} \right.$

Initial condition $\left\{ \begin{array}{l} y_1(0) = c_1, \dots, y_n(0) = c_n \end{array} \right.$

Prop: Given $X \in \mathfrak{X}(M)$

- (1) $p \in M$, $\exists$ integral curve $\gamma: I \rightarrow M$ of X with $\gamma(0) = p$
- any two integral curve $\gamma_1: I_1 \rightarrow M$ which coincide at some time t_0 , must coincide on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$

Rk: Actually $\Rightarrow$ some topology $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$!

pf: Look at:

$$A := \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\} \subseteq I_1 \cap I_2$$

this $\Rightarrow A$ is closed in $I_1 \cap I_2$

But $A = \emptyset$ is open in $I_1 \cap I_2$

But $I_1 \cap I_2$ is connected, $A \neq \emptyset$
 $\Rightarrow A$ is both closed and open in $I_1 \cap I_2$! $\Rightarrow A = I_1 \cap I_2$

Corollary: Given $X \in \mathfrak{X}(M)$, $p \in M$, $\exists!$

$$\gamma_p^X: I_p^X \rightarrow M$$

MAXIMAL integral curve of X with $\gamma_p^X(0) = p$.
 i.e. $\nexists \tilde{\gamma}: \tilde{I} \rightarrow M$ int. curve with $\tilde{I} \supsetneq I_p^X$

Previous examples.

Ex 1: $I_p^X = \mathbb{R}$ $(\forall) p = (a, b) \in \mathbb{R}^2$

Ex 2: $I_p^X = \begin{cases} (-\frac{1}{a}, \infty) & \text{if } a > 0 \\ \mathbb{R} & \text{if } a = 0 \\ (-\infty, -\frac{1}{a}] & \text{if } a < 0 \end{cases}$

$\gamma_p^X(a, b) = \left(\frac{a}{at+1}, be^{-t} \right)$ maximal

Reminder:

- vector fields X on M : $M \ni p \mapsto X_p \in T_p M$ smooth
 i.e. $(\forall) X: U \xrightarrow{\sim} \Omega$ chart, writing the set $\mathfrak{X}(M)$ of vector fields

- On a general M , $(\forall)$ chart $X, U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^m$ induces

9)

$$\gamma_p^X(0) = p$$
$$\supset I_p^X$$
$$\neq I_p^X$$

$\mathbb{R}^2$
 $a > 0$
 $a = 0$
 $a < 0$
anal

In general, given $X \in \mathcal{X}(M)$, given $p \in M$, the problem (7)

$$\boxed{\dot{x}(t) = X(x(t)), x(0) = p}$$

can be looked in a chart $\chi: U \xrightarrow{\sim} \mathcal{O}$ around p .

End up with $\dot{x}^X(t) = F_i^X(x^X(t))$ where F_i^X - coef of X in χ .

Cauchy can be applied inside $\mathcal{O}$



Same statement is true for M, p, X arbitrary

$\exists$ Sol.

local uniqueness

Fix $X \in \mathfrak{X}(M)$.

$$\gamma_p(t) \in M$$

$$p \in M, t \in I_p$$

As a function of p & t , it is defined on

$$\mathcal{D}(X) := \{(p, t) \in M \times \mathbb{R} \mid t \in I_p\} \subseteq M \times \mathbb{R}$$

(contains $M \times \{0\}$!)

& the function is

$$\mathcal{D}(X) \longrightarrow M, (p, t) \longmapsto \gamma_p(t)$$

open (by smoothness in t in Cauchy's theorem)

the flow of X the domain of the flow.

OR: (assume $X = \text{complete}$) for each t we get a function

$$M \longrightarrow M, p \longmapsto \gamma_p(t)$$

Reminder:

- vector fields X on M : $M \ni p \mapsto X_p \in T_p M$ smooth
 i.e., $(\forall) X: U \rightarrow \mathbb{R}^m$ chart, writing
 $X_p = \sum F_i(x(p)) \left(\frac{\partial}{\partial x_i} \right)_p$ then $F_i: U \rightarrow \mathbb{R}$ are smooth and

$(X|_U) \mapsto X|_U \in \mathfrak{X}(U)$ written as $\sum F_i \frac{\partial}{\partial x_i}$

- globally $X \mapsto$ derivation $L_X: C^\infty(M) \rightarrow C^\infty(M)$
 $L_X(f)(p) = \sum F_i(p) \frac{\partial f}{\partial x_i}(p) = \frac{d}{dt} f(\gamma(t))|_{t=0}$

- speeds of curves $\gamma: I \rightarrow M$ at $t=0$: $\dot{\gamma}(0) = \frac{d\gamma}{dt}(0) \in T_{\gamma(0)} M$ (also denoted $\frac{d}{dt}|_{t=0} \gamma(t)$)
 At arbitrary $t_0 \in I$: $\dot{\gamma}(t_0) = \frac{d\gamma}{dt}(t_0) = \frac{d}{dt}|_{t=t_0} \gamma(t) \in T_{\gamma(t_0)} M$

- Ex: for $\Omega \subset \mathbb{R}^m$ open $\Rightarrow$ vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(\Omega)$
 and arbitrary ones $\sum F_i \frac{\partial}{\partial x_i}$, with $F_i \in C^\infty(\Omega)$

For $\gamma: I \rightarrow \Omega$, writing $\gamma = (x_1, \dots, x_m) \Rightarrow \dot{\gamma}(t) = \sum \dot{x}_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)}$

- On a general M , $(\forall)$ chart $X: U \xrightarrow{\sim} \Omega \subset \mathbb{R}^m$ induces $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(U)$
 and, for $\gamma: I \rightarrow M$, at points t with $\gamma(t) \in U$:
 $\dot{\gamma}(t) = \dot{x}_1(t) \left(\frac{\partial}{\partial x_1} \right)_{\gamma(t)} + \dots + \dot{x}_m(t) \left(\frac{\partial}{\partial x_m} \right)_{\gamma(t)}$
 usual derivative of $(x^i)_{i=1}^m: I \rightarrow \mathbb{R}$

Def: Given $X \in \mathfrak{X}(M)$, an integral curve of X is any curve $\gamma: I \rightarrow M$ ($I \subset \mathbb{R}$ open interval) such that: $\dot{\gamma}(t) = X_{\gamma(t)}$ $(\forall) t \in I$
 Say: γ starts at $p \in M$ if $0 \in I, \gamma(0) = p$

Corollary: Given $X \in \mathfrak{X}(M), p \in M, \exists!$ $\gamma_p^X: I_p^X \rightarrow M$
 MAXIMAL integral curve of X with $\gamma_p^X(0) = p$
 i.e. $\nexists \tilde{\gamma}: \tilde{I} \rightarrow M$ int. curve with $\tilde{I} \supsetneq I_p^X$

Previous examples:
 Ex 1: $I_p^X = \mathbb{R}$ $(\forall) p = (a, b) \in \mathbb{R}^2$
 Ex 2: $I_p^X = \begin{cases} (-\frac{1}{a}, \infty) & \text{if } a > 0 \\ \mathbb{R} & \text{if } a = 0 \\ (-\infty, -\frac{1}{a}) & \text{if } a < 0 \end{cases}$
 $p = (a, b)$
 $\gamma_p^X(a, b) = \left(\frac{a}{a+t}, b e^{-t} \right)$ maximal

Prop: Given $X \in \mathfrak{X}(M)$

- $(\forall) p \in M, \exists!$ integral curve $\gamma: I \rightarrow M$ of X with $\gamma(0) = p$
- any two integral curve $\gamma_1: I_1 \rightarrow M$ which coincide at some time t_0 , must coincide on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$

Rk: Actually $\Rightarrow$ some topology $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$!

pf: Look at:

$$A := \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\} \subseteq I_1 \cap I_2$$

this $\Rightarrow A$ is closed in $I_1 \cap I_2$

But $2 \equiv A$ is open in $I_1 \cap I_2$

$\Rightarrow A$ is both closed and open in $I_1 \cap I_2 \Rightarrow A = I_1 \cap I_2$
 But $I_1 \cap I_2$ is connected, $A \neq \emptyset$

Fix $X \in \mathfrak{X}(M)$. $\gamma_p^X(t) \in M$ $p \in M, t \in I_p$

As a function of p & t , it is defined on $\mathcal{D}(X) := \{(p, t) \in M \times \mathbb{R} / t \in I_p\} \subseteq M \times \mathbb{R}$ (contain $M \times \{0\}$!)
 & the function is $\mathcal{D}(X) \rightarrow M, (p, t) \mapsto \gamma_p^X(t)$
 the flow of X the domain of the flow

OR: (assume X = complete) for each t we get a function $M \rightarrow M, p \mapsto \gamma_p^X(t)$

Consider $F = (F_1, \dots, F_m): \Omega \rightarrow \mathbb{R}^m$ smooth.

Look at the equation

$$\dot{\gamma}(t) = F(\gamma(t)), \gamma(0) = x_0 \quad (*)$$

Theorem (Cauchy's local $\exists$ and $!$)

$(\forall) x_0 \in \Omega$ $(*)$ has solutions $\gamma: I \rightarrow \mathbb{R}$ (some $I \ni 0$)
 & $(\forall)$ two solutions $\gamma_1: I_1 \rightarrow \Omega, \gamma_2: I_2 \rightarrow \Omega$ they must coincide on some $I_0 \subseteq I_1 \cap I_2$ of x_0

Moreover, the solution of $(*)$ depends smoothly in x_0 i.e. ...