

Reminder: Given M . Given $X \in \mathfrak{X}(M)$.

- integral curve of X : any $\gamma: I \rightarrow M$ smooth st. $\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases}$ (eg)
- starts at p : $\exists \gamma \in I$ and ...
- two explicit examples in $M = \mathbb{R}^2$, where $(\forall) p = (a, b) \in \mathbb{R}^2$:

Ex 1: $X_1 = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$
 $\Rightarrow \gamma_{a,b}(t) = (t+a, \lambda t+b)$

Ex 2: $X_2 = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$
 $\gamma_{a,b}(t) = (\frac{a}{at+1}, be^{-t})$

- locally, w.r.t a chart $\chi: U \rightarrow \mathbb{R}^m$ around p , we move to $\mathcal{U}$:

$$\gamma^\chi = \chi \circ \gamma, \quad \chi_* X(p) = \sum F_i \frac{\partial}{\partial x_i}, \quad F_i \in C^\infty(\mathcal{U})$$

and (eg) becomes: $\begin{cases} \dot{\gamma}^\chi(t) = F(\gamma^\chi(t)) \\ \gamma^\chi(0) = x_0 \end{cases}$ (unknown γ^χ in $\mathcal{U}$)

↓ (page 0)

① $(\forall) p \in M, \exists$ integral curve of X starting at p

② if $\begin{cases} \gamma_1: I_1 \rightarrow M \\ \gamma_2: I_2 \rightarrow M \end{cases}$ integral curves of X that coincide at some $t_0 \in I_1 \cap I_2$
 $\Downarrow$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

$(\forall) p \in M \exists!$ maximal integral curve γ_p^X of X that starts at p
 $\gamma_p^X: I_p^X \rightarrow M$ ($I_p^X \subseteq \mathbb{R}$ open interval contains 0)

Rearrange $\gamma_p^X(t) =: \phi_X(p, t) \Rightarrow$ the flow of the vector field X ,

$\phi_X: \mathcal{D}(X) \rightarrow M$ defined on $\mathcal{D}(X) = \{(p, t) \in M \times \mathbb{R} : t \in I_p^X\} \subseteq M \times \mathbb{R}$.

The complete case:

Def: X is called complete
 $(\Leftrightarrow)$ all maximal

Theorem: $M = \text{compact}$
 Assuming $X = \text{complete}$

Proposition: Given M , given $X \in \mathfrak{X}(M)$

Integral curve of X along $\gamma: I \rightarrow M$ smooth s.t. $\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases}$

- starts at p , $\forall t \in I$ and

- the explicit example in $M = \mathbb{R}^2$, where X is given by $\mathbb{R}^2$

$$\begin{aligned} \text{Ex 1: } X = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} & \quad \text{Ex 2: } X = -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \\ \Rightarrow X_{\gamma(t)}(t, x(t), y(t)) & \quad X_{\gamma(t)}(t, x(t), y(t)) \end{aligned}$$

- locally, we can chart X as $\frac{\partial}{\partial t}$ around p , we move to $\mathbb{R}^n$

$$t^i = x^i, x = x(t), X(X) = \sum \frac{\partial}{\partial x_i} \quad F_i \in C^\infty(\mathbb{R}^n)$$

and (1) becomes: $\begin{cases} \dot{\gamma}^i(t) = F^i(\gamma(t)) \\ \gamma^i(0) = x_0^i \end{cases}$ (extension of X in $\mathbb{R}^n$)

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③ (1) $p \in M$, integral curve of X starting at p

② if $\gamma_1, \gamma_2: I_1 \rightarrow M$ integral curves of X that coincide at some $t_0 \in I_1 \cap I_2$
 $\downarrow$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

[$\forall p \in M$, $\exists$ maximal integral curve of X that starts at p
 $\gamma: I \rightarrow M$ (I is maximal connected)]

Rearrange $\frac{d}{dt} \gamma(t) = X(\gamma(t)) \Rightarrow$ the flow of the vector field X .

$\Phi_X: \mathbb{R} \times M \rightarrow M$ defined on $\mathcal{D}(X) = \{(t, x) \in \mathbb{R} \times M \mid t \in I_x, x \in M\}$

The complete case.

Def: X is called complete if $\forall p \in M$, $\exists$ has solution $\gamma: \mathbb{R} \rightarrow M$
 $\Leftrightarrow$ all maximal int. curves are defined on $\mathbb{R} \Rightarrow \mathcal{D}(X) = M \times \mathbb{R}$

Theorem: $M = \text{compact} \Rightarrow \forall X \in \mathfrak{X}(M)$ are complete (to be proven)

Assuming X is complete $\Rightarrow$ get $\Phi_X: M \times \mathbb{R} \rightarrow M$

Cauchy's local $\exists \&!$ Given $F = (F_1, \dots, F_m): \mathcal{D} \rightarrow \mathbb{R}^m$ smooth
 $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}^m$
 For each $x_0 \in \mathcal{D}$ consider the ODE $\begin{cases} \dot{\gamma}(t) = F(\gamma(t)) \\ \gamma(0) = x_0 \end{cases}$ (unknown)
 $\gamma: I \rightarrow \mathcal{D}$

Theorem: $(\forall) x_0 \in \mathcal{D}$, $(*)_{x_0}$ has solutions $\gamma: I \rightarrow \mathcal{D}$

$\&$ $(\forall)$ two solutions $\begin{cases} \gamma_1: I_1 \rightarrow \mathcal{D} \\ \gamma_2: I_2 \rightarrow \mathcal{D} \end{cases}$ must coincide on some $I \subset I_1 \cap I_2$

Moreover, "the solution of $(*)_{x_0}$ depends smoothly in x_0 ", i.e.,

$(\forall) x_0 \in \mathcal{D}$, $\exists \Gamma: \mathcal{D}_{x_0} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{D}$ smooth

Some open $\mathcal{D}$ containing x_0

Such that each $\Gamma(x, -) = (-\varepsilon, \varepsilon) \rightarrow \mathcal{D}$ is a solution of $(*)$

$$y^i = f^i(x), \quad x_i = x_i(p), \quad X_*(X) = \sum F_i \frac{\partial}{\partial x_i}, \quad F_i \in C^\infty(\Omega)$$

and (eq) becomes: $\begin{cases} \dot{y}^i(t) = F^i(y(t)) \\ y^i(0) = x_0 \end{cases}$ (unknown $\dot{y}^i$ in Ω)

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$$\begin{cases} \frac{d\phi_p}{dt}(t) = X_{\phi_p(t)} \\ \phi_p(0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi(p, t) = X_{\phi(p, t)} \\ \phi(p, 0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p \end{cases}$$

① $(\forall) p \in M, \exists$ integral curve of X starting at p

② if $\begin{cases} \gamma_1: I_1 \rightarrow M \\ \gamma_2: I_2 \rightarrow M \end{cases}$ integral curves of X that coincide at some $t_0 \in I_1 \cap I_2$
 $\downarrow$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

$(\forall) p \in M \exists!$ maximal integral curve γ_p^X of X that starts at p
 $\gamma_p^X: I_p^X \rightarrow M$ ($I_p^X \subseteq \mathbb{R}$ open interval contains 0)

Rearrange $\gamma_p^X(t) =: \phi_X(p, t) \Rightarrow$ the flow of the vector field X ,
 $\phi_X: \mathcal{D}(X) \rightarrow M$ defined on $\mathcal{D}(X) = \{(p, t) \in M \times \mathbb{R} : t \in I_p^X\} \subseteq M \times \mathbb{R}$.

If we fix t , $\phi_X^t(p) := \phi_X(p, t) = \gamma_p^X(t)$
 ϕ_X^t is called the flow of X at the time t : $\phi_X^t: M \rightarrow M$
 if lucky

ex: Ex 1: $I_p^X = \mathbb{R}$
 Ex 2: $I_p^X = \begin{cases} (-\frac{1}{a}, \infty) & \text{if } a > 0 \\ \mathbb{R} & a = 0 \\ (-\infty, -\frac{1}{a}) & \text{if } a < 0 \end{cases}$
 $p = (a, b)$

The complete case:

Def: X is called complete if $(\forall) p \in M$, (eg) has solution $\gamma: \mathbb{R} \rightarrow M$
 $(\Leftrightarrow)$ all maximal int. curves are defined on $\mathbb{R} \Leftrightarrow \mathcal{D}(X) = M \times \mathbb{R}$.

Theorem: $M = \text{compact} \Rightarrow (\forall) X \in \mathcal{X}(M)$ are complete (to be proven).

Assuming $X = \text{complete} \Rightarrow$ get $\phi_X: M \times \mathbb{R} \rightarrow M$, $\phi_X^t: M \rightarrow M$

Theorem: If $X = \text{complete} \Rightarrow \phi_X$ is smooth, each ϕ_X^t is a diffeom.
 and $\boxed{\phi_X^t \circ \phi_X^s = \phi_X^{t+s} \quad (\forall) t, s \in \mathbb{R}} \quad (*)$

$\begin{cases} \frac{d}{dt} \gamma(t) = X_{\gamma(t)} \\ \gamma(0) = p \end{cases}$	$\begin{cases} \frac{d}{dt} \phi(p, t) = X_{\phi(p, t)} \\ \phi(p, 0) = p \end{cases}$	$\begin{cases} \frac{d}{dt} \phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p \end{cases}$
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pf: Cauchy $\Rightarrow \phi_X = \text{smooth}$.

Claim: $(*) \Rightarrow$ each ϕ_X^t is a diffeo.

pf: $\phi_X^0 = \text{Id}_M$, & taking $s = -t$ in $(*) \Rightarrow \phi_X^{-t}$ is the inverse of ϕ_X^t
 $\Rightarrow \phi_X^t \circ \phi_X^{-t} = \phi_X^0 = \text{Id}_M = \phi_X^{-t} \circ \phi_X^t$

Fix $p \in M$ and prove: $\phi_X^t(\phi_X^s(p)) = \phi_X^{t+s}(p) \quad (\forall) s, t$

Fix s and look at LHS & RHS as functions of t .

$t \xrightarrow{u} \phi_X^t(\phi_X^s(p))$	$\begin{cases} u(0) = \phi_X^s(p) \\ v(0) = \phi_X^s(p) \end{cases}$	$\begin{cases} \frac{du}{dt}(t) = X_{\phi_X^t(\phi_X^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{cases}$
$t \xrightarrow{v} \phi_X^{t+s}(p)$		

$\Rightarrow$ two int. curves of X starting at $\phi_X^s(p) \Rightarrow u = v$. $\square$

Cauchy's local $\exists$ & ! Given $F = (F_1, \dots, F_m): \Omega \rightarrow \mathbb{R}^m$ smooth
 Ω open

for each $x_0 \in \Omega$ consider the ODE $\boxed{\dot{\gamma}(t) = F(\gamma(t))}$ with init. $\gamma(0) = x_0$ / unknown

Reminder: Given M . Given $X \in \mathfrak{X}(M)$.

- integral curve of X : any $\gamma: I \rightarrow M$ smooth st. $\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases}$ (eq)

- starts at p : if $0 \in I$ and...

- two explicit examples in $M = \mathbb{R}^2$, where $\gamma(p) = (a, b) \in \mathbb{R}^2$.

Ex 1: $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ $\gamma_a(t) = (t+a, t+b)$ $\gamma_b(t) = (\frac{a}{a+1}, e^{-t})$ at $t=0$

- locally, wrt a chart $X \xrightarrow{\varphi} \mathbb{R}^2$ around p , we move to $\mathbb{R}^2$.

$\gamma^1 = \gamma, \gamma^2 = \gamma(p), X_i(X) = \sum F_i \frac{\partial}{\partial x_i}, F_i \in C^\infty(\mathbb{R}^2)$

and (eq) becomes: $\begin{cases} \dot{\gamma}^i(t) = F^i(\gamma^1(t)) \\ \gamma^i(0) = \gamma^i_0 \end{cases}$ (unknown γ^i in $\mathbb{R}^2$)

↓ (page 0)

① $(\forall) p \in M, \exists$ integral curve of X starting at p

② if $\gamma_1: I_1 \rightarrow M$ integral curves of X that coincide at some $t_0 \in I_1 \cap I_2$
 $\downarrow$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

$(\forall) p \in M \exists!$ maximal integral curve of X that starts at p
 $\gamma_p^X: I_p^X \rightarrow M$ ($I_p^X \subseteq \mathbb{R}$ open interval containing 0)

Rearrange $\gamma_p^X(t) =: \phi_X^t(p) \Rightarrow$ the flow of the vector field X .

$\phi_X: \mathcal{D}(X) \rightarrow M$ defined on $\mathcal{D}(X) = \{(p, t) \in M \times \mathbb{R} : t \in I_p^X\} \subseteq M \times \mathbb{R}$.

If we fix t , $\phi_X^t(p) =: \phi_X^t(p) = \gamma_p^X(t)$ ($\gamma_p^X(t)$ is defined)

ϕ_X^t is called the flow of X at the time t : $\phi_X^t: M \rightarrow M$ if lucky

The complete case:

Def: X is called complete if $(\forall) p \in M, (eq)$ has solution $\gamma: \mathbb{R} \rightarrow M$
 $(\Leftrightarrow)$ all maximal int. curves are defined on $\mathbb{R} \Rightarrow \mathcal{D}(X) = M \times \mathbb{R}$

Theorem: $M = \text{compact} \Rightarrow (\forall) X \in \mathfrak{X}(M)$ are complete (to be proven)

Assuming X is complete $\Rightarrow$ get $\phi_X: M \times \mathbb{R} \rightarrow M, \phi_X^t: M \rightarrow M$

Theorem: If X is complete $\Rightarrow \phi_X$ is smooth, each ϕ_X^t is a diffeom.
 and $\phi_X^t \circ \phi_X^s = \phi_X^{t+s} \quad (\forall) t, s \in \mathbb{R}$ (*)

$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X^0(p) = p \end{cases}$

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$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X^0(p) = p \end{cases}$

pl: Cauchy $\Rightarrow \phi_X$ is smooth.

Claim: (*) $\Rightarrow$ each ϕ_X^t is a diffeo.
 pl: $\phi_X^0 = \text{Id}_M$, & taking $s = -t$ in (*) $\Rightarrow \phi_X^{-t} \circ \phi_X^t = \text{Id}_M$
 $\Rightarrow \phi_X^t \circ \phi_X^{-t} = \text{Id}_M = \phi_X^0 = \phi_X^{-t} \circ \phi_X^t$

Fix $p \in M$ and prove: $\phi_X^t(\phi_X^s(p)) = \phi_X^{t+s}(p) \quad (\forall) s, t$

Fix s and look at LHS & RHS as functions of t .

$\begin{cases} t \mapsto \phi_X^t(\phi_X^s(p)) \\ t \mapsto \phi_X^{t+s}(p) \end{cases} \quad \begin{cases} u(0) = \phi_X^s(p) \\ v(0) = \phi_X^s(p) \end{cases} \quad \begin{cases} \frac{du}{dt}(t) = X_{\phi_X^t(\phi_X^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{cases}$

$\Rightarrow$ two int. curves of X starting at $\phi_X^s(p) \Rightarrow u = v. \quad \square$

Cauchy's local $\exists!$ Given $F = (F_1, \dots, F_m): \mathcal{O} \rightarrow \mathbb{R}^m$ smooth
 $\mathcal{O}$ open

for each $x_0 \in \mathcal{O}$ consider the ODE $\begin{cases} \dot{\gamma}(t) = F(\gamma(t)) \\ \gamma(0) = x_0 \end{cases}$ (unknown $\gamma: I \rightarrow \mathcal{O}$)

Theorem: $(\forall) x_0 \in \mathcal{O}, (*)_{x_0}$ has solutions $\gamma: I \rightarrow \mathcal{O}$

& $(\forall)$ two solutions $\begin{cases} \gamma_1: I_1 \rightarrow \mathcal{O} \\ \gamma_2: I_2 \rightarrow \mathcal{O} \end{cases}$ must coincide on some $I \subseteq I_1 \cap I_2$

Moreover, "the solution of $(*)_{x_0}$ depends smoothly in x_0 ", i.e.,

$(\forall) x_0 \in \mathcal{O}, \exists \Gamma: \mathcal{O}_{x_0} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{O}$ smooth

such that each $\Gamma(x_0, \cdot): (-\varepsilon, \varepsilon) \rightarrow \mathcal{O}$ is a solution of $(*)_{x_0}$
 $\gamma_{x_0}(\cdot)$

(3)

The complete case:

Def: X is called complete if $(\forall) p \in M$, (eq) has solution $\gamma: \mathbb{R} \rightarrow M$
 $(\Leftrightarrow)$ all maximal int. curves are defined on $\mathbb{R} \Leftrightarrow \mathcal{D}(X) = M \times \mathbb{R}$

Theorem: $M = \text{compact} \Rightarrow (\forall) X \in \mathcal{X}(M)$ are complete (to be proven)

Assuming $X = \text{complete} \Rightarrow$ get $\Phi_X: M \times \mathbb{R} \rightarrow M$, $\Phi_X^t: M \rightarrow M$

Theorem: If $X = \text{complete} \Rightarrow \Phi_X$ is smooth, each Φ_X^t is a diffeom.

and $\Phi_X^t \circ \Phi_X^s = \Phi_X^{t+s} \quad (\forall) t, s \in \mathbb{R} \quad (*)$

$$\begin{cases} \frac{d}{dt} \Phi_X(p, t) = X_{\Phi_X(p, t)} \\ \Phi_X(p, 0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \Phi_X(p, t) = X_{\Phi_X(p, t)} \\ \Phi_X(p, 0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \Phi_X^t(p) = X_{\Phi_X^t(p)} \\ \Phi_X^0(p) = p \end{cases}$$

pl: Cause
Claim

Fix p

Fix s

$$\begin{cases} t \mapsto \Phi_X^t(p) \\ t \mapsto \Phi_X^{t+s}(p) \end{cases}$$

$\Rightarrow$ tw

Reminder

$$L_X(t)$$

The general case (X not assumed complete)

Theorem: (1) $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$
 $\& \Phi_X$ is smooth

(2) $(\forall) t \in \mathbb{R} \Rightarrow \mathcal{D}^t(X) = \{p \in M : (p, t) \in \mathcal{D}(X)\} \subseteq M$ open

$\Phi_X^t: \mathcal{D}^t(X) \rightarrow M$ is actually on

(3) $\Phi_X^t(\Phi_X^s(p)) = \Phi_X^{t+s}(p) \quad \forall p, t, s$ for which the LHS is defined.
 diffeomorphism between $\mathcal{D}^t(X)$ and $\mathcal{D}^{t+s}(X)$

pl: The same as in the complete case, but tediously taking care of where things are defined.

$$\begin{matrix} \Phi_X: \mathcal{D}(X) \rightarrow M \\ \uparrow \\ M \times \mathbb{R} \end{matrix}$$

Cauchy

for each

Theorem

&

More

$\gamma(t)$ (unknown γ^x in $\mathcal{D}$)

$$\begin{cases} \frac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \Phi(p,t) = X_{\Phi(p,t)} \\ \Phi(p,0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \Phi^t(p) = X_{\Phi^t(p)} \\ \Phi^0(p) = p \end{cases}$$

$$L_X(f)(p) = \partial_{\frac{d}{dt}} \bigg|_p (f \circ \gamma_p) = (df^x)_{\gamma_p(0)} \left(\frac{d\gamma_p}{dt}(0) \right) = (df)_p(X_p)$$

ing at p
that coincide at some $t_0 \in I_1 \cap I_2$
 $\Downarrow$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

curve of X that starts at p
($I_p^x \subseteq \mathbb{R}$ open interval contains 0)

$\Rightarrow$ the flow of the vector field X .

on $\mathcal{D}(X) = \{(p,t) \in M \times \mathbb{R} : t \in I_p^x\} \subseteq M \times \mathbb{R}$.
 $\gamma_p(t) = \gamma_p^x(t)$

of X at the time t : $\Phi_X^t: M \rightarrow M$
if lucky

The general case (X not assumed complete)

$$\Phi_X: \mathcal{D}(X) \rightarrow M$$

$\cap$
 $M \times \mathbb{R}$

Theorem: (1) $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$
& Φ_X is smooth

(2) $(\forall) t \in \mathbb{R} \Rightarrow \mathcal{D}^t(X) = \{p \in M : (p,t) \in \mathcal{D}(X)\} \subseteq M$ open

$\Phi_X^t: \mathcal{D}^t(X) \rightarrow M$ is actually a

(3) $\Phi_X^t(\Phi_X^s(p)) = \Phi_X^{t+s}(p) \quad \forall p, t, s$ for which the LHS is defined.
diffeomorphism between $\mathcal{D}^t(X)$ and $\mathcal{D}^{-t}(X)$

Φ : The same as in the complete case, but tediously taking care of where things are defined.

Cauchy's local $\exists$ & ! Given

for each $x_0 \in \mathcal{D}$ consider the OD with initial condition

Theorem: $(\forall) x_0 \in \mathcal{D}$, $(*)_{x_0}$ has

& $(\forall)$ two solutions $\begin{cases} \gamma_1: I_1 \rightarrow M \\ \gamma_2: I_2 \rightarrow M \end{cases}$

Moreover, "the solution of $(*)_{x_0}$ is

$(\forall) x_0 \in \mathcal{D}$, $\exists \Gamma: \mathcal{D}_{x_0} \rightarrow M$

such that each $\gamma \in \mathcal{D}_{x_0}$ $\underbrace{\Gamma(x, -)}_{\gamma_x(-)}$

Theorem: If X is complete $\Rightarrow \phi_X$ is smooth

and $\phi_X^t \circ \phi_X^s = \phi_X^{t+s} \quad (\forall t, s \in \mathbb{R})$ (*)

$$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X^0(p) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi(p, t) = X_{\phi(p, t)} \\ \phi(p, 0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p \end{cases}$$

Reminder:

$$L_X(f)(p) = \partial_{X_p}(f) = \begin{pmatrix} (f \circ \gamma)'(0) \\ (df)_{X_p}(\gamma'(0)) \end{pmatrix}$$

X -chart

$$X_p = \frac{d\gamma}{dt}(0) \quad f: M \rightarrow \mathbb{R}$$

$$(df)_p: T_p M \rightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R}$$

standard

$\Rightarrow$ two int. curves of X starting at $\phi^s(p) \Rightarrow u=0$

The general case (X not assumed complete)

Theorem: (1) $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$ & ϕ_X is smooth

(2) $(\forall t \in \mathbb{R}) \Rightarrow \mathcal{D}^t(X) = \{p \in M : (p, t) \in \mathcal{D}(X)\} \subseteq M$ open

$\phi_X^t: \mathcal{D}^t(X) \rightarrow M$ is actually a

(3) $\phi_X^t(\phi_X^s(p)) = \phi_X^{t+s}(p) \quad \forall p, t, s$ for which the LHS is defined.
 diffeomorphism between $\mathcal{D}^t(X)$ and $\mathcal{D}^{t+s}(X)$

ϕ_X : The same as in the complete case, but tediously taking care of where things are defined.

$$\phi_X: \mathcal{D}(X) \rightarrow M$$

$\cap$
 $M \times \mathbb{R}$

proof of Thm 1: $M = \text{compact}, X \in \mathfrak{X}(M)$

$X = \text{complete?}$

$$M \times \{0\} \subseteq \mathcal{D}(X) \subseteq M \times \mathbb{R}$$

open

$$\mathcal{D}(X) = M \times \mathbb{R}?$$

the tube lemma

$\exists \varepsilon > 0$ s.t. $M \times [-\varepsilon, \varepsilon] \subseteq \mathcal{D}(X)$

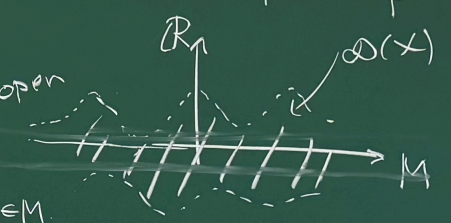
$M \times [-\varepsilon, \varepsilon]$ open

i.e. $\phi_X^t(p)$ is defined $(\forall t \in [-\varepsilon, \varepsilon], \forall p \in M)$

$\Rightarrow \phi_X^t(p) \text{ defined } (\forall t \in [-2\varepsilon, 2\varepsilon], \forall p \in M)$

$\Rightarrow \phi_X^t(p) \text{ defined } (\forall t \in [-2^k \varepsilon, 2^k \varepsilon], \forall p \in M, \forall k)$
i.e. for $(p, t) \in M \times \mathbb{R}$.

Exercise: Show, on general M , that any $X \in \mathfrak{X}(M)$ with compact support is complete.



Theorem 1: M compact $\Rightarrow (M, X) \text{ is compact}$

Assuming X is complete $\Rightarrow$ get $\phi_X: M \times \mathbb{R} \rightarrow M$, $\phi_X^t: M \rightarrow M$

Theorem 2: If X is complete $\Rightarrow \phi_X$ is smooth, each ϕ_X^t is a diffeom.

and $\phi_X^t \circ \phi_X^s = \phi_X^{t+s} \quad (t, s \in \mathbb{R})$ (*)

$$\begin{cases} \frac{d}{dt} \phi_X(t) = X_{\phi_X(t)} \\ \phi_X(0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi_X(p, t) = X_{\phi_X(p, t)} \\ \phi_X(p, 0) = p \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X^0(p) = p \end{cases}$$

Fix s and look at LHS & RHS

$$\begin{cases} t \mapsto \phi_X^t(\phi_X^s(p)) \\ t \mapsto \phi_X^{t+s}(p) \end{cases} \quad \begin{cases} u(0) = \phi_X^s(p) \\ v(0) = \phi_X^s(p) \end{cases} \quad \begin{cases} \frac{du}{dt}(t) = X_{\phi_X^t(\phi_X^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{cases}$$

$\Rightarrow$ two int. curves of X starting at $\phi_X^s(p) \Rightarrow u = v$. $\square$

Reminder:

$$L_X(f)(p) = \frac{d}{dt} f(\phi_X^t(p)) \Big|_{t=0} = \begin{pmatrix} (f \circ \phi_X^t)'(0) \\ (df)_{\phi_X^0(p)} \end{pmatrix} \quad X_p = \frac{d}{dt} \phi_X^t(p) \Big|_{t=0} \quad f: M \rightarrow \mathbb{R} \quad \text{standby } p$$

$$(df)_p: T_p M \rightarrow T_{\phi_X^0(p)} \mathbb{R} \cong \mathbb{R}$$

General Lie derivatives $\mathcal{L}_X$ wrt a vector field $X \in \mathfrak{X}(M)$ of "objects a on M ". Intuitively it describes how a changes when moving along the flow of X .

Slogan: $\mathcal{L}_X(a) = \frac{d}{dt} \Big|_{t=0} (\phi_X^t)^*(a)$ to be made sense of

Example 1: By "object on M " we mean an element $a = f \in C^\infty(M)$

Natural? $\varphi: M \rightarrow N$ $\xrightarrow{f} \mathbb{R}$
 $\varphi^*(f) = f \circ \varphi$

This makes part of formula sense of

Here "object" is a structure that is natural, in the sense that if $\varphi: M \rightarrow N$ is a diffeo and a is an object on N , $\varphi^*(a)$ makes sense on M

$$\phi_X^t: M \rightarrow M$$

$$\mathcal{L}_X(f) = \frac{d}{dt} \Big|_{t=0} (\phi_X^t)^*(f) = \frac{d}{dt} \Big|_{t=0} f \circ \phi_X^t$$

Does make sense:

$$\mathcal{L}_X(f)(p) = \frac{d}{dt} \Big|_{t=0} f(\phi_X^t(p)) = (df)_{\phi_X^0(p)} \left(\frac{d}{dt} \Big|_{t=0} \phi_X^t(p) \right)$$

$$= (df)_p(X_p) = L_X(f)(p)$$

For $f \in C^\infty(M)$, the slogan provides $\mathcal{L}_X: C^\infty(M) \rightarrow C^\infty(M)$

Example 2: By an "object on M " we mean just the old L_X .

Natural? $\varphi: M \rightarrow N$ diffeo, $Y \in \mathfrak{X}(N)$ Implement slogan

$$\varphi^*(Y) = (d\varphi)^{-1}_p(Y_p)$$

$$\begin{matrix} p \in M & \xrightarrow{\varphi} & q \in N \\ \uparrow & & \uparrow \\ T_p M & \xrightarrow{d\varphi_p} & T_q N \end{matrix}$$

$$\Rightarrow \mathcal{L}_X(Y) = [X, Y]$$

Reminder: Given M . Given $X \in \mathfrak{X}(M)$.

- integral curve of X : any $\gamma: I \rightarrow M$ smooth st. $\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases}$ (eg)

- starts at p : if $0 \in I$ and...

- two explicit examples in $M = \mathbb{R}^2$, where $(a,b) \in \mathbb{R}^2$:

Ex1: $X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ Ex2: $X_2 = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$
 $\gamma = \lambda \rightarrow \gamma_{a,b}(t) = (t+a, t+b)$ $\gamma_{a,b}(t) = (\frac{a}{a+t}, b e^{-t})$ at $t \neq 0$
 $-\frac{1}{a}$

- locally, wrt a chart $\chi: U \rightarrow \mathbb{R}^m$ around p , we move to $\mathbb{R}^m$

$\gamma^x = \chi_* \gamma$, $x_* = \chi(p)$, $\chi_*(X) = \sum F_i \frac{\partial}{\partial x_i}$, $F_i \in C^\infty(\mathbb{R}^2)$

and (eg) becomes: $\begin{cases} \dot{\gamma}^x(t) = F(\gamma^x(t)) \\ \gamma^x(0) = x_* \end{cases}$ (unknown γ^x in $\mathbb{R}^2$)

↓ (page 0)

The complete case:

Def: X is called complete if $(\forall) p \in M$, (eg) has solution $\gamma: \mathbb{R} \rightarrow M$
 $(\Leftrightarrow)$ all maximal int. curves are defined on $\mathbb{R} \Leftrightarrow \mathcal{D}(X) = M \times \mathbb{R}$

Theorem 1: $M = \text{compact} \Rightarrow (\forall) X \in \mathfrak{X}(M)$ are complete (to be proven)

Assuming X is complete $\Rightarrow$ get $\phi_X: M \times \mathbb{R} \rightarrow M$, $\phi_X^t: M \rightarrow M$

Theorem 2: If X is complete $\Rightarrow \phi_X$ is smooth, each ϕ_X^t is a diffeom.
 and $\phi_X^t \circ \phi_X^s = \phi_X^{t+s} \quad (\forall) t, s \in \mathbb{R} \quad (*)$

$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X^0(p) = p \end{cases}$

$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi_X(p, 0) = p \end{cases}$

$\begin{cases} \frac{d}{dt} \phi_X^t(p) = X_{\phi_X^t(p)} \\ \phi^0(p) = p \end{cases}$

pl: Cauchy $\Rightarrow \phi_X$ smooth.

Claim: $(*) \Rightarrow$ each ϕ_X^t is a diffeo.

pl: $\phi_X^0 = \text{Id}_M$, & taking $s = -t$ in $(*) \Rightarrow \phi_X^{-t}$ is the inverse of ϕ_X^t
 $\Rightarrow \phi_X^t \circ \phi_X^{-t} = \phi_X^0 = \text{Id}$

Fix $p \in M$ and prove: $\phi_X^t(\phi_X^s(p)) = \phi_X^{t+s}(p) \quad (\forall) s, t$

Fix s and look at LHS & RHS as functions of t

$\begin{cases} t \mapsto \phi_X^t(\phi_X^s(p)) \\ t \mapsto \phi_X^{t+s}(p) \end{cases} \quad \begin{cases} u(0) = \phi_X^s(p) \\ v(0) = \phi_X^s(p) \end{cases} \quad \begin{cases} \frac{du}{dt}(t) = X_{\phi_X^t(\phi_X^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{cases}$

$\Rightarrow$ two int. curves of X starting at $\phi_X^s(p) \Rightarrow u = v$. $\square$

Reminder:

$L_X(f)(p) = \partial_{X_p}(f) = \left(\frac{df}{dt} \right)_{X_p}(u^x)$ $X_p = \frac{d}{dt} \gamma(0)$ $f: M \rightarrow \mathbb{R}$ γ -chart $\downarrow$ standard $\mathbb{R}$
 $(df)_p: T_p M \rightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R}$

① $(\forall) p \in M, \exists$ integral curve of X starting at p

② if $\gamma_1: I_1 \rightarrow M$ integral curves of X that coincide at some $t_0 \in I_1 \cap I_2$
 $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$

↓
 $(\forall) p \in M \exists!$ maximal integral curve of X that starts at p
 $\gamma_p^X: I_p^X \rightarrow M$ ($I_p^X \subseteq \mathbb{R}$ open interval contains 0)

Rearrange $\gamma_p^X(t) =: \phi_X^t(p, t) \Rightarrow$ the flow of the vector field X .

$\phi_X: \mathcal{D}(X) \rightarrow M$ defined on $\mathcal{D}(X) = \{(p, t) \in M \times \mathbb{R} : t \in I_p^X\} \subseteq M \times \mathbb{R}$.

If we fix t , $\phi_X^t(p) := \phi_X(p, t) = \gamma_p^X(t)$ ($\gamma_p^X(t)$ is defined)

ϕ_X^t is called the flow of X at the time t : $\phi_X^t: M \rightarrow M$ if lucky

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$\varphi^*(Y) = (d\varphi)^{-1}_\# (Y)$
 $\varphi^*(Y) = \frac{d}{dt} \bigg|_{t=0} \varphi(\gamma(t))$
 $\Rightarrow \mathcal{L}_X(Y) = [X, Y]$