

• For any M, p : a vector space: the tangent space $T_p M$

• Any chart (U, χ) around p gives rise to a basis of $T_p M$:
 $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^m}\right)_p$

• Vector field X on M means: $M \ni p \mapsto X_p \in T_p M$
 which smooth: (U, χ) , writing $X_p = \sum \text{coeff}_i(p) \left(\frac{\partial}{\partial x^i}\right)_p$, coeff: $U \rightarrow \mathbb{R}$ are smooth.

$\mathcal{X}(M)$: the set of all vector fields on M .

• $\mathcal{X}(M)$ is a vector space: $(X+Y)_p := X_p + Y_p$, $(\lambda X)_p = \lambda X_p$

• $\mathcal{X}(M)$ is a $C^\infty(M)$ -module: $(f \cdot X)_p := f(p) X_p$

• Globally: any $X \in \mathcal{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$, $L_X(f)(p) = dX_p(f)$

& this gives rise to a bijection:

$$\mathcal{X}(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \right\}$$

s.t. L is a derivation

• $\mathcal{X}(M)$ is a Lie algebra: $X, Y \in \mathcal{X}(M) \mapsto [X, Y] \in \mathcal{X}(M)$
 (defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)

• in $\mathbb{R}^n$: all $X \in \mathcal{X}(\mathbb{R}^n)$ are $X = \sum_{i=1}^n F_i \frac{\partial}{\partial x^i}$ with $F_i \in C^\infty(\mathbb{R}^n)$

In general, for $X \in \mathcal{X}(M)$, locally (in a chart): we see m functions:

• on submanifolds $M \subseteq \mathbb{R}^n$: may use the ambient $\mathbb{R}^n$

$\Rightarrow$ any $X \in \mathcal{X}(M)$ can be written as $X_p = \sum F_i(p) \left(\frac{\partial}{\partial x^i}\right)_p$ for $p \in M$

but not all such

expressions are tangent to M !

$$\tilde{X} = \sum F_i \frac{\partial}{\partial x^i} \quad \text{on } M, \quad F_i \in C^\infty(M)$$

Ex. For $M = S^1$:

$$x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathcal{X}(S^1), \quad x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x} \text{ does not } \in \mathcal{X}(S^1) \text{ on } S^1$$

Flow of $X \in \mathcal{X}(M)$:

• for $p \in M \exists!$ maximal integral curve of X starting at p , $\varphi_p: I_p \rightarrow M$
 $\varphi_p(t) = X_{\varphi_p(t)}$ (v.t.)
 $\varphi_p(0) = p$

X complete if $I_p = \mathbb{R} \quad (\forall) p \in M$

• the flow φ_X of X : the function $(p, t) \mapsto \varphi_p(t)$ (Hence $\varphi_X(p, t) = \varphi_p(t)$)

- if X is complete $\Rightarrow$ a fct $\varphi_X: M \times \mathbb{R} \rightarrow M$ (smooth!)

- in general $\Rightarrow \varphi_X: \mathcal{D}(X) \rightarrow M$ (still smooth)
 $\mathcal{D}(X)$ open in $M \times \mathbb{R}$

• the flow φ_X^t of X at time t : the fct $p \mapsto \varphi_p(t)$ (Hence $\varphi_X^t(p) = \varphi_X(p, t)$)

- if X is complete $\Rightarrow (\forall) t: \varphi_X^t: M \rightarrow M$ (a diffeomorphism!)

- in general: φ_X^t defined on the open $\mathcal{D}^t(X)$

• basic property: $\varphi_X^t \circ \varphi_X^s = \varphi_X^{t+s}$
 $\frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)}$
 $\varphi_X^0(p) = p$
 $\frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)}$
 $\varphi_X^0(p) = p$

General Lie derivatives w.r.t. $X \in \mathcal{X}(M)$ of "objects a ": rough formula:

$$\mathcal{L}_X(a) = \frac{d}{dt} \bigg|_{t=0} (\varphi_X^t)^*(a) \quad \left\{ \begin{array}{l} \text{to make sense of it} \\ \text{for each example of objects} \end{array} \right.$$

Ex. For $a = f \in C^\infty(M)$: $\mathcal{L}_X(f) = \frac{d}{dt} \bigg|_{t=0} (\varphi_X^t)^*(f)$

takes precise (making sense): $\mathcal{L}_X(f)(p) = \frac{d}{dt} \bigg|_{t=0} f(\varphi_X^t(p))$ [Lemma: $\mathcal{L}_X(f) = L_X(f)$!]

Ex. For $a = Y \in C^\infty(M)$: $\mathcal{L}_X(Y) = \frac{d}{dt} \bigg|_{t=0} (\varphi_X^t)^*(Y)$

takes the precise

$$\mathcal{L}_X(Y)(p) = \frac{d}{dt} \bigg|_{t=0} \left(\frac{d\varphi_X^t}{dt} \bigg|_p (Y_{\varphi_X^t(p)}) \right) \quad (\in T_p M)$$

[Lemma: $\mathcal{L}_X(Y) = [X, Y]$!]

Corollary:

$L: C^\infty(M) \rightarrow C^\infty(M)$
 s.t. L is a derivation

$[X, Y] \in \mathfrak{X}(M)$
 $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$ (inf)
 $F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(\mathbb{R}^n)$

in a chart: we see n functions.
 may use the ambient $\mathbb{R}^n$

as $X_p = \sum F_i(p) \frac{\partial}{\partial x_i} \Big|_p$ for $p \in M$
 $X = \sum F_i \frac{\partial}{\partial x_i}$ on M , $F_i \in C^\infty(M)$

$\frac{\partial}{\partial y_j} + y \frac{\partial}{\partial x}$ does not $\in \mathfrak{X}(S^1)$
 on S^1

$\frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ on $S^1 \in \mathfrak{X}(S^1)$

General Lie derivatives w.r.t. $X \in \mathfrak{X}(M)$ of "objects a ": rough formula:

$$\mathcal{L}_X(a) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X)^*(a) \quad \left\{ \begin{array}{l} \text{to make sense of it} \\ \text{for each example of objects} \end{array} \right.$$

Ex: For $a = f \in C^\infty(M)$: $\mathcal{L}_X(f) = \frac{d}{dt} \Big|_{t=0} \underbrace{(\varphi_t^X)^*(f)}_{f \circ \varphi_t^X}$

takes precise (making sense): $\boxed{\mathcal{L}_X(f)(p) = \frac{d}{dt} \Big|_{t=0} f(\varphi_t^X(p))}$ [Lemma: $\mathcal{L}_X(f) = L_X(f)$!]

Ex: For $a = Y \in \mathfrak{X}(M)$: $\mathcal{L}_X(Y) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X)^*(Y)$

takes the precise form

defined by ...

$$\boxed{\mathcal{L}_X(Y)(p) = \frac{d}{dt} \Big|_{t=0} \underbrace{(d\varphi_t^X)^{-1}}_{\varphi_t^{X^{-1}}(p)} \underbrace{(Y_{\varphi_t^X(p)})}_{Y_{\varphi_t^X(p)}} \Big|_{p} \in T_p M}$$

[Lemma: $\mathcal{L}_X(Y) = [X, Y]$!]

[Corollary: $[X, Y] = 0 \Leftrightarrow \varphi_t^X \varphi_s^Y = \varphi_s^Y \varphi_t^X$]
 $= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \varphi_t^X \left(\varphi_s^Y \left(\varphi_t^X(p) \right) \right)$

Colours

Def: 7
 E.lts

Ex: $f \in C^\infty(M)$

hence $L_X(f) = \mathcal{L}_X(f)$

Ex: $G \in \mathfrak{X}(M)$

$\Rightarrow$

i.e. the
 associ

Duals of vector spaces: Given $V = n$ -dimensional vector space (5)

its dual is $V^* = \{ \xi: V \rightarrow \mathbb{R} / \xi \text{ linear} \}$ $\left| \begin{array}{l} v \in V \text{ vectors on } V \\ \xi \in V^* \text{ covectors on } V \end{array} \right.$

Remarks:

① V^* = vector space: $(\xi + \xi')(v) = \xi(v) + \xi'(v)$ etc

② covectors pair with vectors $\Rightarrow$ give numbers

$V^* \times V \rightarrow \mathbb{R}, (\xi, v) \mapsto \xi(v) \leftarrow$ sometimes denoted $\langle \xi, v \rangle$

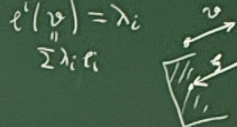
③ V^* = n -dimensional. Actually any basis $\{e_1, \dots, e_n\}$ of V ($\langle \xi, v \rangle$) induces a "dual basis" - a basis of V^* , $\{\xi^1, \dots, \xi^n\}$ given by:

④ Intuitively: $\xi \in V^*$ thought of as $e^i(e_j) = \delta_{ij} = \begin{cases} 1 & j=i \\ 0 & j \neq i \end{cases}$

- a way to measure "oriented length" of vectors $v \in V$

- encoding the hyperplane

$\text{Ker } \xi = \{ v \in V : \xi(v) = 0 \}$



Cotangent spaces: $M, p \in M$ (6)

Def: The cotangent space of M at p is $T_p^* M :=$ the dual of $T_p M$

Elt's $\xi_p \in T_p^* M$: called cotangent vectors (to M) at p . $\{ \xi: T_p M \rightarrow \mathbb{R} / \text{linear} \}$

Ex: $f \in C^\infty(M), p \in M \Rightarrow (df)_p: T_p M \rightarrow \mathbb{R} \quad \left(\frac{df}{dt}(v) \mapsto \frac{df}{dt}(v) \text{ etc etc} \right)$
hence $(df)_p \in T_p^* M$

Ex: Given (U, x) -chart, $p \in U, x = (x_1, \dots, x_n): U \rightarrow \mathbb{R}^n$

$\Rightarrow (dx_1)_p, \dots, (dx_n)_p \in T_p^* M \quad \left(T_p M \ni v \mapsto (dx_i)_p(v) = \lambda_i \right)$

i.e. this is the dual basis associated to $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p$

$$\sum_j \lambda_j \left(\frac{\partial}{\partial x_j} \right)_p$$

Duals of vector spaces: Given $V = n$ -dimensional vector space (5)
 its dual is $V^* = \{ \xi: V \rightarrow \mathbb{R} \mid \xi = \text{linear} \}$ $\left| \begin{array}{l} v \in V \text{ vectors on } V \\ \xi \in V^* \text{ covectors on } V \end{array} \right.$

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$$V^* \times V \rightarrow \mathbb{R}, (\xi, v) \mapsto \xi(v) \leftarrow \text{sometimes denoted } \langle \xi, v \rangle$$

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$$e^i(e_j) = \delta_{ij} = \begin{cases} 1 & j=i \\ 0 & j \neq i \end{cases}$$

$$e^i(v) = \lambda_i$$

$$\sum \lambda_i e_i$$



(6)

Cotangent spaces: $M, p \in M$

Def: The cotangent space of M at p is $T_p^* M :=$ the dual of $T_p M$.
Elt's $\xi_p \in T_p^* M$: called cotangent vectors (to M) at p . $\{\xi: T_p M \rightarrow \mathbb{R} / \text{linear}\}$

Ex: $f \in C^\infty(M), p \in M \rightarrow (df)_p: T_p M \rightarrow \mathbb{R}$ ($\frac{dx}{dt}(t_0) \mapsto \frac{df}{dt}(t_0)$ etc etc)
hence $(df)_p \in T_p^* M$.

Ex: Given (U, x) -chart, $p \in U$, $x = (x_1, \dots, x_m): U \rightarrow \mathbb{R}^m$
 $\Rightarrow (dx_1)_p, \dots, (dx_m)_p \in T_p^* M$ $\left(T_p M \ni v \xrightarrow{(dx_i)_p} x_i \right)$
 $\parallel$
 $\sum_j x_j \left(\frac{\partial}{\partial x_j} \right)_p$

... i.e. this is the dual basis
associated to $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p$

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which smooth: (U, χ) , writing $X_p = \sum F_i \left(\frac{\partial}{\partial x^i}\right)_p$, coeff. $U \rightarrow \mathbb{R}$ are smooth

$\mathfrak{X}(M)$ = the set of all vector fields on M

$\mathfrak{X}(M)$ is a vector space: $(X+Y)_p = X_p + Y_p, (\lambda X)_p = \lambda X_p$

$\mathfrak{X}(M)$ is a $C^\infty(M)$ -module: $(fX)_p = f(p)X_p$

Globally, any $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M), L_X(f) = \text{div}(f)(p)$

& this gives rise to a bijection:

$$\mathfrak{X}(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \mid L \text{ is a derivation} \right\}$$

Flow of $X \in \mathfrak{X}(M)$:

for $p \in M \exists!$ maximal integral curve of X starting at $p, \gamma_p: I_p \rightarrow M$

$$\gamma_p'(t) = X_{\gamma_p(t)} \quad \gamma_p(0) = p \quad \gamma_p(t)$$

X complete if $I_p = \mathbb{R} \quad \forall p \in M$

the flow φ_t of X : the function $(p, t) \mapsto \gamma_p(t)$ (Hence $\varphi_t(p, t) = \gamma_p(t)$)

- if X is complete $\Rightarrow$ a diffeomorphism $M \times \mathbb{R} \rightarrow M$ (smooth!)

- in general $\Rightarrow \varphi_t: \mathcal{D}(X) \rightarrow M$ (still smooth)

the flow φ_t^X of X at time t : the diffeomorphism $p \mapsto \gamma_p(t)$ (Hence $\varphi_t^X(p) = \gamma_p(t)$)

- if X is complete $\Rightarrow \forall t, \varphi_t^X: M \rightarrow M$ (a diffeomorphism!)

- in general: φ_t^X defined on the open $\mathcal{D}^t(X) \subset M$

basic property: $\varphi_t^X \circ \varphi_s^X = \varphi_{t+s}^X$

$$\frac{d}{dt} \varphi_t^X(p) = X_{\varphi_t^X(p)} \quad \frac{d}{dt} \varphi_t^X(p) = X_{\varphi_t^X(p)}$$

Duals of vector spaces: Given $V = n$ -dimensional vector space

its dual is $V^* = \{ \xi: V \rightarrow \mathbb{R} \mid \xi \text{ linear} \}$

$v \in V$ vectors on V

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Remains:

① V^* = vector space: $(\xi + \zeta)(v) = \xi(v) + \zeta(v)$ etc

② covectors pair with vectors $\Rightarrow$ give numbers

$$V^* \times V \rightarrow \mathbb{R}, (\xi, v) \mapsto \xi(v) \text{ --- sometimes denoted } \langle \xi, v \rangle$$

③ $V^* = n$ -dimensional. Actually any basis $\{e^1, \dots, e^n\}$ of V induces a "dual basis" - a basis of $V^* \{e^1, \dots, e^n\}$ given by:

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$\xi(v) = \lambda_i$

$\sum \lambda_i e_i$

$\mathfrak{X}(M)$ is a Lie algebra: $X, Y \in \mathfrak{X}(M) \mapsto [X, Y] \in \mathfrak{X}(M)$

(defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)

$\mathbb{R}^n$ all $X \in \mathfrak{X}(\mathbb{R}^n)$ are $X = \sum F_i \frac{\partial}{\partial x^i}$ with $F_i \in C^\infty(\mathbb{R}^n)$

In general, for $X \in \mathfrak{X}(M)$, locally (in a chart) we see n functions.

on submanifolds $M \subset \mathbb{R}^n$, may use the ambient $\mathbb{R}^n$

$\Rightarrow$ any $X \in \mathfrak{X}(M)$ can be written as $X_p = \sum F_i(p) \left(\frac{\partial}{\partial x^i}\right)_p$ for $p \in M$

but not all such expressions are tangent to M !

Ex For $M = S^1$

$\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \in \mathfrak{X}(S^1)$, $\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ does not $\in \mathfrak{X}(S^1)$

$(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2})^2 = 0$ $\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \in S^1 \in \mathfrak{X}(S^1)$

General Lie derivatives w.r.t $X \in \mathfrak{X}(M)$ of "objects a ": rough formula:

$$L_X(a) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X(a)) \quad \text{(to make sense of it for each example of object)}$$

Ex For $a = f \in C^\infty(M), L_X(f) = \frac{d}{dt} \Big|_{t=0} (f(\varphi_t^X(p)))$

takes precise meaning: $L_X(f)(p) = \frac{d}{dt} \Big|_{t=0} f(\varphi_t^X(p))$ [Lemma: $L_X(f) = L_X(f)$!]

Ex For $a = Y \in \mathfrak{X}(M): L_X(Y) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X(Y))$

takes the precise form

$$L_X(Y)(p) = \frac{d}{dt} \Big|_{t=0} \left(\sum F_i \left(\frac{\partial}{\partial x^i} \right)_{\varphi_t^X(p)} \right)$$

[Lemma: $L_X(Y) = [X, Y]$!]

Corollary: $[X, Y] = L_X(Y) = L_Y(X)$

Cotangent spaces: $M, p \in M$

Def: The cotangent space of M at p is $T_p^* M :=$ the dual of $T_p M$

Elts $\xi_p \in T_p^* M$ called cotangent vectors (to M) at p .

Ex: $f \in C^\infty(M), p \in M \Rightarrow (df)_p: T_p M \rightarrow \mathbb{R}$ ($\frac{d}{dt} f(\gamma(t)) \Big|_{t=0} \mapsto \frac{d}{dt} f(\gamma(t))$ etc etc)

hence $(df)_p \in T_p^* M$

Ex: Given (U, χ) -chart, $p \in U, \chi = (x^1, \dots, x^n): U \rightarrow \mathbb{R}^n$

$\Rightarrow (dx^1)_p, \dots, (dx^n)_p \in T_p^* M$ ($T_p M \ni v \mapsto \sum \lambda_i \frac{\partial}{\partial x^i} \Big|_p$)

i.e. this is the dual basis associated to $(\frac{\partial}{\partial x^1})_p, \dots, (\frac{\partial}{\partial x^n})_p$

- For any M, p : a vector space: the cotangent space T_p^*M
- Any chart (U, χ) around p gives rise to a basis of T_p^*M :
 $(\frac{\partial}{\partial x^1})_p, \dots, (\frac{\partial}{\partial x^n})_p$
- Vector field X on M means: $M \ni p \mapsto \omega_p \in T_p^*M$
 which smooth: $(\forall) (U, \chi)$, writing $\omega_p = \sum_i \text{coeff}_i(p) (dx^i)_p$, $\text{coeff}_i: U \rightarrow \mathbb{R}$ are smooth
- $\Omega^1(M)$:= the set of all vector fields on M .
- $\Omega^1(M)$ is a vector space: $(\omega + \eta)_p = \omega_p + \eta_p, (\lambda \cdot \omega)_p = \lambda \omega_p$
- $\Omega^1(M)$ is a $C^\infty(M)$ -module: $(f \cdot \omega)_p = f(p) \omega_p$
- Globally: any $X \in \mathfrak{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M), L_X(f)(p) = df_p(X_p)$
 & this gives rise to a bijection:

$$\mathfrak{X}(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \right. \\ \left. \text{s.t. } L \text{ is a derivation} \right\}$$

Globally: any $\omega \in \Omega^1(M)$ induces a map $\hat{\omega}: \mathfrak{X}(M) \rightarrow C^\infty(M)$
 $X \mapsto \hat{\omega}(X)$ defined by $\hat{\omega}(X)(p) = \omega_p(X_p)$

Thm: This gives rise to a bijection

$$\Omega^1(M) \xrightarrow{1-1} \left\{ \begin{array}{l} Z: \mathfrak{X}(M) \rightarrow C^\infty(M) \text{ s.t.} \\ C^\infty(M)\text{-linear} \end{array} \right\} \begin{cases} Z(X+Y) = Z(X) + Z(Y) \\ Z(fX) = fZ(X) \end{cases}$$

Surjectivity: Given Z look for $\omega \in \Omega^1(M)$ s.t.

$$Z = \hat{\omega} \quad \text{i.e. } Z(X) = \omega_p(X_p) \quad (\forall X \in \mathfrak{X}(M), \forall p \in M)$$

ω ? For $p \in M, u \in T_pM$ what is $\omega_p(u) \in \mathbb{R}$? Define it by

choosing $X \in \mathfrak{X}(M)$ s.t. $X_p = u$

Setting $\omega_p(u) := Z(X)(p)$

Lemma 1: $(\forall) p \in M \exists X$
Lemma 2: this does not depend on the choice

Duals of vector
 its dual
Remarks:
 ① $V^* = \text{dual of } V$
 ② covered
 ③ $V^* = \text{dual of } V$
 induces
 ④ J

• $\mathfrak{X}(M)$ is a Lie algebra: $X, Y \in \mathfrak{X}(M) \mapsto [X, Y] \in \mathfrak{X}(M)$

(defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)

• (De Rham) differential $d: C^\infty(M) \rightarrow \Omega^1(M), f \mapsto df$

the 1-form

• For any M, p : a vector space: the cotangent space T_p^*M

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$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, (dx_1)_p, \dots, (dx_n)_p$$

• 1-form ω on M means: $M \ni p \mapsto \omega_p \in T_p^*M$

which smooth: (U, χ) , writing $\omega_p = \sum_i \text{coeff}_i(p) (dx_i)_p$, $\text{coeff}_i: U \rightarrow \mathbb{R}$ are smooth.

$\mathcal{O}^1(M)$:= the set of all vector fields on M .

• $\mathcal{O}^1(M)$ is a vector space: $(\omega + \eta)_p = \omega_p + \eta_p, (\lambda \cdot \omega)_p = \lambda \omega_p$

• $\mathcal{O}^1(M)$ is a $C^\infty(M)$ -module: $(f \cdot \omega)_p = f(p) \omega_p$

• Globally: any $X \in \mathcal{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M), L_X(f)(p) = df_p(X_p)$
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• in $\mathbb{R}^n$: all $X \in \mathcal{X}(\mathbb{R}^n)$ are $\omega = \sum_{i=1}^n F_i dx_i$ with $F_i \in C^\infty(\mathbb{R}^n)$

In general, for $X \in \mathcal{X}(M)$, locally (in a chart): we see n functions.

• on submanifolds $M \subseteq \mathbb{R}^n$: may use the ambient $\mathbb{R}^n$

$\Rightarrow$ any $X \in \mathcal{X}(M)$ can be written as $\omega_p = \sum F_i(p) (dx_i)_p$ for $p \in M$

$$\omega = \sum F_i dx_i, F_i \in C^\infty(M)$$

• for submanifolds $M \subseteq N$

$X \in \mathcal{X}(N), X_p \in T_p N$ may fail to $\in T_p M$

For $\omega \in \mathcal{O}^1(N), \omega_p \in T_p^* N$ (i.e. a linear $\omega_p: T_p N \rightarrow \mathbb{R}$)

can be restricted to $T_p M \Rightarrow \omega_p|_{T_p M} \in T_p^* M$. And this

defines $\omega|_M \in \mathcal{O}^1(M)$. (i.e. $(\omega|_M)(X_p) := \omega_p(X_p)$).

& this gives rise to
 $\mathfrak{X}(M) \longleftrightarrow \left\{ \begin{array}{l} L: C^0(M) \rightarrow C^0(M) \\ \text{s.t. } L = \text{a derivation} \end{array} \right\}$

• $\mathfrak{X}(M)$ is a Lie algebra: $X, Y \in \mathfrak{X}(M) \mapsto [X, Y] \in \mathfrak{X}(M)$
 (defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)
 • in $\mathbb{R}^n$: all $X \in \mathfrak{X}(\mathbb{R}^n)$ are $\omega = \sum_{i=1}^n F_i dx_i$ with $F_i \in C^0(\mathbb{R}^n)$

In general, for $X \in \mathfrak{X}(M)$, locally (in a chart): we see n functions.
 • on submanifolds $M \subseteq \mathbb{R}^n$: may use the ambient $\mathbb{R}^n$

$\Rightarrow$ any $X \in \mathfrak{X}(M)$ can be written as $\omega_p = \sum F_i(p) (dx_i)_p$ for $p \in M$
 $\omega = \sum F_i dx_i, F_i \in C^0(M)$

• for submanifolds $M \subseteq N$
 $X \in \mathfrak{X}(N), X_p \in T_p N$ may fail to $\in T_p M$
 For $\omega \in \mathfrak{X}^1(N)$, $\omega_p \in T_p^* N$ (ie a linear $\omega_p: T_p N \rightarrow \mathbb{R}$)
 can be restricted to $T_p M \Rightarrow \omega_p|_{T_p M} \in T_p^* M$. And this
 defines $\omega|_M \in \mathfrak{X}^1(M)$ (i.e. $(\omega|_M)(X_p) = \omega_p(X_p)$).

• (De Rham) differential
 $Rk: \hat{d}f: \mathfrak{X}(M) \rightarrow C^0(M)$
 $X \mapsto L_X(f)$

A 0-form on M is just a function
 $\mathfrak{X}^0(M) := C^0(M)$

any expression $\sum F_i dx_i \in \mathfrak{X}^1(\mathbb{R}^m)$
 makes sense/induces a 1-form on M :
 $(\sum F_i dx_i)|_M$ ← referred to as " $\sum F_i dx_i$ on M "

Particular case of pull-backs
 $(dF)_p: T_p M \rightarrow T_p^* N$
 $F^*: \mathfrak{X}^1(N) \rightarrow \mathfrak{X}^1(M)$
 $F^*(\omega) = \omega \circ dF$
 Should say for $p \in M$
 what $F^*(\omega)_p: T_p M \rightarrow \mathbb{R}$
 $F^*(\omega)(v_p) = \omega_{F(p)}(dF_p(v_p))$

$F: M \rightarrow N$
 $F^*: \mathfrak{X}^1(N) \rightarrow \mathfrak{X}^1(M)$
 $F^*(\omega) = \omega \circ dF$
 For $\omega \in \mathfrak{X}^1(N)$ have
 $\omega_p: T_p N \rightarrow \mathbb{R}$

$F^*: C^0(N) \rightarrow C^0(M)$
 $F^*(f) = f \circ F$
 $F^*: \mathfrak{X}(N) \rightarrow \mathfrak{X}(M)$?
NO!!!

- For any M, p : a vector space: the cotangent space T_p^*M
- Any chart (U, χ) around p gives rise to a basis of T_p^*M :
 $(\frac{\partial}{\partial x^1})_p, \dots, (\frac{\partial}{\partial x^n})_p$
- Vector field X on M means: $M \ni p \mapsto \omega_p \in T_p^*M$
 which smooth: (U, χ) , writing $\omega_p = \sum \text{coeff}_i(p) (dx^i)_p$, $\text{coeff}_i: U \rightarrow \mathbb{R}$ are smooth.
- $\Omega^1(M)$: the set of all vector fields on M .
- $\Omega^1(M)$ is a vector space: $(\lambda X + \mu Y)_p := \lambda X_p + \mu Y_p$, $(\lambda X)_p = \lambda X_p$
- $\Omega^1(M)$ is a $C^\infty(M)$ -module: $(f \cdot X)_p := f(p) X_p$
- Globally, any $X \in \Omega^1(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M)$, $L_X(f)(p) = df_p(X_p)$
 & this gives rise to a bijection:

$$\Omega^1(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \right\}$$

$$\text{s.t. } L = \text{a derivation}$$

Globally: any $\omega \in \Omega^1(M)$ induces a map $\hat{\omega}: \Omega^1(M) \rightarrow C^\infty(M)$
 $X \mapsto \hat{\omega}(X)$ defined by $\hat{\omega}(X)(p) = \omega_p(X_p)$

Thm: This gives rise to a bijection

$$\Omega^1(M) \xrightarrow{1-1} \left\{ \begin{array}{l} L: \Omega^1(M) \rightarrow C^\infty(M) \text{ s.t.} \\ L \text{ is } C^\infty(M)\text{-linear} \end{array} \right\}$$

$$\omega \mapsto \hat{\omega}$$

Surjectivity: Given L look for $\omega \in \Omega^1(M)$ s.t.

$$L = \hat{\omega} \text{ i.e. } L(X) = \omega_p(X_p) \quad (\forall X \in \Omega^1(M), \forall p \in M)$$

ω ? For $p \in M$, $u \in T_p M$ what is $\omega_p(u) \in \mathbb{R}$? Define it by

choosing $X \in \Omega^1(M)$ s.t. $X_p = u$
 Setting $\omega_p(u) := L(X)(p)$

Duals of vector spaces: Given

its dual is $V^* = \{ \xi: V \rightarrow \mathbb{R} \}$

Remarks:

① V^* is a vector space: $(\xi + \eta)(v) = \xi(v) + \eta(v)$

② covector pair with $v \in V$: $V^* \times V \rightarrow \mathbb{R}$, $(\xi, v) \mapsto \xi(v)$

③ V^* is n -dimensional. A basis $\{\xi^i\}$ induces a "dual basis" -

④ Intuitively: $\xi \in V^*$ is a way to measure

- encoding the hypothesis $X_p = 0$

$X_p = \sum f_i \frac{\partial}{\partial x^i}$ with $f_i(p) = 0$

$\xi(X_p) = \xi(\sum f_i \frac{\partial}{\partial x^i}) = \sum f_i \xi(\frac{\partial}{\partial x^i}) = 0$

$\Omega^1(M)$ is a Lie algebra: $X, Y \in \Omega^1(M) \mapsto [X, Y] \in \Omega^1(M)$

(defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)

in $\mathbb{R}^n$: all $X \in \Omega^1(\mathbb{R}^n)$ are $\omega = \sum_{i=1}^n F_i dx^i$ with $F_i \in C^\infty(\mathbb{R}^n)$

In general, for $X \in \Omega^1(M)$, locally (in a chart) we see n_i functions.

on submanifolds $M \subseteq \mathbb{R}^n$: may use the ambient $\mathbb{R}^n$

$\Rightarrow$ any $X \in \Omega^1(M)$ can be written as $\omega_p = \sum F_i(p) (dx^i)_p$ for $p \in M$

$$\omega = \sum F_i dx^i, F_i \in C^\infty(M)$$

For submanifolds $M \subseteq N$

$X \in \Omega^1(N)$, $X_p \in T_p N$ may fail to $\in T_p M$

For $\omega \in \Omega^1(N)$, $\omega_p \in T_p^* N$ (ie a linear $\omega_p: T_p N \rightarrow \mathbb{R}$)

can be restricted to $T_p M \Rightarrow \omega_p|_{T_p M} \in T_p^* M$. And this

defines $\omega|_M \in \Omega^1(M)$. (i.e. $(\omega|_M)(X_p) = \omega_p(X_p)$)

(De Rham) differential $d: C^\infty(M) \rightarrow \Omega^1(M)$, $f \mapsto df$

Rk: $\hat{df}: \Omega^1(M) \rightarrow C^\infty(M)$

$X \mapsto L_X(f)$

the 1-form $M \ni p \mapsto (df)_p \in T_p^* M$

A 0-form on M is just a $f \in C^\infty(M)$

$\Omega^0(M) := C^\infty(M)$

any expression $\sum F_i dx^i \in \Omega^1(\mathbb{R}^n)$

makes sense/induces a 1-form on M :

$(\sum F_i dx^i)|_M \leftarrow$ referred to as " $\sum F_i dx^i$ on M "

Particular case of pull-backs:

$(df)_p: T_p M \rightarrow T_p^* M$

$F^*: \Omega^1(N) \rightarrow \Omega^1(M)$

$F^*(\omega) = \omega \circ F$

$F^*(\omega)(v_p) = \omega_{F(p)}(dF_p(v_p))$

For $g \in N$ have $\omega_g: T_g N \rightarrow \mathbb{R}$ linear

$F^*: \Omega^1(N) \rightarrow \Omega^1(M)$

$F^*(f) = f \circ F$

$F^*: \Omega^1(N) \rightarrow \Omega^1(M)$

NO!!!

Cotangent spaces: M, p

Def: The cotangent space $T_p^* M$ is the dual of $T_p M$

Ex: $f \in C^\infty(M)$, $p \in M$

hence $(df)_p \in T_p^* M$

Ex: Given (U, χ)

$\Rightarrow (dx^i)_p, \dots$

ie this is the dual

associated to $(\frac{\partial}{\partial x^i})_p$

- For any M, p : a vector space: the tangent space $T_p M$
- Any chart (U, x) around p gives rise to a basis of $T_p M$:
 $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^m}\right)_p$
- Vector field X on M means: $M \ni p \mapsto \omega_p \in T_p M$
 which smooth: (i) (U, x) , writing $\omega_p = \sum \text{coeff}_i(p) \left(\frac{\partial}{\partial x^i}\right)_p$, coeff. $U \rightarrow \mathbb{R}$ are smooth
- $\mathcal{X}(M)$: the set of all vector fields on M .
- $\mathcal{X}(M)$ is a vector space: $(\omega + \eta)_p := \omega_p + \eta_p, (\lambda \cdot \omega)_p := \lambda \omega_p$
- $\mathcal{X}(M)$ is a $C^\infty(M)$ -module: $(f \cdot \omega)_p := f(p) \omega_p$
- Globally: any $X \in \mathcal{X}(M)$ induces $L_X: C^\infty(M) \rightarrow C^\infty(M), L_X(f)(p) = df_p(X_p)$
 & this gives rise to a bijection:

$$\mathcal{X}(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \right\}$$

s.t. L is a derivation

Globally: any $\omega \in \Omega^1(M)$ induces a map $\hat{\omega}: \mathcal{X}(M) \rightarrow C^\infty(M)$
 $X \mapsto \hat{\omega}(X)$ defined by $\hat{\omega}(X)(p) = \omega_p(X_p) \in \mathbb{R}$

Thm: This gives rise to a bijection
 $\mathcal{X}(M) \xrightarrow{1-1} \left\{ \begin{array}{l} \mathcal{Z}: \mathcal{X}(M) \rightarrow C^\infty(M) \text{ s.t.} \\ C^\infty(M)\text{-linear} \end{array} \right\}$
 $\omega \mapsto \hat{\omega}$

Surjectivity: Given $\mathcal{Z}$ look for $\omega \in \Omega^1(M)$ s.t.
 $\mathcal{Z} = \hat{\omega}$ i.e. $\mathcal{Z}(X) = \omega_p(X_p) \forall X \in \mathcal{X}(M), p \in M$
 w? For $p \in M, u \in T_p M$ what is $\omega_p(u) \in \mathbb{R}$? Define it by
 choosing $X \in \mathcal{X}(M)$ s.t. $X_p = u$ Lemma: (u) p is $\exists X$
 setting $\omega_p(u) := \mathcal{Z}(X)(p)$ Lemma: this does not depend on the choice

Duals of vector spaces: Given $V = m$ -dimensional vector space
 its dual is $V^* = \{ \xi: V \rightarrow \mathbb{R} / \xi \text{ linear} \}$
 Remarks:
 ① V^* is a vector space: $(\xi + \eta)(v) = \xi(v) + \eta(v)$ etc.
 ② covectors pair with vectors $\Rightarrow$ give numbers
 $V^* \times V \rightarrow \mathbb{R}, (\xi, v) \mapsto \xi(v)$ - sometimes denoted $\langle \xi, v \rangle$
 ③ V^* is m -dimensional. Actually any basis $\{e_1, \dots, e_m\}$ of V induces a "dual basis" - a basis of V^* $\{e^1, \dots, e^m\}$ given by:
 ④ Intuitively: $\xi \in V^*$ thought of as a way to measure "oriented length" of vectors $v \in V$ - encoding the hyperplane $\xi(v) = 0$

Example: $\mathbb{R}^2$ with basis $\{e_1, e_2\}$. Dual basis $\{e^1, e^2\}$ where $e^i(e_j) = \delta^i_j$.
 For $\xi = \xi_1 e^1 + \xi_2 e^2$, $\xi(v) = \xi_1 v_1 + \xi_2 v_2$.
 The hyperplane $\xi(v) = 0$ is the line $\xi_1 x + \xi_2 y = 0$ in $\mathbb{R}^2$.

- $\mathcal{X}(M)$ is a Lie algebra: $X, Y \in \mathcal{X}(M) \mapsto [X, Y] \in \mathcal{X}(M)$
 (defined by condition: $L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)
- in $\mathbb{R}^n$: all $X \in \mathcal{X}(\mathbb{R}^n)$ are $\omega = \sum_{i=1}^n F_i dx^i$ with $F_i \in C^\infty(\mathbb{R}^n)$
 In general, for $X \in \mathcal{X}(M)$, locally (in a chart) we see n functions.
- on submanifolds $M \subseteq \mathbb{R}^n$, may use the ambient $\mathbb{R}^n$
 $\Rightarrow$ any $X \in \mathcal{X}(M)$ can be written as $\omega_p = \sum F_i(p) (dx^i)_p$ for $p \in M$
 $\omega = \sum F_i dx^i, F_i \in C^\infty(M)$

For submanifolds $M \subseteq N$
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 For $\omega \in \Omega^1(N)$, $\omega_p \in T_p^* N$ (i.e. a linear map $T_p N \rightarrow \mathbb{R}$)
 can be restricted to $T_p M \Rightarrow \omega|_{T_p M} \in T_p^* M$. And this defines $\omega|_M \in \Omega^1(M)$ (i.e. $(\omega|_M)(X_p) = \omega_p(X_p)$)

(De Rham) differential $d: C^\infty(M) \rightarrow \Omega^1(M), f \mapsto df$
 Rh: $\hat{df}: \mathcal{X}(M) \rightarrow C^\infty(M)$
 $X \mapsto L_X(f)$
 the 1-form $M \ni p \mapsto (df)_p \in T_p^* M$
 A 0-form on M is just a function $f \in C^\infty(M)$
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 Particular case of pull-backs: $F: M \rightarrow N$
 $(dF)_p: T_p M \rightarrow T_p^* N$
 $F^*: \Omega^1(N) \rightarrow \Omega^1(M)$
 $F^*(f) = f \circ F$
 For $\omega \in \Omega^1(N)$ have $\omega_p: T_p N \rightarrow \mathbb{R}$ but $\omega_p(X_p)$
 $F^*(\omega)(u_p) = \omega_{F(p)}(dF_p(u_p))$
 NO!!!

Cotangent Spaces: $M, p \in M$
 Def: The cotangent space of M at p is $T_p^* M :=$ the dual of $T_p M$
 Elts $\xi_p \in T_p^* M$: called cotangent vectors (to M) at p : $\xi: T_p M \rightarrow \mathbb{R}$

Ex: $f \in C^\infty(M), p \in M \Rightarrow (df)_p: T_p M \rightarrow \mathbb{R}$
 $\left(\frac{df}{dx^i} \right)_p \mapsto \frac{df}{dx^i}(u)$ etc.
 hence $(df)_p \in T_p^* M$
 Ex: Given (U, x) -chart, $p \in U, x = (x_1, \dots, x_m): U \rightarrow \mathbb{R}^m$
 $\Rightarrow (dx^i)_p, \dots, (dx^m)_p \in T_p^* M$
 i.e. this is the dual basis associated to $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^m}\right)_p$
 $\left(\sum \lambda_i \left(\frac{\partial}{\partial x^i}\right)_p \right) \mapsto \lambda_i$

$$X(M) \longleftrightarrow \left\{ L: C^\infty(M) \rightarrow C^\infty(M) \right. \\ \left. \text{s.t. } L = \text{a derivation} \right\}$$

Rham differential $d: C^\infty(M) \rightarrow \Omega^1(M)$, $f \mapsto df$ (8')

$$X = \sum f \cdot 4 \text{ with } f(p) = 0 \quad \text{Ker } \varphi = \{u \in V : \sum (u) = 0\}$$

what $f: M \rightarrow R$
Frobenius I have
 $\omega: TM \rightarrow R$

$$\frac{(\partial \chi_i)_p}{(\partial \chi_m)_p} = \frac{\partial \chi_i}{\partial \chi_m}$$