

• for any $M, p \in M$: the cotangent space $T_p^*M = (T_p M)^* = \{ \xi: T_p M \rightarrow \mathbb{R} \text{ linear} \}$
 • any chart (U, α) around $p \Rightarrow$ a basis of T_p^*M
 $(dx_i)_p, \dots, (dx_n)_p$ $(dx_i)_p$ sends $\left(\frac{\partial}{\partial x_j}\right)_p$ to $\begin{cases} 1 & j=i \\ 0 & j \neq i \end{cases}$
 • a 1-form ω on M means $M \ni p \mapsto \omega_p \in T_p^*M$
 with smoothness: $(\forall) (U, \alpha)$, the coefficients in writing $\omega_p = \sum \text{coeff}_i(p) (dx_i)_p$ smooth on U
 $\Omega^1(M) :=$ the set of all 1-forms, $\Omega^0(M) := C^\infty(M)$
 • $\Omega^1(M)$ is a vector space: $(\omega + \eta)_p := \omega_p + \eta_p$, $(\lambda \cdot \omega)_p := \lambda \omega_p$
 • $\Omega^1(M) = C^\infty(M)$ -module: $(f \cdot \omega)_p := f(p) \omega_p$
 • more global / algebraic viewpoint: bijection
 $\Omega^1(M) \xleftrightarrow{\sim} \begin{cases} \hat{\omega}: \hat{X}(M) \rightarrow C^\infty(M) / \\ \hat{\omega} \text{ is } C^\infty(M)\text{-linear: } \hat{\omega}(fX) = f\hat{\omega}(X) \text{ etc.} \end{cases}$
 $\omega \mapsto \hat{\omega}$ sending $X \mapsto \hat{\omega}(X)$ defined by:
 $\hat{\omega}(X)(p) := \omega_p(X_p)$

• (DeRham) differential $d: C^\infty(M) \rightarrow \Omega^1(M)$
 where $df \in \Omega^1(M)$ also denoted $d\omega(M)$
 is $M \ni p \mapsto (df)_p \in T_p^*M = \{ \xi: T_p M \rightarrow \mathbb{R} \text{ linear} \}$
 • in $\mathbb{R}^n$: all $\omega \in \Omega^1(\mathbb{R}^n)$ are $\omega = \sum_{i=1}^n F_i dx_i$, $F_i \in C^\infty(\mathbb{R}^n)$

On submanifolds $M \subseteq \mathbb{R}^n$ any $\omega \in \Omega^1(M)$ can be written
 $\omega_p = \sum F_i(p) (dx_i)_p$ with $F_i \in C^\infty(M)$
 each such expression defines a 1-form on M , but different
 looking formulas may describe the same 1-form on M !

• restrictions: $M \subseteq N$ submanifold $\Rightarrow$ an operation
 $\Omega^1(N) \rightarrow \Omega^1(M)$, $\omega \mapsto \omega|_M$ where $(\omega|_M)(X_p) = \omega_p(X_p)$

• pull-backs: $F: M \rightarrow N$ smooth induces
 $F^*: \Omega^1(N) \rightarrow \Omega^1(M)$, $(F^*\omega)(X_p) := \omega_{F(p)}(dF_p(X_p))$

Ex: $x dx$

$x dy$

$p = (0, 1)$

OR: $(x dx + y dy)$

$\Rightarrow$

for any $M, p \in M$: the cotangent space $T_p^*M = (T_p M)^* = \{ \xi: T_p M \rightarrow \mathbb{R} \text{ linear} \}$

- any chart (U, α) around $p \Rightarrow$ a basis of T_p^*M
 $(dx_i)_p, (dy_j)_p \quad (dx_i)_p$ sends $\frac{\partial}{\partial x_j}$ to $\begin{cases} 1 & j=i \\ 0 & j \neq i \end{cases}$
- a 1-form ω on M means $M \ni p \mapsto \omega_p \in T_p^*M$
 with smoothness: (U, α) , the coefficients in writing $\omega_p = \sum \text{coeff}_i(p) (dx_i)_p$ smooth on U
- $\Omega^1(M)$ = the set of all 1-forms, $\Omega^1(M) := C^0(M)$
- $\Omega^1(M)$ is a vector space $(\omega + \eta)_p = \omega_p + \eta_p, (\lambda \omega)_p = \lambda \omega_p$
 $\omega_p(X_p) \in \mathbb{R}$
- $\Omega^1(M) = C^0(M)$ -module: $(f \cdot \omega)_p = f(p) \omega_p$
 $\omega(X) \in C^0(M)$
- more global/algebraic viewpoint: by action
 $\Omega^1(M) \xleftrightarrow{\omega} \{ \hat{\omega}: \hat{X}(M) \rightarrow C^0(M) / \hat{\omega} \text{ is } C^0(M)\text{-linear: } \hat{\omega}(fX) = f \hat{\omega}(X) \}$
 $\omega \mapsto \hat{\omega}$ sending $X \mapsto \hat{\omega}(X)$ defined by:
 $\hat{\omega}(X)(p) = \omega_p(X_p)$

• (Derivational differential $d: C^0(M) \rightarrow \Omega^1(M)$)
 where $d\hat{f} \in \Omega^1(M)$
 is $M \ni p \mapsto d\hat{f}_p \in T_p^*M = \{ \xi: T_p M \rightarrow \mathbb{R} \text{ linear} \}$

• $(\Omega^1(M))$ all $\omega \in \Omega^1(M)$ are $\omega = \sum_{i=1}^n F_i d\hat{x}_i, F_i \in C^0(M)$
 On submanifolds $M \subseteq \mathbb{R}^n$ any $\omega \in \Omega^1(M)$ can be written
 $\omega = \sum F_i (dx_i)_p$ with $F_i \in C^0(M)$

each such expression defines a 1-form on M , but different looking formulas may describe the same 1-form on M

• Restrictions: $M \subseteq N$ submanifold $\Rightarrow$ an operation
 $\Omega^1(N) \rightarrow \Omega^1(M), \omega \mapsto \omega|_M$ where $(\omega|_M)_p(X_p) = \omega_p(X_p)$

• pull-backs: $F: M \rightarrow N$ smooth induces
 $F^*: \Omega^1(N) \rightarrow \Omega^1(M), \dots (F^*\omega)(X_p) = \omega(F(p)) (dF_p(X_p))$

More generally, for any $k \in \mathbb{N}, V = \text{vector space}$:
 $\Lambda^k V^* = \{ \xi: \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{R} \mid \begin{array}{l} \xi \text{ linear in each argument} \\ \xi \text{ skew symmetric} \end{array} \}$

Examples: $k=0: \Lambda^0 V^* = \mathbb{R}$
 $k=1: \Lambda^1 V^* = V^*$

Any $\xi_1, \xi_2 \in V^*$ induce an element $\xi \in \Lambda^2 V^*$ by:
 $\xi: V \times V \rightarrow \mathbb{R}, \xi(v_1, v_2) = \xi_1(v_1) \xi_2(v_2) - \xi_1(v_2) \xi_2(v_1)$

Similarly, $\xi_1 \in V^*, \xi_2 \in \Lambda^2 V^* \Rightarrow$ build $\eta \in \Lambda^3 V^*$
 $\eta: V \times V \times V \rightarrow \mathbb{R}, \eta(v_1, v_2, v_3) = \xi_1(v_1) \xi_2(v_2, v_3) - \xi_1(v_2) \xi_2(v_1, v_3) + \xi_1(v_3) \xi_2(v_1, v_2)$

Basis: if $\{e_1, \dots, e_n\}$ - basis of V : get a basis of $\Lambda^k V^*$?
 $(e^i): V \times \dots \times V \rightarrow \mathbb{R}$
 $e^i(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & \text{if } i=j_1, \dots, j_k \\ 0 & \text{otherwise} \end{cases}$
 if $i=j_1, \dots, j_k$ out of $\{e_1, \dots, e_n\}$ more conceptually
 $\{e^I: I \text{ as above}\}$ basis of $\Lambda^k V^*$

$\Rightarrow$ get $\Lambda^{p+q} V^* \times \Lambda^r V^* \rightarrow \Lambda^{p+q+r} V^*$
 with explicit formula:
 $(\omega \wedge \eta)(v_1, \dots, v_{p+q+r}) = \sum_{\sigma \in S_{p+q+r}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q+r)})$
 i.e. σ is a $p+q+r$ -shuffle $\rightarrow \begin{cases} \sigma \in S_k \text{ s.t.} \\ \sigma(1) < \dots < \sigma(p), \sigma(p+1) < \dots < \sigma(p+q) \end{cases}$

Ex: If I have $\xi_1, \dots, \xi_p \in V^*$

Useful in handling $\Lambda^k V^*$ (as a TOOL).

$\Lambda^k V^* \subseteq T^k V^* = \{ \xi: \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{R} \mid \xi \text{ linear in each argument} \}$

Useful 1: While $\Lambda^k V^* \subseteq T^k V^*$ one can also produce, out of any $\omega \in T^k V^*$ an element in $\Lambda^k V^*$: $\text{Alt}(\omega) \in \Lambda^k V^*$
 $\text{Alt}(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \cdot \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

Useful 2: Obvious "product operations" involving $T^k V^*$ for different k
 $\omega \in T^p V^*, \eta \in T^q V^* \Rightarrow$ get $\omega \bullet \eta \in T^{p+q} V^*$ defined by
 $(\omega \bullet \eta)(v_1, \dots, v_{p+q}) = \omega(v_1, \dots, v_p) \eta(v_{p+1}, \dots, v_{p+q})$

Combine the two $\Rightarrow$ for $\omega \in \Lambda^p V^*, \eta \in \Lambda^q V^*$ get
 $\omega \wedge \eta = \frac{(p+q)!}{p!q!} \text{Alt}(\omega \bullet \eta) \in \Lambda^{p+q} V^*$
 to avoid repetitions

Smooth on U

$$\omega_p(X_p) \in \mathbb{R}$$

$$\omega(X) \in C^0(M)$$

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not by:

$$\omega(X)(p) := \omega_p(X_p)$$

More generally, for any $k \in \mathbb{N}$, V = vector space :

$$\Lambda^k V^* := \left\{ \xi : \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{R} \mid \begin{array}{l} \xi = \text{linear in each argument} \\ \xi = \text{skew symmetric} \end{array} \right\}$$

Examples: $k=0$: $\Lambda^0 V^* = \mathbb{R}$
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$$\xi: V \times V \rightarrow \mathbb{R}, \quad \xi(v_1, v_2) = \xi_1(v_1) \xi_2(v_2) - \xi_1(v_2) \xi_2(v_1)$$

Similarly: $\xi_1 \in V^*, \xi_2 \in \Lambda^2 V^* \Rightarrow$ build $\eta \in \Lambda^3 V^*$

$$\eta: V \times V \times V \rightarrow \mathbb{R}, \quad \eta(v_1, v_2, v_3) = \xi_1(v_1) \xi_2(v_2, v_3) - \xi_1(v_2) \xi_2(v_1, v_3) + \xi_1(v_3) \xi_2(v_1, v_2)$$

Basis: If $\{e_1, \dots, e_n\}$ - basis of V . Get a basis of $\Lambda^k V^*$?

$$\{e^I\}: V \times \dots \times V \rightarrow \mathbb{R}$$

$$e^I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & \text{if } i_1 = j_1, \dots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$$

$I = (i_1 < \dots < i_k)$ out of $\{1, 2, \dots, n\}$
 more conceptually
 $\{e^I: I \text{ as above}\}$
 basis of $\Lambda^k V^*$

(3)

Useful in handling $\Lambda^k V^*$ (as a TOOL).

(4)

$$\subseteq T^k V^* := \left\{ \xi : \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{R} \mid \xi = \text{linear in each argument} \right\}$$

Useful 1: While $\Lambda^k V^* \subseteq T^k V^*$ one can also produce, out of any $\omega \in T^k V^*$ an element in $\Lambda^k V^*$: $\text{Alt}(\omega) \in \Lambda^k V^*$

$$\text{Alt}(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \cdot \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

Useful 2: Obvious "product operations" involving $T^k V^*$ for different k 's.

$\omega \in T^p V^*, \eta \in T^q V^*$ get $\omega \bullet \eta \in T^{p+q} V^*$ defined by

$$(\omega \bullet \eta)(v_1, \dots, v_{p+q}) = \omega(v_1, \dots, v_p) \eta(v_{p+1}, \dots, v_{p+q})$$

Combine the two $\Rightarrow$ for $\omega \in \Lambda^p V^*, \eta \in \Lambda^q V^*$ get

$$\omega \wedge \eta := \frac{(p+q)!}{p!q!} \text{Alt}(\omega \bullet \eta) \in \Lambda^{p+q} V^*$$

to avoid repetitions

(3)

$\Rightarrow$ get $\wedge^p V^* \times \wedge^q V^* \longrightarrow \wedge^{p+q} V^*$, $(\omega, \eta) \mapsto \omega \wedge \eta$

with explicit formula:

$$(\omega \wedge \eta)(v_1, \dots, v_{p+q}) = \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$

i.e. σ is a $p+q$ -shuffle $\rightarrow \begin{cases} \sigma \in S_{p+q} \text{ s.t.} \\ \sigma(1) < \sigma(2) < \dots < \sigma(p), & \sigma(p+1) < \dots < \sigma(p+q) \end{cases}$

Ex: If I have $\xi_1, \dots, \xi_p \in V^*$

(5)

sis of V .
all

$$i_1 < \dots < i_k$$

$$V^* = \bigwedge^1 V^*$$

$$I = \{i_1, \dots, i_k\} \in \text{Ord}_k^n$$

$\Rightarrow$ get $\bigwedge^p V^* \times \bigwedge^q V^* \longrightarrow \bigwedge^{p+q} V^*$, $(\omega, \eta) \mapsto \omega \wedge \eta$
with explicit formula:

$$(\omega \wedge \eta)(v_1, \dots, v_{p+q}) = \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}) \quad (*)$$

i.e. σ is a $p+q$ -shuffle $\rightarrow \begin{cases} \sigma \in S_{p+q} \text{ s.t.} \\ \sigma(1) < \sigma(2) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q) \end{cases}$

Ex: If I have $\xi_1, \dots, \xi_p \in V^*$

Basic properties: $\omega \in \bigwedge^p V^*$, $\eta \in \bigwedge^q V^*$, $\theta \in \bigwedge^r V^*$

$$\{ (\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) \quad (\text{associativity}) \quad (1)$$

$$\{ \eta \wedge \omega = (-1)^{p \cdot q} \omega \wedge \eta \quad (2)$$

Consequence 1: expressions of type $\omega_1 \wedge \dots \wedge \omega_k$ make sense

Consequence 2: $\omega \wedge \omega = 0$ if $\omega \in \bigwedge^p V^*$, $p = \text{odd}$.

Consequence 3: $\omega_{\sigma(1)} \wedge \dots \wedge \omega_{\sigma(k)} = \text{sgn}(\sigma) \omega_1 \wedge \dots \wedge \omega_k$ $\omega_i \in V^*$

Back to a basis: Start with $\{e_1, \dots, e_n\}$ basis of V

We consider $\text{Ord}_k(n)$ the collection of all

$$I = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, n\} \text{ with } i_1 < \dots < i_k$$

Make use of the dual basis $\{e^1, \dots, e^n\}$ of $V^* = \Lambda^1 V^*$

and consider

$$e^I = e^{i_1} \wedge \dots \wedge e^{i_k}$$

for any $I = \{i_1, \dots, i_k\} \in \text{Ord}_k(n)$

Prop: These form a basis of $\Lambda^k V^*$
 $\{e^I : I \in \text{Ord}_k(n)\}$

In particular $\dim(\Lambda^k V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\Lambda^k V^* = 0 \quad \text{if } k > \dim(V) !!!$$

$$\omega \in \Lambda^k V^*, \quad k > n = \dim V \Rightarrow \omega = 0$$

even $\omega(e_1, \dots, e_k)$ a repetition
 $e_1, \dots, e_k$

One can also use $\wedge \Rightarrow$ an operation

$\hat{\omega}(X_i)$ defined by:
 $\hat{\omega}(X_i)(p) = \omega_p(X_i(p), X_{k_i})$

$\omega(X_1, \dots, X_k)$
 $u_1, \dots, u_k \in \{1, \dots, n\}$

$d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$
 for each k .
 To be discussed

$F \in C^{\infty}(\mathbb{R})$
 can be written
 $\Omega^k(M)$
 M , but different
 ω form on M !
 $\Rightarrow$ an operation

when $(u, \eta)(X_i) = \omega_p(X_i(p))$
 $\tau_{u, \eta}^M = \tau_{u, \eta}^M \circ \tau_{u, \eta}^M \circ (dF)_p$
 $(X_i)_p = \omega_p(dF(X_i))$

One can also use $\wedge \Rightarrow$ an operation

$$\Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^{p+q}(M), (\omega, \eta) \mapsto \omega \wedge \eta$$

where $\omega \wedge \eta$ is defined by

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p \quad \text{previous one for } V = T_p M$$

Some basic properties (1) & (2) and, on vector fields $X_1, \dots, X_{p+q}$
 Same formula as in (*).

$$\begin{aligned} \text{Ex: } p=0 \quad f \wedge \eta &= f \cdot \eta \\ q=0 \quad \omega \wedge \eta &= \eta \cdot \omega. \end{aligned}$$

$$\Omega^k(M) = \{0\} \quad \text{if } k > \dim(M)$$

Important: Pull-back operation is compatible with $\wedge$ and d .

$$\begin{cases} F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta) \\ F^*(d\omega) = dF^*(\omega) \end{cases}$$

(in particular: $F^*(df) = d(F^*(f))$)

Useful in handling

$$\subseteq T^k V^* = \{ \}$$

Useful 1: While
 $\omega \in T^k V^*$ are
 $\text{Alt}(\omega)$

Useful 2: Obvious
 $\omega \in T^p V^*, \eta \in T^q V^*$
 $(\omega \otimes \eta)(v_1, \dots, v_{p+q})$

Combine the two
 $\omega \wedge \eta :=$

• for any $M, p \in M: \bigwedge^k T_p^* M = \{ \omega: T_p M \times \dots \times T_p M \rightarrow \mathbb{R}, \text{ linear in each argument, skew} \}$
 • any chart (U, α) around $p \Rightarrow$ a basis of $\bigwedge^k T_p^* M$
 $(dx_I)_p := (dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p$ one for each $I = \{i_1, \dots, i_k\} \in \text{Ord}_k(n)$
 • a k-form ω on M means $M \ni p \mapsto \omega_p \in \bigwedge^k T_p^* M$
 with smoothness: (U, α) , the coefficients in writing $\omega_p = \sum_I \text{coeff}_I(p) (dx_I)_p$ smooth on U
 $\mathcal{O}^k(M) :=$ the set of all k-forms, $\mathcal{O}^0(M) := C^\infty(M)$
 • $\mathcal{O}^k(M)$ is a vector space: $(\omega + \eta)_p = \omega_p + \eta_p, (\lambda \omega)_p = \lambda \omega_p$
 • $\mathcal{O}^k(M) = C^k(M)$ -module: $(f \cdot \omega)_p = f(p) \omega_p$
 • more global/algebraic viewpoint: bijection
 $\mathcal{O}^k(M) \xleftrightarrow{\sim} \left\{ \begin{array}{l} \hat{\omega}: \tilde{X}(M) \times \dots \times \tilde{X}(M) \rightarrow C^\infty(M) \\ \hat{\omega} \text{ is } C^\infty(M)\text{-multilinear \& skew} \end{array} \right\}$
 $\omega \mapsto \hat{\omega} \text{ sending } (X_1, \dots, X_k) \mapsto \hat{\omega}(X_1, \dots, X_k) \text{ defined by: } \hat{\omega}(X_1, \dots, X_k)(p) = \omega_p(X_1|_p, \dots, X_k|_p)$

• (DeRham) differential $d: \mathcal{O}^k(M) \rightarrow \mathcal{O}^{k+1}(M)$
 where $df \in \mathcal{O}^1(M)$, also denoted $d: \mathcal{O}^k(M) \rightarrow \mathcal{O}^{k+1}(M)$
 is $M \ni p \mapsto d(\omega)_p \in T_p^* M = \{ \tilde{\omega}: T_p M \rightarrow \mathbb{R} \text{ linear} \}$ for each p .
 • in $\mathbb{R}^n$: all $\omega \in \mathcal{O}^k(\mathbb{R}^n)$ are $\omega = \sum_I F_I d\tilde{x}_I$ with $F_I \in C^\infty(\mathbb{R}^n)$
 On submanifolds $M \subseteq \mathbb{R}^n$ any $\omega \in \mathcal{O}^k(M)$ can be written
 $\omega = \sum_I F_I(p) (d\tilde{x}_I)_p$ with $F_I \in C^\infty(M)$
 each such expression defines a k-form on M , but different
 looking formulas may describe the same k-form on M !
 • restrictions: $M \subseteq N$ submanifold $\Rightarrow$ an operation
 $\mathcal{O}^k(N) \rightarrow \mathcal{O}^k(M), \omega \mapsto \omega|_M$ where $(\omega|_M)_p = \omega_p(X_1|_p, \dots, X_k|_p)$
 • pull-backs: $F: M \rightarrow N$ smooth induces
 $F^*: \mathcal{O}^k(N) \rightarrow \mathcal{O}^k(M), \omega \mapsto (F^*\omega)$ where $(F^*\omega)_p = \omega_{F(p)}(dF_p(X_1), \dots, dF_p(X_k))$

One can also
 $\mathcal{O}^p(M)$
 where $\omega \wedge \eta$
 $(\omega \wedge \eta)$
 Same basic
 Same formula
 Ex: $p=0$ for
 $q=0: \omega$
Important: P
 $\{ F^*(\omega \wedge \eta) \}$
 $\{ F^*(d\omega) \}$

Consequence 3: $\omega_{\sigma(1)} \wedge \dots \wedge \omega_{\sigma(k)} = \underset{\text{sgn}}{\text{sgn}(\sigma)} \omega_1 \wedge \dots \wedge \omega_k$ $\omega_i \in V^*$

$$d: \Omega^k(M) \rightarrow \Omega^{k+1}(M) \quad (7)$$

(DR0) d is linear (over $\mathbb{R}$) and, for $k=0$ is the usual $f \mapsto \underbrace{df}_{\text{old } df}$.

$$(DR1) \quad d \left(\underbrace{\omega}_{\Omega^p(M)} \wedge \underbrace{\eta}_{\Omega^q(M)} \right) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta)$$

$$(DR2) \quad d \circ d = 0 \quad (d(d(\omega)) = 0 \in \Omega^{k+2}(M))$$

Good to know: Compatible

with pull-backs:

$$d(F^*\omega) = F^*(d\omega)$$

$$\underbrace{\Omega^k(M)}_{\Omega^{k+1}(M)}$$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$d(df) = \sum d\left(\frac{\partial f}{\partial x_i}\right) dx_i$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

$$d(df) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} \underbrace{d(dx_j \wedge dx_i)}_{\substack{d(dx_j) \wedge dx_i + dx_j \wedge d(dx_i) \\ = 0}}$$

$$\omega = x dx \wedge dy + yz dx \wedge dz \in \Omega^2(\mathbb{R}^3)$$

$$d\omega = d(x dx \wedge dy) + d(yz dx \wedge dz) =$$

$$= \underbrace{d(x dx) \wedge dy}_{\substack{dx \wedge dx + x d(dx) \\ = 0}} - x dx \wedge \underbrace{d(dy)}_0 + \underbrace{d(yz)}_{yz dz + zy dy} \wedge dx \wedge dz + yz \wedge \underbrace{d(dx \wedge dz)}_{\substack{d(dx) \wedge dz + dx \wedge d(dz) \\ = 0}}$$

$$= y dz \wedge dx \wedge dz + z dy \wedge dx \wedge dz = -z dx \wedge dy \wedge dz$$

Back to a basis: Start with f.e. basis of V
We consider $\text{Ord}(m)$ the collection of all
 $I = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$ with $i_1 < \dots < i_k$.
Make use of the dual basis $\{e^i, e^j\}$ of $V = \wedge^k V$
and consider $e^i = e^{i_1} \wedge \dots \wedge e^{i_k}$ for any $I = \{i_1, \dots, i_k\} \in \text{Ord}(k)$.
Prop: These form a basis of $\wedge^k V$.
 $\{e^I : I \in \text{Ord}(k)\}$
In particular $\dim(\wedge^k V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
 $\wedge^k V = 0$ if $k > \dim(V)$.
 $w \in \wedge^k V, k > n = \dim V \Rightarrow w = 0$
 $e_{i_1} \wedge \dots \wedge e_{i_k}$ a repetition
 $i_1, \dots, i_k \in \{1, \dots, n\}$

$\Rightarrow$ set $\Lambda^p V^* \times \Lambda^q V^* \rightarrow \Lambda^{p+q} V^*$, $(\omega, \eta) \mapsto \omega \wedge \eta$
 with explicit formula:
 $(\omega \wedge \eta)(v_1, \dots, v_{p+q}) = \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$
 $\omega \wedge \eta$ is a $(p+q)$ -tuple $\rightarrow \left\{ \begin{array}{l} \sigma \in S_{p+q} \text{ s.t.} \\ \sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q) \end{array} \right.$
 $\frac{E_k}{E_k} \wedge \frac{J}{J} \wedge \dots \wedge \frac{J}{J} \wedge \frac{J}{J} \wedge \dots \wedge \frac{J}{J} \wedge \frac{J}{J} \wedge \dots \wedge \frac{J}{J} \wedge \frac{J}{J}$
Basic properties: $\omega \in \Lambda^p V^*$, $\eta \in \Lambda^q V^*$, $\theta \in \Lambda^r V^*$
 (associativity) (1)
 $\{ (\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) \}$
 (2)
 $\{ \eta \wedge \omega = (-1)^{pq} \omega \wedge \eta \}$
Consequence 1: expressions of type $\omega_1 \wedge \dots \wedge \omega_k$ make sense
Consequence 2: $\omega \wedge \omega = 0$ if $\omega \in \Lambda^p V^*$, $p = \text{odd}$.
Consequence 3: $\omega_{\sigma(1)} \wedge \dots \wedge \omega_{\sigma(k)} = \text{sgn}(\sigma) \omega_1 \wedge \dots \wedge \omega_k$ $\omega_i \in V^*$

• (Dual Defn) Let $\mathcal{H}$ be a Hilbert space. A linear functional f on $\mathcal{H}$ is a linear map $f: \mathcal{H} \rightarrow \mathbb{C}$.
 • (Riesz Representation Theorem) Let $\mathcal{H}$ be a Hilbert space. For every linear functional f on $\mathcal{H}$, there exists a unique vector $\psi_f \in \mathcal{H}$ such that $f(\phi) = \langle \phi, \psi_f \rangle$ for all $\phi \in \mathcal{H}$.
 • (Proof) Let f be a linear functional on $\mathcal{H}$. Define $N_f = \ker f$. If $N_f = \mathcal{H}$, then $f = 0$ and $\psi_f = 0$. Otherwise, N_f is a proper subspace of $\mathcal{H}$. Let ψ_f be the orthogonal projection of f onto $N_f^\perp$. Then ψ_f is the unique vector such that $f(\phi) = \langle \phi, \psi_f \rangle$ for all $\phi \in \mathcal{H}$.
 • (Corollary) Let $\mathcal{H}$ be a Hilbert space. The dual space $\mathcal{H}^*$ of $\mathcal{H}$ is isometrically isomorphic to $\mathcal{H}$.
 • (Definition) Let $\mathcal{H}$ be a Hilbert space. A linear operator T on $\mathcal{H}$ is a linear map $T: \mathcal{H} \rightarrow \mathcal{H}$.
 • (Adjoint Operator) Let T be a linear operator on $\mathcal{H}$. The adjoint operator T^* of T is the unique linear operator on $\mathcal{H}$ such that $\langle T\phi, \psi \rangle = \langle \phi, T^*\psi \rangle$ for all $\phi, \psi \in \mathcal{H}$.
 • (Properties) Let T be a linear operator on $\mathcal{H}$. Then $(T^*)^* = T$, $(T+S)^* = T^* + S^*$, and $(cT)^* = \bar{c}T^*$ for any scalar c .
 • (Self-Adjoint Operator) A linear operator T on $\mathcal{H}$ is self-adjoint if $T = T^*$.
 • (Normal Operator) A linear operator T on $\mathcal{H}$ is normal if $TT^* = T^*T$.
 • (Spectral Theorem) Let T be a self-adjoint operator on $\mathcal{H}$. Then there exists a unique spectral measure E_T on $\mathcal{H}$ such that $T = \int \lambda dE_T(\lambda)$.
 • (Definition) Let $\mathcal{H}$ be a Hilbert space. A projection P on $\mathcal{H}$ is a linear operator on $\mathcal{H}$ such that $P^2 = P$ and $P = P^*$.
 • (Properties) Let P be a projection on $\mathcal{H}$. Then P is idempotent and self-adjoint. The range of P is a closed subspace of $\mathcal{H}$, and $\mathcal{H} = \text{range}(P) \oplus \ker(P)$.
 • (Definition) Let $\mathcal{H}$ be a Hilbert space. A unitary operator U on $\mathcal{H}$ is a linear operator on $\mathcal{H}$ such that $U^* = U^{-1}$.
 • (Properties) Let U be a unitary operator on $\mathcal{H}$. Then U is invertible and $\|U\phi\| = \|\phi\|$ for all $\phi \in \mathcal{H}$.
 • (Definition) Let $\mathcal{H}$ be a Hilbert space. A compact operator K on $\mathcal{H}$ is a linear operator on $\mathcal{H}$ such that the image of the unit ball under K is compact.

One can also use $\wedge \Rightarrow$ an operation

$$\wedge^2(M) \times \wedge^2(M) \rightarrow \wedge^{2+2}(M), (\omega, \eta) \mapsto \omega \wedge \eta$$

where $\omega \wedge \eta$ is defined by

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p.$$

previous one for $V = T_p M$

Some basic properties (1) $\in \mathbb{R}$ and, on vector fields $X_1, \dots, X_p$

Some formulae as in (A)

$$\begin{aligned} E_1: f \wedge g &= f \wedge g \\ E_2: \omega \wedge f &= f \wedge \omega. \end{aligned}$$

$\wedge^k(M) = \{ \omega \mid k > \dim(M) \}$

Important: Pull-back operation is compatible with $\wedge$ and d.

$$\begin{aligned} F^*(\omega \wedge \eta) &= F^*(\omega) \wedge F^*(\eta) \\ F^*(d\omega) &= dF^*(\omega) \quad (\text{in particular: } F^*(d\omega) = d(F^*\omega)) \end{aligned}$$

$d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

(DR_0) d is linear (over $\mathbb{R}$) and, for $k=0$ is the usual $f \mapsto \frac{df}{dx^i}$

(DR_1) $d(w \wedge \eta) = (dw) \wedge \eta + (-1)^p w \wedge (d\eta)$
 $\Omega^p(M) \quad \Omega^q(M)$

(DR_2) $d \circ d = 0 \quad (d(d(w))) = 0 \in \Omega^{k+2}(M)$

Good to know: Compatible with pull-backs: $d(F^*w) = F^*(dw)$

$w \wedge dx \wedge dy + y \wedge dx \wedge dz \in \Omega^2(\mathbb{R}^3)$
 $dw = d(\underbrace{x \wedge y}_{\frac{\partial x}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_1} dx_1}) + d(\underbrace{y \wedge z}_{y \wedge dz + z \wedge dy}) =$
 $= \underbrace{\frac{\partial x}{\partial x_1} dx_1 \wedge dy - x dx \wedge \frac{\partial y}{\partial x_1} dx_1}_{\frac{\partial x}{\partial x_1} dx_1 \wedge dy - x dx \wedge \frac{\partial y}{\partial x_1} dx_1} + \underbrace{\frac{\partial y}{\partial x_1} dx_1 \wedge dz + y \wedge \frac{\partial z}{\partial x_1} dx_1}_{y \wedge dz + z \wedge dy} =$
 $= \underbrace{y dz \wedge dx \wedge dz}_{0} + \underbrace{z dy \wedge dx \wedge dz}_{0} = -z dx \wedge dy \wedge dz$