

Vector fields $X: M \ni p \mapsto X_p \in T_p M$

→ locally: in any chart $(U, x): X = \sum_i F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(U)$

→ globally: $L_X: C^\infty(M) \rightarrow C^\infty(M)$ ($L_X(f)(p) = \sum_i F_i \frac{\partial f}{\partial x_i} = (df)_p(X_p)$)

→ operations: $X+Y, f \cdot X \Rightarrow \mathfrak{X}(M)$ is an $C^\infty(M)$ -module

$[X, Y] \Rightarrow \mathfrak{X}(M)$ also a Lie algebra ($L_{[X, Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))$)

→ flow of X: $\varphi_X^t(p, t) = \varphi_X^t(p)$ ($p \in M, t \in \mathbb{R}, \dots$) where, by def:

for $p \in M$, $t \mapsto \varphi_X^t(p)$ is the maximal integral curve starting at p

→ Lie derivatives along X of "natural objects":

$$\mathcal{L}_X(\xi) = \frac{d}{dt} \bigg|_{t=0} (\varphi_X^t)^*(\xi)$$

(to be made precise in each example)

Ex 1: For $a = f \in C^\infty(M): \mathcal{L}_X(f) = L_X(f)$

Ex 2: For $a = Y \in \mathfrak{X}(M): \mathcal{L}_X(Y) = [X, Y]$

Prk: $(\varphi_X^t)^*(\xi) = 0 \ (\forall) t \Rightarrow \mathcal{L}_X(\xi) = 0$ (... and conversely!)

(1)

$\Omega^k(M)$ = the set of all k-forms on M (for $k \in \mathbb{N}$). Hence $\omega \in \Omega^k(M)$ means:

for each $p \in M$ we have $\omega_p: \underbrace{T_p M \times \dots \times T_p M}_k \rightarrow \mathbb{R}$ skew, $\mathbb{R}$ -multilinear, smooth

→ more globally: $\omega: \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ skew, $C^\infty(M)$ -multilinear

→ locally, in charts $(U, x): \omega = \sum_{I=(i_1, \dots, i_k) \in \text{Ord}_k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_I F_I dx_I$

(in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = \{0\}$ ($\forall$) $k > \dim M$).

→ operations: $\omega + \eta, f \cdot \omega \dots$ and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

→ wedge products $\wedge$: $\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$
(new operations)

or, explicitly:

$$(\omega \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{1(k)}, \dots, X^{k(k)}) \eta(X^{(k+1)}, \dots, X^{(k+l)})$$

$(\omega, \eta) \mapsto \omega \wedge \eta := \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta)$ and the same formulas at each $p \in M$!

Basic properties:

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$$

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$$

(in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree $\uparrow$ k)

Case $k=0$: $\Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M), \eta \in \Omega^l(M)$:

$$f \wedge \omega = \omega \wedge f = f \cdot \omega$$

De Rham differential

Prop: Manifolds M come with maps $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

which are uniquely determined by the following properties:

(De Rham):

(4)

Lie derivatives L_V

(5)

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$

which are uniquely determined by the following properties:

$\Omega^k(M)$ = the set of all k -forms on M (for $k \in \mathbb{N}$). Hence $\omega \in \Omega^k(M)$ means:

for each $p \in M$ we have $\omega_p: \underbrace{T_p M \times \dots \times T_p M}_k \rightarrow \mathbb{R}$ skew, $\mathbb{R}$ -multilinear, smooth

more globally

→ locally, in charts $(U, x): \omega = \sum_{I=(i_1, \dots, i_k) \in \text{Ord}_k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{I \in \text{Ord}_k(M)} F_I dx_I$

(in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = \{0\}$ ($\forall k > \dim M$)).

→ operations: $\omega + \eta$, $f \cdot \omega$... and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

→ wedge products $\wedge$:
(new operations)

or, explicitly:

$$(\omega \wedge \eta)(x^1, \dots, x^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(x^{\sigma(1)}, \dots, x^{\sigma(k)}) \eta(x^{\sigma(k+1)}, \dots, x^{\sigma(k+l)})$$

Basic properties: $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ (in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree)

Case $k=0$: $\Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M)$, $\eta \in \Omega^k(M)$:

$$f \wedge \omega = \omega \wedge f = f \cdot \omega$$

→ forms are very natural: any smooth map $F: M \rightarrow N$ induces

$$F^*: \Omega^k(N) \rightarrow \Omega^k(M) \quad (\Omega^k(M) \xleftarrow{F^*} \Omega^k(N))$$

For $\omega \in \Omega^k(N)$, the pull-back of ω to M via F , $F^*(\omega) \in \Omega^k(M)$, is given by:

$$F^*(\omega)_p(x_p^1, \dots, x_p^k) := \omega_{F(p)}(dF_p(x_p^1), \dots, dF_p(x_p^k))$$

Useful: F^* is compatible with $\wedge$: $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$
... and with d : $F^*(d\omega) = d(F^*\omega)$

& the chain rule still holds:

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ & \searrow G \circ F & \downarrow G \\ & & P \end{array} \Rightarrow (G \circ F)^*(\omega) = F^*(G^*(\omega)) \quad (\forall \omega \in \Omega^k(P))$$

$$\begin{array}{ccc} \Omega^k(M) & \xleftarrow{F^*} & \Omega^k(N) \\ & \searrow F_* & \uparrow G^* \\ & & \Omega^k(P) \end{array}$$

Next: operations d, i_V, L_V (for $V \in \mathfrak{X}(M)$)

$$\Omega^0(M) \xrightleftharpoons[L_V]{d} \Omega^1(M) \xrightleftharpoons[L_V]{d} \Omega^2(M) \xrightleftharpoons[L_V]{d} \Omega^3(M) \xrightleftharpoons[L_V]{d} \Omega^4(M) \xrightleftharpoons[L_V]{d} \dots$$

& "Cartan magic formula": $d \circ i_V + i_V \circ d = L_V$

Lie derivatives L_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$ which are uniquely determined by the following properties:

Interior products i_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $i_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ which are uniquely determined by the following properties:

Vector fields $X: M \ni p \mapsto X_p \in T_p M$

→ locally: in any chart $(U, \alpha): X = \sum_i F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(U)$

→ globally: $L_X: C^\infty(M) \rightarrow C^\infty(M)$ ($L_X(f)(p) = \partial_X(f) = (df)_p(X_p)$)

→ operations: $X+Y, f \cdot X \Rightarrow \mathfrak{X}(M)$ is an $C^\infty(M)$ -module

$[X, Y] \Rightarrow \mathfrak{X}(M)$ also a Lie algebra ($[L_X(f), L_Y(f)] = L_{[X, Y]}(f)$)

→ flow of X : $\varphi_t(p) = \varphi_t^X(p)$ ($p \in M, t \in \mathbb{R}, \dots$) where, by def:

for $p \in M, t \mapsto \varphi_t^X(p)$ is the maximal integral curve starting at p

→ Lie derivatives along X of "natural objects":

$$\partial_X(\xi) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X)^*(\xi)$$

(to be made precise in each example)

Ex 1: For $a = f \in C^\infty(M): \partial_X(f) = L_X(f)$

Ex 2: For $a = Y \in \mathfrak{X}(M): \partial_X(Y) = [X, Y]$

Prop: $(\varphi_t^X)^*(\xi) = 0 \iff \partial_X(\xi) = 0$ (and conversely!)

$\Omega^k(M)$ = the set of all k -forms on M (for $k \in \mathbb{N}$). Hence $\omega \in \Omega^k(M)$ means:

for each $p \in M$ we have $\omega_p: T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$ skew, $\mathbb{R}$ -multilinear, smooth

more globally $\omega: \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ skew, $C^\infty(M)$ -multilinear

→ locally, in charts $(U, \alpha): \omega = \sum_{I=(i_1, \dots, i_k) \in \text{Ord}_k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_I F_I d\mathbf{x}_I$

(in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = 0$ if $k > \dim M$).

→ operations: $\omega + \eta, f \cdot \omega \dots$ and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

→ wedge products $\wedge$: $\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$

(new operations) $(\omega, \eta) \mapsto \omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta)$ and the same formulas at each $p \in M$!

or, explicitly: $(\omega \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{i_1}, \dots, X^{i_k}) \eta(X^{j_1}, \dots, X^{j_l})$

Basic properties: $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta), \eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ (in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree)

Case $k=0$: $\Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M), \eta \in \Omega^k(M)$:

$$f \wedge \omega = \omega \wedge f = f \cdot \omega$$

→ forms are very natural: any smooth map $F: M \rightarrow N$ induces $F^*: \Omega^k(N) \rightarrow \Omega^k(M)$ ($\Omega^k(M) \xleftarrow{F^*} \Omega^k(N)$)

For $\omega \in \Omega^k(N)$, the pull-back of ω to M via F , $F^*(\omega) \in \Omega^k(M)$, is given by:

$$F^*(\omega)_p(X_p^1, \dots, X_p^k) := \omega_{F(p)}(dF_p(X_p^1), \dots, dF_p(X_p^k))$$

Useful: F^* is compatible with $\wedge$: $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

... and with d : $F^*(d\omega) = d(F^*\omega)$

& the chain rule still holds:

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow \text{GoF} & & \downarrow G \\ & & P \end{array} \Rightarrow \begin{array}{ccc} (G \circ F)^*(\omega) = F^*(G^*(\omega)) & & \Omega^k(M) \xleftarrow{F^*} \Omega^k(N) \\ (V) \omega \in \Omega^k(P) & & \downarrow G^* \\ & & \Omega^k(P) \end{array}$$

Next: operations d, ι_V, L_V (for $V \in \mathfrak{X}(M)$)

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \Omega^4(M) \xrightarrow{d} \dots$$

& "Cartan magic formula": $d \circ \iota_V + \iota_V \circ d = L_V$

De Rham differential

Prop: Manifolds M come with maps $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

which are uniquely determined by the following properties:

(De Rham 0): d is linear (over $\mathbb{R}$) and, for $k=0$: the usual $d: \Omega^0(M) \rightarrow \Omega^1(M)$ ($d(f) = L_X(f)$)

(De Rham 1): d is a derivation of degree 1, i.e. Leibniz:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$$

(De Rham 2): $d \circ d = 0$ (i.e. $d(d(\omega)) = 0$ for ω)

Lie derivatives L_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$

which are uniquely determined by the following properties:

Interior products ι_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $\iota_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

which are uniquely determined by the following properties:

Vector fields $X: M \ni p \mapsto X_p \in T_p M$

→ locally: in any chart $(U, \chi): \chi = \sum_i F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(U)$

→ globally: $L_X: C^\infty(M) \rightarrow C^\infty(M)$ ($L_X(f)(p) = \sum_i F_i(p) \frac{\partial f}{\partial x_i}(p) = (df)_p(X_p)$)

→ operations: $X+Y, f \cdot X \Rightarrow \mathfrak{X}(M) = \text{an } C^\infty(M)\text{-module}$

$[X, Y] \Rightarrow \mathfrak{X}(M)$ also a Lie algebra ($[L_X(f), L_Y(f)] = L_{[X, Y]}(f)$)

→ flow of X: $\varphi_X(p, t) = \varphi_X^t(p)$ ($p \in M, t \in \mathbb{R}, \dots$) where, by def:
for $p \in M, t \mapsto \varphi_X^t(p)$ is the maximal integral curve starting at p

→ Lie derivatives along X of "natural objects":

$\mathcal{L}_X(\xi) = \frac{d}{dt} \Big|_{t=0} (\varphi_X^t)^*(\xi)$ (to be made precise in each example)

Ex 1: For $\alpha = f \in C^\infty(M) = \mathcal{L}_X(f) = X(f)$
Ex 2: For $\alpha = Y \in \mathfrak{X}(M): \mathcal{L}_X(Y) = [X, Y]$
Ex 3: $\mathcal{L}_X(\varphi_X^t)^*(\xi) = \xi \forall t \Rightarrow \mathcal{L}_X(\xi) = 0$ (and conversely!)

$\Omega^k(M)$ = the set of all k -forms on M (for $k \in \mathbb{N}$). Hence $\omega \in \Omega^k(M)$ means:

for each $p \in M$ we have $\omega_p: T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$ skew, $\mathbb{R}$ -multilinear, smooth (1)

more globally $\omega: \mathfrak{X}(M) \wedge \dots \wedge \mathfrak{X}(M) \rightarrow C^\infty(M)$ skew, $C^\infty(M)$ -multilinear (2)

→ locally, in charts $(U, \chi): \omega = \sum_{I \in \text{ord}_k(M)} F_I d\chi_{i_1} \wedge \dots \wedge d\chi_{i_k} = \sum_I F_I d\chi_I$

(in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = \{0\}$ ($\forall k > \dim M$)).

→ operations: $\omega + \eta, f \cdot \omega$... and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

→ wedge products $\wedge$: (new operations)
 $(\omega, \eta) \mapsto \omega \wedge \eta = \frac{(k+l)}{k!l!} \text{Alt}(\omega \otimes \eta)$ and the same formulas at each $p \in M!$
or, explicitly:
 $(\omega \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{1, \dots, k, \sigma(1), \dots, \sigma(k)}) \eta(X^{\sigma(k+1), \dots, \sigma(k+l)})$

Basic properties: $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ (in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree)

Case $k=0$: $\Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M), \eta \in \Omega^k(M)$:
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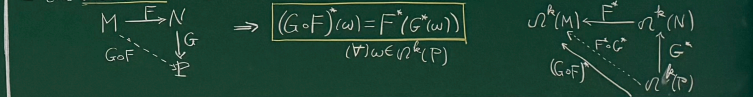
→ forms are very natural: any smooth map $F: M \rightarrow N$ induces

$F^*: \Omega^k(N) \rightarrow \Omega^k(M)$ ($\Omega^k(M) \xleftarrow{F^*} \Omega^k(N)$)

For $\omega \in \Omega^k(N)$, the pull-back of ω to M via $F, F^*(\omega) \in \Omega^k(M)$, is given by:

$F^*(\omega)_p(X_p^1, \dots, X_p^k) := \omega_{F(p)}(dF_p(X_p^1), \dots, dF_p(X_p^k))$ (3)

Useful: F^* is compatible with $\wedge$: $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$
... and with d : $F^*(d\omega) = d(F^*\omega)$



Next: operations $d, \mathcal{L}_V, L_V$ (for $V \in \mathfrak{X}(M)$)

$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{\mathcal{L}_V} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{\mathcal{L}_V} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{\mathcal{L}_V} \Omega^3(M) \xrightarrow{d} \dots$

& "Cartan magic formula": $d \circ \mathcal{L}_V + \mathcal{L}_V \circ d = L_V$

De Rham differential

Prop: Manifolds M come with maps $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

which are uniquely determined by the following properties:

(De Rham): d is linear (over $\mathbb{R}$) and, for $k=0$: the usual $d: \Omega^0(M) \rightarrow \Omega^1(M)$ ($d(f) = L_X(f)$)

(De Rham): d is a derivation of degree 1, i.e. Leibniz: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(De Rham): $d \circ d = 0$ (i.e. $d(d(\omega)) = 0$ for ω)

"cohomology" (6)

need chain complexes:
 $C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \dots$
 $\gamma \circ d = 0$

$H^k_\Delta(M) \cong H^k_{\text{De Rham}}(M)$ (the hairy ball theorem, last lecture)

Lie derivatives $\mathcal{L}_V$

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $\mathcal{L}_V: \Omega^k(M) \rightarrow \Omega^k(M)$

which are uniquely determined by the following properties:

(1) $\mathcal{L}_V$ simply: $(\mathcal{L}_V(X^1, \dots, X^k))(p) := \omega_p(X_p^1, \dots, X_p^k)$ (4)

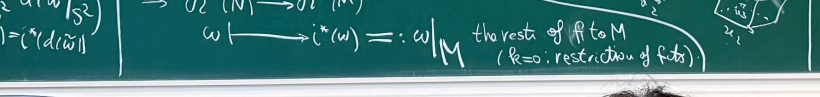
(2) $\mathcal{L}_V$ proposition: $\mathcal{L}_V \omega = \{ \xi: \forall \alpha \in \wedge^k V^* \rightarrow \mathbb{R} \mid \xi = \text{skew, multilinear} \}$ applied to $V = T_p M$

Think of $\xi \in \wedge^k V^*$ as providing a way to measure (signed) volumes of parallelepipeds

Partic. case of pull-backs: $M \subseteq N$ submanifold

take $F := i$ the inclusion $\Rightarrow \Omega^k(N) \rightarrow \Omega^k(M)$

$\omega \mapsto i^*(\omega) = \omega|_M$ the restriction of ω to M ($k=0$: restriction of fct)



Interior products $\mathcal{L}_V$

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which are uniquely determined by the following properties:

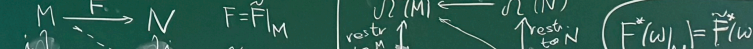
$F: M \rightarrow N$ $p \in M \xrightarrow{\text{push it}} F(p) \in N$ (5)

$(dF)_p: T_p M \rightarrow T_{F(p)} N$ $V_p \in T_p M \xrightarrow{\text{pushed}} (dF)_p(V_p) \in T_{F(p)} N$

$\mathfrak{X}(M) \rightarrow \mathfrak{X}(N)$ cannot be pushed back (only when F is a diffeomorphism!!)

(3) is a formula that really needs (2)

Chain-rule consequence:



$F^* \circ \text{rest}_N = \text{rest}_M \circ F^*$
 $F^*(\omega|_N) = \text{rest}_M^*(F^*\omega)$

Vector fields $X: M \ni p \mapsto X_p \in T_p M$

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→ globally: $L_X: C^\infty(M) \rightarrow C^\infty(M)$ ($L_X(f)(p) = \frac{d}{dt} f(\alpha_t(p)) = (df)_p(X_p)$)

→ operations: $X+Y, f \cdot X \Rightarrow \mathfrak{X}(M) = \text{an } C^\infty(M)\text{-module}$

$[X, Y] \Rightarrow \mathfrak{X}(M)$ also a Lie algebra ($[L_X(f), L_Y(g)] = L_{[X, Y]}(fg)$)

→ flow of $X: \varphi_t(p, t) = \varphi_t^X(p)$ ($p \in M, t \in \mathbb{R}, \dots$) where, by def:

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more globally $\omega: \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$ skew, $C^\infty(M)$ -multilinear (2)

→ locally, in charts $(U, \alpha): \omega = \sum_{I=(i_1, \dots, i_k) \in \mathcal{O}_k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{I \in \mathcal{O}_k(M)} F_I d\alpha_I$

(in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = \{0\}$ ($\forall k > \dim M$))

→ operations: $\omega + \eta, f \cdot \omega, \dots$ and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

→ wedge products $\wedge: \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$

→ forms are very natural: any smooth map $F: M \rightarrow N$ induces $F^*: \Omega^k(N) \rightarrow \Omega^k(M)$ ($\Omega^k(M) \xleftarrow{F^*} \Omega^k(N)$)

For $\omega \in \Omega^k(N)$, the pull-back of ω to M via F , $F^*(\omega) \in \Omega^k(M)$, is given by:

$$F^*(\omega)_p(X_p^1, \dots, X_p^k) = \omega_{F(p)}(dF_p(X_p^1), \dots, dF_p(X_p^k)) \quad (3)$$

Useful: F^* is compatible with $\wedge: F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$
... and with $d: F^*(d\omega) = d(F^*\omega)$

& the chain rule still holds:

De Rham differential

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which are uniquely determined by the following properties:

(De Rham): d is linear (over $\mathbb{R}$) and, for $k=0$: the usual $d: \Omega^0(M) \rightarrow \Omega^1(M)$

(De Rham): d is a derivation of degree 1, i.e. Leibniz: $d(fg) = df \wedge g + f \wedge dg$ ($df(X) = L_X(f)$)

(De Rham): $d \circ d = 0$ (i.e. $d(d\omega) = 0 \forall \omega$)

(Uniqueness): $d(\omega \wedge \eta) = d\omega \wedge \eta + \omega \wedge d\eta$

$d(f_0 df_1 \dots df_k) = df_0 \wedge df_1 \wedge \dots \wedge df_k + f_0 \wedge d(df_1 \wedge \dots \wedge df_k)$

$d(\sum \text{type } f_0 df_1 \dots df_k) \text{ is uniquely determined}$

any form can be written like this (locally).

Existence: A bit tedious

Lie derivatives L_X

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$

which are uniquely determined by the following properties:

Explicit formula for d

• $k=0: df \in \Omega^1(M): df(X) = L_X(f)$

• $k=1, \omega \in \Omega^1(M): d\omega \in \Omega^2(M): d\omega(X, Y) = \omega([X, Y]) + L_X(\omega(Y)) - L_Y(\omega(X))$

• k arbitrary: uses view-point (2)!

$d\omega(X_0, \dots, X_k) = \sum_{i=0}^k (-1)^i \omega(X_0, \dots, \hat{X}_i, \dots, X_k) + \sum_{i=0}^k L_{X_i}(\omega(X_0, \dots, \hat{X}_i, \dots, X_k))$

Interior products ι_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $\iota_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

which are uniquely determined by the following properties:

A bit more motivation: Start with $\theta \in \Omega^1(M)$.

Q: when can I write $\theta = df$ for some f ?

Necessary: $d\theta = 0$!!! (ex: $\theta = x dy$ no df)

Q: When is $\omega \in \Omega^k(M)$ of type dy for some $\eta \in \Omega^{k-1}(M)$?

Could: $d\omega = 0$

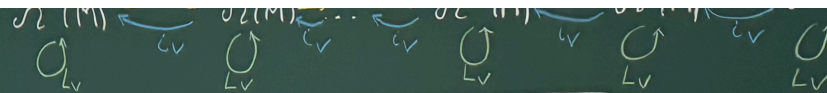
related to top of M

De Rham cohom

Basic properties: $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$, $\eta \wedge \omega = (-1)^k \omega \wedge \eta$ (in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree k)

Case $k=0$: $\Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M)$, $\eta \in \Omega^k(M)$:

$$f \wedge \omega = \omega \wedge f = f \cdot \omega$$



& "Cartan magic formula": $d \circ i_V + i_V \circ d = L_V$

the important one: L_V .

Lie derivative L_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$

which are uniquely determined by the following properties:

(0) $L_V = \text{li}_V$ & for $k=0$: the usual $L_V: C^\infty(M) \rightarrow C^\infty(M)$ (blockaded)

(1) L_V is a derivation of degree 0, i.e.

$$L_V(\omega \wedge \eta) = L_V(\omega) \wedge \eta + \omega \wedge L_V(\eta)$$

(2) $L_V \circ d = d \circ L_V$.

Define $L_V(\omega) := "d_V(\omega)"$ from blackboard & ... making sense of it

$$L_V(\omega) := \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega) \quad \text{Using (3):}$$

$$L_V(\omega)_p(X_1^*, \dots, X_k^*) := \left. \frac{d}{dt} \right|_{t=0} \omega_{\varphi_V^t(p)}(d\varphi_V^t(X_1^*), \dots, d\varphi_V^t(X_k^*))$$

$$\text{Check (2): } L_V(d\omega) = \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(d\omega) = d \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega) = d(L_V(\omega))$$

$$= \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega) + \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(d\omega) = \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega) + \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(d\omega)$$

Interior products i_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $i_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

which are uniquely determined by the following properties:

(0) $i_V = \text{li}_V$ & for $k=1$: $i_V(\omega) = \omega(V)$

(for $k=0$: $i_V(f) = 0$)

(1) i_V is a derivation of degree -1, i.e.

$$i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge i_V(\eta)$$

handy for proofs & for computations

Explicitly: for $\omega \in \Omega^k(M)$, $i_V(\omega) \in \Omega^{k-1}(M)$ given by $i_V(\omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})$

$$\text{Check (1): } L_V(\omega \wedge \eta) = \left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega \wedge \eta) \stackrel{(5)}{=} \left. \frac{d}{dt} \right|_{t=0} \left((\varphi_V^t)^*(\omega) \wedge (\varphi_V^t)^*(\eta) \right)$$

$$= \left(\left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\omega) \right) \wedge (\varphi_V^0)^*(\eta) + (\varphi_V^0)^*(\omega) \wedge \left(\left. \frac{d}{dt} \right|_{t=0} (\varphi_V^t)^*(\eta) \right)$$

$$= L_V(\omega) \wedge \eta + \omega \wedge L_V(\eta) \quad \square$$

$\frac{d}{dt} \left(\frac{1}{t} \right) = -\frac{1}{t^2}$
 (to be made precise in each example)
 $\text{Ex 2. } P_k: (\psi_t^X)^*(\xi) = \sum_{i=1}^k \xi_i(t) \Rightarrow d_X(\xi) = 0$

De Rham differential
 $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

Prop: Manifolds M come with maps which are uniquely determined by the following properties:

(De Rham 0): d is linear (over $\mathbb{R}$) and, for $k=0$: the usual $d: \Omega^0(M) \rightarrow \Omega^1(M)$
 $(df(X)) = L_X(f)$

(De Rham 1): d is a derivation of degree 1, i.e. Leibniz:
 $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$

(De Rham 2): $d \circ d = 0$ (i.e. $d(d(\omega)) = 0$ for ω)

proof of Cartan magic formula: $L_V = \underbrace{d \circ i_V + i_V \circ d}_{\text{call this } K_V}$

Sufficient to check: K_V satisfies (0), (1), (2).

Check (1):

$$\begin{aligned}
 K_V(\omega \wedge \eta) &= d(i_V(\omega \wedge \eta)) + i_V(d(\omega \wedge \eta)) \\
 &= d(i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)) + i_V(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) \\
 &= \underbrace{d(i_V(\omega)) \wedge \eta + (-1)^{k+1} i_V(\omega) \wedge d\eta}_{= L_V(\omega) \wedge \eta + (-1)^{k+1} \omega \wedge d(i_V(\eta))} + (-1)^k (\underbrace{d\omega \wedge i_V(\eta) + (-1)^k \omega \wedge d(i_V(\eta))}_{= L_V(\eta) \wedge \omega + (-1)^{k+1} \omega \wedge d(i_V(\eta))}) + i_V(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta)
 \end{aligned}$$

Lie derivative
Prop: Vector fields $V \in \mathfrak{X}(M)$ which are uniquely determined

- (0) $L_V = \text{linear}$ for $k=0$
- (1) L_V is a derivation of $L_V(\omega \wedge \eta) =$
- (2) $L_V \circ d = d \circ L_V$.

Define $L_V(\omega) := \left. \frac{d}{dt} \right|_{t=0} \psi_t^V(\omega)$

$$L_V(\omega)_p(X_p^1, \dots, X_p^k) := \frac{d}{dt} \bigg|_{t=0} \omega(\psi_t^V(X_p^1, \dots, X_p^k))$$

Check (2): $L_V(d\omega) = \frac{d}{dt} \bigg|_{t=0} d\psi_t^V(\omega) = d \left(\frac{d}{dt} \bigg|_{t=0} \psi_t^V(\omega) \right) = d(L_V(\omega))$

Vector fields $X: M \ni p \mapsto X_p \in T_p M$

→ locally: in any chart $(U, \alpha): X = \sum F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(U)$
 → globally: $L_X: C^\infty(M) \rightarrow C^\infty(M)$ ($L_X(f)(p) = \frac{d}{dt}(f \circ \gamma_p)(0) = (df)_p(X_p)$)

→ operations: $X+Y, f \cdot X \Rightarrow \mathfrak{X}(M) = \text{an } C^\infty(M)\text{-module}$
 $[X, Y] \Rightarrow \mathfrak{X}(M)$ also a Lie algebra ($[L_X(f), L_Y(g)] = L_{[X, Y]}(fg)$)

→ flow of $X: \varphi_X(p, t) = \gamma_p^t(p)$ ($p \in M, t \in \mathbb{R}, \dots$) where, by def:
 for $p \in M, t \mapsto \gamma_p^t(p)$ is the maximal integral curve starting at p

→ Lie derivatives along X of "natural objects":
 $L_X(f) = \frac{d}{dt} \Big|_{t=0} (\varphi_X^t)^*(f)$
 (to be made precise in each example)
 Ex 1: For $a \in C^\infty(M): L_X(a) = X(a)$
 Ex 2: For $a = Y \in \mathfrak{X}(M): L_X(Y) = [X, Y]$
 Ex 3: $L_X(\gamma_p^t) = \frac{d}{dt} \Big|_{t=0} \gamma_p^t = X_p$ (and conversely!)

$\Omega^k(M)$ = the set of all k -forms on M (for $k \in \mathbb{N}$). Hence $\omega \in \Omega^k(M)$ means:
 for each $p \in M$ we have $\omega_p: T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$ skew, $\mathbb{R}$ -multilinear, smooth (1)
 more globally: $\omega: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_k \rightarrow C^\infty(M)$ skew, $C^\infty(M)$ -multilinear (2)
 → locally, in charts $(U, \alpha): \omega = \sum_{I=(i_1, \dots, i_k) \in \text{Ord}_k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{I \in \text{Ord}_k(M)} F_I dx_I$
 (in particular: $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = \{0\}$ for $k > \dim M$).

→ operations: $\omega + \eta, f \cdot \omega \dots$ and $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$
 → wedge products $\wedge: \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$
 (new operations)
 or, explicitly: $(\omega, \eta) \mapsto \omega \wedge \eta = \sum_{i,j} \frac{(k+l)!}{k!l!} \text{Alt}(\omega_i \eta_j)$ and the same formulas at each $p \in M$!
 Ex: $(\omega \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{\sigma(1)}, \dots, X^{\sigma(k)}) \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$

Basic properties: $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta), \eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ (in particular: $\omega \wedge \omega = 0$ whenever ω is of odd degree)
 Case $k=0: \Omega^0(M) = C^\infty(M)$ and, for $f \in C^\infty(M), \eta \in \Omega^l(M):$
 $f \wedge \omega = \omega \wedge f = f \cdot \omega$

→ forms are very natural: any smooth map $F: M \rightarrow N$ induces $F^*: \Omega^k(N) \rightarrow \Omega^k(M)$ (3)
 For $\omega \in \Omega^k(N)$, the pull-back of ω to M via F , $F^*(\omega) \in \Omega^k(M)$, is given by:
 $F^*(\omega)_p(X_p^1, \dots, X_p^k) := \omega_{F(p)}(dF_p(X_p^1), \dots, dF_p(X_p^k))$ (3)

Useful: F^* is compatible with $\wedge: F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$ (5)
 ... and with $d: F^*(d\omega) = d(F^*\omega)$ (6)

& the chain rule still holds:
 $M \xrightarrow{F} N \xrightarrow{G} P \Rightarrow (G \circ F)^*(\omega) = F^*(G^*(\omega))$ ($\forall \omega \in \Omega^k(P)$)
 Diagram: $M \xrightarrow{F} N \xrightarrow{G} P$ with $dF: T_p M \rightarrow T_{F(p)} N$ and $dG: T_{F(p)} N \rightarrow T_{G(F(p))} P$.

Next: operations d, L_V, ι_V (for $V \in \mathfrak{X}(M)$)
 $\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\dim M}(M) \xrightarrow{d} 0$
 (the important one: L_V)
 & "Cartan magic formula": $d \circ \iota_V + \iota_V \circ d = L_V$

DeRham differential

Prop: Manifolds M come with maps $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$
 which are uniquely determined by the following properties:

- (DeRham): d is linear (over $\mathbb{R}$) and, for $k=0$: the usual $d: \Omega^0(M) \rightarrow \Omega^1(M)$ ($d(f) = L_X(f)$)
- (DeRham): d is a derivation of degree +1, i.e. Leibniz: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$
- (DeRham): $d \circ d = 0$ (i.e. $d(d\omega) = 0$ for ω)

proof of Cartan magic formula: $L_V = d \circ \iota_V + \iota_V \circ d$
 Suffices to check: K_V satisfies (0), (1), (2).
 Check (1): $K_V(\omega \wedge \eta) = d(\iota_V(\omega \wedge \eta) + \iota_V(d\omega \wedge \eta))$
 $= d(\iota_V \omega \wedge \eta + (-1)^{|\omega|} \omega \wedge \iota_V \eta) + \iota_V(d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta)$
 $= d\iota_V \omega \wedge \eta + (-1)^{|\omega|} d\omega \wedge \iota_V \eta + \iota_V d\omega \wedge \eta + (-1)^{|\omega|} \iota_V \omega \wedge d\eta$
 $= K_V(\omega) \wedge \eta + \omega \wedge L_V(\eta)$ (1)

Lie derivatives L_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$
 which are uniquely determined by the following properties:

- (0) L_V is linear & for $k=0$: the usual $L_V: \Omega^0(M) \rightarrow \Omega^0(M)$ (block)
- (1) L_V is a derivation of degree 0, i.e. $L_V(\omega \wedge \eta) = L_V \omega \wedge \eta + \omega \wedge L_V \eta$
- (2) $L_V \circ d = d \circ L_V$

Define $L_V(\omega) := "d_V(\omega)"$ from blackboard & ... making sense of it
 $L_V(\omega) := \frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(\omega)$ Using (3).

$L_V(\omega)_p(X_p^1, \dots, X_p^k) := \frac{d}{dt} \Big|_{t=0} \omega_{F(p)}(d\varphi_V^t(X_p^1), \dots, d\varphi_V^t(X_p^k))$
 Check (2): $L_V(d\omega) = \frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(d\omega) = d \frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(\omega) = d(L_V \omega)$

Interior products ι_V

Prop: Vector fields $V \in \mathfrak{X}(M)$ give rise to maps $\iota_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$
 which are uniquely determined by the following properties:

- (0) ι_V is linear & for $k=1: \iota_V(\omega) = \omega(V)$ (for $k=0: \iota_V(f) = 0$)
- (1) ι_V is a derivation of degree -1, i.e. $\iota_V(\omega \wedge \eta) = \iota_V(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge \iota_V(\eta)$ (handy for proofs & for computation)

Explicitly: for $\omega \in \Omega^k(M), \iota_V(\omega) \in \Omega^{k-1}(M)$ given by
 $\iota_V(\omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})$

Check (1): $L_V(\omega \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(\omega \wedge \eta) \stackrel{(5)}{=} \frac{d}{dt} \Big|_{t=0} ((\varphi_V^t)^*(\omega) \wedge (\varphi_V^t)^*(\eta))$
 $= \left(\frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(\omega) \right) \wedge (\varphi_V^0)^*(\eta) + (\varphi_V^0)^*(\omega) \wedge \left(\frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(\eta) \right)$
 $= L_V(\omega) \wedge \eta + \omega \wedge L_V(\eta)$ (1)