

2) Volume forms/orientations  $\mu_p: T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$   
 $p \in M$

- Def: A volume form on  $M$  is any  $\mu \in \Omega^m(M)$  nowhere vanishing  
 $m = \dim M$   $\mu_p \neq 0 \forall p \in M$
- Call  $M$  orientable if such an  $\mu$  exists.
  - Given  $\mu$ , call a chart  $(U, \alpha)$  positively oriented (wrt  $\mu$ ) if  $\mu(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}) > 0$
  - Two volume forms  $\mu, \mu'$  called equivalent if  $\mu' = f\mu$  for  $f > 0$  everywhere
  - An orientation on  $M$  is an equivalence class of volume forms
  - If  $\mu$  vol form the induced orientation is denoted  $\mu$ .
- Recall that  $\wedge^m T_p M$  is 1-dimensional (if  $(U, \alpha)$  chart around  $p$   
 $\Rightarrow (dx_1)_p \wedge \dots \wedge (dx_m)_p$  is a basis)  
Hence  $\mu$ -volume form:  $\mu_p \in \wedge^m T_p M$  can serve as a basis element.  
In particular:  $(\forall) \omega \in \Omega^m(M)$  can be written as  $\omega = f\mu$  for some  $f \in C^\infty(M)$   
And this is still a volume form iff  $f(x) \neq 0 \forall x \in M$

Reminder:  $(U, \alpha)$  chart at  $p \in M \Rightarrow$  various induced bases:

$(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p$  of  $T_p M$   
 $(dx_1)_p, \dots, (dx_m)_p$  of  $T_p^* M$   
 $(dx_1)_p \wedge \dots \wedge (dx_m)_p$  of  $\wedge^m T_p^* M$  ( $i \in \{1, \dots, m\}$ )  
 $(dx_1)_p \wedge \dots \wedge (dx_m)_p$  of  $\wedge^m T_p^* M$

Changing the chart  $(U', \alpha')$  another one, denoting  $c = \alpha' \circ \alpha^{-1} \Rightarrow$   
 $\Rightarrow \frac{\partial}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j}$  and  
dualising it  $\Rightarrow$   
 $(dx'_i)_p = \sum_j \frac{\partial x_j}{\partial x'_i} (dx_j)_p$  (\*)

$f$  fct, chart  $(U, \alpha)$   
 $\frac{\partial f}{\partial x'_i} = df(\frac{\partial}{\partial x'_i})$  is the usual  
 $i$ th derivative of  $f \circ \alpha'$   
 $= \frac{\partial f}{\partial x_j}(\alpha'(p))$   
 $f = x'_j$   
 $\frac{\partial x'_j}{\partial x'_i}(p) = \frac{\partial c_j}{\partial x'_i}(\alpha'(p))$

Standard integration on opens  $\Omega \subseteq \mathbb{R}^m$  (simple version!)  
 $\int_\Omega : \int_\Omega^\infty(\Omega) \rightarrow \mathbb{R}, f \mapsto \int_\Omega f = \int_\Omega f(x) dx = \int_\Omega f(x_1, \dots, x_m) dx_1 \dots dx_m$

with basic properties:  
(Lin)  $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g, \int_\Omega (\lambda f) = \lambda \int_\Omega f$  ( $f, g \in C_c^\infty(\Omega), \lambda \in \mathbb{R}$ )

(Change of variables) if  $c: \Omega' \rightarrow \Omega$  is a diffeo, then:  
 $\int_\Omega f = \int_{\Omega'} |Jac(c)| f \circ c$  ( $f \in C_c^\infty(\Omega)$ )

Here:  $f \in C_c^\infty(\Omega)$  means that  $f$  has compact support i.e.  
 $\exists K \subseteq M$  cpt s.t.  $f=0$  on  $M \setminus K$

$Jac(c): \Omega' \rightarrow \mathbb{R}$  is the fct given by  
 $x \mapsto \det(\frac{\partial c_i}{\partial x'_j}(x))_{1 \leq i, j \leq m}$

Ex 1:  $M = \mathbb{R}^n: \mu_{can} = dx_1 \wedge \dots \wedge dx_n$

Ex 2:  $M = S^1: \mu = y dx - x dy$  (restricted to  $S^1$ )  
To check:  $(\forall) p = (a, b) \in S^1, \mu_p \neq 0$  ( $\forall p \in T_p S^1 \rightarrow \mathbb{R}$ )  
Use  $V = y(\frac{\partial}{\partial x})_p - x(\frac{\partial}{\partial y})_p \in T_p S^1: \mu_p(V) = b + a = 1 \neq 0 \Rightarrow \mu_p \neq 0 \forall p \in S^1$

Ex 3:  $M = S^1, \mu = dx$  not a volume form:  $\mu_p(V_p) = 0$  for  $p = (1, 0)$   
( $\forall V_p \in T_p S^1$ )

Ex 4:  $M = S^2: \mu = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$  is a volume form.  
 $p = (a, b, c) \mu_p: T_p S^2 \times T_p S^2 \rightarrow \mathbb{R}$  is non-zero?  
To show:  $(\forall) p = (a, b, c) \exists V_p, W_p \in T_p S^2$  s.t.  $\mu_p(V_p, W_p) \neq 0$ .  
No global basis of  $T_p S^2$ . Instead we can use:  
 $V = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, W = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \dots$   
 $\mu_p(V, W) = x \wedge y + y \wedge z + z \wedge x = x^2 = a^2 \neq 0$  whenever  $a \neq 0$  but at least one happens  
 $\mu_p(V, W) = \dots \neq 0 \rightarrow a=0, b \neq 0$   
 $\mu_p(V, W) = \dots \neq 0 \rightarrow a=0, c \neq 0$

Focus now on arbitrary top-degree forms  $\omega \in \Omega^m(M)$  the coeff. of  $\omega$  w.r.t.  $(U, \alpha)$   
Locally: use chart  $(U, \alpha)$ , over  $U$  can write  $\omega = \sum_{i_1, \dots, i_m} F_{i_1, \dots, i_m} (dx_{i_1} \wedge \dots \wedge dx_{i_m})$   
 $F \in C^\infty(\Omega)$

How does  $F_x$  change if  $\alpha$  change  $\alpha'$  (while keeping  $\omega$ )? Given  $(U, \alpha')$ :  
 $\omega_p = \sum_{i_1, \dots, i_m} F_{i_1, \dots, i_m}(\alpha'(p)) (dx_{i_1})_p \wedge \dots \wedge (dx_{i_m})_p$   
 $= \sum_{i_1, \dots, i_m} F_{i_1, \dots, i_m}(\alpha'(p)) \frac{\partial x_{i_1}}{\partial x'_1} \dots \frac{\partial x_{i_m}}{\partial x'_m} (dx'_1)_p \wedge \dots \wedge (dx'_m)_p$   
The RHS =  $F_{x'}(\alpha'(p)) \sum_{i_1, \dots, i_m} \frac{\partial x_{i_1}}{\partial x'_1} \dots \frac{\partial x_{i_m}}{\partial x'_m} (dx_{i_1})_p \wedge \dots \wedge (dx_{i_m})_p$   
 $= F_{x'}(\alpha'(p)) \sum_{i_1, \dots, i_m} \frac{\partial x_{i_1}}{\partial x'_1} \dots \frac{\partial x_{i_m}}{\partial x'_m} (dx_{i_1})_p \wedge \dots \wedge (dx_{i_m})_p$   
 $= F_{x'}(\alpha'(p)) \sum_{\sigma \in S_m} \frac{\partial x_{\sigma(1)}}{\partial x'_1} \dots \frac{\partial x_{\sigma(m)}}{\partial x'_m} (dx_{\sigma(1)})_p \wedge \dots \wedge (dx_{\sigma(m)})_p$   
 $= F_{x'}(\alpha'(p)) \sum_{\sigma \in S_m} \text{sgn}(\sigma) \frac{\partial c_1}{\partial x'_1}(\alpha'(p)) \dots \frac{\partial c_m}{\partial x'_m}(\alpha'(p)) (dx_1)_p \wedge \dots \wedge (dx_m)_p$   
 $= F_{x'}(\alpha'(p)) \left( \sum_{\sigma \in S_m} \text{sgn}(\sigma) \frac{\partial c_1}{\partial x'_1}(\alpha'(p)) \dots \frac{\partial c_m}{\partial x'_m}(\alpha'(p)) \right) (dx_1)_p \wedge \dots \wedge (dx_m)_p$   
 $\Rightarrow F_{x'} = Jac(c) \cdot F_x \cdot c$

Problem 1.1. Fix: use an orientation  $\theta$  on  $M$ . Fix  $\theta$ .  
Use now only positively oriented charts  $(U, \alpha)$ !  
For two such  $\alpha, \alpha' \Rightarrow c = \alpha' \circ \alpha^{-1}$  has  $Jac(c) > 0$  (Exercise)

Theorem: On any oriented manifold  $(M, \theta)$   $\exists!$  a map  
 $\int_M : \Omega_c^m(M) \rightarrow \mathbb{R}$  ( $\omega$  s.t.  $\exists K \subseteq M$  cpt  $\omega_p = 0 \forall p \in M \setminus K$ )  
such that:  
(1)  $\int_M$  is linear ( $\int_M (\omega_1 + \omega_2) = \int_M \omega_1 + \int_M \omega_2$ )  
(2) for any positively oriented chart  $\alpha: U \rightarrow \Omega \subseteq \mathbb{R}^m$   
and any  $\omega \in \Omega_c^m(M)$  that is supported in  $U$  then  
 $\int_M \omega = \text{the usual } \int_\Omega F_x^\omega$  where  $F_x^\omega$   
is the coefficient in the writing  
 $\omega|_U = F_x^\omega \alpha^* dx_1 \wedge \dots \wedge dx_m$



$$T_p M \rightarrow \mathbb{R}^m$$

are vanishing  
= 0  $\forall p \in M$

w.r.t.  $\mu$ ) if:

$f > 0$   
everywhere

volume form

$d p$   
is a basis

ut

Some  
 $f \in C^\infty(M)$

Reminder:  $(U, x)$  chart at  $p \in M \Rightarrow$  various induced bases:

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \text{ of } T_p M$$

$$(dx_1)_p, \dots, (dx_m)_p \text{ of } T_p^* M$$

$$(dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p \text{ of } \Lambda^k T_p^* M \quad (1 \leq i_1 < \dots < i_k \leq m)$$

$$(dx_1)_p \wedge \dots \wedge (dx_m)_p \text{ of } \Lambda^m T_p^* M$$

Changing the chart:  $(U', x')$  another one, denoting  $c = x' \circ x^{-1} \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial x_i} = \sum_j \left( \frac{\partial x'_j}{\partial x_i} \right) \frac{\partial}{\partial x'_j} \text{ and,}$$

dualising it  $\Rightarrow$

$$(dx'_i) = \sum_j \frac{\partial x'_i}{\partial x_j} (dx_j) \quad (*)$$

$f$  fct, chart  $(U, x)$

$$\frac{\partial f}{\partial x_i}(p) = df\left(\frac{\partial}{\partial x_i}\right) \dots \text{ is the usual } i\text{th derivative of } f_x = f \circ x^{-1}$$

$$= \frac{\partial f_x}{\partial x_i}(x(p))$$

$$f = x'_j$$

$$\frac{\partial x'_j}{\partial x_i}(p) = \frac{\partial c_{ij}}{\partial x_i}(x(p))$$

Standard

$$\int_\Omega : \int_{\mathbb{R}^n}$$

with basis  
(Lin)

(Char

$$\Omega \xrightarrow{c} \text{foc} \dots$$

Here:



(2) Volume forms / orientations

Def: A volume form on  $M$  is any  $\mu \in \Omega^m(M)$  nowhere vanishing  $\mu_p \neq 0 \forall p \in M$ .

$\mu_p: \underbrace{T_p M \times \dots \times T_p M}_m \rightarrow \mathbb{R}$   
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- Two volume forms  $\mu, \mu'$  called equivalent if  $\mu' = f\mu$  has  $f > 0$  everywhere.
- An orientation on  $M$  is an equivalence class of volume forms.  
If  $\mu = \text{vol. form}$  the induced orientation is denoted  $\mathcal{O}_\mu$ .

Rk: Recall that  $\wedge^m T_p^* M$  is 1-dimensional (if  $(U, x)$  - chart around  $p$   
 $\rightarrow (dx_i)_p \wedge \dots \wedge (dx_m)_p$  is a basis)

Hence  $\mu$ -volume form:  $\mu_p \in \wedge^m T_p^* M$  can serve as a basis element.

In particular:  $(\forall) \omega \in \Omega^m(M)$  can be written as  $\omega = f\mu$  for some  $f \in C^\infty(M)$ .

And this is still a volume form iff  $f(x) \neq 0 \forall x \in M$ .

Ex 1:  $M = \mathbb{R}^m$ :  $\mu_{\text{can}} = dx_1 \wedge \dots \wedge dx_m$ .

Ex 2:  $M = S^1$ :  $\mu = y dx - x dy$  (restricted to  $S^1$ )

To check:  $(\forall) p = (a, b) \in S^1$ ,  $\mu_p \neq 0$  ( $\mu_p: T_p S^1 \rightarrow \mathbb{R}$ )

Use  $V = y(\frac{\partial}{\partial x})_p - x(\frac{\partial}{\partial y})_p \in T_p S^1$ :  $\mu_p(V_p) = b^2 + a^2 = 1 \neq 0 \Rightarrow \mu_p \neq 0 \forall p \in S^1$

Ex 3:  $M = S^1$ ,  $\mu = dx$  not a volume form:  $\mu_p(V_p) = 0$  for  $p = (1, 0)$

Ex 4:  $M = S^2$ :  $\mu = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$  is a volume form.

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No global basis of  $T_p S^2$ . Instead we can use:

$V^1 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ ,  $V^2 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}$ , ...

$\mu_p(V^1, V^2) = x \cdot x + x y^2 + x z^2 = x \big|_p = a \neq 0$  whenever  $a \neq 0$ .

$\mu_p(V^1, V^3) = \dots$

$\mu_p(V^2, V^3) = \dots$

$\neq 0$  —, —  $b \neq 0$  } but  
 $\neq 0$  —, —  $c \neq 0$  } at least one happens

In particular, all  $S^n$  are orientable.  
But for instance  $\mathbb{P}^2$  is not orientable.



$$\Rightarrow \frac{\partial}{\partial x_i} = \sum_j \left( \frac{\partial x'_j}{\partial x_i} \right) \frac{\partial}{\partial x'_j} \text{ and,}$$

dualising it  $\Rightarrow$

$$(dx'_i) = \sum_j \frac{\partial x'_i}{\partial x_j} (dx_j) \quad (*)$$

$\frac{\partial f}{\partial x_i}(p) = df(\frac{\partial}{\partial x_i}) \dots$  is the usual  
ith derivative of  $f_x = f \circ x^{-1}$

$$f = x'_j$$

$$\frac{\partial x'_j}{\partial x_i}(p) = \frac{\partial c_j}{\partial x_i}(x(p))$$

$$\frac{\partial f}{\partial x_i} \rightarrow \frac{\partial f}{\partial x'_i}$$

$$\int_{\Omega'} f = \int_{\Omega} |Jac(c)| \cdot f \circ c \quad (\forall) f \in C_c^\infty(\Omega')$$

Here  $f \in C_c^\infty(\Omega')$  means that  $f$  has compact support i.e.  
 $\exists K \subseteq \Omega'$  cpt s.t.  $f=0$  on  $\Omega' \setminus K$

$Jac(c) : \Omega' \rightarrow \mathbb{R}$  is the fct given by  
 $x \mapsto \det \left( \frac{\partial c_i}{\partial x_j}(x) \right)_{1 \leq i, j \leq m}$

Focus now on arbitrary top-degree forms  $w \in \Omega_c^m(M)$  <sup>(4)</sup> the coeff. of  $w$  w.r.t.  $(u, x)$   
Locally: use chart  $(u, x)$ , over  $U$ : can write  $w|_U = F_x \cdot x(dx_1) \wedge \dots \wedge (dx_m)$   
 $x: U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^m$   
 $F \in C^\infty(\Omega)$

How does  $F_x$  change if  $\mathcal{I}$  change  $x$  (while keeping  $w$ )? Given  $(U', x')$ :  
(for  $p \in U$ :  $w_p = F_x(x(p)) (dx_1)_p \wedge \dots \wedge (dx_m)_p$ )

$$w_p = \left[ \frac{F_x(x(p))}{x} \right] (dx_1)_p \wedge \dots \wedge (dx_m)_p = \left[ F_{x'}(x'(p)) \right] (dx'_1)_p \wedge \dots \wedge (dx'_m)_p$$

$$\begin{aligned} \text{ThoRHS} &= F_{x'}(x'(p)) \left( \sum_{j_1} \frac{\partial x_{j_1}}{\partial x'_{j_1}} dx_{j_1} \right)_p \wedge \dots \wedge \left( \sum_{j_m} \frac{\partial x_{j_m}}{\partial x'_{j_m}} dx_{j_m} \right)_p \quad \text{use } (*) \\ &= F_{x'}(x'(p)) \sum_{j_1, \dots, j_m \in \{1, \dots, m\}} \frac{\partial x_{j_1}}{\partial x'_{j_1}}(p) \dots \frac{\partial x_{j_m}}{\partial x'_{j_m}}(p) (dx_{j_1})_p \wedge \dots \wedge (dx_{j_m})_p \\ &= F_{x'}(x'(p)) \sum_{\sigma \in S_m} \frac{\partial x_{j_1}}{\partial x'_{\sigma(1)}}(p) \dots \frac{\partial x_{j_m}}{\partial x'_{\sigma(m)}}(p) (dx_{\sigma(1)})_p \wedge \dots \wedge (dx_{\sigma(m)})_p \\ &= F_{x'}(x'(p)) \sum_{\sigma \in S_m} \text{sgn}(\sigma) \cdot \frac{\partial x_{j_1}}{\partial x'_{\sigma(1)}}(p) \dots \frac{\partial x_{j_m}}{\partial x'_{\sigma(m)}}(p) (dx_1)_p \wedge \dots \wedge (dx_m)_p \\ &= F_{x'}(x'(p)) \left( \sum_{\sigma \in S_m} \text{sgn}(\sigma) \frac{\partial c_1}{\partial x'_{\sigma(1)}}(x(p)) \dots \frac{\partial c_m}{\partial x'_{\sigma(m)}}(x(p)) \right) (dx_1)_p \wedge \dots \wedge (dx_m)_p \end{aligned}$$

$x'(x'(x)) = c(x)$

$$\Rightarrow F_{x'} = Jac(c) \cdot F_x \circ c$$

Problem: 1.1. Fix: use an orientation  $\theta$  on  $M$ . Fix  $\theta$ .  
Use now only positively oriented charts  $(u, x)$ !  
For two such  $x, x' \Rightarrow c := x' \circ x^{-1}$  has  $Jac(c) > 0$  (Exercise)

Theorem: On any oriented manifold  $(M, \theta)$   $\exists!$  a map  
 $\int_M : \Omega_c^m(M) \rightarrow \mathbb{R}$  ( $w$  s.t.  $\exists K \subseteq M$  cpt  $w_p = 0$  for  $p \in M \setminus K$ )

such that: (1)  $\int_M$  is linear:  $\int_M (w_1 + w_2) = \int_M w_1 + \int_M w_2$  etc.

(2) for any positively oriented chart  $x: U \rightarrow \Omega \subseteq \mathbb{R}^m$   
and any  $w \in \Omega_c^m(M)$  that is supported in  $U$  then  
 $\int_M w = \int_U F_x \cdot x(dx_1) \wedge \dots \wedge (dx_m)$   
is the coefficient in the writing  
 $w|_U = F_x \cdot x(dx_1) \wedge \dots \wedge (dx_m)$



①

Standard integration on opens  $\Omega \subseteq \mathbb{R}^m$  (simple version!)

$$\int_{\Omega} : C_c^{\infty}(\Omega) \rightarrow \mathbb{R}, \quad f \mapsto \int_{\Omega} f = \int_{\Omega} f(x) dx = \int_{\Omega} f(x_1, \dots, x_m) dx_1 \dots dx_m$$

with basic properties:

$$(\text{Lin}) \quad \int_{\Omega} (f+g) = \int_{\Omega} f + \int_{\Omega} g, \quad \int_{\Omega} (\lambda f) = \lambda \int_{\Omega} f \quad (\forall f, g \in C_c^{\infty}(\Omega), \lambda \in \mathbb{R})$$

(change of variables) if  $c: \Omega \rightarrow \Omega'$  is a diffeo, then:

$$\int_{\Omega'} f = \int_{\Omega} |Jac(c)| \cdot f \circ c \quad (\forall f \in C_c^{\infty}(\Omega'))$$

$$\begin{array}{ccc} \Omega & \xrightarrow{c} & \Omega' \\ f \circ c & \searrow & \downarrow f \\ & & \mathbb{R} \end{array}$$

usual

$$f_x = f \circ x^{-1}$$

Here  $f \in C_c^{\infty}(\Omega)$  means that  $f$  has compact support i.e.  
 $\exists K \subseteq \Omega$  cpt s.t.  $f = 0$  on  $\Omega \setminus K$ .

•  $Jac(c): \Omega \rightarrow \mathbb{R}$  is the fct given by

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If  $\mu = \text{vol. form}$  the induced orientation is denoted  $\hat{\mu}$ .

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 $\Rightarrow (dx_1)_p \wedge \dots \wedge (dx_m)_p$  is a basis)

Hence  $\mu$ -volume form:  $\mu_p \in \wedge^m T_p^* M$  can serve as a basis element.

In particular:  $(\forall) \omega \in \mathcal{O}^m(M)$  can be written as  $\omega = f \cdot \mu$  for some  $f \in C^\infty(M)$ .  
 And this is still a volume form iff  $f(x) \neq 0 (\forall) x \in M$ .

dualising it  $\Rightarrow$

$$(dx_i^j) = \sum_j \frac{\partial x_i^j}{\partial x_j} (dx_j) \quad (*)$$

$$f = x_j^i$$

$$\frac{\partial x_j^i}{\partial x_i}(p) = \frac{\partial C_j^i}{\partial x_i}(x(p))$$

Proof: next time, but key idea: show that any  $\omega \in \mathcal{O}_c^m(M)$  can be written as  $\omega = \omega_1 + \dots + \omega_k$  where  $\omega_k \in \mathcal{O}_c^m(M)$  are supported in domains charts  $(U_k, x_k)$ . Issue: the resulting  $\int_M \omega$  does not depend on the choices.

Rk 1: If we fix a volume form  $\mu \Rightarrow$  we can integrate functions:  $\int_M : C_c^0(M) \rightarrow \mathbb{R}, \int_M f := \int_M f \mu$

Rk 2: Given  $\mu$  on compact  $M$  we can talk about the volume of  $M$  (w.r.t.  $\mu$ ):

$$\text{Vol}_\mu(M) := \int_M 1 = \int_M \mu > 0$$

Exercise!

the one using  $\hat{\mu}$ .

Theorem (Stokes):  $(M, \theta)$  oriented manifold. If  $\omega \in \mathcal{O}_c^m(M)$  is of type  $\omega = d\theta$  with  $\theta \in \mathcal{O}^{m-1}(M)$  then  $\int_M \omega = 0$ .

$\theta = d\theta$  & break  $\theta = \theta_1 + \dots + \theta_k$  as before  $\Rightarrow$  reduce everything to  $M = \mathbb{R}^m$ .

$$\theta = \sum f_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m \dots \text{ just assume } \theta = f dx_2 \wedge \dots \wedge dx_m$$

$\Rightarrow \omega = \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2 \wedge \dots \wedge dx_m \Rightarrow$  deal with  $\int_{-\infty}^{\infty} \frac{\partial g}{\partial x_1}$  for  $g$ -compact supp. fcts in  $\mathbb{R}$ :

$$\int_A^B \frac{\partial g}{\partial x_1} = g \Big|_A^B = g(B) - g(A) \Rightarrow$$

$$[a, b] \subseteq \mathbb{R} \\ g \text{ is 0 on } \mathbb{R} \setminus (a, b) \\ \int_a^b \frac{\partial g}{\partial x_1} = 0$$

Corollary: If  $M = \text{cpt. manifold}$   
 $\Rightarrow$  any volume form  $\mu \in \mathcal{O}^m(M)$  cannot be exact, i.e. cannot be written as  $d\theta$  for some  $\theta \in \mathcal{O}^{m-1}(M)$ .

Applic of this when  $M = S^{2k}$ : the hairy ball theorem (to be explained in the next lecture).



Volume forms/orientations

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 $m = \dim M$   
 $\mu \neq 0 \forall p \in M$

- All  $M$  orientable if such an  $\mu$  exists.
- Given  $\mu$ , call a chart  $(U, \chi)$  positively oriented (wrt  $\mu$ ) if  $\mu(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}) > 0$
- Two volume forms  $\mu, \mu'$  called equivalent if  $\mu' = f\mu$  for  $f > 0$  everywhere
- An orientation on  $M$  is an equivalence class of volume forms

If  $\mu$  is vol form then the induced orientation is denoted  $\mathcal{O}_\mu$

Rk: Recall that  $\wedge^m T_p^*M$  is 1-dimensional (if  $(U, \chi)$  - chart around  $p \rightarrow (dx_1)_p, \dots, (dx_m)_p$  is a basis)  
 Hence  $\mu$ -volume form:  $\mu_p \in \wedge^m T_p^*M$  can serve as a basis element.  
 In particular:  $(\forall) w \in \Omega_c^m(M)$  can be written as  $w = f\mu$  for some  $f \in C_c^\infty(M)$   
 And this is still a volume form iff  $f(x) \neq 0 \forall x \in M$

Proof: next time, but key idea: show that any  $w \in \Omega_c^m(M)$  can be written as  $w = w_1 \mu + \dots + w_k \mu$  where  $w_i \in \Omega_c^m(M)$  are supported in domains  $(U_i, \chi_i)$ . Issue: the resulting  $\int w$  does not depend on the choices.

Rk 1: If we fix a volume form  $\mu \Rightarrow$  we can integrate functions:  $\int_M : C_c^0(M) \rightarrow \mathbb{R}, \int_M f := \int_M f\mu$

Rk 2: Given  $\mu$  we can talk about the volume of  $M$  (wrt  $\mu$ ).  
 $\text{Vol}_\mu(M) := \int_M 1 = \int_M \mu > 0$   
 Exercise!

Reminder:  $(U, \chi)$  chart at  $p \in M \Rightarrow$  various induced bases:

- $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p$  of  $T_p M$
- $(dx_1)_p, \dots, (dx_m)_p$  of  $T_p^* M$
- $(dx_1)_p \wedge \dots \wedge (dx_m)_p$  of  $\wedge^m T_p^* M$  (i.e.  $\mathbb{C} \otimes \mathbb{R} = \mathbb{C}$ )
- $\dots$
- $(dx_1)_p, \dots, (dx_m)_p$  of  $\wedge^m T_p^* M$

Changing the chart  $(U', \chi')$  another one, denoting  $c = \chi' \circ \chi^{-1}$   
 $\Rightarrow \frac{\partial}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j}$  and  
 $(dx'_1)_p = \sum_j \frac{\partial x_j}{\partial x'_1} (dx_j)_p$  (\*)  
 dualising it  $\Rightarrow$   
 $f \circ c, \text{ chart } (U, \chi)$   
 $\frac{\partial f}{\partial x'_i} = df(\frac{\partial}{\partial x'_i}) \dots$  i.e. the usual  
 $\Rightarrow \frac{\partial f}{\partial x'_i} = \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x'_i}$  i.e. derivative of  $f \circ c$   
 $f = x'_i$   
 $\frac{\partial x'_i}{\partial x_j}(p) = \frac{\partial c_i}{\partial x_j}(x(p))$

Standard integration on opens  $\mathcal{O} \subseteq \mathbb{R}^m$  (simple version!)  
 $\int_{\mathcal{O}} : C_c^\infty(\mathcal{O}) \rightarrow \mathbb{R}, f \mapsto \int_{\mathcal{O}} f = \int_{\mathcal{O}} f(x) dx = \int_{\mathcal{O}} f(x_1, \dots, x_m) dx_1 \dots dx_m$

with basic properties:  
 (Lin)  $\int_{\mathcal{O}} (f+g) = \int_{\mathcal{O}} f + \int_{\mathcal{O}} g, \int_{\mathcal{O}} (\lambda f) = \lambda \int_{\mathcal{O}} f \quad \forall f, g \in C_c^\infty(\mathcal{O}), \lambda \in \mathbb{R}$   
 (change of variables) if  $c: \mathcal{O} \rightarrow \mathcal{O}'$  is a diffeo, then:  
 $\int_{\mathcal{O}'} f = \int_{\mathcal{O}} |Jac(c)| \cdot f \circ c \quad (\forall) f \in C_c^\infty(\mathcal{O}')$

Here:  $f \in C_c^\infty(\mathcal{O})$  means that  $f$  has compact support i.e.  $\exists K \subseteq M$  cpt st.  $f = 0$  on  $M \setminus K$   
 $\bullet Jac(c): \mathcal{O} \rightarrow \mathbb{R}$  is the fct given by  
 $x \mapsto \det(\frac{\partial c_i}{\partial x_j}(x))_{1 \leq i, j \leq m}$

Theorem (Stokes):  $(M, \mathcal{O})$  oriented manifold. If  $w \in \Omega_c^m(M)$  is of type  $w = d\theta$  with  $\theta \in \Omega_c^{m-1}(M)$  then  $\int_M w = 0$ .

$\theta = d\theta$  & break  $\theta = \theta_1 + \dots + \theta_k$  as before  $\Rightarrow$  reduce everything to  $M = \mathbb{R}^m$   
 $\theta = \sum f_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$  ... just assume  $\theta = f dx_1 \wedge \dots \wedge dx_m$   
 $\Rightarrow w = \frac{\partial f}{\partial x_1} dx_1 \wedge \dots \wedge dx_m \Rightarrow$  deal with  $\int \frac{\partial f}{\partial x_1}$  for  $f$  - compact supp. fct in  $\mathbb{R}^m$ :  
 $\int_A \frac{\partial f}{\partial x_1} = g \Big|_A^B = g(B) - g(A) = 0$   
 $[a, b] \subseteq \mathbb{R}$   
 $g$  is 0 on  $\mathbb{R} \setminus [a, b]$   
 $\int_A \frac{\partial f}{\partial x_1} = \int_a^b \frac{\partial f}{\partial x_1} = f(b) - f(a) = 0$

Corollary: If  $M = \text{cpt. manifold}$   
 $\Rightarrow$  any volume form  $\mu \in \Omega_c^m(M)$  cannot be exact, i.e. cannot be written as  $d\theta$  for some  $\theta \in \Omega_c^{m-1}(M)$

Applic of this when  $M = S^1$ : the hairy ball theorem (to be explained in the next lecture)

Problem 1: Fix: use an orientation  $\mathcal{O}$  on  $M$ . Fix  $\mathcal{O}$ .  
 Use now only positively oriented charts  $(U, \chi)$ !  
 For two such  $\chi, \chi' \Rightarrow c = \chi' \circ \chi^{-1}$  has  $Jac(c) > 0$  (Exercise)

Theorem: On any oriented manifold  $(M, \mathcal{O})$   $\exists!$  a map  $\int_M : \Omega_c^m(M) \rightarrow \mathbb{R}$  (w s.t.  $\exists K \subseteq M$  cpt  $\mu_p = 0 \forall p \in M \setminus K$ )  
 such that: (1)  $\int_M$  is linear:  $\int_M (w_1 + w_2) = \int_M w_1 + \int_M w_2$  etc.  
 (2) for any positively oriented chart  $\chi: U \rightarrow \mathbb{R}^m$  and any  $w \in \Omega_c^m(M)$  that is supported in  $U$  then  $\int_M w =$  the usual  $\int_{\mathbb{R}^m} F_\chi^w$  where  $F_\chi^w$  is the coefficient in the writing  $w|_U = F_\chi^w \chi^* dx_1 \wedge \dots \wedge dx_m$

$F_\chi^w = Jac(\chi) \cdot f_\chi$   
 $f_\chi = \frac{1}{Jac(\chi)} \int_M w$   
 $F_\chi^w = Jac(c) \cdot F_{\chi'}^w$