

• Usual integration: for $\Omega \in \mathbb{R}^m$ open

$$\int_{\Omega} : C_c^\infty(\Omega) \rightarrow \mathbb{R}$$

linear & satisfying CVF: (1) diffeo $c: \Omega \xrightarrow{\sim} \Omega'$

$$\int_{\Omega'} f = \int_{\Omega} f \circ c \cdot |Jac(c)|$$

Rk: Given (M, θ)
 (1) positively oriented
 charts $(U, x), (U', x')$
 one has: $c = x' \circ x^{-1}$
 $\Rightarrow Jac(c) > 0$
 or: $|Jac(c)| = Jac(c)$

• Top-differential forms: $\omega \in \Omega_c^m(M)$ where $m = \dim M$

$\rightarrow$ locally, for chart $x: U \xrightarrow{\sim} \Omega \Rightarrow$ a function $F_x^\omega \in C_c^\infty(\Omega)$ from

$$\omega|_U = F_x^\omega \cdot dx_1 \wedge \dots \wedge dx_m \quad (\text{or } x'_* \omega|_U = F_{x'}^\omega dx_1 \wedge \dots \wedge dx_m)$$

$\rightarrow$ If x' is another chart $\Rightarrow F_{x'}^\omega = F_x^\omega \cdot Jac(c)$ where $c = x' \circ x^{-1}$

• Volume form on M : any $\mu \in \Omega_c^m(M)$ such that $\mu_p \neq 0 \quad \forall p \in M$

Two of them, μ_1 and μ_2 called ~ if $\mu_2 = f \mu_1$ with $f > 0$

• Orientation on M : an equivalence class w.r.t. $\sim$; for μ vol. form $\Rightarrow$ orient $\mathcal{O}_\mu$

Given $\mathcal{O}_\mu \Rightarrow$ can talk about oriented charts: (U, x) st $\mu(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}) > 0$ over U

$\Rightarrow$ on an oriented (M, θ) , we can define $\int_M \omega$ whenever $\omega \in \Omega_c^m(M)$ "is small"
 i.e. $\text{Supp}(\omega) \subseteq U$ for some coordinate chart

Explicitly $\leftarrow$ choose (U, x) oriented
 consider the resulting $F_x^\omega \in C_c^\infty(\Omega)$ | set $\int_M \omega := \int_{\Omega} F_x^\omega \quad (*)$

By the CVF $\Rightarrow$ this does not depend on the choices

Theorem 1: On any oriented (M, θ) , $\exists!$ $\int : \Omega_c^m(M) \rightarrow \mathbb{R}$
 linear which, when ω is small is given by $(*)$

Theorem (Stokes): (M, θ) oriented, $\theta \in \Omega_c^{m-1}(M) \Rightarrow \int_M d\theta = 0$

Applic: M compact, μ = volume form $\Rightarrow$ can talk about

$$\text{Vol}_\mu(M) := \int_M \mu \quad (\text{using } \mathcal{O}_\mu) \dots > 0!$$

Corollary: $M = \text{cpt} \Rightarrow$ any volume form μ on M cannot be exact

Proof Thm 1:

Key idea: For $\omega \in \Omega_c^m(M)$ prove that $\exists \omega_1, \dots, \omega_k \in \Omega_c^m(M)$ small st $\omega = \omega_1 + \dots + \omega_k$

This would imply $\int_M \omega = \int_M \omega_1 + \dots + \int_M \omega_k \Rightarrow$ get uniqueness in Thm 1

Integration: for $\Omega \in \mathbb{R}^m$ open
 $\int_{\Omega} : C_c(\Omega) \rightarrow \mathbb{R}$
 linear & satisfying C.V.F. (1)

Top differential forms: $\omega \in \Omega_c^m(M)$ where $m = \dim M$
 $\rightarrow$ locally, for chart $x: U \xrightarrow{\sim} \Omega \xrightarrow{\sim} \Omega'$
 $\omega|_U = F_x^* \omega'$ where $F_x \in C_c^\infty(\Omega)$ from

Rk: Given (M, θ)
 positively oriented
 charts $(U, x), (U', x')$
 one has: $c = x' \circ x^{-1}$
 $\Rightarrow \text{Jac}(c) > 0$
 or: $|\text{Jac}(c)| = \text{Jac}(c)$

$\Rightarrow$ on an oriented (M, θ) , we can define $\int_M \omega$ whenever $\omega \in \Omega_c^m(M)$ is small
 i.e. $\text{supp}(\omega) \subseteq \text{coord}$

Volume form on M : any $\mu \in \Omega_c^m(M)$ such that $\mu_p \neq 0$ for $p \in M$.
 Two of them, μ_1 and μ_2 called equivalent if $\mu_2 = f \mu_1$ with $f > 0$.
 Orientation on M : an equivalence class w.r.t. $\sim$; for μ vol. form $\Rightarrow$ orient $\mathcal{O}_M$

Explicitly: choose (U, x) oriented
 By the C.V.F. $\Rightarrow$ this does not depend on the choices.
 Theorem 1: On any oriented (M, θ) , $\exists!$ $\int_M : \Omega_c^m(M) \rightarrow \mathbb{R}$
 linear which, when $\omega = \text{small}$ is given by (x) .
 Theorem (Stokes): $(M, \theta) = \text{oriented}$, $\theta \in \Omega_c^{m-1}(M) \Rightarrow \int_M d\theta = 0$

Applic: M compact, $\mu = \text{volume form} \Rightarrow$ can talk about $\int_M \mu$
 Corollary: $M = \text{cpt} \Rightarrow$ any volume form μ on M cannot be zero!
 (using $\mathcal{O}_M$)

Proof Thm 1:

Key idea: For $\omega \in \Omega_c^m(M)$ prove that $\exists w_1, \dots, w_k \in \Omega_c^m(M)$ small st $\omega = w_1 + \dots + w_k$
 This would imply $\int \omega = \int w_1 + \dots + \int w_k \Rightarrow$ get uniqueness in Thm 1.
 & would also tell us how to define $\int \omega$ though we would still have to prove that $\int_M \omega := \int w_1 + \dots + \int w_k$ does not depend on writing $\omega = w_1 + \dots + w_k$

proof: Assume
 Choose

②
 $\in U$ for some
 coordinate chart.
 $\omega = \int_{\partial} F_x^w$ (*)

Thm: $M = \text{manifold}$. Then:

(V) open cover $\{U_1, \dots, U_k\}$ of M
 (E) partition of unity $\{\eta_1, \dots, \eta_k\}$ subordinated to the cover
 i.e. $\eta_i : M \rightarrow [0, 1]$ smooth such that:

(1) $\eta_1 + \dots + \eta_k = 1$ (the oth. fct. 1)
 (2) $\text{supp } \eta_i \subseteq U_i$

(E) $A_i \subseteq U_i$ closed s.t.
 $\eta_i = 0$ on $M \setminus A_i$
 $\Downarrow$
 $\eta_i = 0$ on $M \setminus U_i$

See Inleiding Topologie: $\exists A$ -partitions of unity for $A \subseteq C(M)$
 Satisfying some axioms. Take $A = C^\infty(M)$. To be checked:

(V) $U \subseteq M$ open
 (V) $p \in U$

$\exists f \in C^\infty(M)$ s.t. $f(p) = 1, f = 0$ outside U .

Local! $M = \mathbb{R}^n, p = 0, U = B_1(0) = \{x \in \mathbb{R}^n : \|x\| < 1\}$... write down an explicit f .

be exact.

Now, for $\omega \in \mathcal{D}_c^m(M) \Rightarrow$

(5)

now just choose

Thm: $M = \text{manifold}$: Then:

(4)

(A) open cover $\{U_1, \dots, U_k\}$ of M

(B) partition of unity $\{\eta_1, \dots, \eta_k\}$ subordinated to the cover

i.e. $\eta_i: M \rightarrow [0, 1]$ smooth such that:

(1) $\eta_1 + \dots + \eta_k = 1$ (the const. fct. 1)

(2) $\text{supp } \eta_i \subseteq U_i$

($\exists A_i \subseteq U_i$ closed s.t.
 $\eta_i = 0$ on $M \setminus A_i$)

($\eta_i = 0$ on $M \setminus U_i$)

See Including Topologie: $\exists A$ -partitions of unity for $A \subseteq C(M)$.
Satisfying some axioms. Take $A = C^\infty(M)$. To be checked:

(A) $U \subseteq M$ open $\} \exists f \in C^\infty(M)$ s.t. $f(p) = 1$, $f = 0$ outside U .

(B) $p \in U$

Local! $M = \mathbb{R}^n$, $p = 0$, $U = B_1(0) = \{x \in \mathbb{R}^n : \|x\| < 1\}$ write down an explicit f .

Now, for $\omega \in \mathcal{D}_c^m(M) \Rightarrow$

(5)

$\Rightarrow \omega = 1 \cdot \omega = \eta_1 \cdot \omega + \dots + \eta_k \cdot \omega$... now just choose the cover axisels

Fix $\omega \in \mathcal{D}_c^m(M)$, $K \subseteq M$ cpt s.t. $\omega_p = 0 \Leftrightarrow p \in M \setminus K$

Choose a finite collection $\{U_1, \dots, U_k\}$ with $U_i = \text{domain of a chart}$ and such that $K \subseteq U_1 \cup \dots \cup U_k$. (Ok since $K = \text{cpt}$)

Apply the thm to $\{U_0 = M \setminus K, U_1, \dots, U_k\} \Rightarrow$

$\Rightarrow$ finite partition $\{\eta_0, \eta_1, \dots, \eta_k\}$ with $\eta_0 + \eta_1 + \dots + \eta_k = 1$

Now

$$\omega = \underbrace{\eta_0 \cdot \omega}_0 + \underbrace{\eta_1 \cdot \omega}_{\text{supp in } U_1} + \dots + \underbrace{\eta_k \cdot \omega}_{\text{supp in } U_k}$$

$$\omega = \omega_1 + \dots + \omega_k$$

↑ ↑
small!

• Usual integration: for $R \subseteq \mathbb{R}^m$ open

$$\int_R: C_c^0(R) \rightarrow \mathbb{R}$$

linear & satisfying CVF: (i) diffeo $c: R \xrightarrow{\sim} R'$

$$\int_{R'} f = \int_R f \circ c \cdot |Jc(c)|$$

Ric: Given (M, θ)
(i) positively oriented
charts $(U, x), (U', x')$
one has: $c = x' \circ x^{-1}$
has $Jc(c) > 0$.
or: $|Jc(c)| = Jc(c)$.

• Top-differential forms: $\omega \in \Omega_c^m(M)$ where $m = \dim M$
→ locally, for chart $x: U \xrightarrow{\sim} R \Rightarrow$ a function $F_x \in C_c^\infty(R)$ from
 $\omega|_U = F_x \cdot x^* (dx_1 \wedge \dots \wedge dx_m)$ (or $x'_* \omega|_U = F_x \cdot dx_1 \wedge \dots \wedge dx_m$)
→ If x' is another chart $\Rightarrow F_{x'} = F_x \cdot Jc(c)$ where $c = x' \circ x^{-1}$.

• Volume form on M : any $\mu \in \Omega_c^m(M)$ such that $\mu_p \neq 0 \forall p \in M$.

Two of them, μ_1 and μ_2 called equiv if $\mu_2 = f \mu_1$ with $f > 0$.

• Orientation on M : an equivalence class w.r.t. $\sim$; for μ vol form $\Rightarrow$ orient $\mathcal{O}_M$.

Given $\mathcal{O} \Rightarrow$ can talk about oriented charts: (U, x) st $\mu(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}) > 0$ over U .

$\Rightarrow$ on an oriented (M, θ) , we can define
 $\int_M \omega$ whenever $\omega \in \Omega_c^m(M)$ is "small"
i.e. $\text{supp}(\omega) \subseteq U$ for some coordinate chart

Explicitly: choose (U, x) oriented
consider the resulting $F_x \in C_c^\infty(R)$ | set $\int_M \omega := \int_R F_x$ (*)

By the CVF $\Rightarrow$ this does not depend on the choices.

Theorem 1: On any oriented (M, θ) , $\exists! \int: \Omega_c^m(M) \rightarrow \mathbb{R}$
linear which, when ω is small is given by (*).

Theorem (Stokes) $(M, \theta) = \text{oriented}$, $\theta \in \Omega_c^{m-1}(M) \Rightarrow \int_M d\theta = 0$

Applic: M compact, $\mu = \text{volume form} \Rightarrow$ can talk about
 $\text{vol}_\mu(M) := \int_M \mu$ (using $\mathcal{O}_M$) ... > 0 !

Corollary: $M = \text{cpt} \Rightarrow$ any volume form μ on M cannot be exact.

Thm: $M = \text{manifold}$. Then:

(i) open cover $\{U_1, \dots, U_k\}$ of M

(ii) partition of unity $\{\eta_1, \dots, \eta_k\}$ subordinated to the cover
i.e. $\eta_i: M \rightarrow [0, 1]$ smooth such that:

- (1) $\eta_1 + \dots + \eta_k = 1$ (the cst. fct. 1)
- (2) $\text{supp } \eta_i \subseteq U_i$ ($\exists A_i \subseteq U_i$ closed s.t. $\eta_i = 0$ on $M \setminus A_i$)

See Including Topology: $\exists A$ -partitions of unity for $d \in C(M)$
Satisfying some axioms. Take $d = C_c^\infty(M)$. To be checked:

(i) $U \subseteq M$ open $\Rightarrow \exists f \in C_c^\infty(M)$ s.t. $f|_U = 1$, $f = 0$ outside U .

(ii) $p \in U$

Local! $M = \mathbb{R}^n$, $p = 0$, $U = B_1(0) = \{x \in \mathbb{R}^n : \|x\| < 1\}$, ... write down an explicit f .

Proof Thm 1:

Key idea: For $\omega \in \Omega_c^m(M)$ prove that $\exists \omega_1, \dots, \omega_k \in \Omega_c^m(M)$ small st $\omega = \omega_1 + \dots + \omega_k$.

This would imply $\int_M \omega = \int_M \omega_1 + \dots + \int_M \omega_k \Rightarrow$ get uniqueness in Thm 1.

& would also tell us how to define $\int_M \omega$ though we would still

have to prove that $\int_M \omega := \int_M \omega_1 + \dots + \int_M \omega_k$ does not depend on writing $\omega = \omega_1 + \dots + \omega_k$ small

proof: Assume $\omega = \omega_1 + \dots + \omega_k$

Choose $\{\eta_1, \dots, \eta_k\}$ a part of 1 as before

Look at:

$$\begin{aligned} \int_M \omega_1 + \dots + \int_M \omega_k &= \int_M (\sum \eta_i) \omega_1 + \dots + \int_M (\sum \eta_i) \omega_k \\ &= \sum_{i \in \{1, \dots, k\}} \int_M \eta_i \omega_i = \sum_i \int_M \eta_i \omega = \sum_{i=1}^k \int_M \eta_i \omega \end{aligned}$$

& this does not depend on the writing $\omega = \sum_{j=1}^k \omega_j$

Now, for $\omega \in \Omega_c^m(M) \Rightarrow$

$\Rightarrow \omega = \sum \omega_i = \eta_1 \omega + \dots + \eta_k \omega$... now just choose the cover smaller

Fix $\omega \in \Omega_c^m(M)$, $K \subseteq M$ cpt s.t. $\omega_p = 0 \forall p \in M \setminus K$
Choose a finite collection $\{U_1, \dots, U_k\}$ with $U_i = \text{domain of a chart}$
and such that $K \subseteq U_1 \cup \dots \cup U_k$. (Ok since $K = \text{cpt}$)

Apply the thm to $\{U_0 = M \setminus K, U_1, \dots, U_k\} \Rightarrow$

$\Rightarrow$ finite partition $\{\eta_0, \eta_1, \dots, \eta_k\}$ with $\eta_0 + \eta_1 + \dots + \eta_k = 1$

Now

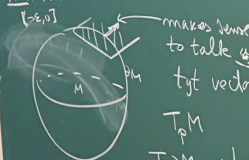
$$\begin{aligned} \omega &= \underbrace{\eta_0 \omega}_0 + \underbrace{\eta_1 \omega}_{\text{supp in } U_1} + \dots + \underbrace{\eta_k \omega}_{\text{supp in } U_k} \\ &= \omega_1 + \dots + \omega_k \\ &\quad \uparrow \quad \quad \uparrow \\ &\quad \text{small!} \end{aligned}$$

Usual integration for $\mathbb{R}^m$ open
 $\int_U f(x) dx = \int_U f(x) dx$
 linear & satisfying CVF: $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^m$
 $\int_U f(x) dx = \int_{\varphi(U)} f(\varphi(x)) |\det D\varphi(x)| dx$

Rk: Given (M, θ)
 (w) positively oriented
 charts $(U, \chi), (U', \chi')$
 one has: $\chi' = \chi \circ \varphi$
 has $\det D\varphi > 0$
 or: $|\det D\chi| = |\det D\chi'|$

Top differential forms $w \in \Omega^m(M)$ where $m = \dim M$
 $\rightarrow$ locally, for chart $\chi: U \rightarrow \mathbb{R}^m \Rightarrow$ a function $F_\chi \in C_c^\infty(\mathbb{R}^m)$ from
 $w|_U = F_\chi \cdot \chi^* (dx_1 \wedge \dots \wedge dx_m)$ (or $\chi^* w|_U = F_\chi dx_1 \wedge \dots \wedge dx_m$)
 $\rightarrow$ If χ' is another chart $\Rightarrow F_{\chi'} = F_\chi \cdot \det D\chi$ where $\chi' = \chi \circ \varphi$

Volume form on M : any $\mu \in \Omega^m(M)$ such that $\mu_p \neq 0 \forall p \in M$
 $\Rightarrow$ the concept of manifold with boundary
 Ex: $\mathbb{D}^m$
 $\partial \mathbb{D}^m = \text{an } (m-1)\text{-manifold}$
 $\Rightarrow$ one makes sense to talk about ∂M
 in charts of M , they go to pts $\in \mathbb{H}^m$
 with $x_m = 0$

Rk: An orientation θ on M induces an orientation on ∂M
 (call it $\partial \theta$)

 makes sense to talk about outward pointing
 tangent vectors at $p \in \partial M$
 $T_p M$
 $T_p \partial M$ a basis $v_1, \dots, v_{m-1}$ is declared oriented
 if $v_1, \dots, v_{m-1}, \nu$ is an oriented basis of $T_p M$
 wrt θ

Thm (Stokes on M with ∂):
 $M = \text{compact, oriented}$
 $\Rightarrow \exists \eta \in \Omega^{m-1}(M)$ one has $\int_M d\eta = \int_{\partial M} \eta$
 along outward

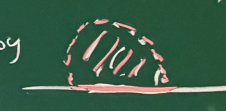
E.g.: assume we are interested
 in $\int_M w$ $\Rightarrow M = \text{usual manifold}$
 $\partial M = \partial N$ and $w = \eta|_M$ for some $\eta \in \Omega^{m-1}(N)$ with $d\eta = 0 \Rightarrow \int_M w = 0$

Explicitly choose (U, χ) oriented
 consider the resulting $F_\chi \in C_c^\infty(\mathbb{R}^m)$
 By the CVF $\Rightarrow$ this does not depend on the choices

Theorem: On any oriented (M, θ) , $\exists!$ $\int: \Omega_c^m(M) \rightarrow \mathbb{R}$
 linear which, when $w = \text{small}$ is given by $(*)$
 Theorem (Stokes) $(M, \theta) = \text{oriented}$, $\theta \in \Omega_c^{m-1}(M) \Rightarrow \int_M d\theta = 0$
 Apply: $M = \text{compact}$, $M = \text{volume}$

General remarks about manifolds $\cong$ integration
 ① the notion of manifold smooth structure: uses
 smooth atlases $\mathcal{A}$: made of topological charts
 that are smoothly compatible
 "Charts": compact opens in M with opens in $\mathbb{R}^m$
 makes use of
 usual smoothness
 in $\mathbb{R}^m$

Variations: ① replace smooth by analytic $\Rightarrow$ analytic structure
 $\cong$ analytic manifolds
 ② $m = 2k$, $\mathbb{R}^m \subseteq \mathbb{C}^k$ and replace smooth
 by holomorphic $\Rightarrow$ complex manifold
 ③ Instead of "opens in $\mathbb{R}^m$ " allow for "opens in $\mathbb{H}^m$ "
 $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$

Note: any open in $\mathbb{R}^m$ is $\cong$ one in $\mathbb{H}^m$
 By using $\mathbb{H}^m$ I allow for more
 general charts: $\chi: U \rightarrow \Omega \leftarrow \mathbb{H}^m$
 Ω open
 $\mathbb{H}^m$
 could be 
 $(\mathbb{R} \cong (0, \infty))$
 $\mathbb{R}^m \cong \text{int}(\mathbb{H}^m)$

See Including
 Satisfying so
 (v) $U \subseteq M$

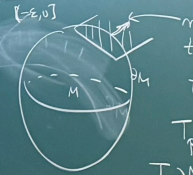
• Usual integration: for $\Omega \subseteq \mathbb{R}^m$ open
 $\int_{\Omega} f: C_c^\infty(\Omega) \rightarrow \mathbb{R}$
 linear & satisfying CVF: (1) diff $c: \Omega \xrightarrow{\sim} \Omega'$
 $\int_{\Omega'} f = \int_{\Omega} f \circ c \cdot |Jac(c)|$

Rk: Given (M, θ)
 (1) positively oriented charts $(U, x), (U', x')$
 one has: $c = x' \circ x^{-1}$
 $\Rightarrow Jac(c) > 0$
 or: $|Jac(c)| = Jac(c)$

$\Rightarrow$ on an oriented manifold M
 $\int_M \omega$ whenever $\omega \in \Omega^m(M)$
 i.e. $\text{Supp}(\omega) \subseteq U$ for some coordinate chart
 Explicitly: choose (U, x) oriented
 consider the resulting $F_x^{\omega} \in C_c^\infty(\Omega)$ | set $\int_M \omega := \int_{\Omega} F_x^{\omega}$ (*)

$\Rightarrow$ the concept of manifold with boundary

Ex (1/1) $M^n \Rightarrow$ one makes sense to talk about ∂M
 (in charts of M , they go to pts $\in \mathbb{H}^m$ with $x_m = 0$)
 $\partial M = \text{an } (m-1)\text{-manifold}$

Rk: An orientation θ on M induces an orientation on ∂M (call it $\partial \theta$)

 makes sense to talk about outward pointing tangent vectors at $p \in \partial M$
 $T_p M$
 $T_p \partial M$ a basis $v_1, \dots, v_{m-1}$ is deduced oriented
 if $v_1, \dots, v_{m-1}, \nu$ is an oriented basis of $T_p M$ wrt θ
 ν along outward

Thm (Stokes on M with ∂):
 $M = \text{compact, oriented}$
 $\Rightarrow \forall \eta \in \Omega^{m-1}(M)$ one has $\int_M d\eta = \int_{\partial M} \eta$
 E.g.: assume we are interested in $\int_M \omega$ $M = \text{usual manifold}$
 if $M = \partial N$ and $\omega = \eta|_M$ for some $\eta \in \Omega^m(N)$ with $d\eta = 0 \Rightarrow \int_M \omega = 0$

General remarks about manifolds & integration (8)
 (1) the notion of manifold smooth structure: uses smooth atlases $\mathcal{A}$: made of topological charts that are smoothly compatible

"Charts": compo opens in M with opens in $\mathbb{R}^m$ makes use of usual smoothness in $\mathbb{R}^m$
Variations: (1) replace smooth by analytic $\Rightarrow$ analytic structure & analytic manifolds
 (2) $m=2k, \mathbb{R}^m \cong \mathbb{C}^k$ and replace smooth by holomorphic $\Rightarrow$ complex manifold
 (3) Instead of "opens in $\mathbb{R}^m$ " allow for "opens in $\mathbb{H}^m$ "
 $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$

Note: any open in $\mathbb{R}^m$ is $\cong$ one in $\mathbb{H}^m$
 By using $\mathbb{H}^m$ I allow for more general charts: $\chi: U \rightarrow \Omega \subseteq \mathbb{H}^m$
 (1) open in $\mathbb{H}^m$
 could be 
 $(\mathbb{R} \cong (0, \infty))$
 $\mathbb{R}^m \cong \text{int}(\mathbb{H}^m)$

i.e. η
 In top
 Thm:
 Recall
 inject
 For

Thm: $M = \text{manifold}$: Then:

(4)

(A) open cover $\{U_1, \dots, U_k\}$ of M

(B) partition of unity $\{\eta_1, \dots, \eta_k\}$ subordinated to the cover

i.e. $\eta_i: M \rightarrow [0, 1]$ smooth such that:

(1) $\eta_1 + \dots + \eta_k = 1$ (the oth. fct. 1)

(2) $\text{supp } \eta_i \subseteq U_i$

$(\exists A_i \subseteq U_i \text{ closed s.t. } \eta_i = 0 \text{ on } M \setminus A_i)$

$\downarrow$
 $(\eta_i = 0 \text{ on } M \setminus U_i)$

See Including Topologic: $\exists A$ -partitions of unity for $A \subseteq C(M)$

Satisfying some axioms. Take $A = C^\infty(M)$. To be checked:

$(\forall) U \subseteq M \text{ open} \} \exists f \in C^\infty(M) \text{ s.t. } f(p) = 1, f = 0 \text{ outside } U.$
 $(\forall) p \in U$

Local! $M = \mathbb{R}^n, p = 0, U = B_1(0) = \{x \in \mathbb{R}^n : \|x\| < 1\}$... write down an explicit f .

In topology:

(3)

Thm2: $M = \text{compact, topological manifold} \Rightarrow M$ can be topologically embedded in $\text{some } \mathbb{R}^n$

Recall: Topological embedding of M in N a continuous map

$F: M \rightarrow N$

injective and, as a map $M \rightarrow F(M) \subseteq N$, it is a homeomorphism

where $F(M)$ - with the topology induced from N .

For Smooth embedding we require:

① $F(M)$ - an embedded submanifold of N

② as a map $M \rightarrow F(M)$, F is a diffeomorphism

Rk: $F = \text{smooth embeddng} \Rightarrow \begin{cases} F = \text{topological embeddng} \\ F = \text{immersion} \end{cases}$

Thm: Actually $\Leftrightarrow$

Two of them, μ_1 and μ_2 called equivalent if $\mu_2 = f \mu_1$ with $f > 0$.

Orientation on M : an equivalence class w.r.t. $\sim$; for μ vol. form $\Rightarrow$ orient M .

Given $\Theta \Rightarrow$ can talk about oriented charts: (U, χ) st. $\mu(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}) > 0$ over U .

Corollary: $M = \text{cpt}$ $\Rightarrow$ any volume form μ on M cannot be exact.

Hence, to prove a given ^{injective} F is a smooth embedding:

$F = \text{top. emb.}$

$F = \text{immersion}$: $dF_p = \text{inj}$ $\Rightarrow$ just check.

$\hookrightarrow$ If M, N -manifold, $M = \text{cpt}$, then that is automatic if $F = \text{injective}$!

"Ex": For the proof of Thm 2 in topology: we construct $F: M \rightarrow \mathbb{R}^L$

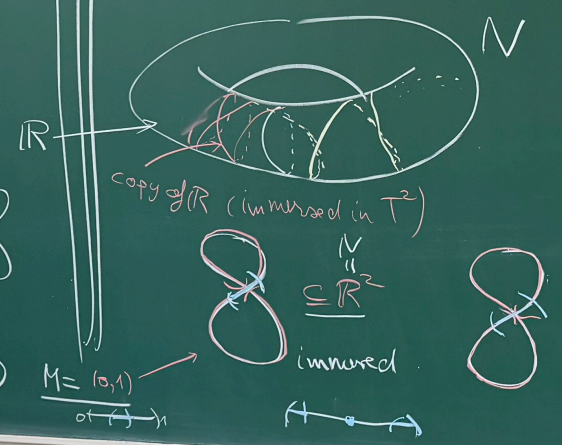
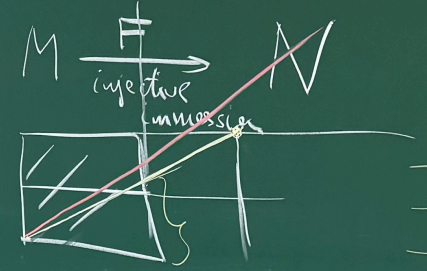
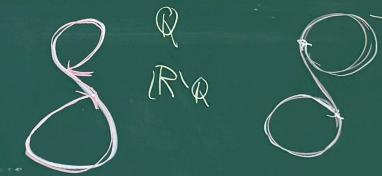
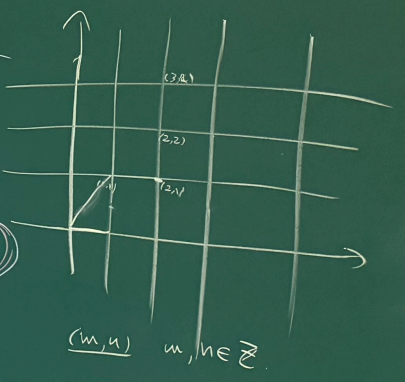
Starting from a partition of unity.

If we use smooth $\text{---} \parallel \text{---}$, the same F just check the immersion cond $\Rightarrow$ $\odot$

the smooth version of Theorem 2.

It is interesting to allow for more general submanifolds of N

immersed submanifolds



• Usual integration for $\mathbb{R}^n$ open. ①
 $f: C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$
 linear & satisfying CVF: $\int_M df \wedge \omega = \int_M f \wedge d\omega$
 $\int_M f = \int_M f(x) \cdot |J(x)|$

Rk: Given (M, θ)
 (a) positively oriented chart $(U, x), (U', x')$
 one has: $C = x' \circ x^{-1}$
 has $J_{ac}(C) > 0$
 or $|J_{ac}(C)| = |J_{ac}(C)|$

⇒ on an oriented (M, θ) , we can define $\int_M \omega$ whenever $\omega \in \Omega_c^m(M)$ is "small" ②
 i.e. $\text{supp}(\omega) \subseteq U$ for some coordinate chart

Explicitly: choose (U, x) oriented, consider the resulting $F_x^\omega \in C_c^\infty(\mathbb{R}^n)$ set $\int_M \omega = \int_{\mathbb{R}^n} F_x^\omega$ (*)

By the CVF ⇒ this does not depend on the choices
 Theorem: On any oriented (M, θ) , $\exists!$ $\int: \Omega_c^m(M) \rightarrow \mathbb{R}$
 linear which, when ω is small is given by (*)

Theorem (Stokes) (M, θ) oriented, $\theta \in \Omega_c^{n-1}(M) \Rightarrow \int_M d\theta = 0$

Applic: M compact, μ volume form ⇒ can talk about $\text{vol}_\mu(M) = \int_M \mu$ (using θ_μ) ... $> 0!$

Corollary: $M = \text{cpt} \Rightarrow$ any volume form μ on M cannot be exact

• Top differential forms: $\omega \in \Omega^m(M)$ where $m = \dim M$
 → locally, for chart $x: U \rightarrow \mathbb{R}^n \Rightarrow$ a function $F_x^\omega \in C_c^\infty(\mathbb{R}^n)$ then
 $\omega|_U = F_x^\omega \cdot dx_1 \wedge \dots \wedge dx_n$ (or $x'_1 \omega|_U = F_{x'}^\omega dx'_1 \wedge \dots \wedge dx'_n$)
 → If x' is another chart $\Rightarrow F_x^\omega = F_{x'}^\omega \cdot J_{ac}(C)$ where $C = x' \circ x^{-1}$

• Volume form on M : any $\mu \in \Omega^n(M)$ such that $\mu_p \neq 0$ ($\forall p \in M$)
 Two of them, μ_1 and μ_2 called equiv if $\mu_2 = f \mu_1$ with $f > 0$

• Orientation on M : an equivalence class w.r.t. $\sim$; for vol form ⇒ orient θ_μ
 Given $\theta_\mu \Rightarrow$ can talk about oriented charts (U, x) st $\mu(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}) > 0$ over U

Thm: $M = \text{manifold}$. Then: ④
 (V) open cover $\{U_i, \dots, U_k\}$ of M
 (E) partition of unity $\{\eta_1, \dots, \eta_k\}$ subordinated to the cover
 i.e. $\eta_i: M \rightarrow [0, 1]$ smooth such that:
 (1) $\eta_1 + \dots + \eta_k = 1$ (the cst. fct. 1)
 (2) $\text{supp } \eta_i \subseteq U_i$ ($\exists A_i \subseteq U_i$ closed st $\eta_i = 0$ on $M \setminus A_i$)
 ($\eta_i = 0$ on $M \setminus U_i$)

See Including Topologie: $\exists$ A-partitions of unity for $d \leq C(M)$
 Satisfying some axioms. Take $A = C^\infty(M)$. To be checked:

(V) $U \subseteq M$ open $\} \exists f \in C^\infty(M)$ s.t. $f(p) = 1, f = 0$ outside U .
 (V) $p \in U$
 Local! $M = \mathbb{R}^n, p = 0, U = B_1(0) = \{x \in \mathbb{R}^n: \|x\| < 1\}$ write down an explicit f .

Hence, to prove a given F is a smooth embedding: ①

$F = \text{top. emb.}$
 $F = \text{immersion: } dF_p \neq 0 \text{ for } p \in M$ just check

⇒ If M, N manifold, $M = \text{cpt}$, then that is automatic if F is injective!

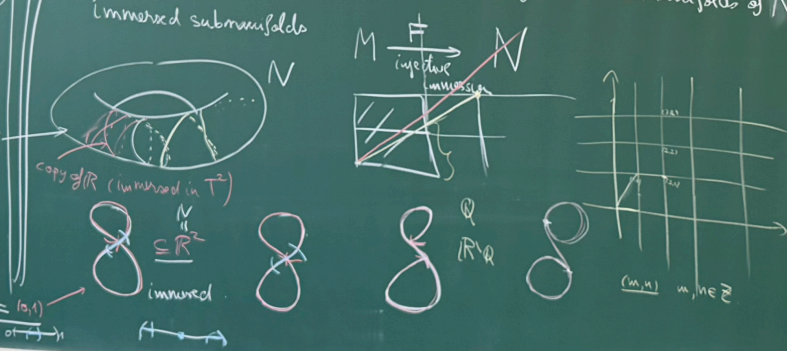
"Ex": For the proof of Thm 2 in topology: one constructs $F: M \rightarrow \mathbb{R}^L$

Starting from a partition of unity. ⑧

If we use smooth " " , the same F just check the immersion cond ⇒

the smooth version of Theorem 2. ①

It is interesting to allow for more general submanifolds of N
 immersed submanifolds



In topology: ③

Thm2: $M = \text{compact}$, topological manifold $\Rightarrow M$ can be topologically embedded in $\text{some } \mathbb{R}^L$

Recall: Topological embedding of M in N a continuous map $F: M \rightarrow N$

injective and, as a map $M \rightarrow F(M) \subseteq N$, it is a homeomorphism

where $F(M)$ - with the topology induced from N .

For smooth embedding we require:
 ① $F(M)$ - an embedded submanifold of N
 ② as a map $M \rightarrow F(M)$, F is a diffeomorphism.

Rk: $F = \text{smooth embedd} \Rightarrow \begin{cases} F = \text{topological embedding} \\ F = \text{immersion} \end{cases}$

Thm: Actually $\Leftrightarrow$