

Reminder: differential forms & DeRham:  $m = \dim M$  ①

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{m-1}(M) \xrightarrow{d} \Omega^m(M) \rightarrow 0$$

$\Omega^k_{cl}(M) = \{w \in \Omega^k(M) : dw = 0\}$   
 $\Omega^k_{ex}(M) = \{w \in \Omega^k(M) : \exists \eta \in \Omega^{k-1}(M) \text{ s.t. } w = d\eta\}$

$d \circ d = 0$  (closed  $w$   $\Rightarrow$  exact  $w$ )  
 $(dw=0) \xrightarrow{\text{closed } w} \Omega^k_{cl}(M) \xrightarrow{\text{exact } w} \Omega^k_{ex}(M)$

• wedge products  
 $\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M), (w, \eta) \mapsto w \wedge \eta$

& Leibniz:  $d(w \wedge \eta) = d(w) \wedge \eta + (-1)^k w \wedge d\eta$

• Volume forms on  $M$ :  $\mu \in \Omega^m(M)$  s.t.  $\mu_p \neq 0$  ( $\forall p \in M$ )  
 Ex:  $\mu_{\mathbb{R}^m} = dx_1 \wedge \dots \wedge dx_m, \mu_{S^m} = \sum_{i=1}^m (-1)^i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$

• Stokes:  $\int_M d\eta = 0$  ( $\forall \eta \in \Omega^{m-1}_c(M)$ )

• Corollary:  $M = \text{cpt}$ ,  $\mu = \text{volume form on } M \Rightarrow \mu$  NOT exact

• Pull-backs:  $F: M \rightarrow N$  smooth map induces ②

$$F^*: \Omega^k(N) \rightarrow \Omega^k(M)$$

and this is compatible with  $\wedge$  and  $d$ :

$$\begin{cases} F^*(dw) = d(F^*w) & (\forall w \in \Omega^k(N)) \\ F^*(w \wedge \eta) = F^*w \wedge F^*\eta & (\forall w, \eta) \end{cases}$$

• May also remember, on opens  $\mathcal{U} \subseteq \mathbb{R}^m$ :  
 given  $F_1, \dots, F_m \in C^\infty(\mathcal{U})$   
 then:

$$\exists F \text{ s.t. } F_i = \frac{\partial F}{\partial x_i} \iff \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

... if  $\mathcal{U}$  = simply connected ( $\forall \gamma_1, \gamma_2$  in  $\mathcal{U}$  can be deformed into each other)

$w = F_1 dx_1 + \dots + F_m dx_m$

$\exists F \text{ s.t. } w = dF \iff dw = 0$  if  $\mathcal{U}$  = simply connected



Reminder: differential forms & DeRham:  $m = \dim M$  (1)

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(M) \rightarrow 0$$

$$\boxed{d \circ d = 0} \quad \begin{array}{c} (dw=0) \\ (w=d\eta) \end{array} \xrightarrow{\text{closed } w} \Omega_{cl}^k(M) = \{w \in \Omega^k(M) : dw=0\}$$

$$\xrightarrow{\text{exact } w} \Omega_{ex}^k(M) = \{w \in \Omega^k(M) : \exists \eta \in \Omega^{k-1}(M) \text{ s.t. } w=d\eta\}$$

• wedge products

$$\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M), (w, \eta) \mapsto w \wedge \eta$$

& Leibniz:  $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$

• Volume forms on  $M$ :  $\mu \in \Omega^m(M)$  s.t.  $\mu_p \neq 0$  ( $\forall$ )  $p \in M$ .

Ex:  $\mu_{\mathbb{R}^m} = dx_1 \wedge \dots \wedge dx_m$ ,  $\mu_{S^m} = \sum_{i=0}^m (-1)^i \hat{x}_i \wedge dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$ .

• Stokes:  $\int_M d\eta = 0$  ( $\forall$ )  $\eta \in \Omega_c^{m-1}(M)$ .

• Corollary:  $M = \text{cpt}$ ,  $\mu = \text{volume form on } M \Rightarrow \mu \text{ NOT exact}$

the difference between them: encoded in the quotient (3)

$$H^k(M) = \Omega_{cl}^k(M) / \Omega_{ex}^k(M) = \{[w] : w \in \Omega^k(M), dw=0\}$$

De Rham cohom. of  $M$  in degree  $k$ .

where  $[w_1] = [w_2] \Leftrightarrow w_1 - w_2 = d\eta$  some  $\eta$ .  
(this is 0 iff  $\Omega_{cl}^k(M) = \Omega_{ex}^k(M)$ )

because of Leibniz:  $H^k(M) \times H^l(M) \rightarrow H^{k+l}(M)$

$$([w], [\theta]) \mapsto [w \wedge \theta]$$

is well defined (e.g. if  $[w_1] = [w_2] \Rightarrow [w_1 \wedge \theta] = [w_2 \wedge \theta]$ )

$$w_1 - w_2 = d(\eta) \Rightarrow (w_1 - w_2) \wedge \theta = d(\eta \wedge \theta) = d(?)$$

try  $\eta \wedge \theta$ :  $d(\eta \wedge \theta) = d\eta \wedge \theta + \eta \wedge d\theta$   
( $w_1 - w_2$ )  $\wedge \theta$

• Pull-backs:  $F: M \rightarrow N$  smooth map induces (2)

$$F^*: \Omega^k(N) \rightarrow \Omega^k(M)$$

and this is compatible with  $\wedge$  and  $d$ :

$$\begin{cases} F^*(dw) = d(F^*w) & (\forall) w \in \Omega^k(N) \\ F^*(w \wedge \eta) = F^*w \wedge F^*\eta & (\forall) w, \eta \end{cases}$$

• May also remember, on opens  $\Omega \subseteq \mathbb{R}^m$ :  
given  $F_1, \dots, F_n \in C^\infty(\Omega)$



forms & De Rham:  $m = \dim M$

1)  $\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M) \rightarrow \dots \rightarrow \Omega^m(M) \rightarrow 0$

$\Omega^0(M) = \{f \in C^\infty(M) : df = 0\}$

$\Omega^1(M) = \{w \in \Omega^1(M) : dw = 0\}$

$\Omega^k(M) = \{w \in \Omega^k(M) : \exists \eta \in \Omega^{k-1}(M) : w = d\eta\}$

the difference between them encoded in the quotient

De Rham cohomology of  $M$  in degree  $k$ :

$$H^k(M) = \Omega^k(M) / \Omega^k_{\text{exact}}(M) = \{[w] : w \in \Omega^k(M), dw = 0\}$$

where  $[w_1] = [w_2] \Leftrightarrow w_1 - w_2 = d\eta$  for some  $\eta$ .

(this is 0 iff  $\Omega^k_{\text{exact}}(M) = \Omega^k(M)$ )

because of Leibniz:  $H^k(M) \wedge H^l(M) \rightarrow H^{k+l}(M)$

$([w], [\eta]) \mapsto [w \wedge \eta]$

is well defined (e.g. if  $[w_1] = [w_2]$  and  $[w_1 - w_2] = d\eta$ , then  $(w_1 - w_2) \wedge \theta = d(\eta \wedge \theta)$ )

defining  $\int_M : H^m(M) \rightarrow \mathbb{R}$ ,  $[w] \mapsto \int_M w$  is well defined!

$\int_M w = \int_M w + \int_M d\eta = \int_M w$

$\mu = \text{volume form on } M \Rightarrow \mu \text{ NOT exist}$

$\int_M \mu \neq 0 \Rightarrow [ \mu ] \in H^m(M)$  is non-zero

maps induced by  $F$

$F^* : H^k(M) \rightarrow H^k(N)$ ,  $F^*([w]) = [F^*w]$

& chain rule still holds:  $(G \circ F)^* = F^* \circ G^*$

This implies:

$F : M \rightarrow N$  diffeom  $\Rightarrow F^* : H^k(M) \rightarrow H^k(N)$  linear isomorphism

$(F^* \circ F^*)^* = (F \circ F)^* = \text{Id}^* = \text{Id}$

Fact:  $M = \text{compact} \Rightarrow$  all  $H^k(M)$  are finite dim. vector spaces

get numbers  $\beta_k(M) := \dim_{\mathbb{R}} H^k(M) \in \mathbb{N}$  Betti #

"numerical invariants"

$\sum_k (-1)^k \beta_k(M) = \chi(M)$  the Euler characteristic of  $M$

(topological invariant!)

$\# \text{vertices} - \# \text{edges} + \# \text{faces} - \dots$

$S^2: 4 - 6 + 4 = 2$

More topology around (singular in AT)



Reminder: differential forms & De Rham

- wedge products  
 $\wedge^k(M) \times \wedge^l(M) \rightarrow \wedge^{k+l}(M)$   
 $(\omega, \eta) \mapsto \omega \wedge \eta$   
 $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
- Volume forms on  $M$ :  $\mu \in \wedge^n(M)$  s.t.  $\mu \neq 0$  on  $\mathcal{O}_p(M)$   
 $E.g.$  edge area,  $\mu_2 = \sum (-1)^i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$
- Stokes:  $\int_M d\eta = \int_{\partial M} \eta$  (if  $\eta \in \wedge^{n-1}(M)$ )
- Corollary:  $M = \text{cpt}$ ,  $\mu = \text{volume form on } M \Rightarrow \int \mu \neq 0$

the difference between them encoded in the quotient

$$H^k(M) = \mathcal{Z}^k(M) / \mathcal{B}^k(M) = \{ [\omega] : \omega \in \mathcal{Z}^k(M), d\omega = 0 \}$$

De Rham cohomology of  $M$  in degree  $k$

what  $[\omega_1] = [\omega_2] \Leftrightarrow \omega_1 - \omega_2 = d\eta$  for some  $\eta$   
 (this is 0 iff  $\mathcal{Z}^k(M) = \mathcal{B}^k(M)$ )

because of Leibniz:  $H^k(M) \wedge H^l(M) \rightarrow H^{k+l}(M)$   
 $([\omega], [\eta]) \mapsto [\omega \wedge \eta]$   
 $[\omega] \cdot [\eta] = [\omega \wedge \eta]$

is well defined (e.g. if  $[\omega_1] = [\omega_2], [\eta_1] = [\eta_2]$   
 $\omega_1 - \omega_2 = d\eta_1, \eta_1 - \eta_2 = d\eta_2$   
 $(\omega_1 - \omega_2) \wedge (\eta_1 - \eta_2) = d(\eta_1 \wedge \eta_2)$   
 $(\omega_1 \wedge \eta_1) - (\omega_1 \wedge \eta_2) - (\omega_2 \wedge \eta_1) + (\omega_2 \wedge \eta_2) = d(\eta_1 \wedge \eta_2)$   
 $\omega_1 \wedge \eta_1 - \omega_2 \wedge \eta_1 - \omega_1 \wedge \eta_2 + \omega_2 \wedge \eta_2 = d(\eta_1 \wedge \eta_2)$   
 $(\omega_1 - \omega_2) \wedge \eta_1 - \omega_1 \wedge (\eta_1 - \eta_2) + \omega_2 \wedge \eta_1 - \omega_1 \wedge \eta_2 + \omega_2 \wedge \eta_2 = d(\eta_1 \wedge \eta_2)$   
 $d\eta_1 \wedge \eta_1 - \omega_1 \wedge d\eta_2 + \omega_2 \wedge \eta_1 - \omega_1 \wedge \eta_2 + \omega_2 \wedge \eta_2 = d(\eta_1 \wedge \eta_2)$   
 $d\eta_1 \wedge \eta_1 - \omega_1 \wedge d\eta_2 + \omega_2 \wedge \eta_1 - \omega_1 \wedge \eta_2 + \omega_2 \wedge \eta_2 = d(\eta_1 \wedge \eta_2)$   
 $d\eta_1 \wedge \eta_1 - \omega_1 \wedge d\eta_2 + \omega_2 \wedge \eta_1 - \omega_1 \wedge \eta_2 + \omega_2 \wedge \eta_2 = d(\eta_1 \wedge \eta_2)$

defining  $\int_M : H^k(M) \rightarrow \mathbb{R}, [\omega] \mapsto \int_M \omega$  is well defined!  
 $\omega \mapsto d\eta \Rightarrow \int_M d\eta = \int_{\partial M} \eta = 0$

try to get  $d\eta = 0 \Rightarrow \int_M d\eta = 0$

if  $\omega \in \mathcal{Z}^k(M)$  s.t.  $\int_M \omega = 0$  then  $[\omega] = 0$  in  $H^k(M)$

Thm: (1) If  $F_0, F_1: M \rightarrow N$  homotopic  
 then  $F_0^*, F_1^*: H^k(N) \rightarrow H^k(M)$  coincide  
 (2) If  $F: M \rightarrow N$  is a homotopy equivalence  
 then  $F^*: H^k(M) \rightarrow H^k(N)$  is an isomorphism

• Pull-backs:  $F: M \rightarrow N$  smooth map induces  
 $F^*: \mathcal{Z}^k(N) \rightarrow \mathcal{Z}^k(M)$   
 and this is compatible with  $\wedge$  and  $d$ :  
 $F^*(d\omega) = d(F^*\omega)$  (if  $\omega \in \mathcal{Z}^k(N)$ )  
 $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$  (if  $\omega, \eta \in \mathcal{Z}^*(N)$ )

• Many also remember, on opens  $U \subseteq \mathbb{R}^n$   
 given  $F_1, F_2 \in C^1(U, \mathbb{R}^n)$   
 then  $\exists F$  s.t.  $F_1 = \frac{\partial F}{\partial x_1}, F_2 = \frac{\partial F}{\partial x_2}$   
 iff  $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$

... if  $U$  is simply connected (if  $x_1, x_2$  in  $U$   
 can be deformed into each other)  
 $\omega = F_1 dx_1 + F_2 dx_2$   
 $\exists F$  s.t.  $\omega = dF \Leftrightarrow d\omega = 0$  (if  $U$  simply connected)

$F$  also induces a well-defined map  $F^*: H^k(N) \rightarrow H^k(M), F^*([\omega]) = [F^*\omega]$   
 & chain rule still holds:  $(G \circ F)^* = F^* \circ G^*$

This implies:  
 $(F: M \rightarrow N \text{ diffeom}) \Rightarrow F^*: H^k(N) \rightarrow H^k(M)$  linear isomorphism  
 $(F^*)^{-1} \circ F^* = \text{Id}$

Fact:  $M = \text{cpt} \Rightarrow$  all  $H^k(M)$  are finite dim vector spaces

get numbers  $\beta_k(M) = \dim_{\mathbb{R}} H^k(M) \in \mathbb{N}$  Betti #  
 "numerical invariants"

$\sum_k (-1)^k \beta_k(M) = \chi(M)$  the Euler characteristic of  $M$   
 (topological invariant!)

$\beta_2(S^1) = 1, \beta_1(S^1) = 0$   
 $S^1: 4 - 6 + 4 = 2$   
 #vertices - #edges + #faces = 2  
 More generalization around (Singular in AT)





③

Thm: (1) If  $F_0, F_1: M \rightarrow N$  homotopic  $F_0 \sim F_1$  (5)  
 then  $F_0^*, F_1^*: H^k(N) \rightarrow H^k(M)$  coincide.

(2) If  $F: M \rightarrow N$  is a homotopy equivalence  $\circ$   
 then  $F^*: H^k(N) \rightarrow H^k(M)$  isomorphism.

$\exists G: N \rightarrow M$   
 st.  $F \circ G$  homotopic to  $\text{Id}_N$   
 $G \circ F \sim \text{Id}_M$   
 $F \circ G \sim \text{Id}_N, G \circ F \sim \text{Id}_M$

$\exists H: M \times \mathbb{R} \rightarrow N$   
 st.  $\begin{cases} H(x, 0) = F_0(x) \\ H(x, 1) = F_1(x) \end{cases} \quad (x \in M)$   
 $t \quad H(-, t): M \rightarrow N$

Ex:  $M = \mathbb{R}^n, N = \mathbb{R}^0 = \{0\}$  are homotopic equivalent. (6)

$\mathbb{R}^n \xrightleftharpoons[G]{F} \{0\}$   $F(x) = 0 \quad \forall x$   
 $G(0) = 0$

$F \circ G: \{0\} \rightarrow \{0\}, 0 \mapsto 0$   
 $G \circ F: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto 0$



③ Thm: (1) If  $F_0, F_1: M \rightarrow N$  homotopic  $F_0 \sim F_1$  (5)  
 then  $F_0^*, F_1^*: H^k(N) \rightarrow H^k(M)$  coincide.  
 (2) If  $F: M \rightarrow N$  is a homotopy equivalence  
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 $G \circ F \sim \text{Id}_M$   
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$\exists H: M \times \mathbb{R} \rightarrow N$   
 st.  $H(x, 0) = F_0(x)$   
 $H(x, 1) = F_1(x)$   
 $t \in H(-, t): M \rightarrow N$

Ex1:  $M = \mathbb{R}^n, N = \mathbb{R}^0 = \{0\}$  are homotopic equivalent. (6)

$\mathbb{R}^n \xrightleftharpoons[G]{F} \{0\}$   
 $F(x) = 0 \forall x$   
 $G(0) = 0$

$F \circ G: \{0\} \rightarrow \{0\}, 0 \mapsto 0$   
 $G \circ F: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto 0$   
 is this  $\sim$  to  $x \mapsto x$   
 $H: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$   
 $H(x, t) = tx$

Ex2:  $M = \mathbb{R}^n \setminus \{0\}, N = S^{n-1}$  are h.e.

$\mathbb{R}^n \setminus \{0\} \xrightleftharpoons[G]{F} S^{n-1}$   
 $F(x) = \frac{x}{\|x\|}$   
 $G(x) = x$

$F \circ G: S^{n-1} \rightarrow S^{n-1} \text{ id!}$   
 $G \circ F: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$   
 $x \mapsto \frac{x}{\|x\|}$   
 This is  $\sim$  to  $x \mapsto x$   
 $H(x, t) = tx + (1-t) \frac{x}{\|x\|}$

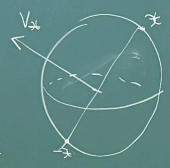
$\text{Id} \Rightarrow G^* \circ F^* = \text{Id} \Rightarrow F^* = \text{Iso}$   
 $\text{Id} \Rightarrow F^* \circ G^* = \text{Id}$



Volume forms on  $M$ :  $\mu \in \Omega^m(M)$  s.t.  $\mu_p \neq 0$  ( $\forall p \in M$ )  
 Ex:  $\mu_{\mathbb{R}^m} = dx_1 \wedge \dots \wedge dx_m$   $\mu_{S^m} = \sum_{i=1}^m (-1)^i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$   
 Stokes:  $\int_M d\eta = 0$  ( $\forall \eta \in \Omega_c^{m-1}(M)$ )

defining on  $M = \text{cpt}$   
 $\int_M H(M \rightarrow \mathbb{R}, [w]) \rightarrow \int_M w$  is well defined  
 $w' - w = dy \rightarrow [w'] = [w] + \int_M dy$   
 $\int_M w' = \int_M w + \int_M dy$

Ex:  $\gamma: S^m \rightarrow S^n, \gamma(x) = -x$  ...  
 could this be  $\sim \text{Id}_{S^n}$   $\gamma(x) = x$ ?  
 Assume  $\exists V \in \mathcal{X}(S^n)$  s.t.  
 $\int_{S^n} V \neq 0$   $\forall V \in S^n$  May assume  $\|V\| = 1$   
 $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$   
 Take  $H: S^m \times [0,1] \rightarrow S^n$   
 $H(x,t) = (\cos \pi t) \cdot x + (\sin \pi t) \cdot V_x$   
 $H(x,0) = x$   
 $H(x, \frac{1}{2}) = V_x$   
 $H(x,1) = -x$   
 If  $\exists V \in \mathcal{X}(S^n)$  nowhere vanishing  $\Rightarrow \gamma \sim \text{Id}_{S^n}$



From Thm on 5  
 $\gamma^* = \text{id}^*$  as  $H^n(S^n) \rightarrow H^n(S^n)$   
 $\gamma^*[w] = \text{id}^*[w] \quad (\forall [w] \in H^n(S^n))$   
 $[\gamma^*w] = [w] \quad (\forall [w] \in H^n(S^n))$   
 For  $w = \mu_{S^n} \Rightarrow [\gamma^* \mu_{S^n}] = [\mu_{S^n}]$   
 $[(-1)^{m+1} \mu_{S^n}] = [\mu_{S^n}]$   
 $\Rightarrow (-1)^{m+1} [\mu_{S^n}] = [\mu_{S^n}]$   
 For  $m = \text{even} \Rightarrow -[\mu_{S^n}] = [\mu_{S^n}] \Rightarrow [\mu_{S^n}] = 0$   
 $\nexists V \in \mathcal{X}(S^n)$  nowhere vanishing!  
 hairy ball theorem.  
 contrast with the consequence to Stokes.

8 Proof of Thm on 5

Lemma 9  $H^k(\mathbb{R}^n) \cong \mathbb{R}$  if  $k=0$  and 0 otherwise  
 (Poincaré lemma)  
 pf: induction by  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$

Observe: can replace  $\mathbb{R}^{n-1}$  by  $\mathbb{R}^n$   
 get:  
 $i_0: M \rightarrow M \times \mathbb{R}, i_0(x) = (x,0)$   
 have  $i_0^* = \text{id}^*$  in cohomology  
 In general: in (i) of Thm 5  
 $H$  is homotopy between  $F_0$  and  $F_1$   
 $H \circ i_0 = F_0, H \circ i_1 = F_1$   
 Now  $F_0^* = i_0^* \circ H^* = \text{id}^* \circ H^* = H^*$   
 $F_1^* = i_1^* \circ H^* = \text{id}^* \circ H^* = H^*$



$$\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M), (\omega, \eta) \mapsto \omega \wedge \eta$$

& Leibniz:  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

• Volume forms on  $M$ :  $\mu \in \Omega^m(M)$  s.t.  $\mu_p \neq 0 \ (\forall) p \in M$

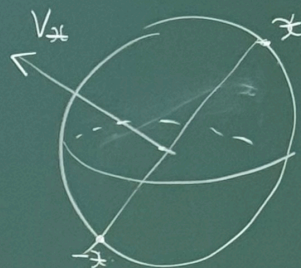
Ex:  $\mu_{\mathbb{R}^m} = dx_1 \wedge \dots \wedge dx_m$   $\mu_{S^m} = \sum_{i=0}^m (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$

Ex:  $\tau: S^m \rightarrow S^m, \tau(x) = -x \dots$

could this be  $\sim \text{Id}_{S^n}$   $\text{id}(x) = x$

(Assume)  $\exists V \in \mathcal{X}(S^n)$  s.t.

$V_x \neq 0 \ (\forall) x \in S^n$  May assume  $\|V_x\| = 1$   
 $T_x S^n \equiv \{v \in \mathbb{R}^n : v \perp x\}$



Take  $H: S^m \times [0, 1] \rightarrow S^m$

$H(x, t) = (\cos \pi t) \cdot x + (\sin \pi t) V_x$   
 $H(x, 0) = x$

$H(x, \frac{1}{2}) = V_x$

$H(x, 1) = -x$

If  $\exists V \in \mathcal{X}(S^n)$  nowhere vanishing  $\Rightarrow$   
 $\Rightarrow \tau \sim \text{id}_{S^n}$



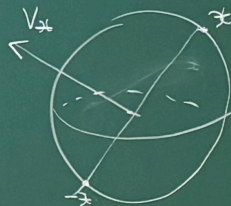
• Volume forms on  $M$ :  $\mu \in \Omega^m(M)$  s.t.  $\mu_p \neq 0$  ( $\forall$ )  $p \in M$ .

Ex:  $\mu_{\mathbb{R}^m} = dx_1 \wedge \dots \wedge dx_m$   $\mu_{S^m} = \sum_{i=0}^m (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_m$

defining  $\int_M : H^m(M) \rightarrow \mathbb{R}$ ,  $[w] \mapsto \int_M w$

Ex:  $\tau: S^m \rightarrow S^m$ ,  $\tau(x) = -x$  ...  
could this be  $\sim \text{Id}_{S^m}$   $\text{id}(x) = x$ .

Assume  $\exists V \in \mathcal{X}(S^m)$  s.t.  
 $V_x \neq 0$   $\forall x \in S^m$  May assume  $\|V_x\| = 1$   
 $T_x S^m \cong \{v \in \mathbb{R}^m : v \perp x\}$



Take  $H: S^m \times [0,1] \rightarrow S^m$   
 $H(x,t) = (\cos \pi t) \cdot x + (\sin \pi t) V_x$   
 $H(x,0) = x$   
 $H(x, \frac{1}{2}) = V_x$   
 $H(x,1) = -x$

If  $\exists V \in \mathcal{X}(S^m)$  nowhere vanishing  $\Rightarrow$   
 $\Rightarrow \tau \sim \text{id}_{S^m}$ .

From Thm on 5

$\tau^* = \text{id}^*$  as  $H^m(S^m) \rightarrow H^m(S^m)$ .

$\tau^*[w] = \text{id}^*[w] \quad (\forall) [w] \in H^m(S^m)$

$[\tau^*w] = [w] \quad (\forall) [w] \in H^m(S^m)$ .

For  $w = \mu_{S^m} \Rightarrow [\tau^*(\mu_{S^m})] = [\mu_{S^m}]$

$[(-1)^{m+1} \mu_{S^m}]$

$\Rightarrow (-1)^{m+1} [\mu_{S^m}] = [\mu_{S^m}]$

For  $m = \text{even} \Rightarrow -[\mu_{S^m}] = [\mu_{S^m}] \Rightarrow [\mu_{S^m}] = 0$

$\nexists V \in \mathcal{X}(S^m)$  nowhere vanishing



Hairy ball theorem.



contrad with the consequence to Stokes.

8 Proof of Thm on 5



$$H^k(M) \times H^l(M) \rightarrow H^{k+l}(M)$$

$$([\omega], [\theta]) \mapsto [\omega \wedge \theta]$$

$$[\omega] \cdot [\theta] = [\omega \wedge \theta]$$

$$\text{if } [\omega_1] = [\omega_2] \Rightarrow [\omega_1 \wedge \theta] = [\omega_2 \wedge \theta]$$

$$\omega_1 - \omega_2 = d(\eta)$$

$$(\omega_1 - \omega_2) \wedge \theta = d(\eta) \wedge \theta$$

$$\text{try } \eta \wedge \theta : d(\eta \wedge \theta) = d\eta \wedge \theta \pm \eta \wedge d\theta$$

$$(M) \rightarrow \mathbb{R}$$

$$[\omega] \rightarrow \int_M \omega$$

$$\int_M \omega$$

is well defined!

$$(\omega_1 - \omega_2) \wedge \theta$$

8

Proof of Thm on 5

Lemma 9 (Poincaré lemma)

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 & \text{otherwise} \end{cases}$$

Ex 1:  $M = \mathbb{R}^n$

$$\mathbb{R}^n \xleftarrow{F} \mathbb{R}^n \xleftarrow{G}$$

pf: induction by n

$$\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$$

place for

Ex 2:  $M = \mathbb{R}^n$

$$\mathbb{R}^n \setminus \{0\}$$

Observe: can replace  $\mathbb{R}^{n-1}$  by an M

get:

$$i_0: M \rightarrow M \times \mathbb{R}, i_1: M \rightarrow M \times \mathbb{R}$$

$$i_0(x) = (x, 0)$$

$$i_1(x) = (x, 1)$$

have  $i_0^* = i_1^*$  in cohomology

In general: in (i) of the Theorem

H is homotopy between  $F_0, F_1$

$$H \circ i_0 = F_0, H \circ i_1 = F_1$$

Now

$$F_0^* = i_0^* \circ H^*$$

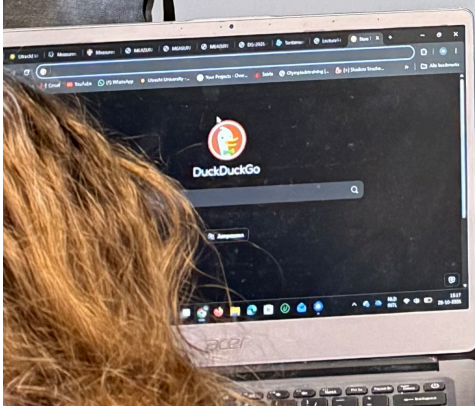
$$F_1^* = i_1^* \circ H^*$$

proves (ii)

For (2) use (1)

$$F_0 \circ G \sim \text{Id} \Rightarrow G^* \circ F^* = \text{Id} \Rightarrow F^* =$$

$$G \circ F \sim \text{Id} \Rightarrow F^* \circ G^* = \text{Id}$$





Reminder: differential forms & De Rham

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \rightarrow 0$$

$$\ker d = \Omega^0(M) = \{f \in C^\infty(M) : df = 0\}$$

$$\operatorname{Im} d = \Omega^1(M) = \{w \in \Omega^1(M) : \exists f \in C^\infty(M) \text{ s.t. } w = df\}$$

• wedge products

$$\Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M), (w, \eta) \mapsto w \wedge \eta$$

$$\text{Leibniz } d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$$

• Volume forms on M:  $\mu \in \Omega^n(M)$  s.t.  $\mu \neq 0$  iff  $M$

$$\text{Ex } \mu_{\mathbb{R}^n} = dx_1 \wedge \dots \wedge dx_n$$

the difference between them, encoded in the quotient

$$H^k(M) = \Omega^k(M) / \operatorname{Im} d = \{[w] : w \in \Omega^k(M), dw = 0\}$$

De Rham cohomology of M in degree k.

$$[w_1] = [w_2] \Leftrightarrow w_1 - w_2 = d\eta \text{ for some } \eta$$

(this is 0 iff  $\Omega^k(M) = \operatorname{Im} d$ )

because of Leibniz

$$H^k(M) \times H^l(M) \rightarrow H^{k+l}(M)$$

$$([w], [\eta]) \mapsto [w \wedge \eta]$$

is well defined (e.g. if  $[w_1] = [w_2]$  then  $w_1 - w_2 = d\eta$ )

$$[w_1 - w_2] \wedge [\eta] = [d\eta \wedge \eta] = [d(\eta \wedge \eta)] = 0$$

defining

$$\int_M : H^m(M) \rightarrow \mathbb{R}, [w] \mapsto \int_M w$$

is well defined!

Thm: (1) If  $F_0, F_1 : M \rightarrow N$  homotopic

then  $F_0^*, F_1^* : H^k(N) \rightarrow H^k(M)$  coincide

(2) If  $F : M \rightarrow N$  is a homotopy equivalence

then  $F^* : H^k(M) \rightarrow H^k(N)$  is isomorphism

$$\exists G : N \rightarrow M \text{ s.t. } F \circ G \sim \operatorname{Id}_N$$

$$G \circ F \sim \operatorname{Id}_M$$

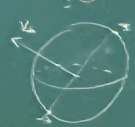
$$\exists H : M \rightarrow N$$

$$\begin{cases} H \circ F = \operatorname{Id}_M \\ H \circ G = \operatorname{Id}_N \end{cases}$$

$$\text{Ex: } \tau : S^1 \rightarrow S^1, \tau(t) = e^{it}$$

could this be  $\sim \operatorname{Id}_{S^1}$ ?

Assume  $\exists V \in \pi(S^1) \neq 0$



$$\text{Take } H : S^1 \times [0, 1] \rightarrow S^1$$

$$H(x, t) = (e^{it}) \cdot x + (e^{it}) \cdot v$$

$$H(x, 0) = x$$

$$H(x, 1) = v$$

$$H(x, 1) = -x$$

$$\nexists \exists V \in \pi(S^1) \text{ nowhere vanishing} \Rightarrow \tau \sim \operatorname{Id}_{S^1}$$

From Thm on

$$\tau^* : H^1(S^1) \rightarrow H^1(S^1)$$

$$\tau^*[w] = [d\tau]$$

$$[d\tau] = [w] \Leftrightarrow [w] = H^1(S^1)$$

$$\text{For } w = \mu_{S^1} \Rightarrow [\tau^* \mu_{S^1}] = [\mu_{S^1}]$$

$$[\tau^* \mu_{S^1}] = [\mu_{S^1}]$$

$$\Rightarrow (1)^{\text{st}} [\mu_{S^1}] = [\mu_{S^1}]$$

$$\text{For } w = \mu_{S^1} \Rightarrow -[\mu_{S^1}] = [\mu_{S^1}] \Rightarrow [\mu_{S^1}] = 0$$

control with the consequence to Stokes.

Having built theorem.

Lemma (Poincaré lemma)

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}$$

Observe: can replace  $\mathbb{R}^n$  by an M

$$i_0 : M \rightarrow M \times \mathbb{R}, i_0(x) = (x, 0)$$

$$i_1 : M \rightarrow M \times \mathbb{R}, i_1(x) = (x, 1)$$

have  $i_1^* = i_0^*$  in cohomology

In general: in (i) of the Thm

H is homotopy between  $F_0, F_1$

$$H \circ i_0 = F_0, H \circ i_1 = F_1$$

$$F_0^* = i_0^* \circ H^*$$

$$F_1^* = i_1^* \circ H^*$$

$$F_0 \sim \operatorname{Id} \Rightarrow F_0^* = \operatorname{Id} \Rightarrow F^* = \operatorname{Id}$$

Ex:  $M = \mathbb{R}^n, N = \mathbb{R}^n$  are homotopic equivalent

$$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^n, F(x) = x$$

$$\mathbb{R}^n \xleftarrow{G} \mathbb{R}^n, G(x) = x$$

$$\text{Ex 2: } M = \mathbb{R}^n, N = S^1$$

$$\mathbb{R}^n \xrightarrow{F} S^1, F(x) = e^{ix}$$

$$\mathbb{R}^n \xleftarrow{G} S^1, G(x) = x$$

$$F \circ G : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x$$

$$G \circ F : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x$$

$$H(x, t) = tx$$

$$F \circ G : S^1 \rightarrow S^1, \operatorname{Id}$$

$$G \circ F : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto \frac{x}{\|x\|}$$

$$H(x, t) = tx$$

$$H(x, t) = tx + (1-t) \frac{x}{\|x\|}$$

$$H(x, t) = tx + (1-t) \frac{x}{\|x\|}$$

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