

An example of computing $\int_M \omega$ for $\omega \in \Omega^k(M)$, using the basic properties of d , namely:

$$(DR0) \quad \boxed{d(\omega + \omega') = d\omega + d\omega'}$$

& for $k=0$, i.e. $\omega = f \in C^\infty(M)$: it is the usual df .
(in particular, on $\mathbb{R}^n$: $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$)

$$(DR1) \quad (\forall \omega \in \Omega^k(M), \eta \in \Omega^\ell(M)):$$

$$\boxed{d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta} \quad \text{"Leibniz"}$$

$$(DR2) \quad \boxed{d(d\omega) = 0} \quad (\forall \omega).$$

Rk1: For 0-forms, i.e. $f \in C^\infty(M)$, and arbitrary $\omega \in \Omega^k(M)$,
 $f \wedge \omega = f\omega$ (basically by definition)

hence (DR1) reads

$$\boxed{d(f\omega) = df \wedge \omega + f \wedge d\omega.}$$

When ω is itself a 0-form, $\omega = g \in C^\infty(M)$ this becomes $d(fg) = f dg + g df$.

$$\left(\begin{array}{l} \text{if } f=g=1 \Rightarrow d(1)=0 \Rightarrow d(\text{constant})=0 \\ \text{when } g=f \Rightarrow d(f^2) = 2f df \text{ etc} \end{array} \right)$$

Rk2: When $M \subseteq N$ submanifold, $\omega \in \Omega^k(N)$ then

$$d(\omega|_M) = (d\omega)|_M$$

(this is useful on submanifolds $M \subseteq \mathbb{R}^n$!)

(2)
Ex 1: Compute $d\theta$ for $\theta = x dx \wedge dy \in \mathcal{O}^2(\mathbb{R}^2)$.

~~Use Leibniz~~ Various ways:

Way 1: Use Leibniz ~~writing~~ $\theta = \underbrace{(x dx)}_w \wedge \underbrace{dy}_\eta \Rightarrow$

$$\Rightarrow d\theta = \underbrace{d(x dx)}_0 \wedge dy - x dx \wedge \underbrace{d(dy)}_0 \text{ by (DR2).}$$

use again Leibniz (this is $x \wedge dx \dots$ see Rk 1),

and get

$$\underbrace{dx \wedge dx}_0 + x \wedge \underbrace{d(dx)}_0 \text{ again.}$$

$$\Rightarrow \boxed{d\theta = 0.}$$

$w \wedge w = 0$ for 1-forms!
 $(\ddot{\omega} \eta = (-1)^{kl} \eta \wedge w)$
 $w \in \mathcal{O}^k, \eta \in \mathcal{O}^l$

Way 2: Use Leibniz but writing $\theta = \underbrace{x}_f \wedge \underbrace{dx \wedge dy}_w$

$$\Rightarrow d\theta = \underbrace{dx \wedge (dx \wedge dy)}_0 + x \wedge \underbrace{d(dx \wedge dy)}$$

Leibniz again $\Rightarrow$

$$\underbrace{d(dx)}_0 \wedge dy - dx \wedge \underbrace{d(dy)}_0$$

$$\Rightarrow \boxed{d\theta = 0}$$

Way 3: $d\theta \in \mathcal{O}^3(\mathbb{R}^2)$ must be 0 ... because
of the dimension $(\mathcal{O}^k(M) = 0 \text{ } \forall k > \dim M)$,

(3)
Ex 2: Compute $d\theta$ for $\theta = yz dx \wedge dz$ on S^2
 (hence $\theta = yz dx \wedge dz|_{S^2} \in \wedge^2(S^2)$).

Again, Leibniz can be applied in various ways, as one can write:

$$\theta = yz \cdot (dx \wedge dz) = (yz dx) \wedge dz = (y dx) \wedge (z dz) = \cancel{yz dx dz} \\ = y \cdot (z dx \wedge dz) \text{ etc etc}$$

By theory $\Rightarrow$ doesn't matter which path you take!
 For instance, with first:

$$d\theta = \underbrace{d(yz)}_{\substack{\text{Leibniz} \\ \downarrow \\ y dz + z dy}} \wedge (dx \wedge dz) + yz \cdot \underbrace{d(dx \wedge dz)}_{\substack{\text{Leibniz} \Rightarrow \\ \frac{d(dx) \wedge dz}{0} - dx \wedge \frac{d(dz)}{0}}} \\ \Rightarrow d\theta = (y dz + z dy) \wedge dx \wedge dz = \underbrace{y dz \wedge dx \wedge dz}_{\text{gives 0.}} + \underbrace{z dy \wedge dx \wedge dz}_{\text{nicer to reorder.}}$$

$$\Rightarrow \boxed{d\theta = -z dx \wedge dy \wedge dz}$$

But this computation is really on $\mathbb{R}^3$ and on S^2

we can go further since: $x^2 + y^2 + z^2 = 1$ on $S^2 \Rightarrow$
 $\Rightarrow x dx + y dy + z dz = 0$ on $S^2 \Rightarrow z dz = -x dx - y dy$ on S^2

hence:

$$d\theta = dx \wedge dy \wedge (x dx + y dy) = \\ = \underbrace{dx \wedge dy \wedge x dx}_{0!} + \underbrace{dx \wedge dy \wedge y dy}_{0!} \\ \Rightarrow \boxed{d\theta = 0 \text{ on } S^2}$$

OR: directly: $d\theta$ is a 3-form on S^2 , $3 > 2 \Rightarrow$ must be 0!

(4)

And now: how to compute pull-backs in concrete examples

$$F^*: \mathcal{R}^k(N) \rightarrow \mathcal{R}^k(M) \text{ induced by } F: M \rightarrow N.$$

Use the basic properties:

(1) F^* is linear: $F^*(w+w') = F^*(w) + F^*(w')$ etc.

(2) F^* preserves $\wedge$: $F^*(w \wedge \eta) = F^*(w) \wedge F^*(\eta)$.

(3) F^* commutes with d : $F^*(dw) = d(F^*w)$.

~~and F^* commutes with restriction to $\mathcal{R}$~~

Example: $F: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $F(x,y) = (x+2y, x^7, \sin(xy))$.

$$\theta = yz \, dx \, dz \in \mathcal{R}^2(\mathbb{R}^3).$$

Actually, for more clarity, ~~let's~~ while we use (x,y) for the coordinates in $\mathbb{R}^2$, let's use (X,Y,Z) for the ones in $\mathbb{R}^3$. Hence

$$\theta = yz \, dx \wedge dz \in \mathcal{R}^2(\mathbb{R}^3).$$

Then:

$$F^*(\underbrace{yz \, dx}_{w} \wedge \underbrace{dz}_{\eta}) = \underbrace{F^*(yz \, dx)}_{(2)} \wedge \underbrace{F^*(dz)}_{\text{again (2) gives}}.$$

$$= F^*(y) F^*(z) \underbrace{F^*(dx)}_{\text{use (3)}} \wedge \underbrace{F^*(dz)}_{\text{use (3)}}$$

$$= F^*(y) F^*(z) d(F^*(x)) \wedge d(F^*(z)).$$

But what is $F^*(x)$? $x \circ F$! The first component of F !

$$\underbrace{(x+2y)}_{F^*(x)}, \underbrace{x^7}_{F^*(y)}, \underbrace{\sin(xy)}_{F^*(z)}.$$

$$\Rightarrow F^*(\theta) = \underbrace{x^7}_{F^*(y)} \cdot \underbrace{\sin(xy)}_{F^*(z)} \underbrace{d(x+2y)}_{dx+2dy} \wedge \underbrace{d(\sin(xy))}_{y \cos(xy) dx + x \cos(xy) dy}$$

... should get $x^7 \sin(xy) (y(xy) \cdot (x-2y) dx \wedge dy$.